Improved Inclusion-Exclusion Identities and Bonferroni Inequalities with Applications to Reliability Analysis of Coherent Systems

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Chapter 1

Introduction and Overview

Many problems in combinatorics, number theory, probability theory, reliability theory and statistics can be solved by applying a unifying method, which is known as the *principle of inclusion-exclusion*. The principle of inclusion-exclusion expresses the indicator function of a union of finitely many sets as an alternating sum of indicator functions of their intersections. More precisely, for any finite family of sets $\{A_v\}_{v\in V}$ the classical principle of inclusion-exclusion states that

$$\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{\substack{I\subseteq V\\ I\neq\emptyset}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right), \tag{1.1}$$

where $\chi(A)$ denotes the indicator function of A with respect to some ground set Ω, that is, $\chi(A)(\omega) = 1$ if $\omega \in A$, and $\chi(A)(\omega) = 0$ if $\omega \in \Omega \setminus A$. Equivalently, (1.1) can be expressed as $\chi\big(\bigcap_{v\in V} \complement A_v\big) = \sum_{I\subseteq V} (-1)^{|I|}\chi\big(\bigcap_{i\in I} A_i\big)$, where $\complement A_v$ denotes the complement of A_v in Ω and $\bigcap_{i\in\emptyset} A_i = \Omega$. A proof by induction on the number of sets is a common task in undergraduate courses. Usually, the A_v's are measurable with respect to some finite measure μ on a σ-field of subsets of Ω. Integration of the indicator function identity (1.1) with respect to μ then gives

$$\mu\left(\bigcup_{v\in V} A_v\right) = \sum_{\substack{I\subseteq V\\ I\neq\emptyset}} (-1)^{|I|-1}\,\mu\left(\bigcap_{i\in I} A_i\right), \tag{1.2}$$

which now expresses the measure of a union of finitely many sets as an alternating sum of measures of their intersections. The step leading from (1.1) to (1.2) is referred to as the *method of indicators* [GS96b]. Naturally, two special cases are of particular interest, namely the case where μ is the counting measure on the power set of Ω and the case where μ is a probability measure on a σ-field of subsets of Ω. These special cases are sometimes attributed to Sylvester (1883) and Poincaré (1896), although the second edition of Montmort's book "Essai d'Analyse sur les Jeux de Hazard", which appeared in 1714, already contains an implicit use of the method, based on a letter by N. Bernoulli in 1710. A first explicit description of the method was given by Da Silva (1854). For references to these sources and additional historical notes, we refer to Takács [Tak67].

Since the classical identities (1.1) and (1.2) contain $2^{|V|}-1$ terms and intersections of up to $|V|$ sets, one often resorts to inequalities like that of Boole [Boo54]:

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\le\; \sum_{i\in V} \chi(A_i)\,. \tag{1.3}$$

A more general result, first discovered by Ch. Jordan [Jor27] and later by Bonferroni [Bon36], states that for any finite family of sets $\{A_v\}_{v\in V}$ and any $r\in\mathbb{N}$,

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\le\; \sum_{\substack{I\subseteq V\\ 0<|I|\le r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) \qquad (r\text{ odd}), \tag{1.4}$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\ge\; \sum_{\substack{I\subseteq V\\ 0<|I|\le r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) \qquad (r\text{ even}). \tag{1.5}$$

Nowadays, these inequalities are usually referred to as *Bonferroni inequalities*. Again, there is no real restriction in using indicator functions rather than measures, since these inequalities can be integrated with respect to any finite measure μ (e.g., a probability measure) on any σ-field containing the sets A_v, $v\in V$.

Numerous extensions of the classical Bonferroni inequalities (1.4) and (1.5) were established in the second half of the 20th century. An excellent survey of the various results, applications and methods of proof is given in the recent book of Galambos and Simonelli [GS96b]. The following inequalities due to Galambos [Gal75], which are valid for any finite collection of sets $\{A_v\}_{v\in V}$, improve (1.4) and (1.5) by including additional terms based on the $(r+1)$-subsets of V:

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\le\; \sum_{\substack{I\subseteq V\\ 0<|I|\le r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) - \frac{r+1}{|V|}\sum_{\substack{I\subseteq V\\ |I|=r+1}} \chi\left(\bigcap_{i\in I} A_i\right) \qquad (r\text{ odd}),$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\ge\; \sum_{\substack{I\subseteq V\\ 0<|I|\le r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) + \frac{r+1}{|V|}\sum_{\substack{I\subseteq V\\ |I|=r+1}} \chi\left(\bigcap_{i\in I} A_i\right) \qquad (r\text{ even}).$$

A related result due to Tomescu [Tom86] states that

$$\chi\left(\bigcup_{v\in V} A_v\right) \le \sum_{\substack{I\subseteq V\\ 0<|I|\le r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) - \sum_{I\in\mathcal{E}_{r+1}} \chi\left(\bigcap_{i\in I} A_i\right) \qquad (r\text{ odd}),$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \ge \sum_{\substack{I\subseteq V\\ 0<|I|\le r}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right) + \sum_{I\in\mathcal{E}_{r+1}} \chi\left(\bigcap_{i\in I} A_i\right) \qquad (r\text{ even}),$$

where $\mathcal{E}_{r+1}$ is the edge-set of a so-called $(r+1)$-hypertree.

Inequalities for the measure or indicator function of a union which are valid for any finite collection of sets like the preceding ones are frequently referred to as *Bonferroni-type inequalities* [GS96a, GS96b] or as inequalities of *Bonferroni-Galambos type* [MS85, Măr89, TX89]. A new inequality of Bonferroni-Galambos type based on chordal graphs will be established in Section 4.3 of this thesis.

The main part of this work deals with improved inclusion-exclusion identities and improved Bonferroni inequalities that require the collection of sets to satisfy some structural restrictions. Examples of such well-structured collections of sets arise in some problems of statistical inference [NW92, NW97], combinatorial reliability theory [Doh98b, Doh99c] and chromatic graph theory [Doh99a, Doh99d]. We shall mainly be interested in inclusion-exclusion identities of the form

$$\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{I\in\mathcal{S}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right)$$

and inequalities of type

$$\chi\left(\bigcup_{v\in V} A_v\right) \leq \sum_{\substack{I\in\mathcal{S}\\ |I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \quad (r \text{ odd}), \tag{1.6}$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \geq \sum_{\substack{I\in\mathcal{S}\\ |I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \quad (r \text{ even}), \tag{1.7}$$

where $\mathcal{S}$ is a restricted set of non-empty subsets of V, and where (1.6) and (1.7) are at least as sharp as their classical counterparts (1.4) and (1.5). A first straightforward approach is to account only for non-empty subsets I of V satisfying $\bigcap_{i\in I} A_i \neq \emptyset$. In fact, Lozinskii [Loz92] demonstrates that a skillful implementation of this approach leads to a reduction of the average running time of the standard inclusion-exclusion algorithm for counting satisfying assignments of propositional formulae in conjunctive normal form. In the present thesis, however, we are interested in more subtle improvements that arise from logical dependencies of the sets involved. Consider for instance the five sets A_1–A_5, whose Venn diagram is shown in Figure 1.1. The classical inclusion-exclusion identity for the indicator function of the union of these sets gives a sum of $2^5 - 1 = 31$ terms, many of which are equal with opposite sign. After cancelling out, we are left with

$$\begin{aligned}\chi(A_1\cup A_2\cup A_3\cup A_4\cup A_5) \;=\;& \chi(A_1)+\chi(A_2)+\chi(A_3)+\chi(A_4)+\chi(A_5)\\ &-\chi(A_1\cap A_2)-\chi(A_2\cap A_3)-\chi(A_3\cap A_4)-\chi(A_4\cap A_5)\,,\end{aligned}$$

which contains only 9 terms. Our purpose is to predict such cancellations and thus to obtain shorter inclusion-exclusion identities and sharper Bonferroni inequalities

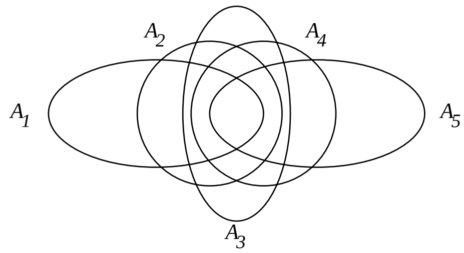

Figure 1.1: A Venn diagram of five sets.

for the indicator function of a union. Although the improved inclusion-exclusion identities follow from the associated improved Bonferroni inequalities, we prefer to treat the identities separately, since they can be proved in an elementary combinatorial way, whereas the inequalities require a topological proof.

The thesis is organized as follows: In Chapter 2 we introduce some terminology on graphs and partially ordered sets that will be repeatedly used in later chapters. In Chapter 3 we bring in some relevant structures and establish several improvements of the classical inclusion-exclusion identity. Several results from the literature such as the semilattice sieve of Narushima [Nar74, Nar82] and the tree sieve of Naiman and Wynn [NW92] are rediscovered in a unified way. In Chapter 4 we give a detailed survey of the recent theory of abstract tubes, which was initiated by Naiman and Wynn [NW92, NW97], and establish some improved Bonferroni inequalities based on the results of this theory. The chapter concludes with a new Bonferroni-Galambos-type inequality based on chordal graphs, itself subsuming several other inequalities. In Chapter 5 the results are applied to reliability analysis of coherent systems such as communication networks, k-out-of-n systems and consecutive k-out-of-n systems. Among other things we rediscover Shier's recursive algorithm and semilattice expression for the reliability of a coherent system [Shi88, Shi91] and establish some related Bonferroni inequalities. We then turn our attention to reliability covering problems and identify a comprehensive class of hypergraphs for which the coverage probability can be computed in polynomial time. In Chapter 6, which is devoted to miscellaneous topics, we give a new and simplified proof of Whitney's broken circuit theorem on the chromatic polynomial of a graph [Whi32] and establish some new inequalities on that polynomial. The results are then extended to a new two-variable polynomial that generalizes both the chromatic polynomial and the independence polynomial of a graph. We finally draw similar conclusions for the Tutte polynomial, the characteristic polynomial and the β invariant of a matroid, the Euler characteristic of an abstract simplicial complex and the Möbius function of a partially ordered set.

Chapter 2

Preliminaries

In this chapter we review some common notions on graphs and posets. Some particular notions are defined later when they are first needed. Readers with background in graph theory and lattice theory may want to skip this chapter.

2.1 Graphs

Unless stated otherwise, all graphs in this thesis are finite, undirected and without loops or multiple edges. When directed edges or loops and multiple edges are allowed, we respectively speak of a digraph or multigraph rather than a graph.

Definition 2.1.1 A *graph* is a pair $G = (V, E)$ where $V = V(G)$ is a finite set, whose elements are the *vertices* of the graph, and where $E = E(G)$ is a set of two-element subsets of V, whose elements are the *edges* of the graph. We write $n(G)$ and $m(G)$ to denote the number of vertices and edges of G, respectively. If $e = \{x, y\}$ is an edge of G, then e is said to *join* x and y, and x and y are called *adjacent in* G. G is *complete* if any two distinct vertices of G are adjacent in G. The *neighborhood* $N_G(v)$ of a vertex v of G is the set of all vertices w of G such that v and w are adjacent in G; the cardinality of $N_G(v)$ is called the *degree* of v. A *subgraph* of G is a graph G' such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. With any subset I of $V(G)$ we associate the *vertex-induced subgraph* $G[I] := (I, \{e \in E(G) \,|\, e \subseteq I\})$. Similarly, with any subset J of $E(G)$ we associate the edge-subgraph $G[J] := \big(\bigcup J, J\big)$ where $\bigcup J$ is a shorthand for $\bigcup_{j \in J} j$. A subset X of $V(G)$ is *independent* or *stable* if $G[X]$ is edgeless. A *clique* of G is a subset X of $V(G)$ such that $G[X]$ is complete. The *clique number* of G is the maximum cardinality of a clique of G. A vertex $v \in V(G)$ is a *simplicial vertex* of G if its neighborhood is a clique. An *path between* x *and* y or x, y*-path* in G is a sequence $(v_1, \ldots, v_k)$ where $k \geq 1$ and where $v_1, \ldots, v_k$ are distinct vertices of G such that $x = v_1$, $y = v_k$ and $\{v_i, v_{i+1}\} \in E(G)$ for $i = 1, \ldots, k-1$. Two paths $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_l)$ are *independent* if $v_i \neq w_j$ for $i = 2, \ldots, k-1$ and $j = 2, \ldots, l-1$. A *cycle* of G is a sequence $(v_1, \ldots, v_k, v_1)$ where $k \geq 3$

and where $v_1, \ldots, v_k$ are distinct vertices of G such that $\{v_i, v_{i+1}\} \in E(G)$ for $i = 1, \ldots, k-1$ and $\{v_k, v_1\} \in E(G)$. The *length* of a path $(v_1, \ldots, v_k)$ resp. cycle $(v_1, \ldots, v_k, v_1)$ is k. The *girth* of a graph which is not cycle-free is the length of a shortest cycle in the graph. Throughout, paths and cycles of G are viewed as subgraphs of G. A *chord* of a path P of G is an edge of G joining two vertices of P that are not adjacent in P; similarly for cycles. G is *chordal* or *triangulated* if any cycle of G of length greater than three has a chord. G is *connected* if there is a path between any pair of vertices of G. A *connected component* of G is a maximal connected subgraph of G. The number of connected components of G is denoted by $c(G)$. G is *biconnected* if there are at least two independent paths between any pair of vertices of G. A maximal biconnected subgraph of G is a *block* of G. G is a *block graph* if every block of G is a complete graph.

A *tree* is a connected graph without cycles. Any connected subgraph of a tree is a *subtree* of that tree. Vertices of degree 0 or 1 in a tree are called *leaves*. A subgraph G' of a graph G is a *spanning tree* of G if G' is a tree and $V(G) = V(G')$.

Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if there is a bijective mapping $\phi : V \to V'$ such that $\{v, w\} \in E(G)$ if and only if $\{\phi(v), \phi(w)\} \in E(G')$.

The *join* $G_1 * G_2$ of two vertex-disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined as $G_1 * G_2 := (V_1 \cup V_2, E_1 \cup E_2 \cup \{\{v_1, v_2\} \mid v_1 \in V_1, v_2 \in V_2\})$. If G_1 and G_2 are not vertex-disjoint, then in the preceding definition they are replaced by vertex-disjoint isomorphic copies G_1' and G_2'. Thus, the join $G_1 * G_2$ is unique up to isomorphism. Finally, for each $n \in \mathbb{N}$ we use K_n, L_n and P_n to denote the complete graph, the edgeless graph and the path on n vertices, respectively.

2.2 Posets

Our terminology on posets is standard and agrees with that of Grätzer [Gra98].

Definition 2.2.1 Let P be a set. A *partial ordering relation* on P is a binary relation which is reflexive, antisymmetric and transitive. Given such a relation, we refer to P as a *partially ordered set* or *poset*. If there are no ambiguities we use $\leq$ to denote the partial ordering relation on P and write $a < b$ if $a \leq b$ and $a \neq b$ for any $a, b \in P$. Two elements $a, b \in P$ are called *comparable* if $a \leq b$ or $b \leq a$; otherwise, they are called *incomparable*. A subset C of P is a *chain* resp. *antichain* of P if any two elements of C are comparable resp. incomparable. In the particular case where P itself is a chain the partial ordering relation on P is called *linear*. The *length* of a chain C of P is one less than the cardinality of C. For any $p \in P$ the length of a longest chain extending upward to p is called the *height* of p. The *height* of P is the maximum height of an element of P. P is called *lower-finite* resp. *upper-finite* if $\{x \in P \mid x \leq p\}$ resp. $\{x \in P \mid x \geq p\}$ is finite for any $p \in P$. A *lower bound* of a subset X of P is an element $p \in P$ such that $p \leq x$ for any $x \in X$. Dually, an *upper bound* of a subset X of P

is an element $p \in P$ such that $p \geq x$ for any $x \in X$. A *lower semilattice* is a partially ordered set P such that P contains a greatest lower bound of any non-empty finite subset X of P, which is called the *infimum* or *meet* of X and abbreviated to $\inf X$ or $\bigwedge X$. Dually, an *upper semilattice* is a partially ordered set P such that P contains a least upper bound of any non-empty finite subset X of P, which is called the *supremum* or *join* of X and abbreviated to $\sup X$ or $\bigvee X$. The *infimum* and *supremum* of any two elements $x, y \in P$ are denoted by $x \wedge y$ and $x \vee y$, respectively. An element p of a lower semilattice P is called *meet-irreducible* if $p = x \wedge y$ implies $p = x$ or $p = y$. Dually, an element p of an upper semilattice P is called *join-irreducible* if $p = x \vee y$ implies $p = x$ or $p = y$. A *lower* resp. *upper subsemilattice* of a lower resp. upper semilattice P is a subset X of P which is closed under $\wedge$ resp. $\vee$. The lower resp. upper subsemilattice of P *generated by a subset* X *of* P is the intersection of all lower resp. upper subsemilattices of P which include X as a subset. A *lattice* is a partially ordered set P which is both a lower and an upper semilattice. We write $L = [\hat{0}, \hat{1}]$ to signify that L has a least element $\hat{0}$ and greatest element $\hat{1}$. A bijective mapping ϕ from a lattice L to a lattice L' is a *lattice isomorphism* if $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$ and $\phi(x \vee y) = \phi(x) \vee \phi(y)$ for any $x, y \in L$.

Usually, a partially ordered set P is represented as a *Hasse diagram* in which elements of P are represented by points in the plane and points associated with distinct elements x and y of P are joined by a line segment ascending from x to y if $x < y$ and there is no $p \in P$ lying strictly between x and y. Thus, relations implied by transitivity are not explicitly shown. Figure 6.3 on page 104 shows a Hasse diagram for a set of five elements a, b, c, d, e where $d < b, e$ and $b < a, c$.

Chapter 3

Improved Inclusion-Exclusion Identities

In this chapter we establish improvements of the classical inclusion-exclusion identity based on closure operators (having the unique basis property) and kernel operators. In a unified way we thus rediscover several results from the literature such as the tree sieve of Naiman and Wynn [NW92] and the semilattice sieve of Narushima [Nar74]. We then establish two recursive schemes for the probability of a union, which have applications in the context of system reliability analysis. Throughout, we use $\mathcal{P}(V)$ to denote the power set of any set V, that is, the set of all subsets of V, and $\mathcal{P}^*(V)$ to denote the set of all non-empty subsets of V.

3.1 Improvements based on closure operators

In what follows it is intuitive to imagine that c is the convex hull operator in $\mathbb{R}^d$.

Definition 3.1.1 Let V be a set. A *closure operator* on V is a mapping c from the power set of V into itself such that for any subsets X and Y of V,

$$\begin{aligned} &\text{(i)} \quad X \subseteq c(X) \quad \text{(extensionality)}, \\ &\text{(ii)} \quad X \subseteq Y \Rightarrow c(X) \subseteq c(Y) \quad \text{(monotonicity)}, \\ &\text{(iii)} \quad c(c(X)) = c(X) \quad \text{(idempotence)}. \end{aligned}$$

If c is a closure operator on V, then a subset X of V is referred to as *c-closed* if $c(X) = X$ and as *c-free* if all subsets of X are c-closed. A *c-basis* of X is a minimal subset B of X such that $c(B) = X$. If there are no ambiguities, we also write *closed* instead of c-closed, *free* instead of c-free, and *basis* instead of c-basis.

The following proposition characterizes the free sets by means of their bases.

Proposition 3.1.2 [Doh00b] *Let V be a finite set, and let c be a closure operator on V. Then, any subset J of V is free if and only if it is a basis of itself.*

Proof. Trivially, if J is free, then J is a basis of itself. Subsequently, the opposite direction is proved by contraposition. Assume that J is not free, that is, $K \subset J$ for some non-closed set K. If J is not closed, then it is not a basis of itself, and we are done. Thus, we may assume that J is closed. For each $k \in c(K) \setminus K$ we find that $k \in c(K) = c(K \setminus \{k\}) \subseteq c(J \setminus \{k\}) \subseteq c(J) = J$ and hence, $J \subseteq c(J \setminus \{k\} \cup \{k\}) \subseteq c(c(J \setminus \{k\}) \cup \{k\}) = c(c(J \setminus \{k\})) = c(J \setminus \{k\}) \subseteq c(J) = J$. Therefore, $k \in J$ and $c(J \setminus \{k\}) = J$, whence J is not a basis of itself. □

The following definition is due to Edelman and Jamison [EJ85].

Definition 3.1.3 A *convex geometry* is a pair (V, c) consisting of a finite set V and a closure operator c on V such that $c(\emptyset) = \emptyset$ and such that any c-closed subset of V has a unique c-basis.

The most prominent example of a convex geometry is the following:

Example 3.1.4 [EJ85] Let V be a finite set of points in $\mathbb{R}^d$, and for any subset $X = \{x_1, \dots, x_n\}$ of V let $\mathrm{conv}(X)$ denote the convex hull of X, that is,

$$\mathrm{conv}(X) := \left\{ \sum_{i=1}^{n} t_i x_i \,\middle|\, t_1, \dots, t_n \geq 0 \text{ and } \sum_{i=1}^{n} t_i = 1 \right\} . \tag{3.1}$$

Then, by $c(X) := \mathrm{conv}(X) \cap V$ a closure operator on V is defined. By the Minkowski-Krein-Milman theorem, any c-closed subset X of V has a unique c-basis, consisting of the vertices of $\mathrm{conv}(X)$. Thus, (V, c) is a convex geometry.

Some further examples associated with graphs and semilattices follow.

Example 3.1.5 [EJ85] For any tree $G = (V, E)$ and any subset X of V define

$$c(X) := \bigcup_{x,y \in X} \{z \in V \mid z \text{ is on the unique path between } x \text{ and } y\} .$$

Then, a subset X of V is c-closed if and only if $G[X]$ is a subtree of G, and c-free if and only if X is an edge or a singleton, that is, if $X = \{v, w\}$ for some $\{v, w\} \in E$ or $X = \{v\}$ for some $v \in V$. Since the leaves of $G[X]$ constitute a unique c-basis for any c-closed subset X of V, (V, c) is a convex geometry.

Example 3.1.6 [EJ85] Let $G = (V, E)$ be a connected block graph, and for any subset X of V let $c(X)$ be the smallest (with respect to inclusion) superset of X that induces a connected subgraph of G. Then, a subset X of V is c-closed if and only if $G[X]$ is connected, and c-free if and only if X is a clique of G, that is, if $G[X]$ is complete. Since the simplicial vertices of $G[X]$ constitute a unique c-basis for any c-closed subset X of V, we conclude that (V, c) is a convex geometry.

Example 3.1.7 [EJ85] Let V be a finite upper (resp. lower) semilattice, and for any subset X of V let $c(X)$ be the upper (resp. lower) subsemilattice of V which is generated by X. Then, the c-closed subsets of V are the upper (resp. lower) subsemilattices of V, while the c-free subsets of V are the chains of V. Since any c-closed subset X of V has a unique c-basis, namely the set of its join-irreducibles (resp. meet-irreducibles), we are again faced with a convex geometry (V, c).

Although not mentioned by Edelman and Jamison [EJ85], the following proposition generalizes a result of Narushima [Nar74, Nar77] on semilattices.

Proposition 3.1.8 [EJ85] *For any closed set J in a convex geometry (V, c),*

$$\sum_{\substack{I\subseteq J\\ c(I)=J}} (-1)^{|I|} \;=\; \begin{cases} (-1)^{|J|} & \textit{if } J \textit{ is free},\\ 0 & \textit{otherwise.}\end{cases}$$

Subsequently, we give our own proof of Proposition 3.1.8. It strongly generalizes Narushima's proof [Nar74, Nar77] for the semilattice case (Example 3.1.7).

Proof. Let J_0 be the unique basis of J. Then, $c(I) = J$ iff $J_0 \subseteq I \subseteq J$. Hence,

$$\sum_{\substack{I\subseteq J\\ c(I)=J}} (-1)^{|I|} \;=\; \begin{cases} (-1)^{|J|} & \text{if } J_0 = J,\\ 0 & \text{otherwise.}\end{cases}$$

From Proposition 3.1.2 it follows that $J_0 = J$ if and only if J is free. $\square$

The following proposition will be used later to derive the main result of this section. It may have applications not only to inclusion-exclusion.

Proposition 3.1.9 [Doh99b] *Let (V, c) be a convex geometry, and let g be a mapping from the power set of V into an abelian group such that $g = g \circ c$. Then,*

$$\sum_{I\subseteq V} (-1)^{|I|}\, g(I) \;=\; \sum_{\substack{J\subseteq V\\ J \text{ free}}} (-1)^{|J|}\, g(J)\,.$$

Proof. By the requirements, $g(I) = g(c(I))$ for any subset I of V. Therefore,

$$\sum_{I\subseteq V} (-1)^{|I|}\, g(I) \;=\; \sum_{I\subseteq V} (-1)^{|I|}\, g(c(I)) \;=\; \sum_{\substack{J\subseteq V\\ c(J)=J}} \sum_{\substack{I\subseteq J\\ c(I)=J}} (-1)^{|I|}\, g(J)\,.$$

Now, by applying Proposition 3.1.8 the statement immediately follows. $\square$

The following corollary contains an unpublished result due to Lawrence.

Corollary 3.1.10 [EJ85] *Let (V, c) be a convex geometry where $V \neq \emptyset$. Then,*

$$\sum_{\substack{J \subseteq V \\ J \text{ free}}} (-1)^{|J|} = 0 .$$

Proof. For any $I \subseteq V$ define $g(I) := 1$ and apply Proposition 3.1.9. □

Although we do not need the following two corollaries, we state them since we feel that they are interesting in their own right.

Corollary 3.1.11 [Doh99b] *In any convex geometry (V, c) there are exactly*

$$\sum_{I \subseteq V} (-1)^{|c(I) \setminus I|}$$

free sets.

Proof. For any $I \subseteq V$ define $g(I) := (-1)^{|c(I)|}$ and apply Proposition 3.1.9. □

Corollary 3.1.12 [Doh99b] *Let (V, c) be a convex geometry. Then,*

$$\sum_{I \subseteq V} (-1)^{|I|} |c(I)| = \sum_{\substack{J \subseteq V \\ J \text{ free}}} (-1)^{|J|} |J| . \tag{3.2}$$

Proof. For any $I \subseteq V$ define $g(I) := |c(I)|$ and apply Proposition 3.1.9. □

Remark. For the convex geometry of Example 3.1.4, where c is derived from the convex hull operator in $\mathbb{R}^d$, Corollary 3.1.12 specializes to a recent result of Gordon [Gor97]. A more recent result of Edelman and Reiner [ER00] states that either side of (3.2) agrees in absolute value with the number of points in V which are in the interior of $\mathrm{conv}(V)$ if $|V| > 1$. This settles a conjecture of Ahrens, Gordon and McMahon [AGM99], who previously gave a proof for $d = 2$.

We continue with a further preliminary proposition.

Proposition 3.1.13 [Doh00b] *Let $\{A_v\}_{v \in V}$ be a finite family of sets and c a closure operator on V such that for any non-empty and non-closed subset X of V,*

$$\bigcap_{x \in X} A_x \subseteq \bigcup_{v \notin X} A_v . \tag{3.3}$$

Then, for any non-empty subset I of V,

$$\bigcap_{i \in I} A_i = \bigcap_{i \in c(I)} A_i .$$

Proof. Fix some $I \subseteq V$, $I \neq \emptyset$. There is nothing to prove if $\bigcap_{i\in I} A_i = \emptyset$. Otherwise choose $\omega \in \bigcap_{i\in I} A_i$ and show that $\omega \in \bigcap_{i\in c(I)} A_i$. By the choice of ω, $I \subseteq V_\omega$ where $V_\omega := \{v \in V \,|\, \omega \in A_v\}$. By the definition of V_ω and (3.3), V_ω is closed and hence, $c(I) \subseteq V_\omega$. Thus, $\omega \in \bigcap_{i\in c(I)} A_i$ and the proof is complete. □

We are now ready to state the main result of this section, which is both a generalization and improvement of the classical inclusion-exclusion identity.

Theorem 3.1.14 [Doh00b] *Let (V, c) be a convex geometry and $\{A_v\}_{v\in V}$ a finite family of sets such that for any non-empty and non-closed subset X of V,*

$$\bigcap_{x\in X} A_x \subseteq \bigcup_{v\notin X} A_v \,.$$

Then,

$$\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{\substack{J\in\mathcal{P}^*(V)\\ J\ free}} (-1)^{|J|-1}\chi\left(\bigcap_{j\in J} A_j\right).$$

Proof. By the classical inclusion-exclusion identity (1.1) we have $\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{I\subseteq V}(-1)^{|I|-1}g(I)$ where $g(I) := \chi\left(\bigcap_{i\in I} A_i\right)$ if $I \neq \emptyset$ and $g(\emptyset) := 0$. By Proposition 3.1.13 and since $c(\emptyset) = \emptyset$ we have $g = g \circ c$. Now apply Proposition 3.1.9. □

Remarks. Note that by setting $c(X) := X$ for any subset X of V, the improved identity of Theorem 3.1.14 specializes to the classical inclusion-exclusion identity.

The reader should also note that the requirements of Theorem 3.1.14 are satisfied if $\bigcap_{x\in X} A_x \subseteq A_v$ for any non-empty subset X of V and any $v \in c(X)$.

We further remark that the improved identity of Theorem 3.1.14 involves intersections of at most $h(c) := \max\{|J| : J\ c\text{-free}\}$ sets. In [JW81] it is shown that $h(c)$ is the Helly number of the family of all c-closed subsets of V, that is, the smallest integer h such that any family of c-closed subsets of V whose intersection is empty has a subfamily of h or fewer sets whose intersection is also empty.

From Theorem 3.1.14 we now deduce some results, which for the first time appear in a common context. Among these results are the semilattice sieve of Narushima [Nar74, Nar77] and the tree sieve of Naiman and Wynn [NW92].

Corollary 3.1.15 [Nar74, Nar77] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where V is an upper semilattice such that $A_x \cap A_y \subseteq A_{x\vee y}$ for any $x, y \in V$. Then,*

$$\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{\substack{I\in\mathcal{P}^*(V)\\ I\ is\ a\ chain}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right). \tag{3.4}$$

Proof. Apply Theorem 3.1.14 in connection with Example 3.1.7. □

Remark. Corollary 3.1.15 reduces to the classical inclusion-exclusion identity if V is a chain. The case where $A_x \cap A_y = A_{x \vee y}$ for any $x, y \in V$ is treated in [Doha].

The following result is due to Naiman and Wynn [NW92]:

Corollary 3.1.16 [NW92] *Let $\{A_v\}_{v \in V}$ be a finite family of sets, where the indices form the vertices of a tree $G = (V, E)$ such that $A_x \cap A_y \subseteq A_z$ for any $x, y \in V$ and any z on the unique path between x and y in G. Then,*

$$\chi\left(\bigcup_{v \in V} A_v\right) = \sum_{i \in V} \chi(A_i) - \sum_{\{i,j\} \in E} \chi(A_i \cap A_j)\,.$$

Proof. Apply Theorem 3.1.14 in connection with Example 3.1.5. □

Remark. The particular case where G is a path on n vertices gives rise to

$$\chi\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \chi(A_i) - \sum_{i=2}^{n} \chi(A_{i-1} \cap A_i)\,,$$

which is valid for all finite sequences of sets $A_1, \ldots, A_n$ that satisfy $A_i \cap A_j \subseteq A_k$ for $i, j = 1, \ldots, n$ and $k = i, \ldots, j$. This latter consequence is again due to Naiman and Wynn [NW92] and explains the cancellations in Figure 1.1.

Since any tree is a connected chordal graph, the following corollary generalizes the preceding one.

Corollary 3.1.17 [Doh99b] *Let $\{A_v\}_{v \in V}$ be a finite family of sets, and let $G = (V, E)$ be a connected chordal graph such that $A_x \cap A_y \subseteq A_z$ for any $x, y \in V$ and any z on any chordless path between x and y in G. Then,*

$$\chi\left(\bigcup_{v \in V} A_v\right) = \sum_{\substack{I \in \mathcal{P}^*(V) \\ I \text{ is a clique}}} (-1)^{|I|-1} \chi\left(\bigcap_{i \in I} A_i\right). \tag{3.5}$$

Proof. We apply Theorem 3.1.14. For any subset X of V define

$$c(X) := \bigcup_{x,y \in X} \{z \in V \mid z \text{ is on a chordless path between } x \text{ and } y\}\,.$$

Then, (V, c) is a convex geometry, where a subset X of V is free if and only if X is a clique of G [EJ85, FJ86]. Theorem 3.1.14 now gives the result. □

Remarks. Note that Corollary 3.1.17 specializes to the classical inclusion-exclusion identity if G is complete, since then all subsets of the vertex-set are cliques.

Howorka [How81] showed that in chordal graphs having the property that all cycles of length five have at least three chords the chordless paths are precisely the shortest paths and that chordal graphs with this property are unions of Ptolemaic graphs and vice versa. Ptolemaic graphs are defined below. For a discussion of this class of graphs and other restricted classes of chordal graphs in connection with convex geometries, the reader is referred to Farber and Jamison [FJ86].

Definition 3.1.18 A graph G is called *Ptolemaic* if it is connected and if for any four vertices v, w, x, y of G, $d(v,w)d(x,y) \leq d(v,x)d(w,y) + d(w,x)d(v,y)$ where $d(a,b)$ denotes the length of a shortest path between a and b in G.

By Corollary 3.1.17 and the preceding remarks, we have the following result:

Corollary 3.1.19 [Doh99b] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, and let $G = (V,E)$ be a Ptolemaic graph such that $A_x \cap A_y \subseteq A_z$ for any $x, y \in V$ and any z on any shortest path between x and y in G. Then, (3.5) holds.*

We proceed with deducing some further consequences of Theorem 3.1.14.

Corollary 3.1.20 *Let $\{A_v\}_{v\in V}$ be a finite family of sets where for any non-empty subset X of V there is a unique minimal non-empty subset Y of X such that*

$$\bigcap_{x\in X} A_x \;=\; \bigcap_{y\in Y} A_y\,. \tag{3.6}$$

Then,

$$\chi\left(\bigcup_{v\in V} A_v\right) \;=\; \sum_I (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right),$$

where the sum is extended over all non-empty subsets I of V such that $\bigcap_{i\in I} A_i \neq \bigcap_{j\in J} A_j$ for all non-empty proper subsets and proper supersets J of I.

Proof. Again, we apply Theorem 3.1.14. It is straightforward to check that

$$c(X) \;:=\; \left\{ v\in V \;\middle|\; \bigcap_{x\in X} A_x \subseteq A_v \right\} \quad (X\neq\emptyset)\,; \qquad c(\emptyset) \;:=\; \emptyset\,;$$

defines a closure operator on V. In order to verify the unique basis property of Definition 3.1.3, let X be a non-empty closed subset of V, and let Y be the unique minimal non-empty subset of X satisfying (3.6). Then, $c(X) = c(Y) \subseteq X$ and hence, $c(Y) = X$. Now, to show that Y is smallest with respect to $c(Y) = X$, suppose that $c(Y') = X$ for some non-empty subset Y' of X. Then, $\bigcap_{x\in X} A_x = \bigcap_{y\in Y'} A_y$ and hence, $Y' \supseteq Y$ by the choice of Y. Thus, Y is the unique c-basis of X. The description of the free sets directly follows from Proposition 3.1.2. □

The dual version of the following corollary has been published in [Doh97] together with a generalization to Möbius inversion over power set lattices.

Corollary 3.1.21 [Doh97] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where V is endowed with a linear ordering relation, and let $\mathcal{X}$ be a set of non-empty subsets of V such that for any $X \in \mathcal{X}$,*

$$\bigcap_{x\in X} A_x \subseteq A_v \quad \textit{for some } v > \max X.$$

Then,

$$\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{\substack{I\in\mathcal{P}^*(V)\\ I\not\supseteq X\,(\forall X\in\mathcal{X})}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right).$$

Proof. By the requirements of the corollary there is a family $\{v_X\}_{X\in\mathcal{X}} \subseteq V$ such that for any $X \in \mathcal{X}$, $\bigcap_{x\in X} A_x \subseteq A_{v_X}$ where $v_X > \max X$. For any $I \subseteq V$ define

$$c(I) := I \cup \{v_X \mid X \in \mathcal{X},\, X \subseteq I\}$$

as well as

$$c^*(I) := c(I) \cup c(c(I)) \cup c(c(c(I))) \cup c(c(c(c(I)))) \cup \cdots.$$

Then, c^* is a closure operator on V. Since $v_X > \max X$ for any $X \in \mathcal{X}$ we find that $I \setminus \{v_X \mid X \in \mathcal{X},\, X \subseteq I\}$ is the unique c^*-basis of any c^*-closed subset I of V. Thus, (V, c^*) is a convex geometry. Evidently, a subset I of V is c^*-free if and only if $I \not\supseteq X$ for any $X \in \mathcal{X}$. Now apply Theorem 3.1.14 with c^* instead of c. □

Remark. Note that the dual version of the preceding corollary is obtained by replacing $v > \max X$ with $v < \min X$. The reader should also note that the corollary reduces to the classical inclusion-exclusion principle if $\mathcal{X}$ is empty.

As a consequence of Corollary 3.1.21 we deduce the following generalization of Corollary 3.1.15, which is again due to Narushima [Nar82].

Corollary 3.1.22 [Nar82] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where V is endowed with a partial ordering relation such that for any $x, y \in V$, $A_x \cap A_y \subseteq A_z$ for some upper bound z of x and y. Then, (3.4) holds.*

Proof. Corollary 3.1.22 follows from Corollary 3.1.21 by defining $\mathcal{X}$ as the set of all unordered pairs of incomparable elements of V and then considering an arbitrary linear extension of the partial ordering relation on V. □

Remark. Note that the requirements of Corollary 3.1.22 are weaker than the original requirements of Narushima [Nar82]. Namely, Narushima requires that for any $x, y \in V$, $A_x \cap A_y \subseteq A_z$ for some *minimal* upper bound z of x and y. In

Corollary 3.1.22, however, the minimality of z is not required. From this point of view, Corollary 3.1.22 is more general than Narushima's original result [Nar82].

