Humboldt-Universität zu Berlin
School of Business and Economics
Ladislaus von Bortkiewicz Chair of Statistics

Multivariate systemic risk: Evidence from a regime-switching factor copula

Master's Thesis submitted

to

Prof. Dr. Wolfgang Karl Härdle
Prof. Dr. Ostap Okhrin

by

Paul Bochmann
(511378)

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Abstract

We propose new dynamic models for the dependence structure of high-dimensional financial data. The models are based on recently proposed factor copula models, which we augment with regime-switching dynamics. We apply the proposed models to a data set of eight major European government debt indices and a Euro Area financial sector index to study government debt systemic risk for the financial sector. As a systemic risk measure we employ a multivariate extension of the CoVaR measure, which takes distress spillovers among government debt markets into account and which we characterize through copulas. We find that a model with multiple skewed and asymmetric common factors and regime-switching dynamics is able to describe the time-variation and non-normal features of the dependence structure of our data well. Our results show a distinct difference in Euro Area government debt systemic risk prior to the financial crisis and thereafter, induced by a regime change in the dependence structure. We find that at the height of the financial crisis the government debt markets of Portugal, Italy, Ireland, Greece and Spain become positively dependent with the financial sector and imply positive systemic risk, whereas the government debt markets of Germany, France and the Netherlands continue to imply negative systemic risk. The total systemic risk, implied by joint distress of all eight debt markets, is highest shortly after the Lehman brothers default and peaks again in the second half of 2011.

Contents

List of Abbreviations

AIC	Akaike information criterion
ARCH	autoregressive conditional heteroscedasticity
ARMA	autoregressive moving average
BIC	bayesian information criterion
cdf	cumulative distribution function
CoVaR	conditional Value-at-Risk
DCC	dynamic conditional correlation
EMU	European Monetary Union
GARCH	generalized autoregressive conditional heteroscedasticity
G5	Germany, France, United Kingdom, United States of America, Japan
LM	Lagrange multiplier
PIIGS	Portugal, Italy, Ireland, Greece, Spain
PIT	probability integral transform
pdf	probability distribution function
VaR	Value-at-Risk

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1 Introduction

How does distress in sovereign debt markets impact on the financial sector in the Euro Area? With the onset of the global financial crisis, and even more during the European debt crisis, this has become a key question to decision makers. Clearly, when banks hold substantial amounts of government debt on their portfolios distress in these markets poses a threat to financial stability. That is, distress in government debt markets is a potential source of systemic risk. Here we measure the systemic risk implied by major Euro Area sovereign debt markets for the Euro Area financial sector throughout the global financial and European debt crisis.

Quantifying systemic risk in this context requires measuring the tail risk interdependence of government debt markets and the financial sector. One systemic risk measure that has been proposed in this context is the CoVaR measure of Adrian and Brunnermeier (2008). CoVaR captures tail risk interdependence through the Value-at-Risk (VaR) of a financial institution[1] conditional on another institution being in distress. An institution's systemic risk contribution, its ΔCoVaR, is measured as the difference of the CoVaR when this institution is in distress and the CoVaR when it is in a state corresponding to normal conditions. In the context of sovereign debt systemic risk we are interested in the VaR of the financial sector conditional on distress in sovereign debt markets. See Bisias et al. (2012) for a survey of CoVaR and alternative systemic risk measures.

One insight from the European debt crisis is how investor concerns with Greek government debt quickly spread to other countries such as Portugal. That is, we observed how multiple government debt markets across Europe became distressed at the same time. In fact, contemporaneous distress of multiple financial institutions is one of the most distinct features of recent financial crisis. However, measuring the systemic risk impact of an institution by considering isolated distress of this institution does not account for the propagation of distress through other institutions being in distress at the same time. In order to account for distress spillovers among multiple financial institutions when assessing their systemic risk contribution, Cao (2013) proposes to extend the CoVaR measure to account for distress of multiple institutions. More specifically, Multi-CoVaR is

[1] Financial institution here is meant in a wider sense than just single financial intermediaries, including markets, sectors, groups of financial intermediaries or whole financial systems.

defined as the VaR of the financial system conditional on distress of multiple institutions, which allows for a more detailed analysis of the systemic risk associated with groups of institutions. The systemic risk contribution of a given group of financial institutions in the Multi-CoVaR framework is computed as the difference of Multi-CoVaR when these institutions are in distress and Multi-CoVaR when these institutions are in a state corresponding to normal conditions.

In the Multi-CoVaR framework the total systemic risk is obtained when all institutions are in distress jointly. In order to obtain individual systemic risk contributions in the Multi-CoVaR framework, we allocate the total systemic risk among the individual financial institutions employing Shapley value methodology (Shapley (1953)), which has originally been proposed in game theory to share the total wealth or cost of a game among the participating players. Shapley value methodology has recently been employed for systemic risk attribution by Tarashev et al. (2010), Cao (2013) and Bernardi et al. (2014). The resulting systemic risk allocation is efficient. That is, the sum of the individual systemic risk contributions equals the total systemic risk. The individual systemic risk contributions obtained through the Shapley value methodology equals the ΔCoVaR of Adrian and Brunnermeier (2008) when the risks of all institutions are orthogonal.

We characterize the Multi-CoVaR systemic risk measure through copulas. Any joint distribution can be decomposed into a copula, which exclusively contains all information about the dependence structure, and marginal distributions. This decomposition provides advantages over working with the joint distribution directly. First, it allows for the construction of very flexible joint distributions by combining arbitrary models for marginal distributions and copulas. Second, it facilitates multi-stage estimation, which can distinctly reduce the computational burden and is therefore particularly useful when working in high dimensions.

Copulas are particularly suited to describe the non-normal features often found in the dependence structure of financial assets such as tail dependence, that is, the dependence of joint extreme events, and an asymmetric dependence structure, with joint crashes being more likely than joint booms. Accounting for these features is crucial in the CoVaR context, given that CoVaR is based on tail risk interdependence.

We obtain copula based Multi-CoVaR through a procedure similar to Reboredo and Ugolini (2015a), which we extend to multivariate dimensions. Most notably, this procedure does not require computation of the individual VaR of an institution but only the nominal level, which is a distinct computational advantage of copulas over multivariate time-series models in the Multi-CoVaR context.

Whereas there is a large number of applications and suitable copula models for two dimensions, suitable copula models for higher dimensions that go beyond copulas implied by elliptical distributions, such as the Gaussian or t copula, or multivariate Archimedian copulas have emerged only recently. Here we employ the class of factor copulas of Oh and Patton (2015). Factor copulas can be understood as copulas implied by factor models, therefore providing a potential dimension reduction, which is crucial when working in high dimension. The difference to factor models arises because marginal distributions are not affected by the factor structure, but modeled separately. Thus the dimension reduction is only applied to the copula, where it is crucial, and flexibility for the margins is retained. The properties of the factor copula depend on the distributions for the common factors and idiosyncratic shocks. Combining a fat tailed and skewed common factor with symmetric idiosyncratic shocks leads to a copula with tail dependence and an asymmetric dependence structure. Factor copulas are scalable, that is, a further dimension reduction can be achieved by placing meaningful and testable restrictions on the parameters.

The dependence structure of financial assets is often characterized by substantial time variation (Longin and Solnik (1995), Engle (2002), Rodriguez (2007)). Here we employ Markov-switching dynamics in order to allow for time-varying dependence. In Markov-switching models the data generating process is assumed to switch between several regimes according to a hidden state variable (Hamilton (1989)), where typically changes of the state are associated with changes in part or all of the model parameters. Markov-switching models allow for sudden changes in regimes that can persist for extended periods of time, thereby closely describing the behavior of financial markets.

When Markov-switching is applied in the context of copulas, not only the copula parameters can be allowed to change through time but, through different copula densities in the different regimes, also the structural characteristics of the dependence such as tail dependence and asymmetric dependence could change. By introducing a Markov-switching factor copula we add to the literature on dynamic high-dimensional copula models for financial data.

We analyze sovereign debt systemic risk for the general Euro Area financial sector of eight Euro Area sovereign debt markets, three core countries (Germany, France and the Netherlands) and the five PIIGS countries (Portugal, Italy, Ireland, Greece and Spain), over the period May 2005 to April 2016 using weekly returns on ten year benchmark government bond indices and a broad Euro Area financial sector index.

Our work is closely related to Reboredo and Ugolini (2015b), who investigate Euro Area sovereign debt systemic risk for individual financial systems using a vine-copula CoVaR approach. They work with a conditioning event including the return on domestic government debt as well as Greek government debt, thereby taking into account dependence between the Greek sovereign debt market and the sovereign and financial systems of other countries. Their approach allows them to analyze the impact of distress in Greek government debt on financial systems of other countries while also taking into account the domestic sovereign debt markets of these countries. We focus instead on the sovereign debt systemic risk for the general Euro Area financial sector implied by eight major government debt markets, while taking spillovers among these markets into account.

Our results show a distinct difference in Euro Area government debt systemic risk prior to the financial crisis and thereafter. We find that, until the height of the global financial crisis all eight government debt markets are characterized by strong positive dependence and are negatively dependent with the Euro Area financial sector. The systemic risk implied by all sovereign debt markets, while accounting for distress spillovers among these markets, over this period is similar and negative in a sense that distress in government debt markets decreased financial sector VaR. This finding can be interpreted in terms of a diversification benefit of government bonds on bank portfolios. The total systemic risk, obtained from joint distress of all eight debt markets, is negative over this period.

At the height of the global financial crisis, with the Lehman Brothers default, we observe a regime shift and the government debt markets decouple. The debt markets of the three core countries remain stable and strongly dependent with each other, whereas the government bond indices of the PIIGS countries show heightened volatility and an increase in yield, reflecting investor concerns with these markets. PIIGS government debt markets become positively dependent with the Euro Area financial sector. Importantly, these markets now imply positive systemic risk, in a sense that distress of these markets increases financial sector VaR. Systemic risk of these markets is highest shortly after the Lehman Brothers default and increased again in the second half of 2011. The debt markets of the core countries remain negatively dependent with the financial sector and continue to imply negative systemic risk. Total government debt systemic risk becomes positive with the Lehman Default, meaning that joint distress of all eight government debt markets increases financial sector VaR. Tail dependence is an important feature of the dependence structure of our data throughout the sample period.

The remainder is organized in five chapters: In chapter two we introduce Markov-switching factor copulas. We first present some basic copula theory and dependence measures, before we explain factor copulas in more detail and show how we model dependence dynamics through Markov-switching in the context of factor copulas. In chapter three we elaborate on the estimation of Markov-switching factor copulas. We first show how the log-likelihood can be obtained and define the estimator. We then describe how we obtained standard errors. Finally, a numerical integration method to obtain the factor copula density is described. In the fourth chapter we present our systemic risk measures. First, we recall the CoVaR measures. We then move on to presenting Multi-CoVaR, a multivariate extension, and characterize it using copulas. Finally, we state how we obtain total systemic risk and efficient individual systemic risk contributions in the Multi-CoVaR framework. In chapter five we present our results. We begin with a short description of the data and show estimation results for univariate models. We then present estimation results from different factor-copula models. Finally, we present our government debt systemic risk risk results. In chapter six we conclude.

2 Markov-switching factor copulas

We begin this chapter with an introduction of some basic copula theory and some dependence measures. We then present factor copulas. Finally, we present the concept of Markov-switching dynamics and formalize it in the context of factor copulas.

2.1 Basic copula theory

Consider a d-variate stochastic process $\{Y\}_{t=1}^{n}$ with $Y_t = (y_{1,t}, ..., y_{d,t})$. Let $F(y_{1,t}, ..., y_{d,t})$ denote the joint distribution and let F_i and f_i denote the marginal distribution and density function of $y_{i,t}$, respectively. From Sklar (1959) it follows that F can be decomposed into d marginal distributions and a copula $C : [0,1]^d \to [0,1]$, which completely contains all the information about the dependence structure, such that:

$$F(Y_t) = C\left(F_1(y_{1,t}), ..., F_d(y_{d,t})\right) \tag{1}$$

The copula can therefore be understood as a multivariate distribution function that combines the marginal distributions $F_1, ..., F_n$ to the joint distribution F. The decomposition offered through Sklar's Theorem is useful for several reasons: First, given univariate distributions $F_1, ..., F_d$ and copula C one can construct a valid joint distribution F with marginal distributions $F_1, ..., F_d$, a finding that allows constructing very flexible joint distributions. Here we focus on new dynamic models for C and draw on existing models for the marginal distributions. Second, Sklar's theorem facilitates multi-stage estimation, which is particularly useful in high-dimensional applications because it distinctly reduces the computational burden. We assume that the marginal distributions are continuous and can be modeled parametrically[2]. The probability integral transform (PIT) is then given by $u_{i,t} = F_i(y_{i,t}; \theta_{m,i})$, where $\theta_{m,i}$ is a vector containing all parameters describing margin i and we define $\theta_m = (\theta'_{m,1}, ..., \theta'_{m,n})'$ as the vector of all marginal parameters. The variable $u_{i,t}$ follows a $Uniform(0,1)$ distribution regardless of F_i, thus the copula can be understood as a joint distribution with $Uniform(0,1)$ margins. The marginal distribution $F_i(y_{i,t}; \theta_{m,i})$ can be a conditional distribution and in the empirical application we model $y_{i,t}$ using an ARMA-GARCH model and treat the resulting residuals as *iid* ran-

[2] This assumption arises because there is no asymptotic theory for models with nonparametric margins as in Chen and Fan (2006) and a dynamic copula.

dom variables. We also assume that the copula is parametric (and absolutely continuous), equation (1) can then be expressed in terms of densities:

$$f(Y_t) = \prod_{i=1}^{d} f_i(y_{i,t}) \cdot c\,(u_{1,t}, ..., u_{d,t}) \tag{2}$$

where $c(u_{1,t}, ..., u_{d,t}) = \frac{\partial^n}{\partial u_{1,t}, ..., \partial u_{d,t}} C(u_{1,t}, ..., u_{d,t})$ denotes the copula density and f is the joint density of Y_t. This expression makes clear why the copula exclusively contains all the information about the dependence structure, because on the right hand side only marginal densities appear in addition to the copula density.

2.2 Dependence measures

We now turn to the explanation of the dependence measures that are used throughout the following analysis. For a detailed discussion of dependence measures see Nelsen (2006) or Patton (2013). First, note that linear correlation is not a suitable measure in the context of copulas. One reason is that it is not scale invariant, and is thus affected by strictly increasing transformations of the data. This means the linear correlation of the data and the PIT can be different, for instance. Linear correlation is thus affected by the marginal distributions and not just a function of the copula, an undesirable property when working with copulas. The dependence measures introduced below are functions of the copula only. Throughout the following consider two random variables $y_{i,t}$ and $y_{j,t}$ with marginal distributions F_i and F_j, respectively, and copula C.

