Construction of Brownian Motions in Enlarged Filtrations and Their Role in Mathematical Models of Insider Trading

Dissertation

zur Erlangung des akademischen Grades
doctor rerum naturalium
im Fach Mathematik

eingereicht am 15. Februar 1999 an der
Mathematisch-Naturwissenschaftlichen Fakultät II
Humboldt-Universität zu Berlin

von
Master of Science Ching-Tang Wu
geb. am 1. Februar 1969 in Keelung, Taiwan

Präsident der Humboldt-Universität zu Berlin
Prof. Dr. Dr. h.c. Hans Meyer

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät II
Prof. Dr. Bodo Krause

Gutachter:
1. Professor Dr. Hans Föllmer (Humboldt-Universität zu Berlin)
2. Professor Dr. Martin Schweizer (Technische Universität Berlin)
3. Professor Dr. Marc Yor (Université Pierre et Marie Currie, Paris)

Tag der mündlichen Prüfung: 8. Juni 1999

Abstract

In this thesis, we study Gaussian processes generated by certain linear transformations of two Gaussian martingales. This class of transformations is motivated by financial equilibrium models with heterogeneous information.

In Chapter 2 we derive the canonical decomposition of such processes, which are constructed in an enlarged filtration, as semimartingales in their own filtration. The resulting drift is described in terms of Volterra kernels. In particular we characterize those processes which are Brownian motions in their own filtration. In Chapter 3 we construct new orthogonal decompositions of Brownian filtrations.

In Chapters 4 to 6 we are concerned with applications of our characterization results in the context of mathematical models of insider trading. We analyze extensions of the financial equilibrium model of Kyle [**42**] and Back [**7**] where the Gaussian martingale describing the insider information is specified in various ways. In particular we discuss the structure of insider strategies which remain inconspicuous in the sense that the resulting cumulative demand is again a Brownian motion.

Zusammenfassung

In dieser Arbeit untersuchen wir die Struktur von Gaußschen Prozessen, die durch gewisse lineare Transformationen von zwei Gaußschen Martingalen erzeugt werden. Die Klasse dieser Transformationen ist durch finanzmathematische Gleichgewichtsmodelle mit heterogener Information motiviert.

In Kapital 2 bestimmen wir für solche Prozesse, die zunächst in einer erweiterten Filtrierung konstruiert werden, die kanonische Zerlegung als Semimartingale in ihrer eigenen Filtrierung. Die resultierende Drift wird durch Volterra-Kerne beschrieben. Insbesondere charakterisieren wir diejenigen Prozesse, die in ihrer eigenen Filtrierung eine Brownsche Bewegung bilden. In Kapital 3 konstruieren wir neue orthogonale Zerlegungen der Brownschen Filtrierungen.

In den Kapitaln 4 bis 6 wenden wir unsere Resultate zur Charakterisierung Brownscher Bewegungen im Kontext finanzmathematischer Modelle an, in denen es Marktteilnehmer mit zusätzlicher Insider-Information gibt. Wir untersuchen Erweiterungen eines Gleichgewichtsmodells von Kyle [**42**] und Back [**7**], in denen die Insider-Information in verschiedener Weise durch Gaußsche Martingale spezifiziert wird. Insbesondere klären wir die Struktur von Insider-Strategien, die insofern unauffällig bleiben, als sich die resultierende Gesamtnachfrage wie eine Brownsche Bewegung verhält.

Contents

CHAPTER 0

Introduction

In 1900, Brownian motion was introduced by Bachelier [**6**] as a model for price fluctuation in the stock market. Since then, the analysis of Brownian motion has become a central topic in the mathematical theory of stochastic processes, quite independent of the original financial motivation. In particular, Brownian motion plays a fundamental role in the theory of continuous martingales, and it is the basis for Itô's development of stochastic calculus.

Since the 60ies, there is a renewed interest in the financial interpretation of Brownian motion. Diffusion models driven by Brownian motion have become the canonical framework for analyzing the structure of financial derivatives. The Black-Scholes formula for the price of an option is derived by computing the cost of a perfect hedging strategy which duplicates the pay-off of the option. The gain of the strategy is computed as a stochastic integral of the underlying price process X. The construction of a suitable hedging strategy involves the methods of Itô's calculus and their connection to partial differential equations. In such models, investors only use the information provided by the canonical filtration of the price process X. Moreover, they are viewed as price takers, i.e., the underlying diffusion process of stock prices is not influenced by the investors' strategies. In recent years, there is a growing literature which departs from these assumptions and introduces additional market microstructure. For a "large investor", the price process may be modified by his investment strategy; see, e.g., Jarrow [**34**], [**35**], Frey [**25**], Cvitanic-Ma [**19**]. Moreover, the information available to the agents may be heterogeneous, i.e., some "insider" may have access to a filtration which is larger than the canonical filtration $(\mathcal{F}_t^X)$; see, e.g., Karatzas-Pikovsky [**39**], Amendinger-Imkeller-Schweizer [**5**], Pikovsvy [**50**], Grorud-Pontier [**28**] and Amendinger [**3**]. From a financial point of view, it is of interest to introduce both effects simultaneously and to analyze their interplay. This has been discussed in the work of Kyle [**42**] and Back [**7**]. Related models appear, e.g., in Glosten-Milgrom [**26**], Kyle [**41**], [**43**], Easley-O'Hara [**22**], Admati-Pfleiderer [**1**], [**2**], Grossman [**29**], Back [**8**], [**9**], O'Hara [**49**], Biais-Rochet [**13**] and Cho-El Karoui [**15**], [**16**].

In this thesis, we follow the approach of Kyle [**42**] and Back [**7**]. Our purpose is to investigate some mathematical problems which appear in this context. From a mathematical point of view, the problems related to insider trading of a large investor belong to the theory of enlargement of filtrations and stochastic filtering. In Chapters 1 to 3 we are going to concentrate on the mathematical analysis of such problems. In Chapters 4 to 6 we will return to the financial interpretation and discuss some application of the results in Chapters 1 to 3 in the context of a financial equilibrium model.

In order to motivate the following discussion, we first consider a well-known classical example. Let $X = (X_t)_{0 \le t \le 1}$ be a Brownian motion with canonical filtration $(\mathcal{F}_t^X)$. In the enlarged filtration generated by $(\mathcal{F}_t^X)$ and the final value X_1, the process X admits the representation

$$X_t = W_t + \int_0^t \frac{X_1 - X_u}{1-u} du, \tag{0.1}$$

where (W_t) is a Brownian motion with respect to the enlarged filtration and is independent of X_1; see, e.g., Jeulin-Yor [**36**]. In our context, we emphasize another aspect of this equation. Let W be a Brownian motion, and let X_1 be an $N(0,1)$-distributed random variable independent of W. The process X defined as the solution of linear stochastic differential equation (0.1) is a Brownian bridge tied to the final value X_1. Furthermore, X is again a Brownian motion with respect to its own filtration. Thus, the linear drift in (0.1) drives the Brownian motion W to the new terminal value X_1, using the information in the enlarged filtration. But it does so in such a way that the law of the process remains unchanged, i.e., the resulting process X is again a Brownian motion. In this sense, the controlling drift in (0.1) remains *inconspicuous*.

From this point of view, some natural questions arise. Replacing the normal random variable X_1 in (0.1) by some independent Gaussian martingale (S_t), can we characterize those drifts in the enlarged filtration $(\mathcal{F}_t^{W,S})$, which are linear in W and S, such that the original Brownian motion (W_t) is driven to the final value S_1? Can we construct such drifts which remain inconspicuous in the sense that the resulting process (X_t) is again a Brownian motion? More generally, can we compute the Doob-Meyer decomposition as a semimartingale in its own filtration? Consider, for example, the process X given by

$$X_t = W_t + \int_0^t \frac{\tilde{W}_u - X_u}{1-u} du,$$

where $\tilde{W}$ is a Wiener process independent of W. In Section 2.1 we will show that this Gaussian process converges to $\tilde{W}_1$ as $t \to 1$, but that it is no longer a Brownian motion with respect to its own filtration. In fact, the canonical decomposition of X in its own filtration is given by

$$X_t = B_t + \int_0^t \int_0^u \frac{(B+1)(1-s)^{-A} - (A+1)(1-s)^{-B}}{A(1-u)^A - B(1-u)^B} dX_s du, \tag{0.2}$$

for $0 \le t < 1$, where $A = (1+\sqrt{5})/2$, $B = (1-\sqrt{5})/2$, and (B_t) is a Brownian motion.

In Chapter 2 we explore the general structure of this problem. This may be viewed as a case study in stochastic filtering. In order to prepare our analysis, we recall in Chapter 1 some basic facts concerning the representation of Gaussian processes in terms of linear transformations of Brownian motion; this is based on Kallianpur [**38**] and Hida-Hitsuda [**32**]. We review the structure of the Kalman-Bucy filter and of the Volterra representation of a Gaussian process X in the form

$$X_t = B_t + \int_0^t \int_0^s l(s,u)dB_u ds, \tag{0.3}$$

where B is a standard Brownian motion and $l(s,u)$ is a Volterra kernel; see Definition 1.3. In Theorem 1.2, we characterize those Volterra kernels which generate a new Brownian motion; this is based on Föllmer-Wu-Yor [**24**]. We also introduce the canonical decomposition of a Gaussian process X of the form

$$X_t = \int_0^t F(t,u)dB_u,$$

where (B_t) is a Brownian motion such that the filtrations $(\mathcal{F}_t^X)$ and $(\mathcal{F}_t^B)$ coincide. In Proposition 1.2 we describe the relation between the canonical decomposition and the Volterra representation.

In the second chapter we study Gaussian processes X defined as solutions of linear stochastic equation driven by a Brownian motion W and an independent Gaussian martingale S. Explicitly, X is given by

$$X_t = W_t + \int_0^t \left(f(s)S_0 + \int_0^s F(s,u)dS_u + \int_0^s H(s,u)dX_u \right) ds, \tag{0.4}$$

where f, F and H satisfy some integrability conditions. First we use some methods of stochastic filtering theory to obtain the canonical decomposition of such processes as semimartingales in their own filtration; see Theorem 2.1. Based on this result, we characterize those transformations which generate a new Brownian motion. In Theorem 2.2 we show that X is a Brownian motion if and only if the kernel $H(s,u)$ satisfies the relation

$$-H(t,s) + \int_0^s H(t,u)H(s,u)du = f(t)f(s)var(S_0) + \int_0^s F(t,u)F(s,u)d(var(S_u)),$$

for almost all $s \leq t$. In particular, if S is a Gaussian martingale with $E[S_1^2] = 1$ and satisfies

$$\int_0^t \frac{u}{(var(S_u)-u)^2}du < \infty \qquad \text{and} \qquad \int_0^t \frac{1}{var(S_u)-u}du < \infty, \tag{0.5}$$

for all $t < 1$, then the process X satisfying

$$X_t = W_t + \int_0^t \frac{S_u - X_u}{var(S_u)-u}du \tag{0.6}$$

is a Brownian motion with respect to its own filtration, and X_t converges to S_1 as $t \to 1$. On the other hand, consider the case where S is a Brownian motion $\tilde{W}$.

Clearly, condition (0.5) does not hold in this case. In fact, it can be shown that there is no Brownian motion of the form

$$X_t = W_t + \int_0^t Y_u du,$$

where Y is an $(\mathcal{F}_t^{W,\tilde{W}})$-adapted drift, which converges to $\tilde{W}_1$ as $t \to 1$; see Proposition 2.3 below and Proposition 5.1 in Föllmer-Wu-Yor [**23**].

In Chapter 3 we investigate another aspect of our basic example 0.1. In fact, equation (0.1) induces an orthogonal decomposition of the Brownian filtration in the form

$$\mathcal{F}_t^X = \mathcal{F}_t^W \oplus \sigma(X_t);$$

see Jeulin-Yor [**37**]. Our purpose is to construct some related orthogonal decomposition; this is based on Wu-Yor [**56**]. First we continue the discussion in Chapter 2 and characterize Brownian motions X of the form

$$X_t = W_t + \int_0^t (f(u)\tilde{W}_u + g(u)X_u)du, \tag{0.7}$$

where f and g satisfy weaker integrability conditions than in Chapter 2. Suppose that the solution X of the equation (0.7) is a Brownian motion, and that one of the functions f and g is not identical to 0. Then the filtration generated by X is strictly smaller than the filtration generated by W and $\tilde{W}$. In addition, we construct a second Brownian motion Y in the natural filtration of W and $\tilde{W}$, which is independent of the Brownian motion X. Using the process Y, we construct two sequences $(X^{(n)})$ and $(Y^{(n)})$ of Brownian motions in $(\mathcal{F}_t^{W,\tilde{W}})$ which are independent of each other, and such that the corresponding natural filtrations decrease. For each $n \geq 1$, this induces an orthogonal decomposition of the filtration generated by X and by Y:

$$\mathcal{F}_t^X = \mathcal{F}_t^{X(0)} = \sigma(X_t^{(0)}) \oplus \sigma(X_t^{(1)}) \oplus \cdots \oplus \sigma(X_t^{(n)}) \oplus \mathcal{F}_t^{X(n+1)},$$

$$\mathcal{F}_t^Y = \mathcal{F}_t^{Y(0)} = \sigma(Y_t^{(0)}) \oplus \sigma(Y_t^{(1)}) \oplus \cdots \oplus \sigma(Y_t^{(n)}) \oplus \mathcal{F}_t^{Y(n+1)}.$$

In Chapter 4 we introduce a simple financial market model with insider trading of large investors; this is based on Kyle [**42**] and Back [**7**]. We recall the definition of equilibrium in the sense of Back. Prices of the underlying stock are set by some "market maker" as a function $P_t = h(X_t, t)$ of the aggregate cumulative demand X_t up to time t. There are "noise traders" whose cumulative demand is given by a Brownian motion (W_t). Moreover, there is an "insider " who has in advance additional information on the price $P_1 = h(S_1, 1)$ at the final time 1, where S_1 is normal random variable with distribution $N(0,1)$ which is independent of (W_t). An insider strategy specifies a cumulative demand (I_t) based on the enlarged filtration generated by (W_t) and S_1. Such a strategy will be called "inconspicuous" if the resulting aggregate demand $X = W + I$ is again a Brownian motion. Using Itô's calculus, it can be shown that the insider's expected gain is maximal as soon as his demand drives the aggregate demand X_t to the final value S_1 as $t \to 1$. An equilibrium is defined by a pricing rule $h(\cdot, t)$ $(0 \le t < 1)$ and by an insider strategy (I_t) such that I is inconspicuous and maximizes the expected gain. As shown in Back [**7**], the mathematical discussion of the basic example in (0.1) proves the existence of such an equilibrium, where the pricing rule $h(\cdot, \cdot)$ is determined as the solution

$$h(x,t) = \frac{1}{\sqrt{2\pi(1-t)}} \int_{-\infty}^{\infty} h(y,1) \exp\left(-\frac{(y-x)^2}{2(1-t)}\right) dy \tag{0.8}$$

of the heat equation

$$\left(\frac{1}{2}\Delta + \frac{\partial}{\partial t}\right) h = 0,$$

with terminal value $h(\cdot, 1)$.

In Section 4.3 we study situation where the insider obtains increasing information by observing a Gaussian martingale (S_t); the discussion in Back [**7**] corresponds to the special case $S_t \equiv S_1$ when the final information is already available at time 0. We restrict our analysis to insider strategies which are linear in S and X. Thus, we can use our results in Chapter 2. In particular, the characterization of Brownian motion in Section 2.3 to describe those strategies which are inconspicuous. Moreover, we examine the existence of equilibrium in the case of increasing information. In particular, we consider the case where the insider information is given by observing a Gaussian martingale S which satisfies (0.5) for all $t < 1$. We show that there is an equilibrium in this case. More precisely, the optimal insider strategy is given by

$$I_t = \int_0^t \frac{S_u - X_u}{var(S_u) - u} du, \tag{0.9}$$

and the pricing rule is again given by (0.8). If the insider information S is a Brownian motion $\tilde{W}$, then it can be shown that there is no equilibrium, i.e., the insider cannot reach his maximal profit without being discovered.

In Chapter 5 we discuss some extension of the insider trading model presented in Chapter 4 and a modified notion of equilibrium. Again we require that the insider strategy is inconspicuous, i.e., the insider uses strategies which turn the cumulative order in the market into a Brownian motion. The pricing rule is assumed to minimize the combined expected profit of the informed and uninformed traders. We give necessary and sufficient conditions on the rational pricing rule and on optimal inconspicuous insider strategies. We show that the rational pricing rule is of the form (0.8), and that a strategy is optimal in a set of inconspicuous strategies if it minimizes the L^2-distance with the final signal S_1 among the strategies in this set. In addition, we study some modified versions of insider information, in particular the case of noisy information and of delayed information, and a model with two insiders with different degrees of information.

In Chapter 6 we introduce information costs. The extra information the insider gets is no longer cost-free. Is it profitable to purchase the information? If yes, how can he invest in an optimal way? Which kind of information should he buy? We analyze some examples in different settings.

Acknowledgment

Foremost I would like to express gratitude to my supervisor Professor Dr. H. Föllmer. Without his guidance and patience I would not have accomplished my thesis so smoothly. Especially, I want to thank him for his careful reading, useful commands and suggestions. Furthermore, I would like to thank Professor Dr. M. Yor for fruitful collaboration, helpful suggestions and for two invitations to Université Paris VI. Also I want to acknowledge the helpful and stimulating discussions with Professor Dr. P. Imkeller. In addition, I am grateful to my colleagues and friends Dr. J. Amendinger, P. Bank, U. Horst and Dr. A. Schied for their help on my thesis. Support of the EU through the TMR contract ERB-FMRX-CT96-0075 "Stochastic Analysis" and support of the Deutsche Forschungsgemeinschaft (SFB 373 "Quantifikation und Simulation ökonomischer Prozesse" and Berliner Graduiertenkolleg "Stochastische Prozesse und probabilistische Analysis") are gratefully acknowledged. Special thanks go to my family in Taiwan for supporting and understanding my study in Germany.

CHAPTER 1

Representations of Gaussian processes

In the present chapter we deal with representations of Gaussian processes in terms of Brownian motion. In Section 1.1 we recall some basic facts about Gaussian processes including the linear Kalman-Bucy filter. In Section 1.2 two representations of Gaussian processes are introduced and analyzed: the canonical representation in terms of linear functionals of Brownian motion, and the Volterra representation as the sum of a Brownian motion and an absolutely continuous process whose density is given by linear functionals of Brownian motion. We discuss the relationship between these two representations and then give some examples. In Section 1.3 we collect some lemmas which will be useful in the sequel. In the last section of this chapter, we shall characterize Volterra representations of a Gaussian process which generate a Brownian motion.

1.1. Gaussian processes and the Kalman-Bucy filter

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. First we recall the definitions and some properties of Gaussian process and Brownian motion.

Definition 1.1. **(1)** A stochastic process $X = (X_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *Gaussian process*, if any finite linear combination $\sum a_i X_{t_i}$, $a_i \in \mathbb{R}$, $t_i \geq 0$, is a Gaussian random variable.
(2) A process X is called a (standard, one-dimensional) *Brownian motion* with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$, if it satisfies the following two conditions:
i) X is a continuous, $(\mathcal{F}_t)$-adapted Gaussian process.
ii) For $s \leq t$ the increment $X_t - X_s$ is independent of $\mathcal{F}_s$ and normally distributed with mean 0 and variance $t - s$.

Note that if X is a Brownian motion with respect to the filtration $(\mathcal{F}_t)$, then it is a Brownian motion relative to the filtration generated by X. For simplicity, we denote σ-algebra generated by stochastic process X up to time t by

$$\mathcal{F}_t^X := \sigma\{X_u; u \leq t\},$$

where the superscript denotes the process which generates this σ-algebra (e.g., $\mathcal{F}^{X,Y}$ denotes the filtration generated by the processes X and Y).

There are several methods to check whether a process is a Brownian motion. The following two will be often used in this thesis:

(1) A process X is a Brownian motion with respect to its natural filtration $(\mathcal{F}_t^X)$ if and only if it is a continuous Gaussian process with covariance $E[X_sX_t] = s\wedge t$.

(2) (**Lévy's Theorem**) A continuous $(\mathcal{F}_t)$-adapted process X is a Brownian motion with respect to $(\mathcal{F}_t)_{t\geq 0}$ if and only if it is a (local) martingale relative to $(\mathcal{F}_t)_{t\geq 0}$ and for all $t \geq 0$ the quadratic variation $\langle X\rangle_t$ is given by t.

The first statement is just a slight variation of the definition of Brownian motions; as to the second, see, e.g., Karatzas-Shreve [**40**], Protter [**51**], Revuz-Yor [**52**].

A key tool of stochastic filtering theory is the Kalman-Bucy filter; see, for instance, Davis [**20**], Liptser-Shiryaev [**46**], Kallianpur [**38**], Rogers-Williams [**54**]. In this section we will describe its basic structure and we single out a special case which is relevant for our discussion in the next chapter.

Suppose X, W and Z are three Gaussian processes. The process Z cannot be directly observed and is called *signal* or *system process*. The process W is an $(\mathcal{F}_t)$-Brownian motion and is called the *noise process*. The process X, which depends on Z and W, is observable, and therefore we call it *observation process*. The goal of the Kalman-Bucy filter is to find the conditional expectation of $f(Z_t)$ relative to the σ-algebra $\mathcal{F}_t^X$, for a real measurable function f. In other words, we try to use the observation process to estimate a function of the signal process. Since this conditional expectation is usually not linear in X, this is called a *nonlinear filtering problem*.

Let us begin with the following standard formulation of the Kalman-Bucy filter:

THEOREM 1.1 (Kallianpur [**38**]). *Suppose the m-dimensional signal process $(Z_t)_{t\geq 0}$ and the n-dimensional observation process $(X_t)_{t\geq 0}$ are given by the stochastic differential equations*

$$dZ_t = [A_0(t) + A_1(t)Z_t + A_2(t)X_t]dt + U(t)d\bar{W}_t \tag{1.1}$$

and

$$dX_t = [C_0(t) + C_1(t)Z_t + C_2(t)X_t]dt + V(t)d\bar{W}_t, \tag{1.2}$$

with initial values $X_0 = Z_0 = 0$, where $(\bar{W}_t)_{t\geq 0}$ is an $(n+m)$-dimensional Brownian motion, A_i, C_i, U and V $(i = 0,1,2)$ are deterministic matrices of appropriate dimensions, the entries in A_i and C_i $(i = 0,1,2)$ are integrable and those in U and V are square-integrable. Then

$$\hat{Z}_t := E[Z_t|\mathcal{F}_t^X],$$

the conditional expectation of Z_t with respect to $\mathcal{F}_t^X$, satisfies

$$d\hat{Z}_t = [A_0(t) + A_1(t)\hat{Z}_t + A_2(t)X_t]dt + [P(t)C_1^T(t) + U(t)V^T(t)][V(t)V^T(t)]^{-\frac{1}{2}}dB_t, \tag{1.3}$$

with $\hat{Z}_0 = 0$. *Here* $(B_t)_{t\geq 0}$ *is an* $(\mathcal{F}_t^X)$*-martingale (called the innovation process), defined by*

$$B_t := X_t - \int_0^t (C_0(u) + C_1(u)\hat{Z}_u + C_2(u)X_u)du. \tag{1.4}$$

Furthermore, the $m \times m$ *matrix* $P(t)$,

$$\begin{aligned} P(t) &= E\left[\left(Z_t - E\left[Z_t \middle| \mathcal{F}_t^X\right]\right)^2 \middle| \mathcal{F}_t^X\right] \\ &= \left(E\left[Z_t^{(i)} Z_t^{(j)} \middle| \mathcal{F}_t^X\right] - E\left[Z_t^{(i)} \middle| \mathcal{F}_t^X\right] E\left[Z_t^{(j)} \middle| \mathcal{F}_t^X\right]\right)_{i,j}, \end{aligned}$$

the conditional variance of $Z = (Z_t^{(1)}, \cdots, Z_t^{(m)})$, *satisfies the ordinary differential equation*

$$\begin{aligned} P'(t) &= A_1(t)P(t) + P(t)A_1^T(t) + U(t)U^T(t) \\ &= [P(t)C_1^T(t) + U(t)V^T(t)][V(t)V^T(t)]^{-1}[C_1(t)P(t) + V(t)U^T(t)], \end{aligned} \tag{1.5}$$

with initial condition $P(0) = 0$.

Consider two independent Brownian motions W and $\tilde{W}$. In the sequel we will consider linear stochastic differential equations

$$dX_t = dW_t + Y_t dt, \tag{1.6}$$

with initial value $X_0 = 0$, where (Y_t) is a linear functional in W, $\tilde{W}$ and X. Explicitly, Y_t is of the form

$$Y_t = \int_0^t F(t,u)dW_u + \int_0^t G(t,u)d\tilde{W}_u + \int_0^t H(t,u)dX_u,$$

where F, G and H satisfy some integrability conditions. The solution X is clearly a Gaussian process.

REMARK 1.1. If the drift term Y is not linear, then the resulting process X given by (1.6) is in general not a Gaussian process. For example, the process $(W_t + \int_0^t |\tilde{W}_u| du)$ is no longer Gaussian. But this does not mean that all resulting processes X of the form (1.6) with a nonlinear drift term are not Gaussian. In Section 1.3 we shall give an example of a Gaussian process with nonlinear drift term.

The following is a simple application of Theorem 1.1.

