Testing Monotonicity of Pricing Kernels

Master Thesis Presented

by

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Declaration of Authorship

I hereby confirm that I have authored this master thesis independently and without use of others than the indicated resources. All passages, which are literally or in general matter taken out of publications or other resources, are marked as such.

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Berlin, 13th September 2007

Abstract

In this master thesis a mechanism to test mononicity of empirical pricing kernels (EPK) is presented. By testing monotonicity of pricing kernel we can determine whether utility function is concave or not. Strictly decreasing pricing kernel corresponds to concave utility function while non-decreasing EPK means that utility function contains some non-concave regions. Risk averse behavior is usually described by concave utility function and considered to be a cornerstone of classical behavioral finance. Agents prefer a fixed profit over insecure choice with the same expected value. Some of the EPKs, obtained from DAX German market, were found to be non-monotone decreasing. These findings show that agents have not always risk averse behavior.

The first part of the thesis describes construction of the test. Pyke's theorem of order statistics is used to reduce the problem to exponential model. On the basis of this model likelihood ratio test is constructed for a fixed interval. Furthemore it is expanded to a test independent from intervals using intersection of test for different intervals. In the second part test performance is evaluated for simulated and observed data. Different cases of data are simulated to estimate power of the test, first and second type errors. Then EPKs, obtained from DAX data in years 2000, 2002 and 2004 are tested for monotonicity.

Keywords: Risk Aversion, Pricing Kernel, Monotonicity

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Chapter 1

Introduction

Behavior of market agents has always been in the focus in economic literature. In the framework von Neumann and Morgenstern (1944) averse behavior is described by concave utility function. Agents prefer a fixed profit over insecure choice with the same expected value. But lately there have been a lot of discussions about the eligibility of this approach. Recent empirical studies by Jackwerth, J. C. (2002) and others showed that overall behavior of agents is risk averse, but there is a reference point near the inital wealth where market utility functions are convex. Detlefsen, et al (2007) raised the same question by recovering utility function through empiricial pricing kernels for different time periods. They observed a bump in empirical pricing kernels which correspond to non-concave utility functions.

On left figure 1.1 there is classical concave utility function obtained from Black Scholes model. On the right side of the figure utility function obtained from empirical pricing kernel on 30th July 2000 is presented. This comparison shows that utility function obtained from pricing kernel is not strictly concave. In this master thesis we construct a test that can verify if pricing kernel is monotone decreasing. Strictly decreasing EPK corresponds to concave utility function and is consistent with classical theory of risk averse behavior, while rejection of monotone decreasing EPK would mean non-concave utility function and as a result non-averse patterns on one or more intervals of the utility function.

The test provides a mechanism to check monotonicity of a function not only as a whole but also indicates on which interval or intervals monotonicity of EPK was rejected. This setup is consistent with the main goal to test with a certain significance level if there is a bump in a pricing kernel around a reference point. Cox, et al (1985) showed that the empirical pricing kernel is defined as a ratio of risk neutral density q and historical (subjective) density p. Risk neutral density is derived from stochastic volatility model which

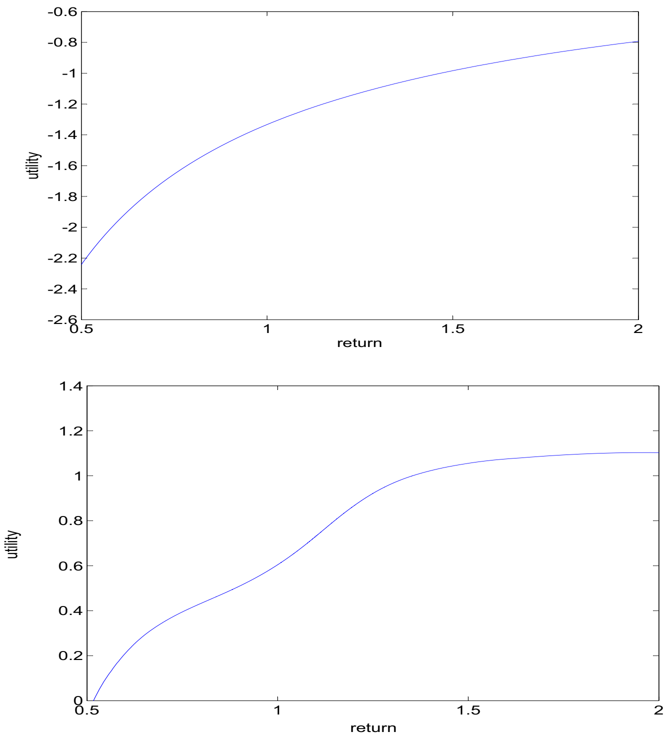

Figure 1.1: Classical utility function produced from Black Scholes model (upper) and market utility function estimated from empirical pricing kernel on 06/30/2000 (lower)

is widely used in industry, see Heston, S. (1993). Due to large number of observations in derivative option market, q can be precisely estimated and considered to be known. Estimation of historical density p is complicated by model specification and data scarcity and therefore is considered to be undefined. Thus we would like to test monotonicity of pricing kernel constructed as a ratio of estimated q and unknown p.

The test is constructed as follows: first the spacing method is used to reduce the problem to exponential model. On the basis of this model likelihood ratio test is applied for a fixed interval. Using test intersection method it is expanded to the test that does not depend on intervals. Finally, test statistics, calculated on observed data, is compared to simulated critical values and final decision about monotonicity is taken.

The thesis is organized as follows. In section 2 we introduce important notations and problem setup and the problem is reduced to exponential model using spacing method. In section 3 we formulate the hypotheses, construct likelihood test for a fixed interval $[I, J]$ and then expand it to independent test using multiple testing technique. We also describe how to simulate critical values using Monte-Carlo method. Section 4 contains performance of the test for simulated data and section 5 - results on DAX data in 2000, 2002 and 2004.

Chapter 2

Conception of the Test

2.1 Problem Setup

Suppose we have at our disposal an *i.i.d.* sample $X_1, \dots, X_n$ with an unknown historical probability density $p(x),\ x \in \mathbb{R}$. Suppose also that we are given a risk neutral probability density $q(x),\ x \in \mathbb{R}$ which is assumed to be known. Let

$$K(x) = \frac{q(x)}{p(x)}$$

We want to check the monotonicity of $K(x),\ x \in \mathbb{R}$. More precisely we would like to check if there exists an interval $[a, b]$, where $K(x)$ is not monotone decreasing. Denote by $X_{(1)}, \dots X_{(n)}$ the order statistics related to $X_1, \dots, X_n$ i.e.

$$X_{(1)} \leq X_{(2)}, \dots, \leq X_{(n)}$$

With these notations we can rephrase our problem as follows: find (if possible) integers I, J such that the sequence

$$K_k = K(X_{(k)}) = \frac{q(X_{(k)})}{p(X_{(k)})}, \quad I \leq k \leq J$$

is not monotone decreasing. The principal difficulty in this problem is related to the fact that p is unknown. To overcome this difficulty we will use three basic ingredients:

- spacing method to reduce the problem to a simple exponential model
- maximum likelihood test to test monotonicity of K_k for given I and J
- multiple-testing procedure to find I and J on the basis of the data at hand.