Spearman's rank correlation measures the relationship between two variables using their concordance and discordance. It can also be understood as the linear correlation of the ranks of the data. Rank correlation can be expressed in terms of the copula:

$$\rho = \text{Corr}(u_{i,t}, u_{j,t}) = 12\,\text{E}(u_{i,t}u_{j,t}) - 3 \tag{3}$$

$$= 12 \int_0^1 \int_0^1 uv dC(u,v) - 3 \tag{4}$$

$$= 12 \int_0^1 \int_0^1 C(u,v) du dv - 3 \tag{5}$$

We have used that $\text{E}(u) = 1/2$ and $\text{V}(u) = 1/12$ for $u \sim Uniform(0,1)$. We can estimate rank correlation as:

$$\widehat{\rho} = \frac{12}{n} \sum_{t=1}^{n} u_{i,t}u_{j,t} - 3 \tag{6}$$

Rank correlation lies in the interval [-1,1] and can therefore provide information on the sign of dependence.

Quantile dependence measures the dependence of two random variables in the joint tails. For quantile q it is defined as:

$$\tau_q = \begin{cases} \mathrm{P}(u_{i,t} \leq q | u_{j,t} \leq q), & q \in (0, 0.5] \\ \mathrm{P}(u_{i,t} > q | u_{j,t} > q), & q \in (0.5, 1) \end{cases} \tag{7}$$

$$= \begin{cases} \frac{C(q,q)}{q}, & q \in (0, 0.5] \\ \frac{1-2q-C(q,q)}{1-q}, & q \in (0.5, 1) \end{cases} \tag{8}$$

Quantile dependence can thus be understood as the conditional probability of observing a realization of one variable in the q-th quantile, given that such an observation has been made for the other variable[3]. Quantile dependence can be estimated as:

$$\widehat{\tau}_q = \begin{cases} \frac{1}{nq} \sum_{t=1}^{n} \mathbf{I}\left\{u_{i,t} \leq q, u_{j,t} \leq q\right\}, & q \in (0, 0.5] \\ \frac{1}{n(1-q)} \sum_{t=1}^{n} \mathbf{I}\left\{u_{i,t} > q, u_{j,t} > q\right\}, & q \in (0.5, 1) \end{cases} \tag{10}$$

where $\mathbf{I}$ denotes the indicator function. Quantile dependence lies in the interval [0,1]. When comparing quantile dependence for the lower and upper tail we can gain insights into the symmetry of the dependence structure.

Tail dependence is defined as the limit of quantile dependence for $q \downarrow 0$ or $q \uparrow 1$:

$$\tau^L = \lim_{q \downarrow 0} \mathrm{P}(u_{i,t} \leq q | u_{j,t} \leq q) = \lim_{q \downarrow 0} \frac{C(q,q)}{q} \tag{11}$$

$$\tau^U = \lim_{q \uparrow 1} \mathrm{P}(u_{i,t} > q | u_{j,t} > q) = \lim_{q \uparrow 1} \frac{1-2q-C(q,q)}{1-q} \tag{12}$$

Tail dependence can thus be understood as the dependence between extreme events. Note that sample tail dependence cannot simply be taken as the limit of equation (10), because this is zero for q being close to the respective boundary. However, for a number of copulas tail dependence is available in closed form, see e.g. Nelsen (2006).

[3]For negatively dependent variables we focus on the counter diagonal, quantile dependence in this case is defined as:

$$\tau_q = \begin{cases} \frac{q-C(1-q,q)}{q}, & q \in (0, 0.5] \\ \frac{q-C(q,1-q)}{1-q}, & q \in (0.5, 1) \end{cases} \tag{9}$$

2.3 Factor copulas

Factor models are commonly used in various disciplines such as economics, finance or statistics to achieve a dimension reduction by summarizing a potentially large number of variables by a smaller number of common factors. The class of factor copulas is introduced by Oh and Patton (2015) and can be understood as copulas generated from a factor model based on latent variables, making them particularly suitable for high-dimensional dependence modeling because of the dimension reduction offered through the factor structure. A simple factor model with one common factor for $X = (x_1, \ldots, x_d)$, which is based on $d+1$ latent variables, is:

$$\begin{aligned} x_i &= \beta_i z + \varepsilon_i, \quad i = 1, 2, ..., d \\ z &\sim F_z(\gamma_z), \quad \varepsilon_i \sim iid\, F_\varepsilon(\gamma_\varepsilon), \quad z \perp \varepsilon_i \; \forall i \\ (x_1, ..., x_d) = X &\sim F_x(\gamma_c) = C(G_1(\gamma_c), ..., G_d(\gamma_c); \gamma_c) \end{aligned} \tag{13}$$

where $\gamma_c = (\beta_1, ..., \beta_d, \gamma_z', \gamma_\varepsilon')'$ denotes the vector of factor copula parameters, z is the latent common factor with distribution F_z, ε_i is a latent idiosyncratic shock with distribution F_ε and G_i denotes the marginal distribution of x_i. The central idea behind factor copulas is to use the above structure exclusively for the copula of Y, $C(Y)$. The marginal distributions of Y are modeled and estimated separately, so F_i does not equal G_i in general. If this assumption is relaxed and margins are modeled jointly with the copula equation (13) becomes a standard factor model with latent factors. However, Oh and Patton (2015) suggest to apply the potential dimension reduction offered through the factor structure only to the copula, where it is crucial, and retain flexibility for the margins. The copula C is in general not known in closed form. One exemption is with Normal distributions for F_z and F_ε, because this results in a multivariate Normal distribution for X, the copula is thus Gaussian with a correlation matrix implied by the factor structure. The properties of C depend on the choices for F_z and F_ε. Choosing the common factor to come from a fat tailed distribution (with at least as fat tails as the distribution for the idiosyncratic shock) allows for tail dependence, whereas combining an asymmetric distribution for the common factor with a symmetric distribution for the idiosyncratic shock leads to an asymmetric dependence structure.

The *Skewt* $t - t$ factor copula has been found by Oh and Patton (2015) and Oh and Patton (2016) to be a good choice for financial data. It is obtained by choosing the

Skew t distribution of Hansen (1994) for F_z and the t distribution (standardized to have unit variance) for F_ε. We now explain the *Skew t* distribution in more detail. First we state the density function:

$$g(z|\nu,\lambda) = \begin{cases} bc\left(1+\frac{1}{\nu-2}\left(\frac{bz+a}{1-\lambda}\right)^2\right)^{-(\nu+1)/2} & z < -a/b \\ bc\left(1+\frac{1}{\nu-2}\left(\frac{bz+a}{1+\lambda}\right)^2\right)^{-(\nu+1)/2} & z \geq -a/b \end{cases} \tag{14}$$

with

$$a = 4\lambda c\left(\frac{\nu-2}{\nu-1}\right), \quad b^2 = 1+3\lambda^2-a^2, \quad c = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi(\nu-2)}\Gamma(\frac{\nu}{2})} \tag{15}$$

and Γ denotes the gamma function. The *Skew t* distribution has zero mean and a variance of one. The degree of freedom parameter $\nu \in (2,\infty)$ determines the fatness of the tails and $\lambda \in (-1,1)$ is a skewness parameter. The density is negatively skewed for $\lambda < 0$, it displays positive skewness for $\lambda > 0$ and for $\lambda = 0$ it is symmetric. For $\lambda = 0$ and $\nu \neq +\infty$ the skew t distribution corresponds to the t distribution, for $\lambda \neq 0$ and $\nu = +\infty$ it corresponds to a skewed Normal distribution and for $\lambda = 0$ and $\nu = +\infty$ it corresponds to the Normal distribution. For the standardized t-distribution that is employed for F_ε we choose the same degree of freedom parameter as for the common factor for simplicity. The t distribution corresponds to a Normal distribution for $\nu = +\infty$. A *Skew* $t-t$ factor copula with a structure as in equation (13) and equal degree of freedom parameters for F_z and F_ε thus has $d+2$ parameters in total - d factor loadings, one degree of freedom parameter, ν, and one skewness parameter, λ.

The *Skewt* $t-t$ factor copula nests a number of other factor copulas. Following the discussion of the involved distributions above, for $\nu = +\infty$ and $\lambda \neq 0$ it becomes the *Skew N* - *N* factor copula with no tail dependence but asymmetric dependence. If $\lambda = 0$ and $\nu \neq +\infty$ the *Skew* $t-t$ factor copula becomes a t - t factor copula, which allows for tail dependence but imposes a symmetric dependence structure. If $\lambda = 0$ and $\nu = +\infty$ the *Skewt* $t-t$ factor copula becomes the Normal factor copula, which corresponds to a Gaussian copula with correlation matrix implied by the factor structure.

We illustrate different factor copulas in figure 1. To this end we present 1000 simulations from bivariate distributions, constructed using the four different factor copulas discussed above. All plots have $N(0,1)$ marginal distributions and a linear correlation

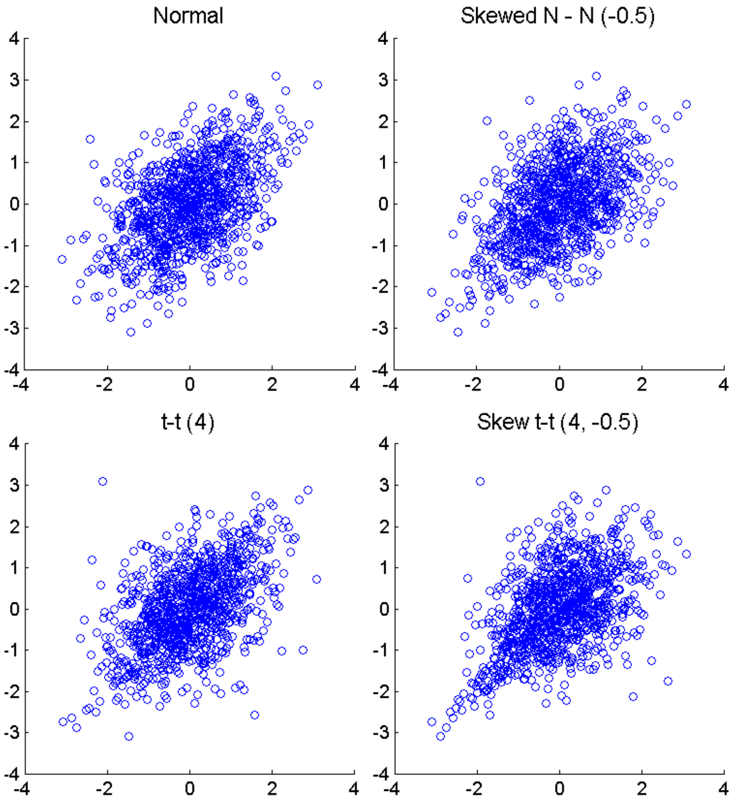

Figure 1: This figure presents scatter plots from four bivariate distributions, all obtained from different factor copulas. All plots have $N(0,1)$ margins and a linear correlation of 0.5.

of 0.5. The plot based on the Normal factor copula, which is obtained by setting $F_z = N(0,1)$ and $F_\varepsilon = N(0,1)$, shows a symmetric dependence structure and compared to the plot based on the t - t factor copula, no observations in the joint tails. The two plots based on skewed distributions for the common shock display a distinct asymmetry in the dependence structure. A comparison of the plot based on the *Skew* $N - N$ factor copula to the plot based on the *Skewt* $t - t$ factor copula reveals a distinct difference in the joint tails. The plots are obtained with $\nu = 4$ and $\lambda = -0.5$.

The model in equation (13) can be extended to include multiple common factors in order to allow for a more flexible dependence structure:

$$x_i = \sum_{j=1}^{k} \beta_{i,j} z_j + \varepsilon_i, \ i = 1, 2, \ldots, d \tag{16}$$

$$\varepsilon_i \sim iid\, F_\varepsilon(\gamma_\varepsilon), \ z_j \sim iid\, F_z(\gamma_z), \ z_j \perp \varepsilon_i \ \forall i, j \tag{17}$$

The common factors need to be independent in order to obtain the copula pdf through the numerical integration procedure described in section 3.3 and we assume them to be identically distributed with equal parameters for simplicity and to reduce the number of parameters in the numerical optimization procedure.

Simulation is required to obtain properties such as rank correlations and quantile dependence for most factor copulas. However, Oh and Patton (2015) derive tail dependence properties of linear factor copulas, in particular for the $Skew\ t - t$ factor copula, in closed form.

2.4 Modeling dependence dynamics through Markov-switching

There is ample evidence that the dependence structure of financial assets, much as financial asset volatility, changes through time. This has first been demonstrated for correlations (Longin and Solnik (1995), Engle (2002)). In particular, there has been a debate if financial asset correlations increase during volatile periods, especially when these are characterized by large negative returns (Longin and Solnik (2001), Ang and Chen (2002), Cappielo et al. (2006)). Given that copulas allow for modeling more general features of the dependence structure than just linear correlation, modeling dependence dynamics through time-varying copulas has attracted a number of researchers, beginning with Patton (2006b) who extends Sklar's theorem to conditional distributions. A survey on time-varying copulas can be found in Manner and Reznikova (2012).

Some of the modeling approaches proposed in the time-varying copula literature are suitable for multivariate applications, including Markov-switching. Markov-switching copula models assume different copulas, or a least different copula parameters, in different regimes between which the the data generating process can switch. Thereby they allow for sudden and potentially recurrent changes in the dependence structure of financial assets, for instance arising from financial crisis, but also for permanent changes such as due to increasing financial integration. We now formalize the concept of Markov-switching in the context of factor copulas.

As in Hamilton (1989) we assume that Y_t depends on a latent state variable $S_t \in (1, 2)$, which determines the regime (or state). We allow S_t to affect the dependence structure, but not the marginal distributions. To be more specific, we assume that in each state a different factor copula determines the dependence structure of the data, whereas the marginal distributions are the same across states. That is, we do not only allow the factor copula parameters γ_c to differ across states, but potentially also the distributions for the common factor and idiosyncratic shocks, F_z and F_ε, and number of common factors. The density of the data conditional on $S_t = j$ is:

$$f^{(j)}(Y_t|Y_{1:(t-1)}, S_t = k) = c^{(j)}(u_{1,t}, \dots, u_{d,t}; \gamma_c^{(j)}) \prod_{i=1}^{d} f_i(y_{i,t}; \theta_{m,i}) \tag{18}$$

Note how the state indexes the copula, but not the marginal distributions. The copula is exclusively driven by S_t, in particular it does not depend on past observations or past realizations of copula parameters. We assume S_t to be generated from a first order Markov chain, that is, S_t depends only on S_{t-1}, with the following transition probability matrix:

$$P = \begin{bmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{bmatrix} \tag{19}$$

where $p_{ij} = \mathrm{P}(S_t = j|S_{t-1} = i)$ denotes the probability of moving from state i to state j the next period. Because the rows must clearly sum to one P is entirely parameterized by p_{11} and p_{22}. A distinct feature of Markov-switching copula models is that they do not only allow copula parameters to change through time but also for changes in the structural characteristics of dependence, through different copula densities in the different regimes. Markov-switching copulas could, for instance, allow for changes from a symmetric to an asymmetric dependence structure or from a dependence structure exhibiting no tail dependence to one with tail dependence.

Alternative approaches for modeling time-varying dependence through copulas are available. However, several researchers modeling time-varying dependence of financial assets in the copula context through so called observation-driven approaches such as time-varying correlations (Jondeau and Rockinger (2006)), ARMA-models (Patton (2006b)), DCC copulas (Christoffersen et al. (2012)) or Generalized Autoregressive Score models (Oh and Patton (2016)) document very high persistence of estimated dependence parameters. This finding is indicative of the presence of large but infrequent breaks in the dependence structure, possibly induced by stochastic regime switches. It has first been pointed out by Jondeau and Rockinger (2006) in the context of copula models, but discussed earlier for volatility models, see Diebold and Inoue (1999) for example, and further motivates the use of a Markov-switching model.