PROPOSITION 1.1. *Let* W, $\tilde{W}$ *be two independent* 1*-dimensional Wiener processes and* X *satisfy*

$$dX_t = dW_t + f(t)\int_0^t g(u)d\tilde{W}_u dt, \tag{1.7}$$

with initial condition $X_0 = 0$, *where* f *and* g *satisfy*

$$\int_0^t \int_0^u f^2(u)g^2(v)dvdu < \infty,$$

for all $t < 1$. *Then*

(1) *the innovation process B is an $(\mathcal{F}_t^X)$-Brownian motion.*

(2) $E\left[\int_0^t g(u)d\tilde{W}_u \middle| \mathcal{F}_t^X\right] = \int_0^t f(u)p(u)dB_u,$

(3) $E\left[\tilde{W}_t \middle| \mathcal{F}_t^X\right] = \int_0^t f(u)q(u)dB_u,$

where p, q are the solutions of the following system of differential equations:

$$\begin{cases} q'(t) + f^2(t)p(t)q(t) = g(t), \\ \\ p'(t) + f^2(t)p^2(t) = g^2(t), \end{cases} \tag{1.8}$$

with initial values $p(0) = q(0) = 0$.

Proof. Consider in Theorem 1.1 the particular case $m = 2$, $n = 1$, $A_i \equiv 0$ for $i \in \{0, 1, 2\}$, $C_0 = C_2 = 0$,

$$C_1(t) = \begin{pmatrix} 0 & f(t) \end{pmatrix},$$

$$U(t) = \begin{pmatrix} 1 & 0 & 0 \\ g(t) & 0 & 0 \end{pmatrix}, \qquad V(t) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix},$$

and a 3-dimensional Brownian motion $\bar{W}_t = \begin{pmatrix} \tilde{W}_t & W_t & \hat{W}_t \end{pmatrix}^T$. It follows from (1.2) and (1.4) that

$$X_t = W_t + \int_0^t f(u)Z_u^{(2)}du = B_t + \int_0^t f(u)\hat{Z}_u^{(2)}du,$$

where $(Z_t^{(i)})$ and $(\hat{Z}_t^{(i)})$ stand for the i-th component of (Z_t) and $(\hat{Z}_t)$, respectively. This implies that the quadratic variation of the $(\mathcal{F}_t^X)$-martingale B equals t. By Lévy's Theorem, B is a Brownian motion relative to the filtration $(\mathcal{F}_t^X)$ and hence the first assertion follows. Let

$$P(t) = \begin{pmatrix} r(t) & q(t) \\ q(t) & p(t) \end{pmatrix}$$

denote the conditional variance matrix. Due to (1.3) the conditional expectation $\hat{Z}_t$ is given by

$$\hat{Z}_t = \begin{pmatrix} \hat{Z}_u^{(1)} \\ \hat{Z}_u^{(2)} \end{pmatrix} = \int_0^t P(u)(C_1(u))^T dB_u = \begin{pmatrix} \int_0^t f(u)q(u)dB_u \\ \int_0^t f(u)p(u)dB_u \end{pmatrix}.$$

By (1.5) we see that the functions $p(t)$ and $q(t)$ are determined by

$$\begin{aligned} \begin{pmatrix} r'(t) & q'(t) \\ q'(t) & p'(t) \end{pmatrix} &= U(t)U^T(t) - P(t)C_1^T(t)C_1(t)P(t) \\ &= \begin{pmatrix} 1 - f^2(t)q^2(t) & g(t) - f^2(t)p(t)q(t) \\ g(t) - f^2(t)p(t)q(t) & g^2(t) - f^2(t)p^2(t) \end{pmatrix}, \end{aligned}$$

which results in the assertions 2 and 3. □

Let us consider the following illustration of this proposition.

EXAMPLE 1.1. Suppose the signal process $(Z_t)_{0\le t\le 1}$ is given by

$$dZ_t = d\tilde{W}_t + aZ_t dt, \tag{1.9}$$

and the observation process $(X_t)_{0\le t\le 1}$ satisfies

$$dX_t = dW_t + cZ_t dt, \tag{1.10}$$

for two independent Brownian motions $(W_t)_{0\le t\le 1}$, $(\tilde{W}_t)_{0\le t\le 1}$ and constants a, c. Solving (1.9) and substituting it into (1.10) yield the representation

$$X_t = W_t + c\int_0^t \int_0^u e^{a(u-v)} d\tilde{W}_v du.$$

Using Proposition 1.1, we can compute the conditional expectations

$$E[\tilde{W}_t|\mathcal{F}_t^X] = \int_0^t \frac{c[c^2e^{2\gamma t} + 2a(a+\gamma)e^{\gamma t} - (a+\gamma)^2]}{\gamma[c^2e^{2\gamma t} + (a+\gamma)^2]} dB_u,$$

and

$$E\left[\int_0^t e^{-au} d\tilde{W}_u \middle| \mathcal{F}_t^X\right] = \int_0^t \frac{ce^{-au}(e^{2\gamma u} - 1)}{(\gamma - a)e^{2\gamma u} + \gamma + a} dB_u,$$

where $\gamma := \sqrt{a^2+c^2}$. The second statement has been shown in Rogers-Williams [**54**] P.327-329.

In Chapter 2 we will discuss an extension of the linear Kalman-Bucy filter and some further applications.

1.2. Canonical representation and Volterra representation

Consider a centered Gaussian process $X = (X_t)_{0\le t\le 1}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this section we focus on some representations for the process X and their relationship. Let us begin with the definition of the canonical representation.

DEFINITION 1.2 (Hida-Hitsuda [**32**]). Suppose there exist a Brownian motion B and a kernel given by a measurable function $F(t,u)$ on $[0,1]\times[0,1]$ satisfying

$$\int_0^t F(t,u)du < \infty,$$

for all t, such that X admits the representation

$$X_t = \int_0^t F(t,u)dB_u. \tag{1.11}$$

If the filtrations generated by X and B are the same, i.e., $\mathcal{F}_t^X = \mathcal{F}_t^B$ for all t, then (1.11) is called the *canonical representation* of X, and $F(t,u)$ is the *canonical kernel.*

Next we introduce the definition of a Volterra kernel and the Volterra representation.

DEFINITION 1.3. (i) A measurable function $l(s,u)$ on $(0,1)\times(0,1)$ is called *Volterra kernel* if $l(s,u)=0$, for $0<s<u<1$. If, furthermore, l is square-integrable on $(0,1)\times(0,1)$, we shall call $l(s,u)$ a *square-integrable* Volterra kernel.
(ii) Suppose that the process X can be represented in the form:

$$X_t = B_t + \int_0^t \int_0^s l(s,u) dB_u ds, \tag{1.12}$$

where B is a Brownian motion with respect to its own filtration $(\mathcal{F}_t^B)$ and $l(s,u)$ is a Volterra kernel satisfying

$$\int_0^1 \left(\int_0^s l^2(s,u) du \right)^{\frac{1}{2}} ds < \infty. \tag{1.13}$$

Then the representation (1.12) is called a *Volterra representation* of X. If $l(s,u)$ is a square-integrable Volterra kernel, then we say that (1.12) is a *square-integrable Volterra representation.*
(iii) A Volterra kernel $l(s,u)$ is said to be *continuously differentiable* if the kernel $\hat{l}$ defined by

$$\hat{l}(s,u) := \begin{cases} l(s,u), & s \geq u, \\ \\ l(u,s), & s < u, \end{cases}$$

is continuously differentiable for all $0 \leq u, s \leq 1$.

REMARK 1.2. Note that the condition (1.13) guarantees that the stochastic integral $\int_0^t \int_0^s l(s,u) dB_u ds$ in (1.12) is well-defined for all $t \leq 1$.

REMARK 1.3. The Volterra representation (1.12) with a square-integrable Volterra kernel specifies the Doob-Meyer decomposition of X as a semimartingale with respect to its own filtration $(\mathcal{F}_t^X)$: the martingale part is given by the Brownian motion B, and the predictable process of bounded variation is given by the absolutely continuous process $(\int_0^t \int_0^s l(s,u) dB_u ds)$.

Hitsuda [**33**] shows that the law of a Gaussian process $(X_t)_{0 \leq t \leq 1}$ with $E(X_t) = 0$ is equivalent to Wiener measure if and only if X admits a Volterra representation with a square-integrable Volterra kernel, i.e., we can construct a Wiener process B on $(\Omega, \mathcal{F}, \mathbb{P})$ and a square-integrable Volterra kernel l such that (1.12) holds. Moreover, this representation is unique in the sense that if X has another square-integrable Volterra representation

$$X_t = \tilde{B}_t + \int_0^t \int_0^s \tilde{l}(s,u) d\tilde{B}_u ds,$$

then $B = \tilde{B}$ and $l(s,u) = \tilde{l}(s,u)$ for almost all $s, u \in (0,1)$; see Hida-Hitsuda [**32**]. But it $l \notin L^2((0,1)\times(0,1))$, this representation is no longer unique. In the last section of this chapter we shall discuss different Volterra representations of a Brownian motion.

Using the stochastic Fubini Theorem (see, e.g., Protter [**51**]) we can write the representation (1.12) as

$$X_t = \int_0^t \left(1 + \int_u^t l(v,u)dv\right) dB_u, \tag{1.14}$$

provided the Volterra kernel $l(t,s)$ is square-integrable. The following theorem identifies the representation (1.14) as the canonical representation (1.11) of X.

PROPOSITION 1.2. *Consider a process X which admits a square-integrable Volterra representation* (1.12). *Then we have* $(\mathcal{F}_t^X) = (\mathcal{F}_t^B)$. *In other words, a process which has a square-integrable Volterra representation* (1.12) *admits a canonical representation* (1.14).

PROOF. Given a square-integrable Volterra kernel l, there is a unique square-integrable Volterra kernel R_l which satisfies the equations

$$\begin{cases} l(t,s) + R_l(t,s) + \displaystyle\int_s^t l(t,u)R_l(u,s)du = 0, \\ l(t,s) + R_l(t,s) + \displaystyle\int_s^t R_l(t,u)l(u,s)du = 0; \end{cases} \tag{1.15}$$

for almost all $s \le t$. We call R_l the resolvent kernel of l; see Chapter 4 in Yosida [**58**] or Hida-Hitsuda [**32**]. As in Hida-Hitsuda [**32**] p.136-137, we can now use the kernel R_l in order to reconstruct B in terms of X:

$$\begin{aligned} & dX_t + \int_0^t R_l(t,u)dX_u dt \\ = \;& dB_t + \int_0^t l(t,u)dB_u dt + \int_0^t R_l(t,u)\left[dB_u + \int_0^u l(u,v)dB_v du\right] dt \\ = \;& dB_t + \int_0^t \left(l(t,u) + R_l(t,u) + \int_u^t R_l(t,v)l(v,u)dv\right) dB_u\, dt \\ = \;& dB_t. \end{aligned}$$

Thus, we have

$$X_t = B_t + \int_0^t \int_0^s l(s,u)dB_u\, ds, \tag{1.16}$$

and

$$B_t = X_t + \int_0^t \int_0^s R_l(s,u)dX_u\, ds. \tag{1.17}$$

Therefore, the filtration generated by X coincides with the one generated by B. Hence (1.12) is the canonical decomposition of X in its own filtration. Thus, we know that the representation (1.14) is a canonical representation. □

REMARK 1.4. Equation (1.17) in the proof shows in particular how the Brownian motion (B_t) in (1.12) can be reconstructed from (X_t) by a linear transformation. This method of reconstruction will often be used in the sequel. We can apply this method not only in the case of a Brownian motion B. For example, the solution of an integral equation

$$m(t) = f(t) + \int_0^t l(t,u)f(u)du$$

is given by

$$f(t) = m(t) + \int_0^t R_l(t,u)m(u)du.$$

Further properties and applications of Volterra kernels can be found in Gohberg-Krein [**27**] and Corduneanu [**18**].

REMARK 1.5. For a Volterra kernel l that satisfies (1.13) but is not square-integrable, there exists also a corresponding resolvent kernel, but the latter is no longer square-integrable. In some cases, the associated resolvent kernel even does not satisfy (1.13). For instance, the resolvent kernel of $l(t,s) = -1/t$ is given by $R_l(t,s) = 1/s$ for $s \le t$, which does not satisfy (1.13). If l and R_l both satisfy (1.13) we know that (1.16) and (1.17) hold and this results in $(\mathcal{F}_t^X) = (\mathcal{F}_t^B)$. For example, the resolvent kernel of a Volterra kernel $l(t,s) = 1/t$ is given by $R_l(t,s) = -s/t^2$, which satisfies (1.13). From this result and the stochastic Fubini Theorem we can represent the random variable

$$\begin{aligned} X_t &= B_t + \int_0^t \frac{B_u}{u}du, & (1.18)\\ &= B_t + \int_0^t \log\frac{t}{u}dB_u = \int_0^t \left(1+\log\frac{t}{u}\right)dB_u & (1.19)\end{aligned}$$

as

$$B_t = X_t - \int_0^t \int_0^u \frac{v}{u^2}dX_v du.$$

Hence, $(\mathcal{F}_t^X) = (\mathcal{F}_t^B)$. Consequently, (1.19) is a canonical representation of X, but (1.18) is *not* a square-integrable Volterra representation, because the Volterra kernel $l(t,s) = 1/t$ is not in $L^2((0,1)\times(0,1))$. From this example we can also see that a square-integrable Volterra representation is not guaranteed to exist even though $(\mathcal{F}_t^X)$ and $(\mathcal{F}_t^B)$ might coincide.

1.3. Some auxiliary lemmas

Denote the space of all continuous functions on $[0,1]$ by $C[0,1]$ and let $(\mathcal{B}_t)$ denote the canonical right continuous filtration generated by the coordinate process. We begin by recalling the definition of "nonanticipative functionals" in Kallianpur [**38**].

DEFINITION 1.4. A $\mathcal{B}_1 \times \mathcal{B}[0,1]$-measurable functional $\gamma : C[0,1] \times [0,1] \longrightarrow \mathbb{R}$ is called *nonanticipative* if the process defined by $(\gamma(\cdot, t))_{0\leq t\leq 1}$ is adapted to the filtration $(\mathcal{B}_t)$.

The next lemma shows how a square-integrable Volterra representation can be constructed for a particular class of Gaussian processes.

LEMMA 1.1 (Theorem 9.4.1 of Kallianpur [**38**]). *Let γ be a nonanticipative functional, $W = (W_t)_{0\leq t\leq 1}$ a Wiener process and $\xi = (\xi_t)_{0\leq t\leq 1}$ a Gaussian process satisfying*

$$\xi_t = W_t + \int_0^t \gamma(\xi, s)ds, \tag{1.20}$$

with

$$\mathbb{P}\left[\omega \in \Omega : \int_0^t \gamma^2(\xi(\omega), s)ds < \infty\right] = 1, \tag{1.21}$$

for all $t < 1$. Then ξ can be expressed in terms of W by the formula

$$\xi_t = W_t + \int_0^t \int_0^s G(s,u)dW_u ds,$$

for all $t < 1$, $\mathbb{P}$-a.s., where G is a square-integrable Volterra kernel. In other words, ξ possesses a square-integrable Volterra representation.

From this Lemma and the discussion of Hitsuda [**33**] in Section 1.2 above, we see that the law of centered Gaussian process ξ of Lemma 1.1 is equivalent to Wiener measure.

LEMMA 1.2 (Lemma 2.3 of Föllmer-Wu-Yor [**23**]). *Suppose the process $(X_t)_{t\geq 0}$ satisfies*

$$X_t = W_t + \int_0^t Y_u du,$$

with an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $(W_t)_{t\geq 0}$ and an $(\mathcal{F}_t)$-adapted process $(Y_t)_{t\geq 0}$ satisfying $\int_0^t E|Y_u|du < \infty$ for all t.
(i) *The Doob-Meyer decomposition of X as a semimartingale in its natural filtration $(\mathcal{F}_t^X)$ is given by*

$$X_t = B_t + \int_0^t E[Y_u|\mathcal{F}_u^X]du, \tag{1.22}$$

where the process B defined by (1.22) is an $(\mathcal{F}_t^X)$-Brownian motion, which is often called the innovation process of X. In particular, $(X_t)_{t\geq 0}$ is a Brownian motion if and only if

$$E[Y_u|\mathcal{F}_u^X] = 0, \qquad d\mathbb{P} \times du - a.s..$$

(ii) *Furthermore, if the function* $s \longmapsto Y_s$ *is* L^1*-continuous on* $(0, \infty)$ *and* $(X_t)_{t \geq 0}$ *is a Gaussian process, then* $(X_t)_{t\geq 0}$ *is a Brownian motion if and only if*

$$E(X_s Y_t) = 0, \tag{1.23}$$

for all $0 < s \leq t$.

The previous lemma provides the construction of the Doob-Meyer decomposition and an alternative characterization of Brownian motion which will be useful in the rest of the thesis. For example, by combining this Lemma and Proposition 1.1 we have the following corollary and example.

Corollary 1.1. *Suppose the process* X *is given by* (1.7). *Then the canonical decomposition of* X *is of the form*

$$X_t = B_t + \int_0^t f(u) \int_0^u f(v)p(v)dB_v du, \tag{1.24}$$

where $p(t)$ *is the solution of* (1.8). *Moreover,* (1.24) *is a square-integrable Volterra representation of* X.

Example 1.2. By the above corollary we get that the canonical decomposition of X mentioned in Example 1.1 is given by

$$X_t = B_t + c^2 \int_0^t \int_0^s \frac{e^{a(s-u)}(e^{2\gamma u} - 1)}{(\gamma - a)e^{2\gamma u} + \gamma + a} dB_u ds,$$

where $(B_t)_{0 \leq t \leq 1}$ is an $(\mathcal{F}_t^X)$-Brownian motion.

With the help of Lemma 1.2 we can also construct a Gaussian process with nonlinear drift term:

Example 1.3. Consider a Wiener process $(X_t)_{0\leq t\leq 1}$ satisfying

$$dX_t = dW_t + \frac{X_1 - X_t}{1-t} dt, \tag{1.25}$$

where $(W_t)_{0\leq t\leq 1}$ is a Brownian motion. Clearly, the process X is a Gaussian semimartingale with respect to the filtration generated by the process (W_t) and the random variable X_1, and it is also one with respect to the enlarged filtration $\bar{\mathcal{F}}_t := \sigma\{X_t, \operatorname{sgn}(X_1)\}$. By Lemma 1.2 the canonical decomposition of X with respect to this σ-algebra is of the form

$$X_t = \tilde{W}_t + \int_0^t \frac{E[X_1|\bar{\mathcal{F}}_u] - X_u}{1-u} du$$

where $(\tilde{W}_t)_{0\leq t\leq 1}$ is an $(\bar{\mathcal{F}}_t)$-Brownian motion. Therefore, we have to calculate the explicit form of the conditional expectation of X_1 with respect to $(\bar{\mathcal{F}}_t)$:

$$\begin{aligned} E[X_1|\bar{\mathcal{F}}_t] &= E[X_1|X_t, \operatorname{sgn}(X_1)] = E[X_1|X_t, X_1 > 0] \\ &= X_t + E[X_1 - X_t|X_t, X_1 - X_t > -X_t]. \end{aligned}$$

Let Y be an $N(0,1)$-distributed random variable. We can rewrite the second term in the above equation as

$$\sqrt{1-t}E[Y|Y > -\frac{X_t}{\sqrt{1-t}}].$$

Moreover,

$$E[Y|Y > a] = \frac{\int_a^\infty y\phi(y)dy}{P[Y > a]} = \frac{\phi(a)}{2(1-\Phi(a))},$$

with normal distribution Φ and its density function ϕ. Taking $a = -\frac{1}{\sqrt{1-t}}X_t$, we get

$$E[X_1|X_t, \mathrm{sgn}(X_1)] = X_t + \frac{\sqrt{1-t}}{2}\frac{\phi(\frac{1}{\sqrt{1-t}}X_t)}{\Phi(\frac{1}{\sqrt{1-t}}X_t)}.$$

Therefore, we can write (1.25) as

$$dX_t = d\tilde{W}_t + \frac{1}{2\sqrt{1-t}}\frac{\phi(\frac{1}{\sqrt{1-t}}X_t)}{\Phi(\frac{1}{\sqrt{1-t}}X_t)}dt.$$

Thus the drift term of the Gaussian process X is clearly non-linear.

1.4. Volterra representations of Brownian motion

Consider a stochastic process $(X_t)_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which admits a Volterra representation

$$X_t = B_t + \int_0^t \int_0^s l(s,u)dB_u ds. \tag{1.26}$$

As already mentioned in Section 1.2, Hitsuda [**33**] shows that the law of X is identical to that of a Brownian motion under some equivalent measure $\tilde{\mathbb{P}} \sim \mathbb{P}$ if and only if X admits a square-integrable Volterra representation. Hence, from the uniqueness of the Doob-Meyer decomposition we know that if X is a Brownian motion admitting a Volterra representation (1.26), then the associated Volterra kernel l is not square-integrable unless $l \equiv 0$. For the case $l \not\equiv 0$, we can conclude that $(\mathcal{F}_t^X) \subsetneqq (\mathcal{F}_t^B)$, i.e., the filtration generated by X is strictly smaller than the one generated by B. Otherwise, the representation (1.26) would be the Doob-Meyer decomposition of X as a semimartingale in its own filtration. Uniqueness of the Doob-Meyer decomposition would imply $l \equiv 0$, which is obviously a contradiction. But is it possible to find a Volterra representation for Brownian motion, where the kernel l is not square-integrable? If so, how does the associated Volterra kernel look like?

THEOREM 1.2. *The process $(X_t)_{t\geq 0}$ satisfying (1.26) is a Brownian motion if and only if the Volterra kernel $-l$ is self-reproducing, i.e.,*

$$l(t,s) + \int_0^s l(t,v)l(s,v)dv = 0, \tag{1.27}$$

for all t and for almost all $s \leq t$. Furthermore, if the process $(X_t)_{t\geq 0}$ is a Brownian motion, then $\{X_s; s \leq t\}$ is independent of $\int_0^t l(t,u)dB_u$ for all $t > 0$.

PROOF. Thanks to the second assertion in Lemma 1.2 we see that (X_t) is a Brownian motion if and only if

$$E\left[X_s \int_0^t l(t,u)dB_u\right] = 0,$$

for all $s \leq t$. It follows from (1.26) that

$$E\left[X_s \int_0^t l(t,u)dB_u\right] = \int_0^s l(t,u)du + \int_0^s \int_0^u l(t,v)l(u,v)dvdu.$$

Taking derivatives with respect to s, we get the first assertion. Furthermore. since both X and $\int_0^t l(t,u)dB_u$ are jointly Gaussian, the second assertion follows. □

REMARK 1.6. The terminology "self-reproducing" is used in Neveu [**48**] in a different context.

REMARK 1.7. If the Volterra kernel $l(t,s)$ is continous, then it satisfies the following properties:

(i) $l(t,t) \leq 0$.
(ii) $|l(t,s)| \leq \sqrt{l(t,t)l(s,s)}$.
(iii) If $l(t,s) \not\equiv 0$, then $l(t,t) \notin L^1(0,1)$, and this implies $l \notin L^2((0,1)\times(0,1))$. This is consistent with the above discussion. In particular we see that a non-zero self-reproducing Volterra kernel l is not square-integrable.

PROOF. Since l is continuous, we get that l satisfies (1.27) for all t and for all $s \leq t$. Taking $s = t$, we have

$$l(t,t) = -\int_0^t l^2(t,u)du, \tag{1.28}$$

which leads to assertion (i). Then it follows from Hölder's inequality that

$$\begin{aligned} |l(t,s)| &\leq \left(\int_0^s l^2(t,v)dv\right)^{\frac{1}{2}} \left(\int_0^s l^2(s,v)dv\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^t l^2(t,v)dv\right)^{\frac{1}{2}} \left(\int_0^s l^2(s,v)dv\right)^{\frac{1}{2}} \\ &= \sqrt{l(t,t)l(s,s)}. \end{aligned}$$

This gives (ii). As for (iii), assume $l \not\equiv 0$. Since l is continuous, we see that $l(t,t) \neq 0$ for some $t \in [0,1]$ due to (1.28). Let us write

$$\{s : l(s,s) \neq 0\} = \bigcup_i (a_i, b_i),$$

with disjoint intervals (a_i, b_i). Substituting (ii) in (1.27), we get

$$|l(t,s)| \leq -\sqrt{l(t,t)l(s,s)} \int_0^s l(v,v)dv.$$

This implies

$$-l(s,s) \le l(s,s) \int_0^s l(v,v)dv,$$

for all s. Since $l(s,s) = 0$ for $s \le a := \inf_i a_i$, we obtain

$$\int_a^s l(v,v)dv = \int_0^s l(v,v)dv \le -1 \tag{1.29}$$

for all $s \in \bigcup_i [a_i, b_i]$. Either we have $a = a_i$ for some i or a is an accumulation point of (a_i). In both cases, (1.29) implies $l(t,t) \notin L^1(0,1)$. In particular, we have

$$\int_0^t \int_0^v l^2(v,u)dvdu = \int_0^t l(v,v)dv = -\infty.$$

□

In order to illustrate Theorem 1.2 more explicitly, let us look at some special cases:

COROLLARY 1.2. *Let X be a process given by*

$$X_t = B_t - \int_0^t a(u) \int_0^u b(v)dB_v du,$$

with $X_0 = 0$, where a, b are deterministic functions with $a \not\equiv 0$, $\int_0^t b^2(u)du \neq 0$ and

$$\int_0^t |a(u)| \left(\int_0^u b^2(v)dv \right)^{\frac{1}{2}} du < \infty,$$

for all $t > 0$. Then X is a Brownian motion if and only if

$$a(t) = -\frac{b(t)}{\int_0^t b^2(v)dv},$$

i.e., the process X is of the form

$$X_t = B_t - \int_0^t \frac{b(u)}{\int_0^u b^2(v)dv} \int_0^u b(r)dB_r du. \tag{1.30}$$

Furthermore, if X is a Brownian motion, then the σ-algebra $\mathcal{F}_t^X$ is orthogonal to the stochastic integral $\int_0^t b(u)dB_u$ for all t.

