

Switching of generalized quadrangles of order s and applications

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Dedicated to A. Barlotti at the occasion of his 80th birthday

Abstract. Let $\mathcal{S} = (P, B, I)$ be a generalized quadrangle of order s , $s \neq 1$, having a flag (x, L) with x and L regular. Then a generalized quadrangle $\mathcal{S}' = (P', B', I')$ of order s can be constructed. We say that \mathcal{S}' is obtained by switching from \mathcal{S} with respect to (x, L) . Examples are given where $\mathcal{S} \not\cong \mathcal{S}'$; e.g., starting from a $T_2(O)$ of Tits, with O an oval of $\text{PG}(2, q)$, q even, with nucleus n , the GQ $T_2((O - \{x\}) \cup \{n\})$ with $x \in O$ can be obtained by switching from $T_2(O)$. Applications to translation generalized quadrangles of order s , s even, and generalized quadrangles of order (s, s^2) , s even, satisfying Property (G), are given.

1 Introduction

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (P, B, I)$ in which P and B are disjoint (nonempty) sets of objects called *points* and *lines*, respectively, and for which $I \subseteq (P \times B) \cup (B \times P)$ is a symmetric point-line *incidence relation* satisfying the following axioms.

- (i) Each point is incident with $1 + t$ lines ($t \geq 1$) and two distinct points are incident with at most one line.
- (ii) Each line is incident with $1 + s$ points ($s \geq 1$) and two distinct lines are incident with at most one point.
- (iii) If x is a point and L is a line not incident with x , then there is a unique pair $(y, M) \in P \times B$ for which $xIMyIL$.

Generalized quadrangles were introduced by Tits [15] in his celebrated work on triality.

The integers s and t are the *parameters* of the GQ and \mathcal{S} is said to have *order* (s, t) ; if $s = t$, then \mathcal{S} is said to have *order* s . There is a point-line duality for GQ (of order (s, t)) for which in any definition or theorem the words “point” and “line” are interchanged and the parameters s and t are interchanged. Hence, we assume without further notice that the dual of a given theorem or definition has also been given.

Let $\mathcal{S} = (P, B, I)$ be a (finite) GQ of order (s, t) . Then \mathcal{S} has $v = |P| =$

$(1 + s)(1 + st)$ points and $b = |B| = (1 + t)(1 + st)$ lines; see 1.2.1 of Payne and Thas [8]. Also, $s + t$ divides $st(1 + s)(1 + t)$, and, for $s \neq 1 \neq t$, we have $t \leq s^2$, and, dually, $s \leq t^2$; see 1.2.2 and 1.2.3 of Payne and Thas [8].

Given two (not necessarily distinct) points x, y of \mathcal{S} , we write $x \sim y$ and say that x and y are *collinear*, provided that there is some line L for which $x \text{ I } L \text{ I } y$. And $x \not\sim y$ means that x and y are not collinear. Dually, for $L, M \in B$, we write $L \sim M$ or $L \not\sim M$ according as L and M are *concurrent* or *non-concurrent*, respectively. The line which is incident with distinct collinear points x, y is denoted by xy ; the point which is incident with distinct concurrent lines L, M is denoted by either LM or $L \cap M$.

For $x \in P$, put $x^\perp = \{y \in P \mid y \sim x\}$, and note that $x \in x^\perp$. If $A \subseteq P$, then $A^\perp = \bigcap \{x^\perp \mid x \in A\}$. Hence, for $x, y \in P$, $x \neq y$, we have $\{x, y\}^\perp = x^\perp \cap y^\perp$; we have $|\{x, y\}^\perp| = s + 1$ or $t + 1$ according as $x \sim y$ or $x \not\sim y$. Further, $\{x, y\}^{\perp\perp} = \{u \in P \mid u \in z^\perp \text{ for all } z \in x^\perp \cap y^\perp\}$; we have $|\{x, y\}^{\perp\perp}| = s + 1$ or $|\{x, y\}^{\perp\perp}| \leq t + 1$ according as $x \sim y$ or $x \not\sim y$. The sets $\{x, y\}^\perp$ and $\{x, y\}^{\perp\perp}$ are respectively called the *trace* and the *span* of the pair $\{x, y\}$.

