Systemic Risk Measure: CoVaR and Copula

Master Thesis submitted

to

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Abstract

We consider an alternative computation methodology of the systemic risk measure (CoVaR) using copula and extend it to high dimension case. In addition, we modify the definition of risk contribution (ΔCoVaR) to make it more reasonable. We investigate the change of ΔCoVaR for eight European sovereign debt markets before and after European debt crisis. Evidences show that crisis markets were highly correlated with system and within each other before crisis, while they decoupled with system after the crisis.
Keywords: VaR,CoVaR,Copula,systemic risk measure

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Abbreviations

VaR	**V**alue-**a**t **R**isk
CoVaR	**Co**nditional **V**alue-**a**t **R**isk
CDF	**C**umulative**D**ensity **F**unction
AMH	**A**li**M**ikhail **H**aq
CDO	**C**ollateralized **D**ebt **O**bligation
CDS	**C**redit **D**efault **S**waps
GIIPS	**G**reece **I**reland **I**taly **P**ortugal **S**pain
ADF	**A**ugmented **D**ickey **F**uller
KPSS	**K**wiatkowski **P**hillips **S**chmidt **S**hin
ARCH	**A**uto **R**egressive **C**onditional **H**eteroskedasticity

Chapter 1

Introduction

The outbreaks of subprime crisis and European debt crisis ignite the discussion of the complexity and fragility of financial system and systemic risk measure both in academia and industry. One lesson that we learn from the crisis is that Value-at-Risk (VaR), which is the most widely used risk measure by financial institutions, is far from enough for risk management. VaR fails to capture the nature of systemic risk — the risk that stability of the financial system as a whole is threatened[1], because it only focuses on individual institution in isolation, and tail comovement and spill-over effect have been ignored. As a result, it is crucial to find a practical method for systemic risk measure to supervise the stability of financial system.

Current scientific and regulatory discussion of systemic risk measure is far from closed and its computing methodology is still under development. CoVaR, shorted for Conditional Value-at-Risk, which is introduced by Adrian and Brunnermeier [1], may be the most popular and widely-used systemic risk measure among various versions in financial institutions (see Board [2], Fong et al. [3]). The general idea of CoVaR is to use the conditional distribution of a random variable X_j conditioning on X_i. They define $CoVaR^{j|i}$ as the VaR of institution j conditioning on institution i being in financial distress, which allows to measure risk contribution of one institution adding to the whole system (or other financial institutions) by taking the difference between CoVaR conditioning on the institution being at stress and CoVaR conditioning on the institution being at normal

[1] Adrian and Brunnermeier [1]

state.

Copula is an elegant concept and a powerful instrument when we have to deal with high-dimension joint CDF. Copula was firstly introduced by Sklar [4], where the famous Sklar's Theorem was given, although similar ideas can be traced back to Hoeffding [5]. The most attractive property of copula is allowing us to separately model margins and dependence structure. The past decade has witnessed the rapidly growing applications of copula, not only in the area of statistic research, but many other disciplines such as finance, geology, engineering. One of early applications of copula is to price the credit derivatives (CDO and CDS). Li [6] was the first to utilize copula in valuation of some credit derivatives, which is known as "the formula that killed Wall Street." Another area of copula's application is risk management. The nature of co-movement between sources of risk creates the demand of estimating high-dimension joint distribution. Embrechts and Höing [7] study the VaR of portfolios and Cousin and Di Bernadino [8] use copula to define VaR in multivariate setting. Basically, CoVaR is the VaR of a conditional distribution and it should capture the dependence between objective variable and conditional variable, which makes it possible to take the advantage of copula to redefine and compute CoVaR.

This thesis attempts to characterize the CoVaR using Copula and to apply Copula-based CoVaR to capturing how systemic risk change as a result of the European debt crisis. The thesis is organized as follows: Chapter 2 discusses copula theories and properties of copula classes, which would be utilized in CoVaR computation and empirical study; In chapter 3, we develop the copula-based CoVaR and extend it to multivariate case. Simulations of bivariate CoVaR and multivariate CoVaR are also given; In chapter 4, we conduct empirical analysis of copula-based CoVaR based on the European sovereign bond markets data. Results indicate that GIIPS markets are closely linked with EMU index (the representative of system) before crisis and decoupled with EMU after crisis.

Chapter 2

Copula

In this Chapter, we will introduce some related definitions and properties of copula. Copula is a important method to model multivariate distribution, because it allows us to model margins and dependence structure respectively, which suggests copula function contains full information of dependence. It is quite clear that linear correlation does not precisely describe the dependence. No correlation does not imply independence and a positive correlation does not mean positive dependence (Lehmann [9]). Therefore, copula-based dependence measures are presented in this chapter. Archimedean copula class is quite popular in the literature because it makes it possible to model dependence with one parameter even in high dimensions. Summary and comparison between Archimedean copulas are also given in this chapter. For details of copula theory, we refer to Joe [10] and Nelsen [11].

2.1 Basic Copula Theory

Definition 2.1.1 (Copula)**.** A d-dimensional copula is a function C from $[0,1]^d \to [0,1]$ with following properties:

(1) $C(u_1,\ldots,u_d)$ is increasing in each component $u_i \in [0,1], i = 1,\ldots,d$.

(2) $C(1,\ldots,1,u_i,1,\ldots,1) = u_i$ for all $u_i \in [0,1], i = 1,\ldots,d$.

(3) For all $(u_i, \ldots, u_d), (u_1^{'}, \ldots, u_d^{'}) \in [0,1]^d$ with $u_i < u_i^{'}$ we have

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d} C(v_{j1}, \ldots, v_{jd}) \geq 0,$$

where $v_{j1} = u_j$ and $v_{j2} = u_j^{'}$,for all $j = 1, \ldots, d$.

Property (1) and (3) in Definition 2.1.1 imply that copula is multivariate cumulative density function. Property (2) suggests that copula has uniform margins. It is apparent that copula is nothing but a multivariate distribution with uniform margins. However, copula has its own power when dealing with multivariate distribution, which is given by Sklar's Theorem.

Theorem 2.1.1 (Sklar's Theorem (Sklar [12]))**.** Let F be a multivariate distribution function with margins $F_1, \ldots, F_d$, the exists the copula C such that

$$F(x_1, \ldots, x_d) = C\{F_1(x_1), \ldots, F_d(x_d)\} = C(u_1, \ldots, u_2), x_1, \ldots, x_d \in \overline{\mathbb{R}}. \tag{2.1}$$

If F_i are continuous for $i = 1, \ldots, d$, the C is unique. Otherwise C is uniquely determined on $\prod_{i=1}^{d} F_i(\overline{\mathbb{R}})$. Conversely, if C is a copula and $F_1, \ldots, F_d$ are univariate distribution functions, the function F defined above is a multivariate distribution function with margins $F_1, \ldots, F_d$.

Sklar's theorem is valuable and important because it associates each multivariate distribution with a copula and allows us to model dependence structure independently. Also, we can derive the joint density from Sklar's Theorem. The joint density $f_{12}(x_1, x_2)$ is

$$\begin{aligned} f_{12}(x_1, x_2) =& \frac{\partial^2 F_{12}(x_1, x_2)}{\partial x_1 \partial x_2} \\ =& \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} \frac{\partial F_1(x_1)}{\partial x_1} \frac{\partial F_2(x_2)}{\partial x_2} \\ =& c(F_1(x_1), F_2(x_2)) f_1(x_1) f_2(x_2), \end{aligned} \tag{2.2}$$

where $c(u_1, u_2) = \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2}$ is the copula density. For independent copula, $c(u_1, u_2) = 1$, and a important property of copula is invariance under increasing and continuous transformation.

Continuing with the equation 2.2, let $X_t = (x_{1,t}, x_{2,t})'$, the log-likelihood function for $X_{t=1}^n$ is

$$\begin{aligned}\mathbb{L}(\theta) &= \sum_{t=1}^{n} \log f_{12}(x_{1,t}, x_{2,t}; \theta) \\ &= \sum_{t=1}^{n} \log f_1(x_{1,t}; \theta_1) + \log f_2(x_{2,t}; \theta_2) + \log c(F_1(x_{1,t}; \theta_1), F_2(x_{2,t}; \theta_2); \theta_3),\end{aligned} \tag{2.3}$$

where $\theta = (\theta_1' \theta_2' \theta_3')'$,$\theta_1$ and θ_2 are the parameters of margins of X_1 and X_2, θ_3 is the copula parameter, and n is the number of the observations.

2.2 Dependence and Copula

As mentioned in Chapter 1, copula allows us to separately model marginal distribution and dependence, which provides a natural way to measure dependence. Pearson's correlation (or linear correlation) is the most frequently-used dependence measure in practice. However, it is often misleading because it is not the copula-based dependence measure. In this section,we recall the definition of Pearson'correlation and continue with copula-based rank correlation and tail dependence.

Definition 2.2.1 (Pearson's Correlation)**.** Let $(X_1, X_2)^T \in \mathbb{R}^2$ be vector of two random variables with $Var(X_d) < \infty, Var(X_d) \neq 0, d \in 1, 2$, the Pearson's correlation ρ_p of X_1 and X_2 can be defined as follows,

$$\rho_p(X_1, X_2) = \frac{CoV(X_1, X_2)}{\sqrt{Var(X_1)}\sqrt{Var(X_2)}}, \tag{2.4}$$

CoV() is the covariance operator and Var() is the variance operator.

Pearson's correlation is a measure of linear dependence and its popularity stems from the simplicity of calculation and that it is a natural scalar measure of dependence in elliptical distribution (see Embrechts et al. [13]). However, plenty of evidences indicate most random variables are not jointly elliptical distribution, especially in the area of finance and economics. Therefore, it is often inappropriate and misleading to take Pearson's correlation as the measure of dependence. Pearson's correlation is relied on the existence of covariance of two variables, but there is possibility that Pearson's correlation does not make sense even they are elliptically distributed, such as t_2 distribution (t_2 distribution has infinite second moment). Moreover, Pearson's correlation is not invariant under non-linear strictly increasing transformation. That's why we need more proper measure of dependence. Two important dependence measures will be introduced in the following part.

Definition 2.2.2 (Kendall's tau)**.** Let X and Y be random variables with joint distribution of F, and (X_1, Y_1) and (X_2, Y_2) be two independent pairs of random variables from F, Kendall's tau is defined as follows:

$$\rho_\tau = \mathbb{P}[(X_1 - X_2)(Y_1 - Y_2) > 0] - \mathbb{P}[(X_1 - X_2)(Y_1 - Y_2) < 0].$$

Kendall's tau is relevant to concept of concordance and discordance. For any pair of observations (x_i, y_i) and (x_j, y_j), if both $x_i > x_j$ and $y_i > y_j$ and if both $x_i < x_j$ and $y_i < y_j$, then we say observations (x_i, y_i) and (x_j, y_j) are concordant; they are said to be discordant if both $x_i > x_j$ and $y_i < y_j$ and if both $x_i < x_j$ and $y_i > y_j$. Kendall's tau tries to measure the dependence as the difference between probability of concordance and probability of discordance, and it can be connected with copula by following theorem.

