

## Constructing simplicial branched covers

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**Abstract.** Izmestiev and Joswig described how to obtain a simplicial covering space (the *partial unfolding*) of a given simplicial complex, thus obtaining a simplicial branched cover [Adv. Geom. 3:191–255, 2003]. We present a large class of branched covers which can be constructed via the partial unfolding. In particular, for  $d \leq 4$  every closed oriented PL  $d$ -manifold is the partial unfolding of some polytopal  $d$ -sphere.

**Key words.** Geometric topology, construction of combinatorial manifolds, branched covers.

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### 1 Introduction

Branched covers are applied frequently in topology – most prominently in the study, construction and classification of closed oriented PL  $d$ -manifolds. First results are by Alexander [1] in 1920, who observed that any closed oriented PL  $d$ -manifold  $M$  is a branched cover of the  $d$ -sphere. Unfortunately Alexander’s proof does not allow for any (reasonable) control over the number of sheets of the branched cover, nor over the topology of the branching set: The number of sheets depends on the size of some triangulation of  $M$  and the branching set is the codimension 2-skeleton of the  $d$ -simplex.

However, in dimension  $d \leq 4$ , the situation is fairly well understood. By results of Hilden [8] and Montesinos [17] any closed oriented 3-manifold  $M$  arises as 3-fold simple branched cover of the 3-sphere branched over a link. In dimension four the situation becomes increasingly difficult. First Piergallini [21] showed how to obtain any closed oriented PL 4-manifold as a 4-fold branched cover of the 4-sphere branched over a transversally immersed PL-surface [21]. Iori & Piergallini [11] then improved the standing result showing that the branching set may be realized locally flat if one allows for a fifth sheet for the branched cover, thus proving a long-standing conjecture by Montesinos [18]. The

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question as to whether any closed oriented PL 4-manifold can be obtained as 4-fold cover of the 4-sphere branched over a locally flat PL-surface is still open.

For the partial unfolding and the construction of closed oriented combinatorial 3-manifolds we recommend Izmitiev & Joswig [14]. Their construction has recently been simplified significantly by Hilden, Montesinos, Tejada & Toro [9]. For those able to read German additional analysis and examples can be found in [24]. The partial unfolding is implemented in the software package `polymake` [6].

This work has been greatly inspired by a paper of Hilden, Montesinos, Tejada & Toro [9] and their bold approach. However, the techniques developed in the following turn out to differ substantially from the ideas in [9], allowing for stronger results in dimension three and generalization to arbitrary dimensions.

**Outline of the paper.** After some basic definitions and notations the partial unfolding  $\widehat{K}$  of a simplicial complex  $K$  is introduced. The partial unfolding defines a projection  $p : \widehat{K} \rightarrow K$  which is a simplicial branched cover if  $K$  meets certain connectivity assumptions. We define combinatorial models of key features of a branched cover, namely the branching set and the monodromy homomorphism.

Sections 2 and 3 are related, yet self-contained. The main result of this paper is presented in Theorem 2.1, and we give an explicit construction of a combinatorial  $d$ -sphere  $S$  such that  $p : \widehat{S} \rightarrow S$  is equivalent to a given simple,  $(d+1)$ -fold branched cover  $r : X \rightarrow \mathbb{S}^d$  (with some additional restriction for the branching set of  $r$ ). Theorem 2.1 is then applied to the construction of closed oriented PL  $d$ -manifolds as branched covers for  $d \leq 4$ . The construction of  $S$  and the proof of its correctness take up the entire Section 2.

Finally, in Section 3 we discuss how to extend a  $k$ -coloring of a subcomplex  $L \subset K$  of a simplicial  $d$ -complex  $K$  to a  $\max\{k, d+1\}$ -coloring of a refinement  $K'$  of  $K$  such that  $L$  is again a subcomplex of  $K'$ . Since  $K'$  is constructed from  $K$  via finitely many stellar subdivisions of edges, all properties invariant under these subdivisions are preserved, e.g. polytopality, regularity, shellability, and others. This improves an earlier result by Izmitiev [12].

**1.1 Basic definitions and notations.** A simplicial complex  $K$  is a *combinatorial  $d$ -sphere* or *combinatorial  $d$ -ball* if it is piecewise linear homeomorphic to the boundary of the  $(d+1)$ -simplex, respectively to the  $d$ -simplex. Equivalently,  $K$  is a combinatorial  $d$ -sphere or  $d$ -ball if there is a common refinement of  $K$  and the boundary of the  $(d+1)$ -simplex, respectively the  $d$ -simplex. A simplicial complex  $K$  is a *combinatorial manifold* if the vertex link of each vertex of  $K$  is a combinatorial sphere or a combinatorial ball. A manifold  $M$  is PL if and only if  $M$  has a triangulation as a combinatorial manifold. For an introduction to PL-topology see Björner [2, Part II], Hudson [10], and Rourke & Sanderson [22].

A finite simplicial complex is *pure* if all the inclusion maximal faces, called the *facets*, have the same dimension. We call a codimension 1-face of a pure simplicial complex  $K$  a *ridge*, and the *dual graph*  $\Gamma^*(K)$  of  $K$  has the facets as its node set, and two nodes are adjacent if they share a ridge. We denote the 1-skeleton of  $K$  by  $\Gamma(K)$ , its *graph*.

Further it is often necessary to restrict ourselves to simplicial complexes with certain connectivity properties: A pure simplicial complex  $K$  is *strongly connected* if its

dual graph  $\Gamma^*(K)$  is connected, and *locally strongly connected* if the star  $\text{st}_K(f)$  of  $f$  is strongly connected for each face  $f \in K$ . If  $K$  is locally strongly connected, then connected and strongly connected coincide. Further we call  $K$  *locally strongly simply connected* if for each face  $f \in K$  with codimension  $\geq 2$  the link  $\text{lk}_K(f)$  of  $f$  is simply connected, and finally,  $K$  is *nice* if it is locally strongly connected and locally strongly simply connected. Observe that combinatorial manifolds are always nice.

Let  $(\sigma_0, \sigma_1, \dots, \sigma_l)$  be an ordering of the facets of a pure simplicial  $d$ -complex  $K$ , and let  $D_i = \bigcup_{0 \leq j \leq i} \sigma_j$  denote the union of the first  $i$  facets. We call the ordering  $(\sigma_0, \sigma_1, \dots, \sigma_l)$  a *shelling* of  $K$  if  $D_{i-1} \cap \sigma_i$  is a pure simplicial  $(d - 1)$ -complex for  $1 \leq i \leq l$ . If  $K$  is the boundary complex of a simplicial  $(d + 1)$ -polytope, then  $K$  admits a shelling order which can be computed efficiently; see Ziegler [27, Chapter 8].

A simplicial complex obtained from a shellable complex by stellar subdivision of a face is again shellable, a shellable sphere or ball is a combinatorial sphere or ball, and for  $1 \leq i \leq l$  the intersection  $D_{i-1} \cap \sigma_i$  is a combinatorial  $(d - 1)$ -ball (or sphere). A shellable simplicial complex  $K$  is a wedge of balls or spheres in general. If  $K$  is a manifold, then  $D_i$  is a combinatorial  $d$ -ball (or sphere) for  $0 \leq i \leq l$ , and in particular we have that  $D_{i-1} \cap \sigma_i$ ,  $D_i$ , and hence  $K$  are nice. We call a face  $f \subset \sigma_i$  *free* if  $f \not\subset D_{i-1}$ . In particular the (inclusion) minimal free faces describe all free faces, and they are also called *restriction sets* in the theory of  $h$ -vectors of simplicial polytopes.

**1.2 The branched cover.** The concept of a covering of a space  $Y$  by another space  $X$  is generalized by Fox [4] to the notion of the branched cover. Here a certain subset  $Y_{\text{sing}} \subset Y$  may violate the conditions of a covering map. This allows for a wider application in the construction of topological spaces. It is essential for a satisfactory theory of (branched) coverings to make certain connectivity assumption for  $X$  and  $Y$ . The spaces mostly considered are Hausdorff, path connected, and locally path connected; see Bredon [3, III.3.1]. Throughout we will restrict our attention to coverings of manifolds, hence they meet the connectivity assumptions in [3].

Consider a continuous map  $h : Z \rightarrow Y$ , and assume the restriction  $h : Z \rightarrow h(Z)$  to be a covering. If  $h(Z)$  is dense in  $Y$  (and meets certain additional connectivity conditions) then there is a surjective map  $p : X \rightarrow Y$  with  $Z \subset X$  and  $p|_Z = h$ . The map  $p$  is called a *completion* of  $h$ , and any two completions  $p : X \rightarrow Y$  and  $p' : X' \rightarrow Y$  are equivalent in the sense that there exists a homeomorphism  $\varphi : X \rightarrow X'$  satisfying  $p' \circ \varphi = p$  and  $\varphi|_Z = \text{Id}$ . The map  $p : X \rightarrow Y$  obtained this way is a *branched cover*, and we call the unique minimal subset  $Y_{\text{sing}} \subset Y$  such that the restriction of  $p$  to the preimage of  $Y \setminus Y_{\text{sing}}$  is a covering the *branching set* of  $p$ . The restriction of  $p$  to  $p^{-1}(Y \setminus Y_{\text{sing}})$  is called the *associated covering* of  $p$ . If  $h : Z \rightarrow Y$  is a covering, then  $X = Z$ , and  $p = h$  is a branched cover with empty branching set.

**Example 1.1.** For  $k \geq 2$  consider the map  $p_k : \mathbb{C} \rightarrow \mathbb{C} : z \mapsto z^k$ . The restriction  $p_k|_{\mathbb{D}^2}$  is a  $k$ -fold branched cover  $\mathbb{D}^2 \rightarrow \mathbb{D}^2$  with the single branch point  $\{0\}$ .