2 Regularity and nets

Let $\mathcal{S} = (P, B, \text{I})$ be a finite GQ of order (s, t) . If $x \sim x', x \neq x'$, or if $x \not\sim x'$ and $|\{x, x'\}^{\perp\perp}| = t + 1$, where $x, x' \in P$, we say the pair $\{x, x'\}$ is *regular*. The point x is *regular* provided $\{x, x'\}$ is regular for all $x' \in P, x' \neq x$. Regularity for lines is defined dually. A point x is *coregular* provided each line incident with x is regular. If $1 < s < t$, then one can show that no pair of distinct points is regular; see 1.3.6 of Payne and Thas [8]. If the GQ \mathcal{S} has even order s and the point x is coregular, then x is regular; see 1.5.2 (iv) of Payne and Thas [8]. A flag (x, L) , hence $x \text{ I } L$, is called *regular* if x and L are regular; for $s \neq 1 \neq t$ this implies that $s = t$.

A (finite) *net of order $k (\geq 2)$ and degree $r (\geq 2)$* is an incidence structure $\mathcal{N} = (P, B, \text{I})$ satisfying

- (i) each point is incident with r lines and two distinct points are incident with at most one line;
- (ii) each line is incident with k points and two distinct lines are incident with at most one point;
- (iii) if x is a point and L is a line not incident with x , then there is a unique line M incident with x and not concurrent with L .

For a net of order k and degree r we have $|P| = k^2$ and $|B| = kr$. Also, $r \leq k + 1$, with $r = k + 1$ if and only if the net is an affine plane; see Dembowski [2].

Theorem 2.1 (1.3.1 of Payne and Thas [8]). *Let x be a regular point of the GQ $\mathcal{S} = (P, B, \text{I})$ of order (s, t) , $s \geq 2$. Then the incidence structure with point set $x^\perp - \{x\}$, with line set the set of all spans $\{y, z\}^{\perp\perp}$, where $y, z \in x^\perp - \{x\}, y \not\sim z$, and with the natural incidence, is the dual of a net of order s and degree $t + 1$. If in particular $s = t > 1$, there arises a dual affine plane of order s . Also, in the case $s = t > 1$ the*

incidence structure π_x with point set x^\perp , with line set the set of all spans $\{y, z\}^{\perp\perp}$, where $y, z \in x^\perp$, $y \neq z$, and with the natural incidence, is a projective plane of order s .

3 Switching of generalized quadrangles

Let $\mathcal{S} = (P, B, I)$ be a GQ of order s , with $s \neq 1$, for which the flag (x, L) is regular. Let $L_0, L_1, L_2, \dots, L_s$ be the lines incident with x , where $L = L_0$.

Define as follows the incidence structure $P(\mathcal{S}, x) = (P_x, B_x, I_x)$. Points of $P(\mathcal{S}, x)$ are the points of $P - x^\perp$. Lines of $P(\mathcal{S}, x)$ are the lines of B not incident with x and the spans $\{x, y\}^{\perp\perp}$, with $y \not\sim x$. If $z \in P_x$ and $M \in B_x$ with $M \in B$, then $z I_x M$ if and only if $z I M$; if $z \in P_x$ and $N = \{x, y\}^{\perp\perp}$, $y \not\sim x$, then $z I N$ if and only if $z \in N$. Then by Payne, see 3.1.4 of Payne and Thas [8], $P(\mathcal{S}, x)$ is a GQ of order $(s-1, s+1)$.

A spread of $P(\mathcal{S}, x)$ is a set S of s^2 lines in B_x such that each point of P_x is incident (for I_x) with exactly one element of S . Spreads of $P(\mathcal{S}, x)$ are the set S_x of all lines of type $\{x, y\}^{\perp\perp}$, $y \not\sim x$, and any set S_i consisting of all lines of \mathcal{S} concurrent with L_i but not incident with x , with $i = 0, 1, \dots, s$. Clearly $B_x = S_x \cup S_0 \cup S_1 \cup \dots \cup S_s$.