Markov-switching copula models have previously been studied by a number of authors, mostly in two dimensions. Jondeau and Rockinger (2006) are among the first to employ Markov-switching in the context of copulas, allowing the parameters of a bivariate t copula to switch between two states. Rodriguez (2007) builds a Markov-switching mixture copula based model in which margins and copula switch jointly, thereby requiring joint estimation of marginal and copula parameters. He does, through regime-switches in the mixture weight, allow for changes in the copula density. He employs his model to a number of pairs of stock indices to investigate financial contagion in international equity markets. To this end he analyzes changes in the dependence structure associated with regime changes from the crisis (high volatility) to the non-crises (low volatility) regime. Okimoto (2008) employs bivariate Markov-switching copula based models to study asymmetric dependence in international equity markets. He follows Rodriguez (2007) in that he allows for different copula densities in different regimes and joint switching of margins and copula.

A few multivariate applications of Markov-switching copula models exist. Chollete et al. (2009) employ Markov-switching copula models with one Gaussian and one canonical vine regime to model two data sets consisting of international equity indices, one for the countries of the G5 and one consisting of four Latin American countries. They compare VaR and Expected Shortfall of different models and also replicate the portfolio selection exercise of Ang and Chen (2002) to investigate the economic costs of ignoring regime-switching. Garcia and Tsafack (2011) propose a Markov-switching mixture copula model with one regime with a symmetric dependence structure and one asymmetric regime in order to study asymmetries in the dependence structure as well as extreme co-movements in international equity and bond markets. They study two equity and two bond indices jointly. Stöber and Czado (2012) employ Markov-switching regular vine copulas to three different data sets, one consisting of returns on ten German stocks. Härdle et al. (2015) employ Markov-switching hierarchical Archimedean copulas to model two different data sets, including one on three foreign exchange rates, for which they also compute VaR. In multivariate Markov-switching copula based models the margins are typically not allowed to be affected by the regime-switching in order to allow for multi-stage estimation, an approach we follow.

3 Estimation of Markov-switching factor copula models

We now turn to the estimation of Markov-switching factor copula models, which is based on multi-stage maximum likelihood. We first present the total log-likelihood of a Markov-switching factor copula based model. We then show algorithms for inference on the latent state, which are required to obtain the log-likelihood. Next we present a decomposition of the log-likelihood in order to facilitate multi-stage estimation, thus making high-dimensional applications feasible. We then define the estimator and show how we obtain standard errors. Finally we present a numerical integration method to obtain the factor copula density.

3.1 Log-likelihood and inference on the hidden state

Let the observed data be denoted as: $Y_{1:n} = (Y_1', ..., Y_n')'$. The log-likelihood function is:

$$L(Y_{1:n}; \theta_m, \theta_c) = \sum_{t=1}^{n} \log f(Y_t; \theta_m, \theta_c | Y_{1:(t-1)}) \tag{20}$$

where $\theta_c = (\gamma_c^{(1)\prime}, \gamma_c^{(2)\prime}, p_{11}, p_{22})'$ contains all parameters of the Markov-switching factor copula, that is, the copula parameters for each regime and the transition probability parameters. The density of Y_t, and therefore the log-likelihood, depends on the hidden state. Given that S_t is unobservable we need to integrate over all possible values:

$$f(Y_t; \theta_m, \theta_c | Y_{1:(t-1)}) = \sum_{i=1}^{2} \mathrm{P}(S_t = i | Y_{1:(t-1)}; \theta_m, \theta_c) \cdot f(Y_t; \theta_m, \theta_c | Y_{1:(t-1)}, S_t = k) \tag{21}$$

where $\mathrm{P}(S_t = i | Y_{1:(t-1)}; \theta_m, \theta_c)$ is the probability of being in a given state at time t given all observations up to $t-1$. This probability is referred to as a predicted probability. We now state the approach of Hamilton (1989) to obtain predicted probabilities in order to evaluate equation (21), adapting the notation in Hamilton (1994) to our factor copula model with two regimes. Let

$$\eta_t = \begin{pmatrix} f^{(1)}(Y_t; \theta_m, \theta_c^{(1)} | Y_{1:(t-1)}) \\ f^{(2)}(Y_t; \theta_m, \theta_c^{(2)} | Y_{1:(t-1)}) \end{pmatrix} \tag{22}$$

denote the (2x1) vector containing the density of the data conditional on the regimes S_t. Next, let $\widehat{\xi}_{t|t-1}$ denote the (2x1) vector that contains the predicted probability for both of the states, that is $\widehat{\xi}_{t|t-1} = (\mathrm{P}(S_t = 1 | Y_{1:(t-1)}; \theta_m, \theta_c), \mathrm{P}(S_t = 2 | Y_{1:(t-1)}; \theta_m, \theta_c))'$.

Similarly, let $\widehat{\xi}_{t|t}$ denote the (2x1) vector that contains the probabilities of being in a given state at time t given all observations up to t for both of the states, that is $\widehat{\xi}_{t|t} = (\mathrm{P}(S_t = 1|Y_{1:t};\theta_m,\theta_c), \mathrm{P}(S_t = 2|Y_{1:t};\theta_m,\theta_c))'$. These probabilities are referred to as filtered probabilities. Given a starting value $\widehat{\xi}_{1|0}$, the following equations now form a filter to obtain optimal $\widehat{\xi}_{t|t}$ and $\widehat{\xi}_{t+1|t}$ for each t:

$$\widehat{\xi}_{t|t} = \frac{\widehat{\xi}_{t|t-1} \odot \eta_t}{\mathbf{1}'(\widehat{\xi}_{t|t-1} \odot \eta_t)} \tag{23}$$

$$\widehat{\xi}_{t+1|t} = P'\widehat{\xi}_{t|t} \tag{24}$$

where $\odot$ denotes the Hadamard, or element by element, product and $\mathbf{1}$ denotes a column vector of ones. For the starting value $\widehat{\xi}_{1|0}$ we follow the suggestion of Hamilton (1994) and choose $\widehat{\xi}_{1|0} = ((1-p_{22})/(2-p_{11}-p_{22}), (1-p_{11})/(2-p_{11}-p_{22}))'$, which corresponds to the ergodic probabilities of the Markov chain. The unconditional density is then evaluated as: $f(Y_t;\theta_m,\theta_c|Y_{1:(t-1)}) = \widehat{\xi}'_{t|t-1}\eta_t$ and the log-likelihood can thus be expressed as:

$$L(Y_{1:n};\theta_m,\theta_c) = \sum_{t=1}^{n} \log(\widehat{\xi}'_{t|t-1}\eta_t) \tag{25}$$

In addition to filtered and predicted probabilities we want to base our regime classification on smoothed probabilities, that is, the probability of being in a certain regime at time t based on the full sample. These can be obtained through the algorithm of Kim (1994):

$$\widehat{\xi}_{t|n} = \widehat{\xi}_{t|t} \odot \left\{ P \cdot \left[\widehat{\xi}_{t+1|n} (\div) \widehat{\xi}_{t+1|t} \right] \right\} \tag{26}$$

where $\widehat{\xi}_{t|n} = (\mathrm{P}(S_t = 1|Y_{1:n};\theta_m,\theta_c), \mathrm{P}(S_t = 2|Y_{1:n};\theta_m,\theta_c))'$ is the (2x1) vector of smoothed probabilities and $(\div)$ denotes element-wise division. Once $\widehat{\xi}_{n|n}$ is obtained from equation (23) for $t = n$, equation (26) can be iterated backward for $t = n-1$ to $t = 1$.

One approach would be to estimate the parameters of the margins and the copula simultaneously by maximizing equation (25). However, given the large number of parameters in our empirical application and strong nonlinearities of the objective function implied by the regime-switching, this is infeasible. Instead we apply a two-stage estimation procedure (Newey and McFadden (1994), Patton (2006a)), which can be employed since the marginal distributions are not affected by the latent state variable. This estimation procedure has been applied in a similar context by Engle (2002) for the DCC model, Pelletier (2006) for a regime-switching DCC model and Chollete et al. (2009) as well as Garcia and Tsafack (2011) for Markov-switching copula based models. In order to facilitate two-stage estimation, the log-likelihood must be decomposed into two parts, one for the marginal distributions and one for the copula. In what follows we present this decomposition:

Proposition (Decomposition of the log-likelihood).

The log-likelihood function can be decomposed into two parts, one for the marginal distribution and one for the copula:

$$L(Y_{1:n}; \theta_m, \theta_c) = \sum_{i=1}^{d} L_{m,i}(\theta_{m,i}) + L_c(\theta_c, \theta_m) \tag{27}$$

where

$$\begin{aligned} L_{m,i}(\theta_{m,i}) &= \sum_{t=1}^{n} \log(f_i(y_{i,t}; \theta_{m,i} | Y_{1:(t-1)}) \\ L_c(\theta_c, \theta_m) &= \sum_{t=1}^{n} \log(\widehat{\xi}'_{t|t-1} \eta_{ct}) \end{aligned}$$

with

$$\eta_{ct} = \begin{pmatrix} c^{(1)}(u_{1,t}, ..., u_{d,t}; \gamma_c^{(1)}) \\ c^{(2)}(u_{1,t}, ..., u_{d,t}; \gamma_c^{(1)}) \end{pmatrix}$$

and $\widehat{\xi}_{t|t-1}$ is obtained from the following filter:

$$\widehat{\xi}_{t|t} = \frac{\widehat{\xi}_{t|t-1} \odot \eta_{ct}}{1'(\widehat{\xi}_{t|t-1} \odot \eta_{ct)}} \tag{28}$$

$$\widehat{\xi}_{t+1|t} = P' \widehat{\xi}_{t|t} \tag{29}$$

Proof: See Chollete et al. (2009) or Garcia and Tsafack (2011).

Therefore we can split the log-likelihood into two parts, $L_c(\theta_c, \theta_m)$ for the copula and $\sum_{i=1}^{d} L_{m,i}(\theta_{m,i})$ for the margins, where the latter can be further split resulting in one term for each of the margins. The copula likelihood depends on θ_m because the marginal parameters are needed to obtain the probability integral transforms. Now we can first estimate the parameters of each of the marginal distributions:

$$\widehat{\theta}_{m,i} = \underset{\theta_{m,i}}{\text{argmax}}\ L_{m,i}(\theta_{m,i}) \tag{30}$$

Given $\widehat{\theta}_m = (\widehat{\theta}_{m,1}, ..., \widehat{\theta}_{m,n})'$ we then estimate the copula parameters in a second step:

$$\widehat{\theta}_c = \underset{\theta_c}{\text{argmax}}\ L_c(\theta_c, \widehat{\theta}_m) \tag{31}$$

This two-stage procedure is less efficient than one-stage estimation but makes estimation of high-dimensional models feasible.

3.2 Obtaining standard errors

We employ results from Patton (2006a) for copula based time-series models, which are based on the findings of Newey and McFadden (1994). Under standard regularity conditions the two-stage maximum likelihood estimator $\widehat{\theta} = (\widehat{\theta}'_m, \widehat{\theta}'_c)'$ is asymptotically normal:

$$\sqrt{T}(\widehat{\theta} - \theta^0) \xrightarrow{\mathcal{L}} N(0, A_T^{-1} B_T (A_T^{-1})') \tag{32}$$

with

$$A = \begin{bmatrix} \nabla_{\theta_m\theta_m} L_m(Y_{1:n}; \theta_m) & 0 \\ \nabla_{\theta_m\theta_c} L_m(Y_{1:n}; \theta_c, \theta_c) & \nabla_{\theta_c\theta_c} L_c(Y_{1:n}; \theta_m, \theta_c) \end{bmatrix}$$

and

$$B = \text{Var}\left[\sum_{t=1}^{n} \left(n^{-1/2} \nabla'_{\theta_m} L_m(Y_t; \theta_m), n^{-1/2} \nabla'_{\theta_c} L_c(Y_t; \theta_m, \theta_c)\right)\right]$$

While the two stage approach simplifies estimation, for the estimation of copula standard errors we have to account for parameter estimation error arising from the estimation of the marginal distribution parameters. In particular this rules out the possibility of simply working with the inverse Hessian of the copula likelihood. Moreover, the information matrix equality does not hold for two-stage maximum likelihood estimation and the above sandwich form asymptotic variance-covariance matrix arises. The variance-covariance matrix for the copula parameters clearly also depends on the parameters of the marginal distributions. Evaluating A and B numerically can be cumbersome, instead we obtain standard errors through the following simulation procedure: First, we simulate from the model for the entire joint distribution, that is we first simulate from the Markov-switching factor copula, we then apply the corresponding inverse *Skew t* distribution with the estimated degree of freedom and skewness parameters and then rescale with the conditional mean and standard deviation from the corresponding ARMA-GARCH model. Second, using the data obtained in the previous step we first estimate the marginal models, obtain standardized residuals and then estimate the Markov-switching factor copula. These steps are repeated 100 times in order to obtain simulation based standard errors. This approach yields correct finite sample results and does not rely on asymptotic theory. See Patton (2013) for details and alternative approaches to estimate standard errors in parametric copula based models.