PROOF. Substituting the Volterra kernel $l(t,s) = a(t)b(s)$ in (1.27) we see that X is a Brownian motion if and only if

$$b(s) + a(s) \int_0^s b^2(u)du = 0,$$

from which we derive (1.30). For $s \le t$, we have

$$E\left[X_s \int_0^t b(u)dB_u \right] = \int_0^s b(u)du - \int_0^s \frac{b(u)}{\int_0^u b^2(v)dv} \int_0^u b^2(r)drdu = 0.$$

This completes the proof. □

If b is as in Corollary 1.2, then it is easy to check that the kernel

$$l(s,u) := -\frac{b(s)b(u)}{\int_0^s b^2(v)dv}$$

satisfies all properties stated in Remark 1.7. In particular, if we take $b(t) = t^k$ for $k > -\frac{1}{2}$, we see that the process

$$X_t = B_t - (2k+1)\int_0^t \int_0^u u^{-k-1} v^k dB_v du$$

is a Brownian motion. This has been discussed in Lévy [**44**], [**45**], Chiu [**14**] and Hibino-Hitsuda-Muraoka [**31**]. Especially, for the case $k = 0$, i.e., the Brownian motion

$$X_t = B_t - \int_0^t \frac{B_u}{u} du,$$

has been studied in a number of papers, e.g., Deheuvels [**21**], Jeulin-Yor [**37**], Yor [**57**]. And we also know that the σ-algebra $\mathcal{F}_t^X$ is strictly smaller than $\mathcal{F}_t^B$ for all $t > 0$. In fact, we have the decomposition

$$\mathcal{F}_t^B = \mathcal{F}_t^X \oplus \sigma(B_t)$$

for all t; see Jeulin-Yor [**37**] or Chapter 1 of Yor [**57**]. In Chapter 3 below we shall also discuss some generalizations of this process and the corresponding orthogonal decompositions of the Brownian filtration.

CHAPTER 2

Brownian motions generated by linear stochastic equations

Let W be a Wiener process, and let S be a continuous square-integrable Gaussian martingale which is independent of W. Consider a Gaussian process X satisfying the stochastic differential equation

$$dX_t = dW_t + Y_t\, dt, \tag{2.1}$$

where the drift term Y is linear in S and X. Explicitly, Y_t is of the form

$$Y_t = f(t)S_0 + \int_0^t F(t,u)dS_u + \int_0^t H(t,u)dX_u, \tag{2.2}$$

where f, F and H are deterministic functions satisfying some suitable integrability conditions. Our aim is to construct the Doob-Meyer decomposition of such processes X as semimartingales in their own filtration. It will also be called the *canonical decomposition of* X. In particular, we characterize the drifts Y such that the resulting process X is a Brownian motion in its own filtration. In addition, we investigate the problem of choosing Y in such a way that X_t converges to the final value S_1 as $t \to 1$. This analysis is motivated by an equilibrium problem in mathematical finance related the role of insider trading. The financial interpretation will be discussed in Chapter 4 and Chapter 5.

We begin in Section 2.1 with the discussion of a special case where S is also a Wiener process. In Section 2.2 we shall derive the canonical decomposition of Gaussian processes with linear drift term. Using this result, we are able to characterize the processes satisfying (2.1) with drift term (2.2) which are Brownian motions in their own filtrations; see Section 2.3. Some examples of such Brownian motions will be given in the last section of this chapter.

2.1. An example of canonical decomposition

Let $(W_t)_{0\le t\le 1}$ be a standard Brownian motion with respect to its canonical filtration $(\mathcal{F}_t^W)_{0\le t\le 1}$. Now let $(\tilde{W}_t)_{0\le t\le 1}$ be another standard Brownian motion independent of $(W_t)_{0\le t\le 1}$. Denote the filtration generated by these two Brownian motions by $(\mathcal{F}_t^{W,\tilde{W}})_{0\le t\le 1}$.

We know that the solution $(\tilde{X}_t)_{0\le t\le 1}$ of the stochastic differential equation

$$d\tilde{X}_t = dW_t + \frac{\tilde{W}_1 - \tilde{X}_t}{1-t}dt, \tag{2.3}$$

with initial value $\tilde{X}_0 = 0$, is a standard Brownian motion with respect to $(\mathcal{F}_t^{\tilde{X}})$ which converges to the final value $\tilde{W}_1$ (cf., for example, Jeulin-Yor [**36**]). Now we look at the process $(X_t)_{0 \le t \le 1}$ starting in $X_0 = 0$ which is defined by a similar stochastic differential equation

$$dX_t = dW_t + \frac{\tilde{W}_t - X_t}{1-t} dt. \tag{2.4}$$

Clearly, for any $t \in [0,1]$, X_t is normally distributed, and (X_t) has quadratic variation $\langle X \rangle_t = t$. The following lemma shows even that X_t approaches $\tilde{W}_1$ as $t \to 1$. However, we will see that $(X_t)_{0 \le t \le 1}$ is no longer a Brownian motion.

LEMMA 2.1. *$X_t \to \tilde{W}_1$ as $t \to 1$.*

PROOF. The explicit solution of (2.4) is given by

$$X_t = (1-t) \int_0^t \frac{\tilde{W}_s}{(1-s)^2} ds + (1-t) \int_0^t \frac{1}{(1-s)} dW_s. \tag{2.5}$$

The first term approaches $\tilde{W}_1$ and the second goes to 0 as $t \to 1$, and this implies the result. Alternatively, we could note that the process $2^{-\frac{1}{2}}(X - \tilde{W})$ satisfies the equation of a Brownian bridge tied down to the final value 0. □

LEMMA 2.2. *For $0 \le s \le t < 1$, we have*

$$E[X_t \tilde{W}_t] = t + (1-t)\log(1-t), \tag{2.6}$$

and the covariance function of X is given by

$$E[X_s X_t] = s + 2s(1-t) + (2-s-t)\log(1-s). \tag{2.7}$$

PROOF. Applying the integration by parts formula to the first integral in (2.5), the solution of (2.4) is given by

$$X_t = (1-t) \int_0^t \frac{dW_s - d\tilde{W}_s}{1-s} + \tilde{W}_t.$$

Since (W_t) and $(\tilde{W}_t)$ are independent, we establish

$$\begin{aligned} E[X_t \tilde{W}_t] &= t + (1-t) E\left[\int_0^t \frac{\tilde{W}_t (dW_s - d\tilde{W}_s)}{1-s}\right] \\ &= t + (1-t)\log(1-t), \end{aligned}$$

and

$$\begin{aligned} E[X_s X_t] &= E[\tilde{W}_s \tilde{W}_t] + (1-t)(1-s) E\left[\left(\int_0^s \frac{dW_u - d\tilde{W}_u}{1-u}\right)^2\right] \\ &+ (1-s) E\left[\tilde{W}_t \int_0^s \frac{dW_u - d\tilde{W}_u}{1-u}\right] + (1-t) E\left[\tilde{W}_s \int_0^t \frac{dW_u - d\tilde{W}_u}{1-u}\right] \\ &= s + 2s(1-t) + (2-s-t)\log(1-s). \end{aligned}$$

□

This Lemma shows that the process $(X_t)_{0\le t\le 1}$ is not a Brownian motion, since its covariance function differs from $s\wedge t$. But from (2.4) we see that it is a semimartingale with respect to the filtration $(\mathcal{F}_t^{W,\tilde{W}})_{0\le t\le 1}$, and therefore it is obviously a semimartingale relative to its natural filtration. A natural question arises: what is the explicit form of its canonical decomposition? This is the purpose we want to carry out in the present section.

A simple application of Lemma 1.2 shows that

COROLLARY 2.1. *Let the process $(X_t)_{0\le t\le 1}$ satisfy (2.4). Then the process B, defined as*

$$B_t := X_t - \int_0^t \frac{E[\tilde{W}_u|\mathcal{F}_u^X] - X_u}{1-u}du, \tag{2.8}$$

is a Brownian motion relative to $(\mathcal{F}_t^X)_{0\le t\le 1}$.

PROOF. Set

$$Y_u = \frac{\tilde{W}_u - X_u}{1-u},$$

then from the first assertion in Lemma 1.2, we obtain the required result. □

Recalling Lemma 1.2 we know that the process B given by (2.8) is the innovation process of X. Therefore, if we desire to get the canonical decomposition of X_t, we only have to compute the conditional expectation of $\tilde{W}_t$ relative to the σ-algebra $\mathcal{F}_t^X$. If this is substituted in Corollary 2.1, then we get the desired result.

LEMMA 2.3. *Set $A := \frac{1}{2}(1+\sqrt{5})$ and $B := \frac{1}{2}(1-\sqrt{5})$. Then for $0\le t<1$,*

$$E[\tilde{W}_t|\mathcal{F}_t^X] = \int_0^t \frac{(B+1)(1-s)^{-A} - (A+1)(1-s)^{-B}}{A(1-t)^{-B} - B(1-t)^{-A}}\, dX_s + X_t. \tag{2.9}$$

PROOF. Due to (2.8), we may choose a nonanticipative functional γ such that

$$X_t = B_t + \int_0^t \gamma(X,s)ds.$$

Using Lemma 1.1, X can be represented by

$$X_t = B_t + \int_0^t\int_0^s G(s,u)dB_u ds,$$

where G is a square-integrable Volterra kernel. Let R_G be the square-integrable resolvent kernel of G, then applying a similar argument as in the proof of Proposition 1.2, we deduce

$$B_t = X_t + \int_0^t\int_0^s R_G(s,u)dX_u ds. \tag{2.10}$$

Comparing (2.8) and (2.10), we may therefore assume that the conditional expectation of $\tilde{W}_t$ with respect to the σ-algebra $\mathcal{F}_t^X$ is of the form

$$E[\tilde{W}_t|\mathcal{F}_t^X] = \int_0^t a(t,u)dX_u,$$

with a continuously differentiable square-integrable Volterra kernel $a(t,u)$. Applying the projection property of the conditional expectation

$$E[X_s(\tilde{W}_t - E[\tilde{W}_t|\mathcal{F}_t^X])] = 0,$$

for all $0 \le s \le t < 1$, as well as the martingale property we obtain

$$E(X_s\tilde{W}_s) - a(t,t)E(X_sX_t) = -\int_0^t a_2(t,u)E(X_sX_u)du.$$

Using (2.6), (2.7) and computing explicitly the left hand side (LHS) and the right hand side (RHS) in this equation, we get

$$\begin{aligned} \text{LHS } &= s + (1-s)\log(1-s) - a(t,t)(s + 2s(1-t) + (2-s-t)\log(1-s)). \\ \text{RHS } &= -\int_0^s a_2(t,u)(u + 2u(1-s) + (2-s-u)\log(1-u))du \\ &\quad - \int_s^t a_2(t,u)(s + 2s(1-u) + (2-s-u)\log(1-s))du. \end{aligned}$$

Taking the second derivatives with respect to s on both sides implies

$$\frac{1}{1-s} - \frac{a(t,t)(t-s)}{(1-s)^2} = a_2(t,s) - \int_s^t \frac{a_2(t,u)(u-s)}{(1-s)^2}du.$$

Multiplication of both sides with $(1-s)^2$ leads to

$$1 - s - a(t,t)(t-s) = a_2(t,s)(1-s)^2 - \int_s^t a_2(t,u)(u-s)du,$$

and this implies

$$1 - s = a_2(t,s)(1-s)^2 + \int_s^t a(t,u)du.$$

Taking further derivatives with respect to s on both sides we get

$$(1-s)^2 a_{22}(t,s) - 2(1-s)a_2(t,s) - a(t,s) + 1 = 0.$$

The solution of this differential equation is given by

$$a(t,s) = c_1(t)(1-s)^{-A} + c_2(t)(1-s)^{-B} + 1.$$

Substituting this equation in RHS and comparing the coefficients of s, $\log(1-s)$ and $s\log(1-s)$ in LHS and RHS, we derive the desired result. □

Therefore, combining Corollary 2.1 and Lemma 2.3 allows us to conclude the following proposition.

PROPOSITION 2.1. *The canonical decomposition of the process X in (2.4) is given by*

$$X_t = B_t + \int_0^t \int_0^u \frac{(B+1)(1-s)^{-A} - (A+1)(1-s)^{-B}}{A(1-u)^A - B(1-u)^B} dX_s\, du, \tag{2.11}$$

for $0 \le t < 1$.

REMARK 2.1. As an alternative, we can use the Kalman-Bucy filter to get the same result. We may set

$$\xi_t := W_t + \int_0^t \frac{\tilde{W}_u}{1-u} du.$$

Applying Proposition 1.1 with $f = 1/(1-t)$ and $g \equiv 1$ and Lemma 1.2 we can get the canonical decomposition of ξ

$$\xi_t = B_t + \int_0^t \int_0^u \frac{1}{2(1-u)} \left(1 - \frac{\sqrt{5}(A(1-s)^A + B(1-s)^B)}{A(1-s)^A - B(1-s)^B}\right) dB_s du.$$

Substituting it into

$$X_t = \xi_t - \int_0^t \frac{X_u}{1-u} du$$

derives the canonical decomposition of X

$$X_t = B_t + \int_0^t \int_0^u \frac{1}{2(1-u)} \left(1 - \frac{\sqrt{5}(A(1-s)^A + B(1-s)^B)}{A(1-s)^A - B(1-s)^B}\right) dB_s du - \int_0^t \frac{X_u}{1-u} du.$$

Applying Itô's product rule and the stochastic Fubini Theorem we obtain

$$\begin{aligned} X_t &= (1-t)\left[\int_0^t \frac{dB_u}{1-u} \right. \\ &\qquad \left. + \int_0^t \int_0^u \frac{1}{2(1-u)^2} \left(1 - \frac{\sqrt{5}(A(1-s)^A + B(1-s)^B)}{A(1-s)^A - B(1-s)^B}\right) dB_s\, du\right] \\ &= B_t - \frac{1-t}{2} \int_0^t \int_0^u \frac{1}{(1-u)^2} \left(1 + \frac{\sqrt{5}(A(1-s)^A + B(1-s)^B)}{A(1-s)^A - B(1-s)^B}\right) dB_s\, du \\ &= B_t - \int_0^t \int_0^u \frac{(A+1)(1-s)^{A-1} - (B+1)(1-s)^{B-1}}{A(1-s)^A - B(1-s)^B} dB_s\, du \end{aligned} \tag{2.12}$$

Comparing these two equations (2.11) and (2.12), we see they are of the same form as (1.16) and (1.17) in the proof of Proposition 1.2, and that their kernels fulfill the relation (1.15). In other words, they are square-integrable resolvent kernels to each other, and (2.12) is the square-integrable Volterra representation of X.

2.2. Linear stochastic equations and canonical decomposition

Let W be a Wiener process, and let S be a continuous square-integrable centered Gaussian martingale which is independent of W. For simplicity, we denote the variance of S_t by $V(t)$ and assume $V(1) = E[S_1^2] = 1$. Furthermore, we assume that $V(t)$ is differentiable. Suppose the process X satisfies the stochastic functional differential equation with linear drift

$$dX_t = dW_t + \left(f(t)S_0 + \int_0^t F(t,u)dS_u + \int_0^t H(t,u)dX_u \right) dt, \tag{2.13}$$

where $f \in L^2(0,1) \cap C(0,1)$ and $F(t,u)\sqrt{V'(u)}$ and $H(t,u)$ are square-integrable Volterra kernels on $(0,1) \times (0,1)$ (Since $V(s)$ is non-decreasing in s, $V'(s) \geq 0$, this implies $\sqrt{V'(s)}$ is well-defined.). We assume $X_0 = 0$. The following theorem will give the canonical decomposition of X, i.e., the Doob-Meyer decomposition of X as a semimartingale in its own filtration.

THEOREM 2.1. *Suppose X satisfies* (2.13)*. Then its canonical decomposition is given by*

$$X_t = B_t + \int_0^t \left(\int_0^s G(s,u)dB_u + \int_0^s H(s,u)dX_u \right) ds, \tag{2.14}$$

where B is a Brownian motion. Here G is a square-integrable Volterra kernel determined by the integral equation

$$G(t,s) + \int_0^s G(t,u)G(s,u)du = f(t)f(s)V(0) + \int_0^s F(t,u)F(s,u)V'(u)du. \tag{2.15}$$

Moreover, we have $(\mathcal{F}_t^X) = (\mathcal{F}_t^B)$, i.e., the filtrations generated by X and B are the same.

PROOF. 1) Define a process ξ by

$$\xi_t := W_t + \int_0^t \left(f(u)S_0 + \int_0^u F(u,v)dS_v \right) du. \tag{2.16}$$

Due to Lemma 1.2 we know that there exists a Brownian motion B such that

$$\xi_t = B_t + \int_0^t E\left[f(u)S_0 + \int_0^u F(u,v)dS_v \middle| \mathcal{F}_u^\xi \right] du.$$

Since $E[f(u)S_0 + \int_0^u F(u,v)dS_v | \mathcal{F}_u^\xi](\omega)$ can be chosen to be both jointly measurable in u and ω and $(\mathcal{F}_u^\xi)$-adapted, we can write

$$E\left[f(u)S_0 + \int_0^u F(u,v)dS_v \middle| \mathcal{F}_u^\xi \right] = \gamma(\xi, u)$$

with a nonanticipative functional γ. Furthermore, it follows from

$$\begin{aligned} E\left[\int_0^t \gamma^2(\xi,u)du\right] &= E\left[\int_0^t \left(E\left[f(u)S_0+\int_0^u F(u,v)dS_v \middle| \mathcal{F}_u^\xi\right]\right)^2 du\right] \\ &\le E\left[\int_0^t \left(f(u)S_0+\int_0^u F(u,v)dS_v\right)^2 du\right] \\ &= \int_0^t \left(f^2(u)+\int_0^u F^2(u,v)dv\right) du < \infty \end{aligned}$$

that

$$\int_0^t \gamma^2(\xi(\omega),s)ds < \infty, \qquad \mathbb{P}-\text{a.s.},$$

for all $t<1$. Applying Lemma 1.1 we know that ξ can be represented as

$$\xi_t = B_t + \int_0^t\int_0^s G(s,u)dB_u ds, \quad \mathbb{P}-\text{a.s.}, \tag{2.17}$$

which is a square-integrable Volterra representation with respect to the Brownian motion B. As for the relation between F and G, we can look at the equations (2.16) and (2.17). For $s\le t$, the covariance function of ξ in (2.16) is given by

$$\begin{aligned} E[\xi_s\xi_t] &= E[W_sW_t] + E\left[\left(\int_0^s \left(f(u)S_0+\int_0^u F(u,v)dS_v\right)du\right)\right. \\ &\qquad \left.\left(\int_0^t \left(f(p)S_0+\int_0^p F(p,q)dS_q\right)dp\right)\right] \\ &= s+\left(\int_0^s f(u)du\right)\left(\int_0^t f(p)dp\right)V(0) \\ &\quad +2\int_0^s\int_0^u\int_0^v F(u,r)F(v,r)V'(r)drdvdu \\ &\quad +\int_s^t\int_0^s\int_0^v F(u,r)F(v,r)V'(r)drdvdu. \end{aligned} \tag{2.18}$$

And the covariance function of ξ satisfying (2.17) is given by

$$\begin{aligned} E[\xi_s\xi_t] &= s+\int_0^s\int_0^u G(u,v)dvdu+\int_s^t\int_0^s G(u,v)dvdu \\ &\quad +2\int_0^s\int_0^u\int_0^v G(u,r)G(v,r)drdvdu \\ &\quad +\int_s^t\int_0^s\int_0^v G(u,r)G(v,r)drdvdu. \end{aligned} \tag{2.19}$$

The right-hand sides of these two equations should coincide. Differentiating first with respect to t, and then with respect to s yield (2.15). From (2.17) and Proposition 1.2, we know that the filtrations generated by ξ and by B are coincide.

2) From (2.13) and (2.16), we see that

$$dX_t = d\xi_t + \int_0^t H(t,u)dX_u dt.$$

Using the same argument as in the proof of Proposition 1.2, we conclude that $(\mathcal{F}_t^X) = (\mathcal{F}_t^\xi)$. This implies $(\mathcal{F}_t^X) = (\mathcal{F}_t^B)$ due to 1). The canonical decomposition of X is therefore given by (2.14). □

REMARK 2.2. In the notation of Kallianpur [**38**] p.235, equation (2.15) can be viewed as the factorization $S = (I+G)(I+G^\star)$ of the integral operator S defined by $I + FF^\star + \tilde{F}\tilde{F}^\star$, where F, $\tilde{F}$, G are integral operators with square-integrable Volterra kernels $f(t)\sqrt{V(0)}$, $F(t,s)\sqrt{V'(s)}$ and $G(t,s)$, respectively. More precisely, for all $g, h \in L^2(0,1)$, we have

$$\langle (I + FF^\star + \tilde{F}\tilde{F}^\star)g, h\rangle = \langle (I+G)(I+G^\star)g, h\rangle.$$

In order to see this, let $g(u) = I_{(0,s)}(u)$ and $h(u) = I_{(0,t)}(u)$ with $0 \le s \le t \le 1$. Using the properties of Volterra kernels, we have

$$\begin{aligned}
&\langle (I + FF^\star + \tilde{F}\tilde{F}^\star)g, h\rangle = \langle g,h\rangle + \langle F^\star g, F^\star h\rangle + \langle \tilde{F}^\star g, \tilde{F}^\star h\rangle \\
= &\int_0^1 g(u)h(u)du + \int_0^1 \left(\int_0^1 f(v)g(v)\sqrt{V(0)}dv\right)\left(\int_0^1 f(r)h(r)\sqrt{V(0)}dr\right)du \\
&\quad + \int_0^1 \left(\int_0^1 F(v,u)g(v)\sqrt{V'(u)}dv\right)\left(\int_0^1 F(r,u)h(r)\sqrt{V'(u)}dr\right)du \\
= &\ s + \left(\int_0^t f(v)dv\right)\left(\int_0^s f(r)dr\right)V(0) \\
&\quad + \int_0^s \left(\int_u^s F(v,u)dv\right)\left(\int_u^t F(r,u)dr\right)V'(u)du,
\end{aligned}$$

which equals the right-hand side of (2.18). On the other hand,

$$\begin{aligned}
&\langle (I+G)(I+G^\star)g,h\rangle = \langle (I+G^\star)g, (I+G^\star)h\rangle \\
= &\int_0^1 \left(g(u) + \int_0^1 G(v,u)g(v)dv)(h(u) + \int_0^1 G(v,u)h(v)dv\right)du \\
= &\int_0^s \left(1 + \int_u^s G(v,u)dv\right)\left(1 + \int_u^t G(v,u)dv\right)du,
\end{aligned}$$

which is exactly the right-hand side of (2.19).

REMARK 2.3. **(i)** Comparing (2.13), (2.14) and Lemma 1.2, we see that

$$E\left[f(t)S_0 + \int_0^t F(t,u)dS_u \,\middle|\, \mathcal{F}_t^X \right] = \int_0^t G(t,u)dB_u, \tag{2.20}$$

for $f \in L^1(0,1) \cap C(0,1)$ and Volterra kernel F satisfying

$$\int_0^1 \int_0^u F^2(u,v)V'(v)dvdu < \infty.$$

(ii) We have obtained the conditional expectation in (2.20) under the assumption that $F(t,u)\sqrt{V'(u)}$ is a square-integrable Volterra kernel. In fact, it is enough to assume that F satisfies

$$\int_0^1 \left(\int_0^u F^2(u,v)V'(v)dv\right)^{\frac{1}{2}} du < \infty.$$

It is not difficult to show that for all $s \leq t$,

$$E\left[X_s\left(f(t)S_0 + \int_0^t F(t,u)dS_u - \int_0^t G(t,u)dB_u\right)\right] = 0,$$

and this result leads to the conclusion.

The representation (2.14) provides a canonical representation, but on the right-hand side there is still a term of X. Can we represent X through B alone? In the following we are going to give another representation for the process X satisfying (2.14).

PROPOSITION 2.2. *The unique strong solution of* (2.14) *is given by*

$$X_t = B_t + \int_0^t \int_0^s \left(G(s,u) + R_{-H}(s,u) + \int_u^s R_{-H}(s,v)G(v,u)dv\right) dB_u ds, \quad (2.21)$$

where R_{-H} is the resolvent kernel of $-H$, i.e., R_{-H} satisfies the equations

$$\begin{cases} R_{-H}(t,s) = H(t,s) + \int_s^t R_{-H}(t,u)H(u,s)du, \\ \\ R_{-H}(t,s) = H(t,s) + \int_s^t H(t,u)R_{-H}(u,s)du. \end{cases}$$

PROOF. Define

$$\xi_t := B_t + \int_0^t \int_0^s G(s,u)dB_u ds = X_t - \int_0^t \int_0^s H(s,u)dX_u ds. \quad (2.22)$$

Therefore similarly to the proof of Proposition 1.2 we get

$$X_t = \xi_t + \int_0^t \int_0^s R_{-H}(s,u)d\xi_u ds.$$

Substituting the first representation of (2.22) into the above equation, we get (2.21). In order to show the uniqueness of the solution, we assume there is another solution Y of (2.14). Then from (2.22) we know that

$$\xi_t = Y_t - \int_0^t \int_0^s H(s,u)dY_u ds,$$

and this implies

$$Y_t = \xi_t + \int_0^t \int_0^s R_{-H}(s,u)d\xi_u ds = X_t, \quad \mathbb{P}-\text{a.s.}.$$

□

Since $F(t,u)\sqrt{V'(u)}$ and $H(t,u)$ are square-integrable and since $(\mathcal{F}_t^X) = (\mathcal{F}_t^B)$ due to Theorem 2.1, we can conclude that (2.21) is a square-integrable Volterra representation of X.

Let us look at the special case where S is a standard Brownian motion $\tilde{W}$. Thus, X satisfies the linear stochastic functional differential equation

$$X_t = W_t + \int_0^t \left(\int_0^s F(s,u)d\tilde{W}_u + \int_0^s H(s,u)dX_u \right) ds, \tag{2.23}$$

with square-integrable Volterra kernels F and H on $(0,1)\times(0,1)$. The following corollary is a direct consequence of Theorem 2.1.