We define the *monodromy homomorphism*

$$\mathfrak{m}_p : \pi_1(Y \setminus Y_{\text{sing}}, y_0) \rightarrow \text{Sym}(p^{-1}(y_0))$$

of a branched cover for a point  $y_0 \in Y \setminus Y_{\text{sing}}$  as the monodromy homomorphism of the associated covering: If  $[\alpha] \in \pi_1(Y \setminus Y_{\text{sing}}, y_0)$  is represented by a closed path  $\alpha$  based at  $y_0$ , then  $m_p$  maps  $[\alpha]$  to the permutation  $(x_i \mapsto \alpha_i(1))$ , where  $\{x_1, x_2, \dots, x_k\} = p^{-1}(y_0)$  is the preimage of  $y_0$  and  $\alpha_i : [0, 1] \rightarrow X$  is the unique lifting of  $\alpha$  with  $p \circ \alpha_i = \alpha$  and  $\alpha_i(0) = x_i$ ; see Munkres [19, Lemma 79.1] and Seifert & Threlfall [23, § 58]. The *monodromy group*  $\mathfrak{M}_p$  is defined as the image of  $m_p$ .

Two branched covers  $p : X \rightarrow Y$  and  $p' : X' \rightarrow Y'$  are *equivalent* if there are homeomorphisms  $\varphi : X \rightarrow X'$  and  $\psi : Y \rightarrow Y'$  with  $\psi(Y_{\text{sing}}) = Y'_{\text{sing}}$  such that  $p' \circ \varphi = \psi \circ p$  holds. The well-known Theorem 1.2 is due to the uniqueness of  $Y_{\text{sing}}$ , and hence the uniqueness of the associated covering; see Piergallini [20, p. 2].

**Theorem 1.2.** *Let  $p : X \rightarrow Y$  be a branched cover of a connected manifold  $Y$ . Then  $p$  is uniquely determined up to equivalence by the branching set  $Y_{\text{sing}}$  and the monodromy homeomorphism  $m_p$ . In particular, the covering space  $X$  is determined up to homeomorphisms.*

Let  $Y$  be a connected manifold and  $Y_{\text{sing}}$  a codimension 2 submanifold, possibly with a finite number of singularities. We call a branched cover  $p$  *simple* if the image  $m_p(m)$  of any meridial loop  $m$  around a non-singular point of the branching set is a transposition in  $\mathfrak{M}_p$ . Note that the  $k$ -fold branched cover  $p_k|_{\mathbb{D}^2} : \mathbb{D}^2 \rightarrow \mathbb{D}^2$  presented in Example 1.1 is not simple for  $k \geq 3$ .

**1.3 The partial unfolding.** The partial unfolding  $\widehat{K}$  of a simplicial complex  $K$  first appeared in a paper by Izmestiev & Joswig [14], with some of the basic notions already developed in Joswig [15]. The partial unfolding is closely related to the complete unfolding, also defined in [14], but we will not discuss the latter. The partial unfolding is a geometric object defined entirely by the combinatorial structure of  $K$ , and comes along with a canonical projection  $p : \widehat{K} \rightarrow K$ .

However, the partial unfolding  $\widehat{K}$  may not be a simplicial complex. In general  $\widehat{K}$  is a pseudo-simplicial complex: Let  $\Sigma$  be a collection of pairwise disjoint geometric simplices with simplicial attaching maps for some pairs  $(\sigma, \tau) \in \Sigma \times \Sigma$ , mapping a subcomplex of  $\sigma$  isomorphically to a subcomplex of  $\tau$ . Identifying the subcomplexes accordingly yields the quotient space  $\Sigma/\sim$ , which is called a *pseudo-simplicial complex* if the quotient map  $\Sigma \rightarrow \Sigma/\sim$  restricted to any  $\sigma \in \Sigma$  is bijective. The last condition ensures that there are no self-identifications within each simplex  $\sigma \in \Sigma$ .

**The group of projectivities.** Let  $\sigma$  and  $\tau$  be neighboring facets of a finite, pure simplicial complex  $K$ , that is,  $\sigma \cap \tau$  is a ridge. Then there is exactly one vertex in  $\sigma$  which is not a vertex of  $\tau$  and vice versa, hence a natural bijection  $\langle \sigma, \tau \rangle$  between the vertex sets of  $\sigma$  and  $\tau$  is given by

$$\langle \sigma, \tau \rangle : V(\sigma) \rightarrow V(\tau) : v \mapsto \begin{cases} v & \text{if } v \in \sigma \cap \tau, \\ \tau \setminus \sigma & \text{if } v = \sigma \setminus \tau. \end{cases}$$

The bijection  $\langle \sigma, \tau \rangle$  is called the *perspectivity* from  $\sigma$  to  $\tau$ .

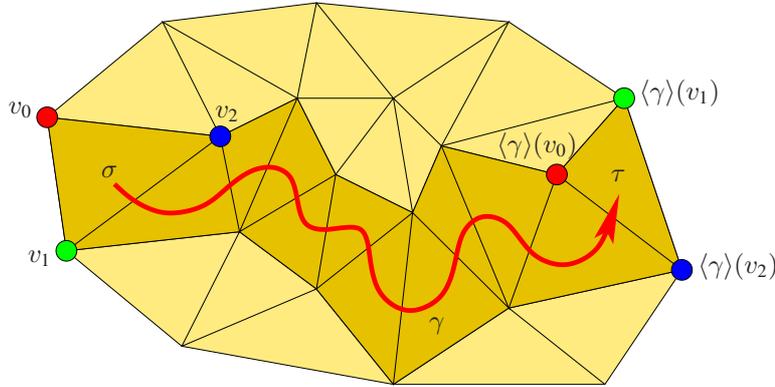


Figure 1. A projectivity from  $\sigma$  to  $\tau$  along the facet path  $\gamma$ .

A *facet path* in  $K$  is a sequence  $\gamma = (\sigma_0, \sigma_1, \dots, \sigma_k)$  of facets such that the corresponding nodes in the dual graph  $\Gamma^*(K)$  form a path, that is,  $\sigma_i \cap \sigma_{i+1}$  is a ridge for all  $0 \leq i < k$ ; see Figure 1. Now the *projectivity*  $\langle \gamma \rangle$  along  $\gamma$  is defined as the composition of perspectivities  $\langle \sigma_i, \sigma_{i+1} \rangle$ , thus  $\langle \gamma \rangle$  maps  $V(\sigma_0)$  to  $V(\sigma_k)$  bijectively via

$$\langle \gamma \rangle = \langle \sigma_{k-1}, \sigma_k \rangle \circ \dots \circ \langle \sigma_1, \sigma_2 \rangle \circ \langle \sigma_0, \sigma_1 \rangle.$$

We write  $\gamma\delta = (\sigma_0, \sigma_1, \dots, \sigma_k, \sigma_{k+1}, \dots, \sigma_{k+l})$  for the *concatenation* of two facet paths  $\gamma = (\sigma_0, \sigma_1, \dots, \sigma_k)$  and  $\delta = (\sigma_k, \sigma_{k+1}, \dots, \sigma_{k+l})$ , denote by  $\gamma^- = (\sigma_k, \sigma_{k-1}, \dots, \sigma_0)$  the *inverse path* of  $\gamma$ , and we call  $\gamma$  a *closed facet path* based at  $\sigma_0$  if  $\sigma_0 = \sigma_k$ . The set of closed facet paths based at  $\sigma_0$  together with the concatenation forms a group, and a closed facet path  $\gamma$  based at  $\sigma_0$  acts on the set  $V(\sigma_0)$  via  $\gamma \cdot v = \langle \gamma \rangle(v)$  for  $v \in V(\sigma_0)$ . Via this action we obtain the *group of projectivities*  $\Pi(K, \sigma_0)$  given by all permutations  $\langle \gamma \rangle$  of  $V(\sigma_0)$ . The group of projectivities is a subgroup of the symmetric group  $\text{Sym}(V(\sigma_0))$  on the vertices of  $\sigma_0$ .

The projectivities along null-homotopic closed facet paths based at  $\sigma_0$  generate the subgroup  $\Pi_0(K, \sigma_0)$  of  $\Pi(K, \sigma_0)$ , which is called the *reduced group of projectivities*. Finally, if  $K$  is strongly connected then  $\Pi(K, \sigma_0)$  and  $\Pi(K, \sigma'_0)$ , respectively  $\Pi_0(K, \sigma_0)$  and  $\Pi_0(K, \sigma'_0)$ , are isomorphic for any two facets  $\sigma_0, \sigma'_0 \in K$ . In this case we usually omit the base facet in the notation of the (reduced) group of projectivities, and write  $\Pi(K) = \Pi(K, \sigma_0)$ , respectively  $\Pi_0(K) = \Pi_0(K, \sigma_0)$ .

**The odd subcomplex.** Let  $K$  be locally strongly connected; in particular,  $K$  is pure. The link of a codimension 2-face  $f$  is a graph which is connected since  $K$  is locally strongly connected, and  $f$  is called *even* if the link  $\text{lk}_K(f)$  of  $f$  is 2-colorable (i.e. bipartite as a graph), and *odd* otherwise. We define the *odd subcomplex* of  $K$  as all odd codimension 2-faces (together with their proper faces), and denote it by  $K_{\text{odd}}$  (or sometimes  $\text{odd}(K)$ ).

Assume that  $K$  is pure and admits a  $(d + 1)$ -coloring of its graph  $\Gamma(K)$ , that is, we assign one color of a set of  $d + 1$  colors to each vertex of  $\Gamma(K)$  such that the two vertices

of any edge carry different colors. Observe that the  $(d+1)$ -coloring of  $K$  is minimal with respect to the number of colors, and is unique up to renaming the colors if  $K$  is strongly connected. Simplicial complexes that are  $(d+1)$ -colorable are called *foldable*, since a  $(d+1)$ -coloring defines a non-degenerate simplicial map of  $K$  to the  $(d+1)$ -simplex. In other places in the literature foldable simplicial complexes are sometimes called balanced.

