Semi-Parametric Estimation of Elliptical Distribution in Case of High Dimensionality

Master's Thesis submitted

to

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Abstract

This paper is devoted to the problem of high dimensionality in finance. We consider a joint multivariate density estimator of elliptical distribution which relies on a non-parametric estimation of a generator function. The factor model is employed in order to obtain a consistent covariance matrix estimator. We provide a simulation study that suggests that the considered estimator significantly outperforms the one based on the sample covariance matrix estimator. We also provide an empirical study using an example of a S&P500 portfolio. The returns of the resulted distribution are fat tailed and have a high peak. The comparison with other distributions illustrates the inappropriateness of normal or Student t distribution to fit the financial returns. Calculations of VaR are provided as an example of possible applications.

Key words: covariance matrix, high dimensionality, factor models, elliptical distributions

Contents

List of Abbreviations

FFL	Fan, Fan, Lv covariance matrix estimator	CAMP	Capital Asset Pricing Model
FF3	Fama-French 3 factor model	ICAMP	Intertemporal CAPM
APT	Arbitrage Pricing Theory	VaR	Value at Risk
SMB	Small Minus Big	HML	High Minus Low
SD	Standard Deviation	OLS	Ordinary Least Squares

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1 Introduction

When dealing with financial problems researchers often have to resort to multivariate density estimation. This problem is relatively well-studied for conventional settings, namely, when the dimensionality p is low while the number of observations n tends to infinity. However, if the basic setting changes, obtaining a good estimator is not such an easy task as it may seem at a first glance. When dealing with some practical problems the big number of observations is usually unaccessible due to the different reasons while the number of dimensions, on the contrary, grows. For instance, financial portfolios often consist of hundreds of assets. One restriction is imposed by the specific features of the dataset: in the world of finance one often can only use a very limited number of observations to produce an estimator. For example, for daily data frequency usually not more that $n = 750$ (that corresponds to 3 years) data points can be used because otherwise the researcher has to deal with a vast bias due to the structural changes in the financial system as a whole. Take a look at the past: in the after-crisis period one can barely rely on the data that stem from 2006 or 2005 because it's natural to expect the change of some data patterns due to the financially unstable period of 2007-2008.
This paper is devoted to the technique that allows to overcome "small n, large p" problem. The considered estimation procedure consists of two steps:

1. Covariance matrix estimation

2. Semi-parametric estimation of elliptical distribution density

Both issues are of great relevance for particular fields of economic analysis: for theoretical problems as well as for practical applications. For example, studying of joint evolution of macroeconomic time series can deliver a great deal of understanding of the key economic process. Careful examination of large panels of data(e.g. home-price data) is used in many spheres of economic analysis. Portfolio optimization problem relies on the covariance matrix estimator. Popular statistical techniques such as principal component analysis or discriminant analysis also require a reliable estimator of the covariance matrix. The knowledge of the underlying distribution is often the backbone of effective risk measurement and risk management. Moreover, similar challenges often arise outside the spheres of finance and economics, so the presented solutions can well be adopted for other fields.

The problem of covariance matrix estimation in case of highly dimensional data is relatively well studied in the literature. Conventional methods often perform poorly in this setting and provide nearly-singular (ill-conditioned) estimators. Possible operations over covariance matrix often amplify the estimation error even further. The alternative estimators developed for high dimensional datasets can be roughly divided into four groups.

First strand of the research concentrates on the dimensionality reduction by imposing some restrictions on the elements of the covariance matrix, namely, assuming that some of them are zero. The unified theory of "sparsistency" was developed by **Lam and Fan (2009)**. One of possible techniques relies on banding to find a consistent estimator (see **Wu and Pourahmadi (2003)**). Some papers apply penalized likelihood method (see **Huang et al. (2006)** and **Rothman et al. (2008)**). Alternative techniques involves thresholding was developed by **Bickel and Levina (2008)** and **Rothman et al. (2009)**.

The second idea mainly relies on the processes that drive the variable of interestes and extensively uses factor models. The dimensionality reduction is achieved because the original p series are replaced with K factors (the number of which is usually much lower) that are expected to capture the crossectional variance of the data. These estimators are especially appealing for finance applications because of an ample strand of literature devoted to the factor models. Notable example of such estimators are given by **Fan et al. (2008)**, **Lam and Yao (2011)**, **Chan et al. (1999)**, **Lam et al. (2009)** and others.

The alternative solution is provided by shrinkage methods that are based on the trade-off between a bias and an estimation error which can be resolved by taking a properly weighted average of biased and unbiased estimators. The idea was first developed by **Stein (1956)** and later adopted for covariance matrix estimators by **Jorion (1986)** , **Ledoit and Wolf (2003)** , **Ledoit and Wolf (2004)** and **Ledoit and Wolf (2010)**. In the covariance matrix estimation framework shrinkage parameter is used to balanced the estimation error due to the ill-conditioned estimator and a bias.

Finally, the covariance matrix estimator can be obtained employing the high frequency data and the concept of realized volatility. In general, the simple realized covariance for multi-dimensional case that was developed by **Barndorff-Nielsen and Shephard (2004)** can't serve as a good proxy for the covariance matrix if the number of dimensions is high (see **Zheng and Li (2011)**). However, other estimation techniques can be developed on the basis of this concept. For example, **Bannouh et al. (2010)** proposed an estimator that makes extensive use of the high-frequency data and factor models. Other prominent examples of the methodology based on the realized covariance can be found in **Barndorff-Nielsen et al. (2011)**, **Zheng and Li (2011)**, **Wang and Zou (2010)**.

All the methods mentioned above can often be combined to improve the estimator performance. For example, **Fan et al. (2011)** developed a estimator based on the factor models as well as sparse estimation techniques.

Much less attention, however, is devoted to the issues of the non-parametric joint multivariate density estimation. Due to the so-called "curse of dimensionality" non-parametric estimation techniques can't be efficiently applied if the highly dimensional dataset is used: the cost grows exponentially with dimension.One way to tackle this issue is to combine the positive effects of both non-parametric and semi-parametric approaches leaving some space for non-parametrics and let "the data speak" but still imposing some structure that is less limiting than an assumption about a particular distribution. We suggest resorting to a family of elliptical distributions for this purpose that can be treated as a generalization of a multivariate normal distribution. Alongside with already mentioned Gaussian distribution, this family also includes such distributions as multivariate Student's t, Laplace, Cauchy, logistic, etc.
The idea of semi-parametric estimation of elliptical distributions first appeared in the papers of **Stute and Werner (1991)** and **Cui and He (1995)**. However, the author assumed that at least some of the parameters of the distribution are given, while we concentrate on the case when all of them should be estimated. This approach was developed by **Fan et al. (2012)** who proposed to combine the idea of semi-parametric density estimation of elliptical distributions and a covariance matrix estimation based on the factor model developed by **Fan et al. (2008)**. The authors also employ the idea of **Liebscher (2005)** that allows for correct estimation of a generator function of elliptical distribution near zero. Alternative estimation procedure based on the finite mixture sieves is presented by **Battey and Linton (2011)**, however, this procedure can only be applied for elliptical distributions whose densities can be expressed as scale mixtures of normal densities. The problem of elliptical copulae estimation was studied by **Sancetta (2008)**.

In this paper we examine the properties of the density estimator developed by **Fan et al. (2012)**. We support the theoretical findings presented in the original paper using a Monte-Carlo simulation technique. The results indicate a clear superiority of the proposed estimator over a benchmark model that employs a sample covariance matrix estimator. However, the simulation study also shows that the Liebscher transformation brings no significant contribution to the estimator when dealing with normal distribution if the number of dimensions is large. Further, we run an empirical study in order to present the possible application of the obtained estimation procedure. The portfolio consisting the some components the S&P500 index is built. Our findings suggest that the resulted distribution has a very specific form which strongly deviates from the normal distribution. It has a higher peak and longer tails than suggested by the Gaussian distribution which is a well-known fact for financial returns. More detailed study of the quantiles reveals that tails of the obtained distribution also diverge from Student t distribution which is usually used to fit fat tails of financial returns.

This paper is organized as follows. Chapter 2 introduces the covariance matrix estimation technique based on factor models developed by **Fan et al. (2008)**, provides a short review of returns models employed in finance and concludes with a simulation study to support the proposed estimator. Chapter 3 presents the semi-parametric density estimator for elliptical distributions as proposed by **Fan et al. (2012)** and develops a theoretical framework for VaR estimation for elliptical distributions. By the means of a Monte-Carlo experiment, Chapter 4 studies the performance of the proposed estimator. Chapter 5 provides an extensive description of the data used to fit the model and obtain a density estimator. Chapter 6 provides an empirical study and presents the estimation results for the portfolio that consists of the S&P500 components. Finally, concluding remarks can be found in chapter 7.

2 Covariance Matrix Estimation

This chapter is devoted to the covariance matrix estimation technique that was developed by **Fan et al. (2008)**. The covariance matrix estimation is one of the key problems of the high-dimensional data analysis which is particularly relevant for the financial science. It is also often the case that the interest of the researcher is not the covariance matrix itself but, for example, the inverse matrix or its derivative. For example, portfolio allocation is defined by the eigenvalues and eigenvectors of matrices.
Given a relatively small number of observation, the question of obtaining a covariance matrix estimator may not be as easy to solve. The estimators obtained by the application of usual techniques often provide very unstable and thus unreliable results. The problem may be even more complicates if the area of interest includes not only the covariance matrix itself. For example, with the growth of dimensionality the sample covariance matrix may often be non-invertible as p is close to n with the determinant tending to zero even if some other measures of this estimator are good enough.

This chapter is mostly based on the solution of the high dimensionality problem that was proposed by **Fan et al. (2008)**. The authors impose a certain structure on the data assuming that the financial returns follow some factor model with number of factors K. They let K grow with dimensionality p, although the number of factors is expected to be much less that p. On the basis of the factor structure covariance matrix estimator is derived. Clearly, this idea may also be applied for other than financial problems where a factor model underlying the variables of interest can be developed.
This chapter first develops the theoretical framework that presents the procedure of obtaining the estimator. Then several of the most widely-used factor models for financial returns that may be considered as an underlying are provided. We conclude with a simulation study that supports the claim about the superiority of the considered estimator over a benchmark sample estimator.

2.1 Theoretical Framework

Consider a multi-factor model that implies that an excess return over a risk free rate for any asset Y_i follows a factor model designed as follows:

$$Y_i = b_{i1} f_1 + \ldots + b_{iK} f_K + \varepsilon_i \quad i = 1, \ldots, p \tag{2.1}$$

- $f_1, \ldots, f_K$ excess returns of K factors that are known and observable
- b_{ij} $i = 1, \ldots, p,\ j = 1, \ldots, K$ factor loading that are unknown and should be estimated
- ε_i $i = 1, \ldots, p$ idiosyncratic errors such that $\text{corr}(\varepsilon_i, \varepsilon_j | f_1, \ldots, f_k) = 0\ \forall i, j \in (1, p)$ and $i \neq j$
- dimensionality p grows with the sample size n and number of factors K increases with the dimensionality p

The key idea of the model suggests that the number of parameters of the covariance matrix that are to be estimated is reduced once the factor model is implemented. It should be noted, however, that the results are only true if the factor model is good enough to capture the returns behavior.

As the natural competitor to the estimator based on the factor model, we consider a sample covariance matrix estimator which is intuitively easy to understand, simple to obtain and also has a property of unbiasedness. The sample estimator also performs very well when dealing with a large number of observations and small dimensionality, however, its performance is expected to deteriorate as the number of dimensions grows.
The factor model described above can also be represented in a matrix form as follows:

$$Y_i = b_{i1} f_1 + \ldots + b_{iK} f_K + \varepsilon_i \quad i = 1, \ldots, p \tag{2.2}$$

- $Y = (Y_1, \ldots, Y_p)^\top$ asset returns
- $B_n = (b_1, \ldots, b_p)^\top$ factor loadings $b_i = (b_{n,i1}, \ldots, b_{n,iK})$ $i = 1, \ldots, p$
- $f = (f_1, \ldots, f_K)^\top$ vector of factors
- $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p)^\top$ errors

We also make several assumptions:

- $(f_1, Y_1), \ldots, (f_n, Y_n)$ are n iid samples of (f, Y)
- Distribution of f is continuous
- $\mathrm{E}[\varepsilon|f] = 0$ and $\mathrm{Cov}(\varepsilon|f) = \Sigma_0$ is diagonal

and use following notations:

- $\Sigma_n = \mathrm{Cov}(y)$ covariance matrix of excess returns
- $X = (f_1, \ldots, f_K)$ matrix of factors
- $Y = (y_1, \ldots, y_n)$ matrix of all observations
- $E = (\varepsilon_1, \ldots, \varepsilon_n)$ matrix of errors

Then if the model holds the covariance matrix is defined by the covariance matrix of factors multiplied by the matrix of factor loadings and the covariance matrix of errors so that:

$$\Sigma_n = B_n \, \mathrm{Cov}(f) B_n^\top + \Sigma_0 \tag{2.3}$$

Once we can estimate all the components (covariance matrix of factors, matrix of factor loadings and covariance matrix of errors), we can also derive an estimator of interest (FFL estimator):

$$\widehat{\Sigma}_n = \widehat{B}_n \widehat{\mathrm{Cov}}(f) \widehat{B}_n^\top + \widehat{\Sigma}_0 \tag{2.4}$$

The easiest way to obtain these estimates is just to use the OLS estimators of corresponding variables, which are defined as the follows:

- $\widehat{B}_n = YX^\top (XX^\top)^{-1}$
- $\widehat{\mathrm{Cov}}(f) = (n-1)^{-1} XX^\top - \{n(n-1)\}^{-1} X\mathbf{I}\mathbf{I}^\top X^\top$
- $\widehat{\Sigma}_0 = diag(n^{-1} \widehat{\varepsilon}\widehat{\varepsilon}^\top)$ with $\widehat{\varepsilon} = Y - \widehat{B}X$

The benchmark sample matrix estimator is obtained as follows:

$$\widehat{\Sigma}_{sam} = (n-1)^{-1}YY^{\top} - \{n(n-1)\}^{-1}Y\mathbf{II}^{\top}Y^{\top} \tag{2.5}$$

The authors claim that the derived estimator under some weak assumptions is asymptotically normal and demonstrates convergence rates that are much faster than those of the benchmark.

2.2 Returns Factor Models

There is a vast amount of economic literature devoted to the development of a model that drives factor returns. The best known model aimed at describing returns as a function of some risk factors is a famous Capital Asset Pricing Model(CAPM) that was independently proposed by **Sharpe (1964)** and **Lintner (1965)** and is largely based on the Markowitz pricing theory(see **Markowitz (1959)** and **Markowitz (1952)**). The CAPM concentrates on the relationship between the asset return and riskiness measured by a sole risk factor β which reflects the correlation between the asset return and the market portfolio. In equilibrium the excess asset return should be proportional to the excess returns of the market portfolio. According to this model, only systematic and non-diversifiable risk matters for investors.

$$Y_i = r_i - R_f = \alpha + \beta_i(R_m - R_f) \tag{2.6}$$

where

- Y_i excess return on the asset i
- r_i returns of the asset i
- R_f risk free rate
- R_m market rate

However, a simply structured CAPM falls short of explaining the complicated reality. At some point, a need of a more advance model became evident (see among others **Graham and Harvey (2001)**). In particular, **Fama and French (1992)** find that the basic correlation of CAPM disappears during the 1963-1990 period in the U.S. These findings encouraged economists to resort to the models based on CAPM but also augmented by other risk factors that aim at predicting returns. The various extensions of CAPM include among others such models as Arbitrage Pricing Theory (APT) of **Ross (1976)** and inter-temporal CAPM(ICAPM) of **Merton (1973)**. While CAPM asserts the dependence on a single factor, APT and ICAPM allow for adding any number of factors without specifying any of them. One another prominent example of the CAPM extension that became a workhorse of the financial literature is a renown Fama French 3 Factor Model(FF3). Based on the empirical study of **Fama and French (1992)**, a 3 factor model that partially is able to correct the inadequacies of the CAPM was developed by **Fama and French (1993)**. The authors argue that the excess returns of an asset is a combination of excess returns on market portfolio, small minus big (SMB) size portfolio and high minus low (HML) value portfolio. The two latter imply additional risk premium which is related to size and distress respectively.

$$Y_i = r_i - R_f = \alpha + \beta_{1i}(R_m - R_f) + \beta_{2i}SMB + \beta_{3i}HML \tag{2.7}$$

where

- SMB the performance of small stocks relative to big stocks (Small [Cap] Minus Big)

$$\begin{aligned} SMB = 1/3(SmallValue + SmallNeutral + SmallGrowth) - \\ 1/3(BigValue + BigNeutral + BigGrowth) \end{aligned} \tag{2.8}$$

- HML the performance of value stocks relative to growth stocks (High [Book/Price Value] Minus Low)

$$HML = 1/2(SmallValue + BigValue) - 1/2(SmallGrowth + BigGrowth) \tag{2.9}$$

- $SmallValue$, $SmallNeutral$, $SmallGrowth$, $BigValue$, $BigNeutral$ and $BigGrowth$ are six book-to-market benchmark portfolios. These factors measure the excess returns of small caps over big caps and of value stocks over growth stocks. To calculate the values of these portfolios, a combination of ranked stocks is used.