Corollary 2.2 (Theorem 4.1 of Föllmer-Wu-Yor [**23**]). *The canonical decomposition of X satisfying* (2.23) *is given by*

$$dX_t = dB_t + \left(\int_0^t G_F(t,u)dB_u + \int_0^t H(t,u)dX_u \right) dt, \tag{2.24}$$

where G_F is determined by

$$G_F(t,s) + \int_0^s G_F(t,u)G_F(s,u)du = \int_0^s F(t,u)F(s,u)du, \tag{2.25}$$

for almost all $s \le t$.

When F admits a factorization $F(t,s) = f(t)g(s)$ for some continuous functions f and g which satisfy

$$\int_0^t \int_0^u f^2(u)g^2(v)dvdu < \infty, \tag{2.26}$$

for all $t < 1$, we can write the above corollary more explicitly.

Corollary 2.3 (Corollary 4.1 of Föllmer-Wu-Yor [**23**]). *Suppose the process X is given by $X_0 = 0$ and*

$$dX_t = dW_t + \left(f(t) \int_0^t g(u)d\tilde{W}_u + \int_0^t H(t,u)dX_u \right) dt, \tag{2.27}$$

where $f, g \in C^1(0,1)$ satisfies (2.26), *$f \neq 0$ a.s., and $H(t,s)$ is a square-integrable Volterra kernel. Then the canonical decomposition of X is of the form*

$$dX_t = dB_t + \left(f(t) \int_0^t \alpha(u)dB_u + \int_0^t H(t,u)dX_u \right) dt, \tag{2.28}$$

where the function $\alpha(t)$ is the solution of the differential equation

$$\left(\frac{\alpha(t)}{f(t)} \right)' + \alpha^2(t) = g^2(t), \tag{2.29}$$

with initial condition $\alpha(0) = 0$.

PROOF. We have only to prove that the solution of (2.25) is given by $G_F(t,s) = f(t)\alpha(s)$, where α satisfies (2.29) with initial value $\alpha(0) = 0$. In fact, the right-hand side of (2.25) is equal to

$$\begin{aligned} G_F(t,s) + \int_0^s G_F(t,u)G_F(s,u)du &= f(t)\alpha(s) + f(t)f(s)\int_0^s \alpha^2(u)du \\ &= f(t)\alpha(s) + f(t)f(s)\left(\int_0^s g^2(u)du - \frac{\alpha(s)}{f(s)}\right) \\ &= f(t)f(s)\int_0^s g^2(u)du = \int_0^s F(t,u)F(s,u)du, \end{aligned}$$

which is exactly the left-hand side of (2.25). □

REMARK 2.4. Due to the special form of (2.27), this corollary can be also proved by the Kalman-Bucy filter. Combining Theorem 1.1 and Proposition 1.1 we can get the canonical decomposition of a process X of the form

$$dX_t = dW_t + f(t)\int_0^t g(u)d\tilde{W}_u dt.$$

With a simple trick we can extend it to (2.27). In fact, the second equation in (1.8) and (2.29) are equivalent, since if we set $p(t) := \alpha(t)/f(t)$ in (1.8), these two equations are identical.

Let $\mathcal{A}(a.b)$ be a class of functions defined by

$$\mathcal{A}(a,b) := \left\{\varphi : \varphi \text{ measurable, } \int_a^t s\varphi^2(s)ds < \infty \text{ for all } a \le t < b\right\}.$$

For the case $g \equiv 1$, we can rewrite the integral condition (2.26) on the square-integrable Volterra kernel F as the condition $f \in \mathcal{A}(0,1)$. Under this condition, the equation (2.29) can be written as

$$\left(\frac{\alpha(t)}{f(t)}\right)' + \alpha^2(t) = 1,$$

with $f \not\equiv 0$ and $\alpha(0) = 0$. The corresponding solution is given by

$$\alpha(t) = \frac{f(t)\Psi(t)}{\Psi'(t)}, \tag{2.30}$$

where $\Psi(t)$ is the solution of the Sturm-Liouville equation

$$\Psi''(t) = f^2(t)\Psi(t), \tag{2.31}$$

with boundary conditions $\Psi(0) = 0$ and $\Psi'(0+) = 1$. Using this result we can construct the following examples.

EXAMPLE 2.1. Consider a process $(X_t)_{0\le t\le 1}$ satisfying the stochastic differential equation

$$dX_t = dW_t + \frac{a}{1-t}(\tilde{W}_t - X_t)dt,$$

that is, $f(t) = -h(t) = a/(1-t)$, with a nonzero constant a. Then the corresponding Sturm-Liouville equation is

$$\Phi''(u) = \frac{a^2}{(1-u)^2}\Phi(u), \tag{2.32}$$

The following argument is due to M. Yor (see, Föllmer-Wu-Yor [**23**]). It is immediate to check that a function $(1-u)^\lambda$ solves (2.32) if and only if $\lambda(\lambda-1) = a^2$, an equation which admits the two solutions: $\lambda_+(a)$ and $\lambda_-(a)$, given by

$$\lambda_\pm(a) := \frac{1}{2} \pm \sqrt{a^2 + \frac{1}{4}}.$$

Clearly, $\lambda_-(a) < 0 < \lambda_+(a)$. Thus, the decreasing solution of (2.32) is

$$\Phi(u) = (1-u)^{\lambda_+(a)}.$$

And from the definition and the boundary condition of $\Psi(u)$ we can get

$$\Psi(u) = \frac{(1-u)^{\lambda_-(a)} - (1-u)^{\lambda_+(a)}}{\sqrt{1+4a^2}}.$$

From (2.20), the conditional expectation of $\tilde{W}_t$ relative to $\mathcal{F}_t^X$ is given by

$$\begin{aligned} E[\tilde{W}_t|\mathcal{F}_t^X] &:= \frac{1}{\Psi'(t)}\int_0^t \Psi(u)\left(\frac{a}{1-u}dX_u + \frac{a^2}{(1-u)^2}X_u du\right) \\ &= X_t + a\int_0^t \frac{(\lambda_-(a)+1)(1-u)^{-\lambda_+(a)} - (\lambda_+(a)+1)(1-u)^{-\lambda_-(a)}}{\lambda_+(a)(1-t)^{-\lambda_-(a)} - \lambda_-(a)(1-t)^{-\lambda_+(a)}} dX_u. \end{aligned}$$

In particular, if $a = 1$, then $\lambda_+(1) = A$ and $\lambda_-(1) = B$, defined as in Lemma 2.3, and we are led to the same result as in Section 2.1.

EXAMPLE 2.2. Consider the simple example. Suppose the process X is given by

$$dX_t = dW_t + a(\tilde{W}_t - X_t)dt,$$

with a nonzero constant a. The desired solution of the corresponding Sturm-Liouville equation is of the form

$$\Psi(t) = \frac{1}{2a}(e^{at} - e^{-at}).$$

Hence, the conditional expectation of $\tilde{W}_t$ with respect to $\mathcal{F}_t^X$ is given by

$$E[\tilde{W}_t|\mathcal{F}_t^X] = X_t - \frac{2}{e^{at}+e^{-at}}\int_0^t e^{-au}dX_u.$$

Therefore, the canonical decomposition of X_t has the form

$$X_t = B_t - \int_0^t \frac{2a}{e^{au}+e^{-au}}\left(\int_0^u e^{-av}dX_v\right)du,$$

where $(B_t)_{0\le t\le 1}$ is a Brownian motion relative to $(\mathcal{F}_t^X)_{0\le t\le 1}$.

2.3. Characterization of Brownian motions

In the last section we have shown how to compute the canonical decomposition of a Gaussian process with linear drift term. In the present section we are concerned with the applications of these theorems. Applying Theorem 2.1 we get the following characterization of Brownian motions.

THEOREM 2.2. *The process X satisfying* (2.13) *is a Brownian motion if and only if the square-integrable Volterra kernel H satisfies*

$$H(t,u) = -G(t,u),$$

where G is determined by (2.15).

PROOF. 1) Suppose X is a Wiener process with respect to its own filtration $(\mathcal{F}_t^X)$. By the uniqueness of the Doob-Meyer decomposition in $(\mathcal{F}_t^X)$ and our representation (2.14), we have $B = X$ and

$$\int_0^t (G(t,u) + H(t,u))\, dX_u = 0, \tag{2.33}$$

$\mathbb{P}$- a.s. for almost all t. But (2.33) implies

$$G(t,u) + H(t,u) = 0,$$

for almost all $u \le t$, since X is a Brownian motion.
2) Conversely, assume that $G(t,u) = -H(t,u)$. The canonical representation (2.14) can be written as

$$X_t = B_t + \int_0^t \left(\int_0^s G(s,u) dB_u - \int_0^s G(s,u) dX_u \right) ds.$$

This implies

$$X_t + \int_0^t \int_0^s G(s,u) dX_u ds = B_t + \int_0^t \int_0^s G(s,u) dB_u ds.$$

We can now apply the reconstruction argument in the proof of Proposition 1.2 to conclude $X = B$. In other words, X is a Brownian motion. □

Using this theorem we see that

$$X_t = W_t + \int_0^t \left(f(s)S_0 + \int_0^s F(s,u) dS_u - \int_0^s G(s,u) dX_u \right) ds, \tag{2.34}$$

with a square-integrable Volterra kernel G satisfying (2.15), is a Brownian motion with respect to its own filtration. From (2.34) we have

$$X_t + \int_0^t \int_0^s G(s,u) dX_u ds = \xi_t := W_t + \int_0^t \left(f(s)S_0 + \int_0^s F(s,u) dS_u \right) ds.$$

As in the proof of Proposition 1.2, the solution of (2.34) is given by

$$\begin{aligned} X_t &= \xi_t + \int_0^t \int_0^s R_G(s,u) d\xi_u ds \\ &= W_t + \int_0^t \int_0^s R_G(s,u) dW_u ds + \int_0^t \left(f(s) S_0 + \int_0^s F(s,u) dS_u \right) ds \\ &\quad + \int_0^t \int_0^s R_G(s,u) \left(f(u) S_0 + \int_0^u F(u,v) dS_v \right) du ds, \qquad (2.35) \end{aligned}$$

where R_G is the resolvent kernel of G. The next theorem characterizes those cases where a Brownian motion X with (2.34) is tied to the final value S_1 of the process S.

THEOREM 2.3. *Let X be a Brownian motion satisfying* (2.34) *with a square-integrable Volterra kernel G given by* (2.15). *This process converges to S_1 if and only if there exists a function $c(t)$ whose integral from* 0 *to* 1 *is equal to* 1 *such that*

$$f(t)V(0) + \int_0^t F(t,u)V'(u)du = c(t) + \int_0^t G(t,u)c(u)du.$$

PROOF. The formula (2.35) ensures the expectation of $X_t S_t$ is given by

$$\begin{aligned} E[X_t S_t] &= \int_0^t \left(f(s)V(0) + \int_0^s F(s,u)V'(u)du \right) ds \\ &\quad + \int_0^t \int_0^s R_G(s,u) \left(f(u)V(0) + \int_0^u F(u,v)V'(v)dv \right) du ds. \end{aligned}$$

Hence, we can calculate the value of

$$E[(X_t - S_t)^2] = E[X_t^2] + E[S_t^2] - 2E[X_t S_t] = t + V(t) - 2E[X_t S_t].$$

The process X_t converges to S_1 if and only if $E[(X_t - S_t)^2] \to 0$ as $t \to 1$. This implies that the necessary and sufficient condition for the process X tied to the final value S_1 is $E[X_1 S_1] = 1$. In other words,

$$\begin{aligned} &\int_0^1 \left(f(s)V(0) + \int_0^s F(s,u)V'(u)du \right) ds \\ &\quad + \int_0^1 \int_0^s R_G(s,u) \left(f(u)V(0) + \int_0^u F(u,v)V'(v)dv \right) du ds = 1. \end{aligned}$$

Let

$$\begin{aligned} c(s) &:= f(s)V(0) + \int_0^s F(s,u)V'(u)du \\ &\quad + \int_0^s R_G(s,u) \left(f(u) + \int_0^u F(u,v)V'(v)dv \right) du, \end{aligned}$$

we get the necessary and sufficient conditions for the convergence of X_t to S_1 are

$$\int_0^1 c(u)du = 1,$$

and

$$f(t)V(0) + \int_0^t F(t,u)V'(u)du = c(t) + \int_0^t G(t,u)c(u)du.$$

□

Using the above two theorems we can derive the following special case.

THEOREM 2.4. (i) *Suppose the process X satisfies*

$$dX_t = dW_t + (f(t)S_t + g(t)X_t)\,dt, \tag{2.36}$$

with initial value $X_0 = 0$, where f and g are two non-zero continuous functions satisfying

$$\int_0^t f^2(u)V(u)du < \infty,$$

for all $t < 1$ and $g \in \mathcal{A}(0,1)$. Then this process $(X_t)_{0\le t\le 1}$ is a Brownian motion if and only if it satisfies the stochastic differential equation

$$dX_t = dW_t + \frac{cS_t - c^2X_t}{V(t) - c^2t}dt, \tag{2.37}$$

with a constant c satisfying the integrability conditions

$$\frac{1}{V(u) - c^2u} \in \mathcal{A}(0,1) \cap L^1_{loc}([0,1)). \tag{2.38}$$

(ii) *If the variance function of S satisfies* (2.38) *with $c = 1$, i.e.,*

$$\frac{1}{V(u) - u} \in \mathcal{A}(0,1) \cap L^1_{loc}([0,1)), \tag{2.39}$$

then the process $(X_t)_{0\le t\le 1}$ given by the stochastic differential equation

$$dX_t = dW_t + \frac{S_t - X_t}{V(t) - t}dt, \tag{2.40}$$

is a standard Brownian motion. Furthermore, X_t converges to S_1 as $t \to 1$.

PROOF. 1) We want to show that (2.37) is the only possible form such that the process X with (2.36) is a Brownian motion with respect to its own filtration. It follows from Theorem 2.2 that the process X is a Brownian motion if and only if the function f and g satisfy

$$f(t)f(s)V(s) = g(t)(sg(s) - 1)$$

for $s \le t$. The associated solution of this equation is given by

$$f(t) = \frac{c}{V(t) - c^2t}, \qquad g(t) = \frac{-c^2}{V(t) - c^2t},$$

for some constant c.

2) Conversely, the solution of (2.37) is given by

$$\begin{aligned} X_t &= \int_0^t \exp\left(-\int_u^t \frac{c^2}{V(v)-c^2v}dv\right)dW_u \\ &\qquad + \int_0^t \exp\left(-\int_u^t \frac{c^2}{V(v)-c^2v}dv\right)\frac{cS_u}{V(u)-c^2u}du. \end{aligned} \tag{2.41}$$

Note that the stochastic integral is well-defined due to condition (2.38). In particular, if the condition (2.39) holds, the solution of (2.40) is given by

$$X_t = \int_0^t \exp\left(-\int_u^t \frac{1}{V(v)-v}dv\right)dW_u + \int_0^t \exp\left(-\int_u^t \frac{1}{V(v)-v}dv\right)\frac{S_u}{V(u)-u}du. \tag{2.42}$$

3) If X is given by (2.40), due to (2.42) we obtain

$$\begin{aligned} E[X_tS_t] &= \int_0^t \exp\left(-\int_u^t \frac{1}{V(v)-v}dv\right)\frac{E[S_uS_t]}{V(u)-u}du \\ &= \int_0^t \exp\left(-\int_u^t \frac{1}{V(v)-v}dv\right)du \\ &\qquad + \int_0^t \exp\left(-\int_u^t \frac{1}{V(v)-v}dv\right)\frac{u}{V(u)-u}du \\ &= t, \end{aligned}$$

using an integration by parts. Therefore,

$$E[(X_t-S_t)^2] = E[X_t^2] + E[S_t^2] - 2E[X_tS_t] = V(t) - t, \tag{2.43}$$

which is non-negative for all $t \leq 1$ and converges to 0 as $t \to 1$. □

Using this theorem, we obtain immediately the following result.

Corollary 2.4. *If there exists a positive constant k such that* (2.38) *holds, then for all $|c| \leq k$, the process X given by* (2.37) *is a Brownian motion.*

What kind of process S does satisfy the condition (2.39)? For a given Gaussian martingale S, which condition on the constant c will guarantee that condition (2.38) holds? In the following we will discuss some examples and some conditions on the process S and the constant c.

Lemma 2.4. (i) *If c satisfies* (2.38), *then*

$$c^2 \leq m := \inf_{0\leq t\leq 1}\left(\frac{V(u)}{u}\right). \tag{2.44}$$

(ii) *If $m < 1$, then* (2.38) *does not hold for $c = \pm\sqrt{m}$.*

Proof. 1) Let c be a constant satisfying

$$\inf_{0\leq u\leq 1}\left(\frac{V(u)}{u}\right) < c^2 < \sup_{0\leq u\leq 1}\left(\frac{V(u)}{u}\right).$$

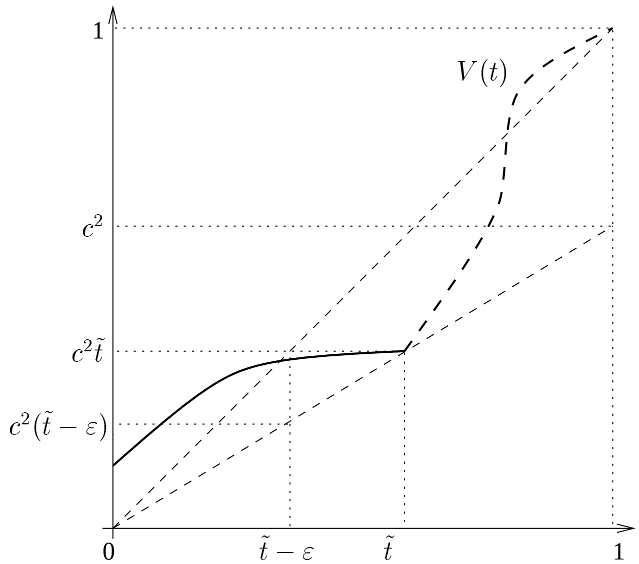

FIGURE 2.1

Then the curves $y = V(t)$ and $y = c^2 t$ intersect at at least one point on $(0,1)$. Let $\tilde{t} \in (0,1)$ be such a point, i.e., $V(\tilde{t}) = c^2\tilde{t}$. We distinguish three cases to discuss.
(a) $V(t) = c^2 t$ for all $t \in (\tilde{t}-\varepsilon, \tilde{t})$ with some $\varepsilon \in (0,\tilde{t})$: In this case the integral of $1/(V(u)-c^2u)$ in this neighborhood is equal to ∞. This implies

$$\int_0^{\tilde{t}} \frac{1}{V(u)-c^2u}du = \infty.$$

(b) $V(t) < c^2 t$ for all $t \in (\tilde{t}-\varepsilon, \tilde{t})$ with some $\varepsilon \in (0,\tilde{t})$: Let

$$t_1 := \max\{t < \tilde{t} : V(t) = c^2 t\}.$$

Since $V(t)$ is nonnegative and nondecreasing, $t_1 \geq 0$ and $c^2 t > V(t) > c^2 t_1$ for all $t \in (t_1, \tilde{t})$. Hence,

$$\int_{t_1}^{\tilde{t}} \frac{1}{V(u)-c^2u}du \leq \int_{t_1}^{\tilde{t}} \frac{1}{c^2(t_1-u)} = -\infty.$$

(c) $V(t) > c^2 t$ for all $t \in (\tilde{t}-\varepsilon, \tilde{t})$ with some $\varepsilon \in (0,\tilde{t})$: Therefore, the variance function $V(t)$ satisfies

$$c^2(\tilde{t}-\varepsilon) < c^2 u \leq V(u) \leq c^2\tilde{t}, \qquad \text{for all } \tilde{t}-\varepsilon < u \leq \tilde{t};$$

see Figure 2.1. Hence,

$$\int_{\tilde{t}-\varepsilon}^{\tilde{t}} \frac{1}{V(u)-c^2u}du \geq \int_{\tilde{t}-\varepsilon}^{\tilde{t}} \frac{1}{c^2\tilde{t}-c^2(\tilde{t}-\varepsilon)}du = \frac{1}{c^2}\int_{\tilde{t}-\varepsilon}^{\tilde{t}} \frac{1}{\varepsilon}du = \frac{1}{c^2},$$

for all $\varepsilon \in (0,\tilde{t})$. This implies

$$\int_0^{\tilde{t}} \frac{1}{V(u)-u}du = \infty,$$

i.e., the condition (2.38) obviously does not hold.

In summary, if condition (2.38) holds, the constant c must satisfy

$$c^2 \leq m := \inf_{0\leq u\leq 1}\left(\frac{V(u)}{u}\right) \qquad \text{or} \qquad c^2 \geq M := \sup_{0\leq u\leq 1}\left(\frac{V(u)}{u}\right).$$

2) Consider a constant c satisfying

$$c^2 \geq M := \sup_{0\leq u\leq 1}\left(\frac{V(u)}{u}\right).$$

We may assume $M < \infty$. Then $V(t) \leq Mt$ for all $t \in [0,1]$. This leads us to

$$\frac{1}{V(u)-c^2u} \geq \left(\frac{1}{M-c^2}\right)\frac{1}{u},$$

which is not integrable at time 0. Therefore, from 1) and 2) we get the assertion (i).
3) If $m < 1$, the curves $y = V(t)$ and $y = mt$ intersect at at least one point. Using the argument 1), we see that (2.38) does not hold. □

Therefore, (2.44) provides only a necessary condition for the integrability condition (2.38). There exist even such cases, that all constant c satisfying (2.44), but none of them, except 0, is valid for (2.38). For example, if S is a Brownian motion, i.e., $V(t) = t$, then for all $c \in \mathbb{R}$, the function $1/(V(t) - c^2t)$ is not integrable, but

$$\sup_{0\leq t\leq 1}\left(\frac{V(t)}{t}\right) = \inf_{0\leq t\leq 1}\left(\frac{V(t)}{t}\right) = 1.$$

LEMMA 2.5. (i) *If $V(t) > t$ for all $t < 1$ (this implies $S_0 \not\equiv 0$), then* (2.39) *holds.*
(ii) *If $V(t) = t^p$ for $0 < p < 1$, then* (2.39) *holds.*
(iii) *If S satisfies the condition* (2.39)*, then $t < V(t) \leq 1$ for all $t \in (0,1)$.*

PROOF. 1) Since $V(u) - u > 0$ for all $u \in [0,t]$, we have

$$V(u) - u \geq \inf_{v\in[0,t]}(V(v) - v) > 0,$$

for all $u \in [0,t]$. Therefore,

$$\begin{aligned}\int_0^t \frac{u}{(V(u)-u)^2}du &= \int_0^t \frac{1}{(V(u)-u)^2}du \leq \int_0^t \frac{1}{\inf_{u\in[0,t]}(V(u)-u)^2}du \\ &= \frac{t}{\inf_{u\in[0,t]}(V(u)-u)^2} < \infty,\end{aligned}$$

and

$$\int_0^t \frac{1}{V(u)-u}du \leq \frac{t}{\inf_{u\in[0,t]}(V(u)-u)} < \infty.$$

2) For $V(t) = t^p$ with $0 < p < 1$, we have

$$\int_0^t \frac{u}{(V(u)-u)^2}du = \int_0^t \frac{1}{u^{2p-1}-2u^p+u}du \leq c_1\int_0^t u^{-(2p-1)}du = c_2 t^{2(1-p)} < \infty,$$

and

$$\int_0^t \frac{1}{V(u)-u}du = \int_0^t \frac{1}{u^p - u}du \leq c_3 t^{1-p} < \infty,$$

where c_i, $i = 1, 2, 3$, are suitable positive constants.
3) From (2.43), we get $V(t) \geq t$, for all $t \in (0,1)$. If there exists $\tilde{t} \in (0,1)$ such that $V(\tilde{t}) = \tilde{t}$, using a similar argument as in the proof 1) (case (c)) of Remark 2.4 we get the desired results. □

REMARK 2.5. If we drop the condition $var(S_1) = 1$, and define

$$t_0 := \inf\{t \geq 0 : t \geq var(S_t)\},$$

then a similar argument of Theorem 2.4 shows that the process $(X_t)_{0 \leq t \leq t_0}$ satisfying (2.37) is a Brownian motion up to time t_0.

Applying Theorem 2.4, Remark 2.5 and above discussion, we can provide the following examples and applications.

EXAMPLE 2.3. Setting $S_t \equiv S_1 \sim N[0,1]$, then from Theorem 2.4 we get that the process X with

$$dX_t = dW_t + \frac{S_1 - X_t}{1-t}dt,$$

is a Brownian motion and converges to S_1. This example is of course well-known, see, e.g., Jeulin-Yor [**36**].

EXAMPLE 2.4. If the martingale $(S_t)_{0 \leq t \leq 1}$ is given by

$$S_t := \frac{\sqrt{\delta}N + \int_0^t g(u)d\tilde{W}_u}{\sqrt{\delta + \int_0^1 g^2(u)du}},$$

where $N \sim N(0,1)$, $\delta > 0$, $g \in L^2(0,1)$ and $\tilde{W}$ is a Wiener process independent of W, condition in Lemma 2.5 (i) amounts to

$$\int_t^1 g^2(u)du < (1-t)\left(\delta + \int_0^1 g^2(u)du\right).$$

Under this condition the process X given

$$dX_t = dW_t + \frac{(\delta + \int_0^1 g^2(u)du)(S_t - X_t)}{\delta + \int_0^t g^2(u)du - (\delta + \int_0^1 g^2(u)du)t}dt,$$

is a Brownian motion converging to S_1.