2.2 The Spacing Method

Our method is based on the Pyke's theorem about the distribution of order statistics, see Pyke, R. (1965). Consider $U_1, \ldots, U_n$ be *i.i.d* with a uniform distribution on $[0,1]$. For the order statistics

$$U_{(1)} \leq U_{(2)}, \ldots, \leq U_{(n)}$$

define *uniform spacings* S_k as

$$S_k = U_{(k+1)} - U_{(k)} \text{ and } S_n = U_{(n)}$$

Theorem 2.2.1. *Let $U_1, \ldots, U_n$ be i.i.d. uniformly distributed on $[0,1]$ and $e_1, \ldots, e_n$ be i.i.d. standard exponentially distributed random variables. Then*

$$\mathcal{L}\left\{S_k, 1 \leq k \leq n\right\} = \mathcal{L}\left\{\frac{e_k}{\sum_{i=1}^{n} e_k}, 1 \leq k \leq n\right\}$$

Using the fact that $E[e_k] = 1$ we obtain the following result:

$$n\left\{U_{(k+1)} - U_{(k)}\right\} = n \cdot S_k \approx e_k. \tag{2.1}$$

Let $P(x) = \int_{-\infty}^{x} p(u)\,du$ be the probability distribution function associated with $p(x)$. Using $U_{(k)} = P(X_{(k)})$ and first order Taylor approximation

$$P(X_{(k+1)}) = P(X_{(k)}) + P'(X_{(k)}) \cdot (X_{(k+1)} - X_{(k)})$$

we derive

$$U_{(k+1)} - U_{(k)} = P(X_{(k+1)}) - P(X_{(k)}) \approx p(X_{(k)}) \cdot \left(X_{(k+1)} - X_{(k)}\right) \tag{2.2}$$

Combining equations (2.2) with (2.1) we obtain

$$n\left\{X_{(k+1)} - X_{(k)}\right\} q(X_{(k)}) \approx \frac{q(X_{(k)})}{p(X_{(k)})} e_k = K_k \cdot e_k.$$

Thus our problem is reduced to the following one: **check monotonicity of** $K(X_{(k)}) = K_k$ **using**

$$Z_k = K_k \cdot e_k, \quad I \leq k \leq J \tag{2.3}$$

Chapter 3

Construction of the Test

3.1 ML test for given I, J

Let $A(I, J)$ be the set of all possible decreasing sequences on a given interval $[I, J]$:

$$A(I, J) = \Big\{ a_k \geq a_{k+1},\ I \leq k < J \Big\}$$

Let us defing the following hypotheses:
Hypothesis H_0**:** $K \subset A(I, J)$ and pricing kernel K is a monotone decreasing function
Hypothesis H_1**:** K is any kind of function.
A nested model of monotone decreasing function under H_0 is compared to a general class of all possible functions under H_1 by calculating maximum of likelihood function for each of the models. If function K is non-monotone in accordance with H_1, maximum likelihood of two models should significantly differ from each other. On the other hand when we remove restriction on monotonicity and it does not bring significant improvement in likelihood, restricted model H_0 should be accepted.
The likelihood ratio monotonicity test is defined by the function

$$\phi(Z) = \mathbf{1} \left\{ \frac{\max_{K \subset A(I,J)} \{p(Z, K)\}}{\max_K \{p(Z, K)\}} - H_\alpha(I, J) \geq 0 \right\}$$

In other words, if $\phi(Z) = 1$ we accept the null hypothesis $H_0 :\ K \in A(I, J)$, otherwise the alternative is accepted. This setup can be simplified with the following monotone transformation:

$$\phi(Z) = \mathbf{1} \left\{ \log \frac{\max_{K \subset A(I,J)} \{p(Z, K)\}}{\max_K \{p(Z, K)\}} - h_\alpha(I, J) \geq 0 \right\}$$

For a given probability of the first kind error α, the critical value $h_\alpha(I,J) = \log H_\alpha(I,J)$ is defined as root of the equation:

$$\mathbf{P}_0\left\{\log\frac{\max_{K\in A(I,J)} p(Z,K)}{\max_K p(Z,K)} - h_\alpha(I,J) \le 0\right\} = \alpha,$$

where $\mathbf{P}_0$ is the probability measure generated by the observations from 2.3 with $K_k \equiv 1,\ I \le k < J$.
Computation of $\max_K \log\{p(Z,K)\}$ is straightforward. Using the results from equation 2.3 that $Z_k = K_k \cdot e_k$ we derive log-likelihood function

$$\log\{p(Z,K)\} = -\sum_{k=I}^{J}\frac{Z_k}{K_k} - \sum_{k=I}^{J}\log(K_k) \qquad (3.1)$$

which gives us analytical result for $\max_K \log\{p(Z,K)\}$ at $K_k = Z_k$:

$$\max_K \log\{p(Z,K)\} = -(J-I) - \textstyle\sum_{k=I}^{J}\log(Z_k)$$

Computation of $\max_{K\subset A(I,J)} \log\{p(Z,K)\}$ is performed with **Newton-Raphson method with the projection on decreasing sequence** $A(I,J)$. The main idea of this approach is to find the maximum likelihood over all possible monotone decreasing sequences by interative optimization via the Newton Raphson algorithm. The result of decreasing sequences that maximizes log-likelihood function is achieved through isotonic regression combined with Newton-Raphson opimization algorithm.
Isotonic regression performs the least square estimation subject to monotonicity contraint with strictly decreasing trend. For a given vectors x, y of size n the following minimization problem is fulfilled:

$$\min_{f_{iso}}\sum_{i=1}^{n}\{y_i - f_{iso}(x_i)\}^2 \text{ s.t. } f_{iso}(x_i) \le f_{iso}(x_j) \text{ where } i > j$$

where f_{iso} is isotonic regression. In practice isotonic regression represents a downward stepwise function, see figure 3.1.1. This procedure is unfortunately very time-consuming. It can be also shown that $\max_{K\subset A(I,J)} \log\{p(Z,K)\}$ is obtained at isotonic regression over Z_k parameters since Z_k gives us $\max_K \log\{p(Z,K)\}$. Thus Newton-Raphson algorithm can be omitted, instead isotonic regression $f_{iso}(Z_k)$ is applied to known Z_k.

$$\max_{K\subset A(I,J)} \log\{p(Z,K)\} = -\textstyle\sum_{k=I}^{J}\frac{Z_k}{f_{iso}(Z_k)} - \sum_{k=I}^{J}\log(f_{iso}(Z_k))$$