The empirical studies about the validity of the FF3 model were carried out for such countries as US(**Fama and French (1993)**), Canada(**Griffin (2002)**), Australia(**Gaunt (2004)**), Sweden(**Asgharian and Hansson (2000)**), Italy(**Silvestri and Veltri (2011)**), Hong Kong (**Lam (2002)**), Thailand(**Homsud et al. (2009)**), Germany, France and Great

Britain(**Malin and Veeraraghavan (2004)**), etc. Although lacking in the theoretical basis, this model proved to be able to successfully explain major market anomalies (see **Fama and French (1996)**) and outperforms many other models(see **Hodrick and Zhang (2001)** for comparison).

However, the FF3 model doesn't not lack of disadvantages as well (as confirmed in **Fama and French (1996)**). So a lot of alternative models and those based on the FF3 can be found in the literature. These improvements among others include liquidity (see, for example, **Amihud and Mendelson (1986)**), momentum (see **Carhart (1997)**), etc. (see also **Avramov and Chordia (2006)** for comparison of augumented models).

Nevertheless, FF3 is a good stepping stone for further models that can easily be applied and is intuitively easy to understand. We restrict ourselves with this model for the purposes of this study. However, the choice of the best model to obtain covariance matrix estimator is yet an open question and an issue for further research.

2.3 Simulation Study

Following the results provided in the original paper we present a simulation study that tends to support the superiority of FFL estimator over the sample covariance matrix estimator. The simulations are structured exactly as in the article. However, while the previous study concentrates exclusively on the Gaussian distribution, we provide the results for Student t distribution that is better fitted for financial returns as it is has fat tails. Also the sample size n is changed from 756 to 250 in order to show that the findings of the paper are also supported if the sample size decreases substantially. Finally, some additional results concerning the determinant are presented.

2.3.1 Simulation Design

We consider the sample size $n = 250$ that approximately corresponds to one year and let the number of dimensions p vary. In order to access the covariance matrix estimation errors for $\widehat{\Sigma_n}$ and $\widehat{\Sigma}_{sam}$, three types of norms are used: Frobenius Norm, the FFL norm as introduced in **Fan et al. (2008)** and Entropy Loss. We also compare the inverse matrices under the Frobenius norm and present a study for differences in determinants.

The number of factors is set $K = 3$, so that the model now takes the following form:

$$Y_{pi} = b_{pi1}f_1 + b_{pi2}f_2 + b_{pi3}f_3 + \varepsilon_i \quad i = 1, \dots, p \tag{2.10}$$

The index p is added in order to underline that the factor loadings differ for different values of p. The authors take the Fama-French 3 factor model as an underlying (see **Fama and French (1993)**) and fit it to the real data to get the idea about the parameters values. We also apply their findings about the parameter values, so that the results are comparable. We keep the number of factors K as well as the sample size n fixed in the simulation study.

The following algorithm is used to carry out a simulation:

1. For each dimensionality p from 10 to 200 by 10:
   (a) Generate a random sample of factors $f = (f_1, f_2, f_3)^\top$ from a Gaussian (Student t with 10 degrees of freedom) distribution with parameters (μ_f, Σ_f) (see Table 2.1 for the values (μ_f, Σ_f))
   (b) Generate a p random samples of factor loading vectors $B = (b_1, \dots, b_p)^\top$ from a Gaussian (or Student t with $d.f = 10$) distribution with parameters (μ_b, Σ_b)
   (c) Generate p random standard deviations $\sigma_1, \dots, \sigma_p$ that characterize error from a Gamma distribution $G(\alpha, \beta)$ with $\alpha = 3.3586$ and $\beta = 0.1876$ (see **Fan et al. (2008)** for derivation of values of α and β)
   (d) Generate a random sample of errors $\varepsilon = (\varepsilon_1, \dots, \varepsilon_p)^\top$ from a Gaussian distribution (Student t with $d.f = 10$) with parameters (0, $\mathrm{diag}(\sigma_1^2, \dots, \sigma_p^2)$). Each vector ε_i should have a length of n.
   (e) Using the model presented above and generated values of factors, factor loadings and errors, get the values of $y = (Y_1, \dots, Y_p)^\top$.
   (f) Estimate results: calculate covariance matrix estimators as described above.
2. Repeat (a)-(f) .

Following the original paper, we use several norms to access the quality of the covariance matrix estimator for true matrix Σ and some estimator $\widehat{\Sigma}$:

- Frobenius Norm:

$$\|\Sigma - \widehat{\Sigma}\| = [\mathrm{tr}\{(\Sigma - \widehat{\Sigma})(\Sigma - \widehat{\Sigma})^\top\}]\} \tag{2.11}$$

Factor	μ_f	Cov_f		
		K_1	K_2	K_3
K_1	0.0236	1.2507	-0.0350	-0.2042
K_2	0.0129	-0.0350	0.3156	-0.0022
K_3	0.0207	-0.2042	-0.0022	0.1930
Factor	μ_b	Cov_b		
K_1	0.7828	0.0291	0.0239	0.0102
K_2	0.5180	0.0239	0.0540	-0.0070
K_3	0.4100	0.0102	-0.0070	0.0869

Table 2.1: Sample means and sample covariance matrices of factors f and factor loadings b

Source: Table 1 from **Fan et al. (2008)**

- Entropy loss function (developed by **James and Stein (1961)**):

$$L_1(\Sigma, \widehat{\Sigma}) = \text{tr}(\widehat{\Sigma}\Sigma^{-1}) - \log|\widehat{\Sigma}\Sigma^{-1}| - p \tag{2.12}$$

- FFL norm

$$\|\Sigma - \widehat{\Sigma}\|_\Sigma = p^{-1/2} L_2(\Sigma, \widehat{\Sigma}) \tag{2.13}$$

where $L_2(\Sigma, \widehat{\Sigma})$ is a quadratic loss $L_2(\Sigma, \widehat{\Sigma}) = \text{tr}(\widehat{\Sigma}\Sigma^{-1} - \mathbf{I})^2$

Further we deal with the accuracy of the determinant estimation. For this purpose the following measure is used:

$$L = \log(|\Sigma|) - \log(|\widehat{\Sigma}|) \tag{2.14}$$

2.3.2 Simulation Results

In general, the simulation results correspond to the findings of **Fan et al. (2008)**(both theoretical and those that follow from the simulation study). The fact that the simulation for Student t distribution mirrors the results for the normal distribution is not unexpected, but this fact supports the claim that the developed approach can be used for other distributions including those with fat tails which is usually the case for financial returns.

We compare the relative performance of two covariance matrix estimators: FFL estimator $\widehat{\Sigma}_n$ and sample estimator $\widehat{\Sigma}_{sam}$. For each measure the averages of calculated errors as well as standard deviations are presented. For all measurements presented standard deviations are relatively low in comparison with the corresponding averages, wo we can conclude that the Monte Carlo simulation we carried out is good enough and errors generated by the calculation procedure can be neglected.

First, we concentrate on the comparison of two estimators under the norms introduced in the previous section (see Figure 4.1). Under the Frobenius norm(see Figure 4.1 (c,d)) both estimators perform equally. However, the appropriateness of this norm to measure the quality of the estimator in the context of factor models is debatable because the factor structure is not taken into account(see **Fan et al. (2008)** for more details and **Horn and Johnson (1990)** for more information about the Frobenius norm). Under FFL norm and entropy loss that are more sensitive to the factor structure the FFL estimator significantly outperforms the benchmark.

The inverse matrix estimator based of FFL estimator performs much better than the competing one even in terms of Frobenius norms (see Figure 2.2). We can observer that FFL estimator performs much better with respect to the determinant estimation than the sample estimator. For the latter the determinant tend to 0 with the growth of p that eventually leads to the almost singular matrix that can't be inverted and may impose severe distortions if the determinant value is of the key interest.

The results for the underlying Student t distribution with 10 degrees of freedom are identical to those demonstrated by the simulation with normal distribution (see Figure 2.3 and 2.4). So, we provide some additional support to the claim that the results can be extrapolated for other distributions, namely, those with fatter tails that are of special interest for us.

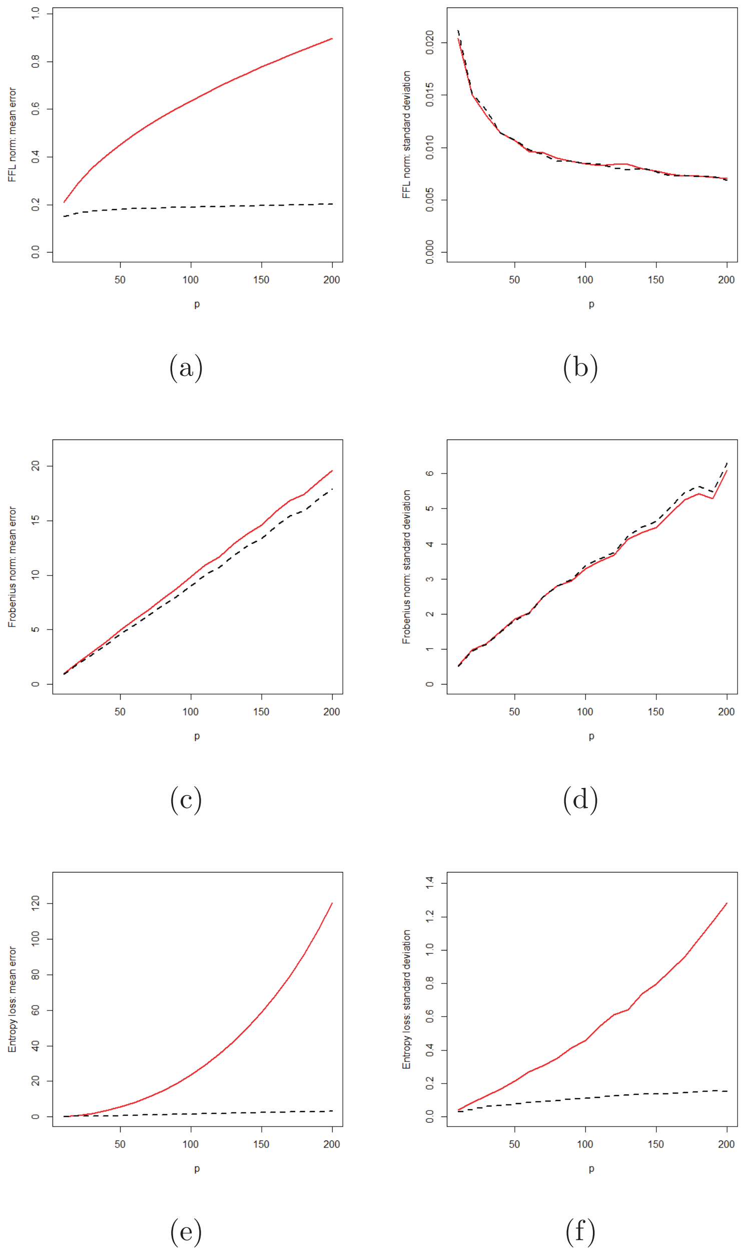

Figure 2.1: Comparison of Covariance Matrix Estimators Performance under FFL, Frobenius norms and Entropy Loss for Normal Distribution

(a) and (b): The mean errors and corresponding standard deviations plotted for FFL estimator $\widehat{\Sigma}_n$(black dashed curve) and sample estimator $\widehat{\Sigma}_{sam}$ (red curve) against p for:(a,b) FFL norm,(c) and (d) Frobenius norm, (e) and (f) Entropy loss. Normal distribution, $n = 250$, 1000 repetitions

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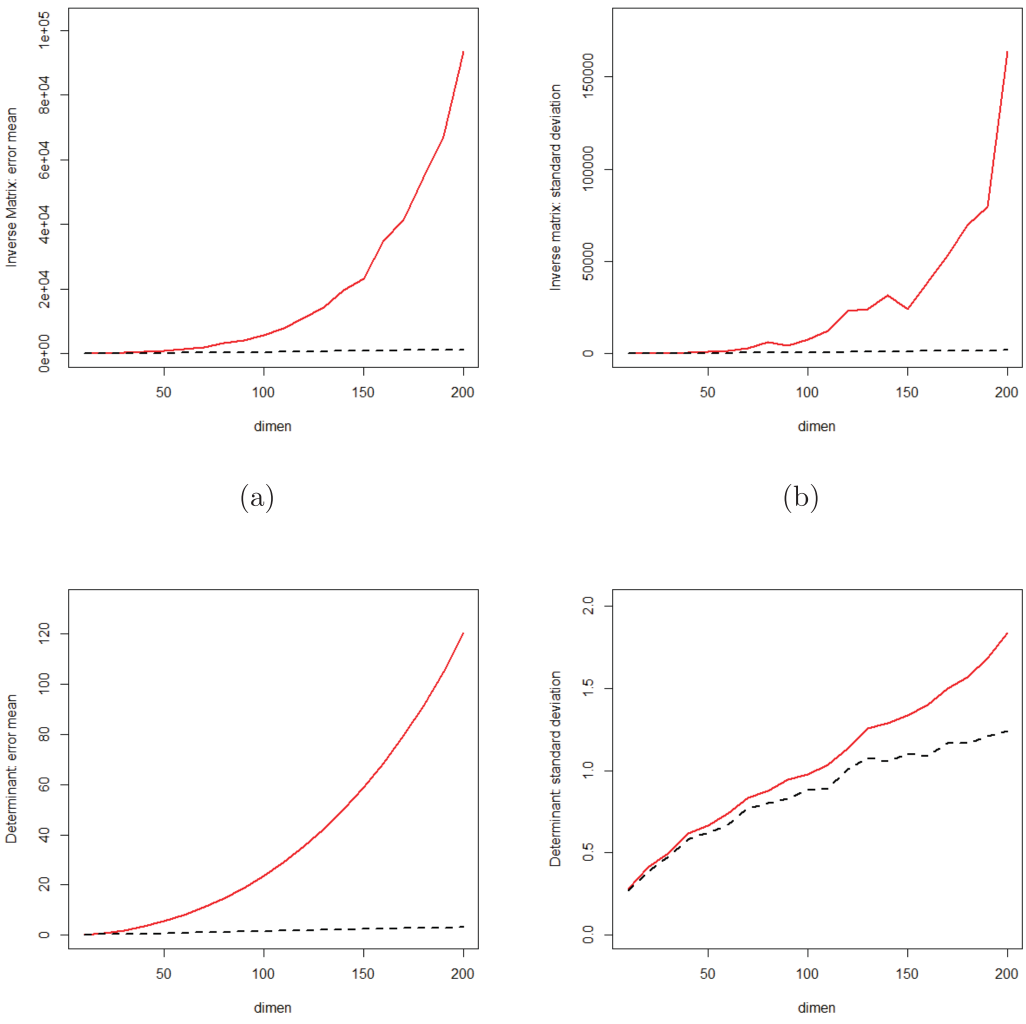

(a) (b)

(c) (d)

Figure 2.2: Comparative Performance of Estimators of Inverse Matrix and Determinant for Normal Distribution

(a) and (b): The mean errors and corresponding standard deviations plotted for FFL estimator $\widehat{\Sigma}_n$(black dashed curve) and sample estimator $\widehat{\Sigma}_{sam}$ (red curve) against p for:(a,b) inverse matrix estimator under Frobenius norm, (c,d) difference of determinants in logs. Normal distribution, $n = 250$, 1000 repetitions