Theorem 2.2.1. Let X and Y be random variables with joint distribution of F and copula C, the Kendall's tau can be calculated as

$$\rho_\tau = 4\int_0^1 \int_0^1 C(u, v) dC(u, v) - 1. \tag{2.5}$$

Another popular rank correlation is Spearman's rho,which is defined as follows,

Definition 2.2.3 (Spearman's rho)**.** Let X and Y be random variables with distribution function F_1 and F_2, and joint distribution of F, spearman's rank correlation is defined as

$$\rho_s(X, Y) = \rho_p(F_1(X), F_2(Y)).$$

Theorem 2.2.2. Let X and Y be random variables with joint distribution of F and copula C, the Spearman's rho can be calculated as

$$\rho_s = 12\int_0^1 \int_0^1 C(u, v) dC(u, v) - 3.$$

The main difference between rank correlation and Pearson' correlation is that both Kendall's tau and Spearman's rho measure the degree of monotonic dependence, while

Pearson's correlation is to measure the degree of linear dependence. The advantages of rank correlations over linear correlation are invariance under monotonic transformation as well as sensitiveness to error and discrepancy in data.

Tail dependence

If we are more concerned about the dependence in the tail and extreme values, tail dependence may be a valuable concept.

Definition 2.2.4 (Upper Tail Dependence)**.** Let X and Y be random variables with distribution function F_1 and F_2, the coefficient of upper tail dependence is defined as

$$\lambda_u = \lim_{\alpha \to 1} \mathbb{P}[Y > F_2^{-1}(\alpha) | X > F_1^{-1}(\alpha)]. \tag{2.6}$$

Provided λ_u ∈[0,1], we call X and Y are asymptotically dependent in upper tail if λ_u ∈(0,1] and asymptotically independent in upper tail if λ_u =0.

If F_1 and F_2 are continuous distribution, we can rewrite the equation 2.6 using copula,

$$\begin{aligned} \lim_{\alpha \to 1} \mathbb{P}[Y > F_2^{-1}(\alpha) | X > F_1^{-1}(\alpha)] &= \lim_{\alpha \to 1} \frac{\mathbb{P}[Y > F_2^{-1}(\alpha), [Y > F_1^{-1}(\alpha)]}{\mathbb{P}[Y > F_1^{-1}(\alpha)]} \\ &= \lim_{\alpha \to 1} \frac{\bar{C}(\alpha, \alpha)}{1 - \alpha} \\ &= \lim_{\alpha \to 1} \frac{2\alpha - 1 + C(1-\alpha, 1-\alpha)}{1 - \alpha}, \end{aligned} \tag{2.7}$$

where $\bar{C}$(u,v) is the survive copula function defined as $\bar{C}$(u,v)=u+v-1+C(1-u,1-v), if copula has explicit form, taking Gumbel copula as example, we can derive the upper tail dependence according to equation 2.7 $\lambda_u = 2 - 2^{\theta}$, and θ is the dependence parameter of Gumbel copula.

Definition 2.2.5 (Lower Tail Dependence)**.** Let X and Y be random variables with distribution function F_1 and F_2, the coefficient of lower tail dependence is defined as

$$\lambda_l = \lim_{\alpha \to 0} \mathbb{P}[Y < F_2(\alpha) | X < F_1(\alpha)].$$

Provided $\lambda_l \in [0, 1]$, we call X and Y are asymptotically dependent in lower tail if $\lambda_l \in$ (0,1] and asymptotically independent in lower tail if λ_l=0.

Adopting the same logics presented in the part of upper tail dependence, we can calculate the lower tail dependence via copula,

$$\lambda_l = \lim_{\alpha \to 0} \frac{C(\alpha, \alpha)}{\alpha}. \tag{2.8}$$

2.3 Copula Class

In this section, elliptical copula class and Archimedean copula class are introduced, which are most popular copula classes in literature. They also will be used in the empirical part of this thesis. Gaussian copula is often used as the benchmark since its dependence parameter is just linear correlation and can not capture the tail dependence. Archimedean copula is an useful copula class because it is convenient to be constructed and has some good analytic properties.

2.3.1 Elliptical Copula

Gaussian Copula

Definition 2.3.1 (Gaussian Copula)**.** For d-dimension Gaussian copula with u=$(u_1, \ldots, u_d)^T \in [0, 1]^d$, the Gaussian copula can be described as

$$C_{Ga}(u_1, \ldots, u_d; \Sigma) = \Phi_{d,\Sigma}(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d)),$$

where Σ is d $\times$ d correlation matrix, $\Phi()$ is standard normal distribution function, $\Phi^{-1}()$ is the inverse function of $\Phi()$. Note that Gaussian copula has no tail dependence unless $\rho = 1$.

Student-t Copula

Definition 2.3.2 (Student-t Copula)**.** For d-dimension Student-t copula with u=$(u_1, \ldots, u_d)^T \in [0, 1]^d$, the Student-t copula can be described as

$$C_t(u_1, \ldots, u_d; \nu, \Sigma) = t_{\nu,d,\Sigma}(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)),$$

where $t_\nu()$ is the student-t distribution function with degree of freedom ν, and σ is the correlation matrix,$t_\nu^{-1}()$ is the inverse function of $t_\nu()$.

Note that student-t copula has tail dependence for all $\nu > 0$.

2.3.2 Archimedean Copula

Definition 2.3.3 (Archimedean Copula)**.**

Let $\phi : [0, 1] \to [0, +\infty]$ be the function which satisfies that $\phi(1) = 0, \phi(+\infty) = 1$, and ϕ is a decreasing function. The function C:$[0, 1] \to [0, +\infty]$ defined as

$$C(u_1, \ldots, u_n; \theta) = \phi\{\phi^{-1}(u_1) + \cdots + \phi^{-1}(u_d)\}, u_1 \ldots u_d \in [0, 1] \tag{2.9}$$

is a d-dimensional Archimedean copula, where ϕ is called the generator of the copula. For Archimedean copula, the relationship with kendall'tau is $\rho_\tau = 1+4\int_0^1\int_0^1 \frac{\phi(u_1,u_2)}{\phi'(u_1,u_2)} du_1 du_2$, which can be derived from equation 2.5.

Table 2.1 gives the summary of some Archimedean copula, including the generator, domain of dependence parameter, rang of attainable tau and tail dependence.

Copula	Generator	θ	τ range	Tail Dependence
Gumbel	$(-\log(t))^\theta$	$[1, +\infty)$	[0,1)	weak λ_l and strong λ_u
Clayton	$1/\theta(t^{-\theta} - 1)$	$\theta \in [-1, +\infty) \setminus \{0\}$	[-1,1)\{0}	weak λ_u and strong λ_l
Joe	$-\log(1 - (1 - t)^\theta)$	$\theta \in [1, +\infty]$	[0,1)	weak λ_l and strong λ_u
Ali-Mikhail-Haq	$\log(\frac{1-\theta(1-t)}{t})$	$\theta \in [-1, 1]$	(0,1/3)	exhibitλ_l only when $\theta = 1$
Frank	$-\log(\frac{exp(-\theta t-1)}{exp(-\theta)-1})$	$\theta \in \mathbb{R} \setminus \{0\}$	(-1,1)	symmetric,weak λ_l and λ_u

Table 2.1: Archimedean copula with corresponding generator ,range of dependence parameter and attainable Kendall's tau

Chapter 3

CoVaR

3.1 CoVaR Background

3.1.1 Definition

Recall the definition of VaR. For a given return X_t^i of institution i and a confidence level 1-α, $VaR_{\alpha,t}^i$ is defined as α-quantile of the return distribution,

$$Pr(X_t^i \leq VaR_{\alpha,t}^i) = \alpha. \tag{3.1}$$

Note that usually $VaR_{\alpha,t}^i$ is a negative number. But in practice X_t^i is often defined as random loss variables so that the positive value of R_t^i represents loss, which switches $VaR_{\alpha,t}^i$ to positive number. We will not follow the convention here for the simplicity of the CoVaR definition in high dimension case. If α =0.05, $VaR_{0.05,t}^i$ represents the probability of X_t^i less than $VaR_{0.05,t}^i$ would not exceed 0.05. The concept of CoVaR is the dependence adjusted of VaR, which was first introduced by Adrian and Brunnermeier [1]. Original definition of CoVaR is just the $\beta-$quantile of the conditional probability distribution,

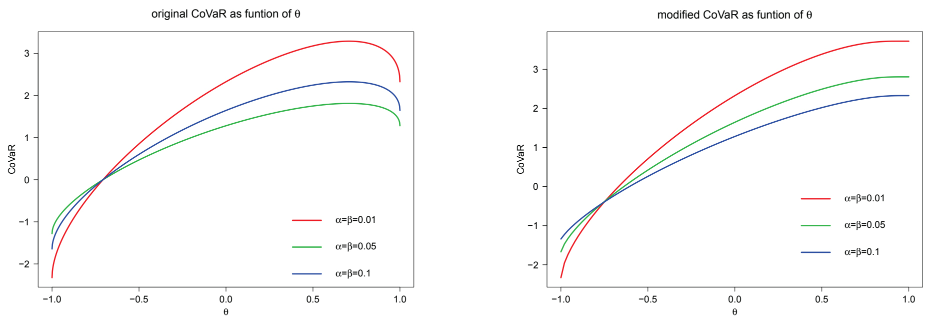
$$Pr(R_t^i \leq CoVaR_{\beta,t}^{i|j} | X_t^j = VaR_{\alpha,t}^j) = \beta. \tag{3.2}$$

However, conditioning on $X_t^j = VaR_{\alpha,t}^j$ has several limitations and drawbacks,which have been discussed by Girardi and Tolga Ergün [14] and Mainik and Schaanning [15]. (1) CoVaR proposed by Adrian and Brunnermeier [1] assumes the conditioning financial stress refers to X_t^i being exactly at its VaR level, which does not consider more severe distress events. (2) It is not consistent with requirements of standard Kupiec [16] and Christoffersen [17] tests to backtest the CoVaR estimates. (3) Mainik and Schaanning [15] argue that, under the definition of Adrian and Brunnermeier [1], CoVaR is not a monotonic function of the dependence parameter (see Figure 3.1a), which is inconsistent with the intuition that systemic risk of one institution should increase with its correlation with financial system. Therefore, CoVaR proposed by Adrian and Brunnermeier [1] fails to detect systematic risk when the correlation is high.

For those reasons above, Girardi and Tolga Ergün [14] propose to generalize the definition by assuming conditioning stress event as its returns being at most at its VaR ($X_t^j \leq VaR_{\alpha,t}^j$). The modified definition of CoVaR is :

$$Pr(X_t^i \leq CoVaR_{\beta,t}^{i|j} | X_t^j \leq VaR_{\alpha,t}^j) = \beta. \tag{3.3}$$

This new-defined conditioning event allows for considering more severe case of losses and facilitates the CoVaR backtesting. More importantly, Mainik and Schaanning [15] find that the modified CoVaR is a continuous and increasing function of dependence parameter (see Figure 3.1b).