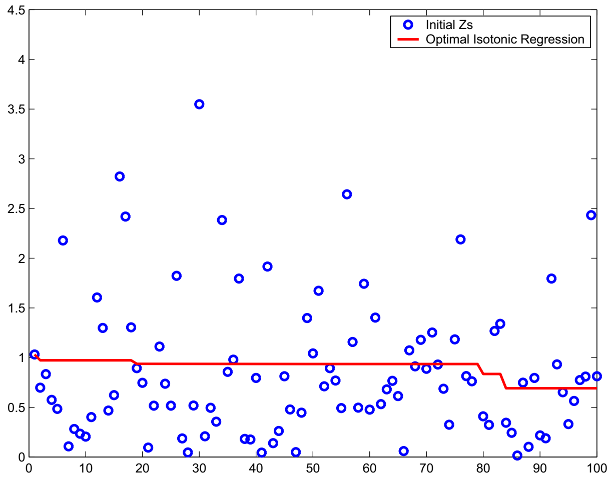

Figure 3.1.1: Isotonic Regression over Z_k generated as *iid* standard exponential

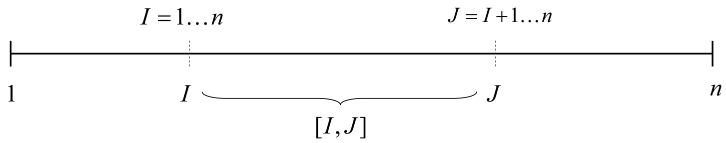

Figure 3.2.2: Multiple testing on intervals I, J

3.2 Multiple-testing

The principal idea in the multiple testing is to construct a test that does not depend on I and J. This problem is typically solved with the help of tests intersection, see Berger (1982). The hypothesis H_0 of monotone decreasing function is rejected if it is rejected at least on one of the interval $[I, J]$, see figure 3.2. It means that we are looking for a minimal critical surface $h(I, J)$ such that:

$$\mathbf{P}_0\left\{\min_{I,J}\left\{\log\frac{\max_{K\in A(I,J)} p(Z,r)}{\max_K p(Z,K)} - h_\alpha(I,J)\right\} \leq 0\right\} = \alpha.$$

Unfortunately the exact solution of this problem is extremely difficult and unknown. Therefore we use the Monte-Carlo simulations to find a reasonable critical surface. We generate "the worst"non-increasing case of the sequence $K_{(k)}$ as a constant:

$$K_{(1)} = K_{(2)} = \ldots = K_{(n)} = 1$$

Then using the result that $Z_k = K_k \cdot e_k$ we generate $Z_k \approx \exp(1)$ as an *iid* standard exponential random variable.
Let us define $\xi(I, J)$ as a test statistics over simulated Z_k:

$$\xi(I, J) = \log \frac{\max_{K \in A(I,J)} p(Z, K)}{\max_K p(Z, K)} = \max_{K \in A(I,J)} \log \{p(Z, K)\} - \max_K \log \{p(Z, K)\} \tag{3.2}$$

Here ξ is a matrix of dimensions I, J with non-positive values. Maximum of value 0 is reached at any monotone decreasing interval I, J.
Define mean $M(I, J)$ and variance $V^2(I, J)$ of test statistics $\xi(I, J)$:

$$M(I, J) = E_0 \xi(I, J)$$
$$V^2(I, J) = E_0 \left\{ \xi^2(I, J) - E_0 \xi(I, J) \right\}^2$$

Parameters $M(I, J)$ and $V(I, J)$ are calculated by Monte-Carlo simulations of Z_k as specified above.
Critical value t_α, where α is a significance level, is calculated as a root of:

$$P_0 \left\{ \min_{I,J} \{\xi(I, J) - M(I, J) + t_\alpha V(I, J)\} \leq 0 \right\} = \alpha \tag{3.3}$$

Equation 3.3 gives us a corresponding critical surface $h_\alpha(I, J)$

$$h_\alpha(I, J) = M(I, J) - t_\alpha \cdot V(I, J)$$

In figure 3.2 the calculation algorithm of critical values t_α is displayed. Over all Monte-Carlo simulations of Z_k should violate α-threshold surfaces $M(I, J) - t_\alpha \cdot V(I, J)$ in α percent cases.

3.3 Multiple testing on blocks

Suppose initial set of Z_k can be divided in m blocks of size b and the remainder $n - b \cdot m$, see figure 3.3.
The idea to introduce blocks is motivated by the variance reduction. Initially we imply that the alternative hypothesis H_1 is a set of all possible functions. By introducing blocks we allow the function to be monotone decreasing on interval of size b and thus we decrease the variance of the distribution. Blocks can be considered as a trade off between the variance reduction and shift parameter. For small size block distribution is shifted, but variance is also big. For large blocks the distribution function is less shifted but at the same time associated with smaller variance.

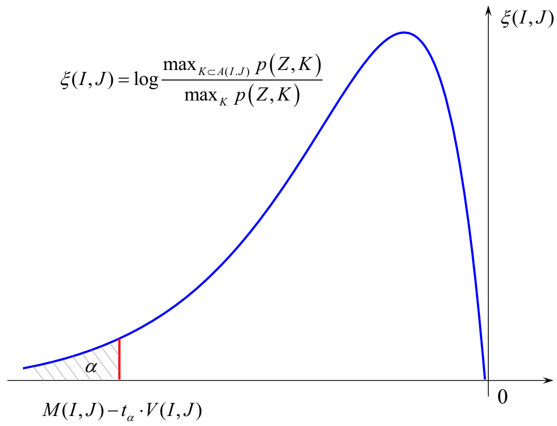

Figure 3.2.3: Calculation of critical value t_α

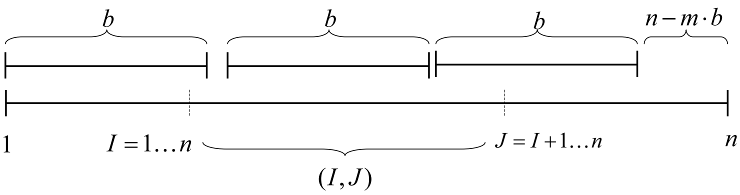

Figure 3.3.4: Multiple testing on blocks

On the upper figure 3.3.5 distribution functions of test statistics without block (blue) and after introduction of block (red) are depicted. First data are generated as linear trend with slope b, constant a and iid exponential errors e_i as $x_i = (a + b\ i) \cdot e_i$. Test statistics is obtained from equation 3.2 then ordered. Shift of distribution function is caused by increase of linear slope b from 0 trend to 0.05. This idea is an underlying principle of the test, non-monotone data shifts the ditribution to the left that should be determined by the test. Lower figure shows the influence of block parameter on variance and shift of cdfs. At best we would like to maximize the shift and minimize the variance, with an increase of block size m both shift and variance of cdf are smaller. The idea of blocks is to test monotonicity not only on each interval I, J but also for all possible block sizes b.
Test statistics $\xi(I, J, b)$ is obtained as a difference between $\max_{K \in A(I,J)} \log \{p(Z, K)\}$ and $\max_K \log \{p(Z, K)\}$ but this time we assume that under H_1 function is monotone decreasing on each of m blocks of size b. Instead of taking each