**Lemma 1.3.** *The odd subcomplex of a foldable simplicial complex  $K$  is empty, and the group of projectivities  $\Pi(K, \sigma_0)$  is trivial. In particular we have  $\langle \gamma \rangle = \langle \delta \rangle$  for any two facet paths  $\gamma$  and  $\delta$  from  $\sigma$  to  $\tau$  for any two facets  $\sigma, \tau \in K$ .*

We leave the proof to the reader. As we will see in Theorem 1.4 the odd subcomplex is of interest in particular for its relation to  $\Pi_0(K, \sigma_0)$  of a nice simplicial complex  $K$ .

Consider a geometric realization  $|K|$  of  $K$ . To a given facet path  $\gamma = (\sigma_0, \sigma_1, \dots, \sigma_k)$  in  $K$  we associate a (piecewise linear) path  $|\gamma|$  in  $|K|$  by connecting the barycenter of  $\sigma_i$  to the barycenters of  $\sigma_i \cap \sigma_{i-1}$  and  $\sigma_i \cap \sigma_{i+1}$  by a straight line for  $1 \leq i < k$ , and connecting the barycenters of  $\sigma_0$  and  $\sigma_0 \cap \sigma_1$ , respectively  $\sigma_k$  and  $\sigma_k \cap \sigma_{k-1}$ . A projectivity *around* a codimension 2-face  $f$  is a projectivity along a facet path  $\gamma\delta\gamma^-$ , where  $\delta$  is a closed facet path in  $\text{st}_K(f)$  (based at some facet  $\sigma \in \text{st}_K(f)$ ) such that  $|\gamma|$  is homotopy equivalent to the boundary of a transversal disc around  $|f| \subset |\text{st}_K(f)|$ , and  $\gamma$  is a facet path from  $\sigma_0$  to  $\sigma$ . The path  $\gamma\delta\gamma^-$  is null-homotopic since  $K$  is locally strongly simply connected.

**Theorem 1.4** (Izmestiev & Joswig [14, Theorem 3.2.2]). *The reduced group of projectivities  $\Pi_0(K, \sigma_0)$  of a nice simplicial complex  $K$  is generated by projectivities around the odd codimension 2-faces. In particular,  $\Pi_0(K, \sigma_0)$  is generated by transpositions.*

The fundamental group  $\pi_1(|K| \setminus |K_{\text{odd}}|, y_0)$  of a nice simplicial complex  $K$  is generated by paths  $|\gamma|$ , where  $\gamma$  is a closed facet path based at  $\sigma_0$ , and  $y_0$  is the barycenter of  $\sigma_0$ ; see [14, Proposition A.2.1]. Furthermore, due to Theorem 1.4 we have the group homomorphism

$$\mathfrak{h}_K : \pi_1(|K| \setminus |K_{\text{odd}}|, y_0) \rightarrow \Pi(K, \sigma_0) : [|\gamma|] \mapsto \langle \gamma \rangle,$$

where  $[|\gamma|]$  is the homotopy class of the path  $|\gamma|$  corresponding to a facet path  $\gamma$ .

**The partial unfolding.** Let  $K$  be a pure simplicial  $d$ -complex and set  $\Sigma$  as the set of all pairs  $(|\sigma|, v)$ , where  $\sigma \in K$  is a facet and  $v \in \sigma$  is a vertex. Thus each pair  $(|\sigma|, v) \in \Sigma$  is a copy of the geometric simplex  $|\sigma|$  labeled by one of its vertices. For neighboring facets  $\sigma$  and  $\tau$  of  $K$  we define the equivalence relation  $\sim$  by attaching  $(|\sigma|, v) \in \Sigma$  and  $(|\tau|, w) \in \Sigma$  along their common ridge  $|\sigma \cap \tau|$  if  $\langle \sigma, \tau \rangle(v) = w$  holds. Now the *partial unfolding*  $\widehat{K}$  is defined as the quotient space  $\widehat{K} = \Sigma / \sim$ . The projection  $p : \widehat{K} \rightarrow K$  is given by the factorization of the map  $\Sigma \rightarrow K : (|\sigma|, v) \mapsto \sigma$ ; see Figure 2.

The partial unfolding of a strongly connected simplicial complex is not strongly connected in general. We denote by  $\widehat{K}_{(|\sigma|, v)}$  the connected component containing the labeled facet  $(|\sigma|, v)$ . Clearly,  $\widehat{K}_{(|\sigma|, v)} = \widehat{K}_{(|\tau|, w)}$  holds if and only if there is a facet path  $\gamma$  from  $\sigma$  to  $\tau$  in  $K$  with  $\langle \gamma \rangle(v) = w$ . It follows that the connected components of  $\widehat{K}$  correspond to the orbits of the action of  $\Pi(K, \sigma_0)$  on  $V(\sigma_0)$ . Note that each connected component of

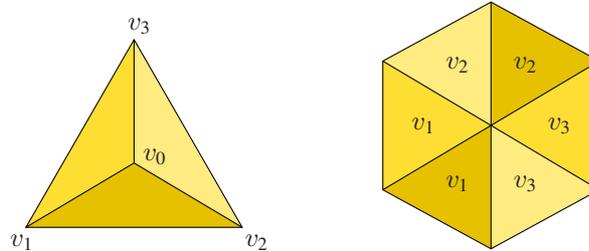


Figure 2. The starred triangle and its partial unfolding. The complex on the right is the non-trivial connected component of the partial unfolding, indicated by the labelling of the facets by the vertices  $v_1, v_2,$  and  $v_3$ . The second connected component is a copy of the starred triangle with all facets labelled  $v_0$ ; see also Example 1.1 for  $k = 2$ .

the partial unfolding is strongly connected and locally strongly connected [24, Satz 3.2.2]. Therefore we do not distinguish between connected and strongly connected components of the partial unfolding.

The problem that the partial unfolding  $\widehat{K}$  may not be a simplicial complex can be addressed in several ways. Izmetiev & Joswig [14] suggest barycentric subdivision of  $\widehat{K}$ , or anti-prismatic subdivision of  $K$ . A more efficient solution (with respect to the size of the resulting triangulations) is given in [24].

**1.4 The partial unfolding as a branched cover.** As preliminaries to this section we state two theorems by Fox [4] and Izmetiev & Joswig [14]. Together they imply that under the “usual connectivity assumptions” the partial unfolding of a simplicial complex is indeed a branched cover as suggested in the heading of this subsection.

**Theorem 1.5** (Izmetiev & Joswig [14, Theorem 3.3.2]). *Let  $K$  be a nice simplicial complex. Then the restriction of  $p : \widehat{K} \rightarrow K$  to the preimage of the complement of the odd subcomplex is a simple covering.*

**Theorem 1.6** (Fox [4, p. 251]; Izmetiev & Joswig [14, Proposition 4.1.2]). *Let  $J$  and  $K$  be nice simplicial complexes and let  $f : J \rightarrow K$  be a simplicial map. Then the map  $f$  is a simplicial branched cover if and only if*

$$\text{codim } K_{\text{sing}} \geq 2.$$

Since the partial unfolding of a nice simplicial complex is nice, Corollary 1.7 follows immediately.

**Corollary 1.7.** *Let  $K$  be a nice simplicial complex. The projection  $p : \widehat{K} \rightarrow K$  is a simple branched cover with the odd subcomplex  $K_{\text{odd}}$  as its branching set.*

For the rest of this section let  $K$  be a nice simplicial complex and let  $y_0$  be the barycenter of a fixed facet  $\sigma_0 \in K$ . The projection  $p : \widehat{K} \rightarrow K$  is a branched cover with  $K_{\text{sing}} = K_{\text{odd}}$  by Corollary 1.7, and Izmitiev & Joswig [14] proved that there is a bijection  $\iota : p^{-1}(y_0) \rightarrow V(\sigma_0)$  that induces a group isomorphism  $\iota_* : \text{Sym}(p^{-1}(y_0)) \rightarrow \text{Sym}(V(\sigma_0))$  such that the following Diagram (1) commutes.

$$\begin{array}{ccc}
 \pi_1(|K| \setminus |K_{\text{odd}}|, y_0) & & (1) \\
 \downarrow m_p & \searrow h_K & \\
 \mathfrak{M}_p & \xrightarrow{\iota_*} & \Pi(K, \sigma_0)
 \end{array}$$

Let  $r : X \rightarrow Y$  be a branched cover and assume that there is a homeomorphism of pairs  $\varphi : (Y, Y_{\text{sing}}) \rightarrow (|K|, |K_{\text{odd}}|)$ , that is,  $\varphi : Y \rightarrow |K|$  is a homeomorphism with  $\varphi(Y_{\text{sing}}) = |K_{\text{odd}}|$ . Then Theorem 1.8 gives sufficient conditions for  $p : \widehat{K} \rightarrow K$  and  $r : X \rightarrow Y$  to be equivalent branched covers. It is the key tool in the proof of the main Theorem 2.1 in Section 2.