We close this section with a generalization of Proposition 3.1.8, Proposition 3.1.9 and Theorem 3.1.14 to the situation where the closure operator does not have the unique basis property. To this end, we need the following definition:

Definition 3.1.23 Let V be a finite set and c a closure operator on V. A *c-formation* of a subset X of V is any non-empty set $\mathcal{B}$ of c-bases of X such that $\bigcup \mathcal{B} = X$. A c-formation $\mathcal{B}$ of X is *odd* resp. *even* if $|\mathcal{B}|$ is odd resp. even. The *c-domination* of X, $\mathrm{dom}_c(X)$, is the number of odd c-formations of X minus the number of even c-formations of X. Evidently, if X is not c-closed, then $\mathrm{dom}_c(X) = 0$, and if X is c-free, then $\mathrm{dom}_c(X) = 1$. If (V, c) is a convex geometry, then $\mathrm{dom}_c(X) = 1$ resp. 0 depending on whether X is c-free or not.

Proposition 3.1.24 [Doh99b] *Let V be a finite set and c a closure operator on V. Then, for any c-closed subset J of V,*

$$\sum_{\substack{I\subseteq J\\ c(I)=J}} (-1)^{|I|} \;=\; (-1)^{|J|}\,\mathrm{dom}_c(J)\,.$$

Proof. Let $J_0, \ldots, J_n$ be the distinct bases of J. Evidently, $c(I) = J$ if and only if $J_k \subseteq I \subseteq J$ for some $k \in \{0, \ldots, n\}$. Thus, by including and excluding terms,

$$\sum_{\substack{I\subseteq J\\ c(I)=J}} (-1)^{|I|} = \sum_{\substack{\mathcal{B}\subseteq\{J_0,\ldots,J_n\}\\ \mathcal{B}\neq\emptyset}} (-1)^{|\mathcal{B}|-1} \sum_{I:\,\bigcup\mathcal{B}\subseteq I\subseteq J} (-1)^{|I|} = \sum_{\substack{\mathcal{B}\subseteq\{J_0,\ldots,J_n\}\\ \mathcal{B}\neq\emptyset,\,\bigcup\mathcal{B}=J}} (-1)^{|\mathcal{B}|-1}(-1)^{|J|}\,.$$

The result now follows from the definition of the c-domination $\mathrm{dom}_c(J)$. □

Proposition 3.1.25 [Doh99b] *Let V be a finite set, c a closure operator on V and g a mapping from $\mathcal{P}(V)$ into an abelian group such that $g = g \circ c$. Then,*

$$\sum_{I\subseteq V} (-1)^{|I|}\, g(I) \;=\; \sum_{\substack{J\subseteq V\\ J\ \mathit{closed}}} (-1)^{|J|}\,\mathrm{dom}_c(J)\, g(J)\,.$$

Proof. Proposition 3.1.25 follows from Proposition 3.1.24 in the same way as Proposition 3.1.9 follows from Proposition 3.1.8. □

Theorem 3.1.26 [Doh99b] *Let $\{A_v\}_{v\in V}$ be a finite family of sets and c a closure operator on V such that for any non-empty and non-closed subset X of V,*

$$\bigcap_{x\in X} A_x \;\subseteq\; \bigcup_{v\notin X} A_v$$

and such that $c(\emptyset) = \emptyset$. *Then,*

$$\chi\left(\bigcup_{v\in V} A_v\right) = \sum_{\substack{J\in\mathcal{P}^*(V)\\ J \text{ closed}}} (-1)^{|J|-1}\,\mathrm{dom}_c(J)\,\chi\left(\bigcap_{j\in J} A_j\right).$$

Proof. Theorem 3.1.26 follows from Proposition 3.1.13 and Proposition 3.1.25 like Theorem 3.1.14 follows from Proposition 3.1.13 and Proposition 3.1.9. □

3.2 Improvements based on kernel operators

Similar inclusion-exclusion results as for closure operators having the unique basis property are now established for kernel operators. There is, however, no duality between the results of the preceding section and the results of the present section.

Definition 3.2.1 Let V be a set. A *kernel operator* on V is a mapping k from the power set of V into itself such that for all subsets X and Y of V,

$$\begin{aligned}
&\text{(i)} && k(X) \subseteq X \quad \text{(intensionality)},\\
&\text{(ii)} && X \subseteq Y \Rightarrow k(X) \subseteq k(Y) \quad \text{(monotonicity)},\\
&\text{(iii)} && k(k(X)) = k(X) \quad \text{(idempotence)}.
\end{aligned}$$

If k is a kernel operator on V, then a subset X of V is called *k-open* if $k(X) = X$.

There is a well-known correspondence between kernel operators on V and union-closed subsets of the power set of V. Similarly, there is a correspondence between closure operators and intersection-closed subsets, which shall not be of interest to us. For a proof of the following proposition, we refer to Erné [Ern82].

Proposition 3.2.2 [Ern82] *Let V be a finite set. If k is a kernel operator on V, then $\{X \subseteq V \mid X\ k\text{-open}\}$ is union-closed. If $\mathcal{X} \subseteq \mathcal{P}(V)$ is union-closed, then*

$$I \mapsto \bigcup\{X \in \mathcal{X} \mid X \subseteq I\} \qquad (I \subseteq V)$$

defines a kernel operator k on V such that X is k-open if and only if $X \in \mathcal{X}$.

Lemma 3.2.3 [Doh99f] *Let V be a set, k a kernel operator on V and X_0 a k-open subset of V. Then, all minimal sets in $\{Y \subseteq V \mid k(Y) \supset X_0\}$ are k-open.*

Proof. Assume that Y is not k-open and $k(Y) \supset X_0$. Then, $k(Y) \subset Y$ and $k(k(Y)) \supset X_0$, thus showing that Y is not minimal in $\{Y \subseteq V \mid k(Y) \supset X_0\}$. □

The following theorem will be used below to prove the main result of this section. It has applications not only to inclusion-exclusion, but also to the Tutte polynomial, the β invariant and the Möbius function (see Sections 6.3 and 6.4).

Theorem 3.2.4 [Doh99f] *Let V be a finite set, and let f and g be mappings from the power set of V into an abelian group such that $f(X) = \sum_{Y \supseteq X} g(Y)$ for any subset X of V. Furthermore, let k be a kernel operator on V, and let X_0 be a k-open subset of V such that $f(X) = 0$ for any k-open $X \supset X_0$. Then,*

$$f(X_0) \;=\; \sum_{Y\,:\,k(Y)=X_0} g(Y)\,.$$

Proof. Obviously, we only have to show that $\sum_{Y\,:\,k(Y)\supset X_0} g(Y) = 0$. Let $\mathcal{Y}_0$ consist of all subsets Y of V which satisfy $k(Y) \supset X_0$ and which are minimal with respect to this property. By including and excluding terms we find that

$$\begin{aligned}\sum_{Y\,:\,k(Y)\supset X_0} g(Y) \;&=\; \sum_{\substack{Y\,\supseteq\,Z \text{ for}\\ \text{some } Z\,\in\,\mathcal{Y}_0}} g(Y) \;=\; \sum_{\mathcal{Z}\subseteq\mathcal{Y}_0,\,\mathcal{Z}\neq\emptyset} (-1)^{|\mathcal{Z}|-1} \sum_{\substack{Y\,\supseteq\,Z \text{ for}\\ \text{any } Z\,\in\,\mathcal{Z}}} g(Y)\\ &=\; \sum_{\mathcal{Z}\subseteq\mathcal{Y}_0,\,\mathcal{Z}\neq\emptyset} (-1)^{|\mathcal{Z}|-1} \sum_{Y\supseteq\bigcup\mathcal{Z}} g(Y) \;=\; \sum_{\mathcal{Z}\subseteq\mathcal{Y}_0,\,\mathcal{Z}\neq\emptyset} (-1)^{|\mathcal{Z}|-1} f\big(\textstyle\bigcup\mathcal{Z}\big)\,.\end{aligned}$$

By the preceding lemma, any $Z \in \mathcal{Z}$ is k-open and hence, $\bigcup\mathcal{Z}$ is k-open. Since obviously $\bigcup\mathcal{Z} \supset X_0$, the requirements give $f\big(\bigcup\mathcal{Z}\big) = 0$, whence the result. □

The main result of this section is the following:

Theorem 3.2.5 [Doh99f] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, and let k be a kernel operator on V such that for any non-empty and k-open subset X of V,*

$$\bigcap_{x\in X} A_x \;\subseteq\; \bigcup_{v\notin X} A_v\,.$$

Then,

$$\chi\left(\bigcup_{v\in V} A_v\right) \;=\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right).$$

Subsequently, we give two different proofs of Theorem 3.2.5. The first proof combines Theorem 3.2.4 with the classical inclusion-exclusion principle, whereas the second proof generalizes Garsia and Milne's bijective proof of the classical inclusion-exclusion principle [GM81, Zei84, Pau86] as well as Blass and Sagan's bijective proof of Whitney's broken circuit theorem on chromatic polynomials [BS86, Whi32]. We come back to Whitney's theorem in Section 6.2, p. 91.

First proof. We apply Theorem 3.2.4. For any subsets X and Y of V define

$$f(X) \;:=\; (-1)^{|X|}\,\chi\left(\bigcap_{x\in X} A_x \cap \bigcap_{v\notin X} \complement A_v\right); \quad g(Y) \;:=\; (-1)^{|Y|}\,\chi\left(\bigcap_{y\in Y} A_y\right)$$

where, by convention, $\bigcap_{x\in\emptyset} A_x = \bigcap_{x\notin V} \complement A_v = \Omega$ (the ground set). By applying the principle of inclusion-exclusion we find that $f(X) = \sum_{Y\supseteq X} g(Y)$ for any subset X of V, and by the requirements of the theorem, $f(X) = 0$ for any non-empty and k-open subset X of V. Hence, by applying Theorem 3.2.4, we obtain $\chi\big(\bigcap_{v\in V} \complement A_v\big) = \sum_{I:k(I)=\emptyset} (-1)^{|I|} \chi\big(\bigcap_{i\in I} A_i\big)$, from which the statement of the theorem follows since, due to a law of De Morgan, $\chi\left(\bigcap \complement A_v\right) = 1 - \chi\big(\bigcup A_v\big)$. □

Second proof. Evidently, it suffices to prove that

$$\chi\left(\bigcup_{v\in V} A_v\right) + \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I| \text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right) = \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I| \text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right). \tag{3.7}$$

For any $\omega \in \bigcup_{v\in V} A_v$ define $V_\omega := \{v \in V \mid \omega \in A_v\}$ as well as

$$\begin{aligned} \mathcal{E}(\omega) &:= \big\{ I \in \mathcal{P}(V_\omega) \,\big|\, k(I) = \emptyset,\ |I| \text{ even} \big\}, \\ \mathcal{O}(\omega) &:= \big\{ I \in \mathcal{P}(V_\omega) \,\big|\, k(I) = \emptyset,\ |I| \text{ odd} \big\}. \end{aligned}$$

Then, (3.7) is equivalent to

$$|\mathcal{E}(\omega)| = |\mathcal{O}(\omega)| \quad \text{for all } \omega \in \bigcup_{v\in V} A_v. \tag{3.8}$$

Now, in order to prove (3.8), fix some $\omega \in \bigcup_{v\in V} A_v$. By the definition of V_ω and the requirements of the theorem we find that V_ω is not k-open, whence some $v \in V_\omega \setminus k(V_\omega)$ can be chosen. It follows that for any subset I of V_ω, $v \notin k(I\cup\{v\})$ since otherwise $v \in k(I \cup \{v\}) \subseteq k(V_\omega \cup \{v\}) = k(V_\omega)$, contradicting $v \notin k(V_\omega)$. Since $v \notin k(I \cup \{v\})$ and $k(I \cup \{v\}) \subseteq I \cup \{v\}$ we obtain $k(I \cup \{v\}) \subseteq I$ and hence, $k(I\cup\{v\}) \subseteq k(I)$. From the latter we conclude that for any subset I of V_ω, $k(I) = \emptyset \Rightarrow k(I \cup \{v\}) = \emptyset$. Hence, $I \mapsto I \bigtriangleup \{v\}$, where $\bigtriangleup$ denotes symmetric difference, is a bijective mapping from $\mathcal{E}(\omega)$ to $\mathcal{O}(\omega)$. Thus, (3.8) is shown. □

Remarks. Note that by setting $k(I) := \emptyset$ for any subset I of V, Theorem 3.2.5 specializes to the classical inclusion-exclusion identity. Similarly, the following theorems specialize to the classical identity if $\mathcal{X}$ is chosen to be empty.

Note that by the correspondence between kernel operators and union-closed sets, the following theorem is equivalent to the preceding one.

Theorem 3.2.6 [Doh00e] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, and let $\mathcal{X}$ be a union-closed set of non-empty subsets of V such that for any $X \in \mathcal{X}$,*

$$\bigcap_{x\in X} A_x \subseteq \bigcup_{v\notin X} A_v.$$

Then,

$$\chi\left(\bigcup_{v\in V} A_v\right) \;=\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ I\not\supseteq X\,(\forall X\in\mathcal{X})}} (-1)^{|I|-1}\,\chi\left(\bigcap_{i\in I} A_i\right). \tag{3.9}$$

Proof. The result follows by combining Proposition 3.2.2 and Theorem 3.2.5. □

Corollary 3.2.7 [Dohc] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, c a closure operator on V and $\mathcal{X}$ a set of non-empty subsets of V such that $\{c(X)\,|\,X\in\mathcal{X}\}$ is a chain and such that for any $X\in\mathcal{X}$,*

$$\bigcap_{x\in X} A_x \;\subseteq\; \bigcup_{v\notin c(X)} A_v\,. \tag{3.10}$$

Then, (3.9) holds.

Proof. Apply Theorem 3.2.6 with $\mathcal{X}' := \{\bigcup\mathcal{I}\,|\,\emptyset\neq\mathcal{I}\subseteq\mathcal{X}\}$ instead of $\mathcal{X}$. Then, $\mathcal{X}'$ is union-closed, and for any $X'\in\mathcal{X}'$ the requirements of the corollary imply

$$X' \;=\; \bigcup\{X\,|\,X\in\mathcal{X},\,X\subseteq X'\} \;\subseteq\; \bigcup\{c(X)\,|\,X\in\mathcal{X},\,X\subseteq X'\} \;=\; c(X_0)$$

for some $X_0\in\mathcal{X}$, $X_0\subseteq X'$. It follows that $X_0\subseteq X'\subseteq c(X_0)$ and therefore,

$$\bigcap_{x\in X'} A_x \;\subseteq\; \bigcap_{x\in X_0} A_x \;\subseteq\; \bigcup_{v\notin c(X_0)} A_v \;\subseteq\; \bigcup_{v\notin X'} A_v\,,$$

whence Theorem 3.2.6 gives the result. □

The following result generalizes Corollary 3.1.21. For an exemplary application of this result to counting arrangements with forbidden positions on a chess-like board, see [Doh99a]. See also [Doh00c] for some additional remarks.

Corollary 3.2.8 [Doh99a] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where V is endowed with a linear ordering relation, and let $\mathcal{X}$ be a set of non-empty subsets of V such that for any $X\in\mathcal{X}$,*

$$\bigcap_{x\in X} A_x \;\subseteq\; \bigcup_{v>\max X} A_v\,. \tag{3.11}$$

Then, (3.9) holds.

Proof. Corollary 3.2.8 follows from Corollary 3.2.7 by using the closure operator $X \mapsto c(X)$ where $c(X) := \{v \in V \mid v \le \max X\}$ if $X \neq \emptyset$, and $c(\emptyset) := \emptyset$. □

We close this section with two self-contained proofs of Corollary 3.2.7 and Corollary 3.2.8, which only require the traditional inclusion-exclusion principle.

Alternative proof of Corollary 3.2.7. [Dohc] By the traditional form of the inclusion-exclusion principle, it suffices to prove that

$$\sum_{I \in \overline{\mathfrak{X}}} (-1)^{|I|-1} \chi\left(\bigcap_{i\in I} A_i\right) \;=\; 0\,, \tag{3.12}$$

where

$$\overline{\mathfrak{X}} \;:=\; \{I \subseteq V \mid I \neq \emptyset \text{ and } I \supseteq X \text{ for some } X \in \mathfrak{X}\}\,. \tag{3.13}$$

The proof employs the following partial ordering relation on the power set of V:

$$A \subseteq_c B \;:\Leftrightarrow\; c(A) \cap B = A \quad \text{for any } A, B \subseteq V.$$

Of course, $\subseteq_c$ is reflexive $[A \subseteq c(A) \Rightarrow c(A) \cap A = A \Rightarrow A \subseteq_c A]$, antisymmetric $[(A \subseteq_c B) \wedge (B \subseteq_c A) \Rightarrow (c(A) \cap B = A) \wedge (c(B) \cap A = B) \Rightarrow A = c(A) \cap B = c(A) \cap c(B) \cap A = c(B) \cap A = B]$ and transitive $[(A \subseteq_c B) \wedge (B \subseteq_c C) \Rightarrow (c(A) \cap B = A) \wedge (c(B) \cap C = B) \Rightarrow c(A) \cap C = c(c(A) \cap B) \cap C \subseteq c(A) \cap c(B) \cap C = c(A) \cap B = A \Rightarrow A \subseteq_c C]$. In order to utilize this partial ordering relation in proving (3.12), we first show that for any $Y_1, Y_2 \in \overline{\mathfrak{X}}$ which are $\subseteq_c$-minimal in $\overline{\mathfrak{X}}$,

$$\{I \subseteq V \mid I \supseteq_c Y_1\} \cap \{I \subseteq V \mid I \supseteq_c Y_2\} \neq \emptyset \;\Rightarrow\; Y_1 = Y_2\,. \tag{3.14}$$

To this end, let $I \supseteq_c Y_1$ and $I \supseteq_c Y_2$. Then, by definition of $\subseteq_c$, $Y_1 = I \cap c(Y_1)$ $(\subseteq I)$ and $Y_2 = I \cap c(Y_2)$ $(\subseteq I)$. It follows that $c(c(Y_1) \cap Y_2) \cap Y_1 \subseteq c(Y_1) \cap c(Y_2) \cap Y_1 = Y_1 \cap c(Y_2) = I \cap c(Y_1) \cap c(Y_2) = c(Y_1) \cap Y_2 = c(Y_1) \cap Y_2 \cap I \cap c(Y_1) = c(Y_1) \cap Y_2 \cap Y_1 \subseteq c(c(Y_1) \cap Y_2) \cap Y_1$ and hence, $c(c(Y_1) \cap Y_2) \cap Y_1 = c(Y_1) \cap Y_2$, that is, $c(Y_1) \cap Y_2 \subseteq_c Y_1$. Since $Y_1, Y_2 \in \overline{\mathfrak{X}}$, there are $X_1, X_2 \in \mathfrak{X}$ such that $Y_1 \supseteq X_1$ and $Y_2 \supseteq X_2$. Since $\{c(X) \mid X \in \mathfrak{X}\}$ is a chain, we may without loss of generality assume that $c(X_1) \supseteq c(X_2)$. Then, $c(Y_1) \cap Y_2 \supseteq c(X_1) \cap X_2 \supseteq c(X_2) \cap X_2 = X_2 \in \mathfrak{X}$ and hence, $c(Y_1) \cap Y_2 \in \overline{\mathfrak{X}}$. This in combination with $c(Y_1) \cap Y_2 \subseteq_c Y_1$ and the $\subseteq_c$-minimality of Y_1 in $\overline{\mathfrak{X}}$ gives $c(Y_1) \cap Y_2 = Y_1$ and hence, $Y_1 \subseteq_c Y_2$. From this and the $\subseteq_c$-minimality of Y_2 in $\overline{\mathfrak{X}}$ we finally deduce $Y_1 = Y_2$, thus establishing implication (3.14). Now, by virtue of (3.14), our claim (3.12) is proved if

$$\sum_{I \supseteq_c Y} (-1)^{|I|-1} \chi\left(\bigcap_{i\in I} A_i\right) \;=\; 0 \tag{3.15}$$

for each $Y \in \overline{\mathfrak{X}}$ which is $\subseteq_c$-minimal in $\overline{\mathfrak{X}}$. By definition of $\subseteq_c$ we obtain

$$(3.16) \qquad \sum_{I \supseteq_c Y} (-1)^{|I|-1} \chi\left(\bigcap_{i \in I} A_i\right) \doteq \sum_{I \subseteq V \setminus c(Y)} (-1)^{|I|-1} \chi\left(\bigcap_{y \in Y} A_y \cap \bigcap_{i \in I} A_i\right)$$

where $\doteq$ means equality up to sign. By inclusion-exclusion, (3.16) becomes

$$\sum_{I \supseteq_c Y} (-1)^{|I|-1} \chi\left(\bigcap_{i \in I} A_i\right) \doteq \chi\left(\bigcap_{y \in Y} A_y \cap \bigcup_{i \notin c(Y)} A_i\right) - \chi\left(\bigcap_{y \in Y} A_y\right).$$

Now, (3.15) (and hence the theorem) is proved if

$$(3.17) \qquad \bigcap_{y \in Y} A_y \subseteq \bigcup_{i \notin c(Y)} A_i\,.$$

Since $Y \in \overline{\mathfrak{X}}$, there is some $X \in \mathfrak{X}$ satisfying $Y \supseteq X$ and hence, $c(X) \cap Y \in \overline{\mathfrak{X}}$. In general, $c(X) \cap Y \subseteq c(c(X) \cap Y) \cap Y \subseteq c(X) \cap c(Y) \cap Y = c(X) \cap Y$ and therefore, $c(c(X) \cap Y) \cap Y = c(X) \cap Y$, or equivalently, $c(X) \cap Y \subseteq_c Y$. By this and the $\subseteq_c$-minimality of Y in $\overline{\mathfrak{X}}$, $c(X) \cap Y = Y$. Hence, $c(Y) = c(c(X) \cap Y) \subseteq c(X) \cap c(Y) \subseteq c(X)$. By (3.10) and since $Y \supseteq X$ and $c(Y) \subseteq c(X)$ we obtain

$$\bigcap_{y \in Y} A_y \subseteq \bigcap_{x \in X} A_x \subseteq \bigcup_{i \notin c(X)} A_i \subseteq \bigcup_{i \notin c(Y)} A_i\,,$$

whence (3.17) holds. Thus, the proof is complete. □

Alternative proof of Corollary 3.2.8. [Doh99a] Again, it suffices to prove (3.12). The idea is to define a partition $\mathfrak{Z}$ of $\overline{\mathfrak{X}}$ such that for any $\mathfrak{Z} \in \mathfrak{Z}$,

$$(3.18) \qquad \sum_{I \in \mathfrak{Z}} (-1)^{|I|-1} \chi\left(\bigcap_{i \in I} A_i\right) = 0\,.$$

We first give the definition of $\mathfrak{Z}$. Define $X' := \{v \in V \mid v > \max X\}$ for any $X \in \mathfrak{X}$. In view of (3.9) and (3.11), we may without loss of generality assume that X' is non-empty for any $X \in \mathfrak{X}$. For non-empty $S, T \subseteq V$ define $S \preccurlyeq T$ if $\max S < \max T$ or $S = T$, and let $\preccurlyeq^*$ be a linear extension of $\preccurlyeq$. For $I \in \overline{\mathfrak{X}}$ let X_I be the first element of $\mathfrak{X}$ in this extension which is included by I, and let $\langle I \rangle$ denote the intervall $[I \setminus X_I', I \cup X_I']$. We now define $\mathfrak{Z} := \{\langle I \rangle \mid I \in \overline{\mathfrak{X}}\}$ and show that $\mathfrak{Z}$ is indeed a partition of $\overline{\mathfrak{X}}$: Of course, $I \in \langle I \rangle$ for any $I \in \overline{\mathfrak{X}}$; moreover, $\langle I \rangle \subseteq \overline{\mathfrak{X}}$ since $J \supseteq X_I$ for any $J \in \langle I \rangle$. Hence, $\overline{\mathfrak{X}} = \bigcup_{I \in \overline{\mathfrak{X}}} \langle I \rangle$. It remains to show that $I \in \langle J \rangle$ if $J \in \langle I \rangle$. For $J \in \langle I \rangle$, $X_J \preccurlyeq^* X_I$ since $J \supseteq X_I$. By this, $\max X_J \leq \max X_I < \min X_I'$ and therefore, $X_J \cap X_I' = \emptyset$. We conclude that $X_J = X_J \setminus X_I' \subseteq J \setminus X_I' \subseteq (I \cup X_I') \setminus X_I' \subseteq I$ and hence, $X_I \preccurlyeq^* X_J$. From this and $X_J \preccurlyeq^* X_I$ it follows that $X_J = X_I$ and hence, $X'_J = X'_I$. Therefore, $J \setminus X'_J = J \setminus X'_I \subseteq (I \cup X'_I) \setminus X'_I \subseteq I \subseteq (I \setminus X'_I) \cup X'_I \subseteq J \cup X'_I = J \cup X'_J$, from which we conclude that $I \in \langle J \rangle$. Thus, it is shown that $\mathfrak{Z}$ is a partition of $\overline{\mathfrak{X}}$.

We finally prove that (3.18) holds for any $\mathcal{Z} \in \mathfrak{Z}$. For $\mathcal{Z} = \langle J \rangle$ we have

$$\sum_{I \in \mathcal{Z}} (-1)^{|I|-1} \chi\left(\bigcap_{i \in I} A_i\right) \;=\; (-1)^{|J \setminus X'_J|-1} \sum_{I \subseteq X'_J} (-1)^{|I|} \chi\left(\bigcap_{j \in J \setminus X'_J} A_j \;\cap\; \bigcap_{i \in I} A_i\right).$$

By applying the inclusion-exclusion principle to the right-hand side we obtain

$$\sum_{I \in \mathcal{Z}} (-1)^{|I|-1} \chi\left(\bigcap_{i \in I} A_i\right) \;=\; (-1)^{|J \setminus X'_J|-1} \chi\left(\bigcap_{j \in J \setminus X'_J} A_j \;\cap\; \bigcap_{x \in X'_J} \complement A_x\right).$$

From $J \setminus X'_J \supseteq X_J$ and (3.11) we conclude that

$$\chi\left(\bigcap_{j \in J \setminus X'_J} A_j \;\cap\; \bigcap_{x \in X'_J} \complement A_x\right) \;\leq\; \chi\left(\bigcap_{x \in X_J} A_x \;\cap\; \bigcap_{x \in X'_J} \complement A_x\right) \;=\; 0\,.$$

Now, (3.18) immediately follows from the preceding two equations. □

3.3 Recursive schemes

From Corollary 3.1.15 we now deduce two recursive schemes for the probability of a union. Our results are formulated for upper semilattices only, although in view of Corollary 3.1.22 an even more general formulation would be possible.

The following theorem strongly generalizes an important result of Shier [Shi88, Shi91] on system reliability. We return to Shier's result in Section 5.1, p. 56.

Theorem 3.3.1 *Let $\{A_v\}_{v \in V}$ be a finite family of sets, where V is an upper semilattice such that $A_x \cap A_y \subseteq A_{x \vee y}$ for any $x, y \in V$. Furthermore, let P be a probability measure on a σ-field containing the events A_v, $v \in V$, such that*

$$P\left(A_{i_2} \cap \cdots \cap A_{i_k}\right) > 0 \quad \text{and} \quad P\left(A_{i_1} \middle| A_{i_2} \cap \cdots \cap A_{i_k}\right) = P\left(A_{i_1} \middle| A_{i_2}\right)$$

for any chain $i_1 < \cdots < i_k$ in V where $k > 1$. Then,

$$P\left(\bigcup_{v \in V} A_v\right) \;=\; \sum_{v \in V} \Lambda(v)\,, \tag{3.19}$$

where Λ is defined by the following recursive scheme:

$$\Lambda(v) \;:=\; P(A_v) - \sum_{w > v} \Lambda(w)\, P(A_v | A_w)\,. \tag{3.20}$$

Proof. By Corollary 3.1.15 it suffices to prove that for any $v \in V$,

$$\Lambda(v) \;=\; \sum_{\substack{I \in \mathcal{P}^*(V) \\ I \text{ is a chain} \\ \min I = v}} (-1)^{|I|-1} \, P\left(\bigcap_{i \in I} A_i\right) . \tag{3.21}$$

We proceed by downward induction on v. If v is maximal in V, then by definition, $\Lambda(v) = P(A_v)$, and the statement is proven. For any non-maximal $v \in V$ the recursive definition, the induction hypothesis and the requirements give

$$\begin{aligned}
\Lambda(v) \;&=\; P(A_v) - \sum_{w > v} \sum_{\substack{I \in \mathcal{P}^*(V) \\ I \text{ is a chain} \\ \min I = w}} (-1)^{|I|-1} \, P\left(\bigcap_{i \in I} A_i\right) P(A_v | A_w) \\
&=\; P(A_v) - \sum_{w > v} \sum_{\substack{I \in \mathcal{P}^*(V) \\ I \text{ is a chain} \\ \min I = w}} (-1)^{|I|-1} \, P\left(\bigcap_{i \in I} A_i\right) \, P\left(A_v \,\middle|\, \bigcap_{i \in I} A_i\right) \\
&=\; P(A_v) + \sum_{w > v} \sum_{\substack{I \in \mathcal{P}^*(V) \\ I \text{ is a chain} \\ \min I = w}} (-1)^{|I \cup \{v\}|-1} \, P\left(\bigcap_{i \in I \cup \{v\}} A_i\right) \\
&=\; P(A_v) + \sum_{\substack{I \in \mathcal{P}^*(V) \\ I \text{ is a chain} \\ \min I = v \\ I \neq \{v\}}} (-1)^{|I|-1} \, P\left(\bigcap_{i \in I} A_i\right) . \qquad \square
\end{aligned}$$

Note that the following theorem is *not* dual to the preceding one:

Theorem 3.3.2 [Doh99d] *Let $\{A_v\}_{v \in V}$ be a finite family of sets, where V is an upper semilattice such that $A_x \cap A_y \subseteq A_{x \vee y}$ for any $x, y \in V$. Furthermore, let P be a probability measure on a σ-field containing the A_v, $v \in V$, such that*

$$P\left(A_{i_{k-1}} \cap \cdots \cap A_{i_1}\right) > 0 \quad \textit{and} \quad P\left(A_{i_k} \middle| A_{i_{k-1}} \cap \cdots \cap A_{i_1}\right) = P\left(A_{i_k} \middle| A_{i_{k-1}}\right)$$

for any chain $i_k > \cdots > i_1$ in V where $k > 1$. Then,

$$P\left(\bigcup_{v \in V} A_v\right) \;=\; \sum_{v \in V} \Lambda(v) \, , \tag{3.22}$$

where Λ is defined by the following recursive scheme:

$$\Lambda(v) \;:=\; P(A_v) - \sum_{w < v} \Lambda(w) \, P(A_v | A_w) \, . \tag{3.23}$$

Proof. Theorem 3.3.2 follows by replacing "downward" with "upward", "min" with "max" and ">" with "<" in the proof of Theorem 3.3.1. □

Remark. By applying the technique of dynamic programming, the preceding recursive schemes can be implemented in a quite efficient manner. See Algorithm I for a dynamic programming implementation of (3.19) and (3.20). It can easily be adapted to implement (3.22) and (3.23) by replacing line 1 respectively 6 with

1': Find an ordering $v_1, \ldots, v_n$ of V such that $v_i < v_j \Rightarrow i < j$ $(i, j = 1, \ldots, n)$
6': **if** $v_j < v_i$ **then**

It is easily seen that Algorithm I has a space complexity of $O(|V|)$ and a time complexity of order $|V|^2 \times T$ where T is the time needed to compute $P(A_v)$ and $P(A_v|A_w)$. A quite different connection between the inclusion-exclusion principle and the dynamic programming technique was established by Karp [Kar82].

Algorithm I Improved IE algorithm for computing the probability of a union

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Require: Same requirements as in Theorem 3.3.1
Ensure: $prob = P\left(\bigcup_{v \in V} A_v\right)$
1: Find an ordering $v_1, \ldots, v_n$ of V such that $v_i < v_j \Rightarrow i > j$ $(i, j = 1, \ldots, n)$
2: $prob \leftarrow 0$
3: **for** $i = 1$ to n **do**
4: $\quad acc \leftarrow 0$
5: $\quad$ **for** $j = 1$ to $i - 1$ **do**
6: $\quad\quad$ **if** $v_j > v_i$ **then**
7: $\quad\quad\quad acc \leftarrow acc + a[j]\, P(A_{v_i}|A_{v_j})$
8: $\quad\quad$ **end if**
9: $\quad$ **end for**
10: $\quad a[i] \leftarrow P(A_{v_i}) - acc$
11: $\quad prob \leftarrow prob + a[i]$
12: **end for**

Under the requirements of Theorem 3.3.1, the following theorem shows that the partial sums of $\sum_{v \in V} \Lambda(v)$ provide lower bounds on $P\left(\bigcup_{v \in V} A_v\right)$. Thus, if Algorithm I is stopped at an arbitrary instant of time, *prob* provides a lower bound to $P\left(\bigcup_{v \in V} A_v\right)$. This is not the case under the requirements of Theorem 3.3.2.

Theorem 3.3.3 *Under the requirements of Theorem 3.3.1,*

$$P\left(\bigcup_{v \in V} A_v\right) \;\geq\; \sum_{v \in V'} \Lambda(v) \quad \textit{for any } V' \subseteq V.$$

Proof. For any $v \in V$ define $V^{>v} := \{i \in V \mid i > v\}$. Then, by (3.21), we see that

$$\Lambda(v) \;=\; P(A_v) - \sum_{\substack{I \in \mathcal{P}^*(V^{>v}) \\ I \text{ is a chain}}} (-1)^{|I|-1}\, P\left(A_v \cap \bigcap_{i \in I} A_i\right) .$$

By applying Corollary 3.1.15 with $V^{>v}$ instead of V and integrating the result with respect to the measure $P(A_v \cap \cdot\,)$, we obtain

$$\Lambda(v) \;=\; P(A_v) - P\left(A_v \cap \bigcup_{i>v} A_i\right) \;=\; P\left(A_v \Big\backslash \bigcup_{i>v} A_i\right) .$$

So $\Lambda(v)$ is non-negative, whence the result follows from Theorem 3.3.1. $\square$

Chapter 4

Improved Bonferroni Inequalities

Recently, Naiman and Wynn [NW92, NW97] introduced the concept of an abstract tube in order to obtain improved Bonferroni inequalities that are at least as sharp as their classical counterparts while at the same time involving fewer terms.

In Section 4.1 of this chapter we review the concept of an abstract tube as well as the main results and some applications of abstract tube theory due to Naiman and Wynn [NW92, NW97]. Then, in Section 4.2 the main results of abstract tube theory are applied in establishing improved Bonferroni inequalities associated with the improved inclusion-exclusion identities of the preceding chapter.

In the final section we establish a new Bonferroni-Galambos type inequality based on chordal graphs and deduce several known results from it. We refer to Chapter 1 of Galambos [Gal78] and to the recent monograph of Galambos and Simonelli [GS96b] for an extensive account of these generally valid inequalities.

4.1 The theory of abstract tubes

This section provides an introduction to the theory of abstract tubes. In order to keep the exposition self-contained, we start with some prerequisites from combinatorial topology as they can be found in the book of Harzheim [Har78].

Definition 4.1.1 An *abstract simplicial complex* is a set $\mathcal{S}$ of non-empty subsets of some finite set V such that $I \in \mathcal{S}$ and $\emptyset \neq J \subset I$ imply $J \in \mathcal{S}$. The elements of $\mathcal{S}$ are the *faces* or *simplices* of $\mathcal{S}$, whereas the elements of $\mathrm{Vert}(\mathcal{S}) := \bigcup_{I \in \mathcal{S}} I$ are the *vertices* of $\mathcal{S}$. The *dimension* of a face I, $\dim I$, is one less than its cardinality. The *dimension* of $\mathcal{S}$, $\dim \mathcal{S}$, is the maximum dimension of a face in $\mathcal{S}$. The *Euler characteristic* $\gamma(\mathcal{S})$ of an abstract simplicial complex $\mathcal{S}$ is defined by

$$\gamma(\mathcal{S}) := c_0(\mathcal{S}) - c_1(\mathcal{S}) + c_2(\mathcal{S}) - c_3(\mathcal{S}) + \cdots \tag{4.1}$$

where $c_k(\mathcal{S})$ denotes the number of faces of dimension k of $\mathcal{S}$ ($k = 0, 1, 2, 3, \ldots$).

A *geometric realization* of a complex $\mathcal{S}$ is any topological space homeomorphic to

$$\mathcal{S}^\pi := \bigcup_{I\in\mathcal{S}} \left\{ \sum_{i\in I} t_i e_{\pi i} \;\middle|\; t_i \geq 0 \,\forall\, i \in I \text{ and } \sum_{i\in I} t_i = 1 \right\}, \tag{4.2}$$

where π is a bijective mapping from $\mathrm{Vert}(\mathcal{S})$ to $\{1,\ldots,n\}$ and where $\{e_1,\ldots,e_n\}$ is the standard basis of $\mathbb{R}^n$ (considered as a vector space). Recall that two topological spaces X and Y are *homeomorphic* if there exists a bijective mapping $\phi : X \to Y$ such that both ϕ and its inverse ϕ^{-1} are continuous. Thus, a geometric realization is unique up to homeomorphism. A topological space X is *contractible* if there is a continuous mapping $F : X \times [0,1] \to X$ such that $F(x,0) = x$ for any $x \in X$ and $F(\,\cdot\,,1) \equiv c$ for some constant $c \in X$. Since contractibility is known to be a homeomorphism invariant, we may call an abstract simplicial complex *contractible* if it has a contractible geometric realization.

Example 4.1.2 For any non-empty finite set V the abstract simplicial complex $\mathcal{P}^*(V)$ consisting of all non-empty subsets of V is contractible. In fact, the geometric realization (4.2) of $\mathcal{P}^*(V)$ is contractible by means of the mapping

$$F : \mathcal{S}^\pi \times [0,1] \to \mathcal{S}^\pi, \;\; (x,t) \mapsto (1-t)x + te_{\pi v},$$

where $v \in V$ can be chosen arbitrarily.

Example 4.1.3 Figure 4.1 shows a realization of the abstract simplicial complex

$$\begin{aligned}\mathcal{S} \;=\; \{&\{1\},\{2\},\{3\},\{4\},\{5\},\{6\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\},\\ &\qquad\qquad\qquad\qquad\{3,4\},\{4,5\},\{4,6\},\{5,6\}\}\,.\end{aligned}$$

Obviously, this complex is not contractible because of the unshaded hole on the right. However, if we fill-in the hole (that is, if we attach the triangle $\{4,5,6\}$ to the complex), then a contractible abstract simplicial complex would result.

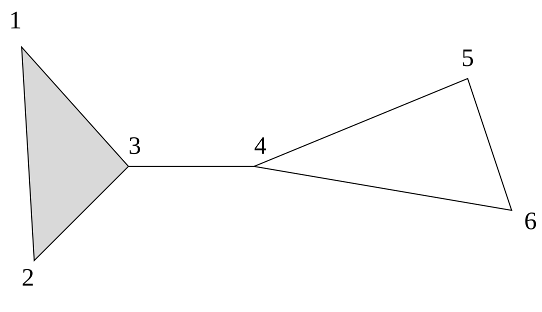

Figure 4.1: A geometric realization of an abstract simplicial complex.