(A) Original CoVaR in the bivariate normal model as function of ρ

(B) Modified CoVaR in the bivariate normal model as function of ρ

FIGURE 3.1: Original CoVaR and modified CoVaR OR-MO-CoVaR.R

3.1.2 Δ CoVaR

Adrian and Brunnermeier [1] define the $\Delta CoVaR_{\beta,t}^{i|j}$ as the difference between the $CoVaR_{\beta,t}^{i|j}$ conditioning j being under stress and $CoVaR_{\beta,t}^{i|j,\alpha=0.5}$conditioning j being at normal state(α=0.5), which would be regarded as measurement of the systematic risk contribution of institution j to the risk of system i,

$$\Delta CoVaR^{i|j}_{\alpha,\beta,t} = CoVaR^{i|j}_{\alpha,\beta,t} - CoVaR^{i|j}_{\alpha=0.5,\beta,t}.$$

For the modified CoVaR,Girardi and Tolga Ergün [14] define the analogical systematic risk contribution measurement

$$\Delta CoVaR^{i|j}_{\alpha,\beta,t} = 100 \cdot (CoVaR^{i|j}_{\beta,t} - CoVaR^{i|b^j}_{\beta,t})/CoVaR^{i|b^j}_{\beta,t},$$

where b^j is a one-standard deviation from the mean event, $\mu^j_t - \sigma^j_t \leq X^j_t \leq \mu^j_t + \sigma^j_t$, μ^j_t and σ^j_t are conditional mean and conditional standard deviation of institution j respectively. Hence, ΔCoVaR proposed by Girardi and Tolga Ergün [14] is the percentage change of CoVaR. Reboredo and Ugolini [18] employ similar definition of systemic risk contribution as Girardi and Tolga Ergün [14].

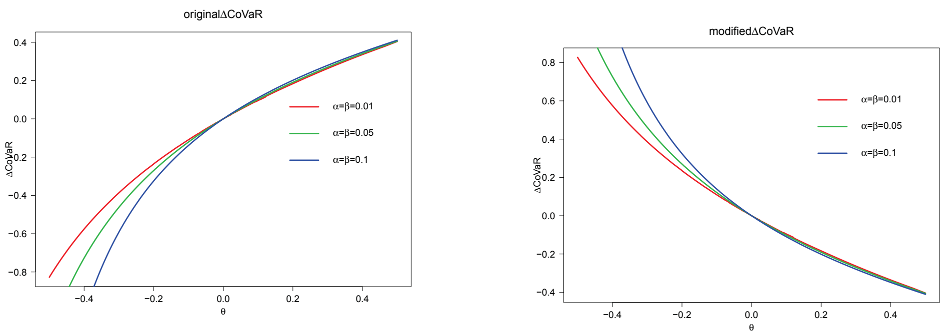
What we are interested in is the change of CoVaR from normal state to distressed state, which measures the risk contribution of j to i. In this thesis, we decide to redefine systemic risk contribution as the percentage change of the CoVaR standardized by absolute value of benchmark state CoVaR (see equation 3.4). There are several reasons to adopt the new-defined systemic risk contribution,

(1) ΔCoVaR defined as simple change of CoVaR in Adrian and Brunnermeier [1] is not standardized, which may be not a proper index for comparison;

(2) ΔCoVaR defined as percentage change of CoVaR in Reboredo and Ugolini [18] allows negative scaling denominator, which may reverse the sign of ΔCoVaR and results in misleading results (see Figure 3.2). Remember that ΔCoVaR should decrease with the dependence parameter;

(3) The new-defined ΔCoVaR allows to capture both positive dependence and negative dependence.

$$\Delta CoVaR^{i|j}_{\alpha,\beta,t} = 100 \cdot \frac{CoVaR^{i|j}_{\alpha,\beta,t} - CoVaR^{i|j}_{\alpha=0.5,\beta,t}}{|CoVaR^{i|j}_{\alpha=0.5,\beta,t}|} \tag{3.4}$$



(A) Original ΔCoVaR defined in Reboredo and Ugolini [18]

(B) New-defined ΔCoVaR

FIGURE 3.2: Comparison of two different ΔCoVaR definitions(Guassian Copula with norm margins N(0.5,1)) New-delta-CoVaR.R

3.1.3 Current Computation Methodology

CoVaR measure can be estimated in many different ways. In this section we will mention three mainstream methods to compute CoVaR, quantile Regression CoVaR (Adrian and Brunnermeier [1]), bivariate GARCH CoVaR (Girardi and Tolga Ergün [14]), and Copula-based CoVaR (Reboredo and Ugolini [18]). We will briefly review the methodology of quantile regression and GARCH to compute CoVaR, because what we are interested in is copula-based CoVaR. We will discuss copula-based CoVaR in details in this section and extend it to high dimension case.

Quantile Regression CoVaR

Considering the fact that $VaR^i_{\alpha,t}$ is α-quantile of the return distribution, and that $CoVaR^{i|j}_{\alpha,\beta,t}$ is just VaR of conditional distribution, quantile regression is a straightforward way to obtain CoVaR. Adrian and Brunnermeier [1] run linear quantile regression of X^i_t on a set of lagged state variables M_{t-1} to get estimated time-varying $VaR^i_{q,t}$, and linear quantile regression of X^j_t on X^i_t together with the same lagged state variables,

$$\begin{aligned} X^i_t =& \alpha^i + \beta^i M_{t-1} + \epsilon^i_t, \\ X^j_t =& \alpha^{j|i} + X^j_t + \beta^{j|i} M_{t-1} + \epsilon^{j|i}_t. \end{aligned} \tag{3.5}$$

Consequently, CoVaR can be obtained in following way:

$$\begin{aligned} VaR^{i}_{q,t} =& \hat{\alpha^{i}} + \hat{\beta^{i}} M_{t-1}, \\ CoVaR^{i|j}_{q,t} =& \hat{\alpha^{j|i}} + VaR^{i}_{q,t} + \hat{\beta^{j|i}} M_{t-1}. \end{aligned} \tag{3.6}$$

Quantile regression proposed by Adrian and Brunnermeier [1] is one of trackable and efficient ways to estimate CoVaR. However, one of the drawbacks is that the effect of VaR on CoVaR stays constant although CoVaR in equation 3.6 is time-varying, which is unrealistic and does not capture the effect of time-varying dependence on CoVaR.

Bivariate GARCH CoVaR

Girardi and Tolga Ergün [14] propose to calculate CoVaR via multivariate DCC model. If we start from CoVaR definition in equation 3.3, given that $Pr(X^i_t \leq VaR^i_\alpha) = \alpha$, CoVaR should satisfy the following equation,

$$Pr(X^j_t \leq CoVaR^{j|i}_{\alpha,\beta,t}, X^i_t \leq VaR^i_{\alpha,t}) = \alpha\beta, \tag{3.7}$$

which requires the knowledge of bivariate joint distribution of (X^i, X^j) and to calculate the $VaR^i_{\alpha,t}$. Girardi and Tolga Ergün [14] run a three-step procedure to estimate CoVaR. First, VaR of each institution i is obtained by estimating a univariate GARCH (1,1) models for each time period; Second, for the return of institution i and j, they set up a bivariate GARCH model with DCC specification to estimate the pdf of (X^i, X^j). Third, according to the equation 3.7, once $VaR^i_{\alpha,t}$ and pdf of (X^i, X^j) have been estimated in previous two steps, CoVaR can be obtained by numerically solving the equation,

$$\int_{-\infty}^{CoVaR^{j|i}_{\alpha,\beta,t}} \int_{-\infty}^{VaR^i_{\alpha,t}} f_t(X^i, X^j) dx dy = \alpha\beta, \tag{3.8}$$

where $f(X^i, X^j)$ is the bivariate density of X^i and X^j.

3.2 Copula-based CoVaR

Methodology

This thesis will employ the modified version of CoVaR propose by Girardi and Tolga Ergün [14]. As the definition of CoVaR shown, the key to compute CoVaR is to find the conditional probability distribution function. Let X^i and X^j be the random variable representing the returns of two financial institutions i and j, then the general case of CoVaR could be express as

$$CoVaR^{j|i}_{\alpha,\beta} = VaR_{\beta}(X_j|X_i \leq VaR^i_{\alpha}), \tag{3.9}$$

which requires to know the conditional cumulative density function $F_{X_j|X_i \leq VaR^i_{\alpha}}(x_j)$, and

$$\begin{aligned} F_{X_j|X_i \leq VaR^i_{\alpha}}(x_j) =& Pr_j \leq x_j|X_i \leq VaR^i_{\alpha}) \\ =& \frac{Pr(X_j \leq x_j, X_i \leq VaR^i_{\alpha})}{Pr(X_i \leq VaR^i_{\alpha})}. \end{aligned}$$

Calculation of $Pr(X_j \leq x_j, X_i \leq VaR^i_{\alpha})$ requires the information of bivariate CDF of X_j and X_i, which give us the opportunity to utilize copula to compute CoVaR .

Let $(U, V) \sim C$, where C is the copula of $F_{X_i,X_j}(x_i, x_j)$ and $U = F_{X_i}(x_i)$, $V = F_{X_j}(x_j)$ are margins of X_j and X_i. We could decompose bivariate distribution function into copula function C and their margins according to Sklar's Theorem

$$F_{X_i,X_j}(x_i, x_j) = C(F_{X_i}(x_i), F_{X_j}(x_j)).$$

If we follow the idea of Mainik and Schaanning [15], we can easily show

$$F_{X_j|X_i \leq VaR^i_{\alpha}}(x_j) = \frac{C(\alpha, v)}{\alpha}.$$

Then the expression of CoVaR based on copula in bivariate case is

$$CoVaR^{j|i}_{\alpha,\beta} = F^{-1}_{X_j}(F_{X_j|X_i \leq VaR^i_{\alpha}}(x_i)) \qquad , \tag{3.10}$$

where $F_{X_j|X_i \leq VaR^i_{\alpha}}(x_i) = \frac{C(\alpha,v)}{\alpha}$.

Alternatively, recalling CoVaR's definition $Pr(R_t^i \leq CoVaR_{beta}^{i|j} | X_t^j \leq VaR_\alpha^j) = \beta$, if the marginal distribution of X_j and copula function are given, CoVaR also can be obtained by numerically solving the following equation:

$$C(F_{x_j}(CoVaR_{\alpha,\beta}^{j|i}), \alpha) = \alpha\beta. \tag{3.11}$$

Similarly, for survived copula, we have

$$\bar{C}(1 - F_{x_j}(CoVaR_{\alpha,\beta}^{j|i}), 1 - \alpha) + \alpha + F_{x_j}(CoVaR_{\alpha,\beta}^{j|i}) - 1 = \alpha\beta. \tag{3.12}$$

Note that copula-based CoVaR only needs the information of cumulative probability of VaR rather than VaR itself according to the equation 3.11, which makes computation of copula-based CoVaR less cumbersome than bivariate GARCH CoVaR. Copula-based CoVaR has several advantages as Reboredo and Ugolini [18] point out.