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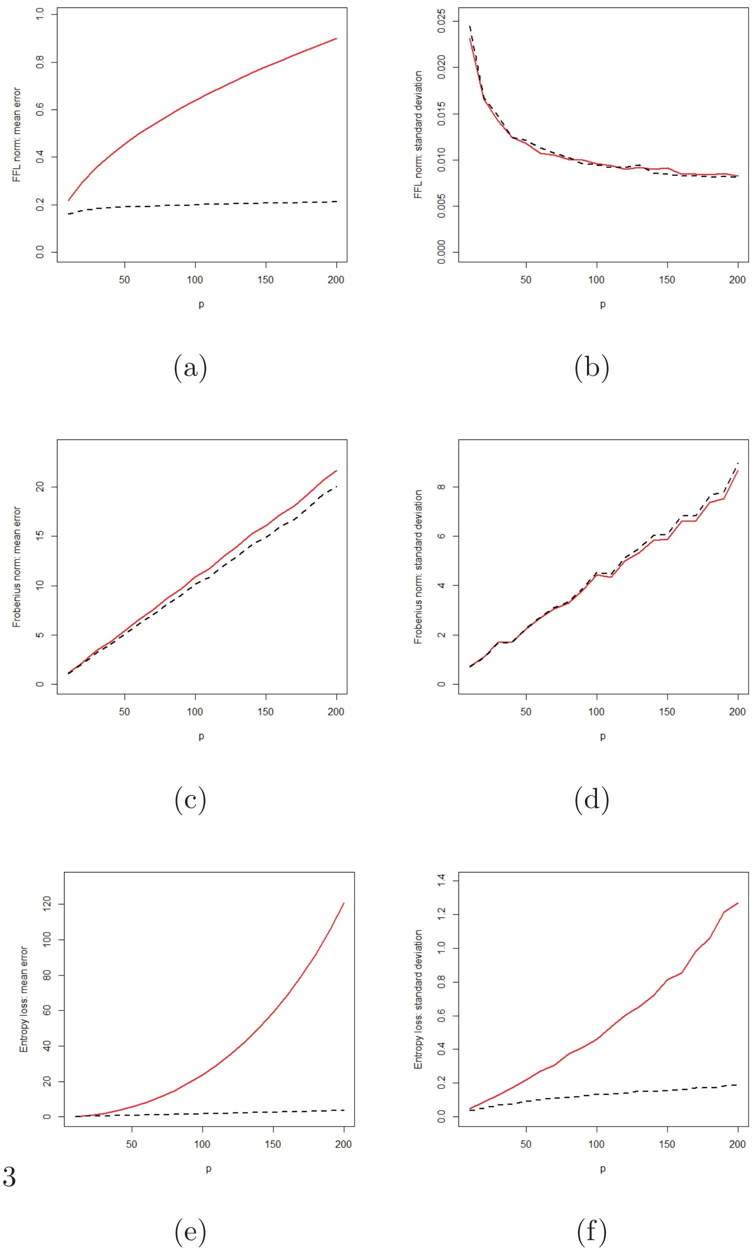


Figure 2.3: Comparative Performance of Covariance Matrix Estimators Performance under FFL, Frobenius norms and Entropy Loss for Student t distribution

(a) and (b): The mean errors and corresponding standard deviations plotted for FFL estimator $\widehat{\Sigma}_n$(black dashed curve) and sample estimator $\widehat{\Sigma}_{sam}$ (red curve) against p for:(a,b) FFL norm,(c,d) Frobenius norm, (e,f) Entropy loss. Student t distribution $d.f = 10$, $n = 250$, 1000 repetitions

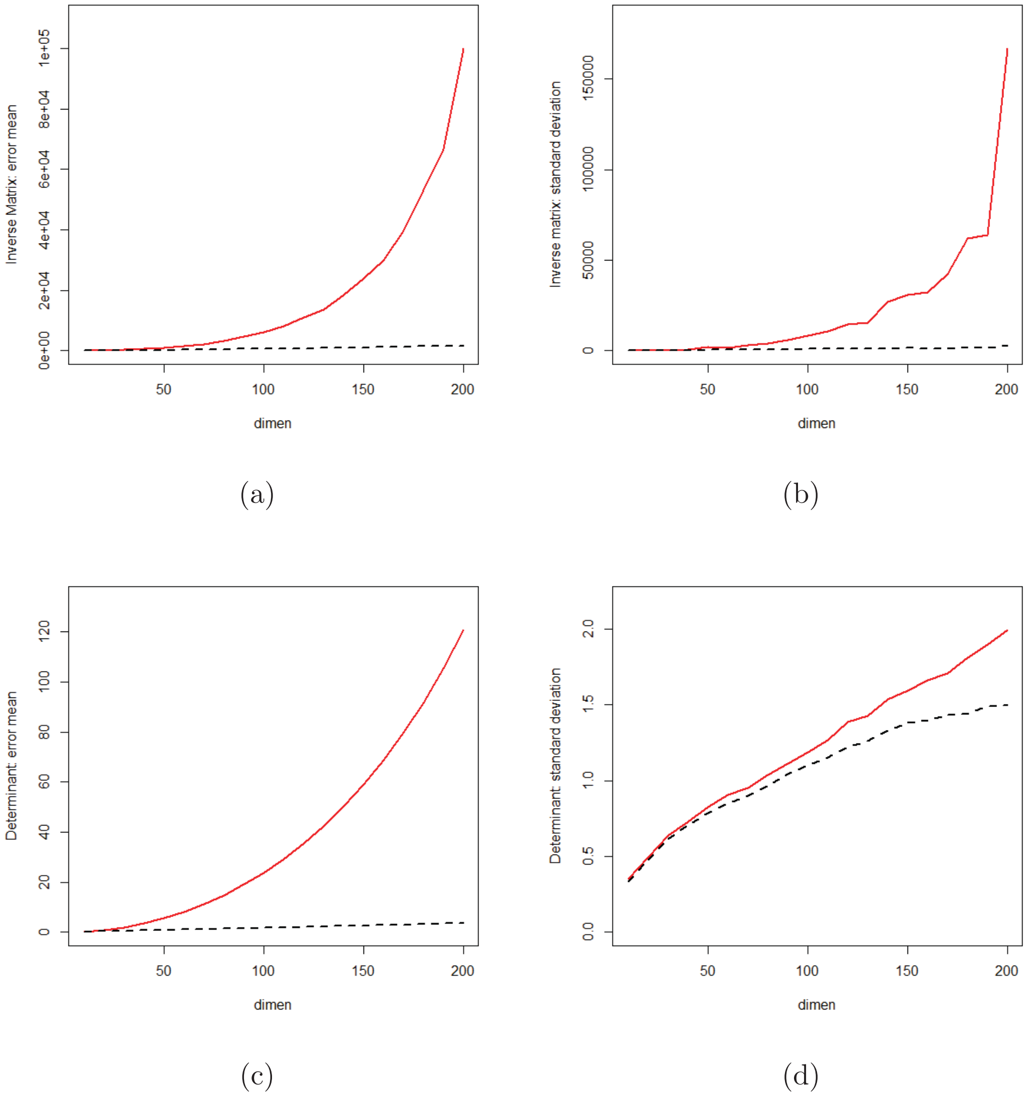


Figure 2.4: Comparative Performance of Estimators of Inverse Matrix and Determinant for Student t distribution

(a) and (b): The mean errors and corresponding standard deviations plotted for FFL estimator $\widehat{\Sigma}_n$(black dashed curve) and sample estimator $\widehat{\Sigma}_{sam}$ (red curve) against p for:(a,b) inverse matrix estimator under Frobenius norm, (c,d) difference of determinants in logs. Student t distribution $d.f = 10$, $n = 250$, 1000 repetitions

3 Estimation of Elliptical Distributions

In this chapter we present the multivariate density estimator based on the elliptical distributions. These distributions depend on a generator function which is one-dimensional and thus can well be estimated by applying non-parametric techniques. We closely follow the ideas developed by **Fan et al. (2012)**. The authors propose to combine a covariance matrix estimator developed above and the idea of **Liebscher (2005)** that guarantees good properties of the estimator in the neighborhood of zero.

3.1 Background of Elliptical Distributions

Let us start with spherical distributions that are closely related to the theory of elliptical distribution and understanding of which is necessary for some of further arguments.

For our purposes we use the definition of the spherical and elliptical distributions that can be found in **Fan et al. (2012)** (see definitions 1 and 2). More detailed explanations can be found in **McNeil et al. (2005)**. Consider a random vector Y with a dimensionality $(p \times 1)$. This vector is said to have a spherical distribution $S_p(\varphi)$ if its characteristic function which satisfies: $\phi_Y(t) = \varphi(t^\top t)$.

Example: if Y follows the standard multivariate normal distribution with uncorrelated components $Y \sim \mathrm{N_p}(0, \mathbf{I}_p)$ it has a sperical distribution because the characteristic function then looks like $\varphi_Y(t) = \mathrm{E}\{exp(it^\top Y)\} = exp(-\frac{1}{2}t^\top t)$

Random vector$Y$ with a dimensionality $(p \times 1)$ is said to follow an elliptical distribution $\mathrm{EC_p}(\mu; \Sigma; \varphi)$ with $\mu(p \times 1)$ and $\Sigma(p \times p)$ and $\mathrm{rank}(\Sigma) = k$ if Y has the same distribution as $\mu + A^\top Z$, where Z follows a spherical distribution $Z \sim S_k(\varphi)$ and $A(k \times p)$ is a a matrix such that $A^\top A = \Sigma$ (see **Fang et al. (2002)** for more details on elliptical distributions). If Σ is not a full rank matrix ($\mathrm{rank}(\Sigma) \neq p$), the density of the elliptical distribution doesn't exist (see **Hult and Lindskog (2002)** for the general case):

$$f_Y(Y) = |\Sigma|^{-1/2} g\{(Y - \mu)^\top \Sigma^{-1} (Y - \mu)\} \tag{3.1}$$

Note that a generator function $g(\bullet)$ often depends on the number of dimensions. For example, for normal distribution $g(r) = \frac{1}{(2\pi)^p} exp(-r/2)$.

Suppose that vector Y is elliptically distributed $Y \sim \mathrm{EC_p}(\mu; \Sigma; \varphi)$, then there exist S, R and A such that (see **Fang et al. (1990)** and **Hult and Lindskog (2002)**):

$$Y = \mu + RA^\top u \tag{3.2}$$

where

- S is uniformely distributed on the unit sphere $s \in \mathbf{R} : \|s^\top s\| = 1$
- $R \geq 0$ is a random variable independent of S
- $A^\top A = \Sigma$

It can be shown that the distribution of R^2 is closely connected to the distribution of Y:

$$\mathcal{L}(Z^\top Z) = \mathcal{L}\{(Y-\mu)^\top \Sigma^{-1}(Y-\mu)\} = \mathcal{L}(R^2) \tag{3.3}$$

R has a following density:

$$g_R(r) = 2 s_d r^{p-1} g(r^2), s_d = \frac{\pi^{p/2}}{\Gamma(p/2)} \tag{3.4}$$

which is closely connected to the density function of R^2

$$g_R^2(r) = \frac{1}{2\sqrt{r}} g_R(\sqrt{r}) = s_d r^{p/2-1} g(r) \tag{3.5}$$

Thus, if $g_R^2(r)$ is known, we can also derive the function $g(r)$

$$g(r) = s_d^{-1} r^{1-p/2} g_R^2(r) \tag{3.6}$$

The derivations above provide an idea about the estimation procedure. After obtaining the estimators of mean and covariance matrix, one can estimate the distribution function of R^2 which can be further transformed to get an estimator of the generator function $g(r)$. This generator function can be used to obtain the multivariate density. However, during such an estimator may be assosiated with some difficulties.

3.2 Liebscher Transformation

As in the paper of **Fan et al. (2012)** we apply the idea first proposed in **Liebscher (2005)** that provides some useful techniques to estimate $g(r)$ non-parametrically and allows to deal with some potential problems that may arise in the neighborhood of 0.

The problem may occur if the function of interest $g(r) \to \infty$ for $r \to 0$ if the estimator $\widehat{g}_R^2(r)$ is bounded away from 0 in the in the neighborhood of 0. **Liebscher (2005)** proposes to use the additional function $\psi : \mathbf{R}^+ \to \mathbf{R}$ that meets follwoing requirements:

- has a derivative $\psi'(x) > 0$ if $x \geq 0$, bounded on $(0, \infty)$
- $\psi''(x)$ is bounded on $(0, \infty)$
- $\psi(0) = 0$.
- $(d + 1)$
- has an inverse function Ψ
- $\lim_{x \to 0} x^{-p/2+1} \phi'(x) h\{\phi(x)\}$ is a positive constant
- $\lim_{x \to \infty} \psi(x)/x = const$

Then the density h of $\xi = \psi((Y - \mu)^\top \Sigma^{-1}(Y - \mu))$ is connected to the generating function $g(r)$ in the following way:

$$h(t) = \Psi'(t) g_R^2 \Psi(t) = s_d \Psi'(t) \Psi(t)^{p/2-1} g(\Psi(t)) \tag{3.7}$$

and

$$g(r) = s_d^{-1} r^{-p/2+1} \psi'(r) h\{\psi(r)\} \tag{3.8}$$

Liebscher (2005) provides an example of such function that meets all the necessary criteria and will be further used in all applications and simulations presented in this paper:

$$\phi(x) = -a + (a^{p/2} + x^{p/2})^{2/p} \quad a = const > 0 \tag{3.9}$$

3.3 Estimation Procedure

Combining the idea of semi-parametric estimation of density with Liebscher transformation and the idea about FFL covariance matrix estimator we can derive a semi-parametric estimator of the density $f_Y(Y)$ following several steps:

1. Estimate covariance matrix employing the idea of **Fan et al. (2008)** $\widehat{\Sigma}_n$
2. Estimate kernel density of transformed variables

$$\widehat{h}_n(x, \omega_n; \widehat{\Sigma}_n) = \frac{1}{n\omega_n} \sum_{i=1}^{n} [\kappa\{(x - \widehat{\xi}_i)\omega_n^{-1}\} + \kappa\{(x + \widehat{\xi}_i)\omega_n^{-1}\} \tag{3.10}$$

3. Transform the resulted density to obtain estimator of $g(r)$

$$\widehat{g}_n(r; \widehat{\Sigma}_n) = s_d^{-1} r^{-p/2+1} \psi'(r) \widehat{h}_n(x, \omega_n; \widehat{\Sigma}_n) \tag{3.11}$$

4. Get estimaton of the density of multivariate elliptical distribution

$$\widehat{f}_Y(Y;\widehat{\Sigma}_n) = |\widehat{\Sigma}_n|^{-1/2}\widehat{g}_n\{(Y-\mu)^\top\Sigma^{-1}(Y-\mu);\widehat{\Sigma}_n\} \tag{3.12}$$

The following notations are used:

- ω_n - bandwidth such that
  - $C_1 b(n) \leq \omega_n \leq C_2 b(n)$ where $C_1, C_2 = const > 0$ and $\{b(n)\}_{n=1,2,\ldots}$ is a sequance of positive real numbers
  - $\lim_{n\to\infty} \log\{\log(n)\} = 0$ and $b(n) \leq C_3 n^{-1/5}$ where $C_3 = const > 0$
- $\kappa : \mathbf{R}^+ \to \mathbf{R}$ - kernel function that satisfies several conditions
  - vanishes outside the interval [-1;1]
  - has derivative on $\mathbf{R}$ which is Liepschitz continuous
  - $\int_{-1}^{1} \kappa(t)dt = 1$
  - $\int_{-1}^{1} t^k\kappa(t)dt = 1 \quad \forall k = 1,\ldots,p-1$

Further we always use Silverman's rule of thumb to calculate the value of bandwidth (bandwidth can be further optimized, see **Härdle et al. (2004)** for more details):

$$\omega_n = 1.06\sqrt{\mathrm{Var}(\widehat{\xi})n^{-1/5}} \tag{3.13}$$

As a kernel density function Epanechnikov kernel is used:

$$\kappa(u) = \frac{3}{4}(1-u^2)\mathbf{I}(|u| \leq 1) \tag{3.14}$$

If all the conditions above as well as the conditions for the function $g(r)$ hold, the authors claim that the estimated density converges to the true one (proofs can be found in **Fan et al. (2012)**).

3.4 VaR for Elliptical Distributions

One of the important spheres of risk management is the calculation of Value at Risk(or VaR) of the portfolion which is defined as the maximum loss that this portfolio can bear over a specified time horizon with a given probability. This risk measure though not lacking in critic is widely used to estimate risks. A more thorough description of VaR as well as computation procedures can be found in **McNeil et al. (2005)**.

When dealing with a linear portfolio(i.e a portfolio returns of which can be presented as a linear function of returns of its components) a Delta-Normal approach that assumes that underlying multivariate distribution is normal is a wide-applicable technique. Note that if a linear approximation is a poor fit, higher order approximations may be employed but the portfolio we deal with can be easily approximated, so we consider only the linear approximation. Similar procedures can also be applied for other representatives of the elliptical distributions family. e.g. Student t distribution.
A logical step ahead is to generalize this framework for elliptical distributions as a whole which is quite a natural expansion of the Delta-Normal model. The resulting model may be expected to provide better fit that the Gaussian distribution because of its flexibility but is still much faster as non-parametric techniques.

The generalization of Delta-Normal VaR for elliptical distributions can be found in **Kamdem (2005)** and the presented derivations follow this paper.