The following definition is due to Naiman and Wynn [NW97].

Definition 4.1.4 An *abstract tube* is a pair $(\mathcal{A}, \mathcal{S})$ consisting of a finite family of sets $\mathcal{A} = \{A_v\}_{v \in V}$ and an abstract simplicial complex $\mathcal{S} \subseteq \mathcal{P}^*(V)$ such that for any $\omega \in \bigcup_{v \in V} A_v$ the abstract simplicial complex

$$\mathcal{S}(\omega) \;:=\; \left\{ I \in \mathcal{S} \;\middle|\; \omega \in \bigcap_{i \in I} A_i \right\} \tag{4.3}$$

is contractible. Given two abstract tubes $(\mathcal{A}_1, \mathcal{S}_1)$ and $(\mathcal{A}_2, \mathcal{S}_2)$, we say that $(\mathcal{A}_1, \mathcal{S}_1)$ is a *subtube* of $(\mathcal{A}_2, \mathcal{S}_2)$ if $\mathcal{A}_1 = \mathcal{A}_2$ and $\mathcal{S}_1 \subseteq \mathcal{S}_2$.

Example 4.1.5 Let $\{A_v\}_{v \in V}$ be a finite family of sets. Then, $(\{A_v\}_{v \in V}, \mathcal{P}^*(V))$ is an abstract tube, where $\mathcal{P}^*(V)$ again denotes the set of non-empty subsets of V.

Example 4.1.6 Consider the sets A_1, A_2, A_3, A_4, A_5, whose Venn diagram is shown in Figure 1.1. It is straightforward to check that

$$\big(\{A_1, A_2, A_3, A_4, A_5\}, \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}\}\big)$$

is an abstract tube.

Example 4.1.7 Let $\{A_v\}_{v \in V}$ be a finite family of sets. Trivially,

$$\left(\{A_v\}_{v \in V}, \left\{ I \in \mathcal{P}^*(V) \;\middle|\; \bigcap_{i \in I} A_i \neq \emptyset \right\} \right)$$

is an abstract tube.

The latter example gives rise to the following definition:

Definition 4.1.8 Let $\mathcal{A} = \{A_v\}_{v \in V}$ be a finite family of sets. Then the set of all $I \in \mathcal{P}^*(V)$ satisfying $\bigcap_{i \in I} A_i \neq \emptyset$ is called the *nerve* of $\mathcal{A}$.

In view of Example 4.1.7, any finite family of sets forms an abstract tube with its nerve. We will discover some more subtle classes of abstract tubes in the course of this and the next section.

In the following, we restate the main results of abstract tube theory due to Naiman and Wynn [NW97] and essentially give their original proofs. We start with two preliminary propositions, which are of vital importance. The first proposition provides a Bonferroni-type generalization of the topological fact that the Euler characteristic of any contractible abstract simplicial complex is equal to 1.

Proposition 4.1.9 [NW97] *Let $\mathcal{S}$ be a contractible abstract simplicial complex. Then, for any $r \in \mathbb{N}$,*

$$\sum_{k=0}^{r-1} (-1)^k \, c_k(\mathcal{S}) \;\leq\; 1 \qquad (r \text{ even}),$$

$$\sum_{k=0}^{r-1} (-1)^k \, c_k(\mathcal{S}) \;\geq\; 1 \qquad (r \text{ odd}).$$

Proof. Let C_k denote the free abelian group generated by the faces of dimension k of $\mathcal{S}$, where by convention, $C_k = (0)$ for $k < 0$ or $k > \dim \mathcal{S}$. Thus, the elements of C_k are formal linear combinations of faces of dimension k of $\mathcal{S}$ with integer coefficients. Obviously, the rank of C_k, that is, the maximum number of linearly independent elements of infinite order in the group, coincides with the number $c_k = c_k(\mathcal{S})$ of faces of dimension k of $\mathcal{S}$. Now, fix an arbitrary linear ordering of the vertices of $\mathcal{S}$, and for any face F of dimension k of $\mathcal{S}$ define $\partial_k F := \sum_{i=0}^{k} (-1)^i F_i$, where F_i is obtained from F by omitting the $(i+1)$-st element of F, when the elements of F are ordered according to the fixed linear ordering of the vertices of $\mathcal{S}$. Extending ∂_k by linearity to all of C_k gives a homomorphism $\partial_k : C_k \to C_{k-1}$ satisfying $\partial_k \circ \partial_{k+1} = 0$ for any k. The *k-th homology group* H_k of $\mathcal{S}$ is the quotient of the kernel Z_k of ∂_k by the image B_k of ∂_{k+1}. Let h_k, z_k and b_k denote the rank of H_k, Z_k and B_k, respectively. (In homology theory, h_k is termed the *k-th Betti number* of $\mathcal{S}$.) From the short exact sequences $0 \to Z_k \xrightarrow{i} C_k \xrightarrow{\partial_k} B_{k-1} \to 0$ $(k = 0, 1, \ldots)$ and $0 \to B_k \xrightarrow{i} Z_k \xrightarrow{\pi_k} H_k \to 0$ $(k = 0, 1, \ldots)$, where π_k is the canonical epimorphism from Z_k onto H_k, we obtain $z_k + b_{k-1} = c_k$ $(k = 0, 1, \ldots)$ resp. $b_k + h_k = z_k$ $(k = 0, 1, \ldots)$. Thus, $h_k = c_k - b_k - b_{k-1}$ $(k = 0, 1, \ldots)$, whence

$$\sum_{k=0}^{r-1} (-1)^k h_k = \sum_{k=0}^{r-1} (-1)^k c_k + (-1)^r b_{r-1}, \tag{4.4}$$

which generalizes the usual Euler-Poincaré formula. Since $b_{r-1} \geq 0$, (4.4) gives

$$\sum_{k=0}^{r-1} (-1)^k h_k \geq \sum_{k=0}^{r-1} (-1)^k c_k \quad (r \text{ even}),$$
$$\sum_{k=0}^{r-1} (-1)^k h_k \leq \sum_{k=0}^{r-1} (-1)^k c_k \quad (r \text{ odd}).$$

Since $\mathcal{S}$ is contractible, $h_0 = 1$ and $h_k = 0$ for $k > 0$. Hence, the result. □

Proposition 4.1.10 [NW97] *Let $\mathcal{S}$ and $\mathcal{S}'$ be contractible abstract simplicial complexes where $\mathcal{S}'$ is a subcomplex of $\mathcal{S}$, that is, $\mathcal{S}' \subseteq \mathcal{S}$. Then, for any $r \in \mathbb{N}$,*

$$\sum_{k=0}^{r-1} (-1)^k c_k(\mathcal{S}) \leq \sum_{k=0}^{r-1} (-1)^k c_k(\mathcal{S}') \quad (r \text{ even}),$$
$$\sum_{k=0}^{r-1} (-1)^k c_k(\mathcal{S}) \geq \sum_{k=0}^{r-1} (-1)^k c_k(\mathcal{S}') \quad (r \text{ odd}).$$

Proof. As in the preceding proof, let C_k, H_k, Z_k and B_k be the groups associated with $\mathcal{S}$ and c_k, h_k, z_k, b_k respectively denote their ranks. Similarly, let c'_k, h'_k, z'_k, b'_k

denote the ranks of the groups C'_k, H'_k, Z'_k, B'_k associated with $\mathcal{S}'$. Note that $c_k = c_k(\mathcal{S})$ and $c'_k = c_k(\mathcal{S}')$. Since $\mathcal{S}$ and $\mathcal{S}'$ are contractible, $h_0 = h'_0 = 1$ and $h_k = h'_k = 0$ for $k > 0$. Putting this into (4.4) and the primed version of it, we obtain

$$\sum_{k=0}^{r-1} (-1)^k c_k \;=\; \sum_{k=0}^{r-1} (-1)^k c'_k \;+\; (-1)^{r-1} \left(b_{r-1} - b'_{r-1}\right) . \tag{4.5}$$

By viewing B'_{r-1} as a subgroup of B_{r-1}, we find that $b'_{r-1} \leq b_{r-1}$, which in combination with (4.5) implies the result. □

Now, the main results of abstract tube theory, which are due to Naiman and Wynn [NW97], are an immediate consequence of the preceding two propositions.

Theorem 4.1.11 [NW97] *Let $(\{A_v\}_{v\in V}, \mathcal{S})$ be an abstract tube. Then, for $r \in \mathbb{N}$,*

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\geq\; \sum_{\substack{I\in\mathcal{S}\\|I|\leq r}} (-1)^{|I|-1} \chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ even}), \tag{4.6}$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \;\leq\; \sum_{\substack{I\in\mathcal{S}\\|I|\leq r}} (-1)^{|I|-1} \chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ odd}). \tag{4.7}$$

Proof. Choose $\omega \in \bigcup_{v\in V} A_v$. By the abstract tube property, $\mathcal{S}(\omega)$ is contractible. The theorem now follows by applying Proposition 4.1.9 with $\mathcal{S}(\omega)$ in place of $\mathcal{S}$. □

Theorem 4.1.12 [NW97] *Let $(\{A_v\}_{v\in V}, \mathcal{S})$ and $(\{A_v\}_{v\in V}, \mathcal{S}')$ be abstract tubes, where $(\{A_v\}_{v\in V}, \mathcal{S}')$ is a subtube of $(\{A_v\}_{v\in V}, \mathcal{S})$. Then, for any $r \in \mathbb{N}$,*

$$\sum_{\substack{I\in\mathcal{S}'\\|I|\leq r}} (-1)^{|I|-1} \chi\left(\bigcap_{i\in I} A_i\right) \;\geq\; \sum_{\substack{I\in\mathcal{S}\\|I|\leq r}} (-1)^{|I|-1} \chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ even}),$$

$$\sum_{\substack{I\in\mathcal{S}'\\|I|\leq r}} (-1)^{|I|-1} \chi\left(\bigcap_{i\in I} A_i\right) \;\leq\; \sum_{\substack{I\in\mathcal{S}\\|I|\leq r}} (-1)^{|I|-1} \chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ odd}).$$

Proof. Choose $\omega \in \bigcup_{v\in V} A_v$. From the abstract tube property it follows that both $\mathcal{S}(\omega)$ and $\mathcal{S}'(\omega)$ are contractible. The theorem now follows by applying Proposition 4.1.10 with $\mathcal{S}(\omega)$ in place of $\mathcal{S}$ and with $\mathcal{S}'(\omega)$ in place of $\mathcal{S}'$. □

Remarks. Since $(\{A_v\}_{v\in V}, \mathcal{P}^*(V))$ is an abstract tube for any finite collection of sets $\{A_v\}_{v\in V}$, the classical Bonferroni inequalities are a particular case of Theorem 4.1.11. Moreover, since any abstract tube $(\{A_v\}_{v\in V}, \mathcal{S})$ is a subtube of $(\{A_v\}_{v\in V}, \mathcal{P}^*(V))$, Theorem 4.1.12 especially states that the bounds provided by Theorem 4.1.11 are at least as sharp as their classical counterparts, although less computational effort is needed to compute them. We further remark that the inequalities in Theorem 4.1.11 become an identity if $r \geq \dim \mathcal{S} + 1$. In particular, any abstract tube $(\{A_v\}_{v\in V}, \mathcal{S})$ gives rise to an improved inclusion-exclusion identity for the indicator function of $\bigcup_{v\in V} A_v$ which does not require intersections of more than $\dim \mathcal{S}+1$ sets, that is, the most complicated intersection is $(\dim \mathcal{S} + 1)$-fold. Thus, in the terminology of Naiman and Wynn [NW97], any abstract tube $(\mathcal{A}, \mathcal{S})$ gives rise to an inclusion-exclusion identity of *depth* $\dim \mathcal{S}+1$.

Due to Naiman and Wynn [NW97], Definition 4.1.4 can be weakened by requiring contractibility of $\mathcal{S}(\omega)$ for almost every ω with respect to some dominating measure μ on the underlying space. In this case, the improved Bonferroni inequalities (and associated inclusion-exclusion identities) of Theorem 4.1.11 and Theorem 4.1.12 hold almost everywhere with respect to μ. We are thus led to

Definition 4.1.13 Let $(\Omega, \mathcal{E}, \mu)$ be a measure space. Then a pair $(\mathcal{A}, \mathcal{S})$ consisting of a finite collection of sets $\mathcal{A} = \{A_v\}_{v\in V} \subseteq \mathcal{E}$ and an abstract simplicial complex $\mathcal{S} \subseteq \mathcal{P}^*(V)$ is called a *weak abstract tube with respect to* μ if $\mathcal{S}(\omega)$, as defined in (4.3), is contractible for almost every $\omega \in \bigcup_{v\in V} A_v$ with respect to μ.

Remark. If $(\Omega, \mathcal{E}, P)$ is a probability space and $(\mathcal{A}, \mathcal{S})$ is a weak abstract tube with respect to P, then the mapping $\omega \mapsto \mathcal{S}(\omega)$ may be considered as a random abstract simplicial complex $\mathcal{S}^{\mathrm{ran}}$ which is required to be P-almost surely contractible.

By the following result, which is stated here for completeness, the improved Bonferroni inequalities of Theorem 4.1.11 give rise to importance sampling schemes for determining the probability content of a union of finitely many events:

Theorem 4.1.14 [NW97] *Let $(\Omega, \mathcal{E}, P)$ be a probability space, $(\{A_v\}_{v\in V}, \mathcal{S})$ an abstract tube with $\{A_v\}_{v\in V} \subseteq \mathcal{E}$, and $r \in \mathbb{N}$. Define $f_r : \Omega \to \mathbb{R}_+$, $h_r : \Omega \to \mathbb{R}$ by*

$$f_r := \frac{1}{Q_r} \sum_{\substack{I\in\mathcal{S}\\|I|=r+1}} \chi\left(\bigcap_{i\in I} A_i\right), \textit{ where } Q_r := \sum_{\substack{I\in\mathcal{S}\\|I|=r+1}} P\left(\bigcap_{i\in I} A_i\right),$$

$$h_r := \chi\left(\bigcup_{v\in V} A_v\right) - \sum_{\substack{I\in\mathcal{S}\\|I|\leq r}} (-1)^{|I|-1} \chi\left(\bigcap_{i\in I} A_i\right),$$

and let $Y_1, Y_2, \ldots$ be independent copies of an $f_r dP$-distributed variable Y. Then,

$$\sum_{\substack{I\in\mathcal{S}\\|I|\leq r}} (-1)^{|I|-1} P\left(\bigcap_{i\in I} A_i\right) + \frac{1}{k}\sum_{i=1}^{k} \frac{h_r(Y_i)}{f_r(Y_i)}$$

is an unbiased estimator for $P\left(\bigcup_{v\in V} A_v\right)$.

Proof. By applying Theorem 4.1.11 first for r and then for $r+1$ we obtain

$$0 \le (-1)^r \left[\chi\left(\bigcup_{v\in V} A_v \right) - \sum_{\substack{I\in\mathcal{S}\\|I|\le r}} (-1)^{|I|-1} \chi\left(\bigcap_{i\in I} A_i \right) \right] \le \sum_{\substack{I\in\mathcal{S}\\|I|=r+1}} \chi\left(\bigcap_{i\in I} A_i \right).$$

Since this is equivalent to $0 \le (-1)^r h_r \le f_r Q_r$, we find that

$$\int h_r\, dP = \int_{f_r>0} h_r\, dP = \int_{f_r>0} \frac{h_r}{f_r} f_r\, dP = E_{f_r}\left[\frac{h_r(Y)}{f_r(Y)} \right],$$

which completes the proof. □

We close this section with some appealing geometric applications of abstract tube theory as they appear in Naiman and Wynn [NW92, NW97]. Related results were obtained by Edelsbrunner [Ede95] and Edelsbrunner and Ramos [ER97].

The original motivation of Naiman and Wynn [NW92, NW97] that led to the theory of abstract tubes was the problem of computing or bounding the volume or probability content of certain geometric objects such as polyhedra and unions of finitely many Euclidean balls or spherical caps, which has applications to the statistical theory of multiple comparisons [NW92] and to computational biology where protein molecules are modeled as unions of balls in $\mathbb{R}^3$ [Ede95].

Definition 4.1.15 A *polyhedron* in $\mathbb{R}^d$ is a set $P = \bigcap_{v\in V} \complement H_v$ where $\{H_v\}_{v\in V}$ is a finite family of open half-spaces in $\mathbb{R}^d$ and $\complement H_v$ denotes the complement of H_v in $\mathbb{R}^d$. P is *d-dimensional* if it contains $d+1$ affinely independent points.

Note that due to a law of De Morgan, $\chi(P) = 1 - \chi\big(\bigcup_{v\in V} H_v\big)$. Thus, by Theorem 4.1.11, the following result of Naiman and Wynn [NW97] gives rise to improved inclusion-exclusion identities and Bonferroni inequalities for the indicator function (and hence for the volume or probability content) of a d-dimensional polyhedron in $\mathbb{R}^d$. For any $H \subseteq \mathbb{R}^d$ we use ∂H to denote the topological boundary of H, that is, the difference between its topological closure and interior.

Theorem 4.1.16 [NW97] *Let $P = \bigcap_{v\in V} \complement H_v$ be a d-dimensional polyhedron in $\mathbb{R}^d$. Then, $\big(\{H_v\}_{v\in V}, \{I \in \mathcal{P}^*(V) | \bigcap_{i\in I}(P\cap \partial H_i) \ne \emptyset\}\big)$ is an abstract tube.*

Proof. (Sketch) Let $\mathcal{S} := \big\{ I \in \mathcal{P}^*(V) \,\big|\, \bigcap_{i\in I}(P\cap\partial H_i) \ne \emptyset \big\}$ and $\omega \in \bigcup_{v\in V} H_v$. By Definition 4.1.4 we have to prove that $\mathcal{S}(\omega)$, with H_i instead of A_i, is contractible. Observe that $\mathcal{S}(\omega)$ is the nerve of $\mathcal{C}(\omega) := \{P\cap\partial H_i \,|\, \omega \in H_i\}$, whence by a classical theorem of Borsuk [Bor48] the contractibility of $\mathcal{S}(\omega)$ follows from that of $\bigcup \mathcal{C}(\omega)$. To prove contractibility of $\bigcup \mathcal{C}(\omega)$, we show that some homeomorphic image of $\bigcup \mathcal{C}(\omega)$ is contractible: Let π be the homeomorphism mapping each $\nu \in \bigcup \mathcal{C}(\omega)$ to the intersection of the line segment $\overline{\nu\omega}$ with some hyperplane K separating ω

from P. Then, the image of $\bigcup \mathcal{C}(\omega)$ under π is convex and hence contractible. $\square$

Remark. Note that if P is in general position, that is, if no $d+1$ facets $P \cap \partial H_i$ share a common point, then the abstract simplicial complex in the abstract tube of Theorem 4.1.16 is at most $(d-1)$-dimensional. In this case, the improved inclusion-exclusion identity associated with the abstract tube of Theorem 4.1.16 involves intersections of up to d half-spaces only and thus contains at most

$$\binom{|V|}{1} + \binom{|V|}{2} + \cdots + \binom{|V|}{d} \;=\; O\big(|V|^d\big)$$

terms, whereas the classical inclusion-exclusion identity still contains $2^{|V|} - 1$ terms and involves intersections of up to $|V|$ half-spaces. If P is not in general position, then due to Naiman and Wynn [NW97] it can be perturbed slightly to give a polyhedron $\tilde{P} = \bigcap_{v \in V} \complement \tilde{H}_v$, which is in general position and which gives rise to a weak abstract tube $\big(\{H_v\}_{v\in V}, \{I \in \mathcal{P}^*(V) \,|\, \bigcap_{i\in I}(\tilde{P} \cap \partial \tilde{H}_i) \neq \emptyset\}\big)$ (involving the original half-spaces H_v) with respect to Lebesgue measure.

Example 4.1.17 Consider the two-dimensional polyhedron $P = \complement H_1 \cap \complement H_2 \cap \complement H_3 \cap \complement H_4 \cap \complement H_5$ which is displayed in Figure 4.2. By combining Theorem 4.1.16 with Theorem 4.1.11 we obtain the improved inclusion-exclusion identity

$$\begin{aligned}\chi(P) \;=\; & 1 - \chi(H_1) - \chi(H_2) - \chi(H_3) - \chi(H_4) - \chi(H_5) \\ & + \chi(H_1 \cap H_2) + \chi(H_2 \cap H_3) + \chi(H_3 \cap H_4) + \chi(H_4 \cap H_5)\,,\end{aligned}$$

which contains ten terms and intersection of up to two sets only. In contrast, the traditional inclusion-exclusion formula for the indicator function of the same polyhedron contains $2^5 = 32$ terms and intersection of up to five sets.

We continue with a further definition.

Definition 4.1.18 Let $d \in \mathbb{N}$. For any $x \in \mathbb{R}^d$ and $r > 0$ let $B_d(x,r)$ denote the open ball in $\mathbb{R}^d$ with center x and radius r, that is, $B_d(x,r) := \{y \in \mathbb{R}^d \,|\, \delta(x,y) < r\}$ where $\delta(\cdot,\cdot)$ denotes Euclidean distance. With any finite set $V \subseteq \mathbb{R}^d$ we associate the *Voronoi subdivision* of $\mathbb{R}^d$ into non-empty closed convex polyhedra

$$D_v \;:=\; \{x \in \mathbb{R}^d \,|\, \delta(x,v) = \min_{u\in V} \delta(x,u)\} \quad (v \in V) \tag{4.8}$$

consisting of points closest in Euclidean distance to v, and the Delauney complex

$$\mathcal{D}(V) \;:=\; \left\{ I \in \mathcal{P}^*(V) \,\middle|\, \bigcap_{i\in I} D_i \neq \emptyset \right\}. \tag{4.9}$$

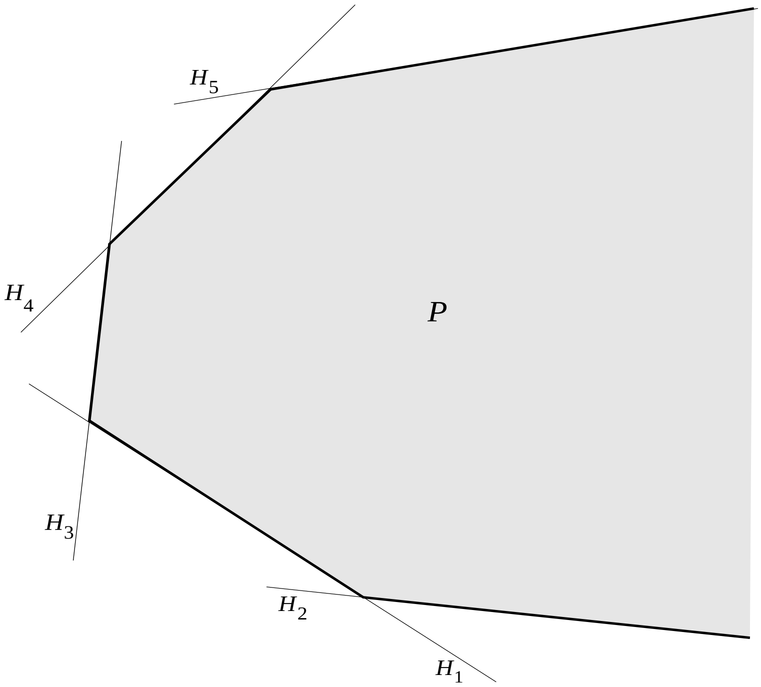

Figure 4.2: A two-dimensional polyhedron.

The following results are implicit in [NW92] and explicit in [NW97].

Theorem 4.1.19 [NW92, NW97] *Let V be a finite set of points in $\mathbb{R}^d$ and $r > 0$. Then, $(\{B_d(v,r)\}_{v\in V}, \mathcal{D}(V))$ is an abstract tube.*

Proof. (Sketch) Let $\mathcal{S} := \mathcal{D}(V)$ and $\omega \in \bigcup_{v\in V} B_d(v,r)$. By Definition 4.1.4 we have to show that $\mathcal{S}(\omega)$, with $B_d(i,r)$ in place of A_i, is contractible. We first observe that $\mathcal{S}(\omega)$ is the nerve of $\mathcal{C}(\omega) := \{D_i \mid \omega \in B_d(i,r)\}$, whence similar to the proof of Theorem 4.1.16 the contractibility of $\mathcal{S}(\omega)$ follows from that of $\bigcup \mathcal{C}(\omega)$. Indeed, $\bigcup \mathcal{C}(\omega)$ is contractible since it is star-shaped with respect to ω. □

Remark. If the centers of the balls are in general position, that is, if no $d+1$ centers of the balls lie in a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$ and no $x \in \mathbb{R}^d$ is equidistant from more than $d+1$ of the centers of the balls, then the intersection of more than $d+1$ of the sets D_v is empty and hence the dimension of $\mathcal{D}(V)$ is at most d, regardless of the radius r. In this case, the improved inclusion-exclusion identity associated with the abstract tube of Theorem 4.1.19 involves intersections of up to $d+1$ balls and therefore contains at most

$$\binom{|V|}{1} + \binom{|V|}{2} + \cdots + \binom{|V|}{d+1} \;=\; O\big(|V|^{d+1}\big)$$

terms, whereas the classical inclusion-exclusion identity still contains $2^{|V|} - 1$ terms and involves intersections of up to $|V|$ balls. As noted in [NW92, NW97], the result can be generalized so that the balls may have different radii.

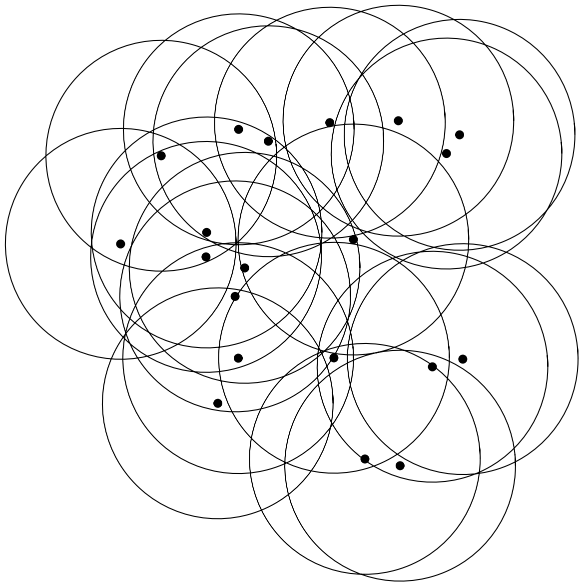

Figure 4.3: Twenty disks of equal radius.

Example 4.1.20 Consider the disks in Figure 4.3. Clearly, the classical inclusion-exclusion identity for the indicator function of their union contains $2^{20} - 1 = 1048575$ terms and involves intersections of up to twenty disks. In order to apply Theorem 4.1.19 we first form the Voronoi subdivision according to (4.8) and then the Delauney complex according to (4.9), see Figure 4.4. The resulting Delauney complex contains 20 vertices, 47 edges and 28 triangles. Hence, the improved inclusion-exclusion identity associated with the abstract tube of Theorem 4.1.19 contains only $20 + 47 + 28 = 95$ terms and intersections of up to three disks.

The next example explains the word *tube* in Definition 4.1.4.

Example 4.1.21 Figure 4.5 shows ten disks in $\mathbb{R}^2$ of equal radius with equidistant centers on a straight line. By Theorem 4.1.19, these ten disks together with the path comprising the centers of the disks constitute an abstract tube.

The preceding theorem has an analogue for the spherical case.

Definition 4.1.22 For any $x \in \mathbb{S}^d$, where $\mathbb{S}^d$ denotes the unit d-sphere in $\mathbb{R}^{d+1}$, and any $r > 0$ we use $B_d^*(x, r)$ to denote the spherical cap in $\mathbb{S}^d$ with center x and radius $r \in (0, \pi/2)$, that is, $B_d^*(x, r) := \{y \in \mathbb{S}^d \,|\, \delta^*(x, y) \leq r\}$ where

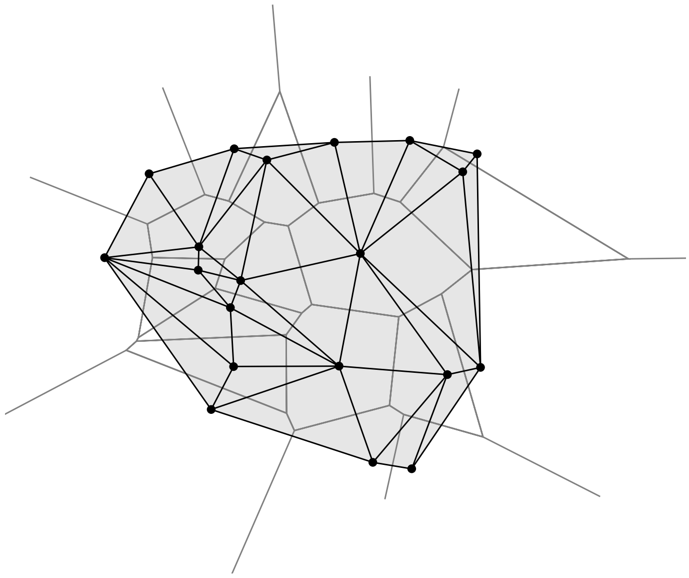

Figure 4.4: Voronoi subdivision and Delauney complex.

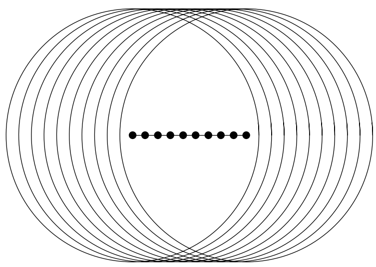

Figure 4.5: Ten disks of equal radius with centers on a straight line.

$\delta^*(x,y) := \cos^{-1}\langle x,y\rangle$ is the angular distance between x and y. With any finite subset V of $\mathbb{S}^d$ we associate the spherical Voronoi subdivision of $\mathbb{S}^d$ into regions

$$D_v^* \;:=\; \{x \in \mathbb{S}^d \,|\, \delta^*(x,v) = \min_{u\in V} \delta^*(x,u)\} \quad (v \in V)$$

and the spherical Delauney complex

$$\mathcal{D}^*(V) \;:=\; \left\{ I \in \mathcal{P}^*(V) \;\middle|\; \bigcap_{i\in I} D_i^* \neq \emptyset \right\}.$$

The spherical analogue of Theorem 4.1.19 follows.

Theorem 4.1.23 [NW92, NW97] *Let V be a finite set of points in $\mathbb{S}^d$ and $r \in (0,\pi/2)$. If $\bigcap_{v\in V} B_d^*(v,r) = \emptyset$, then $(\{B_d^*(v,r)\}_{v\in V}, \mathcal{D}^*(V))$ is an abstract tube.*

Proof. (Sketch) Let $\mathcal{S} := \mathcal{D}^*(V)$ and $\omega \in \bigcup_{v\in V} B_d^*(v,r)$. We have to show that $\mathcal{S}(\omega)$, with $B_d^*(i,r)$ in place of A_i, is contractible. Similar to the preceding proof, $\mathcal{S}(\omega)$ is the nerve of $\mathcal{C}^*(\omega) := \{D_i^* \,|\, \omega \in B_d^*(i,r)\}$, whence the contractibility of $\mathcal{S}(\omega)$ follows from that of $\bigcup \mathcal{C}^*(\omega)$. In fact, it can be shown that $\bigcup \mathcal{C}^*(\omega)$ contains every geodesic arc between any of its points and ω, whence it is contractible or equal to the sphere. The second alternative does not apply, since by the requirements of the theorem the intersection of the spherical caps is empty. □

Remark. Similar remarks as for the Euclidean case apply to the spherical case. In particular, if the centers of the spherical caps are in general position, that is, if no $d+1$ centers lie in a d-dimensional subspace and no $x \in \mathbb{S}^d$ is equidistant from more than $d+1$ of the centers, then the dimension of $\mathcal{D}^*(V)$ is at most d.

4.2 Abstract tubes via closures and kernels

In this section, the results of Section 3.1, Section 3.2 and Section 6.1 are restated and generalized in terms of abstract tubes. Recall from Section 4.1 that any abstract tube gives rise to improved Bonferroni inequalities. We do not mention these inequalities explicitly, since they can easily be read from Theorem 4.1.11.

Our first result is an abstract tube generalization of Theorem 3.1.14.

Theorem 4.2.1 [Doh99b] *Let (V, c) be a convex geometry, and let $\{A_v\}_{v\in V}$ be a finite family of sets such that for any non-empty and non-closed subset X of V,*

$$\bigcap_{x\in X} A_x \subseteq \bigcup_{v\notin X} A_v \,.$$

Then, $\big(\{A_v\}_{v\in V}, \{I \in \mathcal{P}^*(V) \,|\, I \text{ } c\text{-free}\}\big)$ is an abstract tube.

The proof of Theorem 4.2.1 is based on the following statement in Björner and Ziegler [BZ92, Exercise 8.23c]. For a rigorous proof of this statement the reader is referred to the very recent paper of Edelman and Reiner [ER00].

Proposition 4.2.2 [BZ92] *Let (V, c) be a convex geometry. Then the abstract simplicial complex consisting of all non-empty c-free subsets of V is contractible.*

Proof of Theorem 4.2.1. Let $\omega \in \bigcup_{v\in V} A_v$, $V_\omega := \{v \in V \,|\, \omega \in A_v\}$ and $c_\omega(I) := c(I)$ for any $I \subseteq V_\omega$. By the definition of V_ω and the requirements of the theorem, V_ω is c-closed. Thus, (V_ω, c_ω) is a convex geometry. Since moreover

$$\{I \in \mathcal{P}^*(V) \,|\, I \text{ } c\text{-free}\}(\omega) \;=\; \{I \in \mathcal{P}^*(V_\omega) \,|\, I \text{ } c_\omega\text{-free}\}\,,$$

the contractibility of $\{I \in \mathcal{P}^*(V) \,|\, I \text{ } c\text{-free}\}(\omega)$ follows from Proposition 4.2.2. □

Remarks. In view of the remarks in Section 4.1, it is equally easy to prove that $\big(\{A_v\}_{v\in V}, \{I \in \mathcal{P}^*(V)\,|\, I \text{ } c\text{-free}\}\big)$ is a weak abstract tube with respect to any probability measure P on the σ-field generated by $\{A_v\}_{v\in V}$ such that

$$(4.10)\qquad P\left(\bigcap_{x\in X} A_x\right) > 0 \quad\text{and}\quad P\left(\bigcup_{v\notin X} A_v \,\middle|\, \bigcap_{x\in X} A_x\right) = 1$$

for any non-empty and non-closed subset X of V.

We further remark that $\big(\{A_v\}_{v\in V}, \{I \in \mathcal{P}^*(V)\,|\, I \text{ } c'\text{-free}\}\big)$ is a subtube of $\big(\{A_v\}_{v\in V}, \{I \in \mathcal{P}^*(V)\,|\, I \text{ } c\text{-free}\}\big)$ if both c and c' satisfy the requirements of Theorem 4.2.1 and $c' \leq c$, where the partial ordering relation $\leq$ is given by

$$c' \leq c \quad :\Leftrightarrow \quad c(I) \subseteq c'(I) \text{ for any subset } I \text{ of } V$$

or equivalently,

$$c' \leq c \quad :\Leftrightarrow \quad \text{all } c'\text{-closed subsets of } V \text{ are } c\text{-closed.}$$

By this and Theorem 4.1.12, it follows that the improved Bonferroni inequalities associated with c' are at least as sharp as those associated with c if $c' \leq c$. In particular, since the closure operator $I \mapsto I$ on V is largest with respect to $\leq$, the new inequalities are at least as sharp as their classical counterparts.

As a consequence of Theorem 4.2.1 and as an extension of Corollary 3.1.17 and Corollary 3.1.19 we now deduce two results on the clique complex:

Definition 4.2.3 The *clique complex* of a graph G is the abstract simplicial complex of all non-empty cliques of G.

Corollary 4.2.4 [Doh99b] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where the indices form the vertices of a connected chordal graph $G = (V, E)$ such that $A_x \cap A_y \subseteq A_z$ for any $x, y \in V$ and any z on any chordless path between x and y. Then, $\{A_v\}_{v\in V}$ and the clique complex of G constitute an abstract tube.*

Proof. Corollary 4.2.4 follows from Theorem 4.2.1 in the same way as Corollary 3.1.17 follows from Theorem 3.1.14. □

Corollary 4.2.5 *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where the indices form the vertices of a Ptolemaic graph $G = (V, E)$ such that $A_x \cap A_y \subseteq A_z$ for any $x, y \in V$ and any z on any shortest path between x and y in G. Then, $\{A_v\}_{v\in V}$ and the clique complex of G constitute an abstract tube.*

Proof. Corollary 4.2.5 follows from Corollary 4.2.4 and Howorka's result [How81] that in Ptolemaic graphs the chordless paths are precisely the shortest paths. □

The following result is an abstract tube generalization of Theorem 3.2.5:

Theorem 4.2.6 [Doh00a] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, and let k be a kernel operator on V such that for any non-empty and k-open subset X of V,*

$$\bigcap_{x\in X} A_x \subseteq \bigcup_{v\notin X} A_v .$$

Then, $\big(\{A_v\}_{v\in V}, \{I \in \mathcal{P}^*(V) \,|\, k(I) = \emptyset\}\big)$ is an abstract tube.