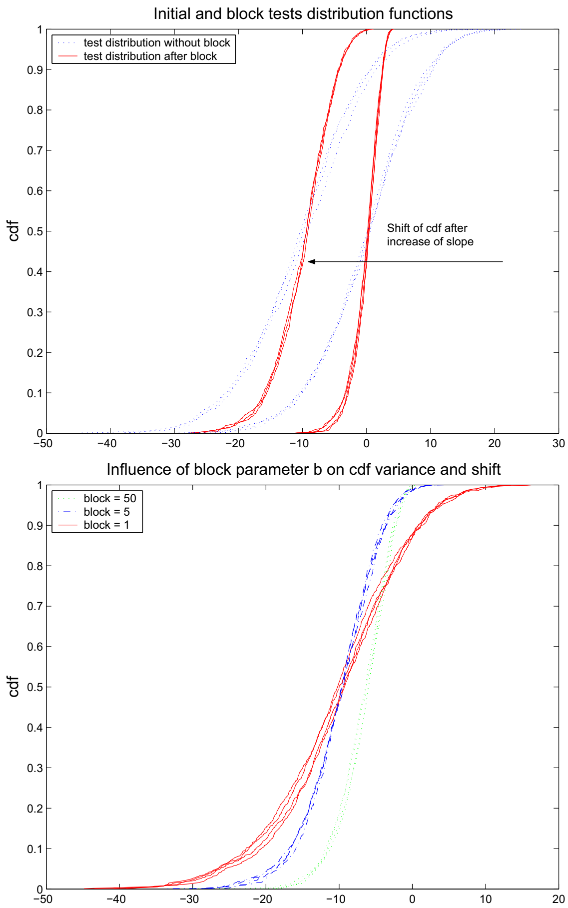

Figure 3.3.5: Multiple testing on blocks

value, average for each block is taken:

$$\max_K \log\{p(Z,K)\} = -m - \sum_{j=1}^{m} \log\left(\frac{\sum_{k=j\cdot b-b+1}^{j\cdot b} Z_k}{b}\right)$$

The same procedure is performed for calculation of $\max_{K\in A(I,J)} \log\{p(Z,K)\}$ but instead of Z_k best monotone decreasing approximation is taken as an isotnic regression $f_{iso}(Z_k)$.
Finally we can formulate hypotheses: H_0 hypothesis about monotonic decreasing function is rejected when monotonicity is rejected at least on one of the intervals I, J with any block size b:

$$\min_{(I,J,b)} \{\xi(I,J,b) - M(I,J,b) + t_{\alpha,b}\cdot V(I,J,b)\} \le 0 \tag{3.4}$$

Critical value $t_{\alpha,b}$ is different for each value of b and obtained from the equation

$$P_0 \left\{ \min_{I,J,b} \{\xi(I,J,b) - M(I,J,b) + t_{\alpha,b} \cdot V(I,J,b)\} \geq 0 \right\} = \alpha \tag{3.5}$$

Now we are ready to summarize the monotonicity test:

1. Compute $Z(X_{(k)}) = n \cdot q(X_{(k)}) \cdot \left\{ X_{(k+1)} - X_{(k)} \right\}$

2. Compute test statistics

$$\xi(I,J,b) = \log \frac{\max_{K \in A(I,J)} p(Z,K)}{\max_K p(Z,K)} =$$
$$\max_{K \in A(I,J)} \log \{p(Z,K)\} - \max_K \log \{p(Z,K)\}$$

3. Take decision: if

$$\min_{I,J,b} \{\xi(I,J,b) - M(I,J,b) + t_{\alpha,b} \cdot V(I,J,b)\} \leq 0$$

then $K(\cdot)$ is a non-monotone decreasing function

Chapter 4

Simulations and Applications

4.1 Simulated Data

In this section the performance of monotonicity test for artifically simulated data is evaluated. We investigate the behavior of the test for different cases: monotone decreasing data, positive linear trend and sudden jumps. Simulated data are generated in accordance with one of the cases multiplied by standard exponential errors e_i. By simulating different errors we can obtain distribution function and then, basing on true function, calculate error probability and evaluate the power of the test.
Before we apply the test to simulated and observed data, important parameters have to be set. The decision about monotonicity is taken basing on sequence of surfaces $\xi(I, J, b) - M(I, J, b) + t_{\alpha,b} \cdot V(I, J, b)$, one surface for each block size b. If at least one surface crosses zero level H_0 hypothesis about monotone decreasing funcion is rejected. If surface is located under zero level it means that calculated test statistics is to the left of threshold value $M(I, J, b) - t_{\alpha,b} \cdot V(I, J, b)$, see figure 3.2. First we set the minimum interval of 10 observations between J and I. This parameter is introduced to approximate test statistics ξ with Gaussian distribution and improve the correlation betweeb statistics $\xi(I_1, J_1)$ and $\xi(I_2, J_2)$. Gaussian approximation is possible due to central limit theorem, the bigger the interval is, the better approximation. Obviously if the approximation is good, critical values t_α should be close to Gaussian critical values. Final goal of this parameter is to improve the power of the test.
The importance of b parameter has been discussed in section 3.3. Large b reduces variance but at the same time decreases shift of the distribution. We start with value $b = 1$ which corresponds to no block until $b = 0.5 \cdot n$ which means the dataset is divided into exactly two blocks. Values more

than 50% of observations would correspond to only one block and remainder and therefore do not make sense.
Calculation of critical values is described in section 3.2. This procedure is very time consuming that is why we use dichotomic method in order to find the root of equation 3.3. This is a method of iterative splitting of intervals into halfs until required precision of solution is found.
First we generate a monotone decreasing sequence and check the performance of the test on this dataset. The “worst”monotone sequence is a constant therefore we simulate $x_1 = x_2 = \cdots = x_i = 1$. On the upper figure 4.1.1 generated sequence x, Z and corresponding isotonic regression over Z_i are displayed. Having fixed $b = 3$ we calculated critical values $t_{\alpha,3}$ from equation 3.3 and corresponding testing surface $M(I,J,3) - t_{\alpha,3} \cdot V(I,J,3) - \xi(I,J,3)$ which are depicted on the lower figure.