(1) Copula makes it flexible to model the marginal distributions. The main advantage of copula is that it could allow separately modelling the margins and dependence structure, which is essential for the computation of CoVaR, because mis-specified marginal model would result in wrong information for copula.

(2) Copula could capture more dependency information than traditional dependence measure given by linear correlation coefficient, especially when the joint distribution is not elliptical.

(3) Finally, Copula-based CoVaR is computationally more tractable than CoVaR proposed by Girardi and Tolga Ergün [14]. The equation 3.8 requires numerically solving of double integral and VaR of conditional variables,Copula-based CoVaR, in contrast, even has explicit form for some Archimedean copulas.

3.3 Multivariate CoVaR

Another advantage of copula-based CoVaR is the convenience to be extended to higher dimension case to quantify more possible risk situations. For example, we may ask questions such as, how does the systemic risk change if more than one institution fall into financial distress at the same times, or how much reserves do we need if the top two risky bonds in our portfolio default simultaneously ? Obviously, bivariate CoVaR is not capable to answer these questions, which requires us to extend CoVaR to higher dimension.

Complying with the same logics and notation of bivariate CoVaR in equation 3.2, if we have d-1 conditional variables, multivariate CoVaR can be defined as follows,

$$Pr(X_t^d \leq CoVaR_{\alpha_1,\ldots,\alpha_d,t} | X_t^1 \leq VaR_{\alpha_1}^1, \ldots, X_t^{d-1} \leq VaR_{\alpha_{d-1}}^{d-1}) = \alpha_d. \tag{3.13}$$

Let $(u_1, \ldots, u_d) \sim C_1$, and $(u_1, \ldots, u_{d-1}) \sim C_2$, where C_1 and C_2 are copula function of $F_{1,\ldots,d}(x_1, \ldots, x_d)$ and $F_{1,\ldots,d-1}(x_1, \ldots, x_{d-1})$ respectively. Then equation 3.13 can be expressed using copula as follows,

$$\frac{C_1(F_d(CoVaR_{\alpha_1,\ldots,\alpha_d,t}), \alpha_1, \ldots, \alpha_{d-1}; \theta_1)}{C_2(\alpha_1, \ldots, \alpha_{d-1}; \theta_2)} = \alpha_d, \tag{3.14}$$

where F_d is the CDF of X_d , θ_1 and θ_2 are parameters of copula C_1 and C_2 respectively. Numerically solving the equation 3.14, we will get the multivariate copula-based CoVaR.

3.4 Simulation

In this section, simulation will be implemented to study the properties of CoVaR. We estimate CoVaR and ΔCoVaR under different margins and copula specification, and investigate the correlation of CoVaR and ΔCoVaR with dependence parameter to guarantee the accuracy of the interpretation of CoVaR and ΔCoVaR.

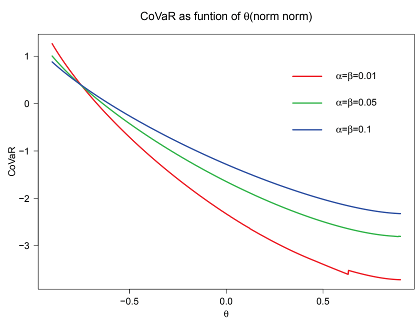
CoVaR and Copula Dependence

Designed as a systemic risk index, CoVaR should be negative correlated with copula dependence. Intuitively, if one of financial institutions in system has higher correlation with the whole system, its CoVaR should be lower when it falls into financial stress (usually CoVaR is a negative value in practice). Gaussian copula is often regarded as the benchmark in literature, so it is reasonable to start with Gaussian copula case. Mainik and Schaanning [15] conclude that CoVaR is always decreasing in dependence parameter for bivariate elliptical copula (see Theorem 3.6 in Mainik and Schaanning [15]

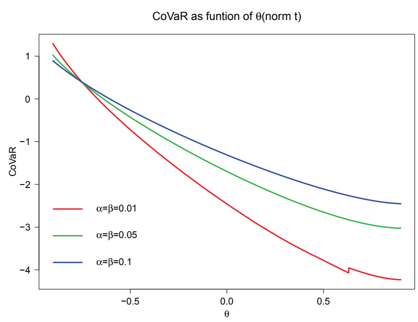
). Figure 3.3 shows the simulation result about the correlation between CoVaR and Gaussian copula dependence parameter θ with different margins, which exactly meets our expectation. The interesting part here is the ordering of CoVaR under different α (or β). From the intuitive point of view, smaller α means that the financial institutions falls in more severe financial stress, which in return would lead to smaller CoVaR. However, Figure 3.3[1] indicates that when dependence parameter is negative,the ordering is reversed at one specified point. Same property has been found in student-t copula case.

In Archimedean copula case, the monotonicity of CoVaR in copula dependence parameter and ordering under different α (or β) are consistent with our expectation. Figure 3.4 displays the results of simulation in Gumbel copula case. Note that Gumbel copula can only capture positive rank correlation. Similar properties have been found in Frank, Joe and Clayton copula cases.

For multivariate CoVaR, simulation results (see Figure 3.5) show CoVaR is still negative correlated with $\theta 1$ in multivariate setting. Similar properties have been found in Frank, Joe and Clayton copula cases.



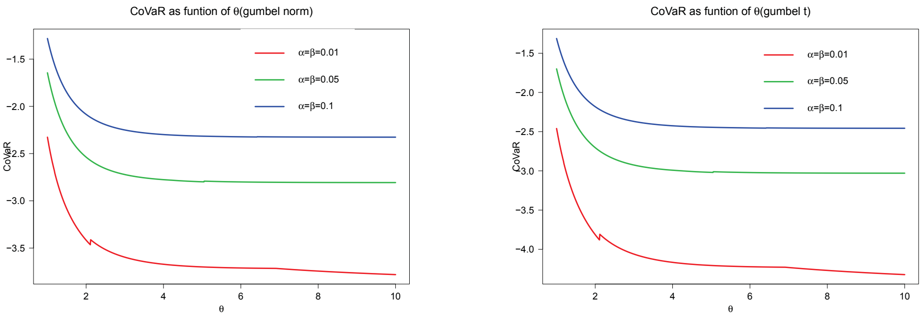
(A) Gaussian Copula CoVaR with standard normal distribution



(B) Gaussian Copula CoVaR with t(30) distribution

FIGURE 3.3: Gaussian Copula CoVaR as function of θ Gau-CoVaR-Sim.R

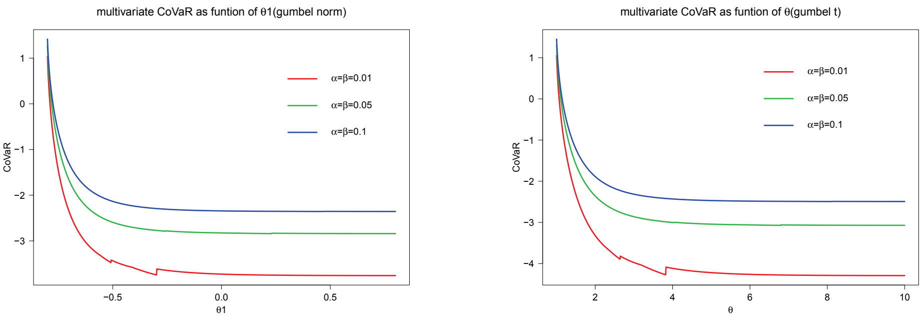
[1]Breaking point is caused by the discontinuity of θ and accuracy of tolerance during numerically solving



(A) Gumbel Copula CoVaR with standard normal distribution

(B) Gumbel Copula CoVaR with t(30) distribution

FIGURE 3.4: Gumbel Copula CoVaR as function of θ Gum-CoVaR-Sim.R



(A) Mulitvariate gumbel CoVaR with normal distribution

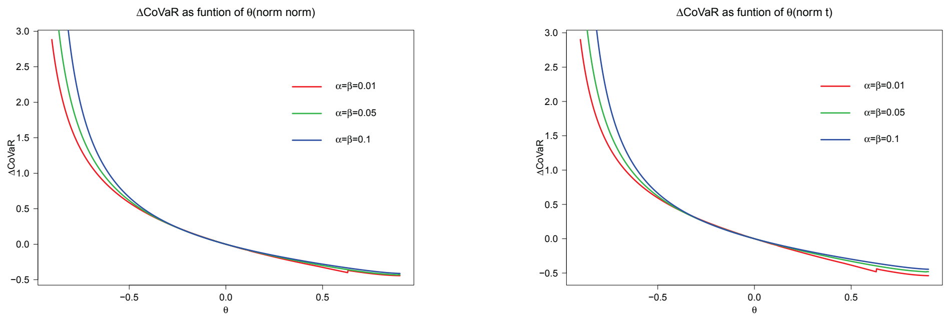
(B) Mulitvariate gumbel CoVaR with t(30) distribution

FIGURE 3.5: Mulitvariate gumbel CoVaR as function of $\theta 1$ Gum-3d-CoVaR-Sim.R

ΔCoVaR and Copula Dependence

The value of CoVaR itself seems not so important and it is not our aim. What is perhaps more important and interesting is the risk contribution measure Δ CoVaR as defined in equation 3.4. Intuitively, we expect ΔCoVaR would be a decreasing function of dependence parameters, since it is obvious that the institution, which has higher dependence with system, should have more risk contribution to the entire system. Figure 3.6 and Figure 3.7 suggest that Δ CoVaR defined in equation 3.4 is consistent with our expectation. Moreover, the Δ CoVaR 's ordering of different confidence level in Gumbel copula case is more reasonable than that in elliptical copula case. Similar properties have been found in Frank, Joe and Clayton copula cases. In multivariate setting, ΔCoVaR is also deceasing function of $\theta 1$ in Gumbel, Joe, Frank and Clayton copula (see Gumbel case 3.8).

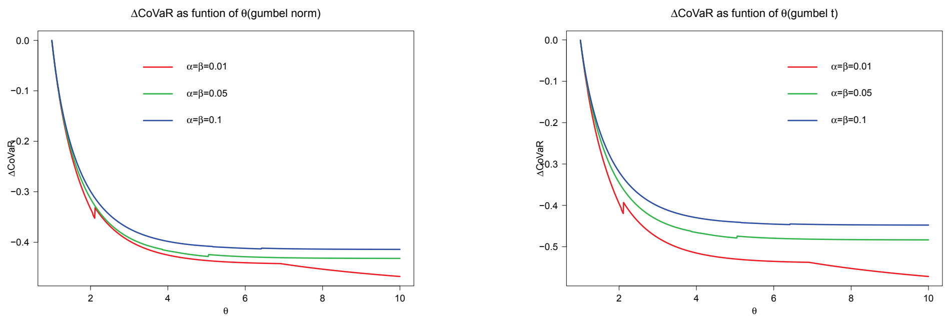
In conclusion, simulation results further confirm that both CoVaR and ΔCoVaR are decreasing function of dependence parameter either in bivariate case or multivariate case, which strengthens the reasonability and validity of copula-based CoVaR designed in this thesis.