The proof of Theorem 4.2.6 is based on the following proposition:

Proposition 4.2.7 *Let V be a finite set, and let k be a kernel operator on V. Then, the complex $\{I \in \mathcal{P}^*(V) \,|\, k(I) = \emptyset\}$ is contractible or V is k-open.*

Proof. Assume that V is not k-open, and let $v \in V \setminus k(V)$. Then, for any $I \subseteq V$, $v \notin k(I \cup \{v\})$, and hence the implication $k(I) = \emptyset \Rightarrow k(I \cup \{v\}) = \emptyset$ holds for any $I \subseteq V$. From this we conclude that v is contained in every maximal face of $\mathcal{S} := \{I \in \mathcal{P}^*(V) \,|\, k(I) = \emptyset\}$, whence $\mathcal{S}$ is a cone and hence contractible. (Recall from topology that a *cone* is an abstract simplicial complex $\mathcal{S}$ having a vertex v which is contained in every maximal face of $\mathcal{S}$. The geometric realization (4.2) of each such complex is contractible by means of $(x, t) \mapsto (1 - t)x + te_{\pi v}$.) □

Proof of Theorem 4.2.6. To obtain a contradiction, assume there is some $\omega \in \bigcup_{v\in V} A_v$ such that $\{I \in \mathcal{P}^*(V) \,|\, k(I) = \emptyset\}(\omega)$ is not contractible. From

$$\{I \in \mathcal{P}^*(V) \,|\, k(I) = \emptyset\}(\omega) = \{I \in \mathcal{P}^*(V_\omega) \,|\, k(I) = \emptyset\} ,$$

where $V_\omega := \{v \in V \,|\, \omega \in A_v\}$, the assumption and Proposition 4.2.7 we conclude that V_ω is k-open. On the contrary, the definition of V_ω and the requirements of the theorem entrain that V_ω is not k-open. Thus, Theorem 4.2.6 is proved. □

Remarks. The remarks concerning Theorem 4.2.1 apply in a similar way to Theorem 4.2.6. If P is a probability measure on the σ-field generated by the family $\{A_v\}_{v\in V}$ such that (4.10) holds for any non-empty and k-open subset X of V, then $\big(\{A_v\}_{v\in V}, \{I \in \mathcal{P}^*(V) \,|\, k(I) = \emptyset\}\big)$ is a weak abstract tube.

In general, $\big(\{A_v\}_{v\in V}, \{I \in \mathcal{P}^*(V) \,|\, k'(I) = \emptyset\}\big)$ is a subtube of $\big(\{A_v\}_{v\in V}, \{I \in \mathcal{P}^*(V) \,|\, k(I) = \emptyset\}\big)$ if k and k' are as required in Theorem 4.2.6 and $k' \leq k$, where

$$k' \leq k \quad :\Leftrightarrow \quad k(I) \subseteq k'(I) \text{ for any subset } I \text{ of } V \tag{4.11}$$

or equivalently,

$$k' \leq k \quad :\Leftrightarrow \quad \text{all } k\text{-open subsets of } V \text{ are } k'\text{-open.} \tag{4.12}$$

By this and Theorem 4.1.12, it follows that the improved Bonferroni inequalities associated with k' are at least as sharp as those associated with k if $k' \leq k$. In particular, since the kernel operator $I \mapsto \emptyset$ on V is largest with respect to $\leq$, the improved inequalities are at least as sharp as their classical counterparts.

Definition 4.2.8 With any finite set V and any system $\mathcal{X}$ of non-empty subsets of V we associate the abstract simplicial complex

$$\mathcal{I}(V,\mathcal{X}) \;:=\; \{I \in \mathcal{P}^*(V) \,|\, I \not\supseteq X \text{ for any } X \in \mathcal{X}\}\,,$$

consisting of all non-empty subsets of V not including any $X \in \mathcal{X}$ as a subset.

The correspondence between kernel operators and union-closed sets discussed in Section 3.2 leads to the following equivalent formulation of Theorem 4.2.6.

Theorem 4.2.9 [Doh00a] *Let $\{A_v\}_{v\in V}$ a finite family of sets, and let $\mathcal{X}$ be a union-closed set of non-empty subsets of V such that for any $X \in \mathcal{X}$,*

$$\bigcap_{x\in X} A_x \;\subseteq\; \bigcup_{v\notin X} A_v\,.$$

Then, $(\{A_v\}_{v\in V}, \mathcal{I}(V,\mathcal{X}))$ is an abstract tube.

Proof. The result follows by combining Proposition 3.2.2 and Theorem 4.2.6. □

Corollary 4.2.10 [Doh00a] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, c a closure operator on V and $\mathcal{X}$ a set of non-empty subsets of V such that $\{c(X)\,|\,X\in\mathcal{X}\}$ is a chain and such that for any $X \in \mathcal{X}$,*

$$\bigcap_{x\in X} A_x \;\subseteq\; \bigcup_{v\notin c(X)} A_v\,.$$

Then, $(\{A_v\}_{v\in V}, \mathcal{I}(V,\mathcal{X}))$ is an abstract tube.

Proof. Corollary 4.2.10 is deduced from Theorem 4.2.9 in the same way as Corollary 3.2.7 is deduced from Theorem 3.2.6. □

Corollary 4.2.11 [Doh99d] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where V is endowed with a linear ordering relation, and let $\mathcal{X}$ be a set of non-empty subsets of V such that for any $X \in \mathcal{X}$,*

$$\bigcap_{x\in X} A_x \;\subseteq\; \bigcup_{v>\max X} A_v\,.$$

Then, $(\{A_v\}_{v\in V}, \mathcal{I}(V,\mathcal{X}))$ is an abstract tube.

Proof. Corollary 4.2.11 follows from Corollary 4.2.10 in the same way as Corollary 3.2.8 follows from Corollary 3.2.7. □

The following definition essentially goes back to Alexandroff [Ale37].

Definition 4.2.12 The *order complex* of a finite partially ordered set V, $\mathcal{C}(V)$ for short, is the abstract simplicial complex of all non-empty chains of V.

Corollary 4.2.13 [Doh99d] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where V is endowed with a partial ordering relation such that for any $x, y \in V$, $A_x \cap A_y \subseteq A_z$ for some upper bound z of x and y. Then, $(\{A_v\}_{v\in V}, \mathcal{C}(V))$ is an abstract tube.*

Proof. Corollary 4.2.13 follows from Corollary 4.2.11 in the same way as Corollary 3.1.22 follows from Corollary 3.1.21. □

Remark. Note that the requirements of Corollary 4.2.13 are already satisfied if V is an upper semilattice such that $A_x \cap A_y \subseteq A_{x\vee y}$ for any $x, y \in V$. In this way, a specialization of Corollary 4.2.13 is obtained, which can also be deduced from Theorem 4.2.1 in connection with the convex geometry of Example 3.1.7.

We close this section with an elementary proof of the improved Bonferroni inequalities associated with the abstract tube of Theorem 4.2.6 and with an elementary proof of the fact that these inequalities are at least as sharp as their classical counterparts. The proofs are obtained by suitably adapting the second proof of Theorem 3.2.5. The proof of the following theorem is new even in the traditional case where $k(I) = \emptyset$ for any subset I of V. In this case, it generalizes Garsia and Milne's bijective proof of the classical inclusion-exclusion principle [GM81, Zei84, Pau86] to an "injective proof" of the traditional Bonferroni inequalities. For an alternative proof via Euler characteristics we refer to [Doh00a].

Theorem 4.2.14 [Doh00a] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, and let k be a kernel operator on V such that for any non-empty and k-open subset X of V,*

$$\bigcap_{x\in X} A_x \subseteq \bigcup_{v\notin X} A_v \,.$$

Then, for any $r \in \mathbb{N}$,

$$\chi\left(\bigcup_{v\in V} A_v\right) \geq \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ even}),$$

$$\chi\left(\bigcup_{v\in V} A_v\right) \leq \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\leq r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \qquad (r \text{ odd}).$$

Proof. It suffices to prove that

$$\chi\left(\bigcup_{v\in V} A_v\right) + \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I|\text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right) \ge \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I|\text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) \quad (r\text{ even}), \tag{4.13}$$

$$\chi\left(\bigcup_{v\in V} A_v\right) + \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I|\text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right) \le \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I|\text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) \quad (r\text{ odd}). \tag{4.14}$$

For any $\omega \in \bigcup_{v\in V} A_v$ and any $r\in\mathbb{N}$ define

$$\begin{aligned}
\mathcal{E}_r(\omega) &:= \left\{I\in\mathcal{P}(V_\omega)\,\middle|\,k(I)=\emptyset,\ |I|\le r,\ |I|\text{ even}\right\},\\
\mathcal{O}_r(\omega) &:= \left\{I\in\mathcal{P}(V_\omega)\,\middle|\,k(I)=\emptyset,\ |I|\le r,\ |I|\text{ odd}\right\},
\end{aligned}$$

where $V_\omega := \{v\in V\,|\,\omega\in A_v\}$. Obviously, (4.13) and (4.14) are equivalent to

$$|\mathcal{E}_r(\omega)| \ge |\mathcal{O}_r(\omega)| \quad \text{for all } \omega\in\bigcup_{v\in V} A_v \quad (r\text{ even}), \tag{4.15}$$

$$|\mathcal{E}_r(\omega)| \le |\mathcal{O}_r(\omega)| \quad \text{for all } \omega\in\bigcup_{v\in V} A_v \quad (r\text{ odd}). \tag{4.16}$$

To prove (4.15) and (4.16), fix $\omega\in\bigcup_{v\in V} A_v$. The definition of V_ω and the requirements of the theorem imply that V_ω is not k-open. Thus, some $v\in V_\omega\setminus k(V_\omega)$ can be chosen. Similar arguments as in the second proof of Theorem 3.2.5 reveal that $I\mapsto I\,\triangle\,\{v\}$, where $\triangle$ denotes symmetric difference, is an injective mapping from $\mathcal{O}_r(\omega)$ into $\mathcal{E}_r(\omega)$ if r is even and from $\mathcal{E}_r(\omega)$ into $\mathcal{O}_r(\omega)$ if r is odd. □

Theorem 4.2.15 *Let $\{A_v\}_{v\in V}$ be a finite family of sets, and let k and k' be kernel operators on V such that $k'\le k$ with respect to (4.11) or (4.12) and such that for any non-empty and k'-open subset X of V,*

$$\bigcap_{x\in X} A_x \subseteq \bigcup_{v\notin X} A_v\,.$$

Then, for any $r\in\mathbb{N}$,

$$\sum_{\substack{I\in\mathcal{P}^*(V)\\ k'(I)=\emptyset\\ |I|\le r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \ge \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \quad (r\ \textit{even}),$$

$$\sum_{\substack{I\in\mathcal{P}^*(V)\\ k'(I)=\emptyset\\ |I|\le r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \le \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right) \quad (r\ \textit{odd}).$$

Proof. It suffices to prove that

$$\sum_{\substack{I\in\mathcal{P}^*(V)\\ k'(I)=\emptyset\\ |I|\le r\\ |I|\text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) + \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I|\text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right) \;\ge\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I|\text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) + \sum_{\substack{I\in\mathcal{P}^*(V)\\ k'(I)=\emptyset\\ |I|\le r\\ |I|\text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right),$$

if r is even, and

$$\sum_{\substack{I\in\mathcal{P}^*(V)\\ k'(I)=\emptyset\\ |I|\le r\\ |I|\text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) + \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I|\text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right) \;\le\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ |I|\le r\\ |I|\text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) + \sum_{\substack{I\in\mathcal{P}^*(V)\\ k'(I)=\emptyset\\ |I|\le r\\ |I|\text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right),$$

if r is odd. Since $k' \le k$ these inequalities are equivalent to

$$\sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ k'(I)\neq\emptyset\\ |I|\le r\\ |I|\text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right) \;\ge\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ k'(I)\neq\emptyset\\ |I|\le r\\ |I|\text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) \quad (r\text{ even}), \tag{4.17}$$

$$\sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ k'(I)\neq\emptyset\\ |I|\le r\\ |I|\text{ even}}} \chi\left(\bigcap_{i\in I} A_i\right) \;\le\; \sum_{\substack{I\in\mathcal{P}^*(V)\\ k(I)=\emptyset\\ k'(I)\neq\emptyset\\ |I|\le r\\ |I|\text{ odd}}} \chi\left(\bigcap_{i\in I} A_i\right) \quad (r\text{ odd}). \tag{4.18}$$

For any $\omega \in \bigcup_{v\in V} A_v$ and any $r \in \mathbb{N}$ define

$$\begin{aligned}
\mathcal{E}^*_r(\omega) &:= \left\{ I \in \mathcal{P}^*(V_\omega) \,\middle|\, k(I)=\emptyset,\ k'(I)\neq\emptyset,\ |I|\le r,\ |I| \text{ even} \right\},\\
\mathcal{O}^*_r(\omega) &:= \left\{ I \in \mathcal{P}^*(V_\omega) \,\middle|\, k(I)=\emptyset,\ k'(I)\neq\emptyset,\ |I|\le r,\ |I| \text{ odd} \right\},
\end{aligned}$$

where $V_\omega := \{v \in V \,|\, \omega \in A_v\}$. Evidently, (4.17) and (4.18) are equivalent to

$$|\mathcal{E}^*_r(\omega)| \;\ge\; |\mathcal{O}^*_r(\omega)| \quad \text{for all } \omega \in \bigcup_{v\in V} A_v \quad (r\text{ even}), \tag{4.19}$$

$$|\mathcal{E}^*_r(\omega)| \;\le\; |\mathcal{O}^*_r(\omega)| \quad \text{for all } \omega \in \bigcup_{v\in V} A_v \quad (r\text{ odd}). \tag{4.20}$$

In order to establish (4.19) and (4.20), fix some $\omega \in \bigcup_{v\in V} A_v$ and choose some arbitrary $v \in V_\omega \setminus k'(V_\omega)$. Since $k' \le k$ it follows that $v \in V_\omega \setminus k(V_\omega)$. By similar arguments as in the second proof of Theorem 3.2.5 it follows that for any subset I of V_ω, $k(I)=\emptyset \Rightarrow k(I\cup\{v\})=\emptyset$ as well as $k'(I)\neq\emptyset \Rightarrow k'(I\setminus\{v\})\neq\emptyset$. Hence, $I \mapsto I \mathbin{\triangle} \{v\}$, where $\triangle$ denotes symmetric difference, is an injective mapping from $\mathcal{O}^*_r(\omega)$ into $\mathcal{E}^*_r(\omega)$ if r is even and from $\mathcal{E}^*_r(\omega)$ into $\mathcal{O}^*_r(\omega)$ if r is odd. □

4.3 The chordal graph sieve

The improved Bonferroni inequalities of the preceding sections are only valid if the collection of sets satisfies some structural restrictions. In this section, we turn our attention to inequalities that are generally valid, meaning that they are valid for any finite collection of sets. We already encountered such inequalities in our first chapter, where we referred to them as *inequalities of Bonferroni-Galambos type*. Here, we study generally valid inequalities where the selection of intersections in the estimates is determined by a graph. In the literature, such inequalities are known as *graph sieves* [GS96a, GS96b]. The first graph sieves were obtained by Rényi [Rén61] and Galambos [Gal66, Gal72], followed by Hunter [Hun76], Worsley [Wor82], Galambos and Simonelli [GS96a, GS96b], McKee [McK97, McK98] and Bukszár and Prékopa [BP99]. Several classes of graphs like trees, stars and joins of an edge and an edgeless graph are known to give rise to graph sieves. The hypertree sieve of Tomescu [Tom86] and related results by Grable [Gra93, Gra94] and Bukszár [Buk99] fall into the more general category of hypergraph sieves, which are not considered here. It is well-known and easy to see that the aforementioned classes of graphs are subclasses of the more comprehensive class of chordal graphs. The following theorem states that any chordal graph gives rise to a graph sieve, where the selection of intersections in the associated inequality corresponds to the non-empty cliques of the graph.

Theorem 4.3.1 *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where the indices form the vertices of a chordal graph $G=(V,E)$. Then, for any odd $r\in\mathbb{N}$,*

$$\chi\left(\bigcup_{v\in V} A_v\right) \le \sum_{\substack{I\in\mathcal{P}^*(V)\\ I \text{ clique of } G\\ |I|\le r}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right). \tag{4.21}$$

In particular,

$$\chi\left(\bigcup_{v\in V} A_v\right) \le \sum_{\substack{I\in\mathcal{P}^*(V)\\ I \text{ clique of } G}} (-1)^{|I|-1}\chi\left(\bigcap_{i\in I} A_i\right). \tag{4.22}$$

Proof. Let $\omega\in\bigcup_{v\in V} A_v$ and $V_\omega := \{v\in V \mid \omega\in A_v\}$. We have to show that

$$1 \le \sum_H \sum_{\substack{I \text{ clique of } H\\ 0<|I|\le r}} (-1)^{|I|-1} \tag{4.23}$$

where H runs over all connected components of $G[V_\omega]$. Since each such H is connected and chordal, we can define a convex geometry (V_H, c_H) as in the proof of Corollary 3.1.17, where V_H is the vertex-set of H and where for any $X\subseteq V_H$,

$$c_H(X) := \bigcup_{x,y\in X} \{z\in V_H \mid z \text{ is on a chordless path between } x \text{ and } y \text{ in } H\}.$$

Then, a subset I of V_H is c_H-free if and only if I is a clique of H. Hence, by Proposition 4.2.2, the abstract simplicial complex $\mathcal{S}_H$ consisting of all non-empty cliques of H is contractible. This in combination with Proposition 4.1.9 shows that the inner sum in (4.23) is at least one, whence (4.23) holds and the proof of (4.21) is complete. For the proof of (4.22) choose some odd $r \geq |V|$. □

As shown subsequently, the second part of the preceding theorem can be proved in an elementary way by employing the fact that the Euler characteristic of the complex of all non-empty cliques of a connected chordal graph equals 1.

Alternative proof of (4.22). Let $\omega \in \bigcup_{v \in V} A_v$. Obviously, it suffices to show that

$$1 \leq \sum_{H} \sum_{I \text{ clique of } H} (-1)^{|I|-1}, \tag{4.24}$$

where H runs over all connected components of $G[V_\omega]$. Corollary 3.1.10, applied to the convex geometry (V_H, c_H) of the preceding proof, shows that the inner sum on the right-hand side of (4.24) is equal to 1 and thus proves the result. □

Remark. (4.22) can also be proved within the general framework of Galambos and Simonelli [GS96a, GS96b]. Thus, (4.22) originates via [GS96a, Theorem II] or [GS96b, Theorem I.2] from the following property of chordal graphs: "for any non-trivial subgraph H of a chordal graph G, the Euler characteristic of the clique complex of H is at least one." A detailed proof is left as an option to the reader.

The following theorem complements the second part of the preceding theorem.

Theorem 4.3.2 *Let $\{A_v\}_{v \in V}$ be a finite family of sets, and let $G = (V, E)$ and $G' = (V, E')$ be two chordal graphs such that E is a subset of E'. Then,*

$$\sum_{\substack{I \in \mathcal{P}^*(V) \\ I \text{ clique of } G'}} (-1)^{|I|-1} \chi\left(\bigcap_{i \in I} A_i\right) \leq \sum_{\substack{I \in \mathcal{P}^*(V) \\ I \text{ clique of } G}} (-1)^{|I|-1} \chi\left(\bigcap_{i \in I} A_i\right).$$

Proof. Again, let $\omega \in \bigcup_{v \in V} A_v$ and $V_\omega := \{v \in V \mid \omega \in A_v\}$. We must show that

$$\sum_{H'} \gamma(\mathcal{S}_{H'}) \leq \sum_{H} \gamma(\mathcal{S}_H) \tag{4.25}$$

where H and H' run over all connected components of $G[V_\omega]$ and $G'[V_\omega]$, respectively, and where $\gamma(\mathcal{S}_H)$ and $\gamma(\mathcal{S}_{H'})$ denote the Euler characteristic of the abstract simplicial complexes $\mathcal{S}_H$ and $\mathcal{S}_{H'}$, which are defined as in the proof of Theorem 4.3.1. Recall from the preceding alternative proof of (4.22) that $\gamma(\mathcal{S}_H) = \gamma(\mathcal{S}_{H'}) = 1$, whence (4.25) holds if and only if the number of connected components of $G'[V_\omega]$ is at most the number of connected components of $G[V_\omega]$. Thus, the result follows from the requirement that E is a subset of E'. □

Remark. Theorem 4.3.1 generalizes several known results. For a complete graph, for instance, we obtain the classical Bonferroni upper bound (1.5), whereas for an edgeless graph we get Boole's inequality (1.3). In case of a tree we rediscover the following prominent result due to Hunter [Hun76] and Worsley [Wor82]:

Corollary 4.3.3 [Hun76, Wor82] *Let $\{A_v\}_{v\in V}$ be a finite family of sets, where the indices form the vertices of a tree $G = (V, E)$. Then,*

$$\chi\left(\bigcup_{v\in V} A_v\right) \leq \sum_{i\in V} \chi(A_i) - \sum_{\{i,j\}\in E} \chi(A_i \cap A_j)\,. \tag{4.26}$$

Proof. Since trees are chordal, the corollary follows from Theorem 4.3.1. □

Remarks. As observed by Hunter [Hun76] and Worsley [Wor82], the best possible upper bound in (4.26) with respect to some probability measure P is obtained by choosing a maximum spanning tree G for the complete weighted graph on V, where each edge $\{i, j\}$ has weight $P(A_i \cap A_j)$. Such a maximum spanning tree can be found efficiently by applying the greedy algorithm [Kru56, Pri57].

By Corollary 3.1.16, the inequality of Corollary 4.3.3 becomes an identity if $A_x \cap A_y \subseteq A_z$ for any $x, y \in V$ and z on the unique path between x and y in G.

By induction on m it follows that all graphs of type $K_m * L_n$ (the join of a complete graph and an edgeless graph) and $K_m * P_n$ (the join of a complete graph and a path) are chordal and thus give rise to inequalities of Bonferroni-Galambos type via Theorem 4.3.1. Some of these graphs are shown in Figures 4.6–4.10. Remarkably, as we will see below, many well-known and new inequalities can be derived in a unified and simplified way by considering these types of graphs.

The following results of Hunter [Hun76] and Worsley [Wor82] and Kounias [Kou68] are well-known consequences of (4.26) and thus of Theorem 4.3.1.

Corollary 4.3.4 [Hun76, Wor82] *Let $A_1, \ldots, A_n$ be sets. Then,*

$$\chi\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \chi(A_i) - \sum_{i=2}^{n} \chi(A_{i-1} \cap A_i)\,. \tag{4.27}$$

Proof. Consider the path $G = P_n$ on n vertices and apply Theorem 4.3.1. □

Corollary 4.3.5 [Kou68] *Let $A_1, \ldots, A_n$ be sets. Then, for any $j \in \{1, \ldots, n\}$,*

$$\chi\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \chi(A_i) - \sum_{\substack{i=1\\ i\neq j}}^{n} \chi(A_i \cap A_j)\,. \tag{4.28}$$

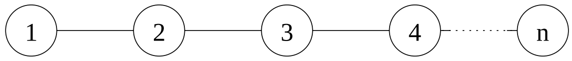

Figure 4.6: The path P_n on n vertices.

Proof. Consider the star $G = K_1 * L_{n-1}$ and apply Theorem 4.3.1. □

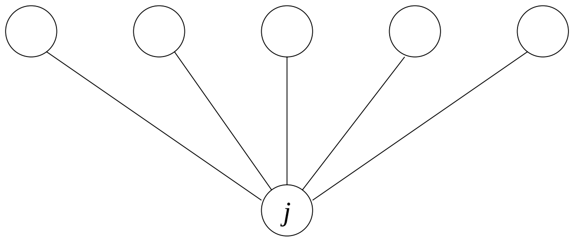

Figure 4.7: The star $K_1 * L_{n-1}$ for $n = 6$.

A well-known consequence of Kounias' inequality is the following result due to Kwerel [Kwe75b]:

Corollary 4.3.6 [Kwe75b] *Let $A_1, \ldots, A_n$ be sets. Then,*

$$\chi\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \chi(A_i) - \frac{2}{n} \sum_{\substack{i,j=1 \\ i<j}}^{n} \chi(A_i \cap A_j)\,.$$

Proof. Take the average over $j = 1, \ldots, n$ in Corollary 4.3.5. □

Our proof of the next inequality due to Seneta [Sen88] is based on the chordality of the graph $K_2 * L_{n-2}$. A self-contained proof of this inequality using essentially the same graph was given by Galambos and Simonelli [GS96a, GS96b]. By Theorem 4.3.2 the inequality is at least as sharp as (4.28) for any $j \in \{1, \ldots, n\}$.

Corollary 4.3.7 [Sen88] *Let $A_1, \ldots, A_n$ be sets and $j, k \in \{1, \ldots, n\}$. Then,*

$$\chi\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \chi(A_i) - \sum_{\substack{i=1 \\ i\neq j}}^{n} \chi(A_i \cap A_j) - \sum_{\substack{i=1 \\ i\neq j,k}}^{n} \chi(A_i \cap A_k) + \sum_{\substack{i=1 \\ i\neq j,k}}^{n} \chi(A_i \cap A_j \cap A_k)\,.$$

Proof. For $j = k$ the inequality coincides with that of Corollary 4.3.5. For $j \neq k$ consider the chordal graph $G = K_2 * L_{n-2}$ and apply Theorem 4.3.1. □

By applying Corollary 4.3.7 we reprove another result of Kwerel [Kwe75a]:

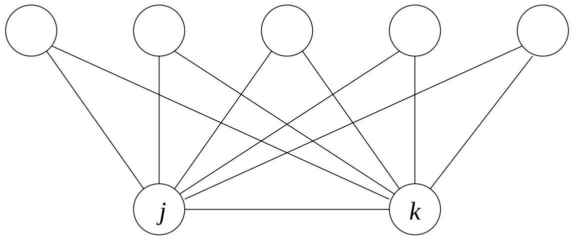

Figure 4.8: The graph $K_2 * L_{n-2}$ for $n = 7$.

Corollary 4.3.8 [Kwe75a] *Let $A_1, \ldots, A_n$ be sets where $n \geq 2$. Then,*

$$\chi\left(\bigcup_{v \in V} A_v\right) \leq \sum_{i=1}^{n} \chi(A_i) - \frac{2n-3}{\binom{n}{2}} \sum_{\substack{i,j=1\\ i<j}}^{n} \chi(A_i \cap A_j) + \frac{3}{\binom{n}{2}} \sum_{\substack{i,j,k=1\\ i<j<k}}^{n} \chi(A_i \cap A_j \cap A_k).$$

Proof. Take the average over all distinct $j, k = 1, \ldots, n$ in Corollary 4.3.7. □

Remark. For a probability measure P, the upper bound in Corollary 4.3.8 is best possible among all upper bounds of type $t_1(n)S_1 + t_2(n)S_2 + t_3(n)S_3$ where

$$S_k = \sum_{i_1 < \cdots < i_k} P\big(A_{i_1} \cap \cdots \cap A_{i_k}\big)\,, \tag{4.29}$$

if $n \leq 3 + \lfloor 3S_3/S_2 \rfloor$, cf. [GS96b, Inequality I.8]. An straightforward calculation shows that the upper bound of Corollary 4.3.8 is at least as sharp as the classical Bonferroni upper bound of degree three if and only if $(n-2)/(n+2) \leq S_3/S_2$.

By Theorem 4.3.2 the following upper bound is at least as sharp as (4.27).

Corollary 4.3.9 *Let $A_1, \ldots, A_n$ be sets. Then,*

$$\begin{aligned}\chi\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \chi(A_i) - \sum_{i=2}^{n} \chi(A_{i-1} \cap A_i) - \sum_{i=2}^{n-1} \chi(A_{i-1} \cap A_n) \\ + \sum_{i=2}^{n-1} \chi(A_{i-1} \cap A_i \cap A_n)\,.\end{aligned}$$

Proof. Consider the graph $G = K_1 * P_{n-1}$ and apply Theorem 4.3.1. □

By Theorem 4.3.2 the next bound is at least as sharp as the preceding one.

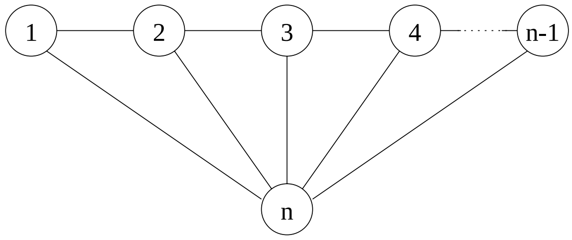

Figure 4.9: The graph $K_1 * P_{n-1}$.

Corollary 4.3.10 *Let $A_1, \ldots, A_n$ be sets. Then,*

$$\chi\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \chi(A_i) - \sum_{i=2}^{n} \chi(A_{i-1} \cap A_i) - \sum_{i=2}^{n-1} \chi(A_{i-1} \cap A_n)$$
$$- \sum_{i=2}^{n-2} \chi(A_{i-1} \cap A_{n-1}) + \sum_{i=2}^{n-2} \chi(A_{i-1} \cap A_i \cap A_n) + \sum_{i=2}^{n-2} \chi(A_{i-1} \cap A_i \cap A_{n-1})$$
$$+ \sum_{i=1}^{n-2} \chi(A_i \cap A_{n-1} \cap A_n) - \sum_{i=2}^{n-2} \chi(A_{i-1} \cap A_i \cap A_{n-1} \cap A_n) .$$

Proof. Consider the graph $G = K_2 * P_{n-2}$ and apply Theorem 4.3.1. □

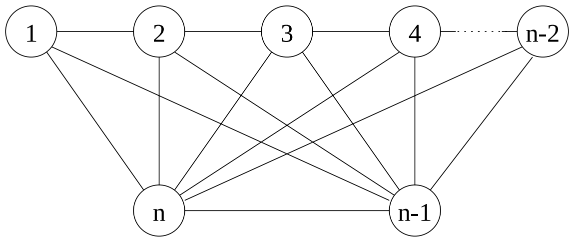

Figure 4.10: The graph $K_2 * P_{n-2}$.

The following inequality is at least as sharp as that of Corollary 4.3.4.

Corollary 4.3.11 *Let $A_1, \ldots, A_n$ be sets. Then,*

$$\chi\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \chi(A_i) - \sum_{i=2}^{n} \chi(A_{i-1} \cap A_i) - \sum_{i=2}^{n-1} \chi(A_{i-1} \cap A_{i+1})$$
$$+ \sum_{i=2}^{n-1} \chi(A_{i-1} \cap A_i \cap A_{i+1}) .$$

Proof. Consider the graph $G = (V, E)$ shown in Figure 4.11 where $V = \{1, \dots, n\}$ and $E = \{\{i, j\} \subseteq \{1, \dots, n\} \mid 1 \le |j - i| \le 2\}$, and apply Theorem 4.3.1. □

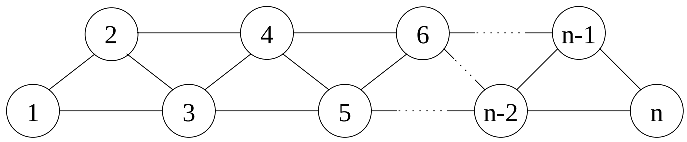

Figure 4.11: The graph in the proof of Corollary 4.3.11 for odd n.

We close this section with a generalization of Boole's inequality (1.3), Corollary 4.3.6 and Corollary 4.3.8. For even m this generalization coincides with a special case of an inequality due to Galambos and Xu [GX90]. Our proof, which is based on the chordal graph sieve of Theorem 4.3.1, is new even in this case.

Theorem 4.3.12 *Let $A_1, \dots, A_n$ be sets. Then, for $m = 0, \dots, n$ we have*

$$\chi\left(\bigcup_{v\in V} A_v\right) \le \sum_{k=1}^{m} (-1)^{k-1} \frac{\binom{m}{k}}{\binom{n}{k}} \frac{nk - (m+1)(k-1)}{m-k+1} \sum_{i_1<\dots<i_k} \chi\big(A_{i_1} \cap \dots \cap A_{i_k}\big)$$
$$+ (-1)^m \frac{m+1}{\binom{n}{m}} \sum_{i_1<\dots<i_{m+1}} \chi\big(A_{i_1} \cap \dots \cap A_{i_{m+1}}\big).$$

Proof. For $m = n$ the right-hand side of the preceding inequality coincides with the classical inclusion-exclusion formula, whence in the sequel we may assume that $m < n$. For any subset M of $\{1, \dots, n\}$ let G_M denote the join of the complete graph on M and the edgeless graph on the complement of M. By averaging the right-hand side of (4.22) over all graphs G_M with $|M| = m$ we obtain

$$\begin{aligned} \chi\left(\bigcup_{v\in V} A_v\right) &\le \frac{1}{\binom{n}{m}} \sum_{\substack{M\subseteq\{1,\dots,n\}\\ |M|=m}} \sum_{\substack{I\in\mathcal{P}^*(\{1,\dots,n\})\\ I \text{ clique of } G_M}} (-1)^{|I|-1} \chi\left(\bigcap_{i\in I} A_i\right) \\ &= \sum_{k=1}^{m+1} (-1)^{k-1} \sum_{\substack{I\in\mathcal{P}^*(\{1,\dots,n\})\\ |I|=k}} \frac{c(n,m,I)}{\binom{n}{m}} \chi\left(\bigcap_{i\in I} A_i\right), \end{aligned}$$

where $c(n, m, I)$ signifies the number of m-subsets M of $\{1, \dots, n\}$ such that I is a clique of G_M. Since I is a clique of G_M iff $I \subseteq M$ or $|I \setminus M| = 1$, we obtain

$$c(n,m,I) = \begin{cases} \binom{n-|I|}{m-|I|} + |I|\binom{n-|I|}{m-|I|+1} & \text{if } |I| \le m, \\ m+1 & \text{if } |I| = m+1, \end{cases}$$

and hence,

$$
\begin{aligned}
\chi\left(\bigcup_{v\in V} A_v\right) \;\le\; & \sum_{k=1}^{m} (-1)^{k-1} \frac{\binom{n-k}{m-k} + k\binom{n-k}{m-k+1}}{\binom{n}{m}} \sum_{i_1<\cdots<i_k} \chi\big(A_{i_1}\cap\cdots\cap A_{i_k}\big) \\
& + (-1)^m \frac{m+1}{\binom{n}{m}} \sum_{i_1<\cdots<i_{m+1}} \chi\big(A_{i_1}\cap\cdots\cap A_{i_{m+1}}\big)\,.
\end{aligned}
$$

The result now follows from the preceding inequality and the binomial identity

$$
\frac{\binom{n-k}{m-k} + k\binom{n-k}{m-k+1}}{\binom{n}{m}} \;=\; \frac{\binom{m}{k}}{\binom{n}{k}}\,\frac{nk-(m+1)(k-1)}{m-k+1}\,,
$$

which is an easy combinatorial exercise. □

Chapter 5

Reliability Applications

In many practical situations one is interested in the probability that a technical system with randomly failing components is operating. Examples include transportation networks, electrical power systems, pipeline networks and nuclear power plants. In recent years, the study of system reliability has received considerable attention from its applicability to computer and telecommunication networks.

One of the standard methods in reliability theory is the principle of inclusion-exclusion and the associated Bonferroni inequalities. This chapter deals with improvements of this method derived from the results of the preceding chapters. In this way, we rediscover Shier's semilattice expression and recursive algorithm for system reliability [Shi88, Shi91] and establish related inequalities based on abstract tubes. The results are then applied in the more specific context of network reliability, k-out-of-n systems, consecutive k-out-of-n systems and covering problems, where several results from the literature are rediscovered in a concise way. Examples demonstrate that the new reliability bounds are much sharper than the usual Bonferroni bounds, although less computational effort is needed to compute them. We finally identify a new class of hypergraphs for which the extremely difficult reliability covering problem can be solved in polynomial time.

For an introduction to reliability theory with emphasis on networks we refer to the monographs of Colbourn [Col87] and Shier [Shi91]. For a general introduction to reliability theory we recommend the monograph of Aven and Jensen [AJ99].

5.1 System reliability

The following definition essentially goes back to Esary and Proschan [EP63].

Definition 5.1.1 A *coherent system* or—to be more precise—a *coherent binary system* is a couple $\Sigma = (E, \phi)$ consisting of a finite set E and a function ϕ from the power set of E into $\{0, 1\}$ such that $\phi(\emptyset) = 0$, $\phi(E) = 1$ and $\phi(X) \leq \phi(Y)$ for any $X, Y \subseteq E$ with $X \subseteq Y$. E and ϕ are respectively called the *component set* and the *structure function* of Σ. It is supposed that each component e of Σ

assumes randomly and independently one of two states, *operating* or *failing*, with known probabilities p_e and $q_e = 1 - p_e$, respectively. Σ is said to be *operating* resp. *failing* if ϕ applied to the set of operating components, which is also referred to as the *state* of Σ, gives 1 resp. 0. The probability that Σ is operating is referred to as the *reliability* of Σ and abbreviated to $\mathrm{Rel}_\Sigma(\mathbf{p})$, where $\mathbf{p} = (p_e)_{e\in E}$.

Remark. Note that by the preceding definition the components of a coherent system operate and fail in a statistically independent fashion. Of course, this definition could be relaxed by allowing dependent component failures, and indeed most of our inclusion-exclusion expressions for system and network reliability can easily be adapted to this more general setting. For ease of presentation, however, we prefer to use the more restrictive definition. A generalization to dependent component failures is (whenever possible) left as an option to the reader.

A key role in calculating $\mathrm{Rel}_\Sigma(\mathbf{p})$ is played by the minpaths and mincuts of Σ:

Definition 5.1.2 A *minpath* of a coherent binary system $\Sigma = (E, \phi)$ is a minimal set $P \subseteq E$ such that $\phi(P) = 1$; that is, $\phi(P) = 1$ and $\phi(Q) = 0$ for any proper subset Q of P. A *mincut* of Σ is a minimal set $C \subseteq E$ such that $\phi(E \setminus C) = 0$; that is, $\phi(E \setminus C) = 0$ and $\phi(E \setminus D) = 1$ for any proper subset D of C.

Thus, with $\mathcal{F}$ denoting the set of minpaths resp. mincuts,

$$\mathrm{Rel}_\Sigma(\mathbf{p}) = P\left(\bigcup_{F\in\mathcal{F}}\{F \text{ operates}\}\right) \text{ resp. } 1 - \mathrm{Rel}_\Sigma(\mathbf{p}) = P\left(\bigcup_{F\in\mathcal{F}}\{F \text{ fails}\}\right),$$

where P denotes the probability measure on the set of system states and where

$$\{F \text{ operates}\} := \bigcap_{e\in F}\{e \text{ operates}\}; \quad \{F \text{ fails}\} := \bigcap_{e\in F}\{e \text{ fails}\}$$

for any $F \in \mathcal{F}$. In the following, $\mathcal{C}(\mathcal{F})$ is used to denote the order complex of $\mathcal{F}$.

Theorem 5.1.3 [Doh99d] *Let $\Sigma = (E, \phi)$ be a coherent binary system, whose set of minpaths resp. mincuts $\mathcal{F}$ is given the structure of a lower semilattice such that $X \wedge Y \subseteq X \cup Y$ for any $X, Y \in \mathcal{F}$. Then,*

$$\left(\{\{F \text{ operates}\}\}_{F\in\mathcal{F}}, \mathcal{C}(\mathcal{F})\right) \quad \textit{resp.} \quad \left(\{\{F \text{ fails}\}\}_{F\in\mathcal{F}}, \mathcal{C}(\mathcal{F})\right)$$

is an abstract tube.