Let x_1, x_2, \dots, x_s, x be the points incident with L and let $S_{x,i}$ be the set of all spans $\{x, y\}^{\perp\perp}$, with $y \sim x_i$, $y \not\sim x$. Then $S_x = S_{x,1} \cup S_{x,2} \cup \dots \cup S_{x,s}$. Further, for $i \in \{1, 2, \dots, s\}$, let $S_{i,j}$, with $j = 1, 2, \dots, s$, be the sets $\{L_i, N\}^\perp - \{L\}$, with $N \sim L$, $x \not\sim N$. Then $S_i = S_{i,1} \cup S_{i,2} \cup \dots \cup S_{i,s}$. It is clear that the sets S_x, S_0, \dots, S_s and $S_{x,j}, S_{1,j}, \dots, S_{s,j}$, with $j = 1, 2, \dots, s$, satisfy the conditions of Theorem 1.1 in Payne [4], and so by Payne [4] a GQ $\mathcal{S}_{(x,L)}$ of order s may be constructed as follows.

Points of $\mathcal{S}_{(x,L)}$ are of three kinds: points of type (i) are just the points of $P(\mathcal{S}, x)$; points of type (ii) are the $s^2 + s$ sets $S_{x,j}, S_{1,j}, \dots, S_{s,j}$, with $j = 1, 2, \dots, s$; there is a unique point of type (iii) denoted by x' . Lines are of two types: lines of type (a) are the lines of $P(\mathcal{S}, x)$ not in the spread S_0 ; lines of type (b) are the spreads S_x, S_1, \dots, S_s . The incidence is as follows. A point of type (i) is incident with a line of type (a) if and only if the two are incident in $P(\mathcal{S}, x)$. A point of type (i) is incident with no line of type (b). A point of type (ii) is incident with each line of type (a) which belongs to it and with the unique line of type (b) of which it is a subset. The unique point of type (iii) is incident with no line of type (a) and all lines of type (b).

It is an easy exercise to show that the flag (x', S_x) of $\mathcal{S}_{(x,L)}$ is regular. Note that $\mathcal{S}_{(x,L)} \cong \mathcal{S}_{(L,x)}$ and that $(\mathcal{S}_{(x,L)})_{(x', \mathcal{S}_x)} \cong \mathcal{S}$. We say that $\mathcal{S}_{(x,L)}$ is obtained by *switching* from \mathcal{S} (with respect to (x, L)) and that \mathcal{S} and $\mathcal{S}_{(x,L)}$ are *switching equivalent*.

We emphasize that the construction of $\mathcal{S}_{(x,L)}$ from \mathcal{S} is a combination of ideas of Payne.

Up to isomorphism the GQ $\mathcal{S}_{(x,L)}$ can also be described as follows. Let $\mathcal{S}' = (P', B', I')$ be the following incidence structure. Points (P') are of four types:

- (i) the s^3 points of P not in x^\perp ;
- (ii) the s^2 spans $\{L, N\}^{\perp\perp}$, with $L \not\sim N$;
- (iii) the s points x_1, x_2, \dots, x_s incident (for I) with L , but distinct from x ;
- (iv) the point x .

Lines (B') are of four types:

- (a) the s^3 lines of B not in L^\perp ;
- (b) the s^2 spans $\{x, y\}^{\perp\perp}$, with $x \not\sim y$;
- (c) the s lines L_1, L_2, \dots, L_s incident (for I) with x , but distinct from L ;
- (d) the line L .

Incidence (I') is defined as follows:

A point of type (i) is incident with the s lines of type (a) incident (for I) with it and with the unique line of type (b) containing it. A point $\{L, N\}^{\perp\perp}$ of type (ii) is incident with the s lines of type (a) contained in it, and with the unique line $L_i \in \{L, N\}^\perp$. A point x_i of type (iii) is incident with the line L and with the s lines of type (b) contained in x_i^\perp . The point x of type (iv) is incident with the $s + 1$ lines L, L_1, \dots, L_s .