3.3 Obtaining the factor copula density

To facilitate maximum likelihood estimation for factor copulas, the factor copula density implied by equation (16), which is in general not available in closed form, must be evaluated. We now present a multi-dimensional extension of the approach of Oh and Patton (2016), who propose to obtain the copula density through a numerical integration procedure. The copula density of X can be presented as:

$$c(u_1, ..., u_d) = \frac{f_X(G_1^{-1}(u_1), ..., G_d^{-1}(u_d))}{g_1(G_1^{-1}(u_1)) \cdot ... \cdot g_d(G_d^{-1}(u_d))} \tag{33}$$

where $f_X(x_1, ..., x_d)$ is the joint density of X, $g_i(x_i)$ is the marginal density of x_i ,$G_i^{-1}(x_i)$ is the inverse marginal distribution of x_i, and $c(u_1, ..., u_d)$ the copula density. Next, we state how to obtain $f_X(x_1, ..., x_d)$, $g_i(x_i)$ and $G_i^{-1}(x_i)$. Due to the independence of $Z = (z_1, ..., z_k)$ and ε_i we have:

$$f_{x_i|Z}(x_i|z_1, ..., z_k) = f_{\varepsilon_i}(x_i - \beta_{i,1}z_1 - ... - \beta_{i,k}z_k) \tag{34}$$

$$F_{x_i|Z}(x_i|z_1, ..., z_k) = F_{\varepsilon_i}(x_i - \beta_{i,1}z_1 - ... - \beta_{i,k}z_k) \tag{35}$$

$$f_{X|Z}(x_1, ..., x_d|z_1, ..., z_k) = \prod_{i=1}^{d} f_{\varepsilon_i}(x_i - \beta_{i,1}z_1 - ... - \beta_{i,k}z_k) \tag{36}$$

Integration over the common factors gives:

$$\begin{aligned} g_i(x_i) &= \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f_{x_i,Z}(x_i, z_1, ..., z_k)\ dz_1...dz_k && (37)\\ &= \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f_{x_i|Z}(x_i|z_1, ..., z_k) f_{z_1}(z_1) \cdot ... \cdot f_{z_k}(z_k)\ dz_1...dz_k && (38)\\ &= \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} f_{\varepsilon_i}(x_i - \beta_{i,1}z_1 - ... - \beta_{i,k}z_k) f_{z_1}(z_1) \cdot ... \cdot f_{z_k}(z_k)\ dz_1...dz_k && (39) \end{aligned}$$

and similarly

$$G_i(x_i) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} F_{\varepsilon_i}(x_i - \beta_{i,1}z_1 - ... - \beta_{i,k}z_k) f_{z_1}(z_1) \cdot ... \cdot f_{z_k}(z_k)\ dz_1...dz_k \tag{40}$$

$$f_x(x_1, ..., x_d) = \int_{-\infty}^{\infty} ... \int_{-\infty}^{\infty} \prod_{i=1}^{d} f_{\varepsilon_i}(x_i - \beta_{i,1}z_1 - ... - \beta_{i,k}z_k) f_{z_1}(z_1) \cdot ... \cdot f_{z_k}(z_k)\ dz_1...dz_k \tag{41}$$

Using change of variable $u_i = F_{z_i}(z_i)$ and that the common factors are independent, we obtain bounded integrals:

$$g_i(x_i) = \int_0^1 ... \int_0^1 f_{\varepsilon_i}(x_i - \beta_{i,1}F_{z_1}^{-1}(u_1) - ... - \beta_{i,k}F_{z_k}^{-1}(u_k))\ du_1...du_k \tag{42}$$

$$G_i(x_i) = \int_0^1 ... \int_{-\infty}^{\infty} F_{\varepsilon_i}(x_i - \beta_{i,1}F_{z_1}^{-1}(u_1) - ... - \beta_{i,k}F_{z_k}^{-1}(u_k))\ du_1...du_k \tag{43}$$

$$f_x(x_1, ..., x_d) = \int_0^1 ... \int_{-\infty}^{\infty} \prod_{i=1}^{d} f_{\varepsilon_i}(x_i - \beta_{i,1}F_{z_1}^{-1}(u_1) - ... - \beta_{i,k}F_{z_k}^{-1}(u_k))\ du_1...du_k \tag{44}$$

Following Oh and Patton (2016) we evaluate these integrals using Gauss-Legendre numerical integration with 50 abscissas. For the inversion of $G_i(x_i)$, which is a function of x_i and $\beta_{i,1}, ..., \beta_{i,k}$, we compute G over a wide range of points and then use piecewise cubic hermite-interpolation to obtain $G_i^{-1}(x_i)$.

4 Copula based Multi-CoVaR and Shapley value

In this chapter we first present an introduction to the bivariate CoVaR systemic risk measure. We then show a multivariate extension and how this Multi-CoVaR measure can be characterized through Markov-switching factor copulas. Finally, we present how total systemic risk and individual systemic risk contributions can be obtained in the Multi-CoVaR framework.

4.1 CoVaR

We begin with recalling the definition of VaR of a random variable $y_{i,t}$:

$$\mathrm{P}(y_{i,t} \leq VaR^{i}_{\alpha,t}) = \alpha \tag{45}$$

That is, the VaR is the α-quantile of the conditional distribution of $y_{i,t}$. VaR is one of the most popular risk measures used by regulators and in the financial industry. In the system risk context $y_{i,t}$ commonly refers to the return on a financial institution. However, VaR as a risk measure of one institution in isolation does not necessarily reflect systemic risk. Adrian and Brunnermeier (2008) introduce CoVaR, where the prefix stands for 'conditional, contagion, comovement' as the VaR of the financial system conditional on some event of an individual financial institution. More specifically, CoVaR of Adrian and Brunnermeier (2008), the VaR of the financial system (with return $y_{d,t}$) conditional on some event $D(y_{i,t})$ defined in terms of the return of institution i, is implicitly defined by:

$$\mathrm{P}(y_{d,t} \leq CoVaR^{D(y_{i,t})}_{\alpha,t} | D(y_{i,t})) = \alpha \tag{46}$$

That is, $CoVaR^{D(y_{i,t})}_{\alpha,t}$ is the α-quantile of the conditional distribution of $y_{d,t}|D(y_{i,t})$. Typically $D(y_{i,t})$ refers to distress of institution i and Adrian and Brunnermeier (2008) use $y_{i,t} = VaR^{i}_{\alpha,t}$, that is, an institution's return being equal to its VaR, as the event for distress in order to obtain their systemic risk measure. Additionally, they define $y_{i,t} = VaR^{i}_{0.5,t}$, that is, an institution's return being equal to its median, as the event for the median state of an institution. Their measure of an institution's marginal contribution to systemic risk, its ΔCoVaR is the difference between CoVaR conditional on the institution being in distress and CoVaR conditional on the median state of the institution. Thus institution i's systemic risk contribution to the financial system is:

$$\Delta CoVar^{i}_{\alpha,t} = CoVaR^{y_{i,t}=VaR^{i}_{\alpha,t}}_{\alpha,t} - CoVaR^{y_{i,t}=VaR^{i}_{0.5,t}}_{\alpha,t} \tag{47}$$

The intuition behind CoVaR and ΔCoVaR is the following: The systemic risk measure CoVaR is high when distress of institution i coincides with distress of the financial system and the systemic risk contribution ΔCoVaR estimates by how much distress of institution i increases systemic distress, thus capturing spillover effects of institution i to the financial system. As Adrian and Brunnermeier (2008) point out, CoVaR is more extreme than unconditional VaR because it conditions on an adverse event with typically lower mean, higher variance, higher negative skewness and higher kurtosis. Since CoVaR focuses on the tail distribution, it reflects changes in these moments.

Mainik and Schaaning (2012) as well as Girardi and Ergün (2013) suggest changing the conditioning event for distress to $y_{i,t} \leq VaR^i_{\alpha,t}$, which allows for a more severe distress definition. Mainik and Schaaning (2012) show that changing the conditioning event in this way has important implications for the consistency of the CoVaR measure with respect to the level of dependence. They show, for a range of distributions, that CoVaR with the modified conditioning event is continuous and that systemic risk increases with the dependence parameter, whereas this does not hold for the CoVaR definition of Adrian and Brunnermeier (2008). Furthermore does the modified definition of CoVaR with $y_{i,t} \leq VaR^i_{\alpha,t}$ as distress event allow for back testing using standard Kupiec- and Christoffersen tests, as shown by Girardi and Ergün (2013). We follow Mainik and Schaaning (2012) and Girardi and Ergün (2013) and choose $y_{i,t} \leq VaR^i_{\alpha,t}$ as event for distress of an institution. Different approaches to estimate bivariate CoVaR have been proposed. Adrian and Brunnermeier (2008) employ quantile regression, Girardi and Ergün (2013) use bivariate DCC-GARCH models, whereas Reboredo and Ugolini (2015a) employ copulas.

4.2 Multi-CoVaR

Financial crisis are often characterized by multiple institutions being in distress at the same time. Therefore, when measuring the systemic risk contribution of an institution this institution should not be considered in isolation because distress might also propagate through other institutions being distressed at the same time. This is why Cao (2013) proposes Multi-CoVar, an extension of the bivariate CoVaR measure with a conditioning event including multiple institutions, thus capturing distress spillovers among these institutions.

In what follows we explain the Multi-CoVaR measure in more detail. Consider a set of $d-1$ institutions $U = \{1, .., d-1\}$, the random variables $y_{1,t}, ..., y_{d-1,t}$, referring to their respective returns, with marginal distributions $F_1, ..., F_{d-1}$. We are interested in the CoVaR with respect to the financial system, which has return $y_{d,t}$ with marginal distribution F_d. Now let S be a set of $s \leq d-1$ institutions indexed as $\{s_1, ..., s_s\}$ with $S \subseteq U$. The conditioning event for the Multi-CoVaR measure is based on the returns of the institutions in S and denoted as $D(y_{s_1,t}, ..., y_{s_s,t})$. Multi-CoVaR is implicitly defined as:

$$\mathrm{P}(y_{d,t} \leq CoVaR_{\alpha,t}^{D(y_{s_1,t},...,y_{s_s,t})} | D(y_{s_1,t}, ..., y_{s_s,t})) = \alpha \tag{48}$$

That is, $CoVaR_{\alpha,t}^{D(y_{s_1,t},...,y_{s_s,t})}$ is the VaR of the financial system conditional on some event of multiple individual financial institutions or equivalently the α-quantile of the conditional distribution of $y_{d,t}|D(y_{s_1,t}, ..., y_{s_s,t})$. This definition is very flexible as it allows constructing various events on different sets of institutions. We construct conditioning events as follows: We consider an institution being in distress when $y_{i,t} \leq VaR_{\alpha,t}^i$, that is, when its return is at or below the individual VaR level, and we consider an institution being in its median state when $VaR_{0.25,t}^i \leq y_{i,t} \leq VaR_{0.75,t}^i$, that is, when its return is within the corresponding interquartile range. For each of the 2^{d-1} distinct sets of institutions S we consider two different events: The first corresponds to all institutions in S being in distress, and in order to simplify the notation we write ACoVaR for the Multi-CoVaR of this event. Thus ACoVaR is implicitly defined as:

$$\mathrm{P}(y_{d,t} \leq ACoVaR_{\alpha,t}^S | y_{s_1,t} \leq VaR_{\alpha,t}^{s_1}, ..., y_{s_s,t} \leq VaR_{\alpha,t}^{s_s})) = \alpha \tag{49}$$

The second event we consider is when all institutions in S are in their median state and in order to simplify the notation, we write NCoVaR for the Multi-CoVaR of this event. Thus NCoVaR is implicitly defined as:

$$\mathrm{P}(y_{d,t} \leq NCoVaR_{\alpha,t}^S | VaR_{0.25,t}^{s_1} \leq y_{s_1,t} \leq VaR_{0.75,t}^{s_1}, ..., VaR_{0.25,t}^{s_s} \leq y_{s_s,t} \leq VaR_{0.75,t}^{s_s})) = \alpha \tag{50}$$

Using Bayes' theorem, equation (49) can be expressed as:

$$\frac{\mathrm{P}(y_{d,t} \leq ACoVaR_{\alpha,t}^S, y_{s_1,t} \leq VaR_{\alpha,t}^{s_1}, ..., y_{s_s,t} \leq VaR_{\alpha,t}^{s_s})}{\mathrm{P}(y_{s_1,t} \leq VaR_{\alpha,t}^{s_1}, ..., y_{s_s,t} \leq VaR_{\alpha,t}^{s_s})} = \alpha \tag{51}$$

Similarly for $NCoVaR^S_\alpha$ we obtain:

$$\frac{\mathrm{P}(y_{d,t} \leq NCoVaR^S_{\alpha,t}, VaR^{s_1}_{0.25,t} \leq y_{s_1,t} \leq VaR^{s_1}_{0.75,t}, ..., VaR^{s_s}_{0.25,t} \leq y_{s_s,t} \leq VaR^{s_s}_{0.75,t})}{\mathrm{P}(VaR^{s_1}_{0.25,t} \leq y_{s_1,t} \leq VaR^{s_1}_{0.75,t}, ..., VaR^{s_s}_{0.25,t} \leq y_{s_s,t} \leq VaR^{s_s}_{0.75,t})} = \alpha \tag{52}$$

Cao (2013) expresses equation (51) as follows:

$$\frac{\int_{-\infty}^{ACoVaR^S_{\alpha,t}} \int_{-\infty}^{VaR^{s_1}_{\alpha,t}} ... \int_{-\infty}^{VaR^{s_s}_{\alpha,t}} f_{s+1}(y_{d,t}, y_{s_1,t}, ..., y_{s_s,t})\ dy_{d,t}dy_{s_1,t}...dy_{s_s,t}}{\int_{-\infty}^{VaR^{s_1}_{\alpha,t}} ... \int_{-\infty}^{VaR^{s_s}_{\alpha,t}} f_s(y_{s_1,t}, ..., y_{s_s,t})\ dy_{s_1,t}...dy_{s_s,t}} = \alpha \tag{53}$$

where f_s and f_{s+1} are the conditional joint densities of $(y_{s_1,t}, ..., y_{s_s,t})$ and $(y_{d,t}, y_{s_1,t}, ..., y_{s_s,t})$, respectively. Equivalently for equation (52)[4]:

$$\frac{\int_{-\infty}^{NCoVaR^S_{\alpha,t}} \int_{VaR^{s_1}_{0.25,t}}^{VaR^{s_1}_{0.75,t}} ... \int_{VaR^{s_s}_{0.25,t}}^{VaR^{s_s}_{0.75,t}} f_{s+1}(y_{d,t}, y_{s_1,t}, ..., y_{s_s,t})\ dy_{d,t}dy_{s_1,t}...dy_{s_s,t}}{\int_{-\infty}^{VaR^{s_1}_{\alpha,t}} ... \int_{VaR^{s_s}_{0.25,t}}^{VaR^{s_s}_{0.75,t}} f_s(y_{s_1,t}, ..., y_{s_s,t})\ dy_{s_1,t}...dy_{s_s,t}} = \alpha \tag{54}$$

Cao (2013) then obtains $ACoVaR^S_{\alpha,t}$ and $NCoVaR^S_{\alpha,t}$ as solutions to equation (53) and equation (54), respectively, which requires numerical evaluation of multi-dimensional integrals. He models the conditional joint density as a multivariate t distribution with conditional variances obtained through univariate GARCH models and conditional correlations through bivariate DCC models. In order to compute $VaR^i_{\alpha,t}$ he employs a nonparametric bootstrap, thereby relaxing the distributional assumption resulting from the multivariate t distribution.

4.3 Copula based estimation of Multi-CoVaR

As opposed to Cao (2013) and Bernardi et al. (2014), who employ multivariate time series models, we employ copula based models for the estimation of Multi-CoVaR. Expressing equation (51) in terms of copulas we obtain:

$$\frac{C_{s+1}(z_{A,t}, \alpha, ..., \alpha)}{C_s(\alpha, ..., \alpha)} = \alpha \tag{55}$$

where C_{s+1} is the copula of $(u_{d,t}, u_{s_1,t}, ..., u_{s_s,t})$, C_s is the copula of $(u_{s_1,t}, ..., u_{s_s,t})$ and $z_{A,t} = F_d(ACoVaR^S_{\alpha,t})$. Similarly, for equation (52) we get:

$$\frac{C_{s+1}(z_{N,t}, 0.75, ..., 0.75) - C_{s+1}(z_{N,t}, 0.25, ..., 0.25)}{C_s(0.75, ..., 0.75) - C_s(0.25, ..., 0.25)} = \alpha \tag{56}$$

[4]Note that the conditioning event for the median state in Cao (2013) is expressed as a one standard deviation around the median event, whereas we use the interquartile range.

where $z_{N,t} = F_d(NCoVaR^S_{\alpha,t})$. Thus $ACoVaR^S_{\alpha,t}$ can be obtained by first solving equation (55) numerically for $z_{A,t}$ and setting $ACoVaR^S_{\alpha,t} = F_d^{-1}(z_{A,t})$ in a second step. $NCoVaR^S_{\alpha,t}$ is obtained in the same way from equation (56).