Consider a linear portfloio which value varies with time t $\Pi(t)$ Due to the linearity assumption the profit and loss function can be expressed in the following way:

$$\Delta\Pi(t) = \Pi(t) - \Pi(t-1) = \delta_1 X_1 + \ldots + \delta_p X_p(t) \tag{3.15}$$

where

- $X = (X_1, \ldots, X_p)$ are profit or losses of the portfolio components which are assumed to be elliptically distributed

$$(X_1, \ldots, X_p) \sim \mathrm{EC_p}(\mu_X; \Sigma_X; \phi_X) \tag{3.16}$$

- $\delta = (\delta_1, \ldots, \delta_p)$ are weights of the constituents

As was already noted if the corresponding covariance matrix if of full rank the density function exist and looks as defined in the Equation 3.1. If $g(r)$ is continuous, integrable and non-zero everywhere, the Value at Risk at the confidence level $1-\alpha$ is defined as

$$\mathrm{P}\{\Delta\Pi(t) < -VaR_\alpha\} = \alpha \tag{3.17}$$

When dealing with elliptical distribution equation above can be transformed as follows:

$$\alpha = |\Sigma_X|^{-1/2} \int_{(\delta x \leq -VaR_\alpha)} g\{(x-\mu_X)^\top \Sigma_X^{-1}(x-\mu_X)\}\, dx \tag{3.18}$$

Solving this equation yields (see Theorem 2.1 in **Kamdem (2005)**):

$$VaR_\alpha = -\delta\mu_X + q_{\alpha,p}^{g_X}\sqrt{\delta^\top \Sigma_X \delta} \tag{3.19}$$

where $s = q_{\alpha,p}^{g}$ is a the unique positive solution for transcendental equation $\alpha = G(s)$:

$$G(s) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_s^\infty \int_{z_1^2}^\infty (u - z_1^2)^{\frac{n-3}{2}} g_X(u)\, du\, dz_1 \tag{3.20}$$

Note that the formula strongly reminds the one for a delta Normal VaR that looks like:

$$VaR_\alpha = -\delta\mu_X + z_\alpha\sqrt{\delta^\top \Sigma_X \delta} \tag{3.21}$$

where z_α is a corresponding quantile of a standard normal distribution. Thus, $q_{\alpha,p}^{g_X}$ can be treated as quantiles of a standardized elliptical distribution that drives the returns.
It should also be mentioned that VaR in both formulas depend on the $\sqrt{\delta^\top \Sigma \delta}$ that has a clear financial interpretation as it represents the volatility of the portfolio.

As in the previous section, we assume that returns of the assets that composite the portfolio and not their profits follow the elliptical distribution and thus the formulas above should be modified. We now assume that portfolio is linear on terms of the returns. This assumption is reasonable when dealing with daily returns because of their small values.

$$R_\Pi(t) = \frac{\Delta\Pi(t)}{\Pi(t)} \approx \delta_1 Y_1 + \ldots + \delta_p Y_p(t) \tag{3.22}$$

where $(Y_1, \ldots, Y_p) \sim \mathrm{EC_p}(\mu; \Sigma; \phi)$ are the returns of the components that follow an elliptical distribution. The final formula for the portfolio then can be rewritten as follows:

$$VaR_\alpha = (-\delta\mu + q_{\alpha,p}^{g}\sqrt{\delta^\top \Sigma \delta})\Pi(t) \tag{3.23}$$

4 Simulation Study

In order to evaluate the quality of the density estimator developed in the chapter 3, we conducted a simulation study. It was based on the study of **Fan et al. (2008)** and designed as we explained in chapter 2, however, we extended it substantially in order to access not only the issues of covariance matrix estimator, but also those of the density estimator. Unlike in the chapter 2, we limited ourselves with the only case based on the normal distribution and didn't explore the properties of other distributions. The number of observations is set $n = 250$ and we keep the underlying factor model with $K = 3$.

4.1 Example

First, let's take a look at an example of a single simulation to get more insight about the drivers of estimators efficiency(see Figure 4.1). Two typical cases of a single simulation are provided: first is an example for a small dimensionality ($p = 5$), while the second one deals with a highly dimensional environment ($p = 150$). We present two function that characterize the estimation results: $\log\{\widehat{g(r)}\}$ and $\widehat{g_R^2}(r)$ (see Equations 3.5 and 3.6 about the connection between the two). On the each graph five lines are presented depending on the covariance matrix estimator used and whether the Liebscher transformation was applied. Concentrate on the Figure 4.1 (a) and (b) first. An interesting(although expected) finding can be observed: for small value of p the fact of Liebscher transformation does matter ($\widehat{log\{g(r)\}} \to \infty$ if it's not employed) while the covariance matrix estimator doesn't contribute so much(results yielded for both types of estimators are identical). However, the results are opposite for big values of p. If $\log\{\widehat{g(r)}\}$ was obtained with the FFL covariance estimator, the fit is much more accurate that for the estimation made with the sample estimator. However, the Liebscher transformation loses its importance. The observation regarding the changes of the relative merit of covariance matrix estimator can be easily explained by our previous results(see Figures and 2.2) while the estimation error for the sample covariance matrix is low and comparable with that of FFL estimator for low dimensionality. In order to get the intuition behind the importance of Liebscher transformation a little bit more effort is required.
Figure 4.1 (c) and (d) shows the estimation of $\widehat{g_R^2}(r)$. This function represents the distribution of $\{(Y-\mu)^\top \widehat{\Sigma}^{-1}(Y-\mu)\}$ that we basically estimate and the transform it to get $\widehat{g(r)}$. This is a distribution of a quadratic form which is closely connected to the χ^2. For $p = 5$ there are some observations in the neighborhood of 0 that can be observed. Thus, the esti-

mation without the Liebscher transformation bears a potential problem that may result into $\widehat{g(r)} \to \infty$ as we discussed in above.

However, when increasing the number of dimensions the number of observations in the neighborhood of 0 is usually also 0 which implies $\widehat{g_R^2(r)} = 0$ (although the theoretical values approach 0 but are never reaches it). So, the problem of the correct estimation of $\widehat{g_R^2}(r)$ 0 is bounded away from 0 in the neighborhood of 0 is not really a matter of concern. We should expect that for high dimensionality that we are interested in for most standard distributions Liebscher transformation is unlikely to provide a considerable improvement.

4.2 Estimation of $f_Y(Y)$

Define the norm L^v that measures the closeness of two distributions in the following way:

$$L^v = [\int_{-\infty}^{\infty} (f(x) - \widehat{f(x)})^v dx]^{1/v} \quad v \in \mathbf{R} \tag{4.1}$$

The multidimensional integration is often not easy to handle. In order to estimate the integral over such a multidimensional function, we apply a Monte Carlo integration with importance sampling.

4.2.1 Methods: Monte Carlo integration with importance sampling

This section heavily relies on the review of Monte Carlo methods by **Weinzierl (2000)**. Suppose we are interested in the value of the following:

$$V = \int f(x)dx \tag{4.2}$$

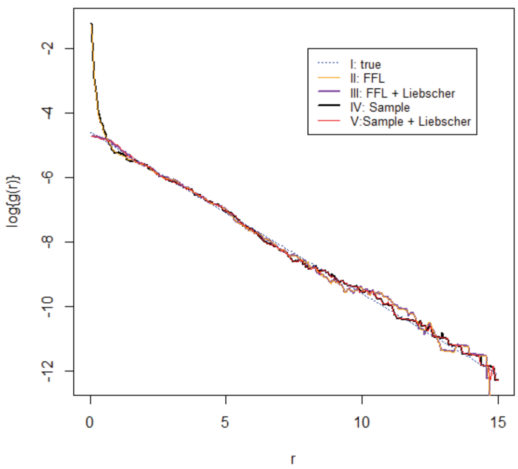
According to the mathematical rules of integration we can change variable so that:

$$V = \int f(x)dx = \int \frac{f(x)}{p(x)} p(x)dx = \int \frac{f(x)}{p(x)} dP(x) \tag{4.3}$$

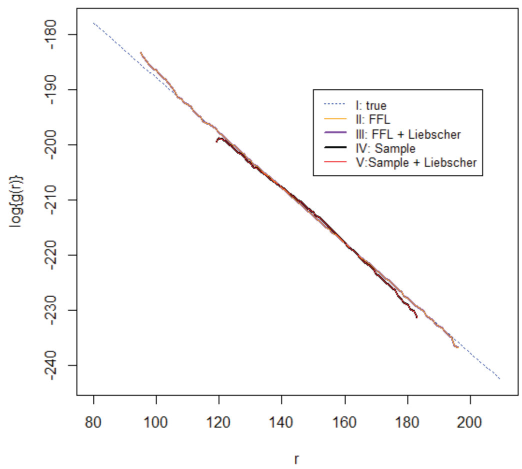
where $P(x) = \frac{\partial^p}{\partial x_1 ... \partial x_p}$ We can treat $p(x)$ as a probability density function if:

1. $p(x) \leq 0$

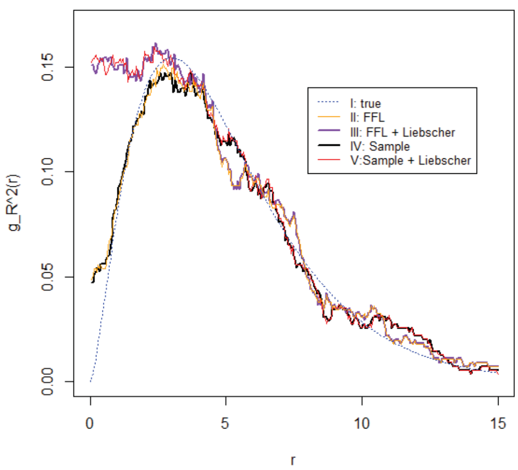
2. $\int p(x)dx = 1$



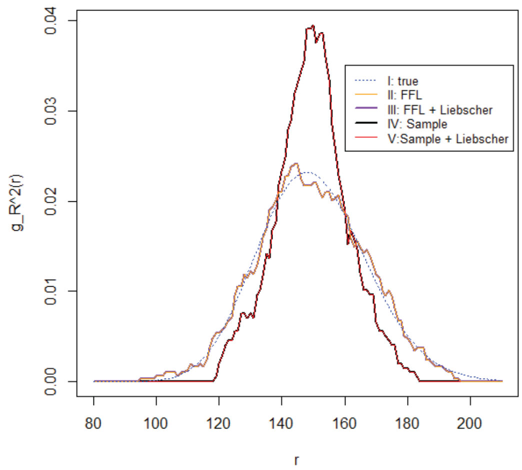
(a)



(b)



(c)



(d)

Figure 4.1: Example for $\log\{\widehat{g(r)}\}$ and $\widehat{g_R}^2(r)$

(a) and (b): $\log\{\widehat{g(r)}\}$ against r for $p = 5$(a) and $p = 100$(b), $n = 250$; (c) and (d): $\widehat{g_R^2}(r)$ against r for $p = 5$(a) and $p = 100$(b), $n = 250$

EllDistrExample

If we can generate a random sample of size M using the density $p(x)$, we may estimate an integral of interest from a sample of random numbers:

$$\int f(x)dx = \int \frac{f(x)}{p(x)}dP(x) \approx \sum_{m=1} M\frac{1}{M}\frac{f(x_m)}{p(x_m)} \tag{4.4}$$

It can be shown that the error is the given by:

$$\sigma(f/p)/\sqrt{M} \tag{4.5}$$

Estimator of the variance $\sigma^2(f/p)/$ can be found as follows

$$\widehat{\sigma}^2(f/p) = \frac{1}{M}\sum_{m=1}^{M}(\frac{f(x_m)}{p(x_m)})^2 - V^2 \tag{4.6}$$

Formula above shows us that variance can be significantly reduced only if the function $p(x)$ is well-chosen. Actually, in order to yield good results, $f(x)/p(x)$ should be a slowly-varying function. So, one should choose $p(x)$ that as closely mimics the shape of $f(x)$ as possible.

This method is beneficial in comparison with other Monte Carlo techniques as it allows to considerably reduce the number of random points needed to estimate the integral if the function of interest has large values in some area, so one can assume that points in this area contribute to the value of the integral more than others. A function of Gaussian distribution can serve a a good example of such a function because of its high peak.

However, we should also be aware of the hidden pitfalls of this method. As mentioned above, the key to success is the right choice of function $p(x)$. If this function becomes 0 (or goes to 0) where the function of interest is relatively large, the variance may become infinite and no reasonable estimate can be derived.

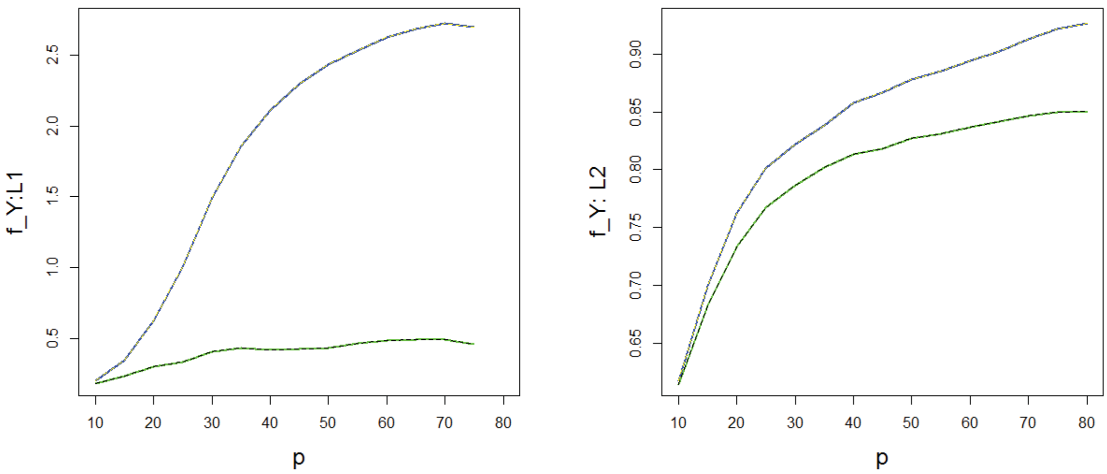
We are interested now in the L^v norm that defines the closeness of two distributions, so we can assume that this function takes the largest values in two peaks of the corresponding distributions. So, choosing $p(x)$ as an equally-weighted mixture of two corresponding distributions should be a good idea. However, it may be difficult to generate random variables that correspond to the estimated distribution. As we generated the original values, we know that the underlying distribution is always normal. So, the suggestion is to use a mixture of two normal distributions, one of which corresponds to the true values of parameters(which are known) $N(\mu, \Sigma)$ and the second corresponds to the estimated values $N(\widehat{\mu}, \widehat{\Sigma})$. The value of random points drawn from this distribution is fixed $M = 10^4$. We expect that the density chosen by this technique should provide stable results.

4.2.2 Results

Using the Monte Carlo integration with importance sample described above, we can calculate the L^1 and L^2 norms for the original density of the distribution $f_Y(Y)$ in order to judge the goodness of the semi-parametric estimation technique presented in this paper. These results are presented on the Figure 4.2.

First, we can see that FFL covariance estimator significantly outperforms the benchmark estimator. The $L1$ norm for the FFL estimator is relatively stable in time, while the one calculated for the covariance estimator demonstrates a high growth. Moreover, the estimated value of the norm for the latter is well over the value of 2 which is theoretically highest possible when we deal with two density functions. This is due to the fact that the value of integral of density function should be equal to 1. This error is originated by the properties of the covariance matrix determinant that significantly declines with growth of p in comparison with the true one. However, we are interested in the determinant directly while the density of elliptic distributions is proportional to the inverse of the square root of the determinant (see Equation 3.1). This implies that if the determinant declines exponentially it may lead to the uncontrollable growth of the density function so that it's integral is no longer equal to 1. This shows that the FFL estimator is clearly beneficial in the context of elliptical density estimation in comparison with the benchmark model.

It should also be noted that for such high values of p as we are interested in, no difference between estimator with and without Liebscher transformation can be spotted (their values coincide completely). The reason for that was discussed above: no points in the neighborhood of 0 eliminate the necessity to control for the density function in this area. However, this result can be only extrapolated with great caution as it is dependent on the form of the underlying distribution. Although we can stipulate the invariance of the estimation with respect to the Liebscher transformation in case of high dimensionality if the normal distribution is employed, some other distributions may be more sensitive.



(a) (b)

Figure 4.2: L^1(a) and L^2(b) norms for $\widehat{f_Y}(Y)$

(a) and (b): L^1 and L^2 norms for $\widehat{f_Y}(Y)$ for sample covariance estimator $\widehat{\Sigma}_sam$ with (solid blue curve) and without Liebscher transformation (dashed yellow curve) and FFL covariance matrix estimator $\widehat{\Sigma}_n$ with (solid green curve) and without Liebscher transformation (dashed black curve) plotted against dimensionality p, $n = 250$, 1000 repetitions. $M = 10^4$

EllDistrFy

Figure 4.2(b) illustrates the L^2 norm for the $\widehat{f_Y}(Y)$. In order for the norms to be comparable over the different number of dimensions p we corrected the resulted values as the following:

$$L^2 = \left\{ \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} (f(x) - \widehat{f(x)})^2 dx \right\}^{1/(2p)} \tag{4.7}$$

This correction is necessary due to the properties of the density function, namely the fact that it's integral should be equal to 1.

Although the results for L^2 norm don't as vividly demonstrate the superiority of FFL estimator as those of the L^1 norm, two important conclusions are valid:

1. FFL estimator outperforms the benchmark model

2. Liebscher transformation doesn't improve the results for the normal distributions in case of high dimensionality

4.2.3 Liebscher Transformation

In order to judge about the importance of the Liebscher transformation for the estimation of the high-density distributions, we provide an estimation of the L^1 norm for small values of dimensionality p (see Figure 4.3).