(A) Gaussian Copula ΔCoVaR with standard normal distribution

(B) Gaussian Copula ΔCoVaR with t(30) distribution

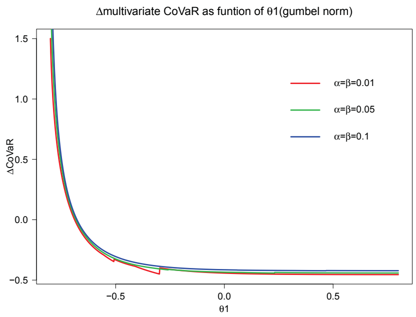
FIGURE 3.6: Gaussian Copula ΔCoVaR as function of θ Gua-delta-CoVaR-Sim.R



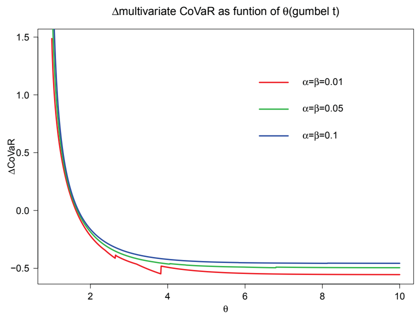
(A) Gumbel Copula ΔCoVaR with standard normal distribution

(B) Gumbel Copula ΔCoVaR with t(30) distribution

FIGURE 3.7: gumbel Copula ΔCoVaR as function of θ Gum-delta-CoVaR-Sim.R



(A) Multivariate Gumbel ΔCoVaR with normal distribution



(B) Multivariate Gumbel ΔCoVaR with t(30) distribution

FIGURE 3.8: Multivariate Gumbel Δ CoVaR as function of $\theta 1$ Gum-3d-delta-CoVaR-Sim.R

Chapter 4

Empirical Study

4.1 Data

We evaluate the systemic risk of eight European countries by considering weekly data of sovereign bond benchmark price indices, including non-crisis markets Germany, France, the Netherlands and GIIPS markets (Greece, Ireland, Italy, Portugal, Spain), and we select the European Economic and Monetary Union Government bond index (EMU) as the representative of the system. All bond price indices are sourced from Datastream for 10 years maturities starting from 7 January 2000 to 1 March 2015.

Table 4.1 gives a report of descriptive statistics for bond price returns (log-return). As we expect, the average returns have slight difference across countries, while the standard deviations of GIIPS markets are much higher than non-crisis markets. The autocorrelation coefficients for squared returns and absolute returns are much higher than those for log-returns for all return series, which suggests ARCH effects may be found for all return series. The ARCH-Lagrange multiplier statistics further confirm our conjecture. High kurtosis for most of return series indicates the fat tails in return distributions, which is consistent with result of Jarque-Bera test (all return series strongly reject the normality hypothesis). Finally, results of ADF test and KPSS test show that all return series are stationary. Table 4.2 shows the rank correlation of the data. All countries in our sample are highly correlated with the EMU index and high positive dependence is also shown within the system.

	GM	IT	FR	SP	GR	NL	IR	PT	EMU
Min	-0.0232	-0.0427	-0.0318	-0.0489	-0.2904	-0.0290	-0.1027	-0.1370	-0.0232
Max	0.0289	0.0817	0.0386	0.0793	0.2933	0.0319	0.1186	0.1266	0.0289
Median	0.0011	0.0007	0.0012	0.0008	0.0006	0.0013	0.0009	0.0008	0.0011
Mean	0.0006	0.0006	0.0007	0.0006	-0.0008	0.0007	0.0006	0.0006	0.0006
Std.dev	0.0080	0.0097	0.0080	0.0111	0.0350	0.0078	0.0145	0.0191	0.0080
ACF	-0.0773	-0.0448	-0.0761	-0.1680	0.0465	-0.0532	0.0531	0.0263	-0.0781
ACFS	0.0882	0.1396	0.1648	0.1123	0.3173	0.1012	0.4085	0.2502	0.0876
ACFABS	0.0813	0.2432	0.0909	0.2253	0.4524	0.0410	0.4470	0.4003	0.0802
Kurtosis	0.3326	9.4338	1.3920	8.9383	24.0191	0.9920	18.5606	13.3512	0.3339
Skewness	-0.1298	0.9312	-0.0943	0.9018	-0.9160	-0.2132	0.0659	-0.1367	-0.1290
JB	6.29*	3110.5*	67.62*	2797.42*	19491.7*	40.06*	11577.7*	5996.7*	6.30*
ADF	-8.951*	-9.743*	-9.331*	-9.848*	-6.469*	-9.150*	-8.827*	-8.212*	-8.939*
KPSS	0.1872	0.2089	0.2376	0.2304	0.1544	0.1763	0.2402	0.2630	0.1852
Autocor	9.068	13.285	8.403	32.057*	22.848*	7.184	11.685	36.198*	9.333
ARCH	124.65*	54.26*	128.88*	72.18*	116.20*	101.43*	161.00*	136.86*	124.38*

This table shows the description statistics of return data, and results of some relevant tests are also given. ACF reports the auto-correlation coefficient of return series; ACFS offers the auto-correlation coefficient of squared return series; ACFABS gives the auto-correlation coefficient of absolute returns; JB provides the results of Jarque-Bera test to test normality; ADF and KPSS are stationarity test . Des-Sta-Return.R

TABLE 4.1: Descriptive statistics for sovereign bond price returns

	GM	IT	FR	SP	GR	NL	IR	PT	EMU
GM	1.0000	0.4880	0.7721	0.5472	0.3922	0.8434	0.4703	0.4571	0.9991
IT	0.4880	1.0000	0.5540	0.7191	0.6051	0.5325	0.6173	0.6253	0.4875
FR	0.7721	0.5540	1.0000	0.6021	0.4657	0.8227	0.5361	0.5201	0.7725
SP	0.5472	0.7191	0.6021	1.0000	0.5952	0.5900	0.6269	0.6332	0.5475
GR	0.3922	0.6051	0.4657	0.5952	1.0000	0.4369	0.6098	0.6481	0.3926
NL	0.8434	0.5325	0.8227	0.5900	0.4369	1.0000	0.5347	0.5027	0.8438
IR	0.4703	0.6173	0.5361	0.6269	0.6098	0.5347	1.0000	0.6591	0.4697
PT	0.4571	0.6253	0.5201	0.6332	0.6481	0.5027	0.6591	1.0000	0.4566
EMU	0.9991	0.4875	0.7725	0.5475	0.3926	0.8438	0.4697	0.4566	1.0000

TABLE 4.2: Kendall'tau matrix of data M-tau.R

4.2 Models for Margins and Copula

For Marginal models, we consider ARMA model for conditional mean and TGARCH model for conditional variance. The marginal model is specified as follows,

$$R_t = \mu + \sum_{i=1}^{m} \phi_i R_{t-i} + \sum_{j=1}^{n} \psi_j \epsilon_{t-j} + \epsilon_t \tag{4.1}$$

This is ARMA(m,n) specification for the conditional mean, where R_t is the return of market return for a European debt market, μ is a constant, ϵ_t is the error term with

$\epsilon_t = z_t \sigma_t$, σ_t is the conditional variance, given by TGARCH(p,q) specification,

$$\begin{aligned} \epsilon_t =& z_t \sigma_t, \\ z_t \sim& D(0,1), \\ \sigma_t^2 =& \omega + \sum_{i=1}^{p} \alpha_i \sigma_{t-i}^2 + \sum_{j=1}^{q} \beta_j \epsilon_{t-j}^2 + \sum_{j=1}^{q} \eta_j 1_{t-j} \epsilon_{t-j}^2, \end{aligned} \tag{4.2}$$

where 1_{t-j}=1 if $\epsilon_{t-j} < 0$, otherwise 1_{t-j}=0; σ_{t-j}^2 is the GARCH component, and ϵ_{t-i}^2 is the ARCH component, z_t is i.i.d random variable with zero mean and unit variance that follows skewed-t distribution, which is suggested by the fact of skewness and fat tail in the data reported in table 4.1.

For copula model, we try several popular copula specifications mentioned in Chapter 1 to capture different types of dependence structure. We also allow dependence parameters to vary with time by following some specified evolution pathes. Therefore, CoVaR becomes time-varying not only because of the time-varying mean and variance in marginal model, but also the time-varying dependence in copula model. Patton [19] consider the time-varying copula with fixed copula form and its dependence parameter vary through time as a function of lagged information, which is very similar with the GARCH specification. For Gaussian copula, the parameter ρ_t is specified as follows:

$$\rho_t = \Lambda(a + b\rho_{t-1} + c \sum_{i=1}^{q} \Phi^{-1}(u_{t-j}) \cdot \Phi^{-1}(v_{t-j})/q), \tag{4.3}$$

where $\Lambda(x) = (1 - \exp(-x))(1 + \exp(-x))^{-1}$ is the modified logistic transformation [1]. The purpose of the transformation is to make ρ_t bound in (-1,1). Φ^{-1} is the quantile transformation of standard normal distribution. For student-t copula, we just replace Φ^{-1} with t^{-1}. Similarly, for Archimedean copula, the evolution path is described as,

$$\psi_t = a + b\psi_{t-1} + c \sum_{i=1}^{q} |u_{t-j} - v_{t-j}|/q. \tag{4.4}$$

The parameters in marginal models and copula models are estimated by two-step procedure called the inference functions for margins proposed by Joe and Xu [20]. According

[1] In Archimedean copula case, we deploy similar transformation to keep dependence parameter in the domain of definition. For example, we take transformation like $\Lambda(x) = (1 + \exp(-x))^{-1}$ for Gumbel copula

to the equation 2.3, the log-likelihood function can be decomposed as the sum of the log-likelihood of margins and the log-likelihood of copula density. Hence,the procedure of estimation can be described as follows,

(1) estimate the parameters of marginal models separately by maximum likelihood method;

(2) transform the return data to presudo-sample observations by probability integral transformation for copula model, namely, $\hat{u}_t = F_{R_{i,t}}(R_{i,t}; \hat{\theta}_i)$ and $\hat{v}_t = F_{R_{j,t}}(R_{j,t}; \hat{\theta}_j)$, which would be used in next step to estimate copula parameters;

(3) estimate the copula parameters by solve the problem,

$$\theta_c = \arg\max_{\theta_c} \sum_{t=1}^{T} \log c(\hat{u}_t, \hat{v}_t; \theta_c), \tag{4.5}$$

where θ_c is the parameter of copula model, $\hat{u}_t$ and $\hat{v}_t$ are pseudo-sample observations getting from step 2;

(4) numerically solve the equation 3.11 to get estimated time-varying CoVaR.