Then it is easily checked that $\mathcal{S}' \cong \mathcal{S}_{(x,L)}$.

The projective plane π_x defined by the regular point x of \mathcal{S} is isomorphic to the dual of the projective plane π_L' defined by the regular line L of \mathcal{S}' ; the projective plane π_L defined by the regular line L of \mathcal{S} is isomorphic to the dual of the projective plane π_x' defined by the regular point x of \mathcal{S}' .

Example. Let $O = \{\pi_0, \pi_1, \dots, \pi_{q^n}\}$ be a *pseudo-oval* of $\text{PG}(3n - 1, q)$, that is, a set of $q^n + 1$ $(n - 1)$ -dimensional subspaces of $\text{PG}(3n - 1, q)$ every three of which generate $\text{PG}(3n - 1, q)$. Then O defines a $\text{GQT}(O)$ of order q^n ; see 8.7 in Payne and Thas [8]. For $n = 1$ the $\text{GQT}(O)$ is the Tits quadrangle $T_2(O)$ arising from the oval O of $\text{PG}(2, q)$. The lines $\pi_0, \pi_1, \dots, \pi_{q^n}$ of type (b) of $T(O)$ are regular, and so, for q even, the point (∞) of type (iii) is also regular. For q even the tangent spaces of O all contain a common $(n - 1)$ -dimensional space π , called the *kernel* or *nucleus* of O ; see Section 4.9 of Thas [9].

Assume that q is even. Then it is easily checked that $T(O)_{((\infty), \pi_i)} \cong T(O_i)$, with $O_i = (O - \{\pi_i\}) \cup \{\pi\}$, $i = 0, 1, \dots, q^n$. The projective plane defined by the regular point (∞) of $T(O)$ is isomorphic to the dual of the projective plane defined by the regular line π of $T(O_i)$, and the projective plane defined by the regular point $(\infty)_i$ of $T(O_i)$ is isomorphic to the dual of the projective plane defined by the regular line π_i of $T(O)$.

4 Application to translation generalized quadrangles

Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, t) , $s \neq 1$, $t \neq 1$. A collineation θ of \mathcal{S} is a *whorl* about the point p provided θ fixes each line incident with p . Let θ be a whorl about p . If $\theta = \text{id}$ or if θ fixes no point of $P - p^\perp$, then θ is an *elation* about p . If there is a group G of elations about p acting regularly on $P - p^\perp$, we say that \mathcal{S} is an *elation generalized quadrangle* (EGQ) with *elation group* G and *base point* p . Briefly, we say that $(\mathcal{S}^{(p)}, G)$ or $\mathcal{S}^{(p)}$ is an EGQ. Most known examples of GQ or their duals

are EGQ, the notable exceptions being those of order $(s - 1, s + 1)$ and their duals. If the group G is abelian, then we say that the EGQ $(\mathcal{S}^{(p)}, G)$ is a *translation generalized quadrangle* (TGQ) with *translation group* G and *base point* p .

The following theorems are taken from Chapter 8 of Payne and Thas [8].

Theorem 4.1. *If $(\mathcal{S}^{(p)}, G)$ is a TGQ then the point p is coregular and so $s \leq t$. Also, G is the complete set of all elations about p .*

Theorem 4.2. *If $(\mathcal{S}^{(p)}, G)$ is a TGQ then the group G is elementary abelian, and s and t must be powers of the same prime. If $s < t$, then there is a prime power q and an odd integer a for which $s = q^a$ and $t = q^{a+1}$. If s (or t) is even then either $s = t$ or $s^2 = t$.*

The following result is easy to prove; see Theorem 4.4 in Thas [13].

Theorem 4.3. *Let $(\mathcal{S}^{(p)}, G)$ be a TGQ of order s . Then for any line L incident with p , the projective plane π_L defined by the regular line L is a translation plane for which the translation line is the line of π_L defined by the point p .*

In Section 5 of Thas [13] the following characterization of TGQ of order s is obtained.