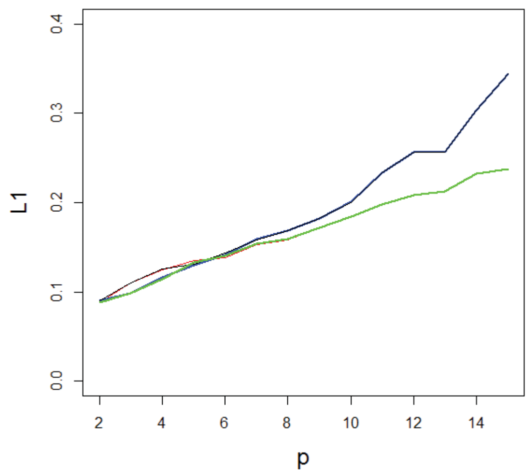


Figure 4.3: L^1 norm for $\widehat{f_Y}(Y)$ for small number of dimensions

L^1 norm for $\widehat{f_Y}(Y)$ for sample covariance estimator $\widehat{\Sigma}_{sam}$ with (blue curve)) and without Liebscher transformation(black curve) and FFL covariance matrix estimator $\widehat{\Sigma}_n$ with(green curve) and without Liebscher transformation(red curve) plotted against dimensionality p, $n = 250$, 500 repetitions. $M = 10^4$

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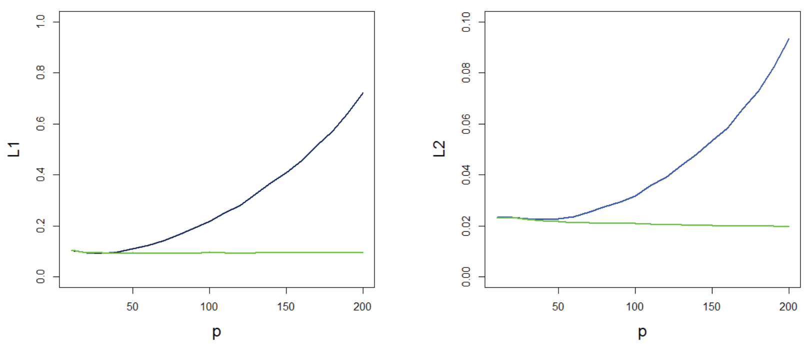
We can conclude that for normal distribution and probably for some other similar distributions(though these results can be extended with great caution only), the Liebscher transformation provides a significant improvement only until the value of dimensionality $p = 6$. After this value is reached there are not enough points close to the values of 0 that are needed to estimate the underlying density $g_R^2(r)$, so we can't estimate $g(r)$ in this area other than 0. This is definitely a weakness of the method used but this weakness makes the problem of estimation of $g(r)$ in the neighborhood of 0 nonessential. However, it must be noted that this result may not hold for other types of distributions, so application of Liebscher transformation is a good way to hedge against the possible problems when having only a vague idea about the underlying distribution.

Surprisingly enough Figure 4.3 also demonstrates that application of the FFL estimator performs better than the benchmark model starting from the values $p = 6$ or $p = 7$. This result should also be only sceptically extrapolated. The design of the simulation study assumes a certain factor model that dives the returns. Moreover, the model specification is exactly known. Although the simulation is based on the real factor loadings and errors estimation, it still imposes such restrictions as stationarity, no change of the underlying model, normality, etc. However, we can see that this estimator is useful even for a relatively modest number of dimensions, so it should at the very least be considered as an alternative when aiming at a density estimation.

4.3 Estimation of $g_R^2(r)$

We estimated the L^1 and L^2 norms for a univariate distribution $g_R^2(r)$ estimation of which serves as a first step towards the estimation of the multivariate density $f_Y(Y)$. The obtained norms serve largely as a support of the ideas impressed in the previous section. We can clearly see that the estimate of the underlying distribution of the quadratic form $\{(Y-\mu)^\top\Sigma^{-1}(Y-\mu)\}$ directly depends on the inverse covariance matrix estimator. According to the Figure 4.4 FFL estimator clearly outperforms the benchmark model even on the first step of the density estimation procedure. Much to our regret, we have to report that the Liebscher transformation plays no significant role for the presented values of dimensionality p.



(a) (b)

Figure 4.4: L^1(a) and L^2(b) norms for $\widehat{g_R^2}(r)$

(a) and (b): L^1 and L^2 norms for $\widehat{f_Y}(Y)$ for sample covariance estimator $\widehat{\Sigma}_s am$(blue curve) and FFL covariance matrix estimator $\widehat{\Sigma}_n$(green curve) plotted against dimensionality p, $n = 250$, 500 repetitions.

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5 Data

In order to illustrate one of possible applications of the method of semi-parametric density estimation an example of the estimation the S&P500 components joint multivariate distribution is developed.

The main data source for the information about the dynamics of index components is Bloomberg database, we thank the Research Data Center of Collaborative Research Center 649: Economic Risk for the provided access.

The data about the risk factors was obtained from the official website of Kenneth French `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`.

5.1 Risk Factors

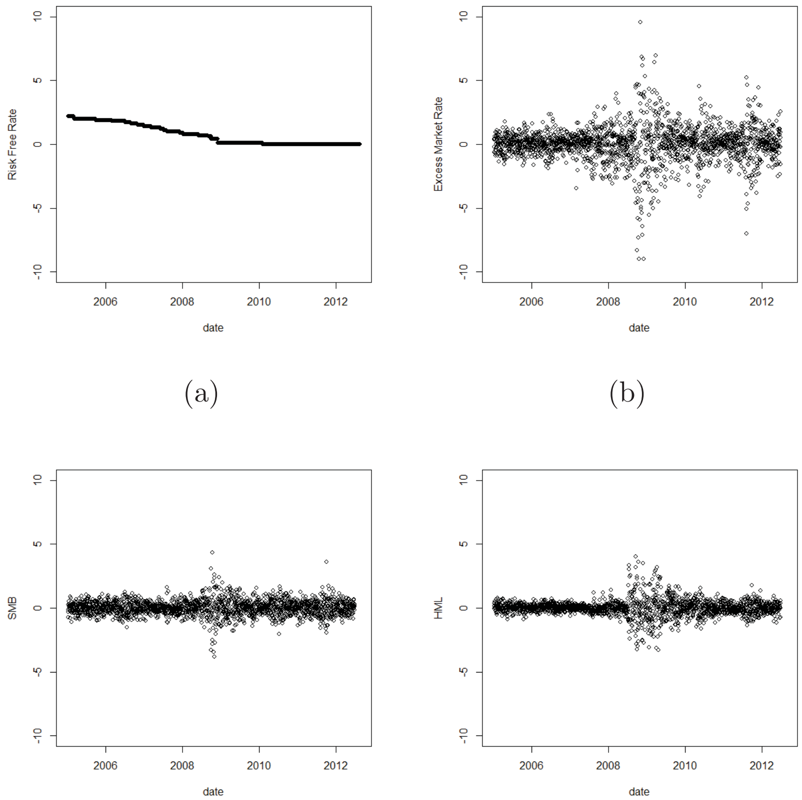
We used daily observations describing the risk factors returns that cover the period from 06 January 2005 to 29 June 2012 (which makes 1881 data points).

For the construction of market risk return covers all firms listed on NYSE, AMEX, and NASDAQ. SMB and HML include all NYSE, AMEX, and NASDAQ stocks that were listed. Information about July of period t to the end of June $t+1$ was based on the stock returns that were listed from December $t-1$ to the end of June t. Both portfolios are quaterly rebalanced. Risk free rate is determined on the basis of the 1 month Treasury bill. HML and SMB factors are defined according to the Equations 2.9 and 2.8 . The size breakpoint is the median NYSE market equity. The ratio of book equity to market equity breakpoints are the 30th and 70th NYSE percentiles. Refer to the website of Kenneth French and to the paper**Fama and French (1993)** for more details about the construction of all portfolios and additional data. The descriptive statistics of the risk factors and risk free rate can be found in Table 5.1

Figure 5.1 illustrates the returns of risk factors. We can observe a gradual decline of the only factor which is controlled by the Federal Reserve System, namely, the risk free rate. During the years considered it was actually pushed to 0 that demonstrates an "easy-money" policy implemented by the US government in the beginning of the considered period and attempts to push the economy out of the crisis in the period after 2008.

Variable	Min	Mean	Max	SD	Skewness	Kurtosis
R_f	0.00	0.75	2.2	0.78	0.48	-1.38
R_m	-9.00	0.02	11.52	1.44	-0.14	8.34
HML	-3.8	0.00	4.31	0.60	0.03	5.05
SMB	-3.32	0.00	4.00	0.63	0.32	7.26

Table 5.1: Summary statistics of daily retursn of Risk Factors and risk free rate over a period 2005/01/06 - 2012/06/29



(a) (b)

(c) (d)

Figure 5.1: Daily retuns of risk factors for a period 2005/01/06 - 2012/06/29

Daily returns of risk factors in % for FF3: (a) Risk Free Rate, (b) Market Risk Rate, (c) SMB and (d) HML

EllDistrData

Market risk is clearly the most volatile of the factors. We can easily spot the periods of financial disturbances with the major one representing the financial crisis starting from 2007 and reaching its peak in the end 2008. The other periods of distress during the financial recession 2008-2012 can be found in the middle of 2010 and in the end of 2011.
The patterns of SMB and HML portfolios are similar to the one demonstrated by the market portfolio though less volatile. It should also be noted that all three risk factors are evidently leptokurtic.

5.2 S&P500 Portfolio

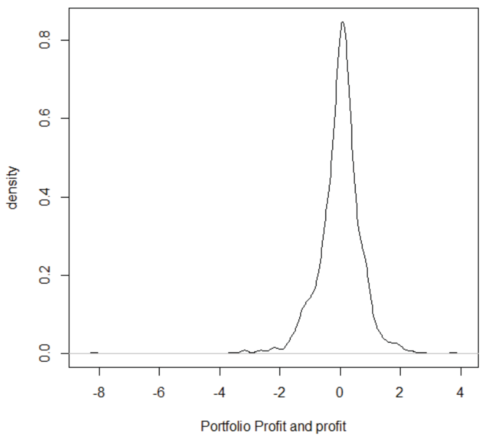
We constructed a portfolio of the components of the S&P500 index. All of them are supposed to be equally weighted $\delta = (1/\mathrm{p}, \dots, 1/\mathrm{p})$ because we concentrate on a long period of time with a plethora of radical changes taking place, so in order to keep the portfolio constant over time we refused from the use of market-value-weighted constituents. The S&P500 index is regularly rebalanced, so we used a snapshot of constituents on the date of 15 August 2012. All the components for which the data was unavailable for some of the data points were excluded. The final portfolio thus consists of $p = 459$ assets the list of which may be found in the appendix.
Further we always use a rolling window approach with a fixed size of the window $n = 750$ that corresponds to 3 years.
Thus, we are interested in that returns of the portfolio that follow the first 3 years of observations, so we provide data for 1331 points for daily returns over a period 2008/01/07 - 2012/06/2.
We provide descriptive statistics for two variables:

- Profit and Loss

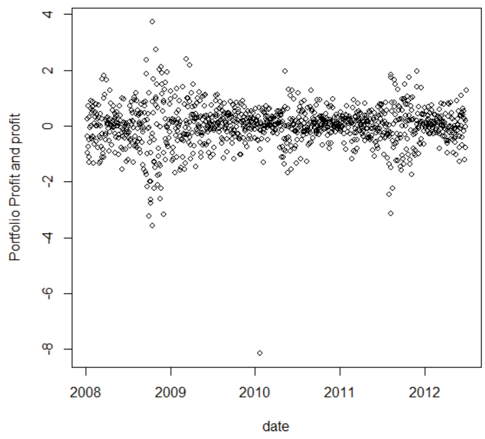
$$\Delta\Pi(t) = \Pi(t) - \Pi(t-1) = \Delta\Pi(t) \tag{5.1}$$

- Returns

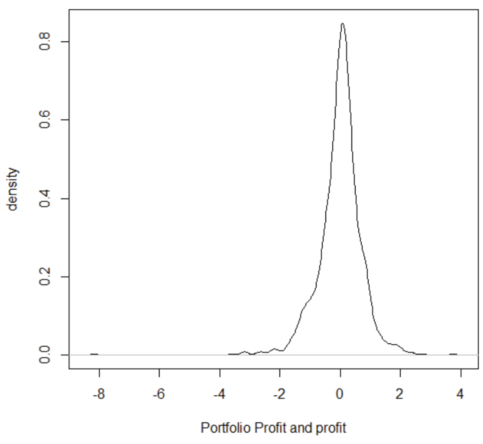
$$R(t) = \frac{\Pi(t) - \Pi(t-1)}{\Pi(t-1)} \tag{5.2}$$



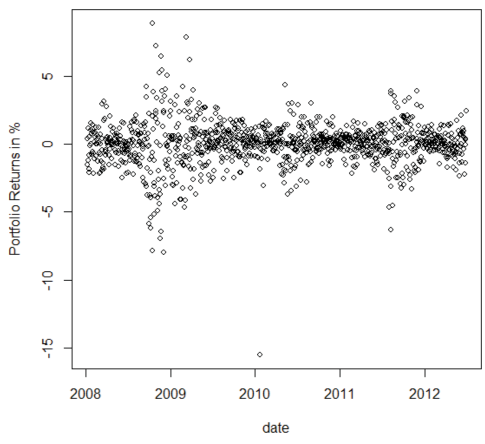
(a)



(b)



(c)



(d)

Figure 5.2: Daily retuns of the S&P500 portfolio for a period 2008/01/07 - 2012/06/29

(a) and (b): Profit and Loss of the S&P500 portfolio as a scatter diagram and kernel density estimation($bandwidth = 0.11$), (c) and (d) Daily returns of the S&P500 portfolio in % as a scatter diagram and kernel density estimation($bandwidth = 0.22$)

EllDistrData

Variable	Min	Mean	Max	SD	Skewness	Kurtosis
Profit and Loss	-8.16	-0.00	3.75	0.75	-1.45	14.28
Returns	-15.54	-0.00	8.93	1.63	-0.89	10.92

Table 5.2: Summary statistics of daily retuns of the S&P500 portfolio for a period 2008/01/07 - 2012/06/29

As follows from the Table 5.2 and Figure 5.2 the returns of the portfolio are leptokutic and fat-talied. We can also observe the bursts of volatily due to the financial distrubances in 2008-2012. Comparing the standard deviations we can find that our portfolio is more volatile than the market portfolio. We also present the desriptive statistics for the profit and loss of the portfolio that will be further compared with VaR.

6 Empirical Study

The techniques of multivariate density estimation can be employed in many spheres. Here we present some results that demonstrate one of possible applications. Namely, the estimation of one of the key financial market indices S&p500 is studied.

6.1 Density Estimation

We use the data described in the previous chapter to fit the FF3 that allows deriving a covariance matrix estimator. Then we get the estimator of the multivariate density on the assumption that we deal with an elliptical distribution using the techniques presented in chapter 3. Due to the fact that we expect the returns to be not iid as assumed, we use a simple AR(1) model to account for a slightly negative autocorrelation that appears.

The following two figures represent the results of the estimation procedure. Figure 6.1 illustrates the estimated function $g(r)$ in terms of logs and compares it with a corresponding function for a Gaussian distribution. On the basis of the comparison of these two functions some important implications about the multivariate distribution that drives the returns can be derived.

It should be noted that the estimated function lies well above and looks more like a hyperbola than a straight line as implied by the normal distribution. This plot illustrates a wide-spread common knowledge: returns distributions have much more fatter tails than the normal distribution which makes the latter a poor model to employ when dealing with financial data. This result can not be originated by the distortions of the estimation procedure which is supported by the fact that the curve takes the same look independent on whether the Liebscher transformation is applied. Unfortunately, we can't present results for the sample covariance matrix estimator for the comparison purposes as the matrix turns out to be nearly singular and usually can't be inverted which is necessary for the estimation procedure. This fact, however, serves as one more advantage of the explored technique that allows producing an estimator when the possible alternative methods just can't be applied.

The dynamics of the estimated function illustrates a striking result. The form of the curve is the same for all the periods under consideration. Moreover, the distribution seems to be relatively stable over time. Some "breaks" in the graphs that appear for certain periods should be caused merely by the lack of data points in this area while they perfectly fit in the form of the curve, appear irregularly and always change their position without an evident pattern.

Figure 6.2 illustrates the estimation of $g_R^2(r)$ which is actually the density that according to the procedure is estimated non-parametrically(and also transformed if we apply Liebscher transformation) and serves as a basis for the estimation of $g(r)$. This function is roughly speaking a density of a sum of squared deviations from means. The estimated function provides a little bit more insight about the nature of the data. The figure also presents the corresponding function if the underlying distribution is supposed to be normal (which behaves very similar to χ^2 but the latter assumes that all variables are independent which is definitely not the case here).