4.3 Empirical Results

Table 4.3 shows the results of estimation for marginal models and table 4.4 reports the results of estimation for different copula models. As for order selection of ARMA(m,n)-GARCH(p,q) model, we try different value combinations of m, n, p, q ranging from zero to six, and select the optimal model according to AIC criteria. We make sure there is no auto-correlation in residuals and squared residuals. Most of estimates in marginal model shown in table 4.3 are significant at 5%, and asymmetric effects are found in all series except for Spain. Significant estimates of parameters in skewed-t distribution indicates error terms are not normally distributed, which is consistent with the facts of fat tail and skewness reported in table 4.1. Estimation results of marginal models indicate our marginal models were not mis-specified and pseudo-observations obtained by probability integral transformation are qualified to be used to estimate the copula model.

Table 4.4 reports the copula model results of the EMU index with eight European countries, and seven different dynamic copula models (including Guassian Copula, Student-t Copula, Gumbel copula, Survive Gumbel Copula, Clayton Copula, Frank Copula and Joe Copula) are tried during estimation. According to AIC value, Survive Gumbel copula are best fitted for all of series except for Germany and Ireland, indicating that there is more stronger lower tail dependence than upper tail dependence with EMU. Table 4.5 gives the statistic summary of estimated ΔCoVaR for eight counties. The first row of table is the mean of ΔCoVaR for each countries. For example, financial distress of Germany bonds market, on average, increases the 1% VaR of Germany by 51.23% over its VaR when it is in the benchmark state. The second row is the standard deviation of ΔCoVaR and the last row is the rank of mean of ΔCoVaR for eight countries. Germany ranks the number one(smallest) in terms of ΔCoVaR values, which is quite reasonable, since Germany is the biggest and most important economy in Europe and any subtle vibration in German market would result in significant shock on the system. Figure 4.1 offers the time-series plot for ΔCoVaR and VaR for each countries and vertical line denotes the outbreak time of European debt crisis (6 November 2009). It is evident that European debt markets are strongly co-moved before European debt crisis and systemic risk contribution index for each market is quite low. However, after the crisis, ΔCoVaR value shots up dramatically for almost every market, especially for GIIPS countries, indicating that crisis countries decoupled with EMU index after crisis. According to Figure 4.1, VaR has no strong correlation with ΔCoVaR although CoVaR of one market is related to its VaR, which is not consistent with Girardi and Tolga Ergün [14].

	GM	IT	FR	SP	GR	NL	IR	PT	EMU
Mean									
ϕ_1	-0.52*	0.98*	-0.47*	-0.60*	-0.30*	-0.49*	0.99*	-0.01	-0.05
	(-41.42)	(335.02)	(-41.81)	(-45.86)	(-11.51)	(-29.90)	(219.16)	(-0.45)	(-1.77)
ϕ_2								0.07*	
								(4.94)	
ψ_1	0.46*	-0.97*	0.41*		0.29*	0.46*		-0.95	
	(35.30)	(-5092)	(36.17)		(12.39)	(29.64)	(-1180.04)		
ψ_2					0.06		0.03*		
					(1.59)				
ψ_3					0.06				
					(1.27)				
ψ_4					0.06				
					(1.87)				
Variance									
ω	0.01*	0.73*	0.02*	0.01*	0.05*	0.01*	0.36*	0.26*	0.01*
	(3.38)	(23.47)	(3.72)	(2.89)	(2.20)	(3.00)	(12.89)	(15.08)	(3.40)
α_1	0.06*	0.21*	0.07*	0.09*	0.21*		0.22*	0.26*	0.06*
	(5.05)	(4.26)	(6.10)	(5.77)	(6.37)		(5.42)	(7.69)	(5.05)
α_2					0.15*		0.38*	0.36*	
					(2.11)		(5.75)	(17.37)	
α_3					0.20*		0.09*	0.23*	
					(2.98)		(2.35)	(8.03)	
α_4					0.13*		0.15*	0.26*	
					(3.37)		(2.71)	(21.25)	
α_5					0.11				
					(1.39)				
α_6					0.05*				
					(3.28)				
β_1	0.92*		0.91*	0.90*		0.91*			0.92
	(78.58)		(76.61)	(56.03)		(78.89)			(78.98)
η_1	-0.30*	0.29*	-0.27*	0.03	0.39*	-0.31*	0.45*	0.11	-0.31*
	(-2.31)	(2.55)	(-2.83)	(0.42)	(2.61)	(-2.45)	(3.76)	(1.02)	(-2.47)
η_2					-0.55*		-0.36*	-0.35*	
					(-2.77)		(-2.12)	(-2.94)	
η_3					-0.52*		-0.06	-0.53*	
					(-2.56)		(-0.20)	(-6.12)	
η_4					-0.65*		-0.22	-0.17	
					(-2.7)		(-0.60)	(-1.88)	
η_5					-0.58				
					(-1.78)				
η_6					0.63*				
					(4.27)				
ξ	0.86*	0.88*	0.86*	0.87*	0.85*	0.84*	0.90*	0.88*	0.86*
	(20.78)	(31.09)	(28.56)	(27.88)	(25.91)	(20.40)	(24.56)		(20.48)
ν	59.99*	5.83*	57.20	11.81*	4.88*	59.17	4.27*	5.52*	59.99*
	(2.65)	(6.24)	(0.72)	(3.43)	(4.69)	(0.46)	(5.81)	(6.29)	(3.08)

the table is to present the maximum likelihood estimates for marginal models described in equation 4.1and equation 4.2, and t-statistics values are also present in parentheses. An asterisk(*) indicates significance at 5%, we use 100*log-return as the estimation sample to avoid convergence problem during estimation

TABLE 4.3: Estimated parameters for marginal models Marg-Est.R

	GM	IT	FR	SP	GR	NL	IR	PT
Gaussian								
a	5.36	3.27	2.10	0.79	4.27	3.18	0.53	0.60
	(291.92)	(0.17)	(6.21)	(0.12)	(0.14)	(7.35)	(0.26)	(0.13)
b	3.29	-0.31	1.01	0.29	0.04	1.17	0.71	0.10
	(292.15)	(0.40)	(6.99)	(0.25)	(0.41)	(7.61)	(0.71)	(0.36)
c	-0.39	-0.07	-0.08	1.81	-0.47	-0.46	1.31	2.00
	(0.13)	(0.21)	(0.13)	(0.14)	(0.24)	(0.14)	(0.34)	(0.21)
AIC	-5463.55	-555.96	-1288.93	-802.07	-671.18	-1965.87	-560.24	-589.51
Student-t								
a	-77.80	1.34	2.33	1.93	1.31	8.85	0.29	0.91
	(115.27)	(0.31)	(2.40)	(0.36)	(1.10)	(2.37)	(0.05)	(0.24)
b	87.12	-0.14	1.64	-0.51	-0.47	-3.02	1.70	0.24
	(115.24)	(0.54)	(2.43)	(0.48)	(2.02)	(2.17)	(0.13)	(0.47)
c	-0.87	0.84	-0.07	1.05	1.31	-1.52	0.65	1.12
	(0.15)	(0.16)	(0.04)	(0.12)	(0.41)	(0.50)	(0.10)	(0.17)
ν	62.26	3.93	2.00	4.58	5.45	7.38	4.98	4.78
	(40.25)	(0.52)	(0.32)	(0.42)	(0.50)	(2.03)	(0.68)	(0.49)
AIC	**-5484.71**	-677.49	-1655.96	-932.75	-687.56	-1995.75	-692.04	-643.72
Gumbel								
a	0.76	0.66	0.68	0.20	0.86	3.03	0.77	0.67
	(0.18)	(0.50)	(0.19)	(0.09)	(0.47)	(0.48)	(0.25)	(0.55)
b	0.80	0.70	0.76	0.91	0.63	-0.06	0.65	0.72
	(0.05)	(0.23)	(0.07)	(0.04)	(0.21)	(0.17)	(0.11)	(0.24)
c	1.02	-4.98	-5.53	-1.33	-6.45	-27.54	-5.54	-5.65
	(4.60)	(3.74)	(1.43)	(0.47)	(3.37)	(4.70)	(1.75)	(4.56)
AIC	-5216.43	-930.81	-1963.09	-1292.96	-964.08	-2213.07	**-973.82**	-902.47
Frank								
a	2.10	1.48	1.57	0.55	0.26	1.60	0.88	0.61
	(0.00)	(0.03)	(0.03)	(0.01)	(0.02)	(0.09)	(0.04)	(0.01)
b	3.26	0.42	0.55	0.81	0.90	0.32	0.67	0.78
	(0.00)	(0.02)	(0.01)	(0.01)	(0.00)	(0.02)	(0.02)	(0.01)
c	-8.12	-3.76	-6.77	-1.53	-0.77	0.97	-2.38	-1.66
	(0.00)	(0.04)	(0.03)	(0.03)	(0.05)	(0.05)	(0.12)	(0.03)
AIC	-5124.72	-942.27	-1941.57	-1296.61	-852.96	-2255.34	-712.09	-926.03
Joe								
a	0.64	0.08	0.14	0.26	0.18	0.80	0.15	0.26
	(0.17)	(0.02)	(0.02)	(0.25)	(0.04)	(0.15)	(0.04)	(0.14)
b	0.80	0.96	0.94	0.87	0.93	0.65	0.92	0.88
	(0.05)	(0.01)	(0.01)	(0.13)	(0.02)	(0.07)	(0.02)	(0.07)
c	0.63	-0.88	-1.42	-2.56	-1.90	-8.48	-1.37	-2.87
	(4.33)	(0.19)	(0.16)	(2.19)	(0.43)	(1.74)	(0.30)	(1.40)
AIC	-4725.51	-739.09	-1588.78	-1076.67	-781.98	-1833.77	-787.49	-708.35
Survive Gumbel								
a	3.61	1.21	1.05	1.39	1.87	3.77	1.02	0.18
	(1.90)	(0.38)	(0.59)	(1.27)	(0.72)	(0.75)	(0.56)	(0.54)
b	0.46	0.50	0.69	0.51	0.26	-0.26	0.54	0.92
	(0.50)	(0.16)	(0.18)	(0.44)	(0.28)	(0.27)	(0.25)	(0.27)
c	0.46	-8.30	-8.25	-10.07	-12.79	-26.68	-7.07	-1.27
	(333.40)	(2.60)	(4.53)	(9.42)	(4.90)	(4.82)	(3.94)	(3.17)
AIC	-5361.23	**-994.83**	**-2141.45**	**-1420.05**	**-1014.77**	**-2348.44**	-924.71	**-943.55**
Clayton								
a	0.79	0.28	0.77	0.22	0.40	0.21	2.57	0.12
	0.21	0.04	0.27	0.02	0.25	0.04	0.27	0.03
b	0.79	0.89	0.73	0.92	0.80	0.93	-0.56	0.94
	0.06	0.02	0.09	0.01	0.12	0.01	0.14	0.02
c	-0.64	-0.73	-6.74	-0.59	-3.43	-1.96	-22.73	-1.06
	6.37	0.10	2.36	0.04	2.16	0.36	2.99	0.30
AIC	-4880.85	-940.74	-1832.26	-1279.03	-689.74	-2062.64	51.60	-713.28

This table displays the ML estimates of copula model and its standard error (in brackets). AIC of the copula models are also provided to select the best fitted model. Standard error is calculated as the square root of Fisher information matrix and Note that some standard error of estimates may be affected by starting value in optimization. The minimum AIC value (in bold) indicates the best copula fit. q in equation 4.3 and 4.4 is set to 10; both α and β are 0.01.