EXAMPLE 2.5. Suppose the variance function $V(t)$ of the Gaussian martingale S is $\frac{1}{4}(2t^2 + t + 1)$. Then we have

$$V(t)\begin{cases} < t, & \text{if } t > \frac{1}{2}, \\ > t, & \text{if } t < \frac{1}{2}. \end{cases}$$

Hence, $V(t)$ does not satisfy (2.39). It is easy to check that

$$m = \inf_{0\le t\le 1}\left(\frac{\frac{1}{4}(2t^2+t+1)}{t}\right) = \frac{1+2\sqrt{2}}{4}.$$

This implies the condition (2.38) is valid, if $c^2 < (1+2\sqrt{2})/4$ (for $c^2 = (1+2\sqrt{2})/4$, the condition (2.38) does not hold). Hence, the process $(X_t)_{0\le t\le 1}$ satisfying the stochastic differential equation

$$dX_t = dW_t + \frac{cS_t - c^2X_t}{\frac{1}{2}t^2 + (\frac{1}{4}-c^2)t + \frac{1}{4}}dt,$$

is a Brownian motion, for $c^2 < (1+2\sqrt{2})/4$.

In the following we consider a process X satisfying a linear stochastic functional differential equation driven by two independent Brownian motions W and $\tilde{W}$. By Corollary 2.2 and Theorem 2.2 we get the following criterion for X to be a Brownian motion.

COROLLARY 2.5 (Theorem 5.1 of Föllmer-Wu-Yor [**23**]). *A process X satisfying*

$$dX_t = dW_t + \left(\int_0^t F(t,u)d\tilde{W}_u + \int_0^t H(t,u)dX_u\right)dt,$$

is a Wiener process with respect to its own filtration $(\mathcal{F}_t^X)$ if and only if $H(t,s) = -G_F(t,s)$, where G_F is the square-integrable Volterra kernel determined by

$$G_F(t,s) + \int_0^s G_F(t,u)G_F(s,u)du = \int_0^s F(t,u)F(s,u)du.$$

Using a similar argument as in the proof of Proposition 2.2, we get the following conclusion.

LEMMA 2.6. *The unique solution of the equation*

$$X_t = W_t + \int_0^t\left(\int_0^s F(s,u)d\tilde{W}_u - \int_0^s G_F(s,u)dX_u\right)ds$$

is given by

$$X_t = W_t + \int_0^t\left\{\int_0^s L_F(s,u)dW_u + \int_0^s (F(s,u) + \int_u^s L_F(s,v)F(v,u)dv)d\tilde{W}_u\right\}ds.$$

Here L_F is the resolvent kernel of G_F, i.e., L_F satisfies the equation

$$\begin{cases} G_F(t,s) + L_F(t,s) + \int_s^t L_F(t,u)G_F(u,s)du = 0, \\ G_F(t,s) + L_F(t,s) + \int_s^t G_F(t,u)L_F(u,s)du = 0, \end{cases}$$

for $s \le t$.

As in the last section we want to look at some special cases. Let $F(t,s) = f(t)g(s)$ with some functions $f, g \in C^1(0,1)$ satisfying

$$\int_0^t \int_0^u f^2(u)g^2(v)dvdu < \infty,$$

for all $t < 1$. Then we obtain immediately from Corollary 2.3, Corollary 2.5 and (2.30) the following result.

COROLLARY 2.6 (Corollary 5.1 and 5.2 of Föllmer-Wu-Yor [**23**]). *Suppose the process $(X_t)_{0\le t\le 1}$ satisfies*

$$dX_t = dW_t + \left(f(t) \int_0^t g(u)d\tilde{W}_u + \int_0^t H(t,u)dX_u \right) dt,$$

with $f, g \in C^1(0,1)$ satisfying (2.26) and $f \neq 0$, a.s. and a square-integrable Volterra kernel $H(t,s)$. Then X is a Brownian motion if and only if

$$H(t,u) = -f(t)\alpha(u),$$

where $\alpha(t)$ is the solution of (2.29) with initial condition $\alpha(0) = 0$. In other words, if $(X_t)_{0\le t\le 1}$ is a Brownian motion with respect to its own filtration, it must be of the form

$$dX_t = dW_t + f(t) \left(\int_0^t g(u)d\tilde{W}_u - \int_0^t \alpha(u)dX_u \right) dt. \tag{2.45}$$

In particular, if $g \equiv 1$, (2.45) can be written as

$$dX_t = dW_t + f(t) \left(\tilde{W}_t - \int_0^t \frac{f(u)\Psi(u)}{\Psi'(u)} dX_u \right) dt, \tag{2.46}$$

where $\Psi(t)$ satisfies (2.31) with initial conditions $\Psi(0) = 0$ and $\Psi'(0+) = 1$.

REMARK 2.6. If X satisfies the stochastic differential equation

$$dX_t = dW_t + (f(t)\tilde{W}_t + h(t)X_t)dt, \tag{2.47}$$

with $f, h \in C^1(0,1) \cap \mathcal{A}(0,1)$, and if one of the functions $f(t)$ and $h(t)$ is not identically 0, then X cannot be a Brownian motion.

PROOF. Suppose X is a Brownian motion satisfying (2.47). Then from (2.46) we know that

$$\int_0^t \frac{f(u)\Psi(u)}{\Psi'(u)} dX_u = cX_t,$$

for some constant c. Consequently, we see that

$$f(t)\Psi(t) = c\Psi'(t). \tag{2.48}$$

Substituting (2.48) into (2.31), we get

$$\Psi''(t) = cf(t)\Psi'(t).$$

Hence, the corresponding solution of the Sturm-Liouville equation is given by

$$\Psi(t) = \int_0^t \exp\left(c\int_0^u f(v)dv\right) du.$$

Substituting this solution again in (2.48), and taking derivatives on both sides with respect to t, we have

$$cf'(t) + (1-c^2)f^2(t) = 0,$$

whose solution is of the form

$$f(t) = \frac{c}{1-c^2}\frac{1}{t}.$$

If $c \neq 0$, this function f does not belong to the class $\mathcal{A}(0,1)$. □

In the above remark it has been shown that any process satisfying (2.47) with $f, h \in C^1(0,1) \cap \mathcal{A}(0,1)$, cannot be a Brownian motion unless $f \equiv h \equiv 0$. But as we shall see in the next chapter, there *does* exist a Brownian motion satisfying the stochastic differential equation (2.47). The main difference is that the functions f and h which we are going to discuss in Chapter 3 do not belong to the class $\mathcal{A}(0,1)$.

Clearly, the condition (2.39) does not hold, if $(S_t)_{0\le t\le 1}$ is a standard Brownian motion $\tilde{W}$. In fact, the following proposition in Föllmer-Wu-Yor [**23**] shows that a processes X of the form

$$X_t = W_t + \int_0^t Y_s ds \tag{2.49}$$

cannot converge to $\tilde{W}_1$, if Y is adapted to $(\mathcal{F}_t^{W,\tilde{W}})$.

PROPOSITION 2.3 (Proposition 5.1 of Föllmer-Wu-Yor [**23**]). *Let X be a Brownian motion of the form* (2.49) *with a drift term $(Y_t)_{0\le t\le 1}$ adapted to $(\mathcal{F}_t)$. If Z is any $(\mathcal{F}_t)$-Brownian motion such that $X_1 = Z_1, \mathbb{P}$-a.s., then we have $Z_t = X_t = W_t$, and in particular, $Y_t = 0$, $dt \times d\mathbb{P}$-a.s..*

2.4. Some examples of Brownian motions

At the end of this chapter we want to give some examples of Brownian motions of the form (2.13).

EXAMPLE 2.6. Let $k \ge 2$ and

$$f(t) = \frac{\sqrt{k(k+1)t^{2k} + k(k+1)^2t^k}}{t^{k+1} + (k+1)t}.$$

Then the solution of the Sturm-Liouville equation is given by

$$\Psi(t) = \frac{1}{k+1}t^{k+1} + t.$$

This implies that the process X satisfying

$$dX_t = dW_t + \frac{\sqrt{k(k+1)t^{2k} + k(k+1)^2t^k}}{t^{k+1} + (k+1)t}\left(\tilde{W}_t - \sqrt{\frac{k}{k+1}}\int_0^t \frac{\sqrt{u^{2k} + (k+1)u^k}}{u^k+1}dX_u\right) dt$$

is a Brownian motion. For the case $k = 2$, we see that the process X satisfying

$$dX_t = dW_t + \frac{\sqrt{6t^2+18}}{t+3}\left(\tilde{W}_t - \sqrt{\frac{2}{3}}\int_0^t \frac{\sqrt{u^4+3u^2}}{u^2+1} dX_u\right) dt$$

is a Brownian motion with respect to its natural filtration.

EXAMPLE 2.7. Consider the case $t \in [0,1)$. Let

$$f(t) = \frac{\pi}{\sqrt{2}} \sec\left(\frac{\pi}{2}t\right).$$

Then the solution of the corresponding Sturm-Liouville equation is given by

$$\Psi(t) = \tan\left(\frac{\pi}{2}t\right).$$

Hence, the process $(X_t)_{0\le t\le 1}$ starting in $X_0 = 0$ and satisfying

$$dX_t = dW_t + \frac{\pi}{\sqrt{2}} \sec\left(\frac{\pi}{2}t\right)\left(\tilde{W}_t - \sqrt{2}\int_0^t \sin\left(\frac{\pi}{2}u\right) dX_u\right) dt,$$

is a Brownian motion.

EXAMPLE 2.8. Setting $\delta \equiv 1$ and $g(t) \equiv c$ in Example 2.4, we see that the process X satisfying

$$dX_t = dW_t + \frac{(1+c^2)(N + c\tilde{W}_t - \sqrt{1+c^2}X_t)}{\sqrt{1+c^2}(1-t)} dt$$

is a Brownian motion.

In the end, we want to give an example of process X of the form

$$dX_t = dW_t + \left(f(t)\tilde{W}_1 + \int_0^t F(t,u)d\tilde{W}_u + \int_0^t H(t,u)dX_u\right) dt,$$

with initial value $X_0 = 0$, $f \in C^1(0,1) \cap L^2(0,1)$ and square-integrable Volterra kernels $F, H \in C^{1,1}((0,1)\times(0,1))$. We are going to apply Theorem 2.4 together with Remark 2.5, and for this we need the following lemma.

LEMMA 2.7. *Suppose the process S is given by*

$$S_t := f(t)\tilde{W}_1 + \int_0^t F(t,u)d\tilde{W}_u,$$

with $f \in L^2(0,1) \cap C^1(0,1)$ and a square-integrable Volterra kernel $F \in C^{1,1}((0,1)\times(0,1))$. Then the following three statements are equivalent:

(i) *S is a martingale with respect to the filtration $(\mathcal{F}_t^{\tilde{W}} \vee \sigma(\tilde{W}_1))$.*

(ii) *The relation between f and F is given by*

$$F(t,u) = -f(t) + (f(u) - f'(u)(1-u)), \tag{2.50}$$

i.e., S is given by

$$S_t = f(t)(\tilde{W}_1 - \tilde{W}_t) + \int_0^t (f(u) - (1-u)f'(u))d\tilde{W}_u. \tag{2.51}$$

(iii) *f and F are given by*

$$\begin{cases} F(t,u) = m(t) + n(u), \\ \\ f(t) = c - m(t), \end{cases} \tag{2.52}$$

where c, k are constants and $m(t)$, $n(t)$ satisfy

$$(1-t)m(t) - \int_0^t n(u)du = k. \tag{2.53}$$

PROOF. (i) ⇒ (ii): Let

$$\mathcal{G}_t := \mathcal{F}_t^{\tilde{W}} \vee \sigma(\tilde{W}_1).$$

Suppose S is a martingale, then

$$\begin{aligned} 0 &= E[S_t - S_s|\mathcal{G}_s] \\ &= E\left[(f(t)-f(s))\tilde{W}_1 + \left(\int_0^t F(t,u)d\tilde{W}_u - \int_0^s F(s,u)d\tilde{W}_u\right)\middle| \mathcal{G}_s\right] \\ &= (f(t)-f(s))\tilde{W}_1 + \int_0^s (F(t,u)-F(s,u))d\tilde{W}_u + F(t,t)E[\tilde{W}_t|\mathcal{G}_s] \\ &\qquad -F(t,s)\tilde{W}_s - \int_s^t F_2(t,u)E[\tilde{W}_u|\mathcal{G}_s]du. \end{aligned} \tag{2.54}$$

Since $\tilde{W}$ is a Brownian motion, we have

$$E[\tilde{W}_t|\mathcal{G}_s] = E[\tilde{W}_t - \tilde{W}_s|\mathcal{G}_s] + \tilde{W}_s = \frac{t-s}{1-s}(\tilde{W}_1 - \tilde{W}_s) + \tilde{W}_s. \tag{2.55}$$

Substituting this formula in (2.54), we see

$$\begin{aligned} 0 &= (f(t)-f(s))\tilde{W}_1 + \int_0^s (F(t,u)-F(s,u))d\tilde{W}_u + \frac{1}{1-s}\int_s^t F(t,u)du(\tilde{W}_1 - \tilde{W}_s) \\ &= (f(t)-f(s) + \frac{1}{1-s}\int_s^t F(t,u)du)(\tilde{W}_1 - \tilde{W}_t) + (f(t)-f(s))\tilde{W}_t \\ &\qquad + \frac{1}{1-s}\int_s^t F(t,u)du(\tilde{W}_t - \tilde{W}_s) + \int_0^s (F(t,u)-F(s,u))d\tilde{W}_u. \end{aligned}$$

From the fact that $\tilde{W}_1 - \tilde{W}_t$ and $\tilde{W}_u$ are independent for all $u \le t$, we know that

$$f(t) - f(s) + \frac{1}{1-s}\int_s^t F(t,u)du = 0.$$

Differentiating with respect to s , we get (2.50).
(ii) ⇒ (iii): Due to (2.50) we obtain the first equation in (2.52). Therefore,

$$m(t) + f(t) = f(u) - f'(u)(1-u) - n(u) = \text{ constant } c,$$

for all $u \le t$. Hence,

$$f(u) - f'(u)(1-u) - n(u) = c,$$

with $f(u) = c - m(u)$, and it implies (2.53).

(iii) ⇒ (i): If F and f are of the form (2.52), we have to check that $E[S_t - S_s|\mathcal{G}_s] = 0$ for all $s \le t$. From (2.54) and (2.55), we see that

$$\begin{aligned} & E[S_t - S_s|\mathcal{G}_s] \\ = \;& (f(t) - f(s))\tilde{W}_1 + \int_0^s (F(t,u) - F(s,u))d\tilde{W}_u + \frac{1}{1-s}\int_s^t F(t,u)du(\tilde{W}_1 - \tilde{W}_s). \end{aligned}$$

Substituting (2.52) and (2.53) in this formula, we get the desired result. □

This result can be seen as a special case of Theorem 3.4 in Amendinger [**4**].

EXAMPLE 2.9. The process $(S_t)_{0\le t\le 1}$ satisfying

$$\begin{aligned} S_t &= \tilde{W}_t - \int_0^t \frac{\tilde{W}_1 - \tilde{W}_u}{1-u}du \\ &= \log(1-t)\tilde{W}_1 + \int_0^t (1 + \log(1-u) - \log(1-t))d\tilde{W}_u, \end{aligned}$$

is a martingale with respect to $(\mathcal{G}_t)$. Furthermore, it is a Brownian motion.

Due to Lemma 2.7 and Remark 2.5 we get the following proposition.

PROPOSITION 2.4. *Let $f \in L^1(0,T)$ for all $T < \infty$ and*

$$t_o := \inf\left\{t \ge 0 : f^2(t)(1-t) + \int_0^t (f(u) - (1-u)f'(u))^2 du \le t\right\}.$$

Then the process X starting in $X_0 = 0$ and satisfying the stochastic differential equation

$$dX_t = dW_t + \frac{f(t)(\tilde{W}_1 - \tilde{W}_t) + \int_0^t (f(u) - (1-u)f'(u))d\tilde{W}_u - X_t}{f^2(t)(1-t) + \int_0^t (f(u) - (1-u)f'(u))^2 du - t}dt$$

is a Brownian motion on $[0, t_0]$.

PROOF. Substituting (2.51) into Remark 2.5, we get the desired result. □

EXAMPLE 2.10. (i) For the case $f(t) \equiv c > 0$, the process $(X_t)_{0\le t\le c^2}$ with $X_0 = 0$ and

$$dX_t = dW_t + \frac{c\tilde{W}_1 - X_t}{c^2 - t}dt,$$

is a Brownian motion. Furthermore, X_t converges to $c\tilde{W}_1$ as $t \to c^2$.
(ii) For $f(t) = 1 + t$, we see that

$$t_0 = \inf\left\{t \ge 0 : \frac{1}{3}t^3 - t^2 + 1 \le 0\right\} = 1 - 2\cos\left(\frac{5}{9}\pi\right) \approx 1.3473.$$

Hence, the process X satisfying

$$dX_t = dW_t + \frac{(1+t)(\tilde{W}_1 - \tilde{W}_t) + 2\int_0^t u d\tilde{W}_u - X_t}{\frac{1}{3}t^3 - t^2 + 1}dt,$$

is a Brownian motion up to time t_0.

CHAPTER 3

Orthogonal decompositions of Brownian filtrations

Consider a process X satisfying the stochastic differential equation

$$dX_t = dW_t + (f(t)\tilde{W}_t + h(t)X_t)dt, \tag{3.1}$$

with $X_0 = 0$, where W, $\tilde{W}$ are two independent Wiener processes. We have seen in Remark 2.6 that X cannot be a Brownian motion if f and h belong to the space $\mathcal{A}(0,1) \cap C(0,1)$ with $f^2 + h^2 \not\equiv 0$. In this chapter, we will show that there exist Brownian motions of the form (3.1) if the conditions on f and h are relaxed.

Following Yor [**57**] we describe in Section 3.1 a basic orthogonal decomposition of the Brownian filtration. In Section 3.2 we will construct a Brownian motion X of the form (3.1), and then another Brownian motion Y which is represented in terms of W and $\tilde{W}$ and is independent of X. Using iteration, we get two sequences of Brownian motions $X^{(n)}$ and $Y^{(n)}$, which are independent of each other. This leads to the construction of new orthogonal decompositions of Brownian filtrations; see Wu-Yor [**56**]. In Section 3.3 a similar decomposition of a Brownian motion related to X will be investigated. In Section 3.4 we replace X by W on the right-hand side of (3.1) and characterize Brownian motions of the form

$$X_t = W_t + \int_0^t (f(u)\tilde{W}_u + g(u)W_u)du.$$

3.1. The basic example of an orthogonal decomposition

Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. In Jeulin-Yor [**37**] and Chapter 1 of Yor [**57**] it has been shown that the natural filtration generated by $(B_t)_{t\geq 0}$ can be decomposed into the direct sum of two independent σ-algebras

$$\mathcal{F}_t^B = \mathcal{G}_t \oplus \sigma(B_t) \tag{3.2}$$

for all $t \geq 0$, where the σ-algebra $\mathcal{G}_t$ is given by

$$\begin{aligned}
\mathcal{G}_t &:= \sigma\left(B_u - \frac{u}{t}B_t; u \leq t\right) \\
&= \sigma\left(\int_0^t f(u)dB_u; f \in L^2[0,t], \int_0^t f(u)du = 0\right) \\
&= \sigma\left(B_u - \int_0^u \frac{B_t - B_v}{t - v}dv; u \leq t\right) \\
&= \sigma\left(B_u - \int_0^u \frac{B_v}{v}dv; u \leq t\right).
\end{aligned}$$

Define an operator T as

$$T(B)_t := B_t - \int_0^t \frac{B_u}{u} du. \tag{3.3}$$

It has been established that the process $(T(B)_t)_{t\geq 0}$ is a Brownian motion; see Deheuvels [**21**] and Chapter 1 in Yor [**57**]. For the sake of convenience, we write $T^0(B) = B$. Consequently, for any non-negative integer n, the process $(T^n(B)_t)_{t\geq 0}$ is a Brownian motion relative to its natural filtration. Using this notation we can rewrite the decomposition (3.2) as

$$\mathcal{F}_t^B = \sigma(B_t) \oplus \sigma(T(B)_u; u \leq t).$$

Using the same argument as above iteratively, we can get an orthogonal decomposition of σ-algebra $\mathcal{F}_t^B$ in the following form:

$$\begin{aligned} \mathcal{F}_t^B &= \sigma(B_t) \oplus \sigma(T(B)_t) \oplus \sigma(T^2(B)_u; u \leq t) \\ &= \sigma(B_t) \oplus \sigma(T(B)_t) \oplus \sigma(T^2(B)_t) \oplus \sigma(T^3(B)_u; u \leq t) = \cdots . \end{aligned} \tag{3.4}$$

From Yor [**57**] Chapter 1, we know that the random variable $T^n(B)_1$ can be represented as

$$T^n(B)_1 = \int_0^1 L_n(\log(\frac{1}{u}))dB_u,$$

where $(L_n(u))_{n\geq 0}$ is the classical Laguerre polynomials defined as

$$L_n(u) = \sum_{k=0}^n \binom{n}{k} \frac{(-u)^k}{k!}, \tag{3.5}$$

which is a sequence of orthonormal polynomials for the measure $e^{-u}du$ in $\mathbb{R}^+$; in other words, for $m, n \in \mathbb{N} \cup \{0\}$,

$$\int_0^\infty L_m(u)L_n(u)e^{-u}du = \delta_{m,n}.$$

REMARK 3.1. Since $(t^{-\frac{1}{2}}T^n(B)_t)_{n\in\mathbb{N}}$ is an orthonormal system in $L^2(\mathbb{P})$, we conclude that for fixed $t \geq 0$, the sequence $(T^n(B)_t)_{n\geq 0}$ is not strongly L^2-convergent, but converges weakly to 0 in L^2.

3.2. Construction of orthogonal decompositions of Brownian filtrations

In this section we want to construct a Brownian motion satisfying the stochastic differential equation

$$dX_t = dW_t + (f(t)\tilde{W}_t + g(t)X_t)dt, \tag{3.6}$$

where f and g satisfy some integral conditions (which we will discuss later). This will involve some orthogonal decompositions of the Brownian filtration similar to (3.4).

Let W, $\tilde{W}$ be two independent Wiener processes. Consider the process X satisfying the stochastic differential equation

$$dX_t = dW_t + \frac{c\tilde{W}_t - c^2 X_t}{(1-c^2)t} dt, \tag{3.7}$$

with $X_0 = 0$ and some constant $|c| < 1$. If $c = 0$, then $X_t = W_t$. If $c \neq 0$, the solution to this equation is given by

$$X_t = \int_0^t \left(\frac{u}{t}\right)^a dW_u + \frac{1}{c}\int_0^t \left(1 - \left(\frac{u}{t}\right)^a\right) d\tilde{W}_u, \tag{3.8}$$

with the constant a defined by

$$a := \frac{c^2}{1-c^2}.$$

Using this formula, we see that X satisfies the identity

$$dX_t = dW_t - at^{-a-1}\int_0^t u^a dW_u dt + \left(\frac{a}{c}\right) t^{-a-1}\int_0^t u^a d\tilde{W}_u dt. \tag{3.9}$$

PROPOSITION 3.1. *For all constant $0 \leq |c| < 1$ the process X satisfying the stochastic differential equation* (3.7) *is a Brownian motion with respect to its own filtration $(\mathcal{F}_t^X)$.*

PROOF. For the case $c = 0$ it is clear, since $X = W$ is a Brownian motion. For $0 < |c| < 1$ and $s \leq t$, we compute the covariance function of X using (3.8)

$$E[X_s X_t] = \int_0^s \left(\frac{u}{s}\right)^a \left(\frac{u}{t}\right)^a du + \frac{1}{c^2}\int_0^s \left(1 - \left(\frac{u}{s}\right)^a\right)\left(1 - \left(\frac{u}{t}\right)^a\right) du = s.$$

This ensures that the Gaussian process X is a standard Brownian motion with respect to its natural filtration. □

REMARK 3.2. In Remark 2.6 we saw that the solution of (3.6) cannot be a Brownian motion, provided that $f, g \in C^1(0,1) \cap \mathcal{A}(0,1)$. However, the function $1/t$ does not belong to $\mathcal{A}(0,1)$, but to $\bar{\mathcal{A}}(0,1)$, where $\bar{\mathcal{A}}(0,T)$ is defined by

$$\bar{\mathcal{A}}(0,T) := \{\varphi(u) : \int_0^t \sqrt{u}|\varphi(u)|du < \infty \text{ for all } t < T\}.$$

In the following discussion of (3.7), we will always exclude the trivial case $c = 0$.

Now, we want to construct a new Brownian motion from W and $\tilde{W}$ which is independent of X. Our first attempt is a Brownian motion, say $\tilde{Y}$, of the form

$$d\tilde{Y}_t = d\tilde{W}_t + \frac{-cW_t - c^2\tilde{Y}_t}{(1-c^2)t} dt.$$

We can easily check that X_t and $\tilde{Y}_t$ are independent for all t. But the processes X and $\tilde{Y}$ are not independent. Hence we have to look for other Brownian motions which might be independent of X. The following proposition gives us one example.

PROPOSITION 3.2. *The process Y satisfying the stochastic differential equation*

$$dY_t = d\tilde{W}_t + \frac{cW_t - (1-c^2)\tilde{W}_t - c^2 Y_t}{(1-c^2)t}dt, \tag{3.10}$$

is a Brownian motion independent of X.