Proof. Apply Corollary 4.2.13 (dualized) with $V := \mathcal{F}$ and $A_F := \{F \text{ operates}\}$ resp. $A_F := \{F \text{ fails}\}$ for any $F \in \mathcal{F}$. □

Remark. The improved inclusion-exclusion identities associated with the abstract tubes of the preceding theorem are due to Shier [Shi88, Shi91], whereas the corresponding inequalities are due to the author [Doh99c]. Subsequently, the improved inclusion-exclusion identities and Bonferroni inequalities are explicitly stated. Note that they specialize to the usual inclusion-exclusion identities and Bonferroni inequalities for system reliability if $\mathcal{F}$ is a chain, that is, if $X \wedge Y = X$ or $X \wedge Y = Y$ for any $X, Y \in \mathcal{F}$. In this case, of course, $\mathcal{C}(\mathcal{F})$ equals $\mathcal{P}^*(\mathcal{F})$.

Theorem 5.1.4 [Doh99c] *Let $\Sigma = (E, \phi)$ be a coherent binary system, whose set of minpaths resp. mincuts $\mathcal{F}$ is a lower semilattice such that $X \wedge Y \subseteq X \cup Y$ for any $X, Y \in \mathcal{F}$, and let $r \in \mathbb{N}$. Then, in case that $\mathcal{F}$ denotes the set of minpaths,*

$$\begin{aligned}
\mathrm{Rel}_\Sigma(\mathbf{p}) &\geq \sum_{\substack{\mathcal{I} \in \mathcal{C}(\mathcal{F}) \\ |\mathcal{I}| \leq r}} (-1)^{|\mathcal{I}|-1} \prod_{e \in \bigcup \mathcal{I}} p_e \quad (r \text{ even}), \\
\mathrm{Rel}_\Sigma(\mathbf{p}) &\leq \sum_{\substack{\mathcal{I} \in \mathcal{C}(\mathcal{F}) \\ |\mathcal{I}| \leq r}} (-1)^{|\mathcal{I}|-1} \prod_{e \in \bigcup \mathcal{I}} p_e \quad (r \text{ odd}),
\end{aligned}$$

and in case that $\mathcal{F}$ denotes the set of mincuts,

$$\begin{aligned}
1 - \mathrm{Rel}_\Sigma(\mathbf{p}) &\geq \sum_{\substack{\mathcal{I} \in \mathcal{C}(\mathcal{F}) \\ |\mathcal{I}| \leq r}} (-1)^{|\mathcal{I}|-1} \prod_{e \in \bigcup \mathcal{I}} q_e \quad (r \text{ even}), \\
1 - \mathrm{Rel}_\Sigma(\mathbf{p}) &\leq \sum_{\substack{\mathcal{I} \in \mathcal{C}(\mathcal{F}) \\ |\mathcal{I}| \leq r}} (-1)^{|\mathcal{I}|-1} \prod_{e \in \bigcup \mathcal{I}} q_e \quad (r \text{ odd}),
\end{aligned}$$

where in both cases $\mathbf{p} = (p_e)_{e \in E} \in [0,1]^E$ and $q_e = 1 - p_e$ for any $e \in E$.

Proof. The result directly follows from Theorem 5.1.3 and Theorem 4.1.11. □

Remarks. In view of Theorem 4.1.12 the inequalities provided by Theorem 5.1.4 are at least as sharp as the usual Bonferroni inequalities for system reliability.

Note that Theorem 5.1.4 can easily be generalized to dependent component failures by replacing $\prod_{e \in \bigcup \mathcal{I}} p_e$ resp. $\prod_{e \in \bigcup \mathcal{I}} q_e$ with the probability that all components in $\bigcup \mathcal{I}$ work resp. fail. The same applies to the following theorem:

Theorem 5.1.5 [Shi88, Shi91] *Let $\Sigma = (E, \phi)$ be a coherent binary system, whose set of minpaths resp. mincuts $\mathcal{F}$ is a lower semilattice such that $X \wedge Y \subseteq X \cup Y$ for any $X, Y \in \mathcal{F}$. Then, in case that $\mathcal{F}$ denotes the set of minpaths,*

$$\mathrm{Rel}_\Sigma(\mathbf{p}) = \sum_{\mathcal{I} \in \mathcal{C}(\mathcal{F})} (-1)^{|\mathcal{I}|-1} \prod_{e \in \bigcup \mathcal{I}} p_e \,,$$

and in case that $\mathcal{F}$ denotes the set of mincuts,

$$1 - \mathrm{Rel}_\Sigma(\mathbf{p}) = \sum_{\mathcal{I} \in \mathcal{C}(\mathcal{F})} (-1)^{|\mathcal{I}|-1} \prod_{e \in \bigcup \mathcal{I}} q_e \,,$$

where in both cases $\mathbf{p} = (p_e)_{e \in E} \in [0,1]^E$ and $q_e = 1 - p_e$ for any $e \in E$.

Proof. Theorem 5.1.5 is an immediate consequence of Theorem 5.1.4. Alternatively, it can be deduced from the dual version of Corollary 3.1.22 in the same way as Theorem 5.1.3 is deduced from the dual version of Corollary 4.2.13. □

The following theorem, which is due to Shier [Shi88, Shi91], generalizes some results of Provan and Ball [PB84, BP87] on two-terminal and source-to-T-terminal network reliability. The theorem requires the following notion of convexity:

Definition 5.1.6 A subset X of a partially ordered set V is *convex* if $[x, y] \subseteq X$ for any $x, y \in X$, where $[x, y]$ denotes the interval $\{z \in V \mid x \leq z \leq y\}$.

Theorem 5.1.7 [Shi88, Shi91] *Let $\Sigma = (E, \phi)$ be a coherent binary system, whose set of minpaths resp. mincuts $\mathcal{F}$ is a lower semilattice such that $X \wedge Y \subseteq X \cup Y$ for any $X, Y \in \mathcal{F}$ and $\{F \in \mathcal{F} \mid e \in F\}$ is convex for any $e \in E$. Then,*

$$\text{(5.1)} \qquad \mathrm{Rel}_\Sigma(\mathbf{p}) = \sum_{F \in \mathcal{F}} \Lambda(F, \mathbf{p}) \quad \textit{resp.} \quad 1 - \mathrm{Rel}_\Sigma(\mathbf{p}) = \sum_{F \in \mathcal{F}} \Lambda(F, \mathbf{q}),$$

where $\mathbf{q} = \mathbf{1} - \mathbf{p} \in [0, 1]^E$ and where in both cases Λ is defined recursively by

$$\text{(5.2)} \qquad \Lambda(F, \mathbf{x}) := \prod_{e \in F} x_e - \sum_{G < F} \Lambda(G, \mathbf{x}) \prod_{e \in F \setminus G} x_e\,; \quad \mathbf{x} = (x_e)_{e \in E} \in [0, 1]^E.$$

Based on Theorem 3.3.1 we now establish a new and simplified proof of Theorem 5.1.7, which does not make use of the disjoint products technique:

Proof. We apply Theorem 3.3.1 (dualized) with $V := \mathcal{F}$ and $A_F := \{F \text{ operates}\}$ resp. $A_F := \{F \text{ fails}\}$. By the assumptions, $A_X \cap A_Y \subseteq A_{X \wedge Y}$ for any $X, Y \in \mathcal{F}$. Define $B_e := \{e \text{ operates}\}$ resp. $B_e := \{e \text{ fails}\}$ for any $e \in E$. Then, for any $F \in \mathcal{F}$, $A_F = \bigcap_{e \in F} B_e$ and thus, by Theorem 3.3.1, we have to check that

$$P\left(\bigcap_{i \in I_1} B_i \,\middle|\, \bigcap_{i \in I_2} B_i \cap \cdots \cap \bigcap_{i \in I_k} B_i\right) = P\left(\bigcap_{i \in I_1} B_i \,\middle|\, \bigcap_{i \in I_2} B_i\right)$$

for any chain $I_1 > \cdots > I_k$ in $\mathcal{F}$ where $k > 1$ and where P again denotes the induced probability measure on the set of system states. Since the components of the system are assumed to operate and fail independently, we find that

$$P\left(\bigcap_{i \in I_1} B_i \,\middle|\, \bigcap_{i \in I_2} B_i \cap \cdots \cap \bigcap_{i \in I_k} B_i\right) = P\left(\bigcap_{i \in I_1 \setminus (I_2 \cup \cdots \cup I_k)} B_i\right).$$

By the convexity assumption, $I_1 \setminus (I_2 \cup \cdots \cup I_k) = I_1 \setminus I_2$ and therefore,

$$P\left(\bigcap_{i \in I_1 \setminus (I_2 \cup \cdots \cup I_k)} B_i\right) = P\left(\bigcap_{i \in I_1 \setminus I_2} B_i\right) = P\left(\bigcap_{i \in I_1} B_i \,\middle|\, \bigcap_{i \in I_2} B_i\right),$$

where the last equals sign again follows from the independence assumption. □

Remarks. In view of Theorem 3.3.2, replacing "<" with ">" in (5.2) would result in a different recursive scheme for $\mathrm{Rel}_\Sigma(\mathbf{p})$. By suitably adapting Algorithm I, we obtain Shier's dynamic programming solution to (5.1) and (5.2), which is restated as Algorithm II. As pointed out by Shier [Shi88, Shi91], this algorithm has a space complexity of $O(|\mathcal{F}|)$ and a time complexity of $O(|E| \times |\mathcal{F}|^2)$, since there are at most $O(|\mathcal{F}|^2)$ products to be calculated in line 7 and the calculation of each requires at most $O(|E|)$ work. Thus, the algorithm is pseudopolynomial, that is, its running time is bounded by a polynomial in the number of components and the number of minpaths resp. mincuts. The classical inclusion-exclusion method for the same problem has a time complexity of $O(|E| \times 2^{|\mathcal{F}|})$. While the classical inclusion-exclusion method and its improvements provided by Theorem 5.1.4 and Theorem 5.1.5 can easily be adapted to deal with statistically dependent component failures, Theorem 5.1.7 strongly relies on the independence assumption.

Algorithm II Pseudopolynomial algorithm for computing system reliability

Require: Same requirements as in Theorem 5.1.7; $\mathbf{x} = (x_e)_{e \in E} \in [0,1]^E$
Ensure: $prob = \sum_{F \in \mathcal{F}} \Lambda(F, \mathbf{x})$
1: Find an ordering $F_1, \ldots, F_n$ of $\mathcal{F}$ such that $F_i < F_j \Rightarrow i < j$ $(i, j = 1, \ldots, n)$
2: $prob \leftarrow 0$
3: **for** $i = 1$ to n **do**
4: $\quad acc \leftarrow 0$
5: $\quad$ **for** $j = 1$ to $i - 1$ **do**
6: $\quad\quad$ **if** $F_j < F_i$ **then**
7: $\quad\quad\quad acc \leftarrow acc + a[j] \prod_{e \in F_i \setminus F_j} x_e$
8: $\quad\quad$ **end if**
9: $\quad$ **end for**
10: $\quad a[i] \leftarrow \prod_{e \in F_i} x_e - acc$
11: $\quad prob \leftarrow prob + a[i]$
12: **end for**

In the same way as Theorem 5.1.7 follows from Theorem 3.3.1, the following inequalities follow from Theorem 3.3.3. These inequalities ensure that during execution of Algorithm II, *prob* provides a lower bound to the desired value.

Theorem 5.1.8 *Under the requirements of Theorem 5.1.7,*

$$\mathrm{Rel}_\Sigma(\mathbf{p}) \geq \sum_{F \in \mathcal{F}'} \Lambda(F, \mathbf{p}) \quad \textit{resp.} \quad 1 - \mathrm{Rel}_\Sigma(\mathbf{p}) \geq \sum_{F \in \mathcal{F}'} \Lambda(F, \mathbf{q})$$

for any subset $\mathcal{F}'$ of $\mathcal{F}$, where $\mathbf{q} = \mathbf{1} - \mathbf{p}$ and Λ is defined recursively as in (5.2).

Remark. Without modifying the proofs, the preceding theorems can be generalized by imposing the somewhat weaker requirement that $\mathcal{F}$ is an extended set of minpaths resp. mincuts of Σ. These concepts are introduced subsequently:

Definition 5.1.9 An *extended set of minpaths* of a coherent binary system Σ is an upper set $\mathcal{F}$ of the set of minpaths of Σ such that any $F \in \mathcal{F}$ is an upper set of some minpath of Σ. An *extended set of mincuts* of Σ is an upper set $\mathcal{F}$ of the set of mincuts of Σ such that any $F \in \mathcal{F}$ is an upper set of some mincut of Σ.

The rest of this section is devoted to general domination theory.

Definition 5.1.10 Let $\Sigma = (E, \phi)$ be a coherent binary system, whose set of minpaths resp. mincuts is denoted by $\mathcal{F}$. An *$\mathcal{F}$-formation* of a subset X of E is any subset $\mathcal{I}$ of $\mathcal{F}$ such that $\bigcup \mathcal{I} = X$. An $\mathcal{F}$-formation $\mathcal{I}$ of X is *odd* resp. *even* if $|\mathcal{I}|$ is odd resp. even. The *$\mathcal{F}$-domination* of X, $\mathrm{dom}_{\mathcal{F}}(X)$, is the number of odd $\mathcal{F}$-formations of X minus the number of even $\mathcal{F}$-formations of X.

The concept of domination was introduced by Satyanarayana and Prabhakar [SP78] in the specific context of source-to-terminal reliability of directed networks. The following well-known proposition yields condensed inclusion-exclusion formulae for system reliability based on the concept of domination. These condensed formulae contain no cancelling terms, and in this sense they are best possible among all inclusion-exclusion expansions. However, apart from some particular measures associated with directed networks [SP78, SH81, Sat82] and consecutively connected systems [KP89, SM91] the determination of $\mathrm{dom}_{\mathcal{F}}(X)$ is a tiresome task and in fact amounts to computing the Möbius function of a lattice.

Proposition 5.1.11 [SP78] *Let $\Sigma = (E, \phi)$ be a coherent binary system, whose set of minpaths resp. mincuts is denoted by $\mathcal{F}$. Then,*

$$\mathrm{Rel}_\Sigma(\mathbf{p}) \;=\; \sum_{X \in \mathcal{P}^*(E)} \mathrm{dom}_{\mathcal{F}}(X) \prod_{e \in X} p_e \,,$$

respectively

$$1 - \mathrm{Rel}_\Sigma(\mathbf{p}) \;=\; \sum_{X \in \mathcal{P}^*(E)} \mathrm{dom}_{\mathcal{F}}(X) \prod_{e \in X} q_e \,,$$

where $\mathbf{p} = (p_e)_{e \in E} \in [0,1]^E$ and $q_e = 1 - p_e$ for any $e \in E$.

Proof. The proposition follows by traditional inclusion-exclusion. $\square$

Definition 5.1.12 For any set $\mathcal{F}$ of minpaths or mincuts we use $\mathcal{F}^*$ to denote the set of all unions of sets in $\mathcal{F}$ including the empty set, that is, $\mathcal{F}^* := \left\{ \bigcup \mathcal{I} \,\middle|\, \mathcal{I} \subseteq \mathcal{F} \right\}$.

Clearly, $\mathcal{F}^*$ is a lattice with respect to the usual inclusion order. The following interpretation of $\mathrm{dom}_{\mathcal{F}}(X)$ in terms of the Möbius function of $\mathcal{F}^*$ is due to Manthei [Man90, Man91] and proved here in a new and simplified way without making use of Rota's crosscut theorem [Rot64]. Let us first define the Möbius function:

Definition 5.1.13 The *Möbius function* of a finite partially ordered set P with least element $\hat{0}$ is the unique $\mathbb{Z}$-valued function μ_P on P such that for any $x \in P$,

$$\sum_{y \leq x} \mu_P(y) \;=\; \delta_{\hat{0}x}\,, \tag{5.3}$$

where δ is the usual Kronecker delta.

Proposition 5.1.14 [Man90, Man91] *Let $\Sigma = (E, \phi)$ be a coherent binary system, whose set of minpaths resp. mincuts is denoted by $\mathcal{F}$. Then, for any $X \in \mathcal{F}^*$,*

$$\mathrm{dom}_{\mathcal{F}}(X) \;=\; -\mu_{\mathcal{F}^*}(X)\,.$$

Proof. By the definition of the Möbius function it suffices to prove that

$$\sum_{\substack{Y \in \mathcal{F}^* \\ Y \subseteq X}} \mathrm{dom}_{\mathcal{F}}(Y) \;=\; \begin{cases} -1 & \text{if } X = \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

which is clear if $X = \emptyset$. Otherwise, $\mathcal{F}|X := \{F \in \mathcal{F} \mid F \subseteq X\} \neq \emptyset$ and hence,

$$\sum_{\substack{Y \in \mathcal{F}^* \\ Y \subseteq X}} \mathrm{dom}_{\mathcal{F}}(Y) \;=\; \sum_{\substack{Y \in \mathcal{F}^* \\ Y \subseteq X}} \sum_{\substack{\mathcal{I} \subseteq \mathcal{F} \\ \bigcup \mathcal{I} = Y}} (-1)^{|\mathcal{I}|-1} \;=\; \sum_{\substack{\mathcal{I} \subseteq \mathcal{F} \\ \bigcup \mathcal{I} \subseteq X}} (-1)^{|\mathcal{I}|-1} \;=\; \sum_{\mathcal{I} \subseteq \mathcal{F}|X} (-1)^{|\mathcal{I}|-1} \;=\; 0\,,$$

which gives the result. □

The c-domination of Section 3.1 is related to the $\mathcal{F}$-domination as follows:

Proposition 5.1.15 *Let $\Sigma = (E, \phi)$ be a coherent binary system, whose set of minpaths resp. mincuts is denoted by $\mathcal{F}$, and let $c_{\mathcal{F}}$ denote the closure operator $\mathcal{I} \mapsto \left\{F \in \mathcal{F} \,\middle|\, F \subseteq \bigcup \mathcal{I}\right\}$ on $\mathcal{F}$. Then, for any $X \in \mathcal{F}^*$,*

$$\mathrm{dom}_{\mathcal{F}}(X) \;=\; (-1)^{|\mathcal{F}|X|-1} \mathrm{dom}_{c_{\mathcal{F}}}(\mathcal{F}|X)\,,$$

where

$$\mathcal{F}|X \;:=\; \{F \in \mathcal{F} \mid F \subseteq X\}\,.$$

Proof. Evidently, $\mathcal{F}|X$ is $c_{\mathcal{F}}$-closed. Hence, by Proposition 3.1.24, we obtain

$$\sum_{\substack{\mathcal{I} \subseteq \mathcal{F}|X \\ c_{\mathcal{F}}(\mathcal{I}) = \mathcal{F}|X}} (-1)^{|\mathcal{I}|-1} \;=\; (-1)^{|\mathcal{F}|X|-1} \mathrm{dom}_{c_{\mathcal{F}}}(\mathcal{F}|X)\,.$$

Now, $c_{\mathcal{F}}(\mathcal{I}) = \mathcal{F}|X$ if and only if $\mathcal{I}$ is an $\mathcal{F}$-formation of X. Thus, the left-hand side of the preceding equation coincides with the $\mathcal{F}$-domination of X. □

5.2 Network reliability

Throughout this section, a network is viewed as a finite graph or digraph, whose nodes or vertices are perfectly reliable and whose edges, which are either undirected or directed, are subject to random and independent failure, where all failure probabilities are assumed to be known in advance. The reader should keep in mind that the improved inclusion-exclusion identities and Bonferroni inequalities in this section can easily be generalized to dependent edge failures provided the joint probability distribution of the edge failures is known. The general objective is to assess the overall reliability of the network relative to a given reliability measure. Some relevant reliability measures are reviewed subsequently.

The *source-to-terminal reliability* or *two-terminal reliability* of a network is the probability that a message can pass from a distinguished source node s to a distinguished terminal node t along a path of operating edges. Here and subsequently, paths are considered as directed if the network is directed and as undirected if the network is undirected, and it is assumed that there is always an operating path from each node to itself. The problem of computing the source-to-terminal reliability for a distinguished source node s and a distinguished terminal node t is usually referred to as the *s,t-connectedness problem*. It has been studied extensively in the literature (see e.g., [Shi91] for a comprehensive account). An appropriate model for the source-to-terminal reliability measure is a coherent binary system $\Sigma = (E, \phi)$, where E is the edge-set of the network and $\phi(X) = 1$ if and only if X contains the edges of an s,t-path. Evidently, the minpaths and mincuts of the system correspond to the s,t-paths and s,t-cutsets (= minimal sets of edges whose removal disconnects s from t) of the network, respectively.

As a generalization of the s,t-connectedness problem, the *s,T-connectedness problem* asks for the probability that a message can be sent from a distinguished source node s to each node of some specified set T along a path of operating edges. This probability is usually referred to as the *source-to-T-terminal reliability*. An appropriate model for dealing with this network reliability measure is a coherent binary system $\Sigma = (E, \phi)$, where E is the edge-set of the network and $\phi(X) = 1$ if and only if X contains the edges of an s,T-tree (= minimal subnetwork containing an s,t-path for all $t \in T$). In this case, the minpaths of the system correspond to the s,T-trees of the network and the mincuts to its s,T-cutsets (= minimal sets of edges whose removal disconnects s from at least one node in T).

A huge literature exists on the *all-terminal reliability* (see e.g., [Col87] for an extensive account), which expresses the probability that a message can be sent between any two nodes of the network along a path of operating edges. In the case of an undirected network, for instance, the appropriate model is a coherent binary system $\Sigma = (E, \phi)$, where E is the edge-set of the network and $\phi(X) = 1$ if and only if the subnetwork induced by X is connected. Thus, in the undirected case the minpaths correspond to the spanning trees of the network and the mincuts to its cutsets (= minimal sets of edges whose removal disconnects the network).

The methods of the preceding section require a generation of the minpaths resp. mincuts as well as an appropriate semilattice structure on the set of these objects. For most relevant network reliability measures these key objects can be generated quite efficiently, that is, their generation time grows only polynomially (or even linearly) with their number. For instance, Tsukiyama *et al.* [TSOA80] devised an efficient algorithm for generating all s,t-cutsets in an undirected network that has a time complexity of only $O((n+m)c_{st})$ where n, m and c_{st} denote the number of nodes, edges and s,t-cutsets of the network, respectively. More recently, Provan and Shier [PS96] established a unifying paradigm for generating all s,t-cutsets and several other classes of cuts in directed and undirected networks that exhibits a worst-case time complexity growing only linearly with the number of objects generated. Efficient algorithms for generating s,t-paths are provided by Read and Tarjan [RT75] as well as Colbourn [Col87]. Shier [Shi91] describes several algebraic enumeration techniques for generating these key objects based on a symbolic version of the Gauss-Jordan elimination algorithm.

Unfortunately, the number of key objects can grow exponentially with the size (= number of nodes and edges) of the network, whence the methods of the preceding section exhibit an exponential time behaviour in the worst case. On the other hand, the computation of nearly all relevant network reliability measures is known to be $\#P$-hard [Val79, Bal86] (see also Garey and Johnson [GJ79] for notions of computational complexity), whence a polynomial time algorithm (that is, an algorithm whose time complexity is bounded by a polynomial in the size of the network) for exactly computing any of these measures is unlikely to exist. Fortunately, though, in many practical situations the networks are sparse and thus do not have too many minpaths or mincuts. In view of this and the fact that the method of inclusion-exclusion is a standard method in system and network reliability analysis, it is reasonable to investigate improvements of this method.

Now, in order to apply the results of the preceding section to a network, an appropriate partial ordering relation on the set of minpaths resp. mincuts (or, more generally, on an extended set of minpaths resp. mincuts) must be imposed. The following partial ordering relations, which are adopted from Shier [Shi88, Shi91], are appropriate for dealing with the source-to-terminal reliability measure:

(i) For edge-sets X and Y of s,t-paths in a planar network define

$$X \leq Y \quad :\Leftrightarrow \quad X \text{ lies below } Y\,.$$

Of course, this partial ordering relation depends on a specific drawing of the network in the plane with no edges crossing and with s and t lying on the boundary of the exterior region. Figure 5.1 clarifies this concept.

(ii) For s,t-cutsets X and Y of an arbitrary network define

$$X \leq Y \quad :\Leftrightarrow \quad N_s(X) \subseteq N_s(Y)\,,$$

where $N_s(X)$ is the set of nodes reachable from s after removing X.

It is easy to see that these partial ordering relations induce a lattice structure, where $X \wedge Y$ and $X \vee Y$ are included in $X \cup Y$. Moreover, both partial ordering relations satisfy the convexity requirement of Theorem 5.1.7. We thus conclude that Theorems 5.1.3–5.1.8 can be applied to networks whose s,t-paths resp. s,t-cutsets are ordered as in (i) resp. (ii). For s,t-paths in planar networks, ordered as in (i), the improved inclusion-exclusion identity associated with the abstract tube of Theorem 5.1.3 is due to Shier [Shi88, Shi91], while the associated improved Bonferroni inequalities are due to the author [Doh99d]. For s,t-cutsets of arbitrary networks, ordered as in (ii), the improved inclusion-exclusion identity associated with the abstract tube of Theorem 5.1.3 coincides with Buzacott's node partition formula [BC84, Buz87], whereas the improved inequalities were first established in [Doh99d]. For complete undirected networks on n nodes the classical and improved inclusion-exclusion identity for two-terminal reliability based on the s,t-cutsets of the network are compared in [Vog99] for some small values of n, see Table 5.2 for details. In [Vog99] similar results were reported for random undirected networks. We further remark that Theorem 5.1.7, when applied to s,t-cutsets, specializes to a well-known result of Provan and Ball [PB84], whereas for s,t-paths it is again due to Shier [Shi88, Shi91]. For further informations on these partial ordering relations, the reader is referred to Shier [Shi88, Shi91].

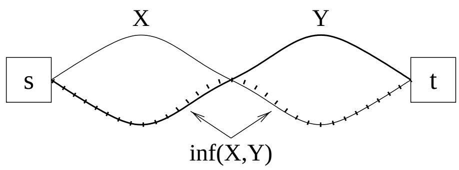

Figure 5.1: Two s,t-paths and their infimum [Shi91].

	number of terms	
n	classical $\left(2^{2^{n-2}}\right)$	improved
3	4	4
4	16	12
5	256	52
6	65535	300
7	4294967296	2164
8	18446744073709551616	18732

Figure 5.2: Classical and improved inclusion-exclusion [Vog99].

In general, it is difficult to find a partial ordering relation on the set of s,t-paths or s,T-cutsets of a directed network such that the requirements of Theorem 5.1.7 are satisfied, since otherwise we could devise an algorithm for computing two-terminal resp. source-to-T-terminal reliability whose time complexity would be bounded by a polynomial in the size of the network and the number of s,t-paths resp. s,T-cutsets. By a well-known result of Provan and Ball [PB84] such an algorithm cannot exist unless the complexity classes P and NP coincide.

For complete networks, however, we subsequently establish a partial ordering relation on the set of s,T-cutsets that satisfies the requirements of Theorem 5.1.7:

(iii) For s,T-cutsets X and Y of a complete network define $X \leq Y$ as in (ii).

Indeed, this partial ordering relation induces a lower semilattice such that $X \wedge Y \subseteq X \cup Y$; see [Doh98b] for details. The convexity requirement is easily verified.

We remark that for s,t-cutsets of arbitrary networks, the recursive scheme of Theorem 5.1.7 is due to Provan and Ball [PB84], whereas for s,T-cutsets of complete networks, it is a special case of a result of Ball and Provan [BP87].

As in [Doh99c], we now establish a partial ordering relation, which is appropriate for dealing with the all-terminal reliability of an undirected network and which leads to an improvement upon the classical inclusion-exclusion identities and inequalities if the network is sufficiently dense. To this end, let G be an undirected network with vertex-set V, and for any non-empty proper subset W of V let $\langle W \rangle$ be the *quasi-cut* associated with W, that is, the set of edges linking some node in W to some node in $V \setminus W$. Then, $\mathcal{F} := \{\langle W \rangle \,|\, \emptyset \neq W \subset V\}$ is an extended set of mincuts of the coherent binary system associated with G and the all-terminal reliability measure. Note that $\mathcal{F}$ coincides with the set of cutsets of the network if and only if the network is complete. Now, fix some $v \in V$, and

(iv) for any non-empty proper subsets W_1 and W_2 of V containing v define

$$\langle W_1 \rangle \leq \langle W_2 \rangle \;:\Leftrightarrow\; W_1 \subseteq W_2\,.$$

In this way, $\mathcal{F}$ becomes a lower semilattice where $\langle W_1 \rangle \wedge \langle W_2 \rangle = \langle W_1 \cap W_2 \rangle \subseteq \langle W_1 \rangle \cup \langle W_2 \rangle$ for any non-empty proper subsets W_1 and W_2 of V containing v. For complete undirected networks, whose cutsets are ordered as in (iv), the improved inclusion-exclusion formula associated with the abstract tube of Theorem 5.1.3 coincides with a particular case of Buzacott's node partition formula [BC84]. As noted in [BC84], the formula contains no cancelling terms in this case and thus is best possible among all cutset-based inclusion-exclusion expansions.

Example 5.2.1 [Doh99c, Doh99d] Consider the network in Figure 5.3. We are interested in bounds for its two-terminal reliability with respect to s and t. For simplicity, assume that all edges fail independently with probability $q = 1 - p$. Let's first consider the classical and improved approach based on the s,t-paths of the network, which are assumed to be partially ordered as proposed in (i). The Hasse diagram corresponding to this partial ordering relation is shown in Figure 5.4, and the corresponding bounds (both classical and improved ones) are

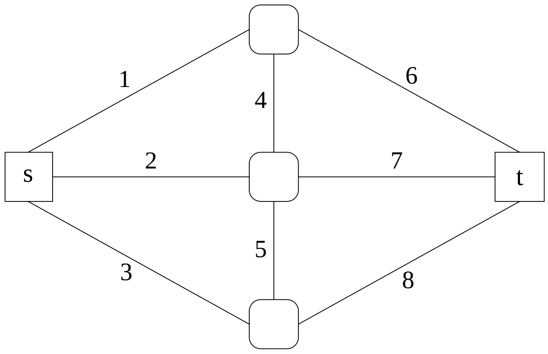

Figure 5.3: A sample network with terminal nodes s and t.

listed in Table 5.1 together with the number of sets inspected during the computation of each bound. Note that the classical bounds come from the classical Bonferroni inequalities, whereas the improved bounds are those provided by Theorem 5.1.4. In a similar way, the s,t-cutsets of the network, which are assumed to be partially ordered as in (ii), give rise to the Hasse diagram in Figure 5.5 and the bounds in Table 5.3. Note that in Table 5.1 even and odd values of r correspond to lower and upper bounds on the reliability of the network, respectively, whereas in Table 5.3 the correspondence is vice versa. In each case, the last bound represents the exact reliability of the network. As expected, the improved bounds employ much fewer sets than their classical counterparts, although they are much closer to the exact reliability value. A numerical comparison of classical and improved bounds is shown in Tables 5.2 and 5.4, and in Figures 5.6 and 5.7 some of the bounds are plotted. We observe that both classical and improved bounds based on the s,t-paths of the network are satisfactory only for small values of p (the less typical case), whereas the bounds based on the s,t-cutsets of the network are satisfactory only for small values of q (the more typical case).

To illustrate the bounds provided by Theorem 5.1.8, we consider the Hasse diagram of s,t-paths in Figure 5.4. (Of course, we could also employ the s,t-cutsets.) Straightforward application of the recursive scheme (5.2) gives

$$\begin{aligned} \Lambda(38) &= p^2, & \Lambda(27) &= p^2 - 3p^4 + 2p^5, \\ \Lambda(258) &= p^3 - p^4, & \Lambda(3546) &= p^4 - 2p^5 + p^6, \\ \Lambda(357) &= p^3 - p^4, & \Lambda(147) &= p^3 - p^4 - 3p^5 + 5p^6 - 2p^7, \\ \Lambda(1458) &= p^4 - 2p^5 + p^6, & \Lambda(246) &= p^3 - p^4 - 3p^5 + 5p^6 - 2p^7, \end{aligned}$$

where the expression $\Lambda(e_1 \dots e_n)$ is used as an abbreviation for $\Lambda(\{e_1, \dots, e_n\}, \mathbf{p})$. Now, by virtue of Theorem 5.1.8, these Λ-values give rise to lower bounds for the two-terminal reliability of our sample network in Figure 5.3. We thus obtain e.g.

the following lower bounds, which are plotted in Figure 5.8:

$$
\begin{aligned}
c_0 &:= \Lambda(38) \;=\; p^2, \\
c_1 &:= c_0 + \Lambda(258) + \Lambda(357) \;=\; p^2 + 2p^3 - 2p^4, \\
c_2 &:= c_1 + \Lambda(1458) + \Lambda(27) + \Lambda(3546) \;=\; 2p^2 + 2p^3 - 3p^4 - 2p^5 + 2p^6, \\
c_3 &:= c_2 + \Lambda(147) + \Lambda(246) \;=\; 2p^2 + 4p^3 - 5p^4 - 8p^5 + 12p^6 - 4p^7.
\end{aligned}
$$

In a similar way, bounds for the all-terminal reliability of the network in Figure 5.3 are obtained by employing the cutsets and quasi-cuts of the network. The Hasse diagram of the quasi-cuts, which are assumed to be partially ordered as in (iv), is shown in Figure 5.9, and the corresponding bounds can be read from Table 5.5 together with the number of sets inspected during the computation of each bound. Here, the classical bounds are the classical Bonferroni bounds based on the cutsets of the network, whereas the improved bounds are obtained by applying Theorem 5.1.4 to the quasi-cuts of the network. Note that in Table 5.5 even and odd values of r correspond to upper and lower bounds on the exact value, respectively. Again, we observe that the improved Bonferroni bounds are much sharper than the classical bounds. Some numerical values are shown in Table 5.6.

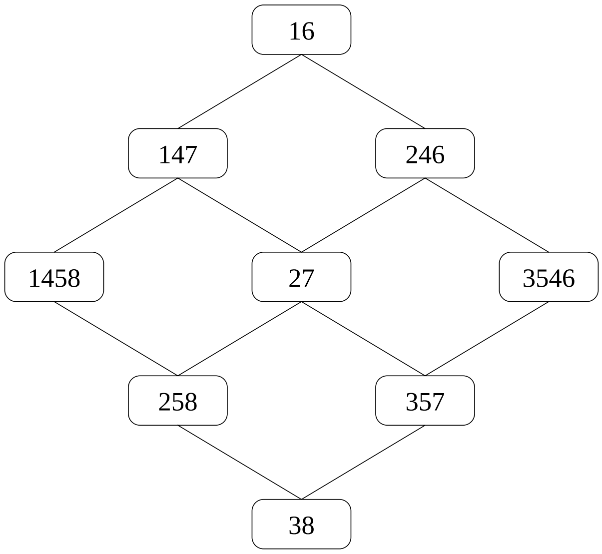

Figure 5.4: Hasse diagram of s, t-paths of the network in Figure 5.3.

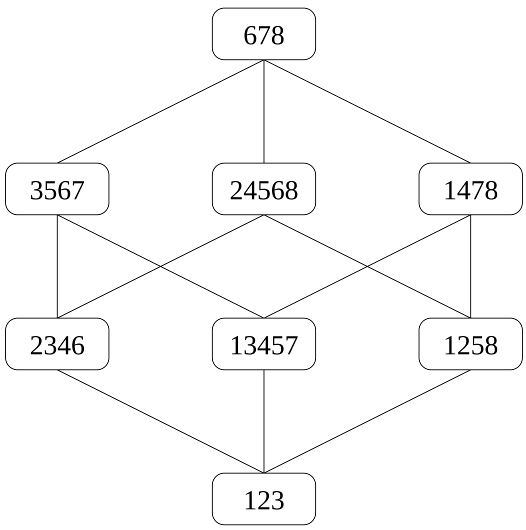

Figure 5.5: Hasse diagram of s, t-cutsets of the network in Figure 5.3.

	classical bounds		improved bounds	
r	$a_r(p)$	# sets	$a^*_r(p)$	# sets
1	$3p^2+4p^3+2p^4$	9	$3p^2+4p^3+2p^4$	9
2	$3p^2+4p^3-9p^4-16p^5-9p^6$	45	$3p^2+4p^3-9p^4-14p^5-2p^6$	36
3	$3p^2+4p^3-9p^4-8p^5+34p^6+30p^7+3p^8$	129	$3p^2+4p^3-9p^4-10p^5+27p^6+4p^7$	73
4	$3p^2+4p^3-9p^4-10p^5+27p^6-50p^7-34p^8$	255	$3p^2+4p^3-9p^4-10p^5+27p^6-18p^7-2p^8$	97
5	$3p^2+4p^3-9p^4-10p^5+27p^6-12p^7+54p^8$	381	$3p^2+4p^3-9p^4-10p^5+27p^6-18p^7+4p^8$	103
6	$3p^2+4p^3-9p^4-10p^5+27p^6-18p^7-24p^8$	465		
7	$3p^2+4p^3-9p^4-10p^5+27p^6-18p^7+12p^8$	501		
8	$3p^2+4p^3-9p^4-10p^5+27p^6-18p^7+3p^8$	510		
9	$3p^2+4p^3-9p^4-10p^5+27p^6-18p^7+4p^8$	511		

Table 5.1: Path-based bounds for the two-terminal reliability of the network in Figure 5.3.

p	$a_2(p)$	$a^*_2(p)$	$a_4(p)$	$a^*_4(p)$	$a_6(p)$	$a^*_6(p)^\dagger$	$a^*_5(p)^\dagger$	$a_5(p)$	$a^*_3(p)$	$a_3(p)$
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	*0.00000*	*0.00000*	0.00000	0.00000	0.00000
0.1	0.03293	0.03296	0.03302	0.03303	0.03303	*0.03303*	*0.03303*	0.03303	0.03303	0.03306
0.2	0.13190	0.13299	0.13540	0.13589	0.13584	*0.13591*	*0.13591*	0.13611	0.13618	0.13761
0.3	0.25966	0.26962	0.28732	0.29642	0.29497	*0.29681*	*0.29681*	0.30140	0.30136	0.31720
0.4	0.3049	0.35405	0.40959	0.48299	0.46857	*0.48692*	*0.48692*	0.52952	0.52035	0.61406
0.5	0.04688	0.21875	0.27344	0.64844	0.56250	*0.67188*	*0.67188*	0.91406	0.82813	1.21484
0.6	-0.88646	-0.40435	-0.71104	0.72224	0.35272	*0.82301*	*0.82301*	1.83078	1.37169	2.63202

†exact network reliability

Table 5.2: Numerical values of the bounds in Table 5.1.

r	classical bounds $b_r(q)$	# sets	improved bounds $b_r^*(q)$	# sets
1	$1-2q^3-4q^4-2q^5$	9	$1-2q^3-4q^4-2q^5$	9
2	$1-2q^3-4q^4+2q^5+13q^6+10q^7+\ \ q^8$	37	$1-2q^3-4q^4+2q^5+13q^6+\ \ 2q^7$	28
3	$1-2q^3-4q^4+2q^5+13q^6-22q^7-23q^8$	93	$1-2q^3-4q^4+2q^5+13q^6-14q^7-2q^8$	46
4	$1-2q^3-4q^4+2q^5+13q^6-14q^7+39q^8$	163	$1-2q^3-4q^4+2q^5+13q^6-14q^7+4q^8$	52
5	$1-2q^3-4q^4+2q^5+13q^6-14q^7-17q^8$	219		
6	$1-2q^3-4q^4+2q^5+13q^6-14q^7+11q^8$	247		
7	$1-2q^3-4q^4+2q^5+13q^6-14q^7+\ \ 3q^8$	255		
8	$1-2q^3-4q^4+2q^5+13q^6-14q^7+\ \ 4q^8$	256		

Table 5.3: Cutset-based bounds for the two-terminal reliability of the network in Figure 5.3.

q	$b_3(q)$	$b_3^*(q)$	$b_5(q)$	$b_5^*(q)^\dagger$	$b_4^*(q)^\dagger$	$b_4(q)$	$b_2^*(q)$	$b_2(q)$
0.0	1.00000	1.00000	1.00000	*1.00000*	*1.00000*	1.00000	1.00000	1.00000
0.1	0.99763	0.99763	0.99763	*0.99763*	*0.99763*	0.99763	0.99763	0.99763
0.2	0.97873	0.97889	0.97885	*0.97890*	*0.97890*	0.97899	0.97910	0.97920
0.3	0.92162	0.92474	0.92376	*0.92514*	*0.92514*	0.92743	0.92837	0.93019
0.4	0.79221	0.81908	0.80925	*0.82301*	*0.82301*	0.84595	0.84661	0.86037
0.5	0.50391	0.64844	0.58984	*0.67188*	*0.67188*	0.80859	0.78125	0.84766
0.6	-0.19052	0.38615	0.13420	*0.48692*	*0.48692*	1.07479	0.86764	1.10838

†exact network reliability

Table 5.4: Numerical values of the bounds in Table 5.3.