Theorem 4.4. *Let $\mathcal{S} = (P, B, I)$ be GQ of order s , $s \neq 1$, with coregular point p . If for at least one line L incident with p the corresponding projective plane π_L is a translation plane with as translation line the set of all lines of \mathcal{S} incident with p , then \mathcal{S} is a TGQ with base point p . If in particular the plane π_L is Desarguesian, then \mathcal{S} is a TGQ with base point p , and also, for s odd, \mathcal{S} is isomorphic to the classical GQ $Q(4, s)$ arising from a non-singular quadric in $PG(4, s)$.*

Next theorem illustrates how switching of GQ can be applied.

Theorem 4.5. *Let $\mathcal{S} = (P, B, I)$ be a GQ of order s , s even, with coregular point x . Then \mathcal{S} is a TGQ with base point x if and only if the projective plane π_x is a dual translation plane with translation point x . If in particular the plane π_x is Desarguesian, then \mathcal{S} is a TGQ with base point x .*

Proof. Let \mathcal{S} be a TGQ of order s , s even, with base point x . Then the point x is coregular and regular. It is easy to check that π_x is indeed a dual translation plane with translation point x .

Conversely, let $\mathcal{S} = (P, B, I)$ be a GQ of order s , s even, with coregular point x and assume that π_x is a dual translation plane with translation point x . Let L, L_1, \dots, L_s be the lines incident with x . Now we consider the GQ $\mathcal{S}_{(x, L)}$ which is obtained by switching from \mathcal{S} with respect to the flag (x, L) . Let \mathcal{S}' be the GQ isomorphic with $\mathcal{S}_{(x, L)}$, described in Section 3. Then (x, L) is a regular flag of \mathcal{S}' and the projective plane π_x is isomorphic to the dual of the projective plane π'_L . Also, all lines L, L_1, L_2, \dots, L_s are regular for \mathcal{S}' , hence x is a coregular point of \mathcal{S}' . The plane π'_L is a

translation plane with $\{L, L_1, L_2, \dots, L_s\}$ as translation line. By Theorem 4.4 \mathcal{S}' is a TGQ with base point x . By 8.7 of Payne and Thas [8] $\mathcal{S}' \cong T(O)$, with O a pseudo-oval in some $\text{PG}(3n - 1, q)$. Let $O = \{\pi, \pi_1, \dots, \pi_s\}$, where π corresponds to L , and let η be the nucleus of O . By the example at the end of Section 4 the GQ $\mathcal{S} \cong \mathcal{S}'_{(x,L)}$ is isomorphic to $T(O')$, with $O' = (O - \{\pi\}) \cup \{\eta\}$. As O' is again a pseudo-oval the GQ $T(O')$ is a TGQ, and so \mathcal{S} is a TGQ with base point x . \square

5 Application to generalized quadrangles satisfying Property (G)

Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, s^2) , $s \neq 1$. Then $|\{x, y, z\}^\perp| = s + 1$ for any triple $\{x, y, z\}$ of pairwise non-collinear points; see 1.2.4 of Payne and Thas [8]. We say $\{x, y, z\}$ is 3-regular provided $|\{x, y, z\}^{\perp\perp}| = s + 1$. The point x is 3-regular if and only if each triple $\{x, y, z\}$ of pairwise non-collinear points is 3-regular.

Let x_1, y_1 be distinct collinear points. We say that the pair $\{x_1, y_1\}$ has Property (G), or that \mathcal{S} has Property (G) at $\{x_1, y_1\}$, if every triple $\{x_1, x_2, x_3\}$ of points, with x_1, x_2, x_3 pairwise non-collinear and $y_1 \in \{x_1, x_2, x_3\}^\perp$, is 3-regular. The GQ \mathcal{S} has Property (G) at the line L , or the line L has Property (G), if each pair of points $\{x, y\}$, $x \neq y$ and $xILy$, has Property (G). If (x, L) is a flag, then we say that \mathcal{S} has Property (G) at (x, L) , or that (x, L) has Property (G), if every pair $\{x, y\}$, $x \neq y$ and yIL , has Property (G). It is clear that the point x is 3-regular if and only if (x, L) has Property (G) for each line L incident with x .