One can observe that the peak for the estimation distribution is located a little bit to the left in comparison with that for a normal distribution. The curve is also much more flat putting more weights for extremely small and large values. It implies that events when all the returns are close to its means or when all of them are, on the opposite, very far from the means are more likely to happen in comparison with the probability implied by the normal distribution. If the change of the density with time is considered, it can be noted that it keeps its general form and all the characteristics described above apply for all periods studied. However, we can observe a clear pattern of the function behavior during different periods of market state. It approaches the normal distribution if the period is relatively "tranquil" (e.g the last years though are not very prosperous for the financial system can be treated as tranquil while not major financial disturbances were present). On the opposite, during the "crisis" and "recovery" periods(e.g. such years as 2008 and 2009 that are marked with an unfolding financial crisis) the facts listed above exacerbate: the curve becomes flatter, the peaks tends away from the "Gaussian" mean and surprisingly becomes lower. Thus the underlying distribution tends to be more fatter-tailed during the "crisis" period which is not an unexpected but still a nice fact to know.

We can also observe that the distribution unlike normal is significant even in the neighborhood of 0. It means that although the presence of the Liebscher transformation may not be of much value when dealing with distributions similar to Gaussian, it may well come into play when true distributions that significantly deviate from the theoretical ones appear. Although no substantial difference when Liebscher transformation is applied can be spotted for the data set studied, we can imagine that it would matter if the data was at a little bit different because already for such data an estimation around 0 is needed.

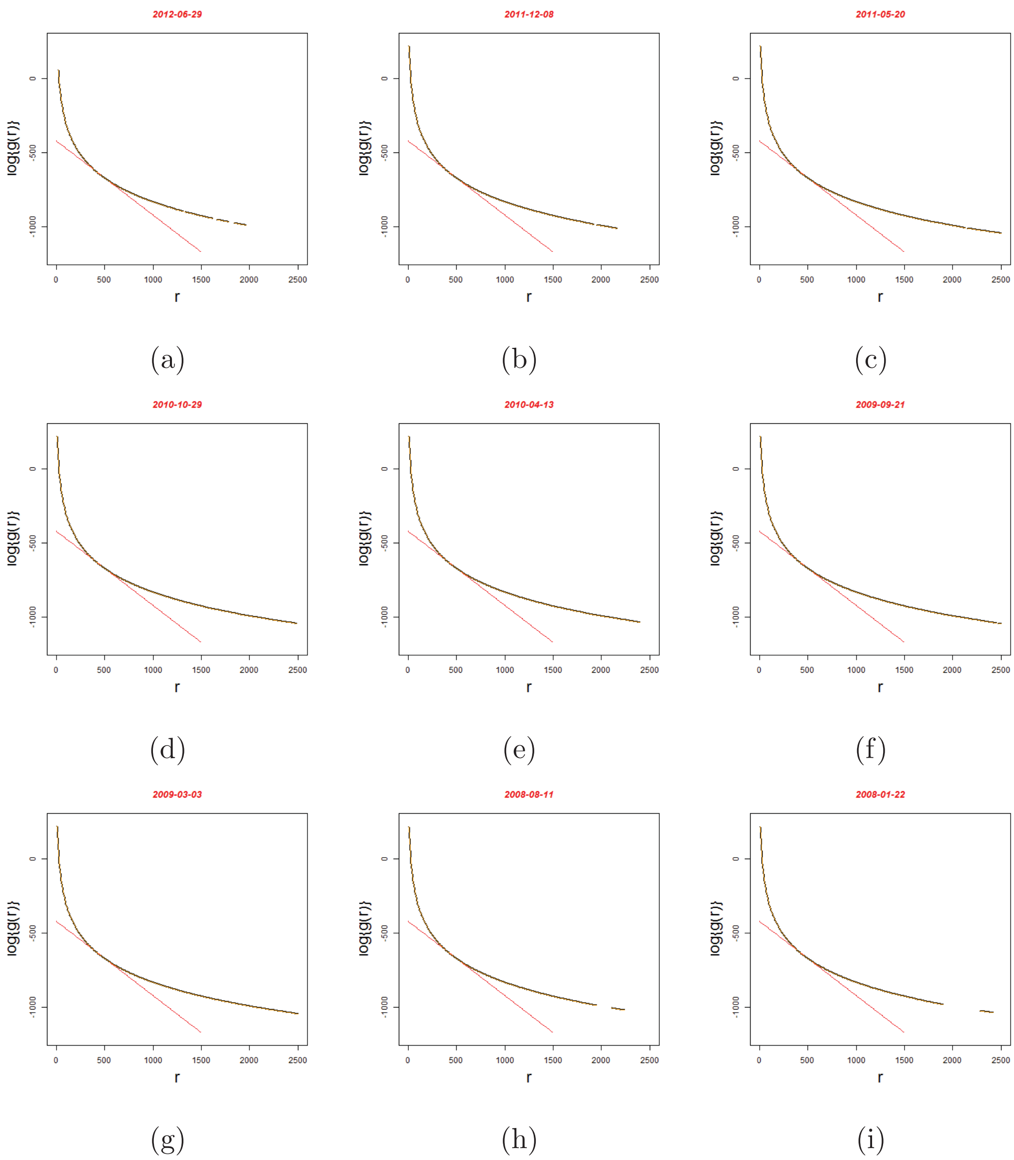


Figure 6.1: Estimated $\log(g(r))$ for S&P500

Normal distribution (red curve) and estimated with $\widehat{\Sigma_n}$ with Liebscher transformation (black curve) and without Liebscher transformation(orange curve) for daily returns with monthly interval, $n = 750$, p=459

EllDistrSP500logg

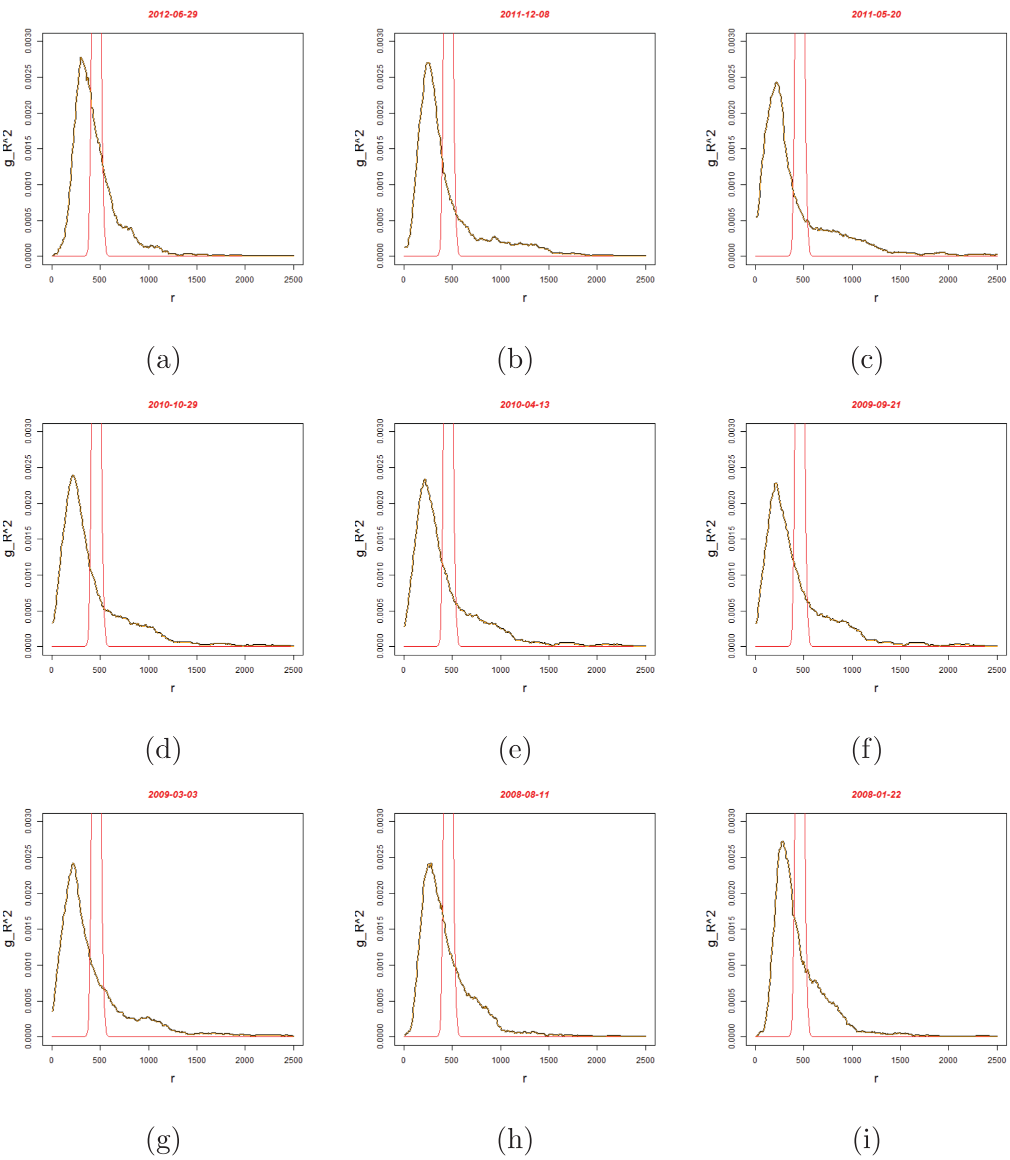


Figure 6.2: Estimated $g_R^2(r)$ for S&P500

Normal distribution (red curve) and estimated with $\widehat{\Sigma_n}$ with Liebscher transformation (black curve) and without Liebscher transformation(orange curve) for daily returns with monthly interval, $n = 750$, p=459

EllDistrSP500Gr2

6.2 VaR

Using the formulas in chapter 3 we also estimated a daily VaR for the corresponding periods using a rolling window approach with $n = 750$ and so we have 1131 data points for which VaR and the actual values of losses may be calculated.

6.2.1 Quantiles

The calculation of VaR for the elliptical distributions consists of 2 steps:

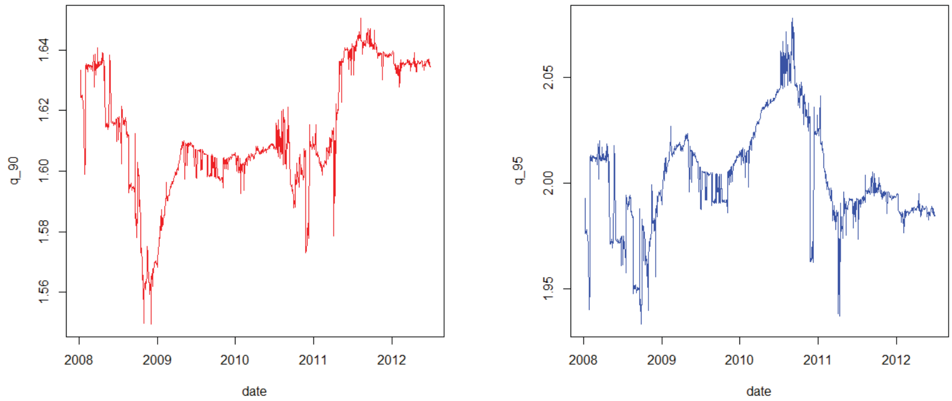
1. calculation of the standard quantile

2. calculation of VaR

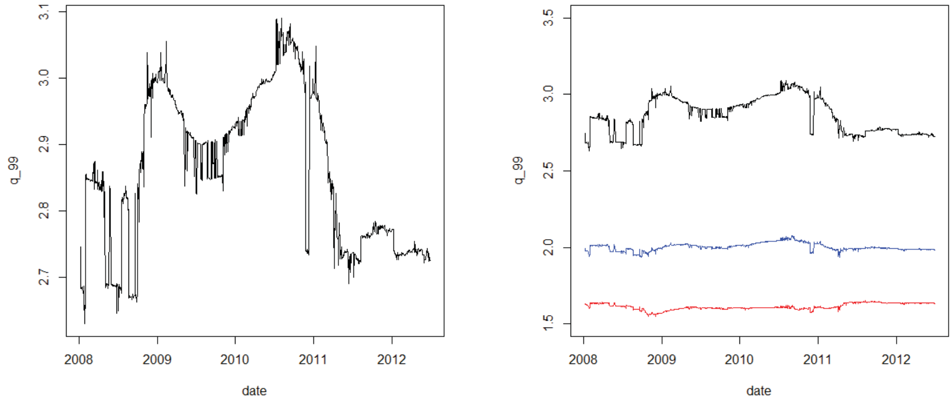
We concentrate ourselves first on the first step and take a closer look on the 90%, 95% and 99% quantiles (note that we deal with two-sided quantiles) that are represented on the Figure 6.3 and the descriptive statistics as well as benchmark quantiles for normal and Student t with 10 degrees of freedom distributions can be found in the Table 6.1. It should be noted that according to our calculations, 90% and 95% quantiles are relatively stable while the 99% percentile is clearly more volatile that the other two although it may be caused by an estimation error that appear on the extremes. The mean values suggest that the distribution we deal with posses very specific tails. The first 90% is low in comparison with those of the Gaussian and Student distribution. The 95% quantile approximately coincides with that of normal but is still lower than that of Student. Finally, the 99% quantile is clearly much higher and even exceeds the comparable value of the Student distribution. These findings mean that non of these two theoretical distributions captures the specific nature of the data we deal with because it represents some type of an intermediate between the two with much higher peak and longer tails than any of them.
The dynamics of the 95% and 99% quantiles can easily be interpreted: they definitely increase after the periods of financial distress, even though this patter is much more vividly expressed for the latter. The interesting observation, however, can be derived from the dynamics of the 90% quantile that not only doesn't follow the pattern that two others take, but moves exactly in the opposite direction. This surprising inconsistence can be intuitively explained by the fact that in "bad" times the probability of the extremely big losses is much bigger that the probability of moderate losses. Financially unstable periods can thus be seen as exacerbating all possible returns: "everything" or "nothing" with no open space for compromise.

Quantile	Min	Mean	Max	SD	Skewness	Kurtosis	Normal	Student t with $d.f. = 10$
90% quantile	1.55	1.61	1.65	0.02	-0.28	-0.34	1.64	1.66
95% quantile	1.93	2.00	2.07	0.02	0.25	0.25	1.96	2.23
99% quantile	2.63	2.86	3.09	0.12	0.00	-1.27	2.58	2.76

Table 6.1: Summary statistics of two-sided quantiles of the estimated multivariate distribution of daily retuns of the S&P500 portfolio for a period 2008/01/07 - 2012/06/29



(a) (b)



(c) (d)

Figure 6.3: Quantiles for the S&P500 portfolio over a period 2008/01/07 - 2012/06/29

(a) 90% quantile, (b): 95% quantile, (c) 99% quantile, (d) all quantiles together

EllDistrSP500VaR

6.2.2 VaR and back-testing

Using the values if the quantiles one can easily calculate the corresponding value of VaR for an elliptical distribution. Note that the following calculations are based on the fact that the portfolio can be linearly approximated. This approximation, however, should be treated as a plausible assumption when we deal with the portfolio which is just a sum of stocks and consider daily returns .

According to the convention profits are usually presented as positive values, losses - as negative values. It contradicts to some extent the convention to present VaR(which is actually a maximum possible loss) as a positive number and that's how we defined it before in Equation 3.23. In order to overcome this problem and get consistent results, we now present VaR as negative value. The results of the estimation are illustrated on the Figure 6.4. The numbers of exceedances compared with the theoretical values can be found in the Table 6.2.

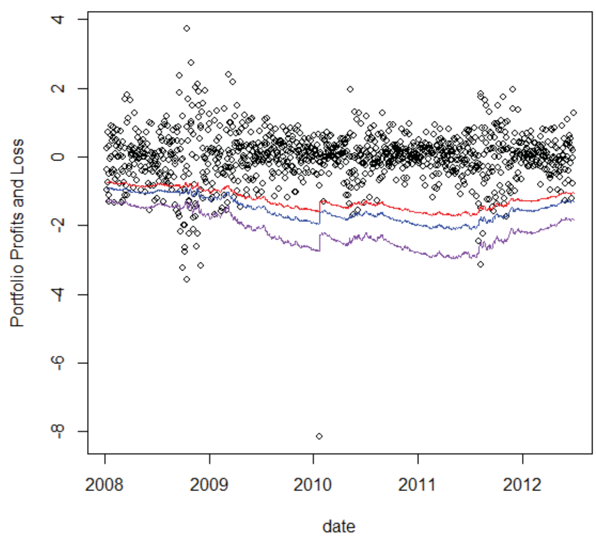


Figure 6.4: VaR for the S&P500 portfolio over a period 2008/01/07 - 2012/06/29

EllDistrSP500VaR

At the first glance, the results are satisfactory for the 90% quantile only. For 95% and 99% quantiles the number of exceedances is definitely too large. The more precise examination of the Figure 6.4, however, offers more insight about the source of the bad performance. It can be observed that the majority of the exceedances occurs during the most volatile period of the end 2008 that is stained by the financial crisis. If the period of the financial

Quantile	Theoretical Number of Exceedances	Number of Exceedances
90% quantile	67	77
95% quantile	33	55
99% quantile	7	25

Table 6.2: Backtesting results for the S&P500 portfolio over a period 2008/01/07 - 2012/06/29

instability 2008/01/07 - 2008/02/14 is excluded (and only 893 VaR predictions are made), the results change completely. On the opposite, during this period VaR could be treated as a conservative risk measure while the theoretical number of excedeences is significantly higher than predicted(see Table 6.3, that is especially true for the 95% quantile).