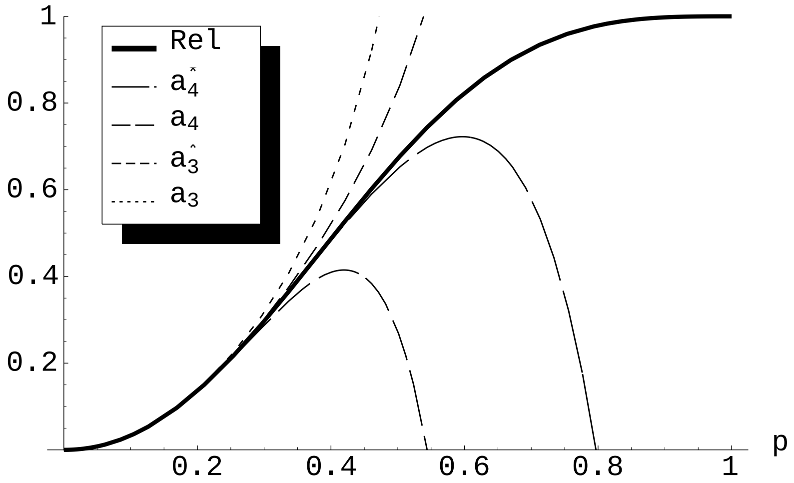

Figure 5.6: A plot of some of the bounds in Table 5.1.

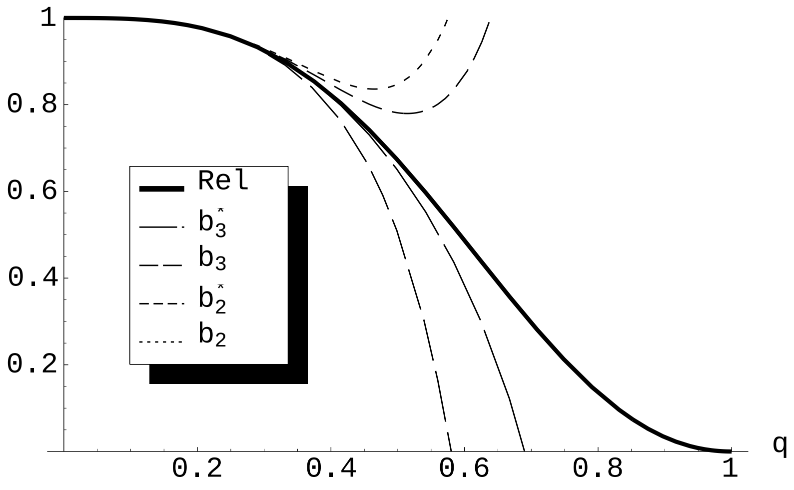

Figure 5.7: A plot of some of the bounds in Table 5.3.

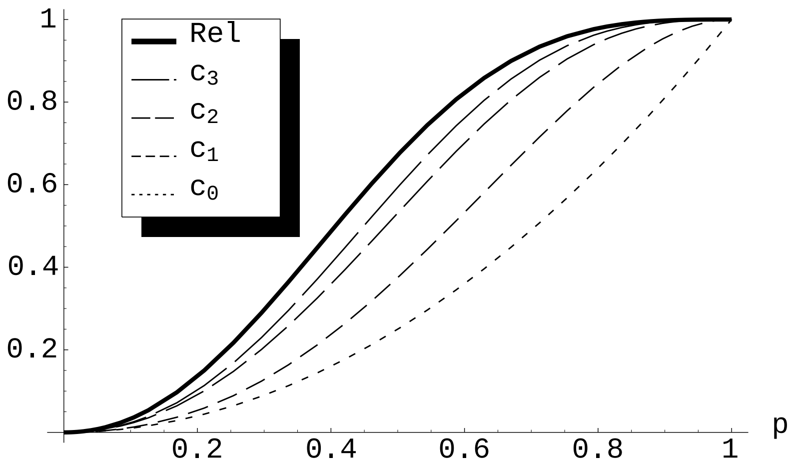

Figure 5.8: A plot of the path-based bounds c_0, c_1, c_2, c_3.

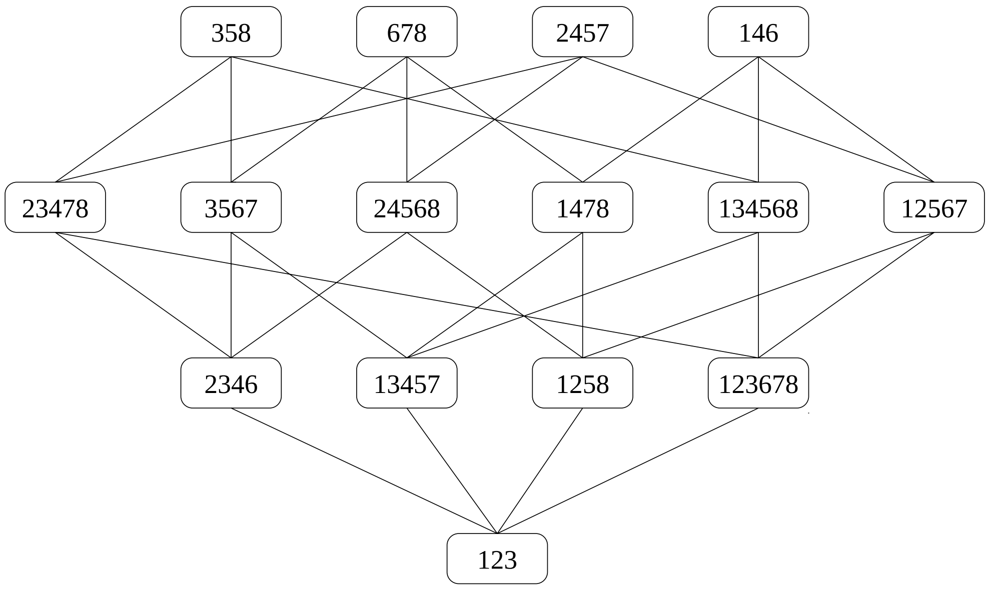

Figure 5.9: Hasse diagram of quasi-cuts of the network in Figure 5.3.

	classical bounds		improved bounds	
r	$d_r(q)$	# sets	$d_r^*(q)$	# sets
1	$1-4q^3-5q^4-4q^5$	13	$1-4q^3-5q^4-4q^5-2q^6$	15
2	$1-4q^3-5q^4+8q^5+44q^6+20q^7+2q^8$	91	$1-4q^3-5q^4+4q^5+30q^6+8q^7+2q^8$	65
3	$1-4q^3-5q^4+4q^5+30q^6-160q^7-86q^8$	377	$1-4q^3-5q^4+4q^5+30q^6-40q^7-10q^8$	125
4	$1-4q^3-5q^4+4q^5+30q^6+40q^7+429q^8$	1092	$1-4q^3-5q^4+4q^5+30q^6-40q^7+14q^8$	149
5	$1-4q^3-5q^4+4q^5+30q^6-68q^7-750q^8$	2379		
6	$1-4q^3-5q^4+4q^5+30q^6-36q^7+934q^8$	4095		
7	$1-4q^3-5q^4+4q^5+30q^6-40q^7-778q^8$	5811		
8	$1-4q^3-5q^4+4q^5+30q^6-40q^7+509q^8$	7098		
9	$1-4q^3-5q^4+4q^5+30q^6-40q^7-206q^8$	7813		
10	$1-4q^3-5q^4+4q^5+30q^6-40q^7+80q^8$	8099		
11	$1-4q^3-5q^4+4q^5+30q^6-40q^7+2q^8$	8177		
12	$1-4q^3-5q^4+4q^5+30q^6-40q^7+15q^8$	8190		
13	$1-4q^3-5q^4+4q^5+30q^6-40q^7+14q^8$	8191		

Table 5.5: Bonferroni bounds for the all-terminal reliability of the network in Figure 5.3.

q	$d_1(q)$	$d_1^*(q)$	$d_3(q)$	$d_3^*(q)$	$d_5(q)$	$d_5^*(q)^\dagger$	$d_4^*(q)^\dagger$	$d_4(q)$	$d_2^*(q)$	$d_2(q)$
0.00	1.00000	1.00000	1.00000	1.00000	1.00000	*1.00000*	*1.00000*	1.00000	1.00000	1.00000
0.10	0.99546	0.99546	0.99555	0.99557	0.99556	*0.99557*	*0.99557*	0.99558	0.99557	0.99563
0.20	0.95872	0.95859	0.96093	0.96266	0.96041	*0.96272*	*0.96272*	0.96481	0.96331	0.96564
0.30	0.84178	0.84032	0.84246	0.87369	0.81901	*0.87526*	*0.87526*	0.91999	0.88497	0.90752
0.40	0.57504	0.56685	0.46134	0.70775	0.17691	*0.72348*	*0.72348*	1.12653	0.79426	0.91222
0.50	0.06250	0.03125	-0.80469	0.42969	-2.67969	*0.52344*	*0.52344*	2.76953	0.85156	1.28906

†exact network reliability

Table 5.6: Numerical values of the bounds in Table 5.5.

5.3 Reliability of k-out-of-n systems

The class of k-out-of-n systems was introduced by Birnbaum, Esary and Saunders [BES61]; see also Rushdi [Rus93] for an extensive survey. A k-out-of-n success (resp. failure) system operates (resp. fails) whenever k or more components operate (resp. fail). As for coherent binary systems, it is again assumed that the components fail randomly and independently (this assumption might be relaxed in some cases). A formal definition in terms of coherent binary systems follows.

Definition 5.3.1 Let $k, n \in \mathbb{N}$ and $1 \leq k \leq n$. A *k-out-of-n success (resp. failure) system* is a coherent binary system $\Sigma = (E, \phi)$ where $|E| = n$ and where for any subset X of E, $\phi(X) = 1$ (resp. $\phi(E \setminus X) = 0$) if and only if $|X| \geq k$.

Remark. The minpaths (resp. mincuts) of a k-out-of-n success (resp. failure) system $\Sigma = (E, \phi)$ are the k-subsets of E. Likewise, the mincuts (resp. minpaths) of a k-out-of-n success (resp. failure) system $\Sigma = (E, \phi)$ are the $(n-k+1)$-subsets of E. Any k-out-of-n success (resp. failure) system is an $(n-k+1)$-out-of-n failure (resp. success) system. Thus, we may restrict ourselves to either type of system.

As in [Doh98b] we establish a partial ordering relation on the set of k-subsets of E, thereby assuming that E is endowed with a linear ordering relation $\leq$.

(v) For k-subsets X and Y of a linearly ordered set E define

$$X \leq Y \quad :\Leftrightarrow \quad x \leq y \text{ for all } x \in X,\, y \in Y \setminus X\,.$$

Thus, a partial ordering relation on the set of k-subsets of E is established. Figure 5.10 shows the associated Hasse diagram for $E = \{1, \ldots, 6\}$ and $k = 3$.

Again, it is easy to see that the convexity requirement of Theorem 5.1.7 is satisfied; moreover, $X \wedge Y \subseteq X \cup Y$, since $X \wedge Y$ consists of the k smallest elements of $X \cup Y$. Therefore, the results of Section 5.1 can be applied to k-out-of-n success or failure systems whose k-subsets are ordered as in (v). We conclude that for fixed k, the time and space complexity of the pseudopolynomial algorithm (Algorithm II), when applied to the k-subsets of an n-set, are $O(n^{2k+1})$, respectively $O(n^k)$. Moreover, since $n-k$ is the height of the partial ordering relation (v), the improved bounds of degree $n-k+1$ already give the exact value.

For k-out-of-n success or failure systems $\Sigma = (E, \phi)$ we subsequently consider the number of terms in the improved inclusion-exclusion identity provided by Theorem 5.1.5, that is, we compute the number of non-empty chains in the poset of k-subsets of E. Note that in the following theorem, $f(t)$ depends on k.

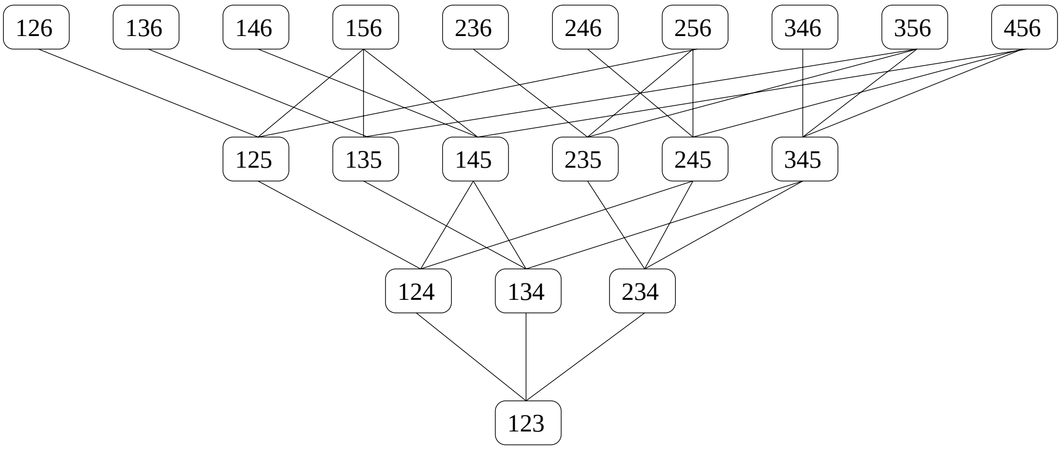

Figure 5.10: Hasse diagram for $E = \{1, \ldots, 6\}$ and $k = 3$.

Theorem 5.3.2 [Doh98b] *Let E be a finite set of cardinality n, whose k-subsets are ordered as in (v). Then, the number of chains in the partially ordered set of k-subsets of E is equal to $2f(n-k)-1$, where $f(0) := 1$ and*

$$f(t) \;:=\; 1 + \sum_{i=0}^{t-1} \binom{t-i+k-1}{k-1} f(i) \qquad (t = 1, \ldots, n-k)\,. \tag{5.4}$$

Proof. For any k-subset P of E, let $c(P)$ denote the number of chains extending upward from P. Then, the total number of chains is $2c(\hat{0})$ where $\hat{0}$ denotes the minimum in the poset of k-subsets of E. Thus, it remains to show that $c(\hat{0}) = f(n-k)$. More generally, by induction on t we prove that $h(P) = n-k-t$ implies $c(P) = f(t)$, where $h(P)$ denotes the height of P. For $t = 0$ this is immediately clear, since $n-k$ is the maximum height. Now let the height of P be $n-k-t$ where $t > 0$. By the induction hypothesis we find that

$$c(P) \;=\; 1 + \sum_{i=0}^{t-1} \sum_{\substack{Q>P \\ h(Q)=n-k-i}} c(Q) \;=\; 1 + \sum_{i=0}^{t-1} \sum_{\substack{Q>P \\ h(Q)=n-k-i}} f(i) \;=\; 1 + \sum_{i=0}^{t-1} s(P,i) f(i)$$

where $s(P,i) := |\{Q > P \,|\, h(Q) = n-k-i\}|$ $(i = 0, \ldots, t-1)$. We conclude that $s(P,i) = s(P,i+1)(t-i+k-1)/(t-i)$, where $s(P,t) := 1$, and therefore,

$$s(P,i) \;=\; \binom{t-i+k-1}{k-1}, \quad c(P) \;=\; 1 + \sum_{i=0}^{t-1} \binom{t-i+k-1}{k-1} f(i) \;=\; f(t)\,,$$

which completes the proof. $\square$

In order to compare the number of terms in the improved inclusion-exclusion identity with the number of terms in the classical inclusion-exclusion expansion for fixed k and increasing n, it seems reasonable to consider the ratio

$$\varrho_k(n) := \frac{2f(n-k)-1}{2^{\binom{n}{k}}-1}, \tag{5.5}$$

where $f(t)$ is defined as in (5.4). It should be clear that the classical inclusion-exclusion method is not a very effective tool for analyzing k-out-of-n systems.

Theorem 5.3.3 [Doh98b] *For any $k > 1$, $\lim_{n\to\infty} \varrho_k(n) = 0$.*

Proof. By Theorem 5.3.2 and since $\binom{t-i+k-1}{k-1} \leq k^{t-i}$ we immediately find that

$$f(t) \leq 1 + \sum_{i=0}^{t-1} k^{t-i} f(i) \quad (t = 0, \dots, n-k),$$

and therefore,

$$f(t) \leq 1 + k \sum_{i=0}^{t-1} (2k)^i = 1 + k\,\frac{1-(2k)^t}{1-2k} \quad (t = 0, \dots, n-k).$$

Hence, for fixed k, there are constants c_1 and c_2 depending only on k such that

$$\varrho_k(n) \leq c_1 \frac{(2k)^n}{2^{\binom{n}{k}}} \sim c_1 2^{c_2 n - n^k},$$

which implies the statement of the theorem. □

Remark. The following numerical values are computed via (5.4) and (5.5):

$$\begin{aligned}
\varrho_2(6) &= 7.0 \times 10^{-3}, & \varrho_3(6) &= 1.7 \times 10^{-4}, & \varrho_4(6) &= 1.8 \times 10^{-3},\\
\varrho_2(8) &= 1.0 \times 10^{-5}, & \varrho_3(8) &= 6.0 \times 10^{-14}, & \varrho_4(8) &= 2.1 \times 10^{-18},\\
\varrho_2(10) &= 9.0 \times 10^{-10}, & \varrho_3(10) &= 7.6 \times 10^{-32}, & \varrho_4(10) &= 5.9 \times 10^{-59}.
\end{aligned}$$

Example 5.3.4 Consider a 3-out-of-6 success system where, for simplicity, all components are assumed to operate with equal probability $p_e \equiv p$. Table 5.7 displays both classical and improved bounds on the reliability of the system together with the number of sets which were inspected during the computation of each bound. Here, even and odd values of r respectively correspond to lower and upper bounds on the reliability of the system. The classical bounds are the usual Bonferroni bounds, whereas the improved bounds are obtained by applying Theorem 5.1.4 in connection with the partial ordering relation (v), whose Hasse diagram is shown in Figure 5.10. In Figure 5.11 some of the bounds are plotted.

	classical bounds		improved bounds	
r	$e_r(p)$	# sets	$e_r^*(p)$	# sets
1	$20p^3$	20	$20p^3$	20
2	$20p^3 - 90p^4 - 90p^5 - 10p^6$	210	$20p^3 - 45p^4 - 18p^5 - p^6$	84
3	$20p^3 - 30p^4 + 510p^5 + 470p^6$	1350	$20p^3 - 45p^4 + 36p^5 + 17p^6$	156
4	$20p^3 - 45p^4 - 720p^5 - 3130p^6$	6195	$20p^3 - 45p^4 + 36p^5 - 10p^6$	183
$\vdots$	$\vdots$	$\vdots$		
20	$20p^3 - 45p^4 + 36p^5 - 10p^6$	1048575		

Table 5.7: Bonferroni bounds for the reliability of a 3-out-of-6 success system.

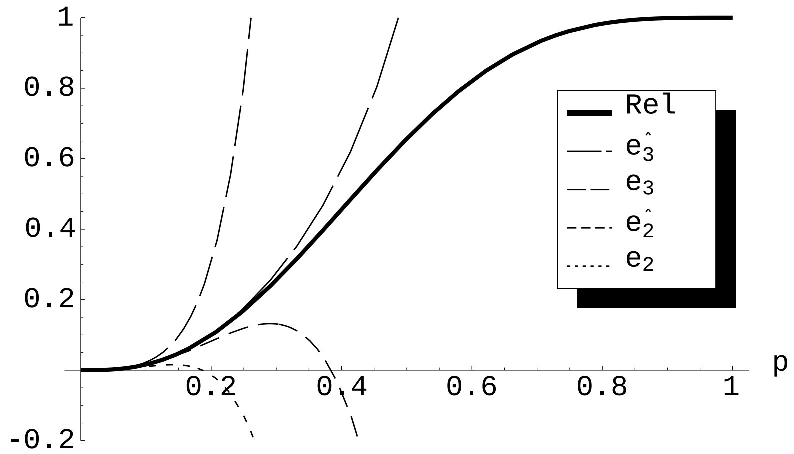

Figure 5.11: A plot of some of the bounds in Table 5.7.

Remarks. Sharper and less time demanding bounds for the reliability of a k-out-of-n success or failure system can be obtained by applying a method of Balagurusamy and Misra [BM75, BM76], rediscovered by Heidtmann [Hei81, Hei82], which uses generalized Bonferroni-Jordan inequalities [Jor27, GS96b] for the probability that at least k out of n events occur. For any k-out-of-n success system $\Sigma = (E, \phi)$ and $r \in \mathbb{N}$, the Balagurusamy-Misra-Heidtmann (BMH) bounds are

$$\mathrm{Rel}_\Sigma(\mathbf{p}) \geq \sum_{i=k}^{r+k-1} (-1)^{i-k} \binom{i-1}{k-1} \sum_{\substack{I \subseteq E \\ |I|=i}} \prod_{e \in I} p_e \quad (r \text{ even}),$$

$$\mathrm{Rel}_\Sigma(\mathbf{p}) \leq \sum_{i=k}^{r+k-1} (-1)^{i-k} \binom{i-1}{k-1} \sum_{\substack{I \subseteq E \\ |I|=i}} \prod_{e \in I} p_e \quad (r \text{ odd}),$$

and for any k-out-of-n failure system $\Sigma = (E, \phi)$ and $r \in \mathbb{N}$ they are given by

$$1 - \mathrm{Rel}_\Sigma(\mathbf{p}) \geq \sum_{i=k}^{r+k-1} (-1)^{i-k} \binom{i-1}{k-1} \sum_{\substack{I \subseteq E \\ |I|=i}} \prod_{e \in I} q_e \quad (r \text{ even}),$$

$$1 - \mathrm{Rel}_\Sigma(\mathbf{p}) \leq \sum_{i=k}^{r+k-1} (-1)^{i-k} \binom{i-1}{k-1} \sum_{\substack{I \subseteq E \\ |I|=i}} \prod_{e \in I} q_e \quad (r \text{ odd}),$$

where in both cases $\mathbf{p} = (p_e)_{e \in E} \in [0,1]^E$ and $q_e = 1 - p_e$ for any $e \in E$.

For the reliability of a 3-out-of-6 success system with equal component reliabilities $p_e \equiv p$, for instance, the Balagurusamy-Misra-Heidtmann bounds are displayed in Table 5.8, where even and odd values of r again correspond to lower and upper bounds, respectively. It turns out that the BMH bounds are closer to the exact value than the improved bounds of Table 5.7, while fewer sets are inspected during their computation. Note, however, that each term $\sum \prod p_e$ has to be multiplied by a binomial coefficient, which increases the computational effort.

r	BMH bounds	# sets
1	$20p^3$	20
2	$20p^3 - 45p^4$	35
3	$20p^3 - 45p^4 + 36p^5$	41
4	$20p^3 - 45p^4 + 36p^5 - 10p^6$	42

Table 5.8: BMH bounds for a 3-out-of-6 success system.

For an exact computation of the reliability of a k-out-of-n system, the BMH method sums $\sum_{i=k}^{n} \binom{n}{i}$ terms (each multiplied by a binomial coefficient), which is exponential in n, if k is fixed. In contrast, as we saw above, the time complexity of the pseudopolynomial algorithm based on our partial ordering relation (v) is $O(n^{2k+1})$ and hence polynomially bounded in n. We therefore conclude that the pseudopolynomial algorithm in connection with our partial ordering relation (v) is superior to the BMH method if the exact reliability is to be computed. It is, however, not superior to the most efficient generating function method of Barlow and Heidtmann [BH84], whose time complexity is linear in n, if k is fixed.

5.4 Reliability of consecutive k-out-of-n systems

Consecutive k-out-of-n success (resp. failure) systems operate (resp. fail) whenever k or more consecutive components operate (resp. fail). Again, it is assumed that the components fail randomly and independently with known probabilities.

Systems of this type were first considered by Kontoleon [Kon80]; the nomenclature goes back to Chiang and Niu [CN81], who also provide several applications of this model. For an account of consecutive k-out-of-n systems, we recommend the survey paper of Papastavridis and Koutras [PK93]. A formal definition follows.

Definition 5.4.1 Let $k, n \in \mathbb{N}$, $1 \leq k \leq n$. A *consecutive k-out-of-n success (resp. failure) system* is a coherent binary system $\Sigma = (E, \phi)$ where E is a linearly ordered finite set of size n and where for any subset X of E, $\phi(X) = 1$ (resp. $\phi(E \setminus X) = 0$) if and only if X contains at least k consecutive elements of E.

As illustrated by the following example, consecutive k-out-of-n failure systems serve as a model for a particular type of communication network.

Example 5.4.2 Consider the communication network displayed in Figure 5.12, where nodes 1–6 are assumed to fail randomly and independently with known probabilities and all other nodes and edges are assumed to be perfectly reliable. It is immediately clear that in this network a message can pass from s to t if and only if no three consecutive nodes among 1–6 simultaneously fail. Thus, the network is appropriately modelled as a consecutive 3-out-of-6 failure system.

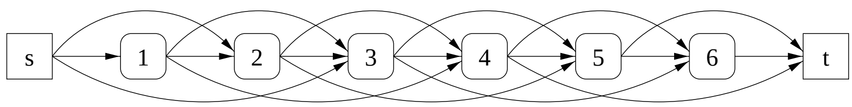

Figure 5.12: A consecutive 3-out-of-6 failure network.

In general, X is a minpath (resp. mincut) of a consecutive k-out-of-n success (resp. failure) system $\Sigma = (E, \phi)$ if and only if X is a consecutive subset of E containing exactly k elements. The following partial ordering relation on the set of consecutive k-subsets of E is adopted from Shier [Shi88, Shi91]:

(vi) For any consecutive k-subsets X and Y of a linearly ordered set E define

$$X \leq Y \quad :\Leftrightarrow \quad \min X \leq \min Y\,.$$

Thus, a partial (in fact: linear) ordering relation on the set of minpaths (resp. mincuts) of a consecutive k-out-of-n success (resp. failure) system is given, which satisfies the requirements of Theorem 5.1.7: If X and Y are two consecutive k-subsets of E, then $X \wedge Y = X$ or $X \wedge Y = Y$ and hence, $X \wedge Y \subseteq X \cup Y$. If $X \leq Y \leq Z$ are three consecutive k-subsets of E and $e \in X \cap Z$, then $\min Y \leq \min Z \leq e \leq \max X \leq \max Y$ and hence $e \in Y$. Therefore, the requirements of Theorem 5.1.7 are satisfied and thus Algorithm II can be applied. Since the number of minpaths (resp. mincuts) is $n - k + 1$, the algorithm has a space complexity of $O(n)$ and a time complexity of $O(n^3)$. In contrast, the time complexity of the classical inclusion-exclusion method for that problem is $O(n2^n)$. Note, however, that the classical method can be adapted to deal with dependent component failures, whereas Algorithm II relies on the independence assumption of Definition 5.1.1. The same remark as for the classical inclusion-exclusion method applies to the improved inclusion-exclusion expansion of Kossow and Preuss [KP89], which involves only $O(n^4)$ non-cancelling terms and which is best possible among all inclusion-exclusion expansions for the reliability of a consecutive k-out-of-n system with unequal component reliabilities. For the restricted case $k \geq n/2$ the following theorem provides an improved inclusion-exclusion expansion that contains only $O(n^2)$ terms none of which cancel. Thus, the expansion is best possible among all inclusion-exclusion expansions for this restricted case.

Theorem 5.4.3 *Let Σ be a consecutive k-out-of-n success system whose component reliabilities are given by the vector $\mathbf{p} = (p_1, \ldots, p_n)$. If $k \geq n/2$, then*

$$\mathrm{Rel}_\Sigma(\mathbf{p}) \;=\; \sum_{i=1}^{n-k+1} \prod_{j=i}^{i+k-1} p_j - \sum_{i=1}^{n-k} \prod_{j=i}^{i+k} p_j \;=\; \sum_{i=1}^{n-k} (1 - p_{i+k}) \prod_{j=i}^{i+k-1} p_j + \prod_{j=n-k+1}^{n} p_j \,.$$

Proof. For $i = 1, \ldots, n-k+1$ let A_i be the event that components $i, \ldots, i+k-1$ operate. Then, $\mathrm{Rel}_\Sigma(\mathbf{p}) = \Pr(A_1 \cup \cdots \cup A_{n-k+1})$. Since $k \geq n/2$, $A_x \cap A_y \subseteq A_z$ for $x, y = 1, \ldots, n-k+1$ and any z between x and y. Thus, by combining Theorem 3.1.14 with Example 3.1.5 (or by applying Corollary 3.1.16) we obtain

$$\mathrm{Rel}_\Sigma(\mathbf{p}) \;=\; \sum_{i=1}^{n-k+1} P(A_i) - \sum_{i=1}^{n-k} P(A_i \cap A_{i+1}) \;=\; \sum_{i=1}^{n-k+1} \prod_{j=i}^{i+k-1} p_j - \sum_{i=1}^{n-k} \prod_{j=i}^{i+k} p_j \;=$$

$$\sum_{i=1}^{n-k} \prod_{j=i}^{i+k-1} p_j - \sum_{i=1}^{n-k} \prod_{j=i}^{i+k} p_j + \prod_{j=n-k+1}^{n} p_j = \sum_{i=1}^{n-k} \prod_{j=i}^{i+k-1} p_j - \sum_{i=1}^{n-k} p_{i+k} \prod_{j=i}^{i+k-1} p_j + \prod_{j=n-k+1}^{n} p_j \,,$$

which gives the result. □

Remark. The first identity in Theorem 5.4.3 can easily be adapted to deal with dependent components. For independent components, the final expression in Theorem 5.4.3 can be computed in $O(n)$ steps by means of Algorithm III.

In the case of equal component reliabilities we even obtain a closed formula:

Corollary 5.4.4 *Let $\Sigma = (E, \phi)$ be a consecutive k-out-of-n success system whose component reliabilities are given by $\mathbf{p} = (p, \ldots, p)$. If $k \geq n/2$, then*

$$\mathrm{Rel}_\Sigma(\mathbf{p}) \;=\; (n-k+1)p^k - (n-k)p^{k+1} \;=\; p^k \left[(n-k)(1-p) + 1\right].$$

Proof. Corollary 5.4.4 is an immediate consequence of Theorem 5.4.3. □

Remarks. Theorem 5.4.3 and Corollary 5.4.4 can equivalently be formulated for consecutive k-out-of-n failure systems by replacing the preceding identities with

$$1 - \mathrm{Rel}_\Sigma(\mathbf{p}) \;=\; \sum_{i=1}^{n-k+1} \prod_{j=i}^{i+k-1} q_j - \sum_{i=1}^{n-k} \prod_{j=i}^{i+k} q_j \;=\; \sum_{i=1}^{n-k} (1 - q_{i+k}) \prod_{j=i}^{i+k-1} q_j + \prod_{j=n-k+1}^{n} q_j$$

in the case of unequal component failure probabilities $q_j = 1 - p_j$, and

$$1 - \mathrm{Rel}_\Sigma(\mathbf{p}) \;=\; (n-k+1)q^k - (n-k)q^{k+1} \;=\; q^k\left[(n-k)(1-q) + 1\right]$$

in the case where all component failure probabilities are equal to $q = 1 - p$. This latter identity was first proved by Shanthikumar [Sha82] without making use of the inclusion-exclusion principle. It should be noted at this point that the algorithms of Shanthikumar [Sha82] and Hwang [Hwa82] are the most efficient algorithms for computing the reliability of a consecutive k-out-of-n system. Note, however, that these most efficient algorithms strongly rely on the assumption that the components of the system fail in a statistically independent fashion.

We finally remark that by Corollary 4.3.4 the right-hand sides of the preceding identities give upper bounds on $\mathrm{Rel}_\Sigma(\mathbf{p})$ resp. $1 - \mathrm{Rel}_\Sigma(\mathbf{p})$ for arbitrary n and k.

Example 5.4.5 As in Example 5.4.2 we consider the consecutive 3-out-of-6 failure system associated with the network in Figure 5.12. In view of the preceding remarks the reliability of this consecutive 3-out-of-6 failure system is computed as

$$\begin{aligned} &1 - q_1q_2q_3 - q_2q_3q_4 - q_3q_4q_5 - q_4q_5q_6 + q_1q_2q_3q_4 + q_2q_3q_4q_5 + q_3q_4q_5q_6 \\ &\qquad = 1 - (1-q_4)q_1q_2q_3 - (1-q_5)q_2q_3q_4 - (1-q_6)q_3q_4q_5 - q_4q_5q_6\,, \end{aligned}$$

where q_i denotes the failure probability of node i ($i = 1, \ldots, 6$). In particular, if the q_i's are all equal to q, then the reliability equals $1 - 4q^3 + 3q^4$. The reader is invited to obtain the same result by the classical inclusion-exclusion method.

Algorithm III Reliability of a consecutive k-out-of-n system where $k \geq n/2$

Require: Components $1, \ldots, n$ work independently with probability $p_1, \ldots, p_n$

Ensure: *acc* gives the probability that at least k consecutive components work

$acc \leftarrow 0$

$h \leftarrow p_1 \ldots p_k$

for $i = 1$ to $n - k$ **do**

  $acc \leftarrow acc + (1 - p_{i+k})h$

  $h \leftarrow hp_{i+k}/p_i$

end for

$acc \leftarrow acc + h$

We close this section with a generalization of Theorem 5.4.3, to which the remarks after Corollary 5.4.4 likewise apply.

Theorem 5.4.6 *Let $t \in \mathbb{N}$ and $n_1, \ldots, n_t, k_1, \ldots, k_t \in \mathbb{N}$ such that $n_1 < \cdots < n_t \leq n_1 + k_1 < \cdots < n_t + k_t$. Let Σ be a coherent binary system with components $n_1, \ldots, n_t + k_t - 1$, minpaths $\{n_1, \ldots, n_1 + k_1 - 1\}, \ldots, \{n_t, \ldots, n_t + k_t - 1\}$ and component reliabilities given by $\mathbf{p} = (p_{n_1}, \ldots, p_{n_t+k_t-1})$. Then,* $\mathrm{Rel}_\Sigma(\mathbf{p})$ *equals*

$$\sum_{i=1}^{t} \prod_{j=n_i}^{n_i+k_i-1} p_j - \sum_{i=1}^{t-1} \prod_{j=n_i}^{n_{i+1}+k_{i+1}-1} p_j = \sum_{i=1}^{t-1} \left(1 - \prod_{j=n_i+k_i}^{n_{i+1}+k_{i+1}-1} p_j\right) \prod_{j=n_i}^{n_i+k_i-1} p_j + \prod_{j=n_t}^{n_t+k_t-1} p_j \,.$$

Proof. For $i = 1, \ldots, t$ let A_i be the event that components $n_i, \ldots, n_i + k_i - 1$ operate. It follows that $\mathrm{Rel}_\Sigma(\mathbf{p}) = P(A_1 \cup \cdots \cup A_t)$ and, by the requirements of the theorem, $A_x \cap A_y \subseteq A_z$ for $x, y = 1, \ldots, t$ and any z between x and y. Therefore, the same argument as in the proof of Theorem 5.4.3 reveals that

$$\begin{aligned}
\mathrm{Rel}_\Sigma(\mathbf{p}) &= \sum_{i=1}^{t} P(A_i) - \sum_{i=1}^{t-1} P(A_i \cap A_{i+1}) = \sum_{i=1}^{t} \prod_{j=n_i}^{n_i+k_i-1} p_j - \sum_{i=1}^{t-1} \prod_{j=n_i}^{n_{i+1}+k_{i+1}-1} p_j \\
&= \sum_{i=1}^{t-1} \prod_{j=n_i}^{n_i+k_i-1} p_j - \sum_{i=1}^{t-1} \prod_{j=n_i}^{n_{i+1}+k_{i+1}-1} p_j + \prod_{j=n_t}^{n_t+k_t-1} p_j \\
&= \sum_{i=1}^{t-1} \prod_{j=n_i}^{n_i+k_i-1} p_j - \sum_{i=1}^{t-1} \prod_{j=n_i}^{n_i+k_i-1} p_j \prod_{j=n_i+k_i}^{n_{i+1}+k_{i+1}-1} p_j + \prod_{j=n_t}^{n_t+k_t-1} p_j \\
&= \sum_{i=1}^{t-1} \left(1 - \prod_{j=n_i+k_i}^{n_{i+1}+k_{i+1}-1} p_j\right) \prod_{j=n_i}^{n_i+k_i-1} p_j + \prod_{j=n_t}^{n_t+k_t-1} p_j \,. \qquad \square
\end{aligned}$$

Example 5.4.7 Consider the network in Figure 5.13, where nodes 1–7 are assumed to fail randomly and independently with probabilities $q_1, \ldots, q_7$ and all other nodes and edges are perfectly reliable. Again, we are interested in the probability that a message can pass from s to t along a path of operating nodes. Thus, an appropriate model is a coherent binary system having components 1–7 and mincuts $\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5, 6\}, \{4, 5, 6, 7\}$. By applying the mincut analogue of Theorem 5.4.6 the reliability of this system is easily seen to be

$$\begin{aligned}
&1 - q_1q_2q_3 - q_2q_3q_4 - q_3q_4q_5q_6 - q_4q_5q_6q_7 + q_1q_2q_3q_4 + q_2q_3q_4q_5q_6 + q_3q_4q_5q_6q_7 \\
&\qquad = 1 - (1 - q_4)q_1q_2q_3 - (1 - q_5q_6)q_2q_3q_4 - (1 - q_7)q_3q_4q_5q_6 - q_4q_5q_6q_7 \,,
\end{aligned}$$

which equals $1 - 2q^3 - q^4 + 2q^5$ if all node failure probabilities are equal to q.