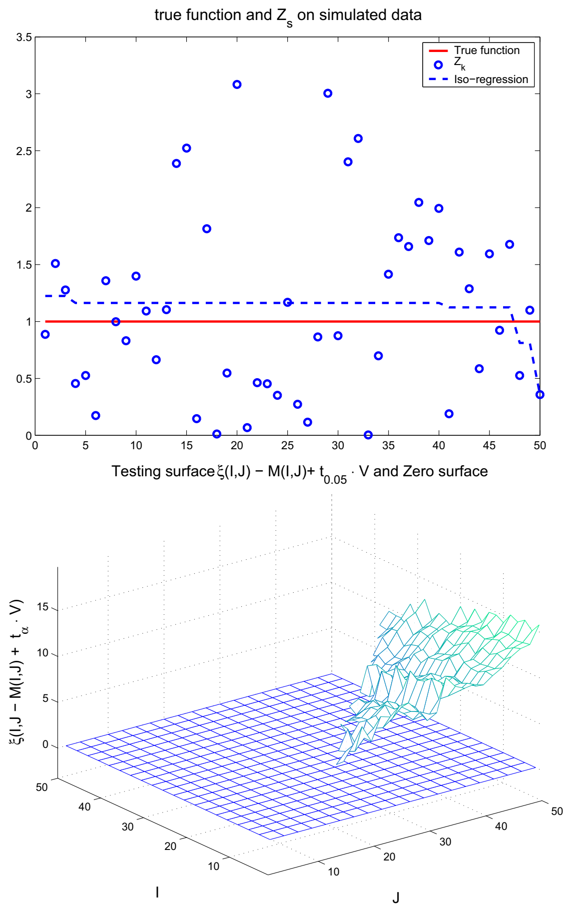

Figure 4.1.1: Simulated monotone data and resulting testing surface, $b = 3$

The entire surface is located above zero level and therefore H_0 hypothesis of monotone decreasing function can not be rejected at 5% sifnificance level. The depicted above surface is a single result of generated errors e_i and fixed parameter b and therefore can not reflect overall performance of the test. In order to demonstrate overall behavior of the test we estimate error probability by generating different errors e_i. In figure 4.1.2 distribution of first type error for different parameter b is plotted, i.e. probability to accept H_1 although data are distributed under H_0. As it can be seen b parameters does not improve the first type error.

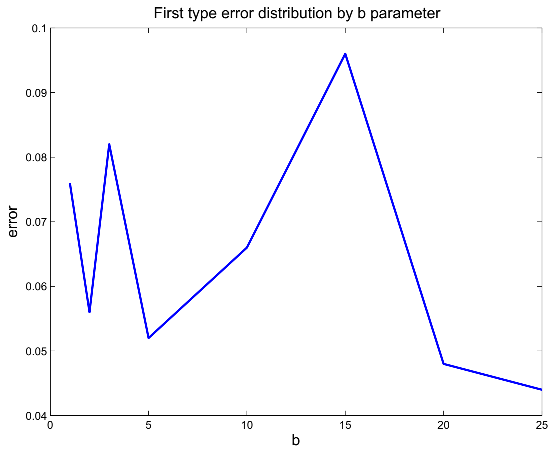

Figure 4.1.2: First type error distribution for different block parameter b

In the next case data are generated with a positive linear trend $x_i = (a+0.05 \cdot i) \cdot e$, where a is a constant and i is an index from 1 to n. Simulated parameters $M(I, J, b)$ and $V(I, J, b)$ do no depend on data but only on parameters b and number of observations n and therefore can be taken from previously simulated example. For fixed $b = 3$ generated data, rejection intervals and resulting surface $\xi(I, J, 3) - M(I, J, 3) + t_{\alpha,3} \cdot V(I, J, 3)$ are given in figures 4.1.3. Rejection invervals show such I and J where testing surface crossed the zero level and H_0 was rejected.

In order to calculate the error probability we calculate number of cases when test failed to identify non-monone structure of the data, i.e. second type error. On figure 4.1.4 there is a distribution of second type erros for different b, starting from no block ($b = 1$) to exactly two intervals $b = 0.5 \cdot n = 25$. This figure shows that introduction of block significantly improves the performance of the test: error probability decreases from 75% for no block to almost 10% for $b = 15$.

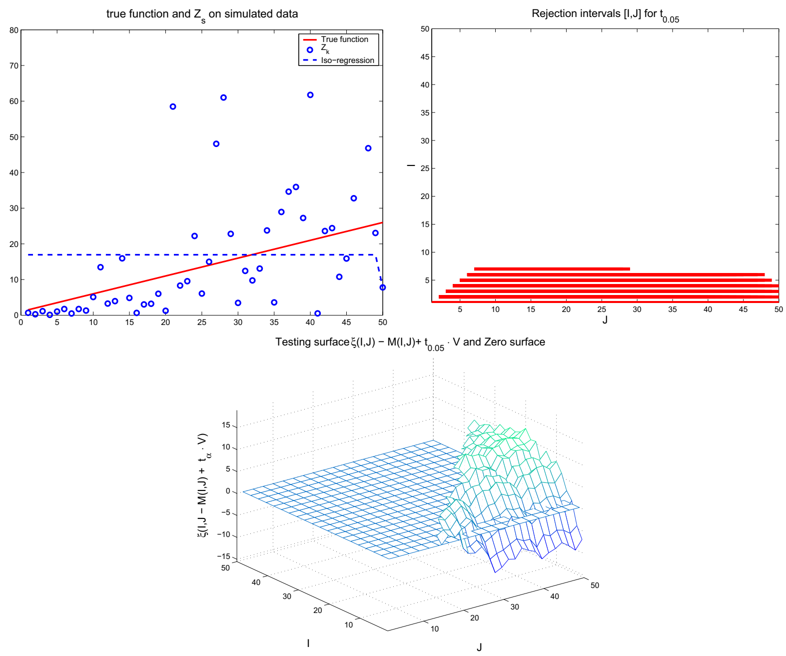

Figure 4.1.3: Simulated increasing data and resulting rejection intervals and testing surface

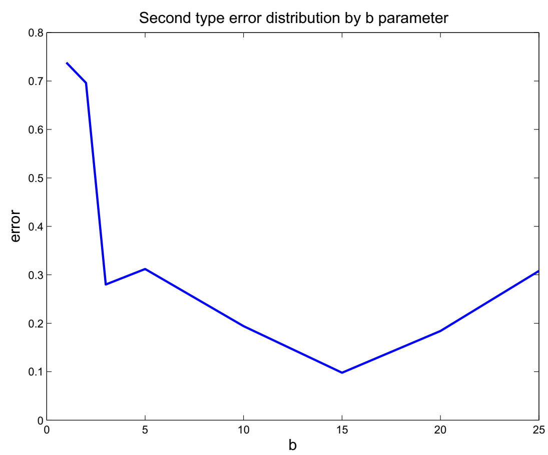

Figure 4.1.4: Second type error distribution for different block parameter b

In next example we simulate an artificial bump, see left figure 4.1.5. Ability of the test to identify jumps or bumps in pricing kernel function is especially important since observed data do not usually have an obvious positive trend.

Instead EPK has various fluctuations, bumps and jumps. Significant bump would correspond to non-concave utility function and contradict to classical theory about risk-averse agents. On middle and right figures 4.1.5 testing surface $\xi(I,J) - M(I,J) + t_\alpha \cdot V(I,J)$ and rejection invervals I, J are given for fixed block size $b = 3$.