PROOF. The solution of (3.10) is given by

$$Y_t = \frac{1}{c}\int_0^t \left(1 - \left(\frac{u}{t}\right)^a\right) dW_u - \int_0^t \left(\frac{1}{a} - \frac{a+1}{a}\left(\frac{u}{t}\right)^a\right) d\tilde{W}_u. \tag{3.11}$$

Then for $s \le t$, it can be shown that $E[Y_s Y_t] = s$. It means that Y is a Brownian motion. Furthermore, we have $E[X_s Y_t] = E[X_t Y_s] = 0$, for all $s \le t$. This implies X and Y are independent. □

REMARK 3.3. At the beginning of Section 3.1 we saw that the process $(T(\tilde{W})_t)_{t\ge 0}$ defined by

$$T(\tilde{W})_t = \tilde{W}_t - \int_0^t \frac{\tilde{W}_u}{u}du,$$

is a Brownian motion and that its natural filtration $(\mathcal{F}_t^{T(\tilde{W})})$ is strictly smaller than $(\mathcal{F}_t^{\tilde{W}})$. Using this notation, (3.10) can be written in the form

$$dY_t = dT(\tilde{W})_t + \frac{cW_t - c^2 Y_t}{(1-c^2)t}dt.$$

Since W and $\tilde{W}$ are independent, the processes $T(\tilde{W})$ and W are clearly also independent. In the same way, we know that the process $(T(W)_t)_{t\ge 0}$ is a Brownian motion independent of $\tilde{W}$ as well as $T(\tilde{W})$, and that $\mathcal{F}_t^{T(W)} \subsetneqq \mathcal{F}_t^{W}$. Using again the same argument as in Proposition 3.2, we know that the process $\tilde{X}$ satisfying

$$d\tilde{X}_t = dT(W)_t + \frac{cT(\tilde{W})_t - c^2 \tilde{X}_t}{(1-c^2)t}dt, \tag{3.12}$$

is a Brownian motion independent of Y. Looking at the processes X and $\tilde{X}$, we see that the equations (3.12) and (3.7) have the same form. Only the σ-algebras generated by the driving Brownian motions $T(W)$ and $T(\tilde{W})$ are strictly smaller than those generated by W and $\tilde{W}$, respectively. Hence, we get also $\mathcal{F}_t^{\tilde{X}} \subseteq \mathcal{F}_t^{X}$. Furthermore, from

$$\begin{aligned}\tilde{X}_s &= \int_0^s \left(\frac{u}{s}\right)^a dT(W)_u + \frac{1}{c}\int_0^s \left(1 - \left(\frac{u}{s}\right)^a\right) dT(\tilde{W})_u \\ &= \int_0^s \left(\frac{u}{s}\right)^a \left(dW_u - \frac{W_u}{u}du\right) + \frac{1}{c}\int_0^s \left(1 - \left(\frac{u}{s}\right)^a\right)\left(d\tilde{W}_u - \frac{\tilde{W}_u}{u}du\right),\end{aligned}$$

we deduce

$$\begin{aligned} E[X_t\tilde{X}_s] &= \int_0^t \left(\frac{u}{s}\right)^a \left(\frac{u}{t}\right)^a du - \int_0^s \frac{1}{u}\left(\frac{u}{s}\right)^a \int_0^u \left(\frac{v}{t}\right)^a dvdu \\ &+\frac{1}{c^2}\int_0^t \left(1-\left(\frac{u}{s}\right)^a\right)\left(1-\left(\frac{u}{t}\right)^a\right) du \\ &-\frac{1}{c^2}\int_0^s \frac{1}{u}\left(1-\left(\frac{u}{s}\right)^a\right)\int_0^u \left(1-\left(\frac{v}{t}\right)^a\right) dvdu \\ &= 0, \end{aligned}$$

for all $s \leq t$. This implies $\mathcal{F}_t^{\tilde{X}} \subsetneqq \mathcal{F}_t^X$.

Iterating this procedure, we define

$$X_t^{(n)} := \int_0^t \left(\frac{u}{t}\right)^a dT^n(W)_u + \frac{1}{c}\int_0^t \left(1-\left(\frac{u}{t}\right)^a\right) dT^n(\tilde{W})_u, \tag{3.13}$$

and

$$Y_t^{(n)} := \frac{1}{c}\int_0^t \left(1-\left(\frac{u}{t}\right)^a\right) dT^n(W)_u + \int_0^t \left(\frac{u}{t}\right)^a dT^{n+1}(\tilde{W})_u, \tag{3.14}$$

for $n \geq 0$ and $0 < |c| < 1$. In other words, the processes $X^{(n)}$ and $Y^{(n)}$ satisfy the stochastic differential equations

$$dX_t^{(n)} = dT^n(W)_t + \frac{cT^n(\tilde{W})_t - c^2X_t^{(n)}}{(1-c^2)t}dt, \tag{3.15}$$

and

$$dY_t^{(n)} = dT^{n+1}(\tilde{W})_t + \frac{cT^n(W)_t - c^2Y_t^{(n)}}{(1-c^2)t}dt. \tag{3.16}$$

From Proposition 3.2 and Remark 3.3 we know that for each $n \geq 0$ the processes $X^{(n)}$ and $Y^{(n)}$ are Brownian motions and

$$\mathcal{F}_t^X = \mathcal{F}_t^{X^{(0)}} \supsetneqq \mathcal{F}_t^{X^{(1)}} \supsetneqq \cdots \supsetneqq \mathcal{F}_t^{X^{(n)}} \supsetneqq \cdots,$$

$$\mathcal{F}_t^Y = \mathcal{F}_t^{Y^{(0)}} \supsetneqq \mathcal{F}_t^{Y^{(1)}} \supsetneqq \cdots \supsetneqq \mathcal{F}_t^{Y^{(n)}} \supsetneqq \cdots.$$

Furthermore, the processes $X^{(n)}$ and $Y^{(n)}$, $Y^{(n)}$ and $X^{(n+1)}$ are mutually independent. The next lemma provides a representation of $X^{(n)}$ and $Y^{(n)}$ as stochastic integrals with respect to W and $\tilde{W}$.

Lemma 3.1. *The processes $X^{(n)}$ and $Y^{(n)}$ can be represented as*

$$\begin{aligned} X_t^{(n)} &= \int_0^t p^{(n)}(\log\frac{t}{u})dW_u + \frac{1}{c}\int_0^t q^{(n)}(\log\frac{t}{u})d\tilde{W}_u, && (3.17) \\ Y_t^{(n)} &= \frac{1}{c}\int_0^t q^{(n)}(\log\frac{t}{u})dW_u + \int_0^t p^{(n+1)}(\log\frac{t}{u})d\tilde{W}_u, && (3.18) \end{aligned}$$

where functions $p^{(n)}(u)$ and $q^{(n)}(u)$ satisfy the recurrence relation

$$\gamma^{(n+1)}(u) = \gamma^{(n)}(u) - \int_0^u \gamma^{(n)}(v)dv,$$

with initial conditions $p^{(0)}(u) = e^{-au}$ *and* $q^{(0)}(u) = 1 - e^{-au}$. *More explicitly,* $p^{(n)}(u)$ *and* $q^{(n)}(u)$ *can be represented in the following form:*

$$p^{(n)}(u) = -\frac{1}{a}\sum_{k=0}^{n-1} r_k^n \frac{(-u)^k}{k!} + \left(\frac{a+1}{a}\right)^n e^{-au}, \tag{3.19}$$

and

$$q^{(n)}(u) = L_n(u) - p^{(n)}(u), \tag{3.20}$$

where the sequence (r_k^n) *satisfies the recurrence relation:*

$$\begin{cases} r_0^{n+1} = r_0^n + \left(\dfrac{a+1}{a}\right)^n, & \forall\, n \geq 0, \\ r_n^{n+1} = 1, & \forall\, n \geq 1, \\ r_k^{n+1} = r_k^n + r_{k-1}^n, & \forall\, 0 < k < n, \\ r_q^p \equiv 0, & for\ p \leq q, \end{cases} \tag{3.21}$$

and $(L_n(u))_{n \geq 0}$ *is the sequence of Laguerre polynomials given by* (3.5).

PROOF. Let $p^{(0)}(u) = e^{-au}$ and $q^{(0)}(u) = 1 - e^{-au}$. From (3.13) and the stochastic Fubini Theorem (see, e.g., Protter [**51**]) we have

$$\begin{aligned} X_t^{(n)} &= \int_0^t p^{(0)}(\log\frac{t}{u})dT^n(W)_u + \frac{1}{c}\int_0^t q^{(0)}(\log\frac{t}{u})dT^n(\tilde{W})_u \\ &= \int_0^t \left(p^{(0)}(\log\frac{t}{u}) - \int_u^t \frac{1}{v}p^{(0)}(\log\frac{t}{v})dv\right) dT^{n-1}(W)_u \\ &\quad + \frac{1}{c}\int_0^t \left(q^{(0)}(\log\frac{t}{u}) - \int_u^t \frac{1}{v}q^{(0)}(\log\frac{t}{v})dv\right) dT^{n-1}(\tilde{W})_u \\ &= \int_0^t p^{(1)}(\log\frac{t}{u})dT^{n-1}(W)_u + \frac{1}{c}\int_0^t q^{(1)}(\log\frac{t}{u})dT^{n-1}(\tilde{W})_u \\ &= \cdots = \int_0^t p^{(n)}(\log\frac{t}{u})dW_u + \frac{1}{c}\int_0^t q^{(n)}(\log\frac{t}{u})d\tilde{W}_u, \end{aligned}$$

where

$$\gamma^{(k+1)}(\log\frac{t}{u}) = \gamma^{(k)}(\log\frac{t}{u}) - \int_u^t \frac{1}{v}\gamma^{(k)}(\log\frac{t}{v})dv,$$

for $\gamma^{(k)} = p^{(k)}$ or $q^{(k)}$, and for all $k \geq 0$. Applying a change of variable, we obtain

$$\gamma^{(n+1)}(u) = \gamma^{(n)}(u) - \int_0^u \gamma^{(n)}(v)dv.$$

The relations (3.19) and (3.20) follow directly by induction. □

REMARK 3.4. We can write the recurrence relation (3.21) as

$$r_m^n = \sum_{i_1=m}^{n-1} \sum_{i_2=m-1}^{i_1-1} \cdots \sum_{i_m=1}^{i_{m-1}-1} \sum_{i_{m+1}=0}^{i_m-1} \left(\frac{a+1}{a}\right)^{i_{m+1}},$$

for $n > m$, with initial conditions $r_{n-1}^n = 1$ and $r_0^n = a(\frac{a+1}{a})^n - a$.

In Section 3.1 it has been shown that

$$T^n(B)_1 = \int_0^1 L_n(\log(\frac{1}{u}))dB_u,$$

for a sequence of orthonormal polynomials $(L_n(u))$ for the measure $e^{-u}du$ in $\mathbb{R}^+$. For our two new sequences of Brownian motions $X^{(n)}$ and $Y^{(n)}$, we want to know whether the corresponding $p^{(n)}(u)$ and $q^{(n)}(u)$ are also orthonormal. The following proposition shows that they are not. But we can see some further properties of these two sequences.

PROPOSITION 3.3. *Let m, n be nonnegative integers, then the sequences of functions $(p^{(n)}(u))_{n\geq 0}$ and $(q^{(n)}(u))_{n\geq 0}$ possess the following properties:*

(a) $$\int_0^\infty p^{(n)}(u)e^{-u}du = \begin{cases} 1-c^2, & n=0, \\ 0, & n\geq 1. \end{cases}$$

(b) $$\int_0^\infty q^{(n)}(u)e^{-u}du = \begin{cases} c^2, & n=0, \\ 0, & n\geq 1. \end{cases}$$

(c) $$\int_0^\infty p^{(n)}(u)p^{(n+m)}(u)e^{-u}du = \frac{c^{2m}(1-c^2)}{1+c^2}, \text{ for all } m\geq 0.$$

(d) $$\int_0^\infty q^{(n)}(u)q^{(n+m)}(u)e^{-u}du = \begin{cases} \frac{2c^4}{1+c^2}, & m=0, \\ -\frac{c^{2(m+1)}(1-c^2)}{1+c^2}, & m\geq 1. \end{cases}$$

(e) $$\int_0^\infty p^{(n+m)}(u)q^{(n)}(u)e^{-u}du = \begin{cases} \frac{c^2(1-c^2)}{1+c^2}, & m=0, \\ -\frac{c^{2m}(1-c^2)}{1+c^2}, & m\geq 1. \end{cases}$$

(f) $$\int_0^\infty p^{(n)}(u)q^{(n+m)}(u)e^{-u}du = \begin{cases} \frac{c^2(1-c^2)}{1+c^2}, & m=0, \\ -\frac{c^{2(m+1)}(1-c^2)}{1+c^2}, & m\geq 1. \end{cases}$$

(g) $$\int_0^\infty p^{(n)}(u)p^{(n+m)}(u)e^{-u}du + \frac{1}{c^2}\int_0^\infty q^{(n)}(u)q^{(n+m)}(u)e^{-u}du = \begin{cases} 1, & m=0, \\ 0, & m\neq 0. \end{cases}$$

PROOF. 1) For $n = 0$,

$$\int_0^\infty p^{(0)}(u)e^{-u}du = \int_0^\infty e^{-(a+1)u}du = \frac{1}{a+1} = 1 - c^2,$$

and

$$\int_0^\infty q^{(0)}(u)e^{-u}du = \int_0^\infty \left(1 - p^{(0)}(u)\right) e^{-u}du = \frac{a}{a+1} = c^2.$$

For $n > 0$, from (3.19), we know that

$$\int_0^\infty p^{(n)}(u)e^{-u}du = -\frac{1}{a}\sum_{k=0}^{n-1} r_k^n(-1)^k + \frac{1}{a+1}\left(\frac{a+1}{a}\right)^n.$$

Due to (3.21), we get

$$\sum_{k=0}^{n-1} r_k^n(-1)^k = r_0^n + \sum_{k=1}^{n-2} r_k^{n-1}(-1)^k - \sum_{k=0}^{n-2} r_k^{n-1}(-1)^k = \left(\frac{a+1}{a}\right)^{n-1},$$

and this implies

$$\int_0^\infty p^{(n)}(u)e^{-u}du = -\frac{1}{a}(\frac{a+1}{a})^{n-1} + \frac{1}{a+1}\left(\frac{a+1}{a}\right)^n = 0.$$

Using (3.20), we have, for $n \geq 1$,

$$\begin{aligned}\int_0^\infty q^{(n)}(u)e^{-u}du &= \int_0^\infty \left(L_n(u) - p^{(n)}(u)\right) e^{-u}du = \int_0^\infty L_n(u)e^{-u}du \\ &= \sum_{k=0}^n \binom{n}{k} \frac{1}{k!}\int_0^\infty (-u)^k e^{-u}du = \sum_{k=0}^n \binom{n}{k}(-1)^k = 0.\end{aligned}$$

These ensure the assertions (a) and (b).

2) Since

$$\begin{aligned}X_t^{(n)} &= \int_0^t p^{(n)}(\log\frac{t}{u})dW_u + \frac{1}{c}\int_0^t q^{(n)}(\log\frac{t}{u})d\tilde{W}_u \\ &= \int_0^t p^{(0)}(\log\frac{t}{u})dT^n(W)_u + \frac{1}{c}\int_0^t q^{(n)}(\log\frac{t}{u})d\tilde{W}_u,\end{aligned}$$

and

$$\begin{aligned}X_t^{(n+m)} &= \int_0^t p^{(n+m)}(\log\frac{t}{u})dW_u + \frac{1}{c}\int_0^t q^{(n+m)}(\log\frac{t}{u})d\tilde{W}_u \\ &= \int_0^t p^{(m)}(\log\frac{t}{u})dT^n(W)_u + \frac{1}{c}\int_0^t q^{(n+m)}(\log\frac{t}{u})d\tilde{W}_u,\end{aligned}$$

for all $n \geq 0$, and the processes $T^n(W)$ and $T^n(\tilde{W})$ are Brownian motions, we get

$$\begin{aligned}E[X_t^{(n)}X_t^{(n+m)}] &= \int_0^t p^{(n)}(\log\frac{t}{u})p^{(n+m)}(\log\frac{t}{u})du + \frac{1}{c^2}\int_0^t q^{(n)}(\log\frac{t}{u})q^{(n+m)}(\log\frac{t}{u})du \\ &= \int_0^t p^{(0)}(\log\frac{t}{u})p^{(m)}(\log\frac{t}{u})du + \frac{1}{c^2}\int_0^t q^{(n)}(\log\frac{t}{u})q^{(n+m)}(\log\frac{t}{u})du.\end{aligned}$$

Hence, due to the change of variables and (3.19), we have

$$\begin{aligned}
&\int_0^\infty p^{(n)}(u)p^{(n+m)}(u)e^{-u}du = \frac{1}{t}\int_0^t p^{(n)}(\log\frac{t}{u})p^{(n+m)}(\log\frac{t}{u})du \\
= \;& \frac{1}{t}\int_0^t p^{(0)}(\log\frac{t}{u})p^{(m)}(\log\frac{t}{u})du = \int_0^\infty p^{(0)}(u)p^{(m)}(u)e^{-u}du \\
= \;& \int_0^\infty \left\{ -\frac{1}{a}\sum_{k=0}^{m-1} r_k^m \frac{(-u)^k}{k!} + \left(\frac{a+1}{a}\right)^m e^{-au} \right\} e^{-(a+1)u}du \\
= \;& -\frac{1}{a}\sum_{k=0}^{m-1} r_k^m \frac{(-1)^k}{(a+1)^{k+1}} + \frac{1}{2a+1}\left(\frac{a+1}{a}\right)^m.
\end{aligned}$$

If $m = 0$, we get

$$\int_0^\infty (p^{(n)}(u))^2 e^{-u}du = \frac{1}{2a+1} = \frac{1-c^2}{1+c^2},$$

which is exactly the same form in assertion (c) for the case $m = 0$. As for $m \geq 1$, we have to compute

$$A(m) := \sum_{k=0}^{m-1} r_k^m \frac{(-1)^k}{(a+1)^{k+1}}.$$

From the recurrence relation (3.21), we conclude that

$$\begin{aligned}
A(m) &= \frac{r_0^m}{a+1} + \sum_{k=1}^{m-1}(r_k^{m-1} + r_{k-1}^{m-1})\frac{(-1)^k}{(a+1)^{k+1}} \\
&= \frac{r_0^m}{a+1} + \left\{ \sum_{k=0}^{m-2} r_k^{m-1}\frac{(-1)^k}{(a+1)^{k+1}} - \frac{r_0^{m-1}}{a+1} \right\} - \frac{1}{a+1}\sum_{k=0}^{m-2} r_k^{m-1}\frac{(-1)^k}{(a+1)^{k+1}} \\
&= \frac{a}{a+1}A(m-1) + \frac{1}{a+1}(r_0^m - r_0^{m-1}) = \frac{a}{a+1}A(m-1) + \frac{(a+1)^{m-2}}{a^{m-1}}.
\end{aligned}$$

By induction, we obtain

$$A(m) = \frac{a}{2a+1}\left[\left(\frac{a+1}{a}\right)^m - \left(\frac{a}{a+1}\right)^m\right].$$

This implies

$$\int_0^\infty p^{(n)}(u)p^{(n+m)}(u)e^{-u}du = \frac{1}{2a+1}\left(\frac{a}{a+1}\right)^m = \frac{c^{2m}(1-c^2)}{1+c^2}.$$

Using the same argument and the fact that $(L_n(u))_{n\geq 0}$ is orthonormal with respect to the measure $e^{-u}du$, we get

$$\begin{aligned}
&\int_0^\infty q^{(n)}(u)q^{(n+m)}(u)e^{-u}du = \int_0^\infty q^{(0)}(u)q^{(m)}(u)e^{-u}du \\
= \;& \int_0^\infty (L_0(u) - p^{(0)}(u))(L_m(u) - p^{(m)}(u))e^{-u}du \\
= \;& \delta_{0,m} + \int_0^\infty p^{(0)}(u)p^{(m)}(u)e^{-u}du - \int_0^\infty p^{(m)}(u)e^{-u}du - \int_0^\infty L_m(u)e^{-(a+1)u}du.
\end{aligned}$$

For the case $m = 0$,

$$\begin{aligned}\int_0^\infty (q^{(n)}(u))^2 e^{-u} du &= 1 + \int_0^\infty (p^{(0)}(u))^2 e^{-u} du - 2\int_0^\infty p^{(0)}(u) e^{-u} du \\ &= 1 + \frac{1}{2a+1} - \frac{2}{a+1} = \frac{2a^2}{(2a+1)(a+1)} = \frac{2c^4}{1+c^2}.\end{aligned}$$

For $m \geq 1$, from the definition of $L_n(u)$ we get

$$\begin{aligned}\int_0^\infty L_m(u) e^{-(a+1)u} du &= \sum_{k=0}^m \binom{m}{k} \frac{1}{k!} \int_0^\infty (-u)^k e^{-(a+1)u} du \\ &= \sum_{k=0}^m \binom{m}{k} \frac{(-1)^k}{(a+1)^{k+1}} = \frac{a^m}{(a+1)^{m+1}}.\end{aligned}$$

Due to assertion (a), we have

$$\int_0^\infty q^{(n)}(u) q^{(n+m)}(u) e^{-u} du = -\frac{1}{2a+1}\left(\frac{a}{a+1}\right)^{m+1} = -\frac{c^{2(m+1)}(1-c^2)}{1+c^2}.$$

Hence, the statements (c) and (d) are proved. And it follows (g).

3) It remains to show the assertions (e) and (f). Since for $m, n \geq 0$, the processes $X^{(n)}$ and $Y^{(n+m)}$ are independent, using the same argument at the beginning of 2), we know that the values of the integrals

$$\int_0^\infty p^{(n+m)}(u) q^{(n)}(u) e^{-u} du \quad \text{and} \quad \int_0^\infty p^{(n)}(u) q^{(n+m)}(u) e^{-u} du$$

are independent of n. It follows from (3.20) and the assertions (a) to (d) that

$$\begin{aligned}\int_0^\infty p^{(n+m)}(u) q^{(n)}(u) e^{-u} du &= \int_0^\infty p^{(m)}(u) q^{(0)}(u) e^{-u} du \\ &= \int_0^\infty p^{(m)}(u) e^{-u} du - \int_0^\infty p^{(m)}(u) p^{(0)}(u) e^{-u} du \\ &= \begin{cases} 1 - c^2 - \dfrac{1-c^2}{1+c^2} = \dfrac{c^2(1-c^2)}{1+c^2}, & m = 0, \\ 0 - \dfrac{c^{2m}(1-c^2)}{1+c^2} = -\dfrac{c^{2m}(1-c^2)}{1+c^2}, & m \geq 1, \end{cases}\end{aligned}$$

as well as

$$\begin{aligned}\int_0^\infty p^{(n)}(u)q^{(n+m)}(u)e^{-u}du &= \int_0^\infty p^{(0)}(u)q^{(m)}(u)e^{-u}du \\ &= \int_0^\infty q^{(m)}(u)e^{-u}du - \int_0^\infty q^{(m)}(u)q^{(0)}(u)e^{-u}du \\ &= \begin{cases} c^2 - \dfrac{2c^4}{1+c^2} = \dfrac{c^2(1-c^2)}{1+c^2}, & m = 0, \\ -\dfrac{c^{2(m+1)}(1-c^2)}{1+c^2}, & m \geq 1. \end{cases}\end{aligned}$$

This completes the proofs. □

REMARK 3.5. From the construction of $X^{(n)}$, $Y^{(n)}$ and the assertion (g) of Proposition 3.3, we know that for every $t > 0$ and every $n \geq 0$, $m \geq 1$, $X_t^{(n)}$, $X_t^{(n+m)}$, $Y_t^{(n)}$ and $Y_t^{(n+m)}$ are mutually independent.

In fact, from this Proposition we even know that

$$E[X_s^{(n+1)}X_t^{(n)}] = E[Y_s^{(n+1)}Y_t^{(n)}] = 0$$

for all $s \leq t$, i.e., $(X_s^{(n+1)})_{s\leq t}$ and $(Y_s^{(n+1)})_{s\leq t}$ are independent of $X_t^{(n)}$ and $Y_t^{(n)}$, respectively. The next proposition gives us more information about these two sequences of stochastic processes.

PROPOSITION 3.4. *For $n \geq 0$ and $t \geq 0$, we have*

$$X_t^{(n+1)} = T(X^{(n)})_t$$

and

$$Y_t^{(n+1)} = T(Y^{(n)})_t.$$

PROOF. From (3.13), (3.14), the definition of $T(X^{(n)})_t$, $T(Y^{(n)})_t$ and the stochastic Fubini Theorem, we get the desired results. □

Due to this Proposition we can rewrite (3.15) and (3.16) as

$$T^n(X)_t = T^n(W)_t + \int_0^t \frac{cT^n(\tilde{W})_u - c^2T^n(X)_u}{(1-c^2)u}du,$$

and

$$T^n(Y)_t = T^{n+1}(\tilde{W})_t + \int_0^t \frac{cT^n(W)_u - c^2T^n(Y)_u}{(1-c^2)u}du.$$

And from these results we see the following corollary.