Remark. The inclusion-exclusion identity of Theorem 5.4.6 contains only non-cancelling terms. The coefficient of each such term is either $+1$ or -1. Due to a result of Shier and McIlwain [SM91], this ± 1 property holds for any coherent binary system whose minpaths (resp. mincuts) are consecutive sets of components. This strongly generalizes a corresponding result of Kossow and Preuss [KP89], who showed that this ± 1 property holds for any consecutive k-out-of-n system.

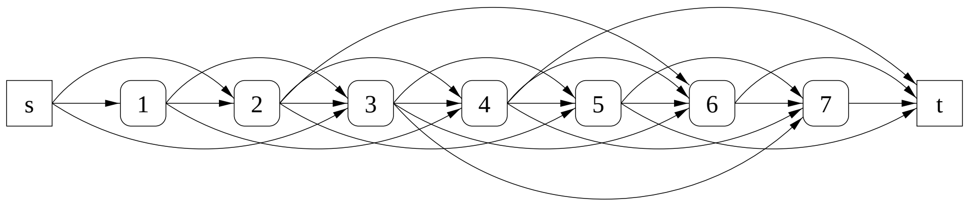

Figure 5.13: A consecutive network.

5.5 Reliability covering problems

Reliability covering problems were introduced by Ball, Provan and Shier [BPS91, Shi91] in order to generalize several types of reliability problems. They serve e.g. as a model for mass transit systems with reliable stops and unreliable routes. The overall reliability of such a system is the probability that each stop is served by an operating route. Further examples include evaluating the reliability of flight schedules for aircraft [BPS91] and determining the reliability of maintaining continuous surveillance of a critical point of a country's border [Shi91]. As in [BPS91, Shi91] we formulate reliability covering problems in terms of hypergraphs:

Definition 5.5.1 A *hypergraph* is a couple $H = (V, \mathcal{E})$ where V is a finite set and $\mathcal{E}$ is a set of non-empty subsets of V. The elements of V resp. $\mathcal{E}$ are the *vertices* resp. *edges* of H. A *covering* of V is a subset $\mathcal{X}$ of $\mathcal{E}$ such that $\bigcup \mathcal{X} = V$.

In case of a mass transit system, the vertices and edges of the hypergraph correspond to the stops and routes of the system, respectively, while the coverings correspond to those sets of routes such that each stop is served by a route.

Throughout, we assume that the vertices of the hypergraph are perfectly reliable, whereas the edges are subject to random and independent failure. The edge operation probabilities are assumed to be given by a vector $\mathbf{p} = (p_E)_{E \in \mathcal{E}} \in [0, 1]^{\mathcal{E}}$. The general objective is to determine or compute bounds on $\mathrm{Cov}(H; \mathbf{p})$, the probability that the vertex-set of H is covered by the operating edges of H.

Besides their practical applicability, reliability covering problems are also interesting from a theoretical point of view. Namely, any coherent binary system gives rise to an equivalent reliability covering problem, and vice versa [BPS91, Shi91]. In this way, the results of the preceding sections (including those on network reliability) can be reformulated as reliability covering problems. One direction of this equivalence is used in deriving the following results on *reduced* hypergraphs, i.e., hypergraphs of type $H = (V, \mathcal{E})$ where the sets $\mathcal{E}(v) := \{E \in \mathcal{E} \mid v \in E\}$ $(v \in V)$ are all distinct. Note that restricting to reduced hypergraphs does not cause any loss of generality, since by deleting vertices any hypergraph H can be transformed efficiently into a reduced hypergraph $R(H)$ such that $\mathrm{Cov}(H; \mathbf{p}) = \mathrm{Cov}(R(H); \mathbf{p})$. Now, using the above sets, the coverage probability can be expressed as

$$\mathrm{Cov}(H; \mathbf{p}) \;=\; 1 - P\left(\bigcup_{v \in V} \bigcap_{E \in \mathcal{E}(v)} \{E \text{ fails}\} \right). \tag{5.6}$$

In connection with Theorem 4.1.11 the first part of the following theorem, which is implicit in [Doh99b], yields improved inclusion-exclusion identities and Bonferroni inequalities for the last term in (5.6) and thus for the coverage probability $\mathrm{Cov}(H; \mathbf{p})$. We do not mention the improved inclusion-exclusion identities explicitly, since they are an immediate consequence of the corresponding inequalities. As in the preceding sections, we use $\mathcal{C}(V)$ to denote the order complex of V.

Theorem 5.5.2 [Doh99b] *Let $H = (V, \mathcal{E})$ be a reduced hypergraph whose edges fail randomly and independently and whose vertex-set is given the structure of a lower semilattice such that the complement of each edge is infimum-closed. Then,*

$$\left(\left\{ \bigcap\nolimits_{E \in \mathcal{E}(v)} \{E \textit{ fails}\} \right\}_{v \in V}, \mathcal{C}(V) \right)$$

is an abstract tube. In particular, for any $\mathbf{p} = (p_E)_{E \in \mathcal{E}} \in [0,1]^{\mathcal{E}}$ and any $r \in \mathbb{N}$,

$$\mathrm{Cov}(H; \mathbf{p}) \;\leq\; \sum_{\substack{I \subseteq V, |I| \leq r \\ I \text{ is a chain}}} (-1)^{|I|} \prod_{\substack{E \in \mathcal{E} \\ E \cap I \neq \emptyset}} q_E \quad (r \textit{ even}), \tag{5.7}$$

$$\mathrm{Cov}(H; \mathbf{p}) \;\geq\; \sum_{\substack{I \subseteq V, |I| \leq r \\ I \text{ is a chain}}} (-1)^{|I|} \prod_{\substack{E \in \mathcal{E} \\ E \cap I \neq \emptyset}} q_E \quad (r \textit{ odd}), \tag{5.8}$$

where $q_E = 1 - p_E$ for any $E \in \mathcal{E}$.

Proof. Consider the coherent binary system $\Sigma = (\mathcal{E}, \phi)$ where for any subset $\mathcal{X}$ of $\mathcal{E}$, $\phi(\mathcal{E} \setminus \mathcal{X}) = 0$ if and only if $\mathcal{X} \supseteq \mathcal{E}(v)$ for some $v \in V$. Clearly $\mathcal{F} := \{\mathcal{E}(v) \mid v \in V\}$ is an extended set of mincuts of this system. Now, by defining $\mathcal{E}(v) \leq \mathcal{E}(w) :\Leftrightarrow v \leq w$ for any $v, w \in V$, a partial ordering relation on $\mathcal{F}$ is established, which satisfies $\mathcal{E}(v) \wedge \mathcal{E}(w) = \mathcal{E}(v \wedge w)$ for any $v, w \in V$. Since the complement of each edge is infimum-closed, $\mathcal{E}(v \wedge w) \subseteq \mathcal{E}(v) \cup \mathcal{E}(w)$ and therefore,

$\mathcal{E}(v) \wedge \mathcal{E}(w) \subseteq \mathcal{E}(v) \cup \mathcal{E}(w)$ for any $v, w \in V$. Hence, the requirements of the extended mincut version of Theorem 5.1.3 are satisfied, and thus the first part of the theorem follows. The second part now follows from Theorem 4.1.11. □

Remarks. Theorem 5.5.2 can easily be generalized to reduced hypergraphs with statistically dependent edge failures: Simply replace the product in (5.7) and (5.8) with the probability that all edges having a non-empty intersection with I fail.

Note that the requirements of Theorem 5.5.2 (as well as those of the next theorem) are satisfied if $H = (\{1, \ldots, n\}, \mathcal{E})$ where $\mathcal{E} \subseteq \{\{k, \ldots, l\} \,|\, 1 \leq k \leq l \leq n\}$ and $n \in \mathbb{N}$. For an application of this type of hypergraph, see Shier [Shi91].

Theorem 5.5.3 [BPS91, Shi91] *Let $H = (V, \mathcal{E})$ be a reduced hypergraph whose edges fail randomly and independently according to some vector $\mathbf{q} = (q_E)_{E\in\mathcal{E}}$ of edge failure probabilities and whose vertex-set is a lower semilattice such that each edge is convex and the complement of each edge is infimum-closed. Then,*

$$\mathrm{Cov}(H; \mathbf{p}) \;=\; 1 - \sum_{v\in V} \Lambda(v, \mathbf{q})\,, \tag{5.9}$$

where $\mathbf{p} = \mathbf{1} - \mathbf{q}$ and where Λ is defined by the following recursive scheme:

$$\Lambda(v, \mathbf{q}) \;:=\; \prod_{E\in\mathcal{E}(v)} q_E \;-\; \sum_{w<v} \Lambda(w, \mathbf{q}) \prod_{\substack{E\in\mathcal{E}(v)\\ E\notin\mathcal{E}(w)}} q_E\,. \tag{5.10}$$

Proof. Theorem 5.5.3 follows from Theorem 5.1.7 in the same way as Theorem 5.5.2 follows from Theorem 5.1.3. □

Remark. In view of equations (5.9) and (5.10), Algorithm II can be adapted to solve the reliability covering problem for the class of hypergraphs $H = (V, \mathcal{E})$ satisfying the requirements of Theorem 5.5.3. The resulting algorithm, which is due to Ball, Provan and Shier [BPS91, Shi91], has a space complexity of $O(|V|)$ and a time complexity of $O(|\mathcal{E}| \times |V|^2)$, that is, its running time is bounded by a polynomial in the number of vertices and edges of the hypergraph. Thus, under the requirements of Theorem 5.5.3, the reliability covering problem can be solved in polynomial time. In general, though, the reliability covering problem is $\#P$-hard, even when restricted to the class of hypergraphs whose vertices are the vertices of a tree and whose edges are paths of length three in the tree [BPS91, Shi91]. However, if the vertices are the elements of a lower tree semilattice (that is, a lower semilattice whose Hasse diagram is a tree) and the edges are of the form $\bigcup_k \{v \in V \,|\, x_k \leq v,\, y_k \not\leq v\}$ where $x_k \leq y_k$ and the x_k's are pairwise incomparable, then the requirements of Theorem 5.5.3 are satisfied and the reliability covering problem can be solved in polynomial time [BPS91, Shi91].

We proceed with establishing a generalization of Theorem 5.5.2.

Theorem 5.5.4 [Doh99b] *Let $H = (V, \mathcal{E})$ be a hypergraph whose edges fail randomly and independently, and let c be a closure operator on V such that (V, c) is a convex geometry and such that the complement of each edge is c-closed. Then,*

$$\left(\left\{\bigcap_{E\in\mathcal{E}(v)}\{E \textit{ fails}\}\right\}_{v\in V}, \mathrm{Free}(V,c)\right)$$

is an abstract tube. In particular, for any $\mathbf{p} = (p_E)_{E\in\mathcal{E}} \in [0,1]^{\mathcal{E}}$ and any $r \in \mathbb{N}$,

$$\begin{aligned}
\mathrm{Cov}(H;\mathbf{p}) &\le \sum_{\substack{I\subseteq V, |I|\le r\\ I \text{ is c-free}}} (-1)^{|I|} \prod_{\substack{E\in\mathcal{E}\\ E\cap I\neq\emptyset}} q_E \quad (r \textit{ even}),\\
\mathrm{Cov}(H;\mathbf{p}) &\ge \sum_{\substack{I\subseteq V, |I|\le r\\ I \text{ is c-free}}} (-1)^{|I|} \prod_{\substack{E\in\mathcal{E}\\ E\cap I\neq\emptyset}} q_E \quad (r \textit{ odd}),
\end{aligned}$$

where $q_E = 1 - p_E$ for any $E \in \mathcal{E}$.

Proof. We apply Theorem 4.2.1 with $A_v := \bigcap_{E\in\mathcal{E}(v)}\{E \text{ fails}\}$ for any $v \in V$. Evidently, the requirements of Theorem 4.2.1 are satisfied if $\bigcap_{x\in X} A_x \subseteq A_v$ for any non-empty subset X of V and any $v \in c(X)$. A sufficient condition for $\bigcap_{x\in X} A_x \subseteq A_v$ is that all edges containing v have a non-empty intersection with X. In order to show that this condition holds, assume that $v \in E$ and $E \cap X = \emptyset$ for some edge E of the hypergraph. Then X would be a subset of the complement $\overline{E}$ of E, and since all complements of edges are required to be c-closed, we would also have $c(X) \subseteq \overline{E}$ and hence $v \in \overline{E}$, contradicting $v \in E$. Now, the first part of Theorem 5.5.4 follows from Theorem 4.2.1 and (5.6). The second part is a direct consequence of the first part and Theorem 4.1.11. □

As already mentioned above, the reliability covering problem is $\#P$-hard, even when restricted to the class of hypergraphs whose vertices are the vertices of a tree and whose edges are paths of length three in the tree [BPS91, Shi91]. A careful reading of the $\#P$-hardness results in [BPS91, Shi91] reveals that the restricted problem remains $\#P$-hard even if the tree is part of the input. Considering complements of paths instead of paths, or more generally, complements of subtrees instead of paths, Theorem 5.5.4 gives rise to the following positive result:

Theorem 5.5.5 [Doh99b] *For hypergraphs whose vertices are the vertices of a tree and whose edges are complements of subtrees of the tree, the coverage probability can be computed in polynomial time from the hypergraph and the tree.*

Proof. Let $G = (V, T)$ be a tree and $H = (V, \mathcal{E})$ be a hypergraph where each edge of H is the complement of a subtree of G. By combining Theorem 5.5.4 with Example 3.1.5 (or by applying Corollary 3.1.16) we obtain the formula

$$\mathrm{Cov}(H;\mathbf{p}) = 1 - \sum_{v\in V} \prod_{E\in\mathcal{E}(v)} q_E + \sum_{\{v,w\}\in T} \prod_{E\in\mathcal{E}(v)\cup\mathcal{E}(w)} q_E\,, \tag{5.11}$$

whose evaluation requires $O(|V| \cdot |\mathcal{E}|)$ time. $\square$

An even more general result is the following. Recall from Section 2.1 that the *clique number* of a graph G is the maximum cardinality of a clique in G.

Theorem 5.5.6 *For hypergraphs whose vertices are those of a connected block graph of bounded clique number and whose edges are complements of connected subgraphs of the connected block graph, the coverage probability can be computed in polynomial time from the hypergraph and the connected block graph.*

Proof. Let G be a connected block graph having clique number at most d, and let $H = (V, \mathcal{E})$ be a hypergraph where each edge of H is the complement of a connected subgraph of G. By applying Theorem 5.5.4 in connection with the convex geometry of Example 3.1.6 we obtain the inclusion-exclusion formula

$$\text{Cov}(H; \mathbf{p}) \; = \; \sum_{\substack{I \text{ is a clique} \\ \text{of } G}} (-1)^{|I|} \prod_{\substack{E \in \mathcal{E} \\ E \cap I \neq \emptyset}} q_E \,, \tag{5.12}$$

whose evaluation requires $O(|V|^d \cdot |\mathcal{E}|)$ time, where d is a constant. $\square$

Remarks. The requirement that the connected block graph is of bounded clique number is essential, since otherwise we could take for instance the complete graph and thus reduce the problem to its unconstrained counterpart, which is $\#P$-hard. Anyway, as in the following example, we can take full advantage of the improved Bonferroni inequalities associated with Theorem 5.5.4 and Example 3.1.6.

A generalization of (5.11) and (5.12) to statistically dependent edge failures is straightforward and left as an option to the reader.

Example 5.5.7 Consider the hypergraph with vertices 1,2,3,4,5,6,7 and edges

$$\{1,2,3,4\}, \{4,5,6,7\}, \{1,6,7\}, \{1,3,6\}, \{2,3,5,7\}, \{2,5,6\}, \{2,6,7\}, \{1,5,7\}.$$

Obviously, the edges of this hypergraph are complements of connected subgraphs of the connected block graph displayed in Figure 5.14. Therefore, we can apply Theorem 5.5.4 in connection with the convex geometry of Example 3.1.6 to obtain improved Bonferroni bounds on the coverage probability $\text{Cov}(H; \mathbf{p})$ of this hypergraph. Under the assumption that the edges of the hypergraph fail randomly and independently with equal probability $q = 1 - p$, the results are shown in Table 5.9. Here, $f_r(q)$ resp. $f_r^*(q)$ denotes the rth classical resp. improved Bonferroni bound, where even and odd values of r correspond to upper and lower bounds, respectively. As in Example 5.2.1, the improved bounds are much sharper than the classical bounds, although much fewer sets are taken into account. Table 5.10 shows some numerical values. Finally, some bounds are plotted in Figure 5.15.

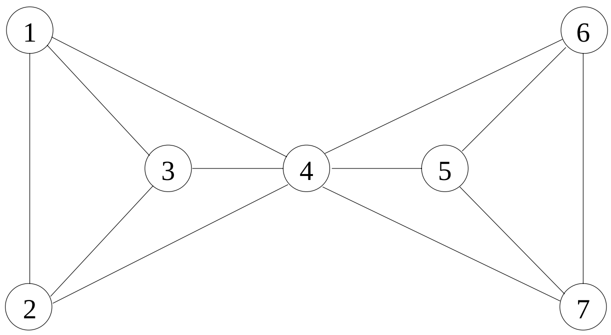

Figure 5.14: A connected block graph.

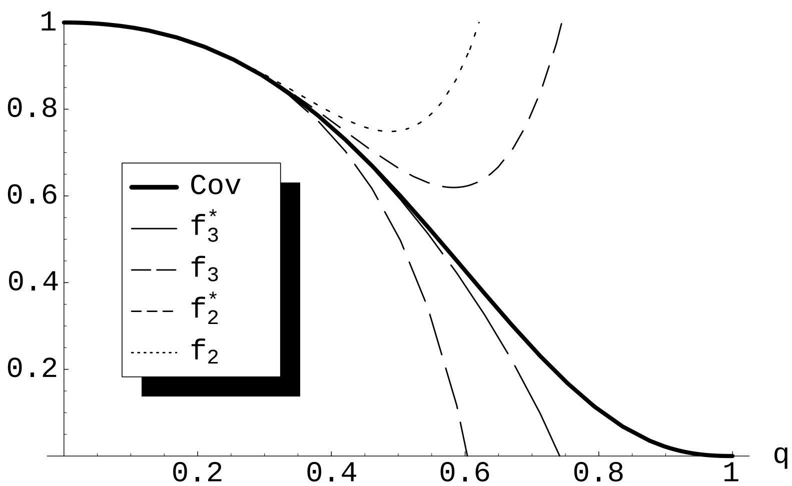

Figure 5.15: A plot of some of the bounds in Table 5.9.

r	classical bounds $f_r(q)$	# sets	improved bounds $f_r^*(q)$	# sets
1	$1 - q^2 - q^3 - 3q^4 - 2q^5$	8	$1 - q^2 - q^3 - 3q^4 - 2q^5$	8
2	$1 - q^2 - q^3 - 2q^4 + 3q^5 + 5q^6 + 10q^7$	29	$1 - q^2 - q^3 - 2q^4 + 3q^5 + 3q^6 + 3q^7$	20
3	$1 - q^2 - q^3 - 2q^4 + 3q^5 + q^6 - 5q^7 - 16q^8$	64	$1 - q^2 - q^3 - 2q^4 + 3q^5 + q^6 - 3q^8$	28
4	$1 - q^2 - q^3 - 2q^4 + 3q^5 + q^6 + 14q^8$	99	$1 - q^2 - q^3 - 2q^4 + 3q^5 + q^6 - q^8$	30
5	$1 - q^2 - q^3 - 2q^4 + 3q^5 + q^6 - 7q^8$	120		
6	$1 - q^2 - q^3 - 2q^4 + 3q^5 + q^6$	127		
7	$1 - q^2 - q^3 - 2q^4 + 3q^5 + q^6 - q^8$	128		

Table 5.9: Bonferroni bounds for the coverage probability of the hypergraph in Example 5.5.7.

q	$f_3(q)$	$f_3^*(q)$	$f_4^*(q)^\dagger$	$f_4(q)$	$f_2^*(q)$	$f_2(q)$
0.0	1.00000	1.00000	*1.00000*	1.00000	1.00000	1.00000
0.1	0.98883	0.98883	*0.98883*	0.98883	0.98883	0.98884
0.2	0.94972	0.94982	*0.94982*	0.94986	0.94999	0.95021
0.3	0.87268	0.87462	*0.87475*	0.87574	0.87693	0.87992
0.4	0.74094	0.75765	*0.75896*	0.76879	0.77272	0.79238
0.5	0.50781	0.59766	*0.60547*	0.66406	0.66406	0.75000
0.6	0.03603	0.39435	*0.42794*	0.67988	0.62203	0.91130
0.7	-1.02548	0.13572	*0.25101*	1.11573	0.79102	1.60280

†exact coverage probability

Table 5.10: Numerical values of the bounds in Table 5.9.

Chapter 6

Miscellaneous Topics

This chapter is devoted to some classical topics of enumerative combinatorics. In the first section we identify an abstract tube associated with partition lattices of finite sets. Then, in the second section we apply our inclusion-exclusion results to the chromatic polynomial of a graph and thus deduce Whitney's broken circuit theorem [Whi32] as well as several of its generalizations. In the remaining sections similar conclusions are drawn for the Tutte polynomial, the characteristic polynomial and the β invariant of a matroid, the Euler characteristic of an abstract simplicial complex and the Möbius function of a partially ordered set. In particular, we rediscover a recent generalization of Rota's crosscut theorem [Rot64] due to Blass and Sagan [BS97] and obtain a new proof of a classical theorem due to Weisner [Wei35]. A key role in proving these results is due to Theorem 3.2.4 and its forthcoming generalization to partially ordered sets.

6.1 Inclusion-exclusion on partition lattices

In this section, we establish an abstract tube generalization of Narushima's principle of inclusion-exclusion on partition lattices [Nar74, Nar77], which turned out as a very useful tool in the enumeration of reduced finite automata [Nar77].

Definition 6.1.1 Let S be a set. A *partition* of S is a set of non-empty and pairwise disjoint subsets of S whose union is S. Each element of the partition is referred to as a *block* of the partition. The set $\Pi(S)$ of all partitions of S is given the structure of a lattice by defining for any $\pi, \tau \in \Pi(S)$,

$$\pi \leq \tau \ :\Leftrightarrow\ \text{each block of } \pi \text{ is included in a block of } \tau.$$

For $s, s' \in S$ we write $s\pi s'$ if s and s' belong to the same block of π. Thus, $\pi \leq \tau$ if and only if for any $s, s' \in S$, $s\pi s'$ implies $s\tau s'$. For any sets S and T, the cartesian product $\Pi(S) \times \Pi(T)$ is given the structure of a lattice by definition of

$$(\pi_1, \tau_1) \leq (\pi_2, \tau_2) \ :\Leftrightarrow\ \pi_1 \leq \pi_2 \text{ and } \tau_1 \leq \tau_2$$

for any $(\pi_1, \tau_1), (\pi_2, \tau_2) \in \Pi(S) \times \Pi(T)$.

The following definitions are non-standard and differ from those of Narushima [Nar74, Nar77], who prefers to use the terminology of Galois correspondences.

Definition 6.1.2 A *partitioned set* is a pair (S, π), consisting of a set S and a partition π of S. Given two partitioned sets (S, π) and (T, τ), a mapping $f : S \to T$ is called a *homomorphism from* (S, π) *to* (T, τ) if for any $s, s' \in S$, $s\pi s'$ implies $f(s)\tau f(s')$. A homomorphism from (S, π) to itself is also referred to as an *endomorphism of* (S, π). For abbreviation, we write $f : (S, \pi) \to (T, \tau)$ if f is a homomorphism from (S, π) to (T, τ), and define

$$\begin{aligned}
\mathrm{Hom}((S, \pi), (T, \tau)) &:= \{f \mid f : (S, \pi) \to (T, \tau)\}, \\
\mathrm{End}(S, \pi) &:= \mathrm{Hom}((S, \pi), (S, \pi)).
\end{aligned}$$

Our abstract tube generalization of Narushima's principle of inclusion-exclusion on partition lattices [Nar74, Nar77] follows. The improved inclusion-exclusion identity associated with this abstract tube is of course due to Narushima [Nar74, Nar77], whereas the associated improved Bonferroni inequalities are new.

Theorem 6.1.3 [Dohb] *Let S and T be finite sets, and let L be a subsemilattice of $\Pi(S) \times \Pi(T)$. Then, $\big(\{\mathrm{Hom}((S, \pi), (T, \tau))\}_{(\pi,\tau)\in L}, \mathcal{C}(L)\big)$ is an abstract tube.*

Proof. It is easy to verify (cf. [Nar74, Nar77]) that for any $(\pi, \tau), (\pi', \tau') \in L$,

$$\begin{aligned}
\mathrm{Hom}((S, \pi), (T, \tau)) \cap \mathrm{Hom}((S, \pi'), (T, \tau')) &\subseteq \mathrm{Hom}((S, \pi \wedge \pi'), (T, \tau \wedge \tau')), \\
\mathrm{Hom}((S, \pi), (T, \tau)) \cap \mathrm{Hom}((S, \pi'), (T, \tau')) &\subseteq \mathrm{Hom}((S, \pi \vee \pi'), (T, \tau \vee \tau')),
\end{aligned}$$

where $\wedge$ and $\vee$ stand for the infimum (greatest lower bound) and supremum (least upper bound) in $\Pi(S)$ and $\Pi(T)$. By Corollary 4.2.13, the result follows. □

Corollary 6.1.4 [Dohb] *Let S be a finite set, and let L be a subsemilattice of $\Pi(S)$. Then, $(\{\mathrm{End}(S, \pi)\}_{\pi\in L}, \mathcal{C}(L))$ is an abstract tube.*

Proof. Since $\pi \mapsto (\pi, \pi)$ is a lattice isomorphism from $\Pi(S)$ to $\Pi(S) \times \Pi(S)$, Corollary 6.1.4 follows from Theorem 6.1.3 by considering the case $S = T$. □

6.2 Chromatic polynomials of graphs

The chromatic polynomial of a graph is a fundamental concept in enumerative combinatorics and chromatic graph theory. It expresses the number of vertex-colorings of a graph in at most λ colors such that no adjacent vertices receive the same color. The chromatic polynomial was first considered by Birkhoff [Bir12] in connection with the four color problem. Since its computation is $\#P$-complete in general [Val79, Wel93], lower bounds and upper bounds are of great value. Some authors [Bye98, Laz90] justify the seeking of bounds by the analysis of Wilf's backtracking algorithm for the graph coloring problem [Wil84, Wil86, BW85].

One of the most important results on the chromatic polynomial is Whitney's broken circuit theorem [Whi32], which relates the coefficients of the chromatic polynomial to the face count numbers of an abstract simplicial complex. Based on our inclusion-exclusion results, we give a new and strongly simplified proof of Whitney's theorem and establish an abstract tube generalization of it. In this way, new Bonferroni inequalities for the chromatic polynomial are obtained that involve the face count numbers of the associated abstract simplicial complex. Finally, a new two-variable generalization of the chromatic polynomial is proposed as well as a corresponding generalization of the results obtained so far.

Definition 6.2.1 Let G be a graph and $\lambda \in \mathbb{N}$. A λ-*coloring* of G is a mapping f from the vertex-set of G into $\{1, \ldots, \lambda\}$. A λ-coloring f of G is *proper* if $f(v) \neq f(w)$ for each edge $\{v, w\}$ of G. We use $P_G(\lambda)$ to denote the number of proper λ-colorings of G and refer to $P_G(\lambda)$ as the *chromatic polynomial* of G.

Remark. The preceding definition is justified by a result of Birkhoff [Bir12], who showed that $P_G(\lambda)$ is a monic polynomial in λ of degree $n(G)$ with integer coefficients. Historically, the definition of the chromatic polynomial was motivated by the four color problem, which asks whether $P_G(4) > 0$ for any planar graph G.

To state Whitney's broken circuit theorem, a further definition is needed:

Definition 6.2.2 Let G be a graph whose edge-set is endowed with a linear ordering relation. A *broken circuit* of G is obtained from the edge-set of a cycle of G by removing its maximum edge. The *broken circuit complex* of G, abbreviated to $\mathcal{BC}(G)$, is the abstract simplicial complex consisting of all non-empty subsets of the edge-set of G that do not include any broken circuit of G as a subset.

The definition of a broken circuit goes back to Whitney [Whi32], while the broken circuit complex was initiated by Wilf [Wil76] (see also [Bry77, BO81, BZ91]).

Example 6.2.3 Consider the graph in Figure 6.1, whose edge-set is linearly ordered according to the labelling of the edges. Obviously, the broken circuits are $\{1, 2\}, \{1, 2, 4\}$ and $\{3, 4\}$, whence the broken circuit complex is equal to

$$\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\},$$
$$\{2, 5\}, \{3, 5\}, \{4, 5\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 5\}\}.$$

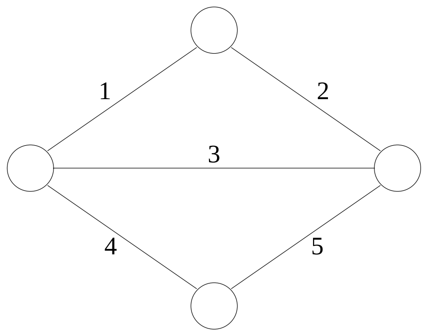

Figure 6.1: A graph with labelled edges.

In the following, we restate and reprove Whitney's famous broken circuit theorem [Whi32]. The reader is invited to compare our proof with the original proof of Whitney [Whi32] or with the bijective proof of Blass and Sagan [BS86].

Theorem 6.2.4 [Whi32] *Let G be a graph whose edge-set is endowed with a linear ordering relation. Then, for any $\lambda \in \mathbb{N}$,*

$$P_G(\lambda) \;=\; \sum_{k=0}^{n(G)} (-1)^k \, b_k(G) \, \lambda^{n(G)-k}, \tag{6.1}$$

where $b_0(G) = 1$ and $b_k(G)$, $k > 0$, counts the faces of cardinality k (dimension $k-1$) in $\mathcal{BC}(G)$.

Proof. Define $\mathcal{X}$ as the set of broken circuits of G, and for any edge e of G define

$$A_e \;:=\; \{f : V(G) \to \{1, \ldots, \lambda\} \mid f(v) = f(w)\}, \qquad e = \{v, w\}.$$

Then, the requirements of Theorem 3.2.8 are satisfied, and thus we obtain

$$P_G(\lambda) \;=\; \lambda^{n(G)} - \left| \bigcup_{e \in E(G)} A_e \right| \;=\; \lambda^{n(G)} + \sum_{k>0} (-1)^k \sum_{\substack{I \in \mathcal{BC}(G) \\ |I|=k}} \left| \bigcap_{i \in I} A_i \right|. \tag{6.2}$$

Since the edge-subgraph $G[I]$ is cycle-free for any $I \in \mathcal{BC}(G)$, it follows that

$$m(G[I]) - n(G[I]) + c(G[I]) \;=\; 0 \tag{6.3}$$

and hence,

$$\left| \bigcap_{i \in I} A_i \right| \;=\; \lambda^{n(G)-n(G[I])} \lambda^{c(G[I])} \;=\; \lambda^{n(G)-m(G[I])} \;=\; \lambda^{n(G)-|I|}.$$

Putting this into (6.2) and collecting powers of λ, the theorem follows. □

Remark. As a consequence of Theorem 6.2.4, $1 + |\mathcal{BC}(G)| = |P_G(-1)|$. By a celebrated result of Stanley [Sta73] the latter quantity coincides with the number of acyclic orientations of G where, by definition, an *acyclic orientation* of G is an acyclic digraph that can be obtained from G by directing the edges of G.

We now restate and generalize Whitney's broken circuit theorem in terms of abstract tubes, which leads to improved Bonferroni inequalities on the chromatic polynomial. These inequalities become an identity if $r \geq \dim \mathcal{BC}(G) + 1$. Since $n(G) \geq \dim \mathcal{BC}(G) + 1$, we thus rediscover Whitney's broken circuit theorem.

Theorem 6.2.5 [Doh99d] *Let G be a graph whose edge-set is endowed with a linear ordering relation. Then, for any $\lambda \in \mathbb{N}$,*

$$\big(\{\{f : V(G) \to \{1, \ldots, \lambda\} \,|\, f(v) = f(w)\}\}_{\{v,w\} \in E(G)}\,, \mathcal{BC}(G)\big)$$

is an abstract tube, and for any $r \in \mathbb{N}$ the following inequalities hold:

$$P_G(\lambda) \;\geq\; \sum_{k=0}^{r} (-1)^k\, b_k(G)\, \lambda^{n(G)-k} \quad (r \text{ odd}),$$
$$P_G(\lambda) \;\leq\; \sum_{k=0}^{r} (-1)^k\, b_k(G)\, \lambda^{n(G)-k} \quad (r \text{ even}),$$

where $b_0(G) = 1$ and $b_k(G)$, $k > 0$, counts the faces of cardinality k (dimension $k-1$) in $\mathcal{BC}(G)$.

Proof. Define $\mathcal{X}$ and A_e as in the proof of Theorem 6.2.4. Then, the first part of Theorem 6.2.5 is an immediate consequence of Theorem 4.2.11. The second part follows from the first one by combining it with Theorem 4.1.11. Thus, one gets

$$P_G(\lambda) \;\geq\; \lambda^{n(G)} + \sum_{k=1}^{r} (-1)^k \sum_{\substack{I \in \mathcal{BC}(G) \\ |I|=k}} \left|\bigcap_{i \in I} A_i\right| \quad (r \text{ odd}),$$
$$P_G(\lambda) \;\leq\; \lambda^{n(G)} + \sum_{k=1}^{r} (-1)^k \sum_{\substack{I \in \mathcal{BC}(G) \\ |I|=k}} \left|\bigcap_{i \in I} A_i\right| \quad (r \text{ even}).$$

Now, the same arguments as in the proof of Theorem 6.2.4 apply. □

Remarks. The first part of Theorem 6.2.5 can be generalized to hypergraphs where each cycle has an edge of cardinality two; see [Doh95a] for a proof of the corresponding identity and [Doh00d] for a generalization of this identity with regard to the number of precoloring extensions of the hypergraph. Further sources on chromatic polynomials of hypergraphs are [Tom98] and [Doh93a, Doh95b].

A restatement and alternative proof of the second part of Theorem 6.2.5 (without supplying an interpretation of the coefficients) is given subsequently.

Theorem 6.2.6 [Doh99e] *Let G be a graph. Then, for any $\lambda \in \mathbb{N}$ and $r \in \mathbb{N}$,*

$$P_G(\lambda) \;\geq\; \sum_{k=0}^{r} (-1)^k\, b_k(G)\, \lambda^{n(G)-k} \quad (r \text{ odd}), \tag{6.4}$$

$$P_G(\lambda) \;\leq\; \sum_{k=0}^{r} (-1)^k\, b_k(G)\, \lambda^{n(G)-k} \quad (r \text{ even}), \tag{6.5}$$

where $(-1)^k b_k(G)$ denotes the coefficient of $\lambda^{n(G)-k}$ in $P_G(\lambda)$ $(k = 0, \ldots, n(G))$.

Proof. We proceed by induction on $m(G)$. If $m(G) = 0$, then $P_G(\lambda) = \lambda^{n(G)}$ and the statement holds. Now let G have at least one edge e, and assume that the statement is true for graphs having fewer edges. Let $G - e$ resp. G/e denote the graph obtained from G by deleting resp. contracting e and then, in the resulting multigraph, replacing each class of parallel edges by a single edge. Note that $n(G) = n(G-e) = n(G/e) + 1$. By the deletion-contraction formula [Rea67],

$$P_G(\lambda) \;=\; P_{G-e}(\lambda) - P_{G/e}(\lambda) \tag{6.6}$$

and consequently,

$$b_k(G) \;=\; b_k(G-e) + b_{k-1}(G/e) \quad (k = 1, \ldots, n(G)). \tag{6.7}$$

The induction hypothesis applied to $G - e$ and G/e gives

$$\begin{aligned} P_{G-e}(\lambda) \;&\leq\; \sum_{k=0}^{r} (-1)^k\, b_k(G-e)\, \lambda^{n(G-e)-k} \;=\; \sum_{k=0}^{r} (-1)^k\, b_k(G-e)\, \lambda^{n(G)-k}\,, \\ P_{G/e}(\lambda) \;&\geq\; \sum_{k=0}^{r-1} (-1)^k\, b_k(G/e)\, \lambda^{n(G/e)-k} \;=\; \sum_{k=1}^{r} (-1)^{k-1}\, b_{k-1}(G/e)\, \lambda^{n(G)-k}\,, \end{aligned}$$

in the case where r is even. From this and (6.6) we conclude that

$$P_G(\lambda) \;\leq\; \lambda^{n(G)} + \sum_{k=1}^{r} (-1)^k\, (b_k(G-e) + b_{k-1}(G/e))\, \lambda^{n(G)-k}\,.$$

Now apply (6.7). The case where r is odd is treated in a similar way. □

Remark. In [Doh99e] some prior bounds for $P_G(\lambda)$ due to the author [Doh93b] are derived from (6.4) and (6.5), which only depend on the girth and the number of vertices and edges of G. Improvements and generalizations of these bounds and a bound due to Lazebnik [Laz90] are established in [Doh95b, Doh96, Doh98a]; see also Byer [Bye96, Bye98] for a discussion and further references.