For the computation of Multi-CoVaR, copulas offer great computational advantages compared to multivariate time series models. This becomes clear when comparing equation (53) and (54) to equation (56) respectively (55), where only nominal VaR levels and not $VaR^i_{\alpha,t}$ itself is required. Furthermore, no multi-dimensional integrals need to be solved as long as the copula cdf is available in closed form. The computational advantages of copulas over time-series models have already been pointed out by Reboredo and Ugolini (2015a) in the context of bivariate CoVaR estimation.

Solving equation (55) and equation (56) for $z_{A,t}$ and $z_{N,t}$, respectively, requires evaluating the copula cdf, which is not available in closed form for most factor copulas. We approximate the factor copula cdf through simulation: First obtain a large sample from equation (16) for which we compute the empirical probability integral transform. That is, we apply the respective empirical marginal cdf to each series to obtain data that has $Uniform(0,1)$ marginal distributions but the dependence structure implied by the factor copula model. The resulting sample is then used to approximate the factor copula cdf C. We found this approach superior to Monte-Carlo integration over the factor copula pdf. Given a two-state Markov-switching process for the factor copula, we obtain C as a mixture of factor copulas as $C = \omega_t C^{(1)} + (1-\omega_t)C^{(2)}$, where $C^{(1)}$ and $C^{(2)}$ are factor copula cdfs for the two regimes and for the mixture weight ω_t we employ the smoothed probability for the first regime, that is, $\omega_t = \mathrm{P}(S_t = 1|Y_{1:n}; \theta_m, \theta_c)$. Therefore we have to simulate two samples to approximate C, one for each state. These samples can be used for each evaluation of the solver and each period t.

4.4 Total systemic risk and individual systemic risk contributions

Multi-ΔCoVaR, the systemic risk contribution, of a set S of institutions is defined as:

$$\Delta CoVaR^S_{\alpha,t} = ACoVaR^S_{\alpha,t} - NCoVaR^S_{\alpha,t} \tag{57}$$

That is, the systemic risk contribution of S is measured as the difference of the financial system VaR conditional on distress and median state of the institutions in S. The total systemic risk based on the Multi-CoVaR measure is naturally obtained when all institutions in the system are in distress jointly, that is, the total systemic risk equals $\Delta CoVaR^{U}_{\alpha,t}$.

In a next step we want to obtain a measure for the individual systemic risk contribution of an institution. Because we want to account for distress spillovers among individual financial institutions, we do not want to simply measure the systemic risk contribution of institution i as $\Delta CoVaR^{\{i\}}_{\alpha,t}$, that is, the ΔCoVaR when only institution i is in distress, which corresponds to the definition of Adrian and Brunnermeier (2008). Instead we employ Shapley value methodology (Shapley (1953)), a general attribution procedure from cooperative game theory, which allows to efficiently allocate the total systemic risk among the individual institutions. It has previously been employed for portfolio (Koylouglu and Stoker (2002)) and systemic risk allocation (Tarashev et al. (2010), Cao (2013), Bernardi et al. (2014)). Shapley value has originally been proposed to share total wealth or cost among a set of players U of a cooperative game. The Shapley value of a player is the share of total wealth or cost that can efficiently be attributed to this player. In order to employ Shapley value methodology a (super)additive characteristic function v, which must be the same across all possible groups of players, is employed. The characteristic function applied to the set of all players, $v(U)$, must equal the total wealth or cost and $v(\emptyset)$ must be zero.

In the context of systemic risk allocation (which can clearly be understood as allocating a cost) the players are financial institutions and v corresponds to a systemic risk measure that is defined on sets of financial institutions. The individual systemic risk contribution of an institution is its Shapley value. Shapley value of institution i is computed as:

$$SV_i(v) = \sum_{S \subseteq U \setminus \{i\}} \frac{|S|!((d-1)-|S|-1)!}{(d-1)!}(v(S \cup \{i\}) - v(S)) \tag{58}$$

where the sum is computed over all subsets $S \subseteq U$ not containing institution i and $d-1$ is the number of players in U. The general idea behind Shapley value is the following: Suppose players are ordered randomly and assume S is the set of players appearing before player i. The term $v(S \cup \{i\}) - v(S)$ then corresponds to the contribution of player i to the cost or wealth of group S. There are $(d-1)!$ possible orderings and the Shapley value of a player is the average contribution over all orderings when all orderings receive equal weight.

Shapley (1953) shows that SV_i uniquely fulfills the following axioms, which carry intuitive meaning for systemic risk allocation:

1. Efficiency/Additivity: $\sum_{i \in U} SV_i(v) = v(U)$. The sum of the individual systemic risk contributions equals the total systemic risk.

2. Dummy axiom: If $v(S \cup \{i\}) - v(S) = v(\{i\}) \; \forall S \subseteq U$ with $i \notin S$ then $SV_i(v) = v(\{i\})$. The systemic risk contribution of institution i equals its ΔCoVaR if institution i is orthogonal to all other institutions. When the orthogonality is not fulfilled, the systemic risk contribution obtained through the Shapley value methodology does not equal $\Delta CoVaR_{\alpha,t}^{\{i\}}$, potentially leading to differing systemic risk rankings of institutions and total systemic risk, as pointed out by Cao (2013).

3. Symmetry/Fairness: If $v(S \cup \{i\}) = v(S \cup \{j\}) \; \forall S \subseteq U$ such that $j \neq i$ and $i, j \notin S$, then $SV_i(v) = SV_j(v)$. This axiom states that the individual systemic risk contribution for two institutions is the same when their marginal systemic risk contributions over any subset S are the same.

4. Linearity: Let $v = \omega_1 v_1 + \omega_2 v_2$ be a linear combination of two characteristic functions, then $SV_i(v) = \omega_1 SV_i(v_1) + \omega_2 SV_i(v_2) \; \forall i \in U$. This axiom states that the individual systemic risk contributions based on a linear combination of systemic risk measures can be obtained as a linear combination of the initial systemic risk contributions. For systemic risk allocation different systemic risk measures can be employed in a single attribution procedure.

5. Zero player: If $v(S \cup \{i\}) - v(S) = 0 \; \forall S \subseteq U$ with $i \notin S$, then $SV_i(v) = 0$. A financial institution that carries no systemic risk has zero systemic risk contribution.

5 Results

In this chapter we turn to a description of our results. We begin with a description of our data set and show results from marginal models. We then move on to studying the dependence structure and presenting estimation results for several Markov-switching factor copula models. Finally, we present results for government debt systemic risk for the Euro Area financial sector.

5.1 Data description and univariate results

We analyze systemic risk for the Euro Area financial sector of eight sovereign debt markets, three core countries (Germany (DE), France (FR) and the Netherlands (NL)) and five PIIGS countries (Portugal (PT), Italy (IT), Ireland (IR), Greece (GR) and Spain (ES)), over the period 04 May 2005 to 27 April 2016 using weekly log-returns on ten year benchmark government bond indices obtained from Datastream. The performance of the Euro Area financial sector is measured using weekly log-returns on the Datastream EMU Financials index (EMU FINA), over the same period.

Figure 2 displays the corresponding price series. The financial sector index shows

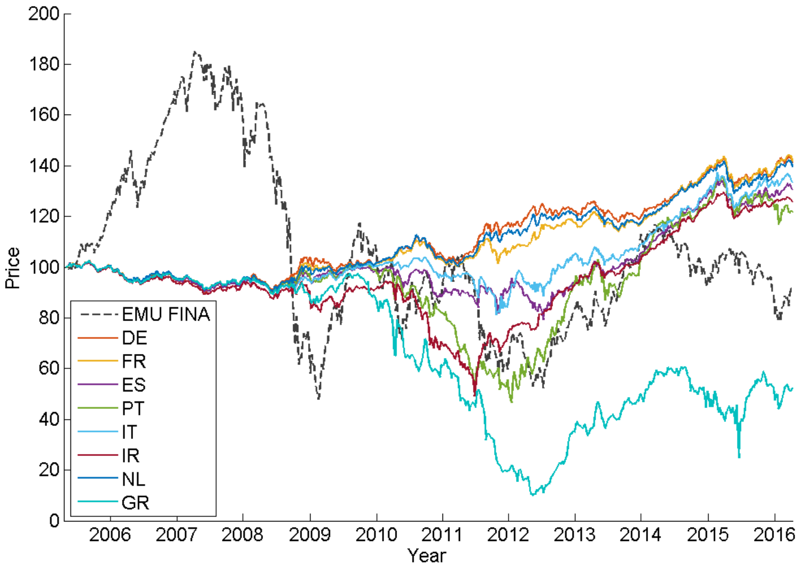

Figure 2: This figure presents price indices of the Euro Area financial sector (EMU FINA) and ten year government bonds over the periods 04 May 2005 to 27 April 2016 ($n = 574$). 04 May 2005 = 100.

heightened volatility from the beginning of 2008 with a strong reduction in value, reflecting the fallout of the global financial crisis in Europe. From September 2008 the government bonds decouple. The indices of the PIIGS countries show heightened volatility, with a distinct reduction in value that is later reversed for all but the Greek bond. The bond indices of the three core countries remain stable and display low volatility throughout the sample period. In table 1 we present summary statistics for the return series. Most series

	EMU FINA	DE	FR	ES	PT	IT	IR	NL	GR
mean	-0.0002	0.0006	0.0006	0.0005	0.0004	0.0005	0.0004	0.0006	-0.0011
std	0.0421	0.0085	0.0082	0.0122	0.0223	0.0119	0.0161	0.0080	0.0514
skew	-0.32	-0.19	-0.52	1.15	-0.02	0.22	-0.09	-0.38	-0.05
kurt	4.70	3.73	4.92	10.75	11.77	12.58	15.98	3.93	22.95
JB	[0.001]	[0.003]	[0.001]	[0.001]	[0.001]	[0.001]	[0.001]	[0.001]	[0.001]

Table 1: This table presents summary statistics for weekly log-returns of a Euro Area financial sector index (EMU FINA) and ten year government bonds indices. JB denotes p-values of Jarque-Bera tests.

have a mean return slightly above zero, only the financial sector index and government bond index of GR display negative mean returns. These two series also have the highest volatility, as measured by the standard deviation. All series but the government bond indices for ES and IT show negative skewness. Kurtosis is well above three for all series, with higher kurtosis for the government bond indices of the PIIGS countries than for the indices of the three core countries. Jarque-Bera tests strongly reject normality for all series.

In order to take time variation in conditional means and variances into account we model each series using univariate ARMA(l,m)-GJR-GARCH(p,o,q) models (Glosten et al. (1993)):

$$y_{i,t} = \mu_i + \sum_{j=1}^{l} \phi_{i,j} y_{i,t-j} + \sum_{k=1}^{m} \psi_{i,k} u_{i,t-k} + u_{i,t} \tag{59}$$

$$u_{i,t} = \sqrt{\sigma^2_{i,t}} z_{i,t} \tag{60}$$

$$\sigma^2_{i,t} = \omega_i + \sum_{j=1}^{p} \alpha_{i,j} u^2_{i,t-j} + \sum_{k=1}^{o} \gamma_{i,k} u^2_{i,t-k} \mathbf{I}\left\{u_{i,t-k} < 0\right\} + \sum_{l=1}^{q} \delta_{i,l} \sigma^2_{i,t-q} \tag{61}$$

$$z_{i,t} \sim Skew\ t(\nu_i, \lambda_i) \tag{62}$$

where $\mathbf{I}\{u_{i,t-k} < 0\} = 1$ if $u_{i,t-k} < 0$ and $\mathbf{I}\{u_{i,t-k} < 0\} = 0$ otherwise. The term $\sum_{k=1}^{o} \gamma_{i,k} u_{i,t-k}^2 \mathbf{I}\{u_{i,t-k} < 0\}$ allows for an asymmetric impact of $u_{i,t-k}$ on volatility, where typically the impact of negative returns on volatility is stronger than the impact of positive returns and therefore $\gamma_{i,k} > 0$, which is commonly referred to as leverage effect. The standardized residuals $z_{i,t}$ are assumed to follow the *Skew t* distribution of Hansen (1994), with density as in equation (14). This distribution can account for the fattailedness and skewness often found in financial time series. Modeling skewness in marginal distributions is important to distinguish skewness from asymmetric dependence, which we want to investigate later. In order to account for the relatively high autocorrelation found in the government bond series we consider a lag order up to three for the ARMA models. For the conditional variance we consider the following specifications: constant, ARCH(1), GARCH(1,1) GJR-GARCH(1,1,1), ARCH(2), GARCH(2,2), GJR-GARCH(2,1,2), GJR-GARCH(2,2,2), GARCH(3,3) and GARCH(4,4). The best model for each series is selected using the AIC. We perform ARCH-LM as well as Ljung-Box tests to test for residual autocorrelation and Kolomogorov-Smirnoff as well as Cramer-von Mieses tests to check that the standardized residuals are well specified by *Skew t* models[5]. We obtain p-values for Kolomogorov-Smirnoff and Cramer-von Mieses tests using simulation, thereby accounting for parameter estimation error, see Patton (2013) for details. All estimation and test results for the marginal models can be found in table 2. The estimated GJR-GARCH models indicate strong persistence in conditional variances and existence of a leverage effect for some series.

[5]See Fermanian and Scaillet (2005) for the consequences of misspecified marginal distribution models in the context of copulas.