**Theorem 1.8.** *Let  $K$  be a nice simplicial complex and let  $r : X \rightarrow Y$  be a (simple) branched cover. Further assume that there is a homeomorphism of pairs  $\varphi : (Y, Y_{\text{sing}}) \rightarrow (|K|, |K_{\text{odd}}|)$ , and let  $y_0 \in Y$  be a point such that  $\varphi(y_0)$  is the barycenter of  $|\sigma_0|$  for some facet  $\sigma_0 \in K$ . The branched covers  $p : \widehat{K} \rightarrow K$  and  $r : X \rightarrow Y$  are equivalent if there is a bijection  $\iota : r^{-1}(y_0) \rightarrow V(\sigma_0)$  that induces a group isomorphism  $\iota_* : \mathfrak{M}_r \rightarrow \Pi(K, \sigma_0)$  such that the diagram*

$$\begin{array}{ccc}
 \pi_1(Y \setminus Y_{\text{sing}}, y_0) & \xrightarrow{\varphi_*} & \pi_1(|K| \setminus |K_{\text{odd}}|, \varphi(y_0)) & (2) \\
 \downarrow m_r & & \downarrow h_K & \\
 \mathfrak{M}_r & \xrightarrow{\iota_*} & \Pi(K, \sigma_0) &
 \end{array}$$

commutes. In particular, we have  $\widehat{K} \cong X$ .

*Proof.* Corollary 1.7 ensures that  $p : \widehat{K} \rightarrow K$  is indeed a branched cover, and commutativity of Diagram (1) and Diagram (2) proves commutativity of their composition:

$$\begin{array}{ccccc}
 \pi_1(Y \setminus Y_{\text{sing}}, y_0) & \xrightarrow{\varphi_*} & \pi_1(|K| \setminus |K_{\text{odd}}|, \varphi(y_0)) & & \\
 \downarrow m_r & & \downarrow h_K & \searrow m_p & \\
 \mathfrak{M}_r & \xrightarrow{\iota_*} & \Pi(K, \sigma_0) & \xleftarrow{\iota_*} & \mathfrak{M}_p
 \end{array}$$

Theorem 1.2 completes the proof. □

## 2 Constructing branched covers

Throughout this section let  $r : X \rightarrow \mathbb{S}^d$  be a branched cover of the  $d$ -sphere with branching set  $F$ . The main objective is to give a large class of branched covers  $r$ , such that there

is a combinatorial sphere  $S$  with  $p : \widehat{S} \rightarrow S$  equivalent to  $r$  as a branched cover. In particular this implies the existence of a homeomorphism of pairs  $\varphi : (\mathbb{S}^d, F) \rightarrow (|S|, |S_{\text{odd}}|)$ . Note that by the nature of the partial unfolding and the projection  $p : \widehat{S} \rightarrow S$ , any branched cover  $r$  equivalent to  $p$  has to be simple and  $(d + 1)$ -fold. A theorem similar to Theorem 2.1 may easily be formulated for branched covers of  $d$ -balls.

Recall that we associate to a facet path  $\gamma$  in  $S$  the (realized) path  $|\gamma|$  in  $|S|$ , and that the square brackets denote the homotopy class of a closed path. Thus we write  $m_r([\varphi^{-1}(|\gamma|)])$  for the image of an element in  $\pi_1(\mathbb{S}^d \setminus F, y_0)$  represented by the closed path  $\varphi^{-1}(|\gamma|)$ , which in turn is obtained from a closed facet path  $\gamma$  based at some facet  $\sigma_0 \in S$  with barycenter  $\varphi(y_0)$  by first considering its realization  $|\gamma|$  and then its preimage under  $\varphi$ .

**Theorem 2.1.** *For  $d \geq 2$  let  $r : X \rightarrow \mathbb{S}^d$  be a  $(d + 1)$ -fold, simple branched cover of the  $d$ -sphere, and assume that the branching set  $F$  of  $r$  can be embedded via a homeomorphism  $\varphi : \mathbb{S}^d \rightarrow |S'|$  into the codimension 2-skeleton of a shellable simplicial  $d$ -sphere  $S'$ . Then there is a shellable simplicial  $d$ -sphere  $S$  such that  $p : \widehat{S} \rightarrow S$  is a branched cover equivalent to  $r$ . Furthermore, the  $d$ -sphere  $S$  can be obtained from  $S'$  by a finite series of stellar subdivision of edges. If  $S'$  is the boundary of a simplicial  $(d + 1)$ -polytope, then also  $S$  is the boundary of a simplicial  $(d + 1)$ -polytope.*

To make the proof of Theorem 2.1 more digestible we first give the (algorithmical) back-bone of the proof and defer some of the more technical aspects to the Lemmas 2.2, 2.3, and 2.4. Fix a point  $y_0 \in \mathbb{S}^d \setminus F$ ; we may assume  $\varphi(y_0)$  to be the barycenter of some facet  $\sigma_0 \in S'$  and  $|\sigma_0| \cap \varphi(F) = \emptyset$  to hold. Further fix a bijection  $\iota$  between the preimage  $\{x_0, x_1, \dots, x_d\} = r^{-1}(y_0)$  of  $y_0$  and the vertices of  $\sigma_0$ , and color the vertices of  $\sigma_0$  via  $\iota$  by the elements in  $r^{-1}(y_0)$ .

The  $d$ -sphere  $S$  is constructed in a finite series ( $S' = S_0, S_1, \dots, S_l = S$ ) of shellable  $d$ -spheres, and each  $d$ -sphere  $S_i$  comes with a shelling of its facet with marked beginning  $(\sigma_{i,0}, \sigma_{i,1}, \dots, \sigma_{i,l_i})$ . The complex  $S_{i+1}$  is obtained from  $S_i$  by (possibly) subdividing  $\sigma_{i,l_i+1}$  in a finite series of stellar subdivisions of edges not contained in any  $\sigma_{i,j}$  for  $0 \leq j \leq l_i$ . Thus we may choose the shelling of  $S_{i+1}$  such that it extends  $(\sigma_{i,0}, \sigma_{i,1}, \dots, \sigma_{i,l_i})$  and we denote the marked beginning of the shelling of  $S_i$  simply by  $(\sigma_0, \sigma_1, \dots, \sigma_{l_i})$ .

Let  $D_i = \bigcup_{0 \leq j \leq l_i} \sigma_j$ . Then the main idea of the proof of Theorem 2.1 is to construct  $S_i$  such that the branched covers  $r : X \rightarrow \mathbb{S}^d$  (restricted to  $\varphi^{-1}(|D_i|)$ ) and  $\widehat{D}_i \rightarrow D_i$  are equivalent. To this end we prove that  $\varphi$  restricted to  $\varphi^{-1}(|D_i|)$  is a homeomorphism of pairs  $(\varphi^{-1}(|D_i|), F \cap \varphi^{-1}(|D_i|)) \rightarrow (|D_i|, |\text{odd}(D_i)|)$  and that the following Diagram (3) commutes; see Figure 3.

$$\begin{array}{ccc}
 \pi_1(\varphi^{-1}(|D_i|) \setminus F, y_0) & \xrightarrow{\varphi_*} & \pi_1(|D_i| \setminus |\text{odd}(D_i)|, \varphi(y_0)) \\
 \downarrow m_r & & \downarrow h_{D_i} \\
 \mathfrak{M}_r & \xrightarrow{\iota_*} & \Pi(D_i, \sigma_0)
 \end{array} \tag{3}$$

Commutativity of Diagram (3) is obtained by ensuring that for each closed facet path  $\gamma$  in  $D_i$  (which is not a facet path in  $D_{i-1}$ ) the projectivity  $\langle \gamma \rangle$  acts on  $V(\sigma_0)$  as  $m_r([\varphi^{-1}(|\gamma|)])$  acts on  $r^{-1}(y_0)$ .

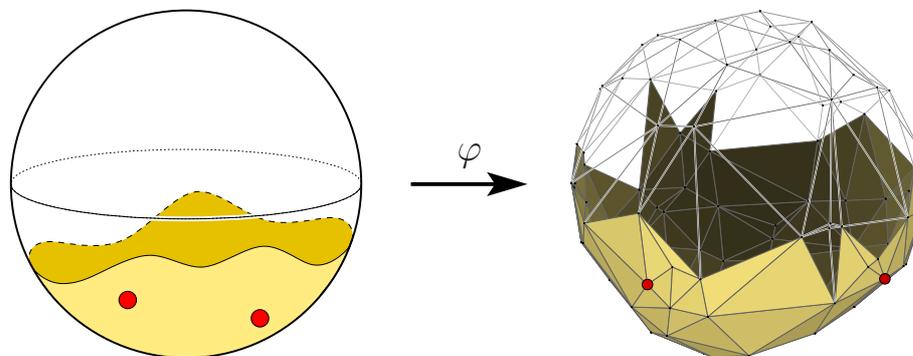


Figure 3. The base space of the branched cover  $r : X \rightarrow \mathbb{S}^2$  (left) and a polytopal 2-sphere  $S_i$  with marked beginning  $(\sigma_j)_{0 \leq j \leq l_i}$  of a shelling (right). On the left the preimage of  $D_i = \bigcup_{0 \leq j \leq l_i} \sigma_j$  under the homomorphism  $\varphi : \mathbb{S}^2 \rightarrow |S_i|$  is shaded and the branching set is marked. The odd subcomplex of  $D_i$  is marked on the right. The branched covers  $r : X \rightarrow \mathbb{S}^2$  (restricted to  $\varphi^{-1}(|D_i|)$ ) and  $\widehat{D}_i \rightarrow D_i$  are equivalent.