COROLLARY 3.1. *For $f_n, g_n \in C(0,1) \cap \bar{\mathcal{A}}(0,\infty)$, the expectations*

$$E\left[X_t^{(n+1)}\left(\int_0^1 f_n(u)dX_u^{(n)} + \int_0^1 g_n(u)dY_u^{(n)}\right)\right] = 0,$$

and

$$E\left[Y_t^{(n+1)}\left(\int_0^1 f_n(u)dX_u^{(n)} + \int_0^1 g_n(u)dY_u^{(n)}\right)\right] = 0,$$

for all $t \leq 1$ *if and only if* f_n *and* g_n *are constant. More precisely, for constants* A, B *and every* $n \geq 0$,

$$E[X_t^{(n+1)}(AX_1^{(n)} + BY_1^{(n)})] = 0,$$

$$E[Y_t^{(n+1)}(AX_1^{(n)} + BY_1^{(n)})] = 0.$$

PROOF. Without loss of generality, we may assume $n = 0$. Furthermore, for the sake of convenience, we denote $f = f_0$ and $g = g_0$. Hence, we have only to check that

$$E\left[T(X)_t \int_0^1 f(u)dX_u\right] = 0 \quad \text{and} \quad E\left[T(Y)_t \int_0^1 g(u)dY_u\right] = 0,$$

for all $t \leq 1$ if and only if f and g are constant. From

$$E\left[T(X)_t \int_0^1 f(u)dX_u\right] = \int_0^t f(u)du - \int_0^t \frac{1}{u}\int_0^u f(v)dvdu,$$

we see that this expectation is equal to 0 if and only if f is a constant. In the same way, we get

$$\left[T(Y)_t \int_0^1 g(u)dY_u\right] = 0 \quad \text{if and only if} \quad g = \text{ constant}.$$

$\square$

From Remark 3.1 we know that the processes $(T^n(X)_t)_{n\geq 0}$ and $(T^n(Y)_t)_{n\geq 0}$ do not L^2-converge strongly, but weakly to 0 in L^2. Furthermore, the orthogonal decompositions of the filtrations generated by $(X_t)_{t\geq 0}$ and by $(Y_t)_{t\geq 0}$, respectively, are given by:

$$\mathcal{F}_t^X = \mathcal{F}_t^{X(0)} = \sigma(X_t^{(0)}) \oplus \sigma(X_t^{(1)}) \oplus \cdots \oplus \sigma(X_t^{(n)}) \oplus \mathcal{F}_t^{X(n+1)},$$

$$\mathcal{F}_t^Y = \mathcal{F}_t^{Y(0)} = \sigma(Y_t^{(0)}) \oplus \sigma(Y_t^{(1)}) \oplus \cdots \oplus \sigma(Y_t^{(n)}) \oplus \mathcal{F}_t^{Y(n+1)},$$

for all $t \geq 0$. Now, we look at some more relations between the natural filtrations of $X^{(n)}$, $Y^{(n)}$, $T^n(W)$ and $T^n(\tilde{W})$.

PROPOSITION 3.5. (i) *The filtration generated by* $X^{(n)}$ *and* $Y^{(n)}$ *is strictly smaller than that generated by* $T^n(W)$ *and* $T^n(\tilde{W})$, *i.e., for all* $n \geq 0$ *and* $t \geq 0$,

$$\mathcal{F}_t^{X^{(n)},Y^{(n)}} = \mathcal{F}_t^{X^{(n)}} \oplus \mathcal{F}_t^{Y^{(n)}} \subsetneqq \mathcal{F}_t^{T^n(W)} \oplus \mathcal{F}_t^{T^n(\tilde{W})} = \mathcal{F}_t^{T^n(W),T^n(\tilde{W})}.$$

Moreover, we have

$$\mathcal{F}_t^{X^{(n+1)},Y^{(n)}} = \mathcal{F}_t^{X^{(n+1)}} \oplus \mathcal{F}_t^{Y^{(n)}} \subsetneqq \mathcal{F}_t^{T^n(W)} \oplus \mathcal{F}_t^{T^{n+1}(\tilde{W})} = \mathcal{F}_t^{T^n(W),T^{n+1}(\tilde{W})}.$$

(ii) *For all* $0 \leq n < m$, *the* σ*-algebras* $\mathcal{F}_t^{X^{(n)}} \oplus \mathcal{F}_t^{Y^{(n)}}$ *and* $\mathcal{F}_t^{T^m(W)} \oplus \mathcal{F}_t^{T^m(\tilde{W})}$ *do not contain each other. The same result is also valid for the* σ*-algebras* $\mathcal{F}_t^{X^{(n+1)}} \oplus \mathcal{F}_t^{Y^{(n)}}$ *and* $\mathcal{F}_t^{T^m(W)} \oplus \mathcal{F}_t^{T^{m+1}(\tilde{W})}$.
(iii) *For* $|c| < 1$ *and* $t \geq 0$,

$$\mathcal{F}_t^{X+cY} = \mathcal{F}_t^{W+c\tilde{W}}.$$

PROOF. Here we prove only the case

$$\mathcal{F}_t^{X^{(1)}} \oplus \mathcal{F}_t^{Y^{(0)}} \subsetneqq \mathcal{F}_t^{W} \oplus \mathcal{F}_t^{T(\tilde{W})}.$$

The general case can be proved by a similar method. We can easily check the inclusion

$$\mathcal{F}_t^{X^{(1)}} \oplus \mathcal{F}_t^{Y^{(0)}} \subseteq \mathcal{F}_t^{W} \oplus \mathcal{F}_t^{T(\tilde{W})},$$

due to the definitions of $(X_t^{(1)})$ and $(Y_t^{(0)})$. Suppose the σ-algebras $\mathcal{F}_t^{X^{(1)}} \oplus \mathcal{F}_t^{Y^{(0)}}$ and $\mathcal{F}_t^{W} \oplus \mathcal{F}_t^{T(\tilde{W})}$ coincide. Then we know from the above proposition that the random variable $X_t^{(0)}$ is independent of $\mathcal{F}_t^{X^{(1)}} \oplus \mathcal{F}_t^{Y^{(0)}}$. It is therefore also independent of $\mathcal{F}_t^{W} \oplus \mathcal{F}_t^{T(\tilde{W})}$. But it is easy to compute

$$\begin{aligned} E[X_t^{(0)} T(\tilde{W})_t] &= \frac{1}{c}\int_0^t \left(1 - (\frac{u}{t})^a\right) du - \frac{1}{c}\int_0^t \frac{1}{u}\int_0^u \left(1 - (\frac{v}{t})^a\right) dv du \\ &= \frac{-a}{c(a+1)^2} t = -c(1-c^2)t, \end{aligned}$$

which obviously contradicts the above assumption. The second assertion follows by the same argument and the property $E[X_t^{(n)} T^{m-1}(W)_t] \neq 0$ for all $m > n$. The last statement follows directly from the relation:

$$X_t + cY_t = W_t + c\tilde{W}_t,$$

for all $t \geq 0$. □

3.3. Some related decompositions

Let us look at some further properties of the process X. Consider the process $(Z_t)_{t\geq 0}$ defined by

$$Z_t := t\int_t^\infty \frac{1}{u} dX_u.$$

From Chapter 1 in Yor [**57**] we see that this process Z is a Brownian motion. Furthermore, it is easy to check that X_t and Z_t are independent for any t, but that the processes X and Z are not. In this section we want to give a representation of Z in terms of W and $\tilde{W}$, and compare it with the representation of X.

LEMMA 3.2. *The process* X *with* (3.7) *satisfies*

$$Z_t = t\int_t^\infty \frac{dX_u}{u} = V_t^1(W) + V_t^2(\tilde{W}), \tag{3.22}$$

where

$$\begin{aligned} V_t^1(W) &:= (1-c^2)t\int_t^\infty \frac{dW_u}{u} - c^2 t^{-a}\int_0^t u^a dW_u, \\ V_t^2(\tilde{W}) &:= \frac{ct}{1-c^2}\int_t^\infty u^{-a-2}\int_0^u v^a d\tilde{W}_v du. \end{aligned}$$

PROOF. It follows from Itô's formula and (3.9) that

$$\begin{aligned} & t^{-a-1}\int_0^t u^a dW_u - s^{-a-1}\int_0^s u^a dW_u = \int_s^t d\left(u^{-a-1}\int_0^u v^a dW_v\right) \\ &= \int_s^t \left(-(a+1)u^{-a-2}\int_0^u v^a dW_v du + u^{-1}dW_u\right) \\ &= -\frac{1}{a}\int_s^t \frac{dW_u}{u} + \frac{a+1}{a}\int_s^t \frac{dX_u}{u} - \frac{a+1}{c}\int_s^t u^{-a-2}\int_0^u v^a d\tilde{W}_v du. \qquad (3.23) \end{aligned}$$

Due to the time-change there exists a standard Brownian motion Γ such that

$$t^{-a-1}\int_0^t u^a dW_u = t^{-a-1}\Gamma_{\int_0^t u^{2a}du} = t^{-(a+1)}\Gamma_{\frac{1}{2a+1}t^{2a+1}}.$$

Applying the law of large numbers we get

$$\lim_{t\to\infty} t^{-(\frac{1}{2}r+\epsilon)}\Gamma_{t^r} = 0,$$

for any $\epsilon > 0$, hence

$$\lim_{t\to\infty} t^{-a-1}\int_0^t u^a dW_u = 0.$$

Letting t go to ∞, it follows that (3.23) can be written in the form (3.22). □

From the decomposition in the previous Lemma, we can derive another representation for (3.22).

PROPOSITION 3.6. *If the process X satisfies (3.7), then*

$$Z_t = t\int_t^\infty \frac{dX_u}{u} = \int_0^t \left(\frac{u}{t}\right)^a dB_u + \frac{1}{c}\int_0^t \left(1-\left(\frac{u}{t}\right)^a\right)d\tilde{B}_u, \qquad (3.24)$$

with B and $\tilde{B}$ are two independent Brownian motions given by

$$B_t = -W_t + \int_0^t\int_u^\infty \frac{dW_v}{v}du,$$

and

$$\tilde{B}_t = -\tilde{W}_t + \int_0^t\int_u^\infty \frac{d\tilde{W}_v}{v}du.$$

PROOF. The covariance function of $(V_t^1(W))_{t\geq 0}$ is given by

$$\begin{aligned} E[V_s^1(W)V_t^1(W)] &= (1-c^2)^2 stE\left[\int_s^\infty \frac{dW_u}{u}\int_t^\infty \frac{dW_u}{u}\right] + c^4 s^{-a}t^{-a}\int_0^s u^{2a}du \\ &\quad -(1-c^2)c^2 st^{-a}E\left[\int_s^\infty \frac{dW_u}{u}\int_0^t u^a dW_u\right] \\ &= (1-c^2)^2 s + \frac{c^4}{2a+1}s^{a+1}t^{-a} - \frac{(1-c^2)c^2}{a}(s - s^{a+1}t^{-a}) \\ &= \left(\frac{1-c^2}{1+c^2}\right) s^{a+1}t^{-a}. \end{aligned}$$

This is exactly the covariance function of the process $(\int_0^t (\frac{u}{t})^a dB_u)_{t\geq 0}$ for some standard Brownian motion B. Similarly, we have

$$E[V_s^2(\tilde{W})V_t^2(\tilde{W})] = s - \left(\frac{1-c^2}{1+c^2}\right) s^{a+1}t^{-a},$$

which coincides with the covariance function of $(\frac{1}{c}\int_0^t (1-(\frac{u}{t})^a)d\tilde{B}_u)_{t\geq 0}$ for some standard Brownian motion $\tilde{B}$. Since the processes $(V_t^1(W))$ and $(V_t^2(\tilde{W}))$ are independent, B and $\tilde{B}$ can therefore be selected to be independent. Hence, we get the representation (3.24). Furthermore, from

$$\int_0^t \left(\frac{u}{t}\right)^a dB_u = V_t^1(W) = (1-c^2)t\int_t^\infty \frac{dW_u}{u} - c^2 t^{-a}\int_0^t u^a dW_u,$$

and

$$\frac{1}{c}\int_0^t \left(1-\left(\frac{u}{t}\right)^a\right) d\tilde{B}_u = V_t^2(\tilde{W}) = \frac{ct}{1-c^2}\int_t^\infty u^{-a-2}\int_0^u v^a d\tilde{W}_v du,$$

we get the representations of B and $\tilde{B}$ in terms of W and $\tilde{W}$, respectively. □

3.4. A related class of Brownian motions

Let $(W_t)_{t\geq 0}$ and $(\tilde{W}_t)_{t\geq 0}$ be two independent Brownian motions. Deheuvels [**21**] has shown that the process $(X_t)_{t\geq 0}$ defined by

$$X_t = W_t + \int_0^t g(u)W_u du,$$

is a Brownian motion if and only if $g(t) \equiv 0$ or $g(t) = -1/t$. Now we want to generalize this result; see also Wu-Yor [**56**]. For two functions f and g in $C(0,\infty)\cap \bar{\mathcal{A}}(0,\infty)$ we consider the process X given by

$$X_t = W_t + \int_0^t (f(u)\tilde{W}_u + g(u)W_u)du, \tag{3.25}$$

and we ask for which functions f and g the resulting process X can be again a Brownian motion.

THEOREM 3.1. *Denote*

$$U_t = \int_0^t \frac{W_s}{s} ds \quad \text{and} \quad \tilde{U}_t = \int_0^t \frac{\tilde{W}_s}{s} ds.$$

For the functions $f, g \in C(0,\infty) \cap \bar{\mathcal{A}}(0,\infty)$, the process $(X_t)_{t\geq 0}$ given by (3.25) *is a Brownian motion if and only if $f(t) = \pm\sqrt{\nu - \nu^2}/t$ and $g(t) = -\nu/t$, for some $\nu \in [0,1]$. In particular, both processes*

$$X_t^{\pm} := W_t - \int_0^t \left(\frac{\nu}{s} W_s \pm \frac{\sqrt{\nu - \nu^2}}{s} \tilde{W}_s \right) ds, \tag{3.26}$$

are Brownian motions.

PROOF. (i) Denote $Z_t = W_t + i\sqrt{\nu - \nu^2}\tilde{U}_t$, and $\Gamma_t = W_t - \nu U_t$. Therefore, essentially from the previous computations, we find

$$E(\Gamma_s \Gamma_t) = E(Z_s Z_t) = E(W_s W_t) - (\nu - \nu^2) E(\tilde{U}_s \tilde{U}_t).$$

Hence, the covariance of the process $(W_t - \nu U_t \pm \sqrt{\nu - \nu^2}\tilde{U}_t)_{t\geq 0}$ is

$$E(\Gamma_s \Gamma_t) + (\nu - \nu^2) E(\tilde{U}_s \tilde{U}_t) = s + (\nu^2 - \nu)(\varphi(s,t) - E[\tilde{U}_s \tilde{U}_t]) = s,$$

which implies that the process $X^{\pm}$ are Brownian motions.
(ii) Conversely, since in Deheuvels [**21**] the case $f \equiv 0$ has been proved, here we may assume $f \not\equiv 0$. Suppose $(X_t)_{t\geq 0}$ is a Brownian motion. Then from Lemma 1.2, we know that, for $s \leq t$,

$$f(t) E(X_s \tilde{W}_t) + g(t) E(X_s W_t) = 0.$$

Due to (3.25) we can compute $E(X_s \tilde{W}_t)$ and $E(X_s W_t)$, which yields:

$$f(t) \int_0^s u f(u) du + g(t) \left(s + \int_0^s u g(u) du \right) = 0.$$

Taking derivatives with respect to s, we get

$$s f(s) f(t) + (1 + s g(s)) g(t) = 0. \tag{3.27}$$

Since f is continuous, there exists a countable collection of disjoint component intervals $\{(a_i, b_i) : i \in \mathbb{N}\}$ in $(0,\infty)$, such that

$$f(t) \begin{cases} \neq 0, & \forall t \in \bigcup_{i=1}^{\infty} (a_i, b_i), \\ = 0, & \forall t \in (0,\infty) \setminus \bigcup_{i=1}^{\infty} (a_i, b_i). \end{cases}$$

Without loss of generality, we have only to look at the case: $f \in C(0,\infty) \cap \bar{\mathcal{A}}(0,\infty)$, $f(t) \neq 0$ for all $t \in (a,b)$, and $f(t) \equiv 0$ on the set $(0,\infty) \setminus (a,b)$. Then for all $s, t \in (a,b)$, $s < t$, we can rewrite (3.27) as

$$s f(s) + \frac{g(t)}{f(t)} (1 + s g(s)) = 0,$$

which implies $g(t) = cf(t)$ for some nonzero constant c, for all $t \in (a, b)$. Plugging this result into (3.27) it follows that

$$f(s) = -\frac{c}{(1+c^2)s},$$

which is not equal to 0 on $\mathbb{R}^+$. Since f is continuous, we get $(a, b) = (0, \infty)$, which gives the results. □

REMARK 3.6. The intersection of the class of all processes X of the form (3.26) and of the class of processes satisfying (3.7) for some $c \in (-1, 1)$ contains exactly one element, namely, the Brownian motion $X = W$.

PROPOSITION 3.7. *The processes $Y_t^{\pm}$ defined by*

$$Y_t^{\pm} := \tilde{W}_t - \int_0^t \left(\frac{1-\nu}{s} \tilde{W}_s \pm \frac{\sqrt{\nu - \nu^2}}{s} W_s \right) ds,$$

are Brownian motions independent of the processes $X_t^{\pm}$, respectively.

PROOF. If we change the roles of W and $\tilde{W}$ in the process $X^{\pm}$, we get that the resulting process is still a Brownian motion. Hence, we see that the process $Y^{\pm}$ is a Brownian motion. Furthermore, it is easy to check $E[X_t^{\pm} Y_s^{\pm}] = E[X_s^{\pm} Y_t^{\pm}] = 0$ for all $s \leq t$. □

CHAPTER 4

An equilibrium model of insider trading

From this chapter on we are concerned with insider trading of a large investor. We want to investigate a mathematical model of a financial market, in which some investors have more information than others, and where the investment of the traders can influence the price of the stock. Several such models have been proposed by a number of authors, e.g., Back [**7**], Bagehot [**12**], Copeland-Galai [**17**], Easley-O'Hara [**22**], Glosten-Milgram [**26**], Grossman [**30**], Kyle [**42**]. Our study will be based on the model introduced by Kyle [**42**] and Back [**7**]. They develop a model with a risk neutral informed trader who knows in advance the final stock price at time 1. As an extension of their study, we also consider the case where the insider obtains more information as time increases. This idea already appears in Back-Pedersen [**11**]. Our purpose is to analyze the structure of insider strategies in such an extended setting. Our analysis will be based on the results on stochastic filtering in Chapter 2.

In Section 4.1 we specify the basic assumptions on our model and a definition of equilibrium introduced by K. Back. We shall review some results of Kyle [**42**] and Back [**7**] in Section 4.2. In Section 4.3 we derive generalized versions of the results in Back-Pedersen [**11**]. Moreover, we consider special cases of sequential information where information is obtained by observing some Gaussian martingale, and in particular a standard Brownian motion.

4.1. Definition of equilibrium in the sense of K. Back

In this section we introduce a simple model of insider trading and formulate a notion of equilibrium due to K. Back [**7**].

Assume that there are only one bond and one stock in the financial market. The interest rate of the bond is equal to 0. Trading occurs continuously during the time interval $[0, 1]$. Before the trading begins, the final stock price P_1 at time 1 is already determined by the outcome of some $N(0, 1)$-distributed random variable S_1. More precisely, we assume

$$P_1 = h(S_1, 1), \tag{4.1}$$

where $h(\cdot, 1)$ is some continuous, strictly increasing function satisfying

$$E[h^2(S_1, 1)] < \infty. \tag{4.2}$$

Suppose all market participants are risk neutral. We can classify the agents in the market into three groups: uninformed traders, informed trader and market maker.

(1) **uninformed traders**: The uninformed traders have no information about the future price of the stock, and they can only observe their own cumulative demands. Their cumulative demand at time t is a standard Brownian motion (W_t), which is price-inelastic and independent of the final price P_1. We call the uninformed traders also noise traders or liquidity traders.

(2) **informed trader**: There is only one informed trader in the market, also called insider. He gains continuously some extra information about the final price of the stock. This information process is a continuous centered square-integrable Gaussian martingale, denoted by $(S_t)_{0\leq t\leq 1}$. Furthermore, the informed trader can observe the cumulative orders in the whole market, from which he can derive, in particular, the cumulative orders W of the noise traders. Hence, the insider can choose his cumulative orders, denoted by (I_t), depending on his additional information flow (S_t) and the cumulative demands by the noise traders W. Technically, this means that the insider strategy (I_t) is $(\mathcal{F}_t^{W,S})$-adapted. In the sequel, we assume that the process (I_t) is a semimartingale with respect to the filtration $(\mathcal{F}_t^{W,S})$. We denote the collection of such strategies by $\mathcal{I}$, i.e.,

$$\mathcal{I} := \{(I_t)_{0\leq t\leq 1} : I \text{ is an } (\mathcal{F}_t^{W,S})\text{-adapted semimartingale}\}.$$

(3) **market maker**: There is only one market maker in the market. He knows from the very beginning of the trading the distribution of the stock price at time 1. He decides the stock price according to the cumulative orders in the whole market. Thus, at each time $t \in [0, 1)$ the stock price is a functional of the demand process $(X_u)_{u\leq t}$, where $X_t := W_t + I_t$ is the sum of cumulative demands of the uninformed and informed traders. This is a semimartingale with respect to its own filtration.

In this chapter we discuss only the case where the market maker uses a price functional of the form

$$P_t = h(X_t, t), \qquad (0 \leq t < 1) \tag{4.3}$$

for some continuous function $h(\cdot, t)$. Since stock prices rise with increasing demand, we assume that the function $h(x, t)$ is strictly increasing in x for each $t \in [0, 1]$. Due to this assumption the inverse function $h^{-1}(\cdot, t)$ exists for every fixed time t. Furthermore, we suppose that $h(x, t)$ is twice continuously differentiable with respect to x, and once with respect to $t \in [0, 1)$. We also assume that $h(x, t)$ satisfies the integrability condition

$$E\left[\int_0^1 h^2(W_u, u)du\right] < \infty. \tag{4.4}$$

DEFINITION 4.1. A function $h(\cdot, \cdot)$ satisfying the preceding conditions will be called a *pricing rule*. The pricing rule is called *space-time harmonic* if it is a solution of the heat equation

$$\frac{1}{2} h_{xx}(x,t) + h_t(x,t) = 0, \tag{4.5}$$

for all $0 \leq t < 1$.

LEMMA 4.1. *There is exactly one space-time harmonic pricing rule $h(\cdot, \cdot)$ whose boundary values $h(\cdot, 1)$ are given by the function in (4.1), namely the function defined by*

$$\begin{aligned} h(x,t) &= E\left[h(x + W_1 - W_t, 1)\right] \\ &= \frac{1}{\sqrt{2\pi(1-t)}} \int_{-\infty}^{\infty} h(y,1) \exp\left(-\frac{(y-x)^2}{2(1-t)}\right) dy. \end{aligned} \tag{4.6}$$

PROOF. Assumption (4.2) means that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h^2(x,1) \exp\left(-\frac{x^2}{2}\right) dx < \infty. \tag{4.7}$$

Under condition (4.7), equation (4.6) defines a smooth function $h(x,t)$ which satisfies (4.4) and (4.5); this is well known from the theory of the heat equation. Moreover,

$$h_x(x,t) = \frac{1}{\sqrt{2\pi(1-t)}} \int_{-\infty}^{\infty} h_x(y,1) \exp\left(-\frac{(y-x)^2}{2(1-t)}\right) dy > 0, \tag{4.8}$$

and this implies that $h(\cdot, t)$ is strictly increasing in x for all $t \in [0,1)$. □

Recall that at the terminal time $t = 1$ we assume the stock price to be given by a function of the final signal S_1, i.e., $P_1 = h(S_1, 1)$. Thus, given the development (X_t) of the cumulated demand in the whole market, the resulting price process (P_t) is given by

$$P_t = \begin{cases} h(X_t, t), & \text{for } 0 \leq t < 1, \\ h(S_1, 1), & \text{for } t = 1. \end{cases}$$

Note that although the price process can (and will) be influenced by the trading activities over the time interval $[0,1)$, at the terminal time 1 it will assume to the value $h(S_1, 1)$ regardless of what has happened before.

With this formalization of the price process at hand we are now in a position to define the profit resulting from a given process of cumulative demands ξ by

$$(P_1 - P_{1-})\xi_{1-} + \int_0^{1-} \xi_{u-} dP_u, \tag{4.9}$$

where we assume, for simplicity, that all agents have initial capital 0. For an informed trader using strategy $I = (I_t) \in \mathcal{I}$, his final profit is given by

$$\begin{aligned}\Lambda_1 := \Lambda_1(h, I) &:= (h(S_1, 1) - h(X_{1-}, 1-))I_1 + \int_0^{1-} I_{u-} dh(X_u, u) \\ &= (P_1 - P_{1-})I_1 + \int_0^{1-} I_{u-} dP_u \\ &= \int_0^1 (P_1 - P_{u-}) dI_u - [P, I]_{1-}, \qquad (4.10)\end{aligned}$$

where for the last equality we used Itô's product rule and where $([P, I]_t)_{0 \le t \le 1}$ denotes the optional quadratic variation of the processes (P_t) and (I_t).

Let us now recall the definition of equilibrium in Back [**7**].

DEFINITION 4.2. **(1)** (**Market Efficiency**) Given an insider strategy $I = (I_t) \in \mathcal{I}$, a pricing rule $h(x, t)$ is called *rational* given I if it satisfies

$$h(X_t, t) = E[h(S_1, 1)|\mathcal{F}_t^X], \qquad (4.11)$$

for all $t \le 1$.
(2) (**Profit Maximization**) Given a pricing rule $h(x, t)$, the insider strategy $I^\star = (I_t^\star)_{0 \le t \le 1} \in \mathcal{I}$ is said to be *optimal* given h if it maximizes the corresponding expected final profit

$$E[\Lambda_1(h, I^\star)] = \max_{I \in \mathcal{I}} E[\Lambda_1(h, I)].$$

(3) (**Equilibrium**) A pricing rule h and a strategy $I^\star$ for the insider define an equilibrium $(h, I^\star)$ if h is a rational pricing rule given $I^\star$ and $I^\star$ is an optimal insider strategy given h. In this case, we call h an *equilibrium pricing rule.*

Let us now address the question of existence of an equilibrium in the above sense. First we investigate the case of an insider with full information (Section 4.2) and then the case of an insider with increasing extra information (Section 4.3).

4.2. Equilibrium in the case of full information

In this section we assume, as in Kyle [**42**] and Back [**7**], that the insider has full information about the final stock price already at the beginning of the trading. It is also assumed that the final signal S_1 is independent of the Brownian motion W, the cumulative orders of the uninformed traders.