As a corollary we obtain that in absolute value each coefficient of the chromatic polynomial is at most the sum of the coefficients of its neighbours:

Corollary 6.2.7 *Let the chromatic polynomial of G be given by (6.1). Then,*

$$b_k(G) \;\leq\; b_{k-1}(G) + b_{k+1}(G) \qquad (k = 1, \ldots, n(G) - 1).$$

Proof. We only consider the case where k is odd. Then, by applying Theorem 6.2.5 or Theorem 6.2.6 with $\lambda = 1$ and $r = k - 2$ resp. $r = k + 1$ we get

$$\sum_{i=0}^{k-2} (-1)^i \, b_i(G) \;\leq\; \sum_{i=0}^{k+1} (-1)^i \, b_i(G)\,,$$

which implies the result. □

We close this section with a new two-variable generalization of the chromatic polynomial and an associated generalization of Theorem 6.2.5.

Definition 6.2.8 Let G be a graph. For any $\lambda \in \mathbb{N}$ and $\mu \in \{0, \ldots, \lambda\}$, we use $P_G(\lambda, \mu)$ to denote the number of all mappings f from the vertex-set of G into $\{1, \ldots, \lambda\}$ such that $f(v) \neq f(w)$ or $f(v) = f(w) > \mu$ for each edge $\{v, w\}$ of G.

Remark. Evidently, $P_G(\lambda, 0) = \lambda^{n(G)}$ and $P_G(\lambda, \lambda) = P_G(\lambda)$. Moreover, $P_G(2, 1)$ gives the number of independent sets of G. Recall that a subset $W \subseteq V(G)$ is an *independent set* of G if the vertex-induced subgraph $G[W]$ has no edges.

Theorem 6.2.9 *Let G be a graph whose edge-set is endowed with a linear ordering relation, and let $\lambda \in \mathbb{N}$ and $\mu \in \{0, \ldots, \lambda\}$. Then*

$$\left(\left\{\{f : V(G) \to \{1, \ldots, \lambda\} \,|\, f(v) = f(w) \leq \mu\}\right\}_{\{v,w\} \in E(G)}\,,\, \mathcal{BC}(G)\right)$$

is an abstract tube, and for any $r \in \mathbb{N}$ the following inequalities hold:

$$P_G(\lambda, \mu) \;\geq\; \sum_{k=0}^{r} \sum_{l=0}^{k} (-1)^k \, b_{k,l}(G) \, \lambda^{n(G)-k-l} \mu^l \quad (r \text{ odd}),$$
$$P_G(\lambda, \mu) \;\leq\; \sum_{k=0}^{r} \sum_{l=0}^{k} (-1)^k \, b_{k,l}(G) \, \lambda^{n(G)-k-l} \mu^l \quad (r \text{ even}),$$

where $b_{0,0}(G) = 1$ and $b_{k,l}(G)$, $k > 0$, counts all faces I of cardinality k (dimension $k - 1$) in $\mathcal{BC}(G)$ such that the edge-subgraph $G[I]$ has l connected components.

Proof. Define $\mathcal{X}$ as the set of broken circuits of G, and for any edge e of G define

$$A_e \;:=\; \{f : V(G) \to \{1, \ldots, \lambda\} \,|\, f(v) = f(w) \leq \mu\}\,, \qquad e = \{v, w\}.$$

As in the proof of Theorem 6.2.5, the first part is an immediate consequence of Theorem 4.2.11. In combination with Theorem 4.1.11 the first part gives

$$P_G(\lambda,\mu) \;\geq\; \lambda^{n(G)} + \sum_{k=1}^{r} (-1)^k \sum_{\substack{I\in\mathcal{BC}(G)\\ |I|=k}} \left|\bigcap_{i\in I} A_i\right| \quad (r \text{ odd}), \tag{6.8}$$

$$P_G(\lambda,\mu) \;\leq\; \lambda^{n(G)} + \sum_{k=1}^{r} (-1)^k \sum_{\substack{I\in\mathcal{BC}(G)\\ |I|=k}} \left|\bigcap_{i\in I} A_i\right| \quad (r \text{ even}). \tag{6.9}$$

Since $G[I]$ is cycle-free for any $I \in \mathcal{BC}(G)$, we again have (6.3) and hence,

$$\left|\bigcap_{i\in I} A_i\right| \;=\; \lambda^{n(G)-n(G[I])}\mu^{c(G[I])} \;=\; \lambda^{n(G)-m(G[I])-c(G[I])}\mu^{c(G[I])}\,.$$

Putting this into (6.8) and (6.9) and taking account of $c(G[I]) \leq m(G[I]) = |I|$, we complete the proof. □

Corollary 6.2.10 *Let G be a graph whose edge-set is endowed with a linear ordering relation. Then, for any $\lambda \in \mathbb{N}$ and any $\mu \in \{0,\ldots,\lambda\}$,*

$$P_G(\lambda,\mu) \;=\; \sum_{k=0}^{m(G)} \sum_{l=0}^{k} (-1)^k\, b_{k,l}(G)\, \lambda^{n(G)-k-l}\mu^l\,, \tag{6.10}$$

where $b_{0,0}(G) = 1$ and $b_{k,l}(G)$, $k > 0$, counts all faces I of cardinality k (dimension $k-1$) in $\mathcal{BC}(G)$ such that the edge-subgraph $G[I]$ has l connected components.

Proof. Corollary 6.2.10 is an immediate consequence of Theorem 6.2.9. □

Remarks. Note that the preceding equation (6.10) can equivalently be stated as

$$P_G(\lambda,\mu) \;=\; \lambda^{n(G)}\, Q\left(G;\, -\lambda^{-1}, \mu\lambda^{-1}\right)\,,$$

where Q is defined by

$$Q(G;\, x,y) \;=\; \sum_{k,l} b_{k,l}(G)\, x^k\, y^l\,.$$

Thus, $Q(G;\, -\lambda^{-1}, \mu\lambda^{-1})$ expresses the probability that a λ-coloring of G, which is chosen uniformly at random, is admissible in the sense of Definition 6.2.8.

Statements on the number of independent sets are obtained from the preceding results by putting $\lambda = 2$ and $\mu = 1$. The corresponding abstract tube is

$$\left(\left\{\{W \,|\, W \subseteq V(G),\, e \subseteq W\}\right\}_{e\in E(G)}, \mathcal{BC}(G)\right).$$

We further remark that non-isomorphic trees on the same number of vertices may have different polynomials in λ and μ. This contrasts the situation for the usual chromatic polynomial, which equals $\lambda(\lambda-1)^{n-1}$ for all trees on n vertices. Consider for instance a path G and a star G', both on four vertices. Then,

$$\begin{array}{llll} b_{0,0}(G)=1, & b_{2,2}(G)=1, & b_{0,0}(G')=1, & b_{2,2}(G')=0, \\ b_{1,0}(G)=0, & b_{3,0}(G)=0, & b_{1,0}(G')=0, & b_{3,0}(G')=0, \\ b_{1,1}(G)=3, & b_{3,1}(G)=1, & b_{1,1}(G')=3, & b_{3,1}(G')=1, \\ b_{2,0}(G)=0, & b_{3,2}(G)=0, & b_{2,0}(G')=0, & b_{3,2}(G')=0, \\ b_{2,1}(G)=2, & b_{3,3}(G)=0, & b_{2,1}(G')=3, & b_{3,3}(G')=0. \end{array}$$

Putting these values into (6.10) we obtain

$$\begin{aligned} P_G(\lambda,\mu) &= \lambda^4-3\lambda^2\mu+2\lambda\mu+\mu^2-\mu, \\ P_{G'}(\lambda,\mu) &= \lambda^4-3\lambda^2\mu+3\lambda\mu-\mu. \end{aligned}$$

Evidently, these two polynomials differ unless $\mu=\lambda$ or $\mu=0$. As a consequence, the generalized chromatic polynomial $P_G(\lambda,\mu)$ is not an evaluation of the Tutte polynomial [Tut47, Tut54], which is the same for all trees on a given number of vertices. We shall be concerned with the Tutte polynomial in the next section.

Very recently, Tittmann [Tit] showed that the generalized chromatic polynomial of a tree can be computed in polynomial time, and he also gave general formulae for complete graphs, complete bipartite graphs, paths and cycles. These formulae are much more complicated than their counterparts for the usual chromatic polynomial. For instance, if P_n denotes the path on n vertices, then

$$P_{P_n}(\lambda,\mu) \;=\; \lambda^n+\sum_{k=0}^{n}\sum_{l=1}^{\lfloor\frac{n-k}{2}\rfloor}(-1)^{n+k+l}\binom{k+l}{k}\binom{n-k-l-1}{l-1}\lambda^k\mu^l\,.$$

Tittmann [Tit] also observed that

$$P_G(\lambda+1,1) \;=\; \sum_{k=0}^{n(G)}\alpha_k(G)\lambda^{n(G)-k} \tag{6.11}$$

where $\alpha_k(G)$ denotes the number of independent sets of cardinality k in G. The polynomial on the right-hand side of (6.11) is known as the *independence polynomial* of G. Thus, our new two-variable polynomial $P_G(\lambda,\mu)$ generalizes both the chromatic polynomial and the independence polynomial.

The results on the new two-variable polynomial will appear in a joint paper by Dohmen and Tittmann [DT].

6.3 Matroid polynomials and the β invariant

Similar results as for the chromatic polynomial of a graph can be devised for the Tutte polynomial, the characteristic polynomial and the β invariant of a matroid. To this end, we briefly review some basic notions and facts of matroid theory. For a detailed exposition, the reader is referred to the textbook of Welsh [Wel76].

Definition 6.3.1 A *matroid* is a pair $M = (E, r)$ consisting of a finite set E and a $\mathbb{Z}$-valued function r on the power set of E such that for any $A, B \subseteq E$,

$$\begin{aligned} &\text{(i)} && 0 \le r(A) \le |A|, \\ &\text{(ii)} && A \subseteq B \Rightarrow r(A) \le r(B), \\ &\text{(iii)} && r(A \cup B) + r(A \cap B) \le r(A) + r(B). \end{aligned}$$

Example 6.3.2 If G is a graph and $r(I) = n(G[I]) - c(G[I])$ for any $I \subseteq E(G)$, then $M(G) := (E(G), r)$ is a matroid, which is called the *cycle matroid* of G.

Definition 6.3.3 The *Tutte polynomial* $T(M; a, b)$ of a matroid $M = (E, r)$ is defined by

$$T(M;\, a, b) \ :=\ \sum_{I \subseteq E} (a-1)^{r(E)-r(I)} (b-1)^{|I|-r(I)},$$

where a and b are independent variables.

Specializations of the Tutte polynomial $T(M(G);\, a, b)$ count various objects associated with a graph G, e.g., subgraphs, spanning trees, acyclic orientations and proper λ-colorings. It is also related to the all-terminal reliability $R(G)$. Namely, if G is a connected graph whose nodes are perfectly reliable and whose edges fail randomly and independently with equal probability $q = 1 - p$, then

$$R(G) \ =\ q^{m(G)-n(G)+1}\, p^{n(G)+1}\, T(M(G);\, 1, q^{-1}).$$

For further applications of the Tutte polynomial, we refer to [BO92, Wel93].

Definition 6.3.4 For any matroid $M = (E, r)$ and any subset X of E, the *contraction of X from M* is given by $M/X := (E \setminus X, r_X)$ where $r_X(I) := r(X \cup I) - r(X)$ for any $I \subseteq E \setminus X$.

Our result on the Tutte polynomial is stated below. An equivalent formulation using union-closed sets instead of kernel operators is left to the reader.

Theorem 6.3.5 [Doh99f] *Let $M = (E, r)$ be a matroid, k a kernel operator on E and $a, b \in \mathbb{C}$ such that $T(M/X;\, a, b) = 0$ for any k-open $X \in \mathcal{P}^*(E)$. Then,*

$$T(M;\, a, b) \ =\ \sum_{I\,:\,k(I)=\emptyset} (a-1)^{r(E)-r(I)} (b-1)^{|I|-r(I)}.$$

Proof. For any $X \subseteq E$ define $f(X) := \sum_{I \supseteq X} (a-1)^{r(E)-r(I)} (b-1)^{|I|-r(I)}$. Then, by Theorem 3.2.4, it suffices to prove that $f(X) = 0$ for any non-empty and k-open subset X of E. With $\overline{X}$ denoting the complement of X in E we obtain

$$\begin{aligned} f(X) &= \sum_{I \subseteq \overline{X}} (a-1)^{r(E)-r(X \cup I)} (b-1)^{|X \cup I| - r(X \cup I)} \\ &= (a-1)^{r(E)-r(X)-r_X(\overline{X})} (b-1)^{|X|-r(X)} \sum_{I \subseteq \overline{X}} (a-1)^{r_X(\overline{X})-r_X(I)} (b-1)^{|I|-r_X(I)} \\ &= (a-1)^{r(E)-r(X)-r_X(\overline{X})} (b-1)^{|X|-r(X)}\, T(M/X; a, b) \;=\; 0\,. \quad \square \end{aligned}$$

From Theorem 6.3.5 we subsequently deduce Heron's broken circuit theorem [Her72], which generalizes Whitney's broken circuit theorem [Whi32] from chromatic polynomials of graphs to characteristic polynomials of matroids.

Definition 6.3.6 The *characteristic polynomial* $C(M; \lambda)$ of a matroid $M = (E, r)$ is defined by

$$C(M; \lambda) \;:=\; (-1)^{r(E)}\, T(M;\, 1-\lambda, 0) \;=\; \sum_{I \subseteq E} (-1)^{|I|}\, \lambda^{r(E)-r(I)}\,. \tag{6.12}$$

A *circuit* of a matroid $M = (E, r)$ is set $C \in \mathcal{P}^*(E)$ such that $r(C \backslash \{c\}) = |C| - 1 = r(C)$ for any $c \in C$. A *loop* is a circuit of cardinality 1. If E is linearly ordered and C a circuit of M, then $C \setminus \{\max C\}$ is referred to as a *broken circuit* of M.

Remark. By the classical inclusion-exclusion principle, the chromatic polynomial of a graph G is related to the characteristic polynomial of $M(G)$ by

$$P_G(\lambda) \;=\; \lambda^{c(G)}\, C(M(G); \lambda)\,.$$

We are now ready to state Heron's broken circuit theorem [Her72].

Corollary 6.3.7 [Her72] *Let $M = (E, r)$ be a matroid, where E is endowed with a linear ordering relation. Then,*

$$C(M; \lambda) \;=\; \sum_{k=0}^{r(E)} (-1)^k\, b_k(M)\, \lambda^{r(E)-k}$$

where $b_k(M)$ is the number of k-subsets of E which do not include a broken circuit of M as a subset. In particular, the coefficients of $C(M; \lambda)$ alternate in sign.

Proof. [Doh00e] It is easy to see (cf. Lemma 1.4 of [Her72]) that, in general,

$$C(M;\lambda) \;=\; 0 \quad \text{if } M \text{ contains a loop.} \tag{6.13}$$

For any $X \subseteq E$ define $k(X) := \bigcup\{C \subseteq X \mid C \text{ is a broken circuit of } M\}$. Then, for any non-empty and k-open subset X of E there is some $e > \max X$ such that $X \cup \{e\}$ includes a circuit of M. Therefore, $r(X \cup \{e\}) = r(X)$, or equivalently, $r_X(\{e\}) = 0$. From this we conclude that e is a loop of M/X and hence by (6.13), $C(M/X;\,\lambda) = 0$. By applying Theorem 6.3.5, using identity (6.12) and utilizing the fact that $r(I) = |I|$ for any I including no broken circuit of M, we obtain

$$C(M;\lambda) \;=\; \sum_{\substack{I\subseteq E\\ I\not\supseteq X\,(\forall X\in\mathcal{X})}} (-1)^{|I|}\lambda^{r(E)-|I|}\,,$$

where $\mathcal{X}$ is the set of broken circuits of M. □

Results similar to Theorem 6.3.5 and Corollary 6.3.7 can also be established for Crapo's β invariant [Cra67], which among other things indicates whether M is connected and whether M is the cycle matroid of a series-parallel network.

Definition 6.3.8 The β *invariant* of a matroid $M = (E, r)$ is defined by

$$\beta(M) \;:=\; (-1)^{r(E)} \sum_{I\subseteq E} (-1)^{|I|}\, r(I)\,. \tag{6.14}$$

A matroid $M = (E, r)$ is called *disconnected* if there is a pair of distinct elements of E that are not jointly contained by a circuit of M.

Our result on the β invariant is given below. Again, an equivalent formulation using union-closed sets instead of kernel operators is left to the reader.

Theorem 6.3.9 [Doh99f] *Let $M = (E, r)$ be a matroid, and let k be a kernel operator on E such that E is not k-open and M/X is disconnected or a loop for any non-empty and k-open subset X of E. Then,*

$$\beta(M) \;=\; (-1)^{r(E)} \sum_{I\,:\,k(I)=\emptyset} (-1)^{|I|}\, r(I)\,.$$

Proof. By Lemma II of Crapo [Cra67], $\beta(M) = 0$ if M is disconnected or a loop. By this and the assumptions, $\beta(M/X) = 0$ for any non-empty, k-open subset X of E. By Theorem 3.2.4 it suffices to prove that $f(X) = 0$ for each such X, where $f(X) := \sum_{I\supseteq X}(-1)^{|I|}r(I)$. Since the assumptions entrain $\overline{X} \neq \emptyset$, we obtain

$$f(X) \;=\; \sum_{I\subseteq\overline{X}}(-1)^{|X\cup I|}r(X\cup I) \;\doteq\; \sum_{I\subseteq\overline{X}}(-1)^{|I|}\,(r_X(I) + r(X)) \;\doteq\; \beta(M/X) \;=\; 0\,,$$

where $\doteq$ means equality up to sign. □

The following result is well known:

Corollary 6.3.10 *Let $M = (E, r)$ be a matroid, where E is endowed with a linear ordering relation. Then,*

$$\beta(M) \;=\; (-1)^{r(E)} \sum_{k=1}^{|E|} (-1)^k \, k \, b_k(M)$$

where again $b_k(M)$ is the number of k-subsets of E including no broken circuit.

Proof. Corollary 6.3.10 follows from Theorem 6.3.9 in nearly the same way as Corollary 6.3.7 follows from Theorem 6.3.5. Alternatively, consider the identity

$$\beta(M) \;=\; (-1)^{r(E)-1} \left.\frac{\partial C(M;\lambda)}{\partial \lambda}\right|_{\lambda=1}$$

for $E \neq \emptyset$ and apply Corollary 6.3.7. For $E = \emptyset$ the statement is obvious. □

6.4 Euler characteristics and Möbius functions

In this last section we establish some results on the Euler characteristic of an abstract simplicial complex and the Möbius function of a partially ordered set. Our first result concerns the Euler characteristic and requires the following definition:

Definition 6.4.1 For any abstract simplicial complex $\mathcal{S}$ and any simplex $X \in \mathcal{S}$ the *link* $\mathcal{S}/X$ is the abstract simplicial complex defined by

$$\mathcal{S}/X \;:=\; \{I \in \mathcal{S} \,|\, I \cap X = \emptyset,\, I \cup X \in \mathcal{S}\}\,.$$

Theorem 6.4.2 *Let k be a kernel operator on the vertex-set of an abstract simplicial complex $\mathcal{S}$ such that $\gamma(\mathcal{S}/X) = 1$ for any k-open simplex $X \in \mathcal{S}$. Then,*

$$\gamma(\mathcal{S}) \;=\; \gamma\left(\{I \in \mathcal{S} \,|\, k(I) = \emptyset\}\right)\,.$$

Proof. Let $\mathcal{S}^+ := \mathcal{S} \cup \{\emptyset\}$, and for any subset X of the vertex-set of $\mathcal{S}$ define

$$f(X) \;:=\; \sum_{Y \supseteq X} g(Y), \quad \text{where} \quad g(Y) \;:=\; \begin{cases} (-1)^{|Y|-1} & \text{if } Y \in \mathcal{S}^+, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $f(X) = 0$ for any non-empty $X \subseteq \mathrm{Vert}(\mathcal{S})$ which is not a simplex of $\mathcal{S}$, and

$$f(X) \;=\; (-1)^{|X|-1} + \sum_{\substack{Y \in \mathcal{S} \\ Y \supset X}} (-1)^{|Y|-1} \;=\; (-1)^{|X|-1} + (-1)^{|X|}\, \gamma(\mathcal{S}/X) \;=\; 0$$

for any k-open simplex $X \in \mathcal{S}$. Hence, by applying Theorem 3.2.4,

$$\sum_{I \in \mathcal{S}^+} (-1)^{|I|-1} \;=\; \sum_{\substack{I \in \mathcal{S}^+ \\ k(I)=\emptyset}} (-1)^{|I|-1}\,.$$

Now, by subtracting the term for $I = \emptyset$ from both sides the result follows. □

By applying Theorem 6.4.2 with $\mathcal{S} = \mathcal{P}^*(V_\omega)$ one obtains yet another proof of Theorem 3.2.5. We omit the details and pose the following conjecture. For the notion of homotopy equivalence we refer to the textbook of Harzheim [Har78].

Conjecture 6.4.3 *Let k be a kernel operator on the vertex-set of an abstract simplicial complex $\mathcal{S}$ such that $\mathcal{S}/X$ is contractible for any k-open simplex $X \in \mathcal{S}$. Then, the complexes $\mathcal{S}$ and $\{I \in \mathcal{S} \,|\, k(I) = \emptyset\}$ are homotopy equivalent.*

The following terminologies are adopted from Rota [Rot64].

Definition 6.4.4 Let $L = [\hat{0}, \hat{1}]$ be a finite lattice. L is called *non-trivial* if $L \setminus \{\hat{0}, \hat{1}\} \neq \emptyset$. An element $a \in L$ is an *atom* of L if $a > \hat{0}$ and no $a' \in L$ satisfies $a > a' > \hat{0}$. The set of atoms of L is denoted by $A(L)$. A *crosscut* of L is an antichain $C \subseteq L \setminus \{\hat{0}, \hat{1}\}$ having a non-empty intersection with any maximal chain from $\hat{0}$ to $\hat{1}$ in L. The *crosscut complex* of L associated with C is defined by

$$\Gamma(L, C) \;:=\; \left\{ I \in \mathcal{P}^*(C) \;\middle|\; \bigwedge I > \hat{0} \text{ or } \bigvee I < \hat{1} \right\}$$

and is easily seen to be an abstract simplicial complex. For any abstract simplicial complex $\mathcal{S}$, we refer to $\tilde{\gamma}(\mathcal{S}) := \gamma(\mathcal{S}) - 1$ as the *reduced Euler characteristic* of $\mathcal{S}$.

Example 6.4.5 $A(L)$ is a crosscut for any non-trivial finite lattice L.

The following proposition, which is stated without proof, is known as *Rota's crosscut theorem* [Rot64]. A proof of Rota's crosscut theorem can also be found in the textbook of Aigner [Aig79]. In the following, we write $\mu(L)$ instead of $\mu_L(\hat{1})$, where μ_L denotes the Möbius function of L (see Definition 5.1.13).

Proposition 6.4.6 [Rot64] *Let $L = [\hat{0}, \hat{1}]$ be a non-trivial finite lattice, and let C be a crosscut of L. Then, $\mu(L)$ equals the reduced Euler characteristic of the crosscut complex of L associated with C, or equivalently,*

$$\mu(L) \;=\; \sum_{\substack{I \in \mathcal{P}^*(C) \\ \bigwedge I = \hat{0},\, \bigvee I = \hat{1}}} (-1)^{|I|}\,. \tag{6.15}$$

Example 6.4.7 Let L be the lattice shown in Figure 6.2. Evidently, $C = \{a, b, c, d, e\}$ is a crosscut of L. The associated crosscut complex $\Gamma(L, C)$ equals

$$\{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\},$$
$$\{b, d\}, \{b, e\}, \{c, d\}, \{c, e\}, \{d, e\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\},$$
$$\{b, c, d\}, \{b, c, e\}, \{b, d, e\}, \{c, d, e\}, \{a, b, c, d\}, \{b, c, d, e\}\}$$

and contains 23 simplices each of which contributes to the reduced Euler characteristic $\tilde{\gamma}(\Gamma(L,C)) = \mu(L)$. Alternatively, we can compute $\mu(L)$ via (6.15), which gives a sum involving only eight terms. Anyway, it turns out that $\mu(L) = 0$.

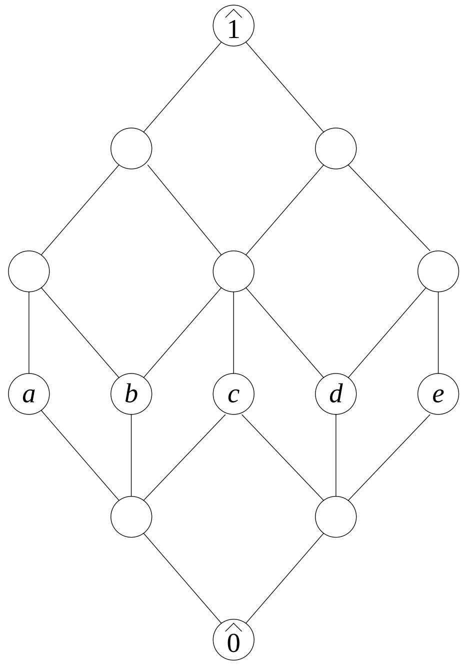

Figure 6.2: A lattice with crosscut $\{a, b, c, d, e\}$.

The following theorem shows that under reasonable assumptions $\mu(L)$ may be considered as the reduced Euler characteristic of a subcomplex of $\Gamma(L,C)$. The theorem may be viewed as an extension of Rota's crosscut theorem [Rot64] and as a generalization of a recent result of Blass and Sagan [BS97].

Theorem 6.4.8 *Let $L = [\hat{0}, \hat{1}]$ be a non-trivial finite lattice, and let C be a crosscut of L, which is endowed with a partial order which is denoted by $\trianglelefteq$ to distinguish it from the partial order $\leq$ in L. Furthermore, let k be a kernel operator on C such that for any non-empty and k-open subset X of C and any $x \in X$ there is some $c \in C$ satisfying $c \triangleleft x$ and $\bigwedge X < c < \bigvee X$. Then,*

$$\mu(L) \;=\; \tilde{\gamma}\left(\{I \in \Gamma(L,C) \,|\, k(I) = \emptyset\}\right) \tag{6.16}$$

or equivalently,

$$\mu(L) \;=\; \sum_{\substack{I \in \mathcal{P}^*(C) \\ \bigwedge I = \hat{0}, \bigvee I = \hat{1} \\ k(I) = \emptyset}} (-1)^{|I|}\,. \tag{6.17}$$

Proof. By Theorem 6.4.2 and Proposition 6.4.6, (6.16) is proved if

$$\gamma\,(\Gamma(L,C)/X) \;=\; 1 \tag{6.18}$$

for any k-open simplex $X \in \Gamma(L,C)$. We first observe that

$$\Gamma(L,C)/X \;=\; \left\{ I \in \mathcal{P}^*(C \setminus X) \;\middle|\; \bigwedge(I \cup X) > \hat{0} \text{ or } \bigvee(I \cup X) < \hat{1} \right\}. \tag{6.19}$$

Now, let c_X be $\trianglelefteq$-minimal in $\{c \in C \mid \bigwedge X < c < \bigvee X\}$. Then, $c_X \notin X$ since otherwise $c \lhd c_X$ and $\bigwedge X < c < \bigvee X$ for some $c \in C$, contradicting the minimality of c_X. Now, by (6.19) and since $\bigwedge X < c_X < \bigvee X$ and $c_X \notin X$, it follows that $Y \mapsto Y \vartriangle \{c_X\}$ is a sign-reversing involution on $\Gamma(L,C)/X \cup \{\emptyset\}$, whence (6.18) and thus (6.16) is shown. To establish (6.17) we first note that

$$\tilde{\gamma}(\{I \in \Gamma(L,C) | k(I) = \emptyset\}) = \tilde{\gamma}(\{I \in \mathcal{P}^*(C) | k(I) = \emptyset\}) + \sum_{\substack{I \in \mathcal{P}^*(C) \\ \bigwedge I = \hat{0}, \bigvee I = \hat{1} \\ k(I) = \emptyset}} (-1)^{|I|}. \tag{6.20}$$

The assumptions of the theorem imply that C is not k-open, whence $\mathcal{S} = \mathcal{P}^*(C)$ satisfies the requirements of Theorem 6.4.2. Thus, by applying Theorem 6.4.2,

$$\tilde{\gamma}(\{I \in \mathcal{P}^*(C) \mid k(I) = \emptyset\}) \;=\; \tilde{\gamma}(\mathcal{P}^*(C)) \;=\; 0\,. \tag{6.21}$$

In view of (6.20) and (6.21) the equivalence of (6.16) and (6.17) is obvious. □

Corollary 6.4.9 *Let $L = [\hat{0}, \hat{1}]$ be a non-trivial finite lattice, and let C be a crosscut of L, which is given a partial order denoted by $\trianglelefteq$ to distinguish it from the partial order $\leq$ in L. Let $\mathcal{X}$ consist of all non-empty subsets X of C such that for any $x \in X$ there is some $c \in C$ satisfying $c \lhd x$ and $\bigwedge X < c < \bigvee X$. Then,*

$$\mu(L) \;=\; \tilde{\gamma}\,(\{I \in \Gamma(L,C) \mid I \not\supseteq X \textit{ for any } X \in \mathcal{X}\}) \tag{6.22}$$

or equivalently,

$$\mu(L) \;=\; \sum_{\substack{I \in \mathcal{P}^*(C) \\ \bigwedge I = \hat{0}, \bigvee I = \hat{1} \\ I \not\supseteq X\,(\forall X \in \mathcal{X})}} (-1)^{|I|}\,. \tag{6.23}$$

Proof. Since $\mathcal{X}$ turns out as union-closed, the corollary follows from Theorem 6.4.8 and the correspondence between kernel operators and union-closed sets. □

Example 6.4.10 Let L and C be as in Example 6.4.7. By putting $d \lhd b$, $d \lhd e$, $b \lhd a$ and $b \lhd c$ a partial ordering relation $\trianglelefteq$ on C is defined, whose Hasse diagram is shown in Figure 6.3. In connection with this partial ordering relation, $\mathcal{X} =$

$\{\{a,c\},\{b,e\},\{c,e\}\}$ satisfies the requirements of Corollary 6.4.9, and thus $\mu(L)$ can be expressed via (6.22) as the reduced Euler characteristic of the subcomplex

$$\{\{a\},\{b\},\{c\},\{d\},\{e\},\{a,b\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{d,e\},\{a,b,d\},\{b,c,d\}\}$$

of $\Gamma(L,C)$, which contains only 13 of the total 23 simplices of $\Gamma(L,C)$. Moreover, for the present choice of $\mathfrak{X}$ it turns out that the sum on the right-hand side of (6.23) contains only two of the eight terms that appear in the sum of (6.15).

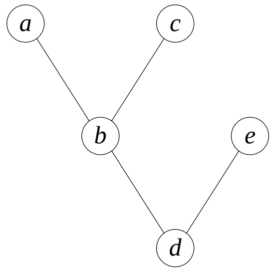

Figure 6.3: Hasse diagram of $\{a,b,c,d,e\}$ with respect to $\trianglelefteq$.

Remarks. In view of Theorem 6.4.8 the corollary can be stated more generally by requiring $\mathfrak{X}$ to be a union-closed set of non-empty subsets of the above type.

By putting $C = A(L)$ the corollary specializes to a result of Blass and Sagan [BS97], which they used in computing and combinatorially explaining the Möbius function of various lattices and in generalizing Stanley's well-known theorem [Sta72] that the characteristic polynomial of a semimodular supersolvable lattice factors over the integers. As pointed out by Blass and Sagan [BS97], their result generalizes a particular case of Rota's broken circuit theorem [Rot64] as well as a prior generalization of that particular case due to Sagan [Sag95]. For a restatement of Rota's result [Rot64], the following definitions are necessary:

Definition 6.4.11 Let $L = [\hat{0},\hat{1}]$ be a finite lattice. A *rank function* of L is a function $r : L \to \mathbb{N} \cup \{0\}$ such that $r(\hat{0}) = 0$ and $r(a) = r(b) + 1$ whenever a is an immediate successor of b. It is well known and straightforward to prove that the rank function is unique if it exists. A *geometric lattice* is a finite lattice L whose rank function exists and satisfies $r(a \wedge b) + r(a \vee b) \leq r(a) + r(b)$ for any $a, b \in L$. A subset I of $A(L)$ is *independent* if $r(\bigvee I) = |I|$, and *dependent* otherwise. Each minimal dependent subset of $A(L)$ is called a *circuit* of L. Given a linear ordering relation on $A(L)$, each set $C \setminus \{\max C\}$, where C is a circuit of L, is referred to as a *broken circuit* of L. Note that this depends on the ordering of the atoms.

Corollary 6.4.12 [Rot64] *Let $L = [\hat{0}, \hat{1}]$ be a non-trivial geometric lattice with rank function r. Furthermore, let the set of atoms $A(L)$ be endowed with a linear ordering relation, and let $\mathcal{B}$ consist of all broken circuits of L. Then,*

$$\mu(L) \;=\; (-1)^{r(\hat{1})} \times \left|\left\{I \subseteq A(L) \,\middle|\, \bigvee I = \hat{1} \text{ and } I \not\supseteq B \text{ for any } B \in \mathcal{B}\right\}\right| .$$

Proof. [BS97] (Sketch) Let $C = A(L)$ and $\mathcal{X}$ as in Corollary 6.4.9. Then, $\mathcal{X} \supseteq \mathcal{B}$ and any $X \in \mathcal{X}$ includes some $B \in \mathcal{B}$. Thus, for any $I \subseteq A(L)$, $I \not\supseteq X$ for any $X \in \mathcal{X}$ if and only if $I \not\supseteq B$ for any $B \in \mathcal{B}$. In this case, I is independent and hence $r(\bigvee I) = |I|$. Thus, Corollary 6.4.12 follows from Corollary 6.4.9. □

So far, the results of this section as well as several of our earlier results rest upon Theorem 3.2.4. It is therefore natural to generalize Theorem 3.2.4 from power sets to partially ordered sets. We are thus led to the following definition:

Definition 6.4.13 Let P be a partially ordered set. A mapping $k : P \to P$ is a *kernel operator* if for any $x, y \in P$,

$$\begin{array}{rl} \text{(i)} & k(x) \leq x \quad \text{(intensionality)}, \\ \text{(ii)} & x \leq y \;\Rightarrow\; k(x) \leq k(y) \quad \text{(monotonicity)}, \\ \text{(iii)} & k(k(x)) = k(x) \quad \text{(idempotence)}. \end{array}$$

Dually, a mapping $c : P \to P$ is a *closure operator* if for any $x, y \in P$,

$$\begin{array}{rl} \text{(i)} & x \leq c(x) \quad \text{(extensionality)}, \\ \text{(ii)} & x \leq y \;\Rightarrow\; c(x) \leq c(y) \quad \text{(monotonicity)}, \\ \text{(iii)} & c(c(x)) = c(x) \quad \text{(idempotence)}. \end{array}$$

An element $x \in P$ is called *k-open* if $k(x) = x$ and *c-closed* if $c(x) = x$.

Kernel and closure operators for partially ordered sets were introduced by Ward [War42] and extensively studied by Rota [Rot64]; see also Aigner [Aig79].

The following result generalizes Theorem 3.2.4 to partially ordered sets. Consequently, it also generalizes Whitney's broken circuit theorem [Whi32] as well as all results obtained in Section 3.2, Section 6.3 and the present section. As we will see below, it additionally generalizes one of the most important classical results of enumerative combinatorics, which is known as *Weisner's theorem* [Wei35].

Theorem 6.4.14 [Doh99f] *Let P be an upper-finite partially ordered set, and let f and g be mappings from P into an abelian group such that $f(x) = \sum_{y \geq x} g(y)$ for any $x \in P$. Furthermore, let $k : P \to P$ be a kernel operator, and let x_0 be a k-open element of P such that $f(x) = 0$ for any k-open $x > x_0$. Then,*

$$f(x_0) \;=\; \sum_{y \,:\, k(y) = x_0} g(y)\,.$$

The proof given below is due to an anonymous referee. For the author's original proof, which closely follows the proof of Theorem 3.2.4, see [Doh99f].

Proof. It suffices to show that $\sum_{y\,:\,k(y)>x_0} g(y) = 0$. Evidently, if x_0 is maximal in P, then this sum is empty, and hence the statement trivially holds. We proceed by downward induction on x_0. In this way, we obtain

$$\sum_{y\,:\,k(y)>x_0} g(y) \;=\; \sum_{\substack{x>x_0 \\ x\ k\text{-open}}} \;\sum_{y\,:\,k(y)=x} g(y) \;=\; \sum_{\substack{x>x_0 \\ x\ k\text{-open}}} f(x) \;=\; 0\,,$$

where the second equality comes from the induction hypothesis and the third from the hypothesis of the theorem. □

Dualizing Theorem 6.4.14 we obtain

Theorem 6.4.15 [Doh99f] *Let P be a lower-finite partially ordered set, and let f and g be mappings from P into an abelian group such that $f(x) = \sum_{y\leq x} g(y)$ for any $x \in P$. Furthermore, let $c : P \to P$ be a closure operator and x_0 a c-closed element of P such that $f(x) = 0$ for any c-closed $x < x_0$. Then,*

$$f(x_0) \;=\; \sum_{y\,:\,c(y)=x_0} g(y)\,.$$

From the preceding theorem we now deduce a prominent result of Rota [Rot64] on the Möbius function of a lower-finite partially ordered set, which in turn specializes to Weisner's theorem [Wei35]. The proof is adopted from [Doh99f].

Corollary 6.4.16 [Rot64] *Let P be a lower-finite partially ordered set with least element $\hat{0}$ and $c : P \to P$ a closure operator where $c(\hat{0}) > \hat{0}$. Then, for all $x_0 \in P$,*

$$\sum_{y\,:\,c(y)=x_0} \mu_P(y) \;=\; 0\,.$$

Proof. For any $x \in P$ define $f(x) := \sum_{y\leq x} \mu_P(y)$. There is nothing to prove if x_0 is not c-closed. Otherwise, $x_0 \neq \hat{0}$ and hence by (5.3), $f(x_0) = 0$. Likewise, one finds that $f(x) = 0$ for any c-closed $x < x_0$. Now apply Theorem 6.4.15. □

Corollary 6.4.17 [Wei35] *Let P be a lower-finite partially ordered set with least element $\hat{0}$. Then, for any $a > \hat{0}$ and all $x_0 \in P$,*

$$\sum_{y\,:\,y\vee a=x_0} \mu_P(y) \;=\; 0\,.$$

Proof. Define $c(y) := y \vee a$ for any $y \in P$ and apply Corollary 6.4.16. □

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