	EMU FINA	DE	FR	ES	PT	IT	IR	NL	GR
μ_i	-0.0001	0.0006	0.0006	0.0009	0.0003	0.0005	0.00	0.0006	-0.0011
	(-0.11)	(1.75)	(1.84)	(0.99)	(0.43)	(1.07)	(0.57)	(1.79)	(-0.51)
$\phi_{i,1}$	-0.3135			-0.9315	0.8911	-0.1524			0.0489
	(-2.73)			(-20.76)	(4.32)	(-3.66)			(1.18)
$\phi_{i,2}$	0.1015				-1.0778	0.1152			-0.1571
	(0.74)				(-9.17)	(2.77)			(-3.85)
$\phi_{i,3}$	0.8821				0.3613				0.1689
	(7.88)				(1.86)				(4.09)
$\psi_{i,1}$	0.3146			0.8876	-0.8825		0.09		
	(2.42)			(14.92)	(-4.41)		(4.67)		
$\psi_{i,2}$	-0.0769				1.0468				
	(-0.51)				(10.28)				
$\psi_{i,3}$	-0.8228				-0.4557				
	(-6.73)				(-2.60)				
ω_i	3.4E-05	2.6E-06	7.6E-06	2.6E-06	4.2E-07	1.4E-06	8.3E-06	3.7E-06	2.4E-06
	(2.08)	(1.89)	(2.40)	(1.38)	(0.80)	(1.73)	(2.86)	(1.63)	(1.96)
$\alpha_{i,1}$	9.8E-08	0.0814	0.1286	0.0504	0.0083	0.0802	0.0425	0.0805	0.0512
	(0.71)	(3.28)	(2.88)	(1.83)	(0.29)	(3.17)	(1.15)	(2.49)	(1.58)
$\gamma_{i,1}$	0.2042			0.1303	0.0812		0.3178		0.1197
	(2.74)			(1.80)	(3.37)		(2.93)		(2.79)
$\delta_{i,1}$	0.8779	0.8825	0.7547	0.8708	0.9509	0.9088	0.7750	0.8621	0.8888
	(20.35)	(24.95)	(9.89)	(17.31)	(43.05)	(36.42)	(16.87)	(14.59)	(27.64)
ν_i	76.8818	24.5513	14.6371	8.1577	7.8090	7.3696	3.8850	19.0068	4.5863
	(0.98)	(1.17)	(1.82)	(3.40)	(3.67)	(3.77)	(5.98)	(1.50)	(6.17)
λ_i	-0.1406	-0.0323	-0.0730	0.0401	-0.1116	-0.1242	-0.1067	-0.1050	-0.0766
	(-2.50)	(-0.50)	(-1.19)	(0.67)	(-1.75)	(-2.06)	(-2.18)	(-1.56)	(-1.57)
LB	[0.70]	[0.53]	[0.68]	[0.22]	[0.95]	[0.70]	[0.91]	[0.28]	[0.99]
ARCH	[0.84]	[0.71]	[0.09]	[0.92]	[0.56]	[0.13]	[0.65]	[0.64]	[0.67]
LB^2	[0.86]	[0.69]	[0.08]	[0.92]	[0.52]	[0.17]	[0.61]	[0.63]	[0.63]
CvM	[0.45]	[0.33]	[0.69]	[0.59]	[0.28]	[0.08]	[1.00]	[0.18]	[0.80]
KS	[0.65]	[0.53]	[0.88]	[0.82]	[0.17]	[0.18]	[1.00]	[0.33]	[0.84]

Table 2: This table presents parameter estimates and t-statistics (in parentheses) for the ARMA-GJR-GARCH-*Skew t* marginal distribution models for log-returns of a Euro Area financial sector index (EMU FINA) and ten year government bond indices. LB denotes the Ljung-Box test for autocorrelation in ARMA residuals. ARCH denotes Engle's LM test for ARCH effects in standardized residuals and LB^2 denotes the Ljung-Box test for autocorrelation in squared standardized residuals. CvM and KS denote Cramer-von Mieses and Kolmogorov-Smirnov specification tests for *Skew t* distribution, respectively. We present p-values of all tests in square brackets.

5.2 Multivariate results

We now want to analyze the dependence structure of our data set. An inspection of the price series in figure 2 already reveals a first impression of the dependence structure. All government bond series closely comove until September 2008, when they decouple and dependence of the financial sector index and the PIIGS bonds increases. We now present some dependence measures of our data. In table 3 we display the unconditional rank correlation matrix and quantile dependence matrices for $q = 0.1$ and $q = 0.9$. Rank correlations of the financial sector index and the core countries are negative, for the financial sector and the PIIGS countries we observe low positive rank correlations.

	EMU FINA	DE	FR	ES	PT	IT	IR	NL	GR
					Rank correlations				
EMU FINA									
DE	-0.41								
FR	-0.25	0.87							
ES	0.11	0.46	0.60						
PT	0.10	0.34	0.45	0.63					
IT	0.12	0.47	0.63	0.84	0.66				
IR	0.05	0.42	0.55	0.69	0.68	0.70			
NL	-0.34	0.95	0.91	0.53	0.41	0.55	0.50		
GR	0.23	0.18	0.32	0.52	0.58	0.53	0.54	0.26	
					Quantile dependence				
EMU FINA		0.31	0.23	0.21	0.16	0.19	0.16	0.26	0.17
DE	0.21		0.68	0.30	0.28	0.28	0.31	0.73	0.17
FR	0.12	0.71		0.44	0.31	0.40	0.45	0.77	0.26
ES	0.19	0.44	0.49		0.47	0.68	0.59	0.38	0.30
PT	0.21	0.33	0.37	0.44		0.38	0.47	0.28	0.35
IT	0.19	0.42	0.49	0.56	0.49		0.51	0.35	0.37
IR	0.14	0.49	0.44	0.51	0.56	0.45		0.40	0.33
NL	0.16	0.78	0.82	0.49	0.37	0.45	0.49		0.23
GR	0.24	0.21	0.26	0.38	0.40	0.35	0.37	0.24	

Table 3: This table presents rank correlations and quantile dependence for $q = 0.1$ (lower triangular part) and $q = 0.9$ (upper triangular part) of the data.

All rank correlations of government bonds are positive, with highest values for DE, FR and NL. The quantile dependence measures are higher between bond indices than between the financial sector index and the bond indices.[6] The difference of $\tau_{0.1}$ and $\tau_{0.9}$ reveals some mild asymmetry.

We now turn to modeling the dependence of the government bond return series and the returns on the EMU financial sector index with Markov-switching factor copulas. We employ the multifactor-model as in equation (16), which we estimate as described as in chapter 3. Given that we employ maximum likelihood estimation, different models can always be compared on the basis of model selection criteria such as the AIC and BIC or likelihood-ratio tests. However, having some prior information on the number of factors necessary to describe the data can be useful and even necessary in high-dimensional applications. To this end, Oh and Patton (2015) suggest a graphical tool, namely scree plots. Scree plots display the ordered eigenvalues of a correlation matrix, in the context of factor copulas the *rank* correlation matrix of the data. The number of factors is estimated as the number of eigenvalues larger one, see Oh and Patton (2015) for details about this estimator. We display a scree plot of the sample rank correlation matrix in figure 3.

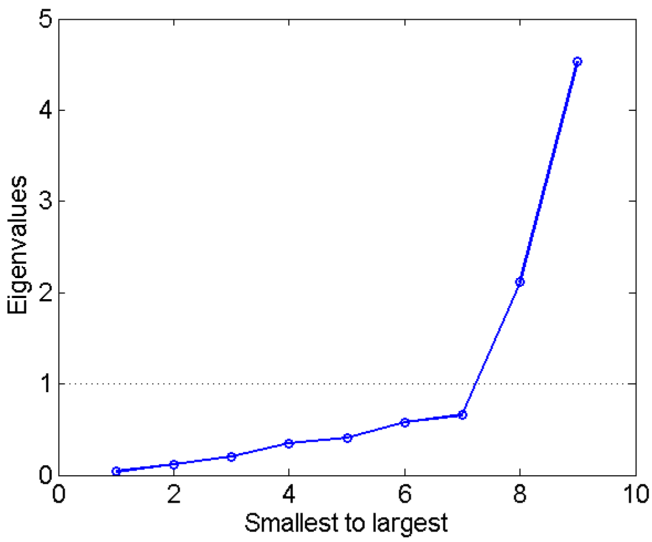

Figure 3: This figure presents a plot of the ordered eigenvalues of the sample rank correlation matrix.

[6] For the three negatively dependent pairs (EMU FINA, DE), (EMU FINA, FR) and (EMU FINA, NL) we compute quantile dependence measures along the counter diagonal, see section 2.2 for details.

The scree plot shows that there are two relatively large eigenvalues, both distinctly larger than one, while the remaining eigenvalues are considerably smaller and gradually tail off, with the third largest eigenvalue already being distinctly below one. Thus the scree plot indicates two common factors.

We now present estimation results for several factor copula models. We employ the Normal factor copula density as well as the *Skew* $t-t$ factor copula density, which is able to account for potential non-normal features of the data, such as tail dependence and asymmetric dependence, through fat tailed and asymmetric common factors. The simplest specification we estimate is a static model with just one common factor. We also estimate models with two common factors, as indicated by the scree plot above. In order to account for time-variation in the dependence structure we also estimate a number of Markov-switching factor copula models, where we limit our analysis to models with two regimes. We consider Markov-switching models with one common factor in both regimes, one factor in one regime and two factors in the second regime, and two factors in both regimes. A summary of the estimation results can be found in table 4, which contains the number of parameters for each model, the copula log-likelihood as well as the model selection criteria AIC and BIC.

For all specifications the *Skew* $t-t$ models show distinctly lower values for the model selection criteria than the corresponding Normal models, providing evidence against the

		Normal				*Skew t-t*			
$k^{(1)}$	$k^{(2)}$	# param.	L_c	AIC	BIC	# param.	L_c	AIC	BIC
1		9	1532.4	-3046.8	-3007.6	11	1910.7	-3799.4	-3751.5
2		18	2306.7	-4577.4	-4499.1	20	2591.0	-5142.0	-5054.9
1	1	20	2798.7	-5557.4	-5470.3	24	2851.8	-5655.6	-5551.1
1	2	29	3320.9	-6583.8	-6457.6	33	3400.3	-6734.6	-6591.0
2	2	38	3349.2	-6622.4	-6457.0	42	**3451.5**	**-6819.0**	**-6636.2**

Table 4: This table presents the number of parameters, log-likelihood (L_c), model selection criteria (AIC and BIC) of Markov-switching and static factor copula models with different numbers of common factors ($k^{(1)}$ is the number of common factors in the first regime and $k^{(2)}$ the number of commons factor in the second regime).

Normal factor copula and showing that accounting for non-normal features of the dependence structure of the data is important. We can see that including a second factor in the static model distinctly lowers AIC and BIC values. When moving from static to Markov-switching models we also see a distinct improvement in selection criteria, which shows that allowing for dependence dynamics is important for this data set. Including a second factor in only one regime increases the log-likelihood far more than moving from this model to a model with two factors in both regimes. This could be interpreted as some evidence for time-variation in the number of common factors. In fact, when considering the Normal models, the lowest value for the BIC is achieved by the model with one factor in the first regime and two factors in the second regime. However, AIC is lowest for the model with two factors in both regimes. When considering the *Skew* $t-t$ models the best model according to both criteria is the *Skew* $t-t$ factor copula with two common factors, which thus is the best model overall according to these criteria.

We now turn to a detailed discussion of the estimation results for the model which performed best according to model selection criteria, namely the Markov-switching *Skew* $t-t$ factor copula with two common factors in both regimes. We first display smoothed and filtered probabilities of being in the first regime in figure 4. These prob-

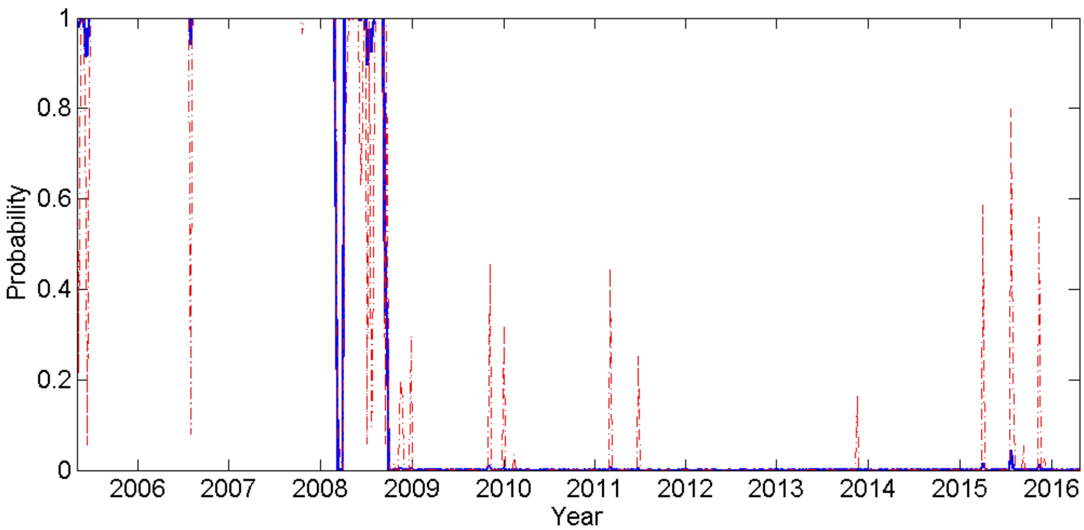

Figure 4: This figure presents smoothed (solid) and filtered (dashed) probabilities of the first regime based on a Markov-switching *Skew* $t-t$ factor copula with two common factors and two regimes.

abilities are obtained through the algorithms of Hamilton and Kim, described in section 3.1. We base the following discussion on the smoothed probability, that is, the probability of being in a given regime based on the full sample up to $t = n$. We observe a clear pattern: With only one very short exception in the beginning of 2008, the smoothed probability for the first regime is very close to one until 17 September 2008, the week of the Lehman Brothers bankruptcy. The smoothed probability of being in the first regime then remains very close to zero until the end of the sample period.

Table 5 displays the estimated copula parameters and t-statistics. We use the sim-

	Regime one		Regime two	
	param.	t-stat.	param.	t-stat.
ν^{-1}	0.15	(6.04)	0.17	(6.17)
λ	-0.20	(-2.64)	-0.04	(-0.72)
Factor 1				
$\beta_{\text{EMU FINA},1}$	-0.14	(-2.15)	-0.06	(-0.97)
$\beta_{\text{DE},1}$	9.40	(6.01)	2.79	(7.37)
$\beta_{\text{FR},1}$	19.69	(7.35)	2.59	(10.74)
$\beta_{\text{ES},1}$	3.56	(11.14)	1.34	(7.63)
$\beta_{\text{PT},1}$	2.39	(8.36)	0.71	(6.55)
$\beta_{\text{IT},1}$	3.14	(11.05)	1.53	(8.27)
$\beta_{\text{IR},1}$	17.48	(11.26)	0.88	(6.76)
$\beta_{\text{NL},1}$	16.25	(7.32)	4.22	(7.52)
$\beta_{\text{GR},1}$	1.75	(10.65)	0.33	(3.79)
Factor 2				
$\beta_{\text{EMU FINA},2}$	-0.48	(-7.32)	0.73	(7.78)
$\beta_{\text{DE},2}$	8.87	(6.34)	-2.13	(-4.95)
$\beta_{\text{FR},2}$	18.62	(7.21)	-1.04	(-3.33)
$\beta_{\text{ES},2}$	3.90	(9.75)	0.77	(4.73)
$\beta_{\text{PT},2}$	2.64	(8.65)	0.53	(5.12)
$\beta_{\text{IT},2}$	2.64	(10.47)	0.83	(4.24)
$\beta_{\text{IR},2}$	22.17	(10.18)	0.54	(4.28)
$\beta_{\text{NL},2}$	15.86	(8.02)	-2.32	(-4.90)
$\beta_{\text{GR},2}$	1.95	(12.06)	0.48	(4.99)
p_{11}	0.9895	(729.24)		
p_{22}			0.9947	(636.84)

Table 5: This table presents parameters estimates and t-statistics (in parentheses) for a Markov-switching *Skew* $t - t$ factor copula with two common factors applied to weekly returns on a Euro Area financial sector index (EMU FINA) and government bonds.

ulation based approach described in section 3.2 with 100 simulations to obtain standard errors. The transition matrix probabilities p_{11} and p_{22} are close to one, indicating high persistence of the corresponding regimes, which confirms the impression from figure 4. The estimated inverse degree of freedom parameters are similar in both regimes, implying around seven degrees of freedom. Both are significant, implying non-normality in the form of tail dependence. The estimated skewness parameter for the first regime is significantly negative, indicating asymmetries in the dependence structure of the data. For the second regime this parameter estimate is insignificant.