TABLE 4.4: Estimated parameters for time-varying copula models Cop-Est.R

	GM	IT	FR	SP	GR	NL	IR	PT
Mean	-0.5123	-0.3276	-0.4414	-0.3521	-0.3013	-0.5082	-0.3157	-0.3233
SD	0.0097	0.2091	0.1307	0.2217	0.2265	0.0447	0.2137	0.2155
Max	-0.4855	0.0000	-0.0184	0.0000	0.0000	-0.2598	-0.0007	-0.0040
Min	-0.5424	-0.5436	-0.5903	-0.5726	-0.5382	-0.5868	-0.5342	-0.5384
Rank	1	5	3	4	8	2	7	6

Rank is the rank of mean by increasing order

TABLE 4.5: Statistics summary of estimated copula-based CoVaR Sum-delta-CoVaR.R

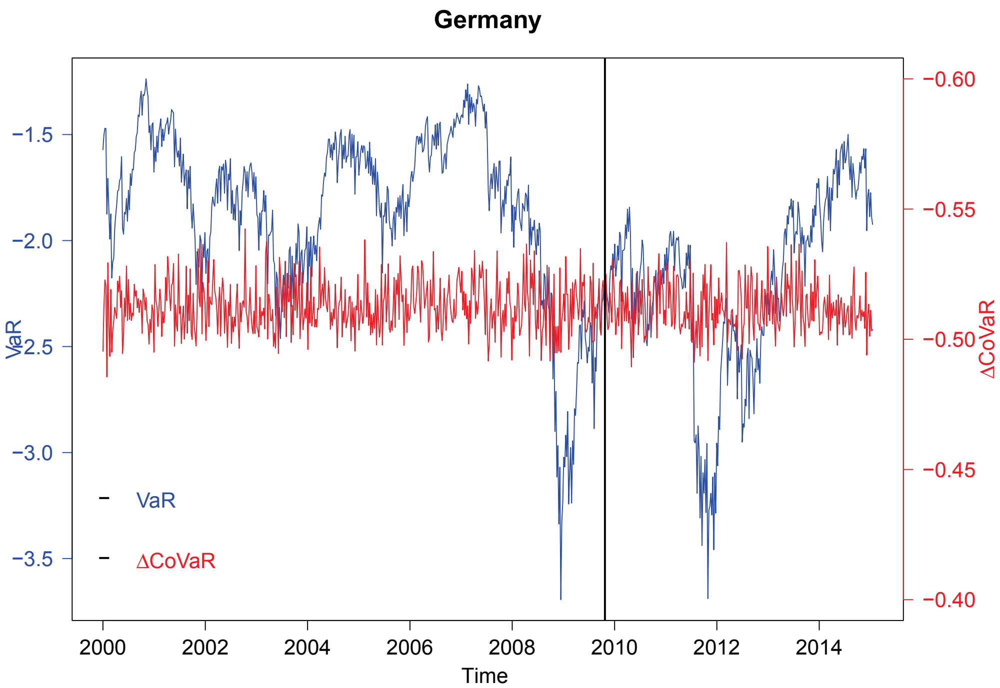


FIGURE 4.1: Estimated time-varying ΔCoVaR and VaR for eight European countries VaR-delta-CoVaR.R

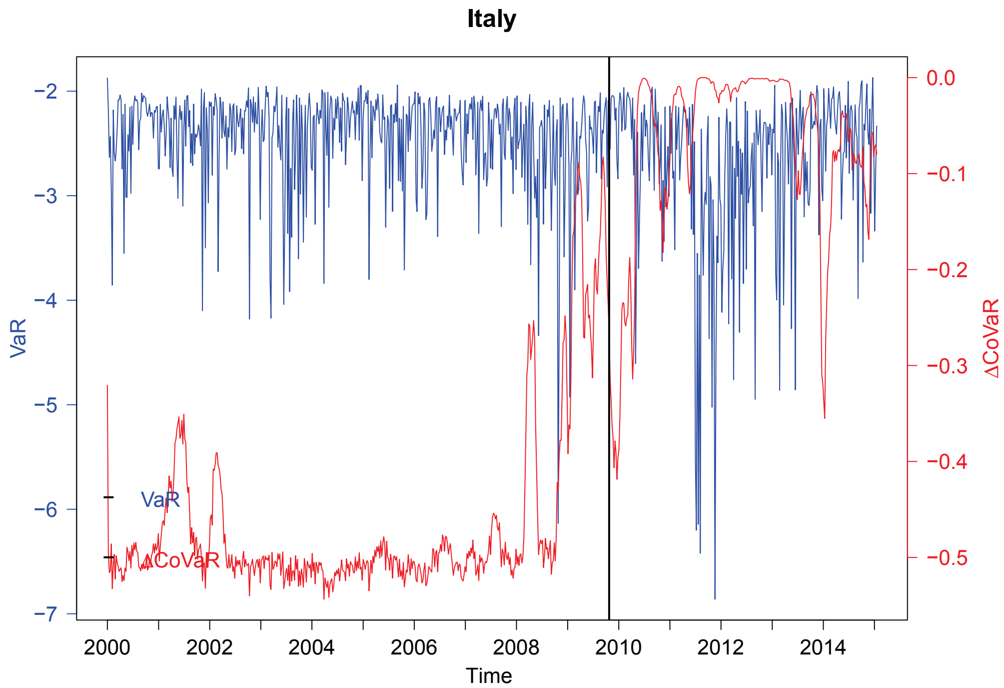


FIGURE 4.1: Estimated time-varying ΔCoVaR and VaR for eight European countries*(cont.)*

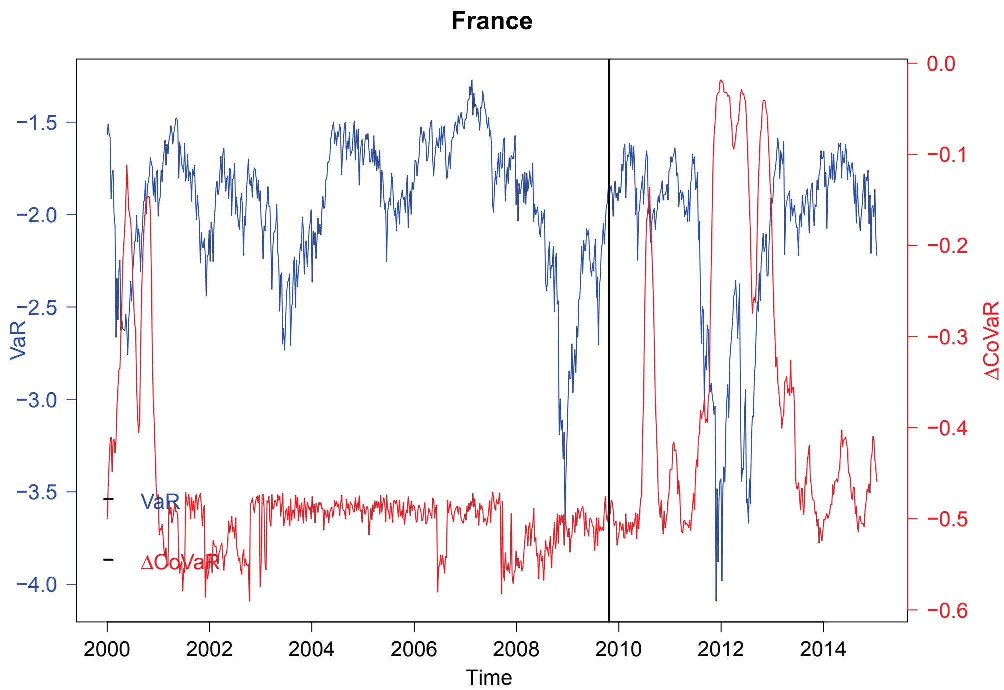


FIGURE 4.1: Estimated time-varying ΔCoVaR and VaR for eight European countries*(cont.)*

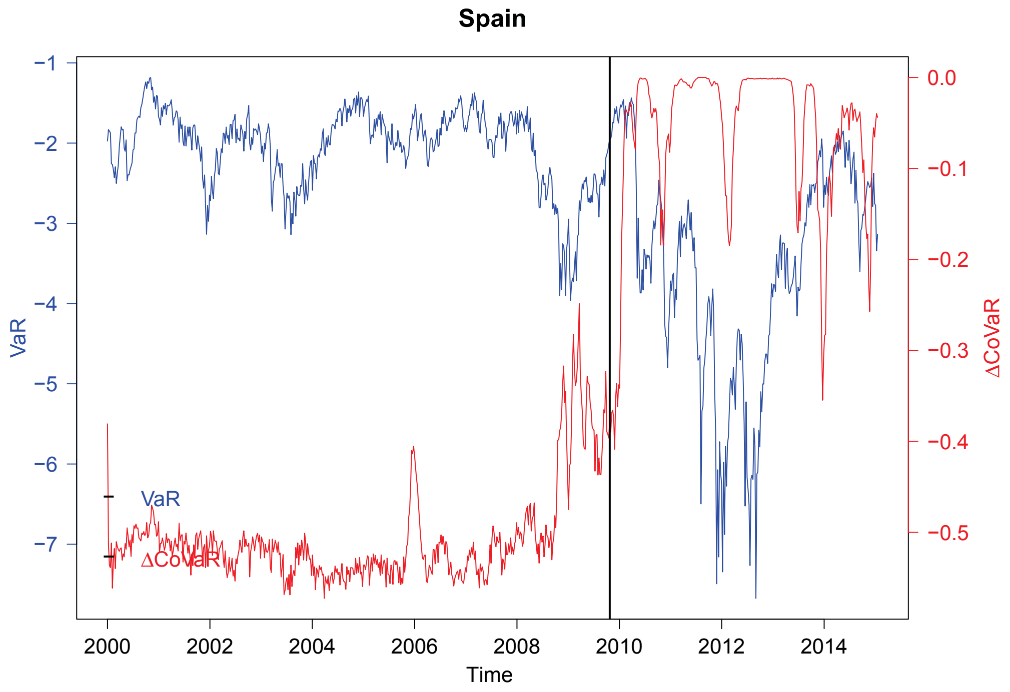


FIGURE 4.1: Estimated time-varying ΔCoVaR and VaR for eight European countries*(cont.)*