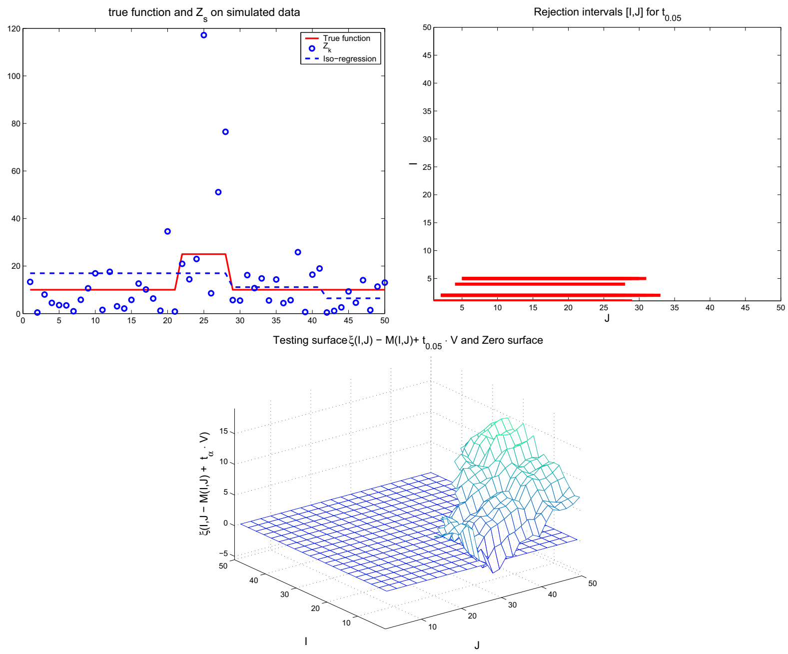

Figure 4.1.5: Simulated data with a bump and resulting rejection intervals and testing surface

Distribution of second type erros for different b is given on figure 4.1.6. We can see that there exists an optimal block size b which corresponds to a trade off between shift and variance of distribution, see section 3.3. Optimal b is different for each dataset and therefore we consider a sequence of surfaces $\xi(I,J,b) - M(I,J,b) + t_{\alpha,b} \cdot V(I,J,b)$ for each block size b. H_0 hypothesis of monotonic decreasing function is rejected when at least one of these surfaces crosses zero level.

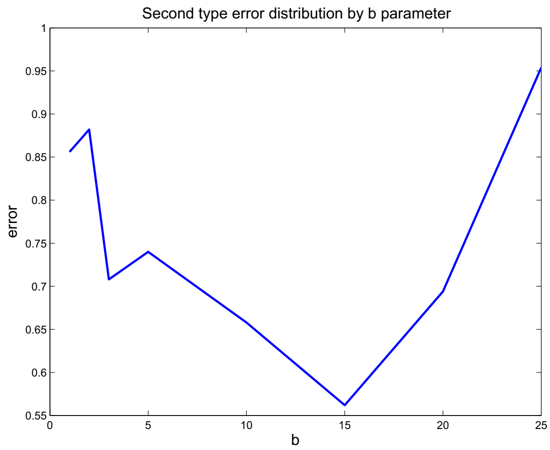

Figure 4.1.6: Second type error distribution for different block parameter b

4.2 Monotonicity of DAX Empirical Pricing Kernel

Final goal of this work is to test empirical pricing kernel obtained from observed data. For the analysis we take data used in Detlefsen, et al (2007) where the pricing kernels and the risk aversion are analyzed in summer 2000, summer 2002 and summer 2004 in order to consider different market regimes. According to our test design the decision about monotonicity of pricing kernel is made on the basis of generated $Z_k = n \cdot (X_{(k+1)} - X_{(k)}) \cdot q(X_{(k)})$ where X are DAX returns and q is risk neutral density. DAX returns are calculated on half year basis $X_i = \frac{X_i - X_{i-126}}{X_{i-126}}$ and then ordered to $X_{(k)}$. Corresponding ordered returns differences $X_{(k+1)} - X_{(k)}$ for years 2000, 2002 and 2004 are displayed in figure 4.2.1.
Risk neutral density q (see figure 4.2.2) is estimated using Heston model (1993) calibrated on observed implied volatility surfaces with half year maturity. Fore more details on estimation of risk neutral density refer to Detlefsen, et al (2007).
Resulting Z_k values are displayed in figure 4.2.3. For each set of Z_k an isotonic regression was constructed which represents $\max_{K \subset A(I,J)} \log\{p(Z,K)\}$ in equation 3.2. Numerous simulations showed that in order to compute maximum likelihood for restricted model $\max_{K \subset A(I,J)} \log\{p(Z,K)\}$ we have to take isotonic regression over optimal parameters which maximize $\log\{p(Z,K)\}$ for all possible K. $\max_K \log\{p(Z,K)\}$ is reached at $K_k = Z_k$ and equal to $-n - \sum_{k=1}^{n} \log(Z_k)$, so isotonic regression over observed Z_k maximizes

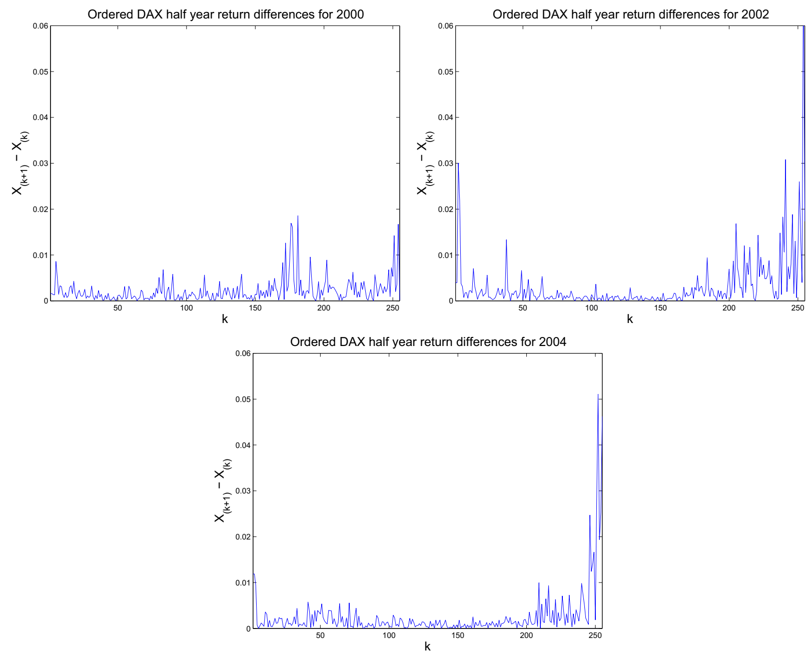

Figure 4.2.1: Half year ordered returns differences $X_{(k+1)} - X_{(k)}$ for years 2000, 2002 and 2004

$\max_{K \subset A(I,J)} \log \{p(Z, K)\}$.
In order to take a final decision about the motonocity sequence of surfaces $M(I, I, b)$ and $V(I, J, b)$ has to be computed. M and V^2 are mean and variance parameters of test statistics ξ obtained via Monte Carlo simulations of Z_k as *iid* standard exponential random variable. Each value of the matrixes M and V represent correspondingly mean and standard error of ξ for a fixed parameter b and interval I, J and calculated as $\max_{K \subset A(I,J)} \log \{p(Z(I, J), K)\} - \max_K \log \{p(Z(I, J), K)\}$. Matrix M has non-positive values with maximum at 0, V is non-negative. Both matrixes exist only for $J > I$, see section 3.2 for details. Since surfaces M and V do not depend on observed data but only on the number of observations n and block size b they are computed once for all years. In figure 4.2.4 corresponding surfaces $M(I, J)$ and $V(I, J)$ are plotted for $b = 1$, M is linear increasing in I, J; V is increasing in I, J at square root speed.
Next important step is to calculate critical values $t_{\alpha,b}$ which are defined as a root to equation 3.5. This procedure is time consuming, but at the same time does not rely on data and has to be simulated once for a fixed number of observation n and block size b. We use dichotomic method of iterative split-