The pair  $(S_{i+1}, (\sigma_j)_{0 \leq j \leq l_{i+1}})$  is constructed from the pair  $(S_i, (\sigma_j)_{0 \leq j \leq l_i})$  as follows. Let  $\sigma = \sigma_{l_{i+1}}$  be the first facet in the shelling of  $S_i$  not contained in  $D_i$ , let  $\gamma$  be a facet path in  $D_i \cup \sigma$  from  $\sigma_0$  to  $\sigma$ , and let  $f \subset \sigma$  be a face. Further let  $H_{f,\gamma}$  be the subgroup of  $\mathfrak{M}_r$  which is induced via  $m_r$  by all elements of  $\pi_1(\mathbb{S}^d \setminus F, y_0)$  of the form  $[\varphi^{-1}(|\gamma\delta\gamma^{-1}|)]$ , where  $\delta$  is any closed facet path in  $\text{st}_{S_i}(f)$  based at  $\sigma$ . The subgroup  $H_{f,\gamma}$  has at least  $\dim(f) + 1$  trivial orbits, namely, the orbits corresponding to the vertices of  $f$ , and for  $g \subset f$  we have that the set of trivial orbits of  $H_{f,\gamma}$  contains the trivial orbits of  $H_{g,\gamma}$ . We consider the following three case:

- (i) The intersection  $\sigma \cap D_i$  is a ridge  $f$ . Let  $\gamma$  be a facet path in  $D_i \cup \sigma$  from  $\sigma_0$  to  $\sigma$ , and color  $\sigma$  (and hence  $f$ ) by the coloring induced along  $\gamma$  by the fixed coloring of  $\sigma_0$ . Now keep the coloring of  $f$ , but change the color of the remaining vertex  $v = \sigma \setminus f$  to any trivial orbit of  $H_{v,\gamma}$ ; see Figure 4 (right).
- (ii) The intersection  $\sigma \cap D_i$  equals two ridges  $f \cup v$  and  $f \cup w$  with a common codimension 2-face  $f$ . Let  $\sigma_v \in D_i$  be the facet intersecting  $\sigma$  in  $f \cup v$ , let  $\sigma_w \in D_i$  be the facet intersecting  $\sigma$  in  $f \cup w$ , and choose facet paths  $\gamma$  from  $\sigma_0$  to  $\sigma_v$  in  $D_i$  and  $\delta$  from  $\sigma_v$  to  $\sigma_w$  in  $\text{st}_{D_i}(f)$ . The fixed coloring of  $\sigma_0$  induces along  $\gamma$ , respectively  $\gamma\delta$ , colorings on  $f \cup v$  and  $f \cup w$ , and the colorings coincide on  $f$ . Now we change the color of  $w$  according to  $m_r([\varphi^{-1}(|\gamma\delta(\sigma_w, \sigma, \sigma_v)\gamma^{-1}|)])$ , which is either a transposition (changing the color of  $w$ ) or the identity; see Figure 4 (left).
- (iii) Otherwise set  $S_{i+1} = S_i$  and let  $(\sigma_0, \sigma_1, \dots, \sigma_{l_i}, \sigma)$  be the marked beginning of a shelling of  $S_{i+1}$ .

We obtained a (possibly inconsistent) coloring of the vertices of  $\sigma$  in the Cases (i) and (ii). Note that the coloring of  $\sigma$  induces a consistent coloring on  $D_i \cap \sigma$ , and that there is at most one *conflicting edge*  $\{v, w\}$ , that is,  $v$  and  $w$  are colored the same. A consistently colored subdivision of  $\sigma$  is constructed in at most  $d - 1$  subdivisions of  $\sigma$  with exactly



This completes the construction of  $S_{i+1}$  in the Cases (i) and (ii), and we define the marked beginning of a shelling of  $S_{i+1}$  by  $(\sigma_0, \sigma_1, \dots, \sigma_{i_i})$  followed by the facets of the refinement of  $\sigma$  in an appropriate order.

It remains to prove that the algorithm described above terminates and that  $p : \widehat{S} \rightarrow S$  is a branched cover equivalent to  $r : X \rightarrow \mathbb{S}^d$ . Since  $S$  is shellable and hence nice,  $p$  is a branched cover by see Corollary 1.7. The following Lemmas 2.2 and 2.3 prove the equivalence of  $p$  and  $r$ , while termination of the construction above is provided by Lemma 2.4.

**Lemma 2.2.** *The branched covers  $p : \widehat{S} \rightarrow S$  and  $r : X \rightarrow \mathbb{S}^d$  are equivalent.*

*Proof.* In order to show the equivalence of the branched covers  $p$  and  $r$  we prove by induction that the following holds for  $0 \leq i \leq l$ :

- (I) For any closed facet path  $\gamma$  based at  $\sigma_0$  in  $D_i$  we have  $\langle \gamma \rangle = \iota_* \circ m_r([\varphi^{-1}(|\gamma|)])$ .
- (II) Let  $v \in D_i$  be a vertex, and let  $\gamma$  be a facet path in  $D_i$  from  $\sigma_0$  to a facet  $\sigma$  containing  $v$ . Then the color induced on  $v$  along  $\gamma$  by the fixed coloring of  $\sigma_0$  is a trivial orbit of  $H_{v,\gamma}$ .

We remark that (I) implies that  $\varphi$  restricted to  $\varphi^{-1}(|D_i|)$  is a homeomorphism of pairs  $(\varphi^{-1}(|D_i|), F \cap \varphi^{-1}(|D_i|)) \rightarrow (|D_i|, |\text{odd}(D_i)|)$  and that Diagram (3) commutes. Finally, (I) and (II) are met for the pair  $(S_0, D_0) = (S', \sigma_0)$ , and commutativity of Diagram (3) proves the equivalence of  $r : X \rightarrow \mathbb{S}^d$  and  $p : \widehat{S} \rightarrow S$  for  $i = l$ ; see Theorem 1.8.

We show that (I) and (II) hold for the pair  $(S_{i+1}, D_{i+1})$  provided they hold for the pair  $(S_i, D_i)$ . Recall that we denote the first facet  $\sigma_{i+1}$  of the shelling of  $S_i$  not contained in  $D_i$  by  $\sigma$ . The simplicial complex  $D_i$  is contractible and hence  $\Pi_0(D_i, \sigma_0) = \Pi(D_i, \sigma_0)$  is generated by closed facet paths around (odd) codimension 2-faces by Theorem 1.4. Thus it suffices to verify (I) for closed facet paths around (interior) codimension 2-faces by examining the three Cases (i), (ii), and (iii).

- (i) The intersection  $\sigma \cap D_i$  is a ridge  $f$ . New interior codimension 2-faces in  $D_{i+1}$  arise only in the refinement of  $\sigma$ , which is foldable by construction. Since  $\varphi(F)$  does not intersect the interior of  $|\sigma|$ , any facet path around a new interior codimension 2-face corresponds to the identity of  $\mathfrak{M}_r$  and (I) holds by Lemma 1.3.
- (ii) The intersection  $\sigma \cap D_i$  equals two ridges  $f \cup v$  and  $f \cup w$  with a common codimension 2-face  $f$ . By induction hypothesis (II) holds for the vertices of  $f$  in  $D_i$  and thus (I) follows for the new interior codimension 2-face  $f$  of  $D_{i+1}$  by construction. As for any new interior codimension 2-face in the refinement of  $\sigma$ , (I) holds (as in Case (i)) since the refinement is foldable and  $\varphi(F)$  does not intersect the interior of  $|\sigma|$ .
- (iii) Otherwise there is no codimension 2-faces  $f \subset \sigma$  with a free corresponding edge  $e_f = \sigma \setminus f$  and (I) follows from Lemma 2.3.

Having established (I), it suffices to verify (II) for a single facet path  $\gamma$  in  $D_{i+1}$  from  $\sigma_0$  to any facet containing a given vertex  $v$ . Thus (II) holds by choice of color for any vertex added to  $D_i$  in the construction of the pair  $(S_{i+1}, D_{i+1})$ . □

**Lemma 2.3.** *If  $f \in \sigma$  is a codimension 2-face with a non-free corresponding edge  $e_f = \sigma \setminus f$ , then (I) holds for any closed facet path based at  $\sigma_0$  around  $f$  in  $D_{i+1}$ .*

*Proof.* Let  $\gamma\delta\gamma^{-1}$  be a closed facet path based at  $\sigma_0$  around  $f$  in  $D_{i+1}$ , where  $\delta$  is a closed path around  $f$  in  $\text{st}_{D_{i+1}}(f)$ . Since  $\{v, w\} = e_f$  is a non-free edge, there is a facet path  $\delta'$  in  $D_i$  with  $|\delta'|$  homotopy equivalent to  $|\{f_e, f \cup v, f \cup w\}|$  in  $|D_i| \setminus |\text{odd}(D_i)|$ , and we assume  $\delta$  and  $\delta'$  to have the same orientation; see Figure 6. Note that the complex  $\{f_e, f \cup v, f \cup w\}$  itself is homotopy equivalent to  $\mathbb{S}^1$ .

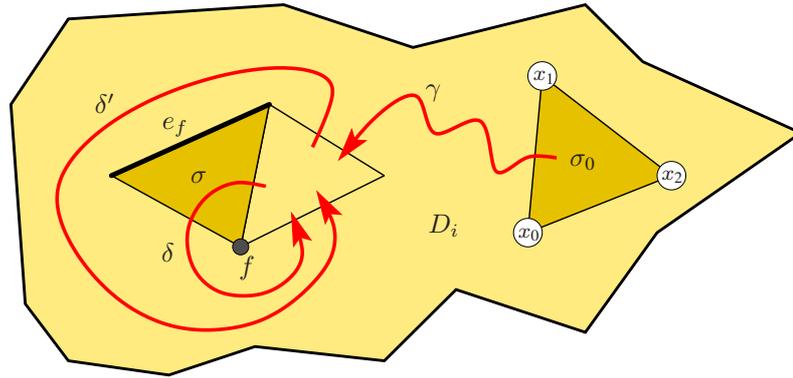


Figure 6. Case (iii): The paths  $\gamma$ ,  $\delta$ , and  $\delta'$  if the corresponding edge  $e_f$  of a codimension 2-face  $f$  is non-free.