Let F be a flock of the quadratic cone K with vertex x of $\text{PG}(3, q)$, that is, a partition of $K - \{x\}$ into q disjoint irreducible conics. Then, relying on work of Payne [5, 6] and Kantor [3], Thas [10] proves that with F corresponds a GQ $\mathcal{S}(F)$ of order (q^2, q) . In Payne [7] it was shown that $\mathcal{S}(F)$ satisfies Property (G) at its point (∞) . The following fundamental result is taken from Thas [12].

Theorem 5.1. *A GQ $\mathcal{S} = (P, B, I)$ of order (q, q^2) , q odd and $q \neq 1$, satisfies Property (G) at some flag (x, L) if and only if \mathcal{S} is the dual of a flock GQ.*

In Thas [12] also several strong results for q even are proved. It follows that Theorem 5.1 also holds for $q \in \{2, 4, 16\}$. Further it is conjectured that a GQ of order (q, q^2) , q even, satisfies Property (G) at some line L if and only if \mathcal{S} is the dual of a flock GQ.

Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, s^2) , s even, which satisfies Property (G) at some line L . Let x_0, x_1, \dots, x_s be the points incident with L . Further, let $\{x_i, y, z\}$ be a triple of pairwise non-collinear points for which $\{x_i, y, z\}^\perp$ contains an element incident with L , say x_j , with $i, j \in \{0, 1, \dots, s\}$. Let $T = \{x_i, y, z\}$ and let P' be the set of all points incident with lines of the form uv , $u \in T^\perp$, and $v \in T^{\perp\perp}$. If B' is the set of all lines in B which are incident with at least two points in P' , and if I' is the restriction of I to $(P' \times B') \cup (B' \times P')$, then $\mathcal{S}' = (P', B', I')$ is a subquadrangle of \mathcal{S} of order s ; moreover $\{x_i, y\}$ is a regular pair of \mathcal{S}' , with $\{x_i, y\}^{\perp'} = T^\perp$ and $\{x_i, y\}^{\perp'\perp'} = T^{\perp\perp}$ (see 2.6.2 of Payne and Thas [8]). In this way there arise $s^3 + s^2$ subquadrangles of order s in \mathcal{S} ; see Theorem 3.1.5 of Thas [11]. This is the maximum number of sub-

quadrangles of order s of \mathcal{S} containing the line L ; see e.g. Lemma 2.4 of Brown and Thas [1].

Theorem 5.2. *Let $\mathcal{S} = (P, B, I)$ be a GQ of order (s, s^2) , s even, which satisfies Property (G) at some line L . Then the $s^3 + s^2$ subquadrangles of order s of \mathcal{S} containing the line L are TGQ with base line L .*

Proof. Let \mathcal{S}' be one of the subquadrangles of order s of \mathcal{S} containing the line L . By Theorem 3.2.1 of Thas [11] the line L is a coregular line of \mathcal{S}' and a regular line of \mathcal{S} . By Thas and Van Maldeghem [14] the dual net \mathcal{N}_L^* defined by the regular line L of \mathcal{S} (see also Section 2) is isomorphic to the dual net H_s^3 , with point set the set of all points of $\text{PG}(3, s)$ not on a given line M of $\text{PG}(3, s)$, with line set the set of all lines of $\text{PG}(3, s)$ skew to M , and where incidence is induced by $\text{PG}(3, s)$. The dual affine plane \mathcal{A} defined by the regular line L of \mathcal{S}' is a subgeometry of \mathcal{N}_L^* , hence is easily seen to be isomorphic to the dual of $\text{AG}(2, s)$. Now by Theorem 4.5 the GQ \mathcal{S}' is a TGQ with base line L . \square

We hope Theorem 5.2 is a step towards the classification of all GQ of order (s, s^2) , s even, satisfying Property (G) at some line L .

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