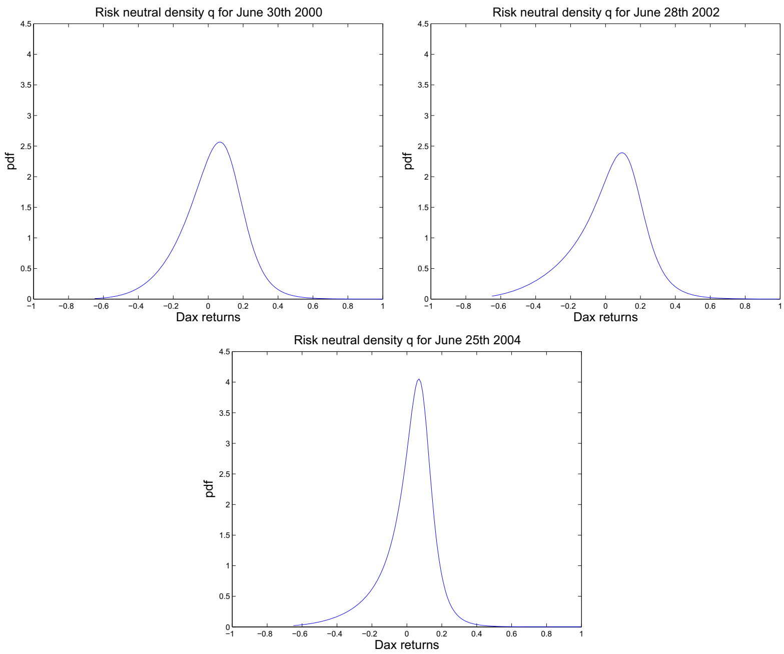

Figure 4.2.2: Estimated risk neutral densities q for years 2000, 2002 and 2004

ting intervals. In our analysis we start with intervals $[0.0, 20.0]$ then calculate correponding α for the mean of the interval. Depending on calculated α one of two resulting intervals $[0.0, 10.0]$ and $[10.0, 20.0]$ is chosen. This procedure is repeated for selected interval until solution of required precision is found. Resulting critical values are presented in figure 4.2.5. It can be seen that critical values are changing for different parameter b.

Finally testing surfaces $\xi(I, J, b) - M(I, J, b) + t_{0.05,b} \cdot V(I, J, b)$ for years 2000, 2002 and 2004 are produced. For fixed $b = 50$ corresponding surfaces are presented in figure 4.2.6. They show the differences between simulated 5% threshold surface $M - t_{0.05,50} \cdot V$ calculated via Monte Carlo simulations and test statistics ξ obtained from observed data in years 2000, 2002 and 2004. Hypothesis H_0 of motononic descreasing EPK is rejected at 5% significance level if test statistics ξ is smaller than threshold value $M(I, J) - t_{0.05}V(I, J)$. For each interval I, J where surface $\xi(I, J, 50) - M(I, J, 50) + t_{0.05,50}V(I, J, 50)$ is negative, a corresponding rejection interval is plotted in figure 4.2.7. Summary of results for three years is presented in table 4.2.1. In addition to accepted hypothesis, value of $\min_{I,J,b} \{\xi(I, J, b) - M(I, J, b) + t_{\alpha,b} \cdot V(I, J, b)\}$ is givenin the table. By evaluating this values we can estimate the significance of accepted hypotheses. Test significantly rejectes monotone decreasing

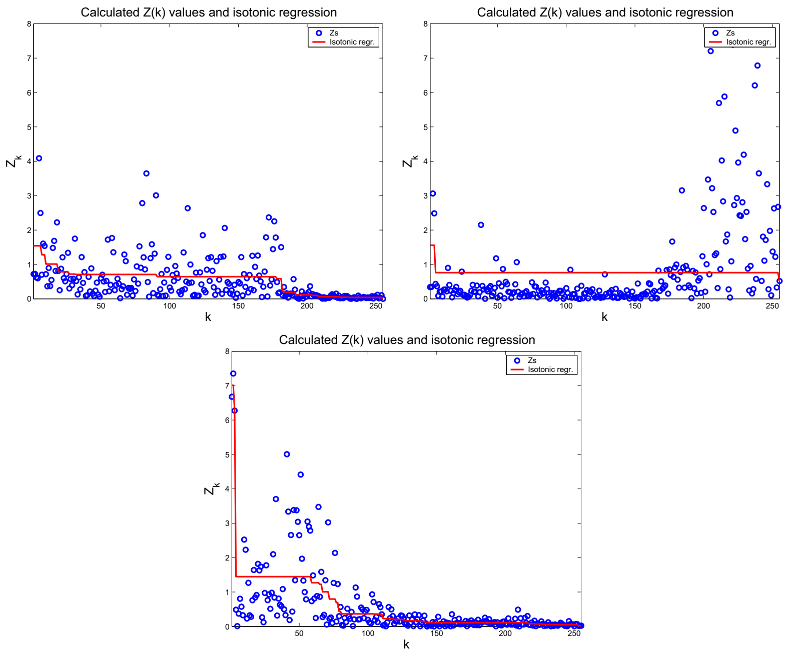

Figure 4.2.3: Calculated Z_k for years 2000, 2002 and 2004

EPK in 2002 as well as can not reject strictly decreasing EPK in 2004 for 5% and 10% significance level. Situation in 2002 is on the verge: H_0 can not be rejected with 5% critical values, but rejected at 10% signifiance level.

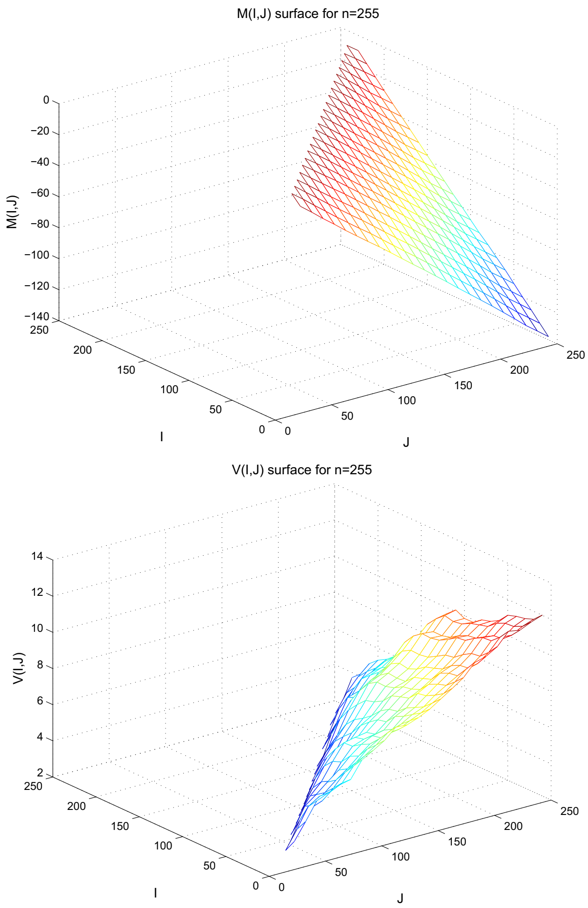

Figure 4.2.4: Surfaces M and V for 255 observations, $b = 1$

Sign. level/Year of analysis	2000	2002	2004
5% Significance level			
$\min_{I,J,b}$	0.5437	-133.78	3.7935
Accepted H_0	H_0	H_1	H_0
10% Significance level			
$\min_{I,J,b}$	-0.1840	-134.42	3.1685
Accepted H_1	H_1	H_1	H_0

Table 4.2.1: Summary of results on monotonicity of EPK in 2000, 2002 and 2004.