Without loss of generality let  $m_r([\varphi^{-1}(|\gamma\delta\gamma^{-1}|)])$  either be the identity or the transposition  $(x_0, x_1) \in \mathfrak{M}_r$ . Each transposition  $(x_i, x_j)$ , for  $i \neq j$ , appears at most once in the (unique) reduced representation of the element  $a = m_r([\varphi^{-1}(|A|)]) \in \mathfrak{M}_r$  corresponding to the facet path  $A = \gamma\delta'\gamma^{-1}$ , since  $A$  is composed from facet paths around codimension 2-faces of  $\sigma$ . Let  $b = m_r([\varphi^{-1}(|B|)]) \in \mathfrak{M}_r$  denote the element corresponding to the facet path  $B = \gamma\delta'\delta^{-1}\gamma^{-1}$ , then  $a = (x_0, x_1) \circ b$  holds if and only if  $(x_0, x_1)$  is in the reduced representation of  $a$ , and we have  $a = b$  otherwise. Since (I) holds for  $D_i$  and hence in particular for the facet path  $A$ , and with

$$A = \gamma\delta'\gamma^{-1} = \gamma\delta'\delta^{-1}\gamma^{-1}\gamma\delta\gamma^{-1} = B\gamma\delta\gamma^{-1},$$

we conclude that the projectivity along  $\gamma\delta\gamma^{-1}$  is the identity on the vertices of  $\sigma_0$  if and only if  $\gamma\delta\gamma^{-1}$  corresponds via  $m_r \circ \varphi^{-1}$  to the identity in  $\mathfrak{M}_r$ , and exchanges exactly the vertices colored  $x_0$  and  $x_1$  otherwise.  $\square$

The following Lemma 2.4 proves termination of the construction of the shellable  $d$ -sphere  $S$  and completes the proof of Theorem 2.1.

**Lemma 2.4.** *The shellable  $d$ -sphere  $S$  is obtained by finitely many stellar subdivisions of edges.*

*Proof.* We prove that no facet will be subdivided more than a finite number of times in the construction of  $S$ . The facet  $\sigma_{l_i+i}$  is subdivided at most  $d - 1$  times in the construction

$S_{i+1}$  from  $S_i$ , and no facet in  $D_i$  is subdivided. The refinement of  $\sigma_{l_{i+1}}$  is added to  $D_i$  to define  $D_{i+1}$  and no facet in the refinement will be subdivided any further.

Problems may accrue since subdividing  $\sigma_{l_{i+1}}$  results in subdividing other facets (not in  $D_i$ ) intersecting  $\sigma_{l_{i+1}}$ , and each facet of the refinement of an intersecting facet appears in the shelling, yet is not in  $D_{i+1}$ . Thus a facet might get subdivided over and over again.

For a face  $f \in S'$  let  $L_{f,i} \subset S_i$  denote the refinement of  $f$  in  $S_i$ . Without loss of generality we may assume that the facets of the refinement  $L_{\sigma,i}$  of any facet  $\sigma \in S'$  appear consecutively in the shelling order of  $S_i$ . Let  $\sigma \in S'$  be a fixed facet and let  $i_0$  be the number such that  $S_{i_0}$  is the  $d$ -sphere with  $\sigma_{l_{i_0+1}}$  is the facet of  $L_{\sigma,i_0}$  appearing first in the shelling order, that is,  $S_{i_0+1}$  is constructed by adding (a refinement) of the first facet of  $L_{\sigma,i_0}$  to  $D_{i_0}$ . Thus we obtain an induced coloring of the boundary vertices of  $L_{\sigma,i_0}$  which is consistent on  $D_{i_0} \cap L_{\sigma,i_0}$  by construction. Since  $\varphi(F)$  does not intersect the interior of  $|L_{\sigma,i_0}|$  and by Lemma 1.3, it remains to prove that this coloring of  $D_{i_0} \cap L_{\sigma,i_0}$  extends to a foldable refinement of  $L_{\sigma,i_0}$  obtained via a finite series of stellar subdivisions.

Observe that each facet of  $L_{\sigma,i_0}$  is the cone over a  $(d-1)$ -simplex in the boundary of  $L_{\sigma,i_0}$  and that  $L_{\sigma,i_0}$  has no interior vertices: This is obviously true for  $L_{\sigma,0} = \sigma$ . For  $1 \leq i \leq i_0$  let  $\text{cone}(f)$  be a facet of  $L_{\sigma,i-1}$  with  $f$  a boundary  $(d-1)$ -simplex. Now if  $\text{cone}(f)$  is subdivided via stellarly subdividing an edge  $e \in f$ , both facets replacing  $\text{cone}(f)$  are cones over boundary  $(d-1)$ -simplices which in turn are obtained from  $f$  by replacing one vertex of  $e$  by the new vertex subdividing  $e$ .

We strengthen the statement above and claim that each facet of  $L_{\sigma,i_0}$  is the cone over a  $(d-1)$ -simplex in  $D_{i_0} \cap L_{\sigma,i_0}$ . To this end note the trivial fact that if  $e \in L_{g,i}$  is an edge of the subdivision of a boundary  $k$ -face  $g \in \sigma$  and if  $\{f_j\}_{1 \leq j \leq d-k}$  are the boundary  $(d-1)$ -faces of  $\sigma$  with  $g = \bigcap_{1 \leq j \leq d-k} f_j$ , then there is a  $(d-1)$ -simplex in each  $L_{f_j,i}$  containing  $e$ . Thus if for some  $i < i_0$  an edge  $e$  is subdivided when adding the simplex  $\sigma_{l_{i+1}}$  to  $D_i$  which intersects  $L_{\sigma,i}$  in a low dimensional face, then at least one of the boundary  $(d-1)$ -simplices of  $L_{\sigma,i}$  containing  $e$  will be added to  $D_{i'} \cap L_{\sigma,i'}$  at some point  $i < i' \leq i_0$ .

Returning to the consistent coloring of  $D_{i_0} \cap L_{\sigma,i_0}$  we conclude that all vertices of  $L_{\sigma,i_0}$  are colored since there are no interior vertices, and that each facet  $\text{cone}(f)$  of  $L_{\sigma,i_0}$  has at most one conflicting edge since the boundary  $(d-1)$ -simplex  $f \subset D_{i_0} \cap L_{\sigma,i_0}$  is consistently colored. Hence  $\text{st}_{L_{\sigma,i_0}}(e)$  of a conflicting edge  $e$  does not contain any other conflicting edges and we consider  $\text{st}_{L_{\sigma,i_0}}(e)$  independently.

Now  $\text{st}_{L_{\sigma,i_0}}(e)$  is subdivided only finitely many times since  $H_{v,\gamma}$  is trivial for any new vertex  $v$  (except for finitely many vertices in the boundary of  $|\text{st}_{L_{\sigma,i_0}}(e)|$ ) and hence the construction (Case (i) and (ii)) induces a linear order on the colors used to color the new vertices.  $\square$

**Remark 2.5.** It appears as if the shellable  $d$ -sphere  $S$  may be constructed along a spanning tree of the dual graph  $\Gamma^*(S')$  instead of a shelling, though the construction would become substantially more complicated. Using a spanning tree of  $\Gamma^*(S')$  would eliminate the somehow (to the theory of branched covers) alien concept of a shelling, and would allow for more general base spaces, e.g. PL  $d$ -manifolds.

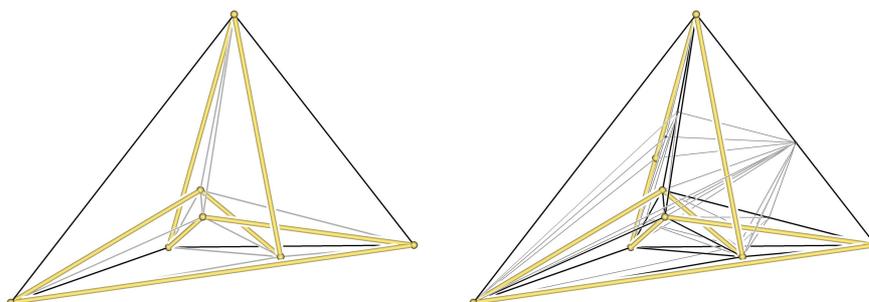


Figure 7. Constructing the trefoil as odd subcomplex of a 3-sphere  $S$  with  $\Pi(S)$  isomorphic to the symmetric group on three elements. On the left the Schlegel diagram of the cyclic 4-polytope  $C_{4,7}$  on seven vertices with the trefoil embedded in the 1-skeleton. On the right  $S$  as a subdivision of the Schlegel diagram after stellary subdividing eight edges of  $C_{4,7}$ . The odd subcomplex is marked and the  $f$ -vector of  $S$  reads  $(15, 63, 96, 48)$ ; watch [25].

Applying Theorem 2.1 to the results of Hilden [8] and Montesinos [17], Piergallini [21], and Iori & Piergallini [11] we obtain the following three corollaries.

**Corollary 2.6.** *Let  $d = 2$  or  $d = 3$ . For every closed oriented  $d$ -manifold  $M$  there is a polytopal  $d$ -sphere  $S$  such that one of the connected components  $\hat{S}$  of the partial unfolding of  $S$  is a combinatorial  $d$ -manifold homeomorphic to  $M$ . The projection  $\hat{S} \rightarrow S$  is a simple  $d$ -fold branched cover branched over finitely many points for  $d = 2$ , respectively a link for  $d = 3$ .*

**Corollary 2.7.** *For every closed oriented PL 4-manifold  $M$  there is a polytopal 4-sphere  $S$  such that one of the connected components  $\hat{S}$  of the partial unfolding of  $S$  is a combinatorial 4-manifold PL-homeomorphic to  $M$ . The projection  $\hat{S} \rightarrow S$  is a simple 4-fold branched cover branched over a PL-surface with a finite number of cusp and node singularities.*

**Corollary 2.8.** *For every closed oriented PL 4-manifold  $M$  there is a polytopal 4-sphere  $S$  such that the partial unfolding  $\hat{S}$  of  $S$  is a combinatorial 4-manifold PL-homeomorphic to  $M$ . The projection  $\hat{S} \rightarrow S$  is a simple 5-fold branched cover branched over a locally flat PL-surface.*

A weaker version of Corollary 2.6 was already established by Izmistiev & Joswig [14] and later by Hilden, Montesinos, Tejada & Toro [9]. A weaker version of Corollary 2.7 can be found in [26].