To complement table 5, we also display the estimated factor loadings graphically in figure 5. The estimated loadings for the first regime are similar for both factors. All

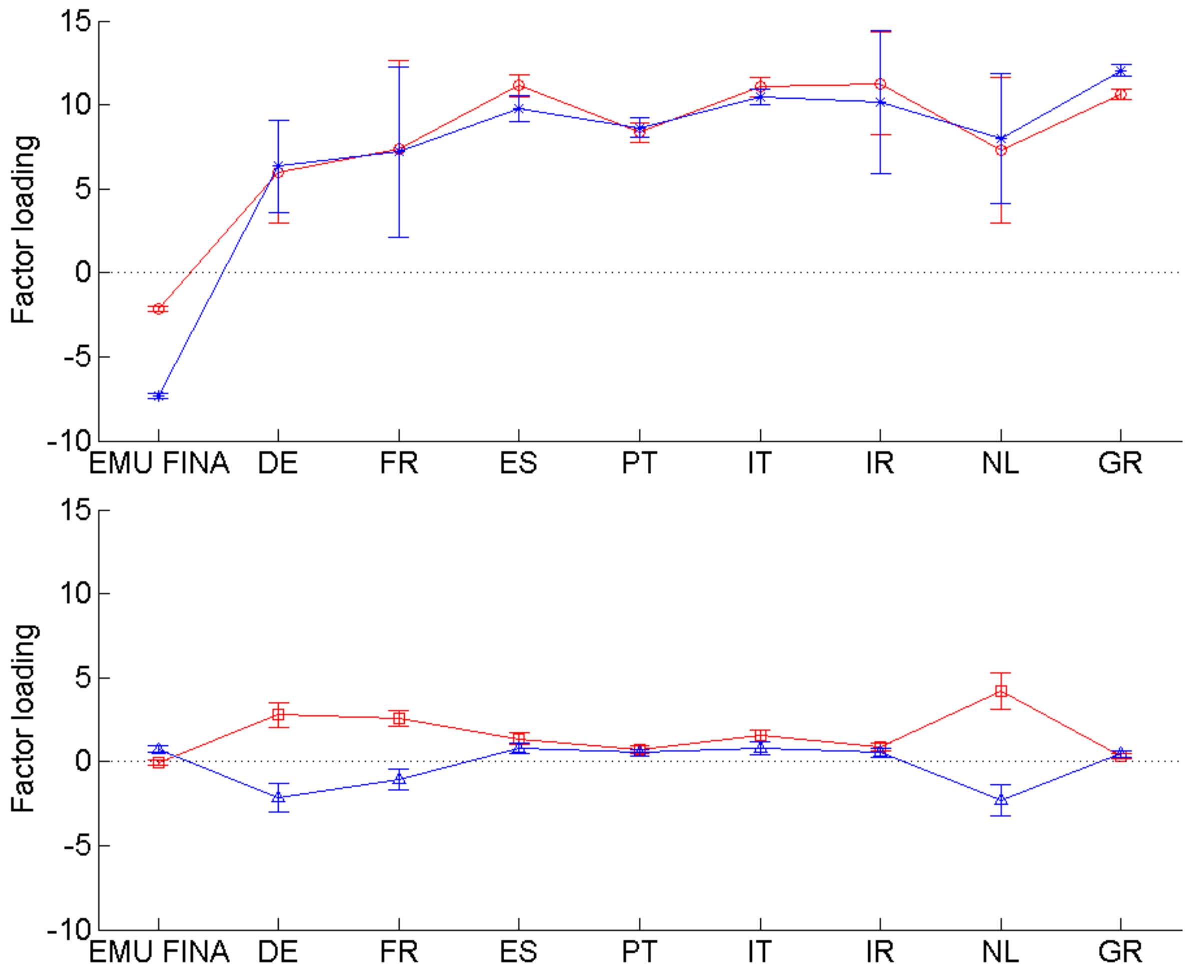

Figure 5: This figure presents estimated factor loadings of a Markov-switching *Skew* $t - t$ factor copula with two common factors for the first regime (top) and the second regime (bottom). Bars indicate 95 % confidence intervals.

estimated loadings are positive and relatively large but for the financial sector index, for which both factor loadings are negative. We find all estimated factor loadings for the first regime to be significant at the usual significance levels. We study the model implied dependence structure in more detail below, but the estimated factor loadings already point to a dependence structure with high positive dependence for the government bond indices and negative dependence for the government bond indices and the financial sector index.

The estimated loadings for the second regime, which is prevalent from 17 September 2008 onward, indicate greater heterogeneity. For one factor the estimated loading of the financial sector index is not significantly different from zero, whereas all estimated loadings for the government bond indices for this factor are significant and positive. Therefore this factor introduces positive dependence for the government bond indices, whereas it implies zero dependence of government bond indices and the financial sector. The other has negative loadings for the three core countries and positive loadings for all other bond indices and the financial sector index. Thus, it introduces positive dependence for the financial sector and the PIIGS government bond indices, but negative dependence for the financial sector index with the three core countries. The second regime thus implies positive dependence for the financial sector index and the PIIGS government bond indices. Therefore, we see a distinct change in the dependence structure from the first to the second regime, or equivalently with the Lehman Brothers default at the height of the global financial crisis. Most notably the financial sector index and the PIIGS government bonds become positively dependent.

We now want to study the dependence structure implied by the Markov-switching $Skew\ t-t$ factor copula model in more detail and analyze how well the model can replicate the dependence structure of the data. We perform this analysis conditional on the regime and focus on quantile dependence and rank correlations. In order to obtain conditional dependence measures of the data we classify observations according to the smoothed probability. More specifically, an observation is classified to belong to a given regime if the corresponding smoothed probability is higher than 0.5. This conditional approach is valid given the very strong regime classification with smoothed probabilities being close to zero or one almost all the time. Neither quantile dependence nor rank correlations are available in closed form for most factor copulas, including the $Skew\ t-t$ factor copula.

We obtain these quantities through simulation using a sample of size $n = 500000$ for each regime.

We display the rank correlation matrices implied by the model and the data in table 6 for both regimes. For the first regime the model implies very high rank correlations close

	EMU FINA	DE	FR	ES	PT	IT	IR	NL	GR
				Regime one					
EMU FINA		-0.26	-0.24	-0.24	-0.24	-0.21	-0.27	-0.24	-0.24
DE	-0.38		1.00	0.97	0.95	0.96	0.99	0.99	0.93
FR	-0.38	1.00		0.97	0.95	0.96	0.99	1.00	0.93
ES	-0.39	0.97	0.97		0.93	0.95	0.96	0.97	0.90
PT	-0.38	0.95	0.95	0.93		0.94	0.95	0.95	0.91
IT	-0.36	0.96	0.96	0.94	0.91		0.94	0.96	0.91
IR	-0.41	0.98	0.99	0.98	0.95	0.94		0.99	0.94
NL	-0.39	0.99	1.00	0.98	0.95	0.96	0.99		0.93
GR	-0.37	0.92	0.92	0.90	0.88	0.88	0.92	0.92	
				Regime two					
EMU FINA		-0.46	-0.26	0.24	0.23	0.25	0.18	-0.38	0.40
DE	-0.37		0.82	0.27	0.11	0.27	0.21	0.94	-0.09
FR	-0.25	0.85		0.47	0.26	0.50	0.39	0.88	0.11
ES	0.20	0.28	0.44		0.52	0.80	0.60	0.37	0.40
PT	0.20	0.16	0.29	0.54		0.55	0.59	0.21	0.47
IT	0.20	0.31	0.47	0.70	0.55		0.61	0.39	0.41
IR	0.19	0.22	0.36	0.58	0.46	0.60		0.31	0.40
NL	-0.32	0.92	0.90	0.38	0.24	0.41	0.31		0.03
GR	0.23	-0.03	0.09	0.38	0.32	0.39	0.33	0.04	

Table 6: This table presents conditional rank correlations implied by a Markov-switching *Skew* $t - t$ factor copula with two common factors (lower triangular part, based on 500000 simulations). We also display rank correlations of the data (upper triangular part), where we allocate observations to the regimes based on the smoothed probability.

to one for all government bond indices, and negative rank correlations of the government bond indices with the financial sector index, confirming the impression from figure 2 and being in line with our discussion of the estimated factor loadings from above. The rank correlation matrix implied by the second regime is distinctly different, the financial index is still negatively correlated with the government bond indices of the three core countries, but positively correlated with the PIIGS government bond indices. The rank correlation between the core countries is still high, but the rank correlation between core countries and the PIIGS countries and also between the PIIGS countries is distinctly lower than in the first regime, that is, the government bond markets decouple with the regime change. The rank correlations implied by the model are in general very close to the rank correlations of the data, with notable differences mostly in the first regime for the rank correlations of the financial sector index and the government bonds indices.

We now turn to a discussion of the quantile dependence implied by the model, and how well this replicates the data. Figures 6 and 7 plot the quantile dependence for $q = 0.01, 0.02, ..., 0.25$ and $q = 0.75, 0.02, ..., 0.99$, respectively, for each pair of variables. For the quantile dependence of the data we also add 95% bootstrap confidence intervals based on 500 bootstrap samples. The quantile dependence implied by the model is usually very close to the data, thus the Markov-switching $Skew\ t - t$ factor copula model with two common factors is able to replicate the quantile dependence structure of the data. For some pairs of variables there appears to be a deviation for the first regime and very small quantiles, but here one should keep the small sample size ($n^{(1)} = 171$ for the first regime) in mind. As for the rank correlations we observe distinct differences between the two regimes: In the first regime the model implies low quantile dependence between the financial index and the government bond indices,[7] both for the lower and upper joint tails, whereas all quantile dependence measures between the government bond indices are very high, in fact close to the boundary for some pairs of variables. There is a mild degree of asymmetry in the sense that dependence is higher for the lower joint tails than for the upper joint tails for most variables. For the eight negatively dependent pairs observations with high return for the financial sector and low return for the bond index are more likely than observations with low return for the financial sector and high return for the bond index. This observed asymmetry is in line with the negative estimate for the skewness parameter for the first regime.

[7] Given the negative rank correlations of these eight pairs we compute the quantile dependence measures along the counter diagonal, as described in section 2.2.

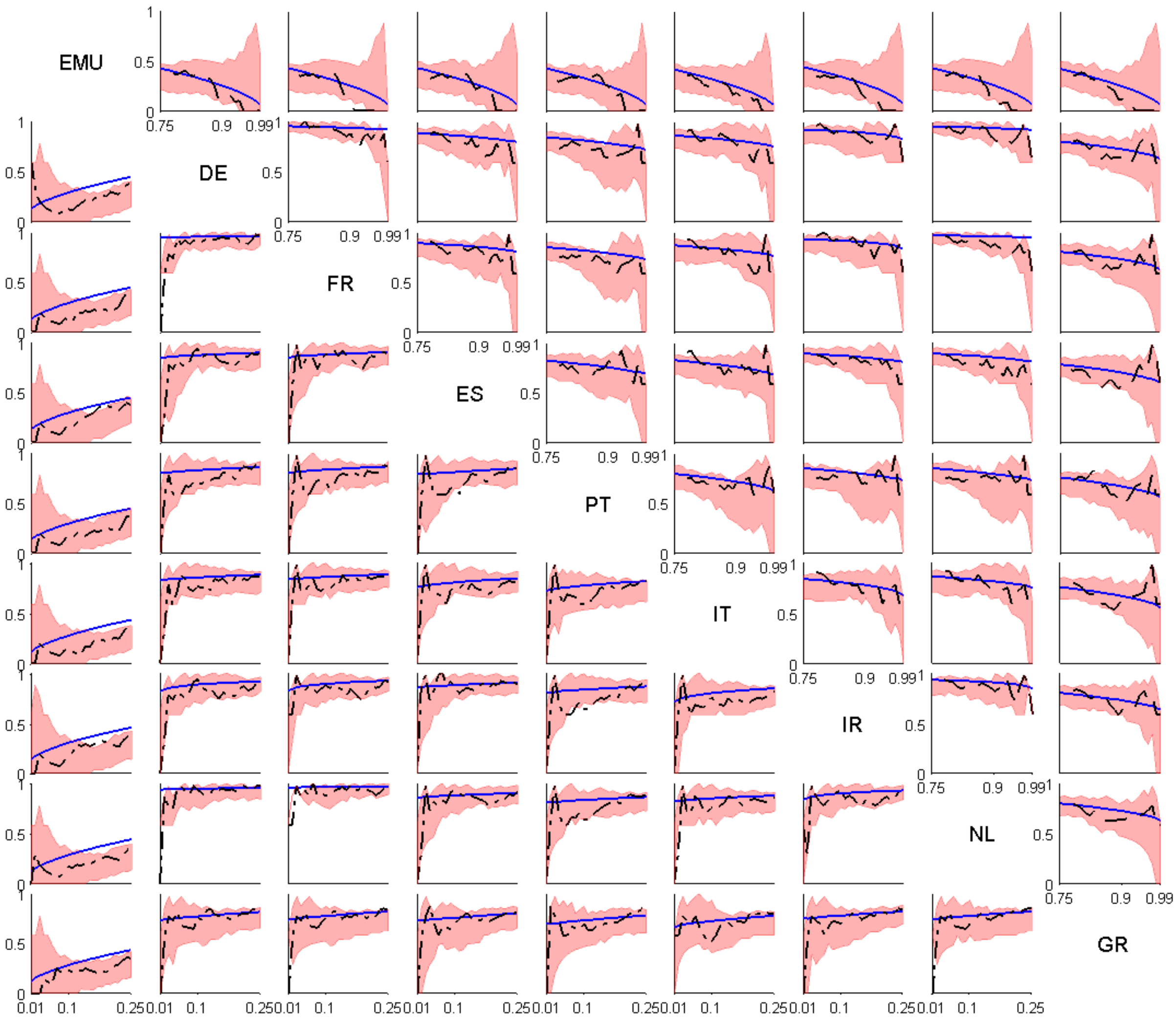

Figure 6: This figure presents quantile dependence implied by a Markov-switching *Skew* $t-t$ factor copula model with two common factors based on 500000 simulations (solid) and the data (dashed) for the first regime. We also show 95 % bootstrap confidence intervals based on 500 replications. We allocate obervations based on the smoothed probability to obtain conditional quantile dependence of the data. EMU is the Euro Area financial sector index.