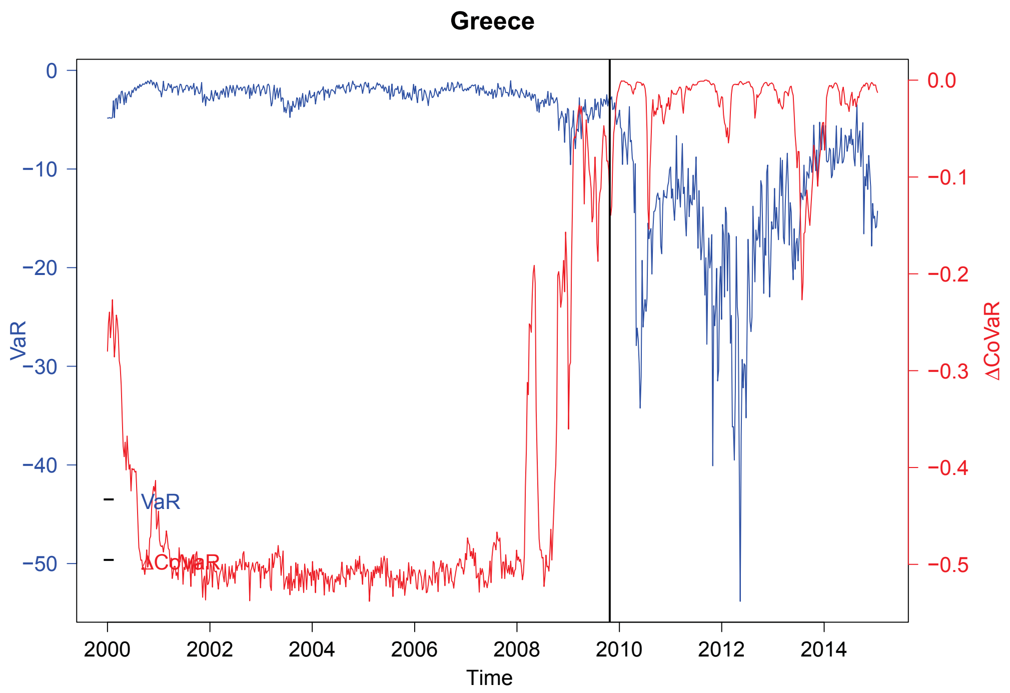


FIGURE 4.1: Estimated time-varying ΔCoVaR and VaR for eight European countries*(cont.)*

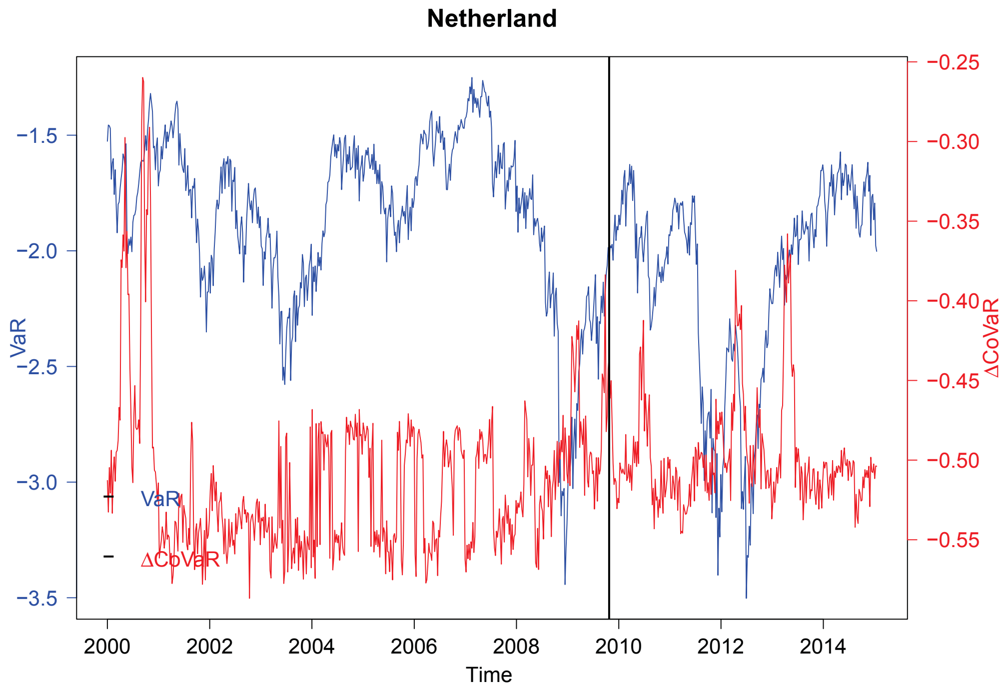


FIGURE 4.1: Estimated time-varying ΔCoVaR and VaR for eight European countries*(cont.)*

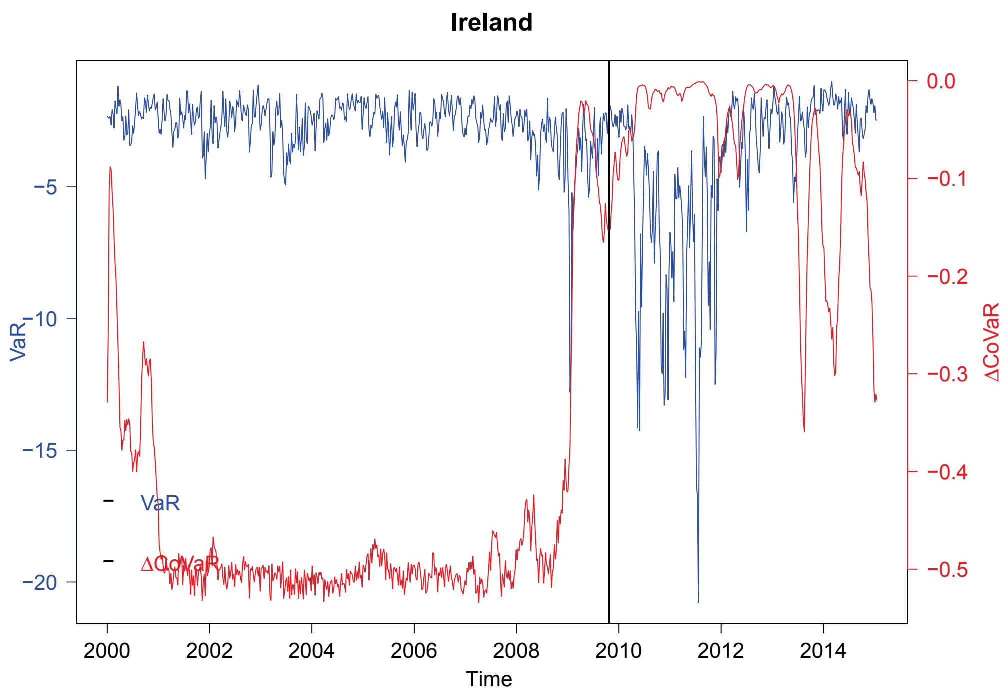


FIGURE 4.1: Estimated time-varying ΔCoVaR and VaR for eight European countries*(cont.)*

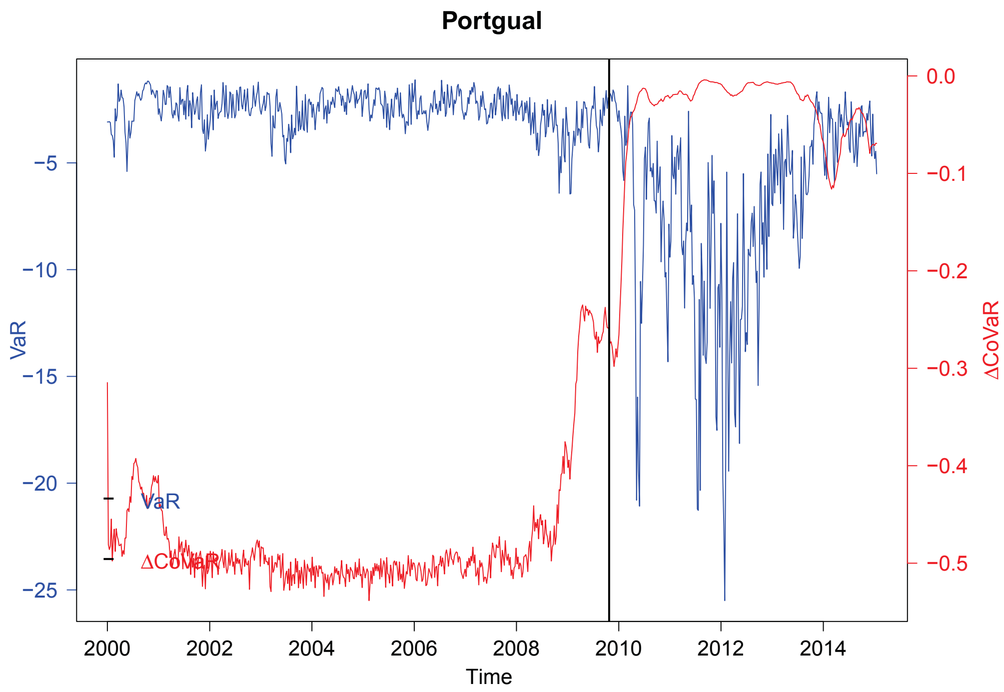


FIGURE 4.1: Estimated time-varying ΔCoVaR and VaR for eight European countries*(cont.)*

Chapter 5

Conclusion

The European debt crisis has raised the concerns of investors and regulators about the stability of financial system and risk contagion among Europe, but the crisis and concerns are far from over, which stimulates the demand of supplementary risk management tools besides VaR. CoVaR is the VaR of a market conditioning on the financial distress of another market, which was firstly introduced by Adrian and Brunnermeier [1] and generalized by Girardi and Tolga Ergün [14]. In this thesis, we introduce current computation methodology of CoVaR and develop the CoVaR measure using copula. Furthermore,we allow copula parameter to vary over time to construct more reasonable dynamic copula-based CoVaR. Copula-based CoVaR is less cumbersome in computation and more flexible to extend to multivariate case. Moreover, we modify the systemic risk contribution index ΔCoVaR as the percentage change of CoVaR scaled by absolute benchmark CoVaR, which avoids the possibility of misleading sign of ΔCoVaR.

In empirical part, we attempt to calculate the systemic risk measure ΔCoVaR to capture how the systemic risk change during European debt crisis. Results find that GIIPS markets in our sample shown high co-movement before the crisis and decoupled with the system index EMU after crisis, while non-crisis countries stay relatively stable in ΔCoVaR, although they have higher risk contribution to system on average. Reasonable rank order of estimated ΔCoVaR for eight countries further verify CoVaR's ability to capture the systemic risk change. Although the discussion of CoVaR is still open, CoVaR could be regard as another powerful risk management tool, together with VaR , to improve and enrich current risk management system.

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Declaration of Authorship

I hereby confirm that I have authored this Master's thesis independently and without use of others than the indicated sources. All passages which are literally or in general matter taken out of publications or other sources are marked as such.

Berlin, March 31, 2015

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