Stellar subdivision of an edge  $e \in S$  of a combinatorial  $d$ -manifold  $S$  changes the parity of the codimension 2-faces in  $\text{lk}_S(e)$ . Since the link of an edge of  $S$  is a (combinatorial)  $(d - 2)$ -sphere, we obtain the following Corollary 2.9. A topological proof for arbitrary simple branched covers is available by Izmistiev [13].

**Corollary 2.9.** *The branching set of a branched cover  $r : X \rightarrow \mathbb{S}^d$  as described in Theorem 2.1 is the symmetric difference of finitely many  $(d - 2)$ -spheres.*

We conclude this section by a remark and a conjecture as to which branched covers  $r : X \rightarrow \mathbb{S}^d$  may be obtained via the method presented above. In other words, which branching sets can be embedded via a homeomorphism  $\varphi : \mathbb{S}^d \rightarrow |S'|$  into the codimension 2-skeleton of a shellable simplicial  $d$ -sphere  $S'$ .

**Remark 2.10.** For  $d \geq 6$  there are branching sets non-embeddable into the codimension 2-skeleton of a shellable simplicial  $d$ -sphere: Freedman & Quinn [5] constructed a 4-manifold which does not have a triangulation as a combinatorial manifold. In fact, there are 4-manifolds which cannot be triangulated at all [16, p. 9].

The branching set of a branched cover  $r : X \rightarrow \mathbb{S}^d$  for  $d \leq 5$  is at most 3-dimensional and since there is no difference between PL and non-PL topology up to dimension three, we conjecture the following.

**Conjecture 2.11.** For  $d \leq 5$  every branched cover  $r : X \rightarrow \mathbb{S}^d$  can be obtained via the partial unfolding of some polytopal  $d$ -sphere.

### 3 Extending triangulations

A first assault on how to extend triangulation and coloring is by Goodman & Onishi [7], who proved that a 4-colorable triangulation of the 2-sphere may be extended to a 4-colorable triangulation of the 3-ball. Their result was improved independently by Izmistiev [12] and [24] to arbitrary dimensions. Here we generalize the construction to arbitrary simplicial complexes with  $k$ -colored subcomplexes.

**Theorem 3.1.** *Given a simplicial  $d$ -complex  $K$  and a  $k$ -colored induced subcomplex  $L$ , then there is a finite series of stellar subdivisions of edges such that the resulting simplicial complex  $K'$  has a  $\max\{k, d + 1\}$ -coloring,  $K'$  contains  $L$  as an induced subcomplex, and the  $\max\{k, d + 1\}$ -coloring of  $K'$  induces the original  $k$ -coloring on  $L$ .*

*Proof.* We may assume  $K$  to be pure. Let  $K_0 = K$  and assign 0 to all vertices not in  $L$ . For  $1 \leq i \leq d$  we obtain the simplicial complex  $K_i$  from  $K_{i-1}$  by stellarly subdividing all conflicting edges with both vertices colored  $i - 1$  in an arbitrary order. The new vertices are colored  $i$ . We prove by induction that for  $0 \leq j \leq i - 1$  and each facet  $\sigma \in K_i$  there is exactly one vertex  $v_j \in \sigma$  colored  $j$ . The assumption holds for  $K_0$  and completes the proof for  $K' = K_d$ . Note that since  $L$  is properly colored, no edges in  $L$  are subdivided and  $L$  is an induced subcomplex of any  $K_i$  for  $0 \leq i \leq d$ .

To prove the induction hypothesis for  $K_i$ , we again use an inductive argument: Let  $\sigma$  be a facet of a subdivision of  $K_{i-1}$  produced in the making of  $K_i$ . Assume that each color less than  $i - 1$  appears exactly once in  $\sigma$ , and let  $l \geq 2$  be the number of  $(i - 1)$ -colored vertices of  $\sigma$ . This assumption clearly holds for any facet of  $K_{i-1}$  for some  $l \leq d - i + 2$ . After subdividing an  $(i - 1)$ -colored conflicting edge of  $\sigma$  and assigning the color  $i$  to the

new vertex, each of the two new facets has  $l - 1$  vertices colored  $i - 1$ , and each color less than  $i - 1$  appears exactly once. Thus any facet of  $K_{i-1}$  has to be subdivided into at most  $2^{d-i+1}$  simplices in order for  $K_i$  to meet the induction hypothesis.  $\square$

Izmestiev gives a result similar to Theorem 3.1 in [12], but the following Remark 3.2 points out the advantage of using only stellar subdivisions of edges.

**Remark 3.2.** Since only stellar subdivisions of edges are used to construct  $K'$  from  $K$ , all properties invariant under these subdivisions are preserved, e.g. polytopality, regularity, shellability, and others. In the case that  $L$  is not induced, stellarly subdivide all edges  $\{v, w\} \in K \setminus L$  with  $v, w \in L$ . In order to obtain a small triangulation, one can try to (greedily)  $(d + 1)$ -color a (large) foldable subcomplex first.

**Corollary 3.3.** *The odd subcomplex of a closed combinatorial  $d$ -manifold is the symmetric difference of finitely many  $(d - 2)$ -spheres.*

**Corollary 3.4.** *Given a  $k$ -colored simplicial  $(d - 1)$ -sphere  $S$ , then there is a simplicial  $d$ -ball  $D$  with boundary equal to  $S$  such that there is a  $\max\{k, d + 1\}$ -coloring of  $D$  which induces the original  $k$ -coloring on  $S$ . The  $d$ -ball  $D$  is obtained from  $\text{cone}(S)$  by a finite series of stellar subdivision of edges. In particular  $D$  is a combinatorial  $d$ -ball if  $S$  is a combinatorial  $(d - 1)$ -sphere, shellable if  $S$  is shellable, and regular if  $S$  is polytopal; see Figure 8.*

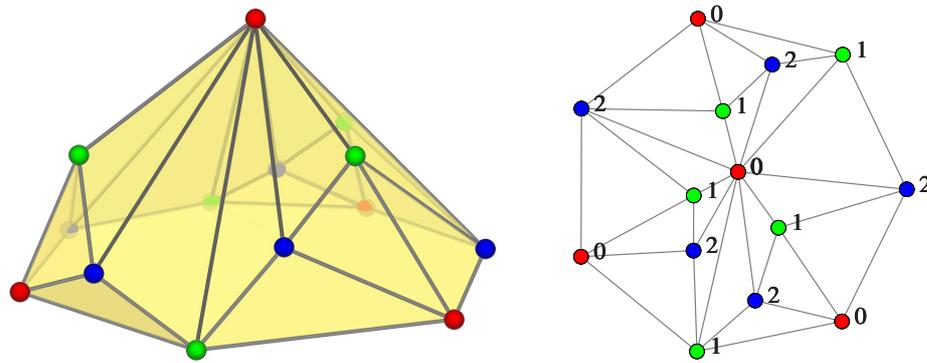


Figure 8. Convex hull of the extended triangulation of a 3-colored 7-gon and its Schlegel diagram.

**Remark 3.5.** Similar results as Corollary 3.4 may easily be obtained for partial triangulations of a CW-complex and relative handle-body decompositions of a PL-manifold (with boundary).

The partial unfoldings of two homeomorphic simplicial complexes  $K$  and  $K'$  need not to be homeomorphic in general. We present a notion of equivalence of simplicial complexes which agrees with their unfolding behavior, that is, we give sufficient criteria such that if  $p : \widehat{K} \rightarrow K$  and  $p' : \widehat{K}' \rightarrow K'$  are branched covers, then  $p$  and  $p'$  are equivalent.

Assume  $K$  and  $K'$  to be strongly connected and that the odd subcomplexes  $K_{\text{odd}}$  and  $K'_{\text{odd}}$  are *equivalent*, that is, there is a homeomorphism of pairs  $\varphi : (|K|, |K_{\text{odd}}|) \rightarrow (|K'|, |K'_{\text{odd}}|)$ . Let  $\sigma_0 \in K$  be a facet, and  $y_0$  the barycenter of  $\sigma_0$ , and assume that the image  $y'_0 = \varphi(y_0)$  is the barycenter of  $|\sigma'_0|$  for some facet  $\sigma'_0 \in K'$ . Now  $K$  and  $K'$  are *color equivalent* if there is a bijection  $\psi : V(\sigma_0) \rightarrow V(\sigma'_0)$ , such that

$$\psi_* \circ \mathfrak{h}_K = \mathfrak{h}_{K'} \circ \varphi_* \tag{4}$$

holds, where the maps  $\varphi_* : \pi_1(|K| \setminus |K_{\text{odd}}|, y_0) \rightarrow \pi_1(|K'| \setminus |K'_{\text{odd}}|, y'_0)$  and  $\psi_* : \text{Sym}(V(\sigma_0)) \rightarrow \text{Sym}(V(\sigma'_0))$  are the group isomorphisms induced by  $\varphi$  and  $\psi$ , respectively.

Observe that this is indeed an equivalence relation. The name ‘‘color equivalent’’ suggests that the pairs  $(K, K_{\text{odd}})$  and  $(K', K'_{\text{odd}})$  are equivalent, and that the ‘‘colorings’’ of  $K_{\text{odd}}$  and  $K'_{\text{odd}}$  by the  $\Pi(K)$ -action, respectively  $\Pi(K')$ -action, of projectivities around odd faces are equivalent. Lemma 3.6 justifies this name.