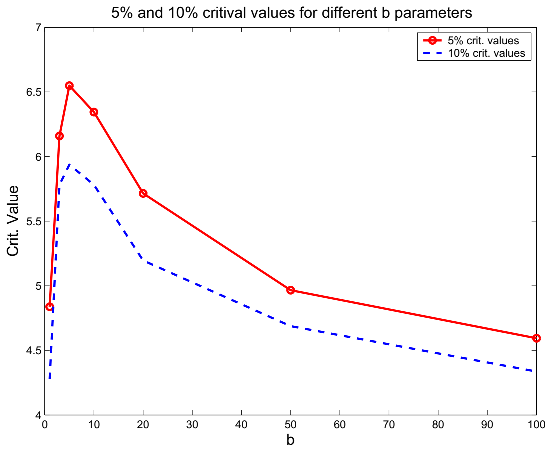

Figure 4.2.5: 5% and 10% distribution over b

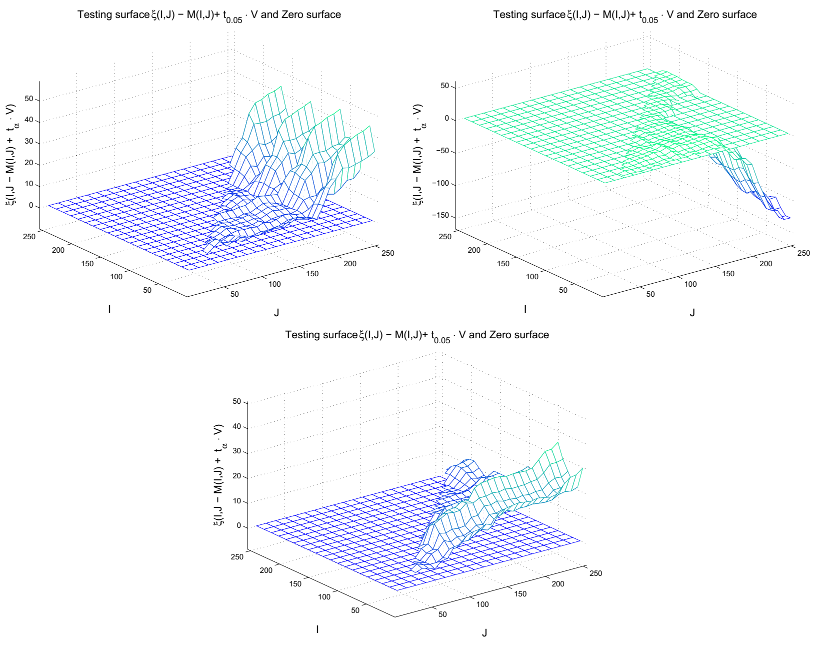

Figure 4.2.6: Surface $\xi(I, J, 50) - M(I, J, 50) - t_{0.05,50} \cdot V(I, J, 50)$ for years 2000, 2002, 2004

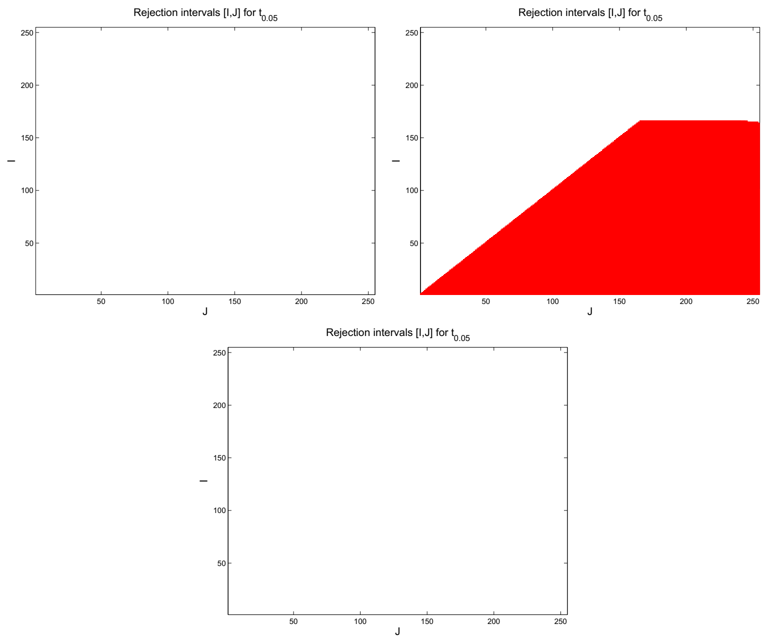

Figure 4.2.7: Rejection intervals (I, J) for years 2000, 2002 and 2004, $b = 50$

Chapter 5

Conclusion

In this master thesis we describe the test that checks monotonicity of pricing kernels. By testing monotonicity of pricing kernel we can determine whether utility function is concave or not. Strictly decreasing pricing kernel corresponds to concave utility function while non-decreasing EPK means that utility function contains some non-concave regions.

Pricing kernels are constructed as a ratio of risk neutral density q and subjective density p. Density q is obtained from derivative market and due to large number of obsrevations can precisely estimated. p is usually estimated from historical information, but due to scaricity of data is considered to be unknown. Therefore we test ratio of two densities $\frac{q}{p}$, where q is given and p is unknown. Using Pyke's theorem (see Pyke, R. (1965)) this problem is reduced to simple exponential problem. The test isteslf is constructed on the basis of likelihood ratio test for a fixed interval. By using intersection of tests for different intervals we can expand it to the variant which is independent from intervals.

We investigated EPK for German DAX data in years 2000, 2002 and 2004. We found the evidence of non-concave utility function: H_0 hypothesis of monotone decreasing pricing kernel function was rejected at 5% and 10% significance level in 2002; in 2000 H_0 was rejected at 10% significance level. This result is consistent with work of Detlefsen, et al (2007) who observed non-cocavity in utility functions obtained from German DAX market. For year 2004 hypothesis of decreasing EPK could not be rejected. These findings also support the idea of Giacomini and Haerdle (2007) who wrote the structure of pricing kernel may vary over time.

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