Quantile	Theoretical Number of Exceedances	Number of Exceedances
90% quantile	45	26
95% quantile	22	15
99% quantile	4	4

Table 6.3: Backtesting results for the S&P500 portfolio over a period 2008/02/15 - 2012/06/29 (excluding crisis period)

Twol potential sources of this mismatch can be mentioned:

1. Rolling Window Size

   The first possible problem is the size of the rolling window. We use $n = 750$ and we don't control for the changes of conditional volatility. However, in the economic reality the system can be hardly treated so stable. The used window is large. Probably, the estimator is just too "slow", so that it can't incorporate the recent changes in the system for the current data fast enough to capture the volatility burst. One possible solution would be to apply weighted covariance matrix estimator.

   Another assumption is based on exactly the opposite point of view: the rolling window we use never includes other shock periods. It may be the case that in order to incorporate the possible shock in the estimator, even larger dataset should be examined.

2. Factor Model

   We implicitly assume that the FF3 factor model holds during the entire period considered. However, it's well known that crisis periods should be rather treated as "dragon kings" which means that the financial system follows totally different processes during the times of recession (see among others **Sornette and Ouillon (2012)**). Putting this idea differently, the process we deal with is not ergodic, so the same factor structure can't be applied. (It's still an open question though how the alternative model during the financial distress should look like and whether it can be derived). Some papers argue that FF3 model doesn't hold during the periods of financial instability. As well as CAPM FF3 tends to describe market at the "equilibrium" which is definitely not true for the recession (tests of the FF3 performance during the financial instability periods are provided in **Pesaran and Yamagata (2012)**).

It should also be noted that other VaR models didn't actually perform much better during the crisis period (see, e.g. **Halbleib-Chiriac and Pohlmeier (2011)** for comparison).

However, the results presented above may be portfolio-sensitive, so no general conclusions can be derived as we deal with just one case now. Nevertheless, the methodology of calculating VaR we apply is almost the only feasible approach that can be employed in the case of high-dimensionality with the limited number of data points available. This approach combines non- and parametric methods which makes it application fast in comparison with other alternatives one can think of.

7 Conclusions

In this paper we analyzed the multivariate joint density estimator proposed by **Fan et al. (2012)**. This estimator relies on the assumption that the underlying distribution is elliptical. The generator function of the distribution is estimated non-parametrically. Liebscher transformation is applied in order to avoid possible problems in the neighborhood of 0. The covariance matrix estimator is the one derived by **Fan et al. (2008)** that employs the FF3 factor model. The theoretical derivations prove the convergence of the estimated function towards the true one. Our findings suggest that the FFL covariance matrix estimator indeed outperforms the sample covariance matrix estimator if Gaussian or Student t distribution are assumed. This results in significantly more reliable estimates of the density function starting from $p = 6$ or $p = 7$ if the underlying distribution is normal. The error of the covariance matrix estimator appears even during to the first step which is a non-parametric estimation of the generator function. During the final density estimation, the error is amplified even further due to the error of the determinant estimation. However, much to our regret the Liebscher transformation contributes significantly only if the number of dimensions is low(less than 6) if the underlying distribution is normal or close to normal.

An empirical study is presented to support the theoretical findings. Based on the portfolio constructed of the components of the S&P500 index, we could derive several observations about the underlying distribution. The sample covariance matrix turned out to be non-invertible in most of the cases which reflects the superiority of the FFL estimator. As expected, the underlying distribution has fat tails and a high peak. Moreover, based on the quantiles we can conclude that the distribution is poorly approximated both by Gaussian and Student t distributions. The form of the distribution seems to be relatively stable over time. Nevertheless, during the periods of financial instability it becomes even thicker tails while during the tranquility periods normal distribution can serve as a relatively good approximation. During the financial distress periods, the distribution tends to give more probability to extremely small or extremely large values of returns.

We also present an example of VaR calculation that provides us some insights about the possible weak sides of the approach. We could observe that the burst of volatility are not captured fast enough which probably stems from the fact that the factor model doesn't hold any more during the financial instability periods. This results into a relatively poor performance of VaR.

Possible limitations of the study mostly stem from the imposed assumptions. Factor models don't work well when major changes in the economic system take place which leads to the poor performance of the indicators based on them. Also the increased correlations of the returns during such periods may not be well captured as the covariance matrix of errors of the factor model is supposed to be diagonal.

However, although the considered estimation procedure doesn't lack of shortcomings, it also has a plethora of appealing features. First, it makes it possible to get an idea about the distribution of the returns in case of high dimensionality and avoid putting too much structure which is inevitable when a particular distribution is used. Second, the reliable estimations of this distribution is can be only obtained because the FFL covariance matrix estimator is employed because sample covariance matrix estimator often can not be applied at all. Finally, risk measures(such as VaR) can be derived analytically without resorting to simulations or historical data which may not be longer valid. Such calculations are much faster than Monte Carlo methods and feasible when such methods as GARCH models can't be applied due to the high dimensionality.

The estimation procedure discussed in this paper also offers a lot of opportunities for further research. For example, the choice of the best factor model to capture the variance was not studied yet. The literature also offers a lot of alternative models to estimate a covariance matrix, so it remains an open question which of them should be preferred.

References

Amihud, Y. and H. Mendelson (1986): "Asset pricing and the bid-ask spread," *Journal of financial Economics*, 17, 223–249.

Asgharian, H. and B. Hansson (2000): "Cross-sectional analysis of Swedish stock returns with time-varying beta: the Swedish stock market 1983–96," *European Financial Management*, 6, 213–233.

Avramov, D. and T. Chordia (2006): "Predicting stock returns," *Journal of Financial Economics*, 82, 387–415.

Bannouh, K., M. Martens, R. Oomen, and D. van Dijk (2010): "Realized mixed frequency factor models for vast dimensional covariance estimation," *Erasmus University Rotterdam*.

Barndorff-Nielsen, O., P. Hansen, A. Lunde, and N. Shephard (2011): "Multivariate realised kernels: consistent positive semi-definite estimators of the covariation of equity prices with noise and non-synchronous trading," *Journal of Econometrics*, 162, 149–169.

Barndorff-Nielsen, O. and N. Shephard (2004): "Econometric analysis of realized covariation: High frequency based covariance, regression, and correlation in financial economics," *Econometrica*, 72, 885–925.

Battey, H. and O. Linton (2011): "Nonparametric estimation of multivariate elliptic densities via finite mixture sieves," .

Bickel, P. and E. Levina (2008): "Regularized estimation of large covariance matrices," *The Annals of Statistics*, 36, 199–227.

Carhart, M. (1997): "On Persistence in Mutual Fund Performance," *Journal of Finance*, 52, 57–82.

Chan, L., J. Karceski, and J. Lakonishok (1999): "On portfolio optimization: Forecasting covariances and choosing the risk model," Tech. rep., National Bureau of Economic Research.

Cui, H. and X. He (1995): "The consistence of semiparametric estimation of elliptic densities." *Acta Mathematica Sinica. New Series*, 11, 44–58.

FAMA, E. AND K. FRENCH (1992): "The Cross-Section of Expected Stock Returns," *The Journal of Finance*, 47, 427–465.

——— (1993): "Common risk factors in the returns on stocks and bonds," *Journal of financial economics*, 33, 3–56.

——— (1996): "Multifactor Explanations of Asset Pricing Anomalies," *The Journal of Finance*, 51.

FAN, J., Y. FAN, AND J. LV (2008): "High dimensional covariance matrix estimation using a factor model," *Journal of Econometrics*, 147, 186–197.

FAN, J., W. HÄRDLE, AND O. OKHRIN (2012): "Semiparametric Estimation for very high-dimensional Elliptical Distributions," Forthcoming.

FAN, J., Y. LIAO, AND M. MINCHEVA (2011): "High dimensional covariance matrix estimation in approximate factor models," *Annals of statistics*, 39, 3320.

FANG, H., K. FANG, AND S. KOTZ (2002): "The meta-elliptical distributions with given marginals," *Journal of Multivariate Analysis*, 82, 1–16.

FANG, K., S. KOTZ, AND K. NG (1990): *Symmetric Multivariate and Related Distributions Monographs on Statistics and Applied Probability*, London: Chapman and Hall Ltd. MR1071174.

GAUNT, C. (2004): "Size and book to market effects and the Fama French three factor asset pricing model: evidence from the Australian stockmarket," *Accounting & Finance*, 44, 27–44.

GRAHAM, J. AND C. HARVEY (2001): "The theory and practice of corporate finance: Evidence from the field," *Journal of financial economics*, 60, 187–243.

GRIFFIN, J. (2002): "Are the Fama and French factors global or country specific?" *Review of Financial Studies*, 15, 783–803.

HALBLEIB-CHIRIAC, R. AND W. POHLMEIER (2011): "Improving the Value at Risk Forecasts: Theory and Evidence from the Financial Crisis," *Journal of Economic Dynamics and Control, Forthcoming.*

HÄRDLE, W., A. WERWATZ, M. MÜLLER, AND S. SPERLICH (2004): "Introduction," *Nonparametric and semiparametric models.*

Hodrick, R. and X. Zhang (2001): "Evaluating the specification errors of asset pricing models," *Journal of Financial Economics*, 62, 327–376.

Homsud, N., J. Wasunsakul, S. Phuangnark, and J. Joongpong (2009): "A study of Fama and French three factors model and capital asset pricing model in the Stock exchange of Thailand," *International Research Journal of Finance and Economics*, 25, 31–40.

Horn, R. and C. Johnson (1990): *Matrix analysis*, Cambridge university press.

Huang, J., N. Liu, M. Pourahmadi, and L. Liu (2006): "Covariance matrix selection and estimation via penalised normal likelihood," *Biometrika*, 93, 85–98.

Hult, H. and F. Lindskog (2002): "Multivariate extremes, aggregation and dependence in elliptical distributions," *Advances in Applied probability*, 34, 587–608.

James, W. and C. Stein (1961): "Estimation with quadratic loss," in *Proceedings of the fourth Berkeley symposium on mathematical statistics and probability*, vol. 1, 1–379.

Jorion, P. (1986): "Bayes-Stein estimation for portfolio analysis," *Journal of Financial and Quantitative Analysis*, 21, 279–292.

Kamdem, J. (2005): "Value-at-Risk and expected shortfall for linear portfolios with elliptically distributed risk factors," *International Journal of Theoretical and Applied Finance*, 8, 537–551.

Lam, C. and J. Fan (2009): "Sparsistency and rates of convergence in large covariance matrix estimation," *Annals of statistics*, 37, 4254.

Lam, C. and Q. Yao (2011): "Factor modelling for high-dimensional time series: A dimension-reduction approach," Tech. rep., Technical Report.

Lam, K. (2002): "The relationship between size, book-to-market equity ratio, earnings–price ratio, and return for the Hong Kong stock market," *Global Finance Journal*, 13, 163–179.

Lam, L., L. Fung, and I. Yu (2009): "Forecasting a large dimensional covariance matrix of a portfolio of different asset classes," *Hong Kong Monetary Authority Working Paper*.

Ledoit, O. and M. Wolf (2003): "Improved estimation of the covariance matrix of stock returns with an application to portfolio selection," *Journal of Empirical Finance*, 10, 603–621.

——— (2004): "A well-conditioned estimator for large-dimensional covariance matrices," *Journal of multivariate analysis*, 88, 365–411.

——— (2010): "Nonlinear shrinkage estimation of large-dimensional covariance matrices," *Institute for Empirical Research in Economics University of Zurich Working Paper*.

Liebscher, E. (2005): "A semiparametric density estimator based on elliptical distributions," *Journal of multivariate analysis*, 92, 205–225.

Lintner, J. (1965): "The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets," *The review of economics and statistics*, 47, 13–37.

Malin, M. and M. Veeraraghavan (2004): "On the robustness of the Fama and French multifactor model: evidence from France, Germany, and the United Kingdom," *International Journal of Business and Economics*, 3, 155–176.

Markowitz, H. (1952): "Portfolio Selection," *The Journal of Finance*, 7, 77–91.

——— (1959): "Portfolio selection: efficient diversification of investments," *New Haven, CT: Cowles Foundation*, 94.

McNeil, A., R. Frey, and P. Embrechts (2005): *Quantitative risk management: concepts, techniques, and tools*, Princeton university press.

Merton, R. (1973): "An intertemporal capital asset pricing model," *Econometrica: Journal of the Econometric Society*, 867–887.

Pesaran, M. and T. Yamagata (2012): "Testing CAPM with a large number of assets," .

Ross, S. (1976): "The Arbitrage Theory of Capital Asset Pricing," *Journal of Economic Theory*, 13, 341–360.

Rothman, A., P. Bickel, E. Levina, and J. Zhu (2008): "Sparse permutation invariant covariance estimation," *Electronic Journal of Statistics*, 2, 494–515.

Rothman, A., E. Levina, and J. Zhu (2009): "Generalized thresholding of large covariance matrices," *Journal of the American Statistical Association*, 104, 177–186.

Sancetta, A. (2008): "Bayesian Semiparametric Estimation of Elliptic Copulae," .

Sharpe, W. (1964): "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," *The Journal of Finance*, 19, 425–442.

Silvestri, A. and S. Veltri (2011): "On the robustness of Fama and French Model: evidence from Italy," *Journal of Applied Finance & Banking*, 1, 201–221.

Sornette, D. and G. Ouillon (2012): "Dragon-kings: Mechanisms, statistical methods and empirical evidence," *The European Physical Journal-Special Topics*, 205, 1–26.

Stein, C. (1956): "Inadmissibility of the usual estimator for the mean of a multivariate normal distribution," in *Proceedings of the Third Berkeley symposium on mathematical statistics and probability*, vol. 1, 197–206.

Stute, W. and U. Werner (1991): "Nonparametric estimation of elliptically contoured densities," *Nonparametric Functional Estimation and Related Topics, Kluwer, Amsterdam.*

Wang, Y. and J. Zou (2010): "Vast volatility matrix estimation for high-frequency financial data," *The Annals of Statistics*, 38, 943–978.

Weinzierl, S. (2000): "Introduction to monte carlo methods," *arXiv preprint hep-ph/0006269.*

Wu, W. and M. Pourahmadi (2003): "Nonparametric estimation of large covariance matrices of longitudinal data," *Biometrika*, 90, 831–844.