**Lemma 3.6.** *Let  $K$  and  $K'$  be color equivalent nice simplicial complexes. Then the branched covers  $p : \widehat{K} \rightarrow K$  and  $p' : \widehat{K}' \rightarrow K'$  are equivalent.*

*Proof.* With the notation of Equation (4) we have that

$$\begin{array}{ccccc}
 \pi_1(|K| \setminus |K_{\text{odd}}|, y_0) & \xrightarrow{\varphi_*} & \pi_1(|K'| \setminus |K'_{\text{odd}}|, y'_0) & & \\
 \swarrow m_p & & \downarrow \mathfrak{h}_{K'} & \swarrow m_{p'} & \\
 \mathfrak{M}_p & \xrightarrow{\psi_*} & \Pi(K, \sigma_0) & \xrightarrow{\psi_*} & \Pi(K', \sigma'_0) & \xleftarrow{\psi'_*} & \mathfrak{M}_{p'} \\
 \uparrow i_* & & \downarrow \mathfrak{h}_K & & \downarrow \mathfrak{h}_{K'} & & \uparrow i'_*
 \end{array}$$

commutes, since Diagram (1) commutes and Equation (4) holds. Theorem 1.2 completes the proof.  $\square$

**Proposition 3.7.** *For every strongly connected simplicial complex  $K$  there is a simplicial complex  $K'$ , obtained from a foldable simplicial complex via a finite series of stellar subdivision of edges, such that  $K$  and  $K'$  are color equivalent.*

Theorem 2.1 proves Proposition 3.7 above for shellable spheres. We will not prove the general case and only give a sketch of the construction for general  $K$ .

Let  $L$  be a foldable simplicial complex obtained from  $K$  via a finite series of stellar subdivisions according to Theorem 3.1, that is, there is a series  $(K = K_0, K_1, \dots, K_l = L)$  where  $K_i$  is obtained from  $K_{i-1}$  by stellarly subdividing a single edge  $e_{i-1} \in K_{i-1}$ . The idea is to reverse the effect of the stellar subdivisions by subdividing each edge  $e$  a second time in the reversed order, since stellarly subdividing  $e$  twice yields the anti-prismatic subdivision of  $e$  (which does not alter the color equivalence class).

We construct  $K'$  from  $L$  inductively in a series ( $L = L_l, L_{l-1}, \dots, L_0 = K'$ ) of simplicial complexes, where  $L_i$  is obtained from  $L_{i+1}$  by a finite series of stellar subdivisions of edges. The complexes  $L_i$  and  $K_i$  are color equivalent: For a facet path  $(\sigma'_j)_{j \in J}$  in  $L_i$  associate the facet path  $(\sigma_j)_{j \in J}$  in  $K_i$ , where  $\sigma_j$  is the unique facet such that  $|\sigma'_j|$  lies in  $|\sigma_j|$ .

We fix some notation in order to describe the construction of  $L_i$  from  $L_{i+1}$ . Subdividing the edge  $e_i \in K_i$  in order to construct  $K_{i+1}$  replaces  $e_i$  by two edges in  $K_{i+1}$ , and we call one of these two edges  $e'_i$ . A facet in  $\text{st}_{K_{i+1}}(e'_i)$  might get subdivided further in the process of constructing  $K_{i+2}, K_{i+3}, \dots, K_l = L_l, L_{l-1}, \dots, L_{i+1}$ , and we define  $L_{e'_i}$  as the subcomplex of  $L_{i+1}$  which refines  $\text{st}_{K_{i+1}}(e'_i)$ .

Note that  $e'_i$  is an edge of  $L_{e'_i}$ , and that  $L_{e'_i}$  and  $\text{st}_{K_{i+1}}(e'_i)$  are color equivalent. It follows that the group of projectivities of  $L_{e'_i}$  has at least two trivial orbits corresponding to the vertices of  $e'_i$ . Now  $L_i$  is obtained from  $L_{i+1}$  by stellarly subdividing all edges with vertices belonging to the same two trivial orbits as the vertices of  $e'_i$ .

## References

- [1] J. W. Alexander, Note on Riemann spaces. *Bull. Amer. Math. Soc.* **26** (1920), 370–372. [MR1560318](#) [Zbl 47.0529.02](#)
- [2] A. Björner, Topological methods. In: *Handbook of combinatorics, Vol. 1, 2*, 1819–1872, Elsevier, Amsterdam 1995. [MR1373690](#) (96m:52012) [Zbl 0851.52016](#)
- [3] G. E. Bredon, *Topology and geometry*. Springer 1997. [MR1700700](#) (2000b:55001) [Zbl 0934.55001](#)
- [4] R. H. Fox, Covering spaces with singularities. In: *A symposium in honor of S. Lefschetz*, 243–257, Princeton Univ. Press 1957. [MR0123298](#) (23 #A626) [Zbl 0079.16505](#)
- [5] M. H. Freedman, F. Quinn, *Topology of 4-manifolds*. Princeton Univ. Press 1990. [MR1201584](#) (94b:57021) [Zbl 0705.57001](#)
- [6] E. Gawrilow, M. Joswig, `polymake`, version 2.1. with contributions by Rörig, T. and Witte, N., free software, 1997–2004. <http://www.math.tu-berlin.de/polymake>
- [7] J. E. Goodman, H. Onishi, Even triangulations of  $S^3$  and the coloring of graphs. *Trans. Amer. Math. Soc.* **246** (1978), 501–510. [MR515556](#) (80a:05092) [Zbl 0397.05021](#)
- [8] H. M. Hilden, Three-fold branched coverings of  $S^3$ . *Amer. J. Math.* **98** (1976), 989–997. [MR0425968](#) (54 #13917) [Zbl 0342.57002](#)
- [9] M. Hilden, J. M. Montesinos, D. Tejada, M. Toro, Representing 3-manifolds by triangulations of  $S^3$ : a constructive approach. *Rev. Colombiana Mat.* **39** (2005), 63–86. [MR2218370](#) (2007c:57029) [Zbl 1130.57008](#)
- [10] J. F. P. Hudson, *Piecewise linear topology*. Benjamin, New York 1969. [MR0248844](#) (40 #2094) [Zbl 0189.54507](#)
- [11] M. Iori, R. Piergallini, 4-manifolds as covers of the 4-sphere branched over non-singular surfaces. *Geom. Topol.* **6** (2002), 393–401. [MR1914574](#) (2003f:57007) [Zbl 1021.57003](#)
- [12] I. Zmestiev, Extension of colorings. *European J. Combin.* **26** (2005), 779–781. [MR2127696](#) (2005j:05032) [Zbl 1066.05065](#)

- [13] I. Izmistiev, Addendum to “Branched coverings, triangulations, and 3-manifolds”. Personal communication, 2007.
- [14] I. Izmistiev, M. Joswig, Branched coverings, triangulations, and 3-manifolds. *Adv. Geom.* **3** (2003), 191–225. [MR1967999 \(2004b:57004\)](#) [Zbl 1025.57008](#)
- [15] M. Joswig, Projectivities in simplicial complexes and colorings of simple polytopes. *Math. Z.* **240** (2002), 243–259. [MR1900311 \(2003f:05047\)](#) [Zbl 1054.05039](#)
- [16] F. H. Lutz, *Triangulated manifolds with few vertices and vertex-transitive group actions*. Verlag Shaker, Aachen 1999. [MR1866007 \(2002h:57034\)](#) [Zbl 0977.57030](#)
- [17] J. M. Montesinos, Three-manifolds as 3-fold branched covers of  $S^3$ . *Quart. J. Math. Oxford Ser. (2)* **27** (1976), 85–94. [MR0394630 \(52 #15431\)](#)
- [18] J. M. Montesinos, 4-manifolds, 3-fold covering spaces and ribbons. *Trans. Amer. Math. Soc.* **245** (1978), 453–467. [MR511423 \(80k:57001\)](#) [Zbl 0359.55002](#)
- [19] J. R. Munkres, *Topology: a first course*. Prentice-Hall Inc., Englewood Cliffs, N.J. 1975. [MR0464128 \(57 #4063\)](#) [Zbl 0306.54001](#)
- [20] R. Piergallini, Manifolds as branched covers of spheres. In: *Proceedings of the Eleventh International Conference of Topology (Trieste, 1993)*, volume 25, 419–439 (1994), 1993. [MR1346337 \(96g:57004\)](#) [Zbl 0861.57004](#)
- [21] R. Piergallini, Four-manifolds as 4-fold branched covers of  $S^4$ . *Topology* **34** (1995), 497–508. [MR1341805 \(96g:57003\)](#) [Zbl 0869.57002](#)
- [22] C. P. Rourke, B. J. Sanderson, *Introduction to piecewise-linear topology*. Springer 1972. [MR0350744 \(50 #3236\)](#) [Zbl 0254.57010](#)
- [23] H. Seifert, W. Threlfall, *Seifert and Threlfall: a textbook of topology*, volume 89 of *Pure and Applied Mathematics*. Academic Press 1980. [MR575168 \(82b:55001\)](#) [Zbl 0469.55001](#)
- [24] N. Witte, Entfaltung simplizialer Sphären. Diplomarbeit, TU Berlin, 2004. <http://www.math.tu-berlin.de/~witte/diplom/>
- [25] N. Witte, Constructing the trefoil as odd subcomplex of a 3-sphere (movie). 2007. [http://www.math.tu-berlin.de/~witte/witte\\_files/cyc.mov](http://www.math.tu-berlin.de/~witte/witte_files/cyc.mov)
- [26] N. Witte, Foldable triangulations. Dissertation, TU Darmstadt, 2007. <http://elib.tu-darmstadt.de/diss/000788>
- [27] G. M. Ziegler, *Lectures on polytopes*. Springer 1995. [MR1311028 \(96a:52011\)](#) [Zbl 0823.52002](#)

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