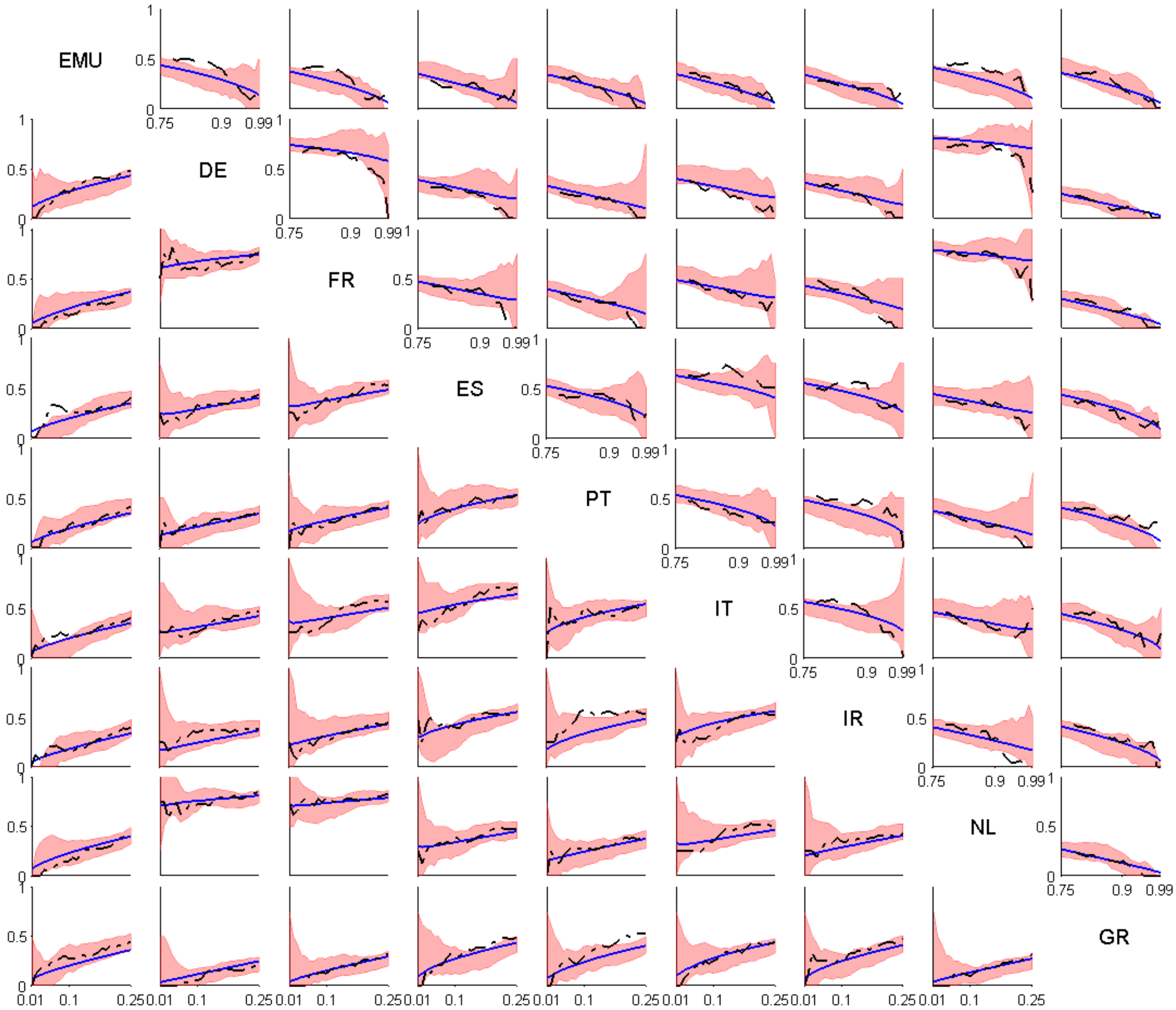

Figure 7: This figure presents quantile dependence implied by a Markov-switching *Skew* $t-t$ factor copula model with two common factors based on 500000 simulations (solid) and the data (dashed) for the second regime. We also show 95 % bootstrap confidence intervals based on 500 replications. We allocate obervations based on the smoothed probability to obtain conditional quantile dependence of the data. EMU is the Euro Area financial sector index.

For the second regime we observe that quantile dependence is low between the financial index and the government bond indices.[8] The quantile dependence measures between the government bond indices are distinctly lower than in the first regime, with highest values for the three core countries. We do not observe any systematic asymmetry for the second regime, in line with an insignificant skewness parameter for this regime.

5.3 Systemic risk results

We now turn to our analysis of government debt systemic risk for the Euro Area financial sector. To this end we employ the Multi-CoVaR measure introduced in section 4.2, which we compute using the Markov-switching *Skew* $t-t$ factor copula with two common factors from section 5.2. This model performs best among the models we compare according to model selection criteria and we demonstrate that the model is able to replicate the rank correlations and quantile dependence of the data. Two common factors are indicated by a scree plot of the rank correlation matrix of our data.

We obtain total systemic risk as the ΔCoVaR when all eight government debt markets are in distress jointly. We then employ the Shapley value methodology introduced in section 4.4 to efficiently allocate the total systemic risk among the debt markets and obtain the individual systemic risk contribution of each debt market, thereby accounting for distress spillovers among these markets. Specifically, we employ the systemic risk measure $\Delta CoVaR^{S}_{\alpha,t}$ as a characteristic function for the Shapley value computation. That is, $v(S)$ refers to $\Delta CoVaR^{S}_{\alpha,t}$ within our framework of systemic risk allocation. The subsets $S \subseteq U \setminus \{i\}$ of players refer to sets of government debt markets. We now turn to a discussion of our results.

For our model, time variation in the systemic risk measures occurs because of two reasons. First, due to time variation in the conditional distribution of the financial sector index, that is, the conditional mean and, to a greater extent, variance. To guide the following discussion, we display the VaR of the financial sector index in figure 8. We observe that Euro Area financial sector VaR is particularly high at the height of the global financial crisis around the Lehman Brothers default and again in the second half of

[8] For the pairs involving PIIGS bonds we now measure quantile dependence along the main diagonal because the corresponding rank correlations are positive.

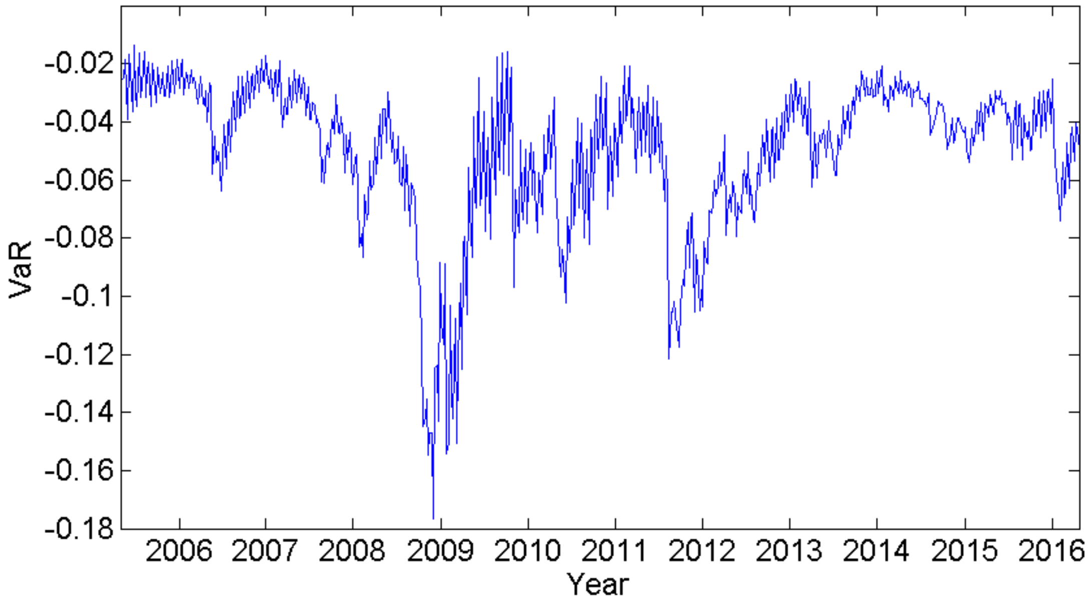

Figure 8: This figure presents VaR of the Euro Area financial sector. $\alpha = 0.05$.

2011. The second reason why time variation in the systemic risk measures occurs is due to changes in the dependence structure, that is, regime switching in the copula. The regime classification from the Markov-switching *Skew* $t-t$ factor copula is displayed in figure 4, with a clear regime change occurring at the height of the financial crisis, with the Lehman Brothers default in September 2008. Section 5.2 reveals that with the regime change the government bonds decouple and the financial index becomes positively dependent with the indices of the PIIGS countries, whereas the government debt indices of the three core countries, DE, FR and NL, remain negatively dependent with the fincial sector index.

Figure 9 displays the total systemic risk as well as the individual systemic risk contributions, the Shapley values, of all sovereign debt markets over the sample period for $\alpha = 0.05$. The systemic risk measures clearly reflect the time variation in the conditional mean and variance of the financial sector index, as measured by time variation in the VaR: We observe that, conditional on a given regime, Shapley values and total systemic risk are high in absolute value when financial sector VaR is high. The systemic risk measures also clearly reflect the regime change in the dependence structure: With a short exception in the beginning of 2008, we observe positive and very similar Shapley values until the Lehman Brothers default. That means that the individual systemic risk impact, while accounting for systemic risk spillovers among debt markets, is negative for each

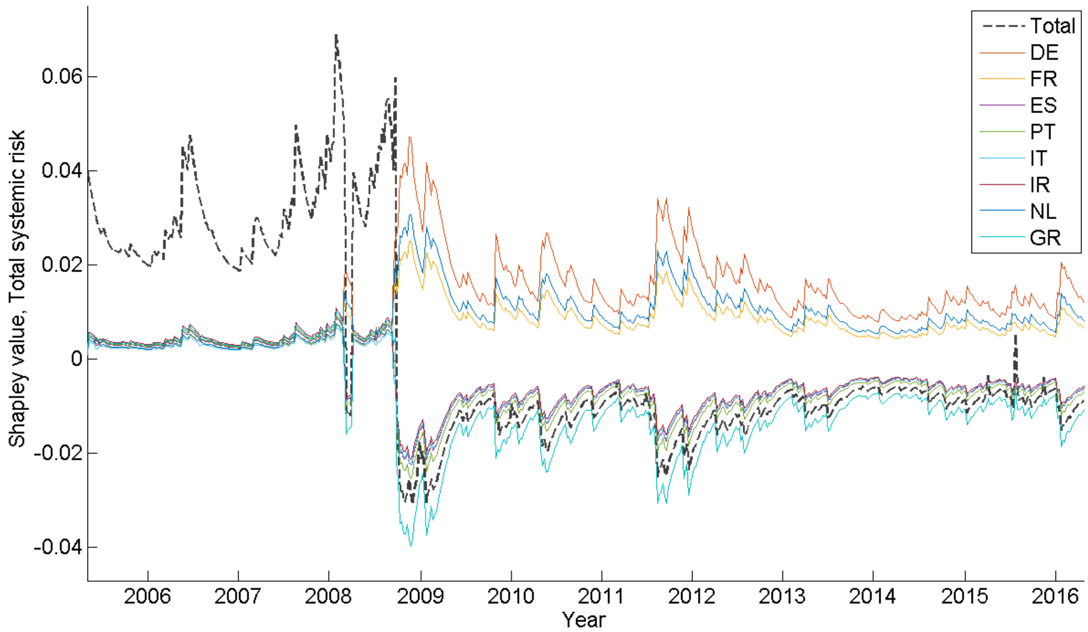

Figure 9: This figure displays total systemic risk (dashed) and individual systemic risk contributions (solid) of Euro Area government debt markets for the financial sector implied by a Markov-switching *Skew* $t - t$ factor copula model with two common factors. $\alpha = 0.05$.

government debt market in a sense that distress in government debt markets reduces the VaR of the financial sector. Importantly, government debt systemic risk changes with the regime shift in September 2008. Shapley values of the three core countries are still positive, and distinctly larger than prior to the regime change, with DE having the largest value. However, the individual systemic risk impact turns negative for all PIIGS countries, meaning that distress in these markets implies positive systemic risk for the Euro Area financial sector, in the sense of an increased financial sector VaR when these markets become distressed. This impact is highest shortly after the Lehman default and peaks again in the second half of 2011, where financial sector VaR is high.

With a short exception in the beginning of 2008, the systemic risk implied by joint distress of all eight government debt markets, as measured by ΔCoVaR based on all eight markets or equivalently by the sum of their Shapley values, is negative until the Lehman default, after which it turned positive. That is, joint distress of all eight debt markets decreased financial sector VaR until the height of the financial crisis, but increased the financial sector VaR thereafter. Total systemic risk is highest shortly after the Lehman Brothers default and peaked again in the second half of 2011.

In other words, a portfolio of these eight government bonds has a diversification benefit for bank portfolios until the height of the financial crisis. This diversification benefit is greatest during the months shortly before the Lehman default, when the financial sector experiences heightened volatility but was is negatively dependent with all eight government bonds. After the Lehman default a portfolio of these government bonds no longer offers a diversification benefit on bank portfolios, but amplifies large losses.

6 Conclusion

We study systemic government debt risk of eight major government debt markets for the Euro Area financial sector before and after the financial crisis. To this end we employ the Multi-CoVar measure of Cao (2013), which is a multivariate extension of the bivariate CoVaR measure of Adrian and Brunnermeier (2008). Multi-CoVaR is defined as the VaR of the financial system conditional on sets instead of single financial institutions being in distress. Multi-CoVaR can therefore explicitly account for distress spillovers among financial institutions when determining their systemic risk impact. Efficient individual systemic risk contributions are obtained by allocating the total systemic risk, arising from joint distress of all institutions, through the Shapley value methodology, a concept from game theory.

We propose to characterize Multi-CoVaR through copulas. Copula based models are capable of describing the features often found in the dependence structure of financial assets, such as tail dependence and asymmetric dependence. Accounting for these features is crucial for the computation of systemic risk measures that require accurately measuring the tail risk interdependence of financial assets, such as CoVaR. Additionally, we show that in the Multi-CoVaR context copulas provide computational advantages over multivariate time-series models. We employ the factor copula model of Oh and Patton (2015) for modeling the dependence structure of our data. Factor copulas are implied by factor models and thus provide a potential dimension reduction, which is crucial when working in high dimension. We propose to augment factor copulas by Markov-switching dynamics to allow for time-varying dependence of government bond markets and the financial sector. Thereby we add to the existing literature on time-varying copula models for high-dimensional financial data.

For a sample of weekly returns on eight major sovereign bond indices and a Euro Area financial sector index we observe a clear regime shift in the dependence structure and systemic risk at the height of the financial crisis in September 2008, with the Lehman Brothers default. Prior to the Lehman Brothers default we observe very strong dependence of all bond markets as measured by very high rank correlations and quantile dependence measures. All government bond markets are negatively dependent to the financial sector index over this period. At the height of the financial crisis the bond markets decouple and the dependence of the government bond indices of the PIIGS countries with the financial sector becomes positive. This regime shift in the dependence structure of these assets occurring during the financial crisis is clearly reflected in systemic risk measures. Prior to this event, we observe a negative systemic risk impact for all eight debt markets, in a sense that distress of debt markets lowers systemic risk for the financial sector. At the height of the financial crisis the systemic risk impact of government debt markets of the PIIGS countries becomes positive, in a sense that distress in these markets now increases VaR of the financial sector, whereas it remains negative for Germany, France and the Netherlands. The systemic risk from joint distress of all eights markets is negative until the Lehman Brothers default, but becomes positive thereafter. It is highest shortly after the default event and again in the second half of 2011. Besides distinct time variation we find that pronounced heterogeneity as well as tail dependence and asymmetric dependence are important features of the dependence structure of these financial assets, which a Markov-switching factor copula model with multiple fat tailed and asymmetric common factors is able to account for.

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Declaration of Authorship

I hereby confirm that I have authored this Master's thesis independently and without use of others than the indicated sources. All passages which are literally or in general matter taken out of publications or other sources are marked as such.

Berlin, October 28, 2016

Paul Bochmann