Zheng, X. and Y. Li (2011): "On the estimation of integrated covariance matrices of high dimensional diffusion processes," *Annals of Statistics, Forthcoming.*

A List of S&P500 stocks used

1. 3M Co (MMM)
2. ACE Limited (ACE)
3. AES Corp (AES)
4. AFLAC Inc (AFL)
5. AGL Resources (GAS)
6. AT&T Inc (T)
7. Abbott Laboratories (ABT)
8. Abercrombie & Fitch Company A (ANF)
9. Accenture plc (ACN)
10. Adobe Systems Inc (ADBE)
11. Advanced Micro Devices (AMD)
12. Aetna Inc (AET)
13. Agilent Technologies Inc (A)
14. Air Products & Chemicals Inc (APD)
15. Airgas Inc (ARG)
16. Akamai Technologies Inc (AKAM)
17. Alcoa Inc (AA)
18. Alexion Pharmaceuticals Inc (ALXN)
19. Allegheny Technologies Inc (ATI)
20. Allergan Inc (AGN)
21. Allstate Corp (ALL)
22. Altera Corp (ALTR)
23. Altria Group Inc (MO)
24. Amazon.com Inc (AMZN)
25. Ameren Corp (AEE)
26. American Electric Power (AEP)
27. American Express Co (AXP)
28. American Intl Group Inc (AIG)
29. American Tower Corp A (AMT)
30. AmerisourceBergen Corp (ABC)
31. Amgen Inc (AMGN)
32. Amphenol Corp A (APH)
33. Anadarko Petroleum Corp (APC)
34. Analog Devices Inc (ADI)
35. Aon plc (AON)
36. Apache Corp (APA)
37. Apartment Investment & Mgmt (AIV)
38. Apollo Group Inc (APOL)
39. Apple Inc. (AAPL)
40. Applied Materials Inc (AMAT)
41. Archer-Daniels-Midland Co (ADM)
42. Assurant Inc (AIZ)
43. AutoNation Inc (AN)
44. AutoZone Inc (AZO)

45. Autodesk Inc (ADSK)
46. Automatic Data Processing (ADP)
47. AvalonBay Communities Inc (AVB)
48. Avery Dennison Corp (AVY)
49. Avon Products (AVP)
50. BB&T Corp (BBT)
51. BMC Software Inc (BMC)
52. Baker Hughes Inc (BHI)
53. Ball Corp (BLL)
54. Bank of America Corp (BAC)
55. Bard, C.R. Inc (BCR)
56. Baxter Intl Inc (BAX)
57. Beam Inc (BEAM)
58. Becton, Dickinson & Co (BDX)
59. Bed Bath & Beyond Inc (BBBY)
60. Bemis Co Inc (BMS)
61. Berkshire Hathaway B (BRK/B)
62. Best Buy Co Inc (BBY)
63. Big Lots Inc (BIG)
64. Biogen Idec Inc (BIIB)
65. BlackRock Inc (BLK)
66. Block H & R Inc (HRB)
67. Boeing Co (BA)
68. Boston Properties Inc (BXP)
69. Boston Scientific Corp (BSX)
70. Bristol-Myers Squibb (BMY)
71. Broadcom Corp A (BRCM)
72. Brown-Forman Corp B (BF/B)
73. CA Inc (CA)
74. CBRE Group, Inc. (CBG)
75. CH Robinson Worldwide Inc (CHRW)
76. CMS Energy Corp (CMS)
77. CONSOL Energy Inc (CNX)
78. CSX Corp (CSX)
79. CVS Caremark Corp. (CVS)
80. Cablevision Systems Co A (CVC)
81. Cabot Oil & Gas A (COG)
82. Cameron International Corp (CAM)
83. Campbell Soup Co (CPB)
84. Capital One Financial (COF)
85. Cardinal Health Inc (CAH)
86. Carmax Inc (KMX)
87. Carnival Corp (CCL)
88. Caterpillar Inc (CAT)
89. Celgene Corp (CELG)
90. Centerpoint Energy Inc (CNP)
91. CenturyLink Inc (CTL)
92. Cerner Corp (CERN)

93. Chesapeake Energy Corp (CHK)
94. Chevron Corp (CVX)
95. Chicago Mercantile Exchange (CME)
96. Chubb Corp (CB)
97. Cigna Corporation (CI)
98. Cincinnati Financial Corp (CINF)
99. Cintas Corp (CTAS)
100. Cisco Systems Inc (CSCO)
101. Citigroup Inc (C)
102. Citrix Systems Inc (CTXS)
103. Cliffs Natural Resources Inc (CLF)
104. Clorox Co (CLX)
105. Coach Inc (COH)
106. Coca-Cola Co (KO)
107. Coca-Cola Enterprises (CCE)
108. Cognizant Tech Solutions Corp (CTSH)
109. Colgate-Palmolive Co (CL)
110. Comcast Corp (CMCSA)
111. Comerica Inc (MI) (CMA)
112. Computer Sciences (CSC)
113. ConAgra Foods Inc (CAG)
114. ConocoPhillips (COP)
115. Consolidated Edison Inc (ED)
116. Constellation Brands Inc A (STZ)
117. Cooper Industries Plc (CBE)
118. Corning Inc (GLW)
119. Costco Wholesale Corp (COST)
120. Coventry Health Care Inc (CVH)
121. Crown Castle Intl Corp (CCI)
122. Cummins Inc (CMI)
123. DIRECTV Class A (DTV)
124. DTE Energy Co (DTE)
125. Danaher Corp (DHR)
126. Darden Restaurants Inc (DRI)
127. Davita Inc (DVA)
128. DeVry Inc (DV)
129. Dean Foods Co (DF)
130. Deere & Co (DE)
131. Dell Inc (DELL)
132. Denbury Resources Inc (DNR)
133. Dentsply Intl (XRAY)
134. Devon Energy Corp (DVN)
135. Diamond Offshore Drilling (DO)
136. Dollar Tree Inc (DLTR)
137. Dominion Resources Inc (D)
138. Donnelley, R.R. & Sons (RRD)
139. Dover Corp (DOV)
140. Dow Chemical (DOW)

141. DuPont, E.I. de Nemours (DD)
142. Duke Energy Corp (DUK)
143. Dun & Bradstreet Corp (DNB)
144. E*TRADE Financial Corp (ETFC)
145. EMC Corp (EMC)
146. EOG Resources (EOG)
147. EQT Corporation (EQT)
148. Eastman Chemical Co (EMN)
149. Eaton Corp (ETN)
150. Ecolab Inc (ECL)
151. Edison Intl (EIX)
152. Edwards Lifesciences Corp (EW)
153. Electronic Arts (EA)
154. Emerson Electric Co (EMR)
155. Ensco PLC - CL A (ESV)
156. Entergy Corp (ETR)
157. Equifax Inc (EFX)
158. Equity Residential (EQR)
159. Estee Lauder Cos. (EL)
160. Exelon Corp (EXC)
161. Expeditors Intl of WA Inc (EXPD)
162. Express Scripts Holding Co. (ESRX)
163. Exxon Mobil Corp (XOM)
164. F5 Networks Inc (FFIV)
165. FLIR Systems Inc (FLIR)
166. FMC Corp (FMC)
167. FMC Technologies Inc (FTI)
168. Family Dollar Stores Inc (FDO)
169. Fastenal Co (FAST)
170. FedEx Corp (FDX)
171. Federated Investors Inc B (FII)
172. Fidelity National Information (FIS)
173. Fifth Third Bancorp (OH) (FITB)
174. First Horizon National Corp (FHN)
175. FirstEnergy Corp (FE)
176. Fiserv Inc (FISV)
177. Flowserve Corp (FLS)
178. Fluor Corp (FLR)
179. Ford Motor Co (F)
180. Forest Laboratories (FRX)
181. Fossil Inc (FOSL)
182. Franklin Resources Inc (BEN)
183. Freeport McMoRan Copper & Gold (FCX)
184. Frontier Communications Corp (FTR)
185. GameStop Corp A (GME)
186. Gannett Co Inc (GCI)
187. Gap Inc (GPS)
188. General Dynamics (GD)

189. General Electric Co (GE)
190. General Mills Inc (GIS)
191. Genuine Parts Co (GPC)
192. Genworth Financial Inc (GNW)
193. Gilead Sciences Inc (GILD)
194. Goldman Sachs Group Inc (GS)
195. Goodyear Tire & Rubber Co (GT)
196. Google Inc (GOOG)
197. Grainger, W.W. Inc (GWW)
198. HCP Inc (HCP)
199. Halliburton Co (HAL)
200. Harley-Davidson Inc (HOG)
201. Harman Intl Industries Inc (HAR)
202. Harris Corp (HRS)
203. Hartford Finl Services Group (HIG)
204. Hasbro Inc (HAS)
205. Health Care REIT Inc (HCN)
206. Heinz, H.J. Co (HNZ)
207. Helmerich & Payne Inc (HP)
208. Hershey Foods Corp (HSY)
209. Hess Corp (HES)
210. Hewlett-Packard Co (HPQ)
211. Home Depot Inc (HD)
212. Honeywell Intl Inc (HON)
213. Hormel Foods Corp (HRL)
214. Horton, D.R. Inc (DHI)
215. Hospira, Inc (HSP)
216. Host Hotels & Resorts Inc (HST)
217. Hudson City Bancorp (HCBK)
218. Humana Inc (HUM)
219. Huntington Bancshares (OH) (HBAN)
220. Illinois Tool Works Inc (ITW)
221. Ingersoll-Rand Plc (IR)
222. Integrys Energy Group Inc (TEG)
223. Intel Corp (INTC)
224. Interpublic Group Cos (IPG)
225. Intl Business Machines Corp (IBM)
226. Intl Flavors & Fragrances (IFF)
227. Intl Game Technology (IGT)
228. Intl Paper Co (IP)
229. Intuit Inc (INTU)
230. Intuitive Surgical Inc (ISRG)
231. Invesco Ltd (IVZ)
232. Iron Mountain Inc (IRM)
233. JDS Uniphase Corp (JDSU)
234. JP Morgan Chase & Co (JPM)
235. Jabil Circuit Inc (JBL)
236. Jacobs Engineering Group Inc (JEC)

237. Johnson & Johnson (JNJ)
238. Johnson Controls Inc (JCI)
239. Joy Global Inc (JOY)
240. Juniper Networks Inc (JNPR)
241. KLA-Tencor Corporation (KLAC)
242. Kellogg Co (K)
243. KeyCorp (KEY)
244. Kimberly-Clark (KMB)
245. Kimco Realty Corp (KIM)
246. Kohl's Corp (KSS)
247. Kraft Foods Inc A (KFT)
248. Kroger Co (KR)
249. L-3 Communications Holdings (LLL)
250. LSI Corporation (LSI)
251. Lab Corp of America Hldgs (LH)
252. Lam Research Corp (LRCX)
253. Legg Mason Inc (LM)
254. Leggett & Platt (LEG)
255. Lennar Corp (LEN)
256. Leucadia National Corp (NY) (LUK)
257. Lexmark International Inc (LXK)
258. Life Technologies Corp (LIFE)
259. Lilly, Eli & Co (LLY)
260. Limited Brands Inc (LTD)
261. Lincoln National Corp (LNC)
262. Linear Technology Corp (LLTC)
263. Lockheed Martin (LMT)
264. Loews Corp (L)
265. Lowe's Cos Inc (LOW)
266. M&T Bank Corp (MTB)
267. Macy's Inc (M)
268. Marathon Oil Corp (MRO)
269. Marriott Intl A (MAR)
270. Marsh & McLennan Companies (MMC)
271. Masco Corp (MAS)
272. Mattel Inc (MAT)
273. McCormick & Co (MKC)
274. McDonald's Corp (MCD)
275. McGraw-Hill Cos Inc (MHP)
276. McKesson Corp (MCK)
277. MeadWestvaco Corp (MWV)
278. Medtronic Inc (MDT)
279. Merck & Co Inc (MRK)
280. Metlife Inc (MET)
281. Microchip Technology Inc (MCHP)
282. Micron Technology Inc (MU)
283. Microsoft Corp (MSFT)
284. Molex Inc (MOLX)

285. Molson Coors Brewing Co B (TAP)
286. Monsanto Co. (MON)
287. Monster Beverage Corp (MNST)
288. Moody's Corp (MCO)
289. Morgan Stanley (MS)
290. Mosaic Co (MOS)
291. Motorola Solutions, Inc (MSI)
292. Murphy Oil Corp (MUR)
293. Mylan Inc. (MYL)
294. NIKE Inc B (NKE)
295. NRG Energy (NRG)
296. NYSE Euronext (NYX)
297. Nabors Industries Ltd (NBR)
298. Nasdaq OMX Group/The (NDAQ)
299. National Oilwell Varco Inc (NOV)
300. NetApp Inc (NTAP)
301. NetFlix Inc (NFLX)
302. Newell Rubbermaid Inc (NWL)
303. Newfield Exploration Co (NFX)
304. Newmont Mining Corp (NEM)
305. News Corporation (NWSA)
306. NextEra Energy Inc (NEE)
307. Nisource Inc (NI)
308. Noble Corp (NE)
309. Noble Energy Inc (NBL)
310. Nordstrom Inc (JWN)
311. Norfolk Southern Corp (NSC)
312. Northeast Utilities (NU)
313. Northern Trust Corp (IL) (NTRS)
314. Northrop Grumman Corp (NOC)
315. Nucor Corp (NUE)
316. Nvidia Corp (NVDA)
317. O'Reilly Automotive (ORLY)
318. ONEOK Inc (OKE)
319. Occidental Petroleum (OXY)
320. Omnicom Group (OMC)
321. Oracle Corp (ORCL)
322. Owens-Illinois Inc (OI)
323. PACCAR Inc (PCAR)
324. PG&E Corporation (PCG)
325. PNC Finl Services Group (PNC)
326. PPG Industries Inc (PPG)
327. PPL Corp (PPL)
328. Pall Corp (PLL)
329. Parker-Hannifin Corp (PH)
330. Patterson Cos Inc (PDCO)
331. Paychex Inc (PAYX)
332. Peabody Energy Corp (BTU)

333. Penney, J.C. Inc (JCP)
334. People's United Financial Inc (PBCT)
335. Pepco Holdings Inc (POM)
336. PepsiCo Inc (PEP)
337. PerkinElmer Inc (PKI)
338. Perrigo Co (PRGO)
339. Pfizer Inc (PFE)
340. Pinnacle West Capital (AZ) (PNW)
341. Pioneer Natural Resources (PXD)
342. Plum Creek Timber Co (PCL)
343. Praxair Inc (PX)
344. Precision Castparts Corp (PCP)
345. Priceline.com Inc (PCLN)
346. Principal Financial Group (PFG)
347. ProLogis, Inc (PLD)
348. Procter & Gamble (PG)
349. Progressive Corp (PGR)
350. Prudential Financial Inc (PRU)
351. Public Storage (PSA)
352. Pulte Group Inc (PHM)
353. QUALCOMM Inc (QCOM)
354. Quanta Services Inc (PWR)
355. Quest Diagnostics (DGX)
356. Ralph Lauren Corp (RL)
357. Range Resources Corp (RRC)
358. Raytheon Co (RTN)
359. Red Hat Inc (RHT)
360. Regions Financial Corp (RF)
361. Republic Services Inc (RSG)
362. Reynolds American Inc (RAI)
363. Robert Half Intl Inc (RHI)
364. Rockwell Automation Inc (ROK)
365. Rockwell Collins (COL)
366. Roper Industries Inc (ROP)
367. Ross Stores Inc (ROST)
368. Rowan Cos Plc (RDC)
369. Ryder System Inc (R)
370. SCANA Corp (SCG)
371. SLM Corp (SLM)
372. Safeway Inc (SWY)
373. Salesforce.com (CRM)
374. SanDisk Corp (SNDK)
375. Schlumberger Ltd (SLB)
376. Schwab, Charles Corp (SCHW)
377. Seagate Technology (STX)
378. Sealed Air Corp (SEE)
379. Sears Holdings Corp (SHLD)
380. Sempra Energy (SRE)

381. Sigma-Aldrich Corp (SIAL)
382. Simon Property Group (SPG)
383. Smucker, J.M. Co (SJM)
384. Snap On Inc (SNA)
385. Southern Co (SO)
386. Southwest Airlines Co (LUV)
387. Southwestern Energy Co (SWN)
388. Sprint Nextel Corp (S)
389. St Jude Medical Inc (STJ)
390. Stanley Black & Decker (SWK)
391. Staples Inc (SPLS)
392. Starbucks Corp (SBUX)
393. Starwood Hotel & Resort World (HOT)
394. State Street Corp (STT)
395. Stericycle Inc (SRCL)
396. Stryker Corp (SYK)
397. SunTrust Banks Inc (GA) (STI)
398. Sunoco Inc (SUN)
399. Symantec Corp (SYMC)
400. Sysco Corp (SYY)
401. T Rowe Price Group Inc (TROW)
402. TECO Energy Inc (TE)
403. TJX Cos Inc (TJX)
404. Target Corp (TGT)
405. Tenet Healthcare (THC)
406. Teradyne Inc (TER)
407. Tesoro Corp (TSO)
408. Texas Instruments Inc (TXN)
409. Textron Inc (TXT)
410. The Bank of New York Mellon Corp (BK)
411. The Williams Companies Inc (WMB)
412. Thermo Fisher Scientific (TMO)
413. Tiffany & Co (TIF)
414. Time Warner Inc (TWX)
415. Titanium Metals Corp (TIE)
416. Torchmark Corp (TMK)
417. Total System Services Inc (TSS)
418. Travelers Cos Inc (TRV)
419. Tyco Intl (TYC)
420. Tyson Foods Inc A (TSN)
421. US Bancorp (USB)
422. Union Pacific Corp (UNP)
423. United Parcel Service Inc B (UPS)
424. United States Steel Corp (X)
425. United Technologies Corp (UTX)
426. Unitedhealth Group Inc (UNH)
427. Unum Group (UNM)

428. Urban Outfitters (URBN)
429. VF Corp (VFC)
430. Valero Energy Corp (VLO)
431. Varian Medical Systems Inc (VAR)
432. Ventas Inc (VTR)
433. VeriSign Inc (VRSN)
434. Verizon Communications Inc (VZ)
435. Vulcan Materials Co (VMC)
436. Wal-Mart Stores (WMT)
437. Walgreen Co (WAG)
438. Walt Disney Co (DIS)
439. Washington Post Co B (WPO)
440. Waste Management Inc (WM)
441. Waters Corp (WAT)
442. Watson Pharmaceuticals (WPI)
443. WellPoint Inc (WLP)
444. Wells Fargo & Co (WFC)
445. Western Digital Corp (WDC)
446. Weyerhaeuser Co (WY)
447. Whirlpool Corp (WHR)
448. Whole Foods Market Inc (WFM)
449. Wisconsin Energy Corp (WEC)
450. Wynn Resorts Ltd (WYNN)
451. XL Group Plc (XL)
452. Xcel Energy Inc (XEL)
453. Xerox Corp (XRX)
454. Xilinx Inc (XLNX)
455. Yahoo Inc (YHOO)
456. Yum! Brands Inc (YUM)
457. Zimmer Holdings Inc (ZMH)
458. Zions Bancorp (UT) (ZION)
459. eBay Inc. (EBAY)

Declaration of Authorship

I hereby confirm that I have authored this Master's thesis independently and without use of others than the indicated sources. All passages which are literally or in general matter taken out of publications or other sources are marked as such.

Berlin, November 6, 2012

Irina Pimenova