By using a discrete time approximation, Kyle [**42**] shows that there exists a unique linear equilibrium $(h, I^\star)$ for which the function $h(x, t)$ does not depend on t and is linear in x, provided the final stock price is normally distributed. Note that Kyle [**42**] uses a definition of equilibrium different from our Definition 4.2. Rochet-Vila [**53**] construct a nonlinear equilibrium model in the sense of Kyle. In the more general setup reviewed in the previous section, Back [**7**] shows that there exists an equilibrium in the sense of Definition 4.2. He proves that the insider reaches his maximal profit if and only if he drives the cumulative orders in the whole market to the final value $X_1 = S_1$. He also gives an explicit description of the equilibrium:

(B1) The rational pricing rule is given by

$$dP_t = h_x(X_t, t)dX_t, \tag{4.12}$$

where $h(\cdot,\cdot)$ is the space-time harmonic pricing rule defined in (4.6).

(B2) The optimal strategy for the insider is given by

$$I_t^\star = \int_0^t \frac{S_1 - X_s}{1-s} ds, \tag{4.13}$$

which is continuous and of bounded variation. Therefore, the optional quadratic variation of P and $I^\star$ vanishes identically.

(B3) The cumulative demand in the whole market $(X_t)_{0\leq t\leq 1}$ forms a Brownian motion with respect to its own filtration (see Example 2.3), and it converges to S_1 as $t \to 1$. In fact, equation(4.13) shows that the optimal strategy consists in constructing a Brownian bridge tied to the final value S_1.

Recall that Back [**7**] does not assume the insider strategy I to be absolutely continuous a priori. Using Bellman's equation and an optimization argument he shows that any optimal strategy must have this property.

Remark 4.1. Property (B3) means, in particular, that the market maker cannot discover that there is an insider in the market. This is due to our assumption that he can only observe the cumulative orders in the whole market. But these cumulative orders evolve like a Brownian motion whether there is an insider (using an optimal strategy) or not.

4.3. Equilibrium with increasing information

In this section we would like to consider a more general setup. We assume that instead of possessing the full information from the beginning the informed trader now gains an increasing amount of information by observing a signal process $(S_t)_{0\leq t\leq 1}$. Here S is supposed to be a continuous centered square-integrable Gaussian martingale with respect to its own filtration and with final value $S_1 \sim N(0,1)$. Back-Pedersen [**11**] consider the special case

$$S_t = S_0 + \int_0^t \sigma(u)d\tilde{W}_u,$$

where S_0 is normally distributed with mean 0, and $(\tilde{W}_t)_{0\leq t\leq 1}$ is a Wiener process, both independent of W. Moreover, $\sigma(u)$ is assumed to be a deterministic continuous function satisfying

$$var(S_0) + \int_0^1 \sigma^2(u)du = 1,$$

and

$$\int_t^1 \sigma^2(u)du < \frac{1-t}{1+\epsilon}, \tag{4.14}$$

for all $0 \le t < 1$, with some constant $\epsilon > 0$. Furthermore, Back-Pedersen [**11**] restrict the insider to use only absolutely continuous strategies. Under these assumptions Back-Pedersen [**11**] show that there exists an equilibrium in the sense of Definition 4.2. More precisely, the optimal strategy for the insider is of the form

$$I_t^\star = \int_0^t \frac{S_u - X_u}{\int_u^1 (1-\sigma^2(v))dv} du, \tag{4.15}$$

and the pricing rule is given by (4.6). Again the cumulative demand in the whole market turns out to be a Brownian motion. Hence, we are again in a situation where the market maker will not be able to discover the insider (c.f. Remark 4.1).

Let us now consider the existence of equilibrium in our extended model where S is only a continuous centered square-integrable Gaussian martingale. We follow Back-Pedersen [**11**] in restricting the insider to absolutely continuous strategies.

Next we are going to discuss the insider's optimization problem under this assumptions. Using the same argument as in Back [**7**] we get the following result.

LEMMA 4.2. *Let $I^\star \in \mathcal{I}$ be an absolutely continuous insider strategy and let h be the space-time harmonic pricing rule of* (4.6). *If $I^\star$ satisfies $S_1 = W_1 + I_1^\star$, then it is an optimal strategy, i.e.,*

$$E[\Lambda_1(h, I^\star)] \ge E[\Lambda_1(h, I)],$$

for all $I \in \mathcal{I}_0$. Moreover, if $I^{\star\star}$ is another optimal insider strategy, then $I_1^\star = I_1^{\star\star}$.

PROOF. For a constant a in the range of $h(\cdot, 1)$, let

$$G^a(z,1) = \int_{h^{-1}(\cdot,1)(a)}^z (h(y,1) - a)dy, \tag{4.16}$$

so that

$$G_x^a(x,1) = h(x,1) - a. \tag{4.17}$$

Since $h(\cdot,1)$ is strictly increasing, $G_x^a(\cdot,1)$ is strictly increasing. Hence, $G^a(\cdot,1)$ is strictly convex, and has its minimum at the point $h^{-1}(\cdot,1)(a)$. It follows from $G^a(h^{-1}(\cdot,1)(a),1) = 0$ that $G^a(\cdot,1)$ is nonnegative. We define

$$\begin{aligned} G^a(x,t) &= E\left[G^a(x + W_1 - W_t, 1)\right] \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G^a(x + \sqrt{1-t}\, y, 1) \exp\left(-\frac{y^2}{2}\right) dy \end{aligned}$$

for $0 \le t < 1$. Then G^a is a solution of the heat equation

$$\left(\frac{1}{2}\Delta + \frac{\partial}{\partial t}\right) G^a = 0 \tag{4.18}$$

on the strip $\mathbb{R} \times [0,1)$ with boundary condition (4.16). As in (4.8), we have

$$G_x^a(x,t) = E[G_x^a(x + W_1 - W_t, 1)] = E[h(x + W_1 - W_t, 1)] - a,$$

hence

$$G_x^a(x,t) = h(x,t) - a,$$

since h is assumed to be space-time harmonic. Alternatively, we could directly define the function

$$G^a(x,t) = \int_{h^{-1}(\cdot,1)(a)}^{x} (h(y,t) - a)dy + \frac{1}{2}\int_t^1 h_x(h^{-1}(\cdot,1)(a), s)ds,$$

and check that this function does satisfy conditions (4.17) and (4.18). Let us now apply this construction pathwise for $a = h(S_1, 1)$. It follows from (4.17) and Itô's formula that

$$\begin{aligned}
& G^{h(S_1,1)}(X_t, t) - G^{h(S_1,1)}(0,0) \\
= & \int_0^t G_x^{h(S_1,1)}(X_s, s)dX_s + \int_0^t G_t^{h(S_1,1)}(X_s, s)ds + \frac{1}{2}\int_0^t G_{xx}^{h(S_1,1)}(X_s, s)d\langle X\rangle_s \\
= & \int_0^t (h(X_s, s) - h(S_1, 1))dX_s + \int_0^t \left(\frac{1}{2}\Delta + \frac{\partial}{\partial t}\right) G^{h(S_1,1)}(X_s, s)ds \\
= & \int_0^t (h(X_s, s) - h(S_1, 1))dX_s. \qquad (4.19)
\end{aligned}$$

Thanks to (4.18) the last term on the third line vanishes. Since W is a Brownian motion with respect to its natural filtration, and since W and S_1 are independent, we know that W is a Brownian motion relative to the enlarged filtration $(\mathcal{F}_t^W \vee \sigma(S_1))$. Hence, for all $t \leq 1$,

$$E\left[\int_0^t (h(X_s, s) - h(S_1, 1))dW_s\right] = 0, \qquad (4.20)$$

due to (4.2) and (4.3). Using (4.20), (4.17) and (4.19), we can rewrite the expected profit of the insider as

$$\begin{aligned}
E[\Lambda_1] &= E\left[\int_0^t (h(S_1, 1) - h(X_s, s))dI_s\right] \\
&= E\left[\int_0^t (h(S_1, 1) - h(X_s, s))dX_s\right] \\
&= E[G^{h(S_1,1)}(0,0)] - E[G^{h(S_1,1)}(X_1, 1)].
\end{aligned}$$

The first term on the third line is fixed, since the final signal S_1 is determined before the trading begins. Therefore, we can reformulate the optimization problem as

$$E[G^{h(S_1,1)}(X_1, 1)] \overset{!}{=} \min_{I \in \mathcal{I}}. \qquad (4.21)$$

Since $G^{h(S_1,1)}(\cdot,1)$ is nonnegative and achieves its minimum 0 in S_1, (4.21) certainly holds if $X_1 = S_1$. This means that the insider can reach his maximal profit if he can drive the cumulative orders X_t to the final value S_1 as $t \to 1$. □

From now on, we assume that the informed trader does not want to be discovered. From Remark 4.1 we infer that this is assured if we allow the insider to use only strategies such that the resulting demand process $(W_t + I_t)$ is again a Wiener process; such strategies will also be called *inconspicuous*. More precisely, let us introduce the class of strategies

$$\mathcal{I}_0 := \Bigg\{ (I_t)_{0\le t\le 1} : I_t = \int_0^t Y_u du, \text{ where } (Y_u) \text{ is } (\mathcal{F}_t^{W,S})\text{-adapted,} \\ E\left[\int_0^t Y_u^2 du\right] < \infty, \text{ for all } t < 1, \text{ and } (W_t + I_t)_{0\le t\le 1} \text{ is a Wiener process} \Bigg\}.$$

PROPOSITION 4.1. *Consider an inconspicuous insider strategy* $I^\star \in \mathcal{I}_0$. *If* $X_t^\star = W_t + I_t^\star$ *converges to* S_1 *as* $t \to 1$, *then there exists an equilibrium. The pricing rule* $h(x,t)$ *is given by* (4.6). *Furthermore, there is no jump of the stock price at time* 1, *i.e.,* $P_{1-} = P_1$.

PROOF. Lemma 4.2 shows that the absolutely continuous strategy $I^\star$ is optimal as soon as $X_t^\star \to S_1$. Moreover, $X^\star$ is a Brownian motion for $I^\star \in \mathcal{I}_0$. Market efficiency (4.11) now follows from the fact that $h(X_t,t)$ is a martingale, since $h(\cdot,\cdot)$ is space-time harmonic. □

Therefore if the insider can use a strategy $I^\star$ driving the terminal cumulative demand $W_1 + I_1^\star$ to S_1, this strategy $I^\star$ is optimal, and $(h, I^\star)$ is an equilibrium, where $h(\cdot,t)$ is defined by (4.6). But conversely, if $I^\star$ yields

$$E[\Lambda_1(h,I^\star)] = \max_{I\in\mathcal{I}} E[\Lambda(h,I)],$$

we cannot say that $W_1 + I_1^\star$ must be equal to S_1, because it can happen that there is no process $I \in \mathcal{I}$ such that $W_1 + I_1 = S_1$. In Subsection 4.3.2 and in the next chapter we will give several examples for such a situation. Therefore the converse of the lemma fails to be true in general.

In Proposition 4.1, we consider just those insider strategies which turn the cumulative order process into a Brownian motion. In other words, the primary goal of the insider is not to be discovered (cf. Remark 4.1). The discussion in Section 2.3 suggests several classes of such strategies defined by certain linear transformations of W and S.

In the following two subsections, we discuss existence of equilibria in the case where $(S_t)_{0\le t\le 1}$ is a Gaussian square-integrable martingale whose variance satisfies some integrability conditions, and in the case where $S_t = \tilde{W}_t$, a standard Brownian motion.

4.3.1. Partial information given by observing a Gaussian martingale. Assume that the insider's extra information is given by observing a continuous centered square-integrable Gaussian martingale $(S_t)_{0\leq t\leq 1}$. We assume that its variance function $V(t) := var(S_t)$ satisfies

$$V(1) = E[S_1^2] = 1 \tag{4.22}$$

and the two integrability conditions

$$\frac{1}{V(u)-u} \in \mathcal{A}(0,1) \cap L^1_{loc}([0,1)). \tag{4.23}$$

PROPOSITION 4.2. *Assume condition* (4.22) *and* (4.23) *hold. Then there exists an equilibrium* $(h, I^\star)$. *Explicitly, the rational pricing rule is given by* (4.6), *and*

$$I_t^\star := \int_0^t \frac{S_u - X_u}{V(u)-u} du \tag{4.24}$$

is an optimal insider strategy.

PROOF. It follows from the second assertion in Theorem 2.4 that the process X satisfying

$$X_t = W_t + \int_0^t \frac{S_u - X_u}{V(u)-u} du,$$

is a Wiener process and converges to S_1 as $t \to 1$. Due to Proposition 4.1 we get the desired result. □

This proposition shows the existence of an equilibrium in our present setting. But regarding uniqueness, in the terminology in Cho-El Karoui [**15**] we are only sure that there exists a *weakly unique equilibrium* (i.e., there is a unique pricing rule, but there may be multiple optimal strategies in the class $\mathcal{I}_0$). Explicitly, due to Lemma 4.1, we derive the uniqueness of the rational pricing rule $(h(\cdot, t))$. But we cannot prove uniqueness of the optimal insider strategy. However, the insider strategy (4.24) is the unique optimal strategy in the collection of linear strategies

$$\mathcal{I}_1 := \left\{ \begin{array}{l} I = (I_t)_{0\leq t\leq 1} \in \mathcal{I}_0 : I_t = \displaystyle\int_0^t (f(u)S_u + g(u)X_u)\, du, \\ \\ \text{where } f(t)\sqrt{V(t)}, g(t) \in C^1(0,1) \cap L^2_{loc}([0,1)), \end{array} \right\}.$$

PROPOSITION 4.3. $I^\star$ *is the unique optimal strategy in* $\mathcal{I}_1$.

PROOF. From Corollary 2.4 we see that $I \in \mathcal{I}_1$ if and only if

$$I_t = \int_0^t \frac{cS_u - c^2 X_u}{V(u) - c^2 u} du,$$

for some $|c| \le 1$. Hence,

$$\begin{aligned} E[(W_1 + I_1 - S_1)^2] &= 2 - 2E[(W_1 + I_1)S_1] \\ &= 2 - 2\int_0^1 \exp\left(-\int_u^1 \frac{c^2}{V(v) - c^2 v} dv\right) \frac{cV(u)}{V(u) - c^2 u} du \\ &= 2(1 - c), \end{aligned}$$

which is equal to 0 if and only if $c = 1$. □

REMARK 4.2. Condition (4.23) holds if either there is enough insider information at time 0 or the additional information at time 0 increases quickly enough. In the next subsection and in the next chapter we will relax condition (4.23) and discuss more general forms of insider information.

The following remark shows that Proposition 4.2 is an extension of the results in Back-Pedersen [**11**].

REMARK 4.3. Assume that the Gaussian martingale S is given by

$$S_t = S_0 + \int_0^t \sigma(u) d\tilde{W}_u, \tag{4.25}$$

where $\tilde{W}$ is a Wiener process and $\sigma(t)$ satisfies

$$V(0) + \int_0^1 \sigma^2(u) du = 1. \tag{4.26}$$

To derive the results in Back-Pedersen [**11**] as a special case of Proposition 4.2 we can apply the above discussion as follows. From (4.25) and (4.26) we see that

$$V(t) = V(0) + \int_0^t \sigma^2(u) du = 1 - \int_t^1 \sigma^2(u) du.$$

Thus, (4.15) coincides with our insider strategy (4.24). The condition (4.23) can be written as

$$\int_0^t \frac{u}{(V(0) + \int_0^u \sigma^2(v) dv)^2} du = \int_0^t \frac{u}{(1 - \int_u^1 \sigma^2(v) dv)^2} du < \infty \tag{4.27}$$

for all $t < 1$. Thus, the condition (4.14) as considered in Back-Pedersen [**11**] is a special case of our condition (4.27). For instance, the case where $S_0 = 0$ and $\sigma(t) = \sqrt{p\, t^{p-1}}$ does not satisfy (4.14), but we still have (4.27). Furthermore, not all continuous centered Gaussian martingales (S_t) can be represented in the form (4.25) with a Brownian motion $\tilde{W}$ and a deterministic function σ. For example, consider the process

$$S_t = S_0 + B_{g(t)}, \tag{4.28}$$

where S_0 is an $N(0, 1/4)$-distributed random variable and B is a Brownian motion independent of S_0. Moreover, suppose that g is given by

$$g(t) := \begin{cases} f(t), & 0 \le t \le \frac{1}{2}, \\ \frac{1}{4} + \frac{1}{2}\,t, & \frac{1}{2} < t \le 1, \end{cases}$$

where $f(t)$ is the Cantor function. In this case we have $\langle S \rangle_t = g(t)$ for all $t \le 1$. And the variance

$$V(t) = E[S_t^2] = E[S_0^2] + g(t) = \frac{1}{4} + g(t) = \begin{cases} \frac{1}{4} + f(t), & 0 \le t \le \frac{1}{2}, \\ \frac{1}{2} + \frac{1}{2}\,t, & \frac{1}{2} < t \le 1, \end{cases}$$

is strictly larger than t for all $t < 1$. Suppose S can be represented in the form (4.25). From (4.25) we see that

$$E[(S_t - S_0)^2] = \int_0^t \sigma^2(u)du,$$

which is absolutely continuous for all t. On the other hand, by (4.28), we conclude that for $0 \le t \le 1/2$,

$$E[(S_t - S_0)^2] = E\left[B_{g(t)}^2\right] = g(t) = \frac{1}{4} + f(t).$$

This is clearly a contradiction, since the Cantor function $f(t)$ is not absolutely continuous.

Now let us consider two typical examples.

Example 4.1. As in Kyle [**42**], we suppose the final price of the stock is given by

$$P_1 = m + \sigma S_1,$$

where $m \in \mathbb{R}$ and $\sigma > 0$. From Proposition 4.2 and the above discussion we see that there exists a weakly unique equilibrium in this model. Explicitly, an optimal strategy of the insider is given by

$$I_t^\star = \int_0^t \frac{S_u - X_u}{V(u) - u} du.$$

Using this strategy the informed trader can drive the cumulative demands X_t to S_1. Therefore, in equilibrium, the pricing rule of the stock is given by

$$P_t = h(X_t, t) = E[P_1|\mathcal{F}_t^X] = m + \sigma E[S_1|\mathcal{F}_t^X] = m + \sigma E[X_1|\mathcal{F}_t^X] = m + \sigma X_t.$$

Thus, the price process P satisfies the stochastic differential equation

$$dP_t = \sigma dX_t,$$

with initial value $P_0 = m$. Since P_1 and W_1 are independent, the processes X and W are both Brownian motions. The insider's expected final profit is therefore given by

$$\begin{aligned} E[\Lambda_1] &= E\left[P_1 I_1 - \int_0^1 P_t dI_t\right] = E\left[P_1(X_1 - W_1) - \int_0^1 P_u d(X_u - W_u)\right] \\ &= E[(m + \sigma S_1)S_1] = \sigma. \end{aligned}$$

This value is independent of the expected value of the price m. Moreover, it coincides with the expected final profit of an informed trader who owns the full information. Furthermore, the noise traders' expected final profit is given by

$$\begin{aligned} E\left[(P_1 - P_{1-})W_1 + \int_0^1 W_u dP_u\right] &= E\left[P_1 W_1 - \int_0^1 P_u dW_u - \langle P, W\rangle_{1-}\right] \\ &= E[(m + \sigma S_1)W_1] - E[\langle m + \sigma X, W\rangle_{1-}] \\ &= -E[\langle \sigma W, W\rangle_{1-}] = -\sigma. \end{aligned}$$

Thus, the expected profit of the insider is at the expense of the noise traders.

EXAMPLE 4.2. Suppose the final price P_1 is log-normally distributed, i.e., P_1 is given by

$$\tilde{v} = P_1 = \exp(m + \sigma S_1),$$

where $S_1 \sim N(0,1)$, $m \in \mathbb{R}$ and $\sigma > 0$. Due to Proposition 4.2 we know that there exists an equilibrium. Thus the price process of the stock P_t is of the form

$$\begin{aligned} P(X_t, t) &= E[P_1|\mathcal{F}_t^X] = \exp\left(m + \frac{1}{2}\sigma^2\right) E\left[\exp\left(\sigma S_1 - \frac{1}{2}\sigma^2\right)\middle| \mathcal{F}_t^X\right] \\ &= \exp\left(\sigma X_t + m + \frac{1}{2}\sigma^2(1-t)\right), \end{aligned}$$

i.e., the equilibrium pricing rule is given by

$$h(x,t) = \exp\left(\sigma x + m + \frac{1}{2}\sigma^2(1-t)\right).$$

Therefore, the price process P satisfies the stochastic differential equation of geometric Brownian motion

$$dP_t = \sigma P_t dX_t,$$

with initial condition $P_0 = \exp(m + \frac{1}{2}\sigma^2)$, i.e., we are in the context of the standard Black-Scholes models. Furthermore, from the discussion above, we conclude that the optimal insider strategy is of the form

$$I_t^\star = \int_0^t \frac{S_u - X_u}{V(u) - u} du,$$

and that $(h, I^\star)$ forms an equilibrium. Hence, the expectation of the final profit of the insider is given by

$$E[\Lambda_1] = E[X_1 h(S_1, 1)] = E[S_1 \exp(m + \sigma S_1)] = \sigma \exp\left(m + \frac{\sigma^2}{2}\right),$$

as in the case where the insider has full information. Using a similar argument as in the last example, we see that the expected final profit of the noise traders amounts to $-\sigma \exp(m + \sigma^2/2)$.

REMARK 4.4. From the two examples above, we see that the expected profit of the insider does not depend on whether he has full or only partial information, provided the latter satisfies the integrability condition (4.23). But what is the difference between these two cases? Let us compare the insider strategy in case of full information

$$I_t^f := \int_0^t \frac{S_1 - X_u^f}{1-u} du,$$

and that in case of partial information:

$$I_t^p := \int_0^t \frac{S_u - X_u^p}{V(u) - u} du.$$

Here the processes X^f and X^p denote the cumulative demands in the whole market in the cases of full information and of sequential information, respectively. We know that both processes X^f and X^p are Brownian motions with respect to their own filtrations, and that $S_1 = X_1^f = X_1^p$. Measuring the activity of the insider strategies as

$$\begin{aligned} F^f(t) &:= \int_0^t E\left[\left(\frac{S_1 - X_u^f}{1-u}\right)^2\right] du = \int_0^t \frac{E[S_1^2] - 2E[S_1 X_u^f] + E[(X_u^f)^2]}{(1-u)^2} du \\ &= \int_0^t \frac{1 - 2E[X_1^f X_u^f] + u}{(1-u)^2} du = \int_0^t \frac{1}{1-u} du = \log\left(\frac{1}{1-t}\right), \end{aligned}$$

and

$$\begin{aligned} F^p(t) &:= \int_0^t E\left[\left(\frac{S_u - X_u^p}{V(u) - u}\right)^2\right] du = \int_0^t \frac{E[S_u^2] - 2E[S_u X_u^p] + E[(X_u^p)^2]}{(V(u)-u)^2} du \\ &= \int_0^t \frac{V(u) - 2E[S_1 X_u^p] + u}{(V(u)-u)^2} du = \int_0^t \frac{V(u) - 2E[X_1^p X_u^p] + u}{(V(u)-u)^2} du \\ &= \int_0^t \frac{1}{V(u) - u} du \\ &> \int_0^t \frac{1}{1-u} du = F^f(t), \end{aligned}$$

we see that the activity for the insider with partial information is strictly larger than that of the insider with full information.

REMARK 4.5. Up to now we have only discussed the insider information as a Gaussian martingale with variance function satisfying (4.23). Due to the first assertion in Theorem 2.4 we can relax condition (4.23) to get Brownian motions driven by a more general class of Gaussian martingales. But these extra Brownian motions may not converge to S_1. In the end of Section 5.1 we shall investigate this general situation.

4.3.2. Partial information given by a Brownian motion. In this subsection we want to investigate the case where the insider's additional information consists in observing a standard Brownian motion $\tilde{W}$. Clearly, $\tilde{W}$ does not satisfy the integrability conditions (4.23). Therefore, the discussion in the last subsection is not valid for this case. We want to ask whether there is an equilibrium in the model. If so, what is the associated optimal insider strategy and the equilibrium pricing rule? If not, which strategies can the insider apply to get a positive expected profit?

Suppose the pricing rule $h(x,t)$ is space-time harmonic. If the cumulative orders in the whole market X_t converge to $\tilde{W}_1$ as $t \to 1$, we know from the discussion in Section 4.2 that the insider reaches his maximal expected profit. But suppose that X is a Brownian motion satisfying the stochastic differential equation

$$dX_t = dW_t + Y_t dt, \tag{4.29}$$

with initial value $X_0 = 0$, and where (Y_t) is an $(\mathcal{F}_t^{W,\tilde{W}})$-adapted process. Then X does not converge to $\tilde{W}_1$ as $t \to 1$ as has been shown in Föllmer-Wu-Yor [**23**] (see Proposition 2.3 above). But this does yet not imply that there is no equilibrium in this model. A priori, it might happen that there exists an insider strategy $(I_t^\star)$ such that

$$E[\Lambda_1(h, I^\star)] = \max_{I \in \mathcal{I}} E[\Lambda_1(h, I)],$$

but $W_1 + I_1^\star \neq \tilde{W}_1$.

DEFINITION 4.3. We say a strategy $I \in \mathcal{I}_0$ belongs to $\mathcal{I}_2$ if I is given by

$$I_t = \int_0^t (f(u)\tilde{W}_u + \int_0^u G(u,v)dX_v)du,$$

where $f \in \mathcal{A}(0,1) \cap C(0,1)$, G is a continuous square-integrable Volterra kernel.

In the following we first want to give an explicit representation of such strategies $I \in \mathcal{I}_2$, then to check if there exists an insider strategy $I^\star \in \mathcal{I}_2$ such that

$$E[\Lambda_1(P, I^\star)] = \max_{I \in \mathcal{I}_2} E[\Lambda_1(P, I)].$$

Due to Corollary 2.6, we see that a strategy $I \in \mathcal{I}_0$ belongs to $\mathcal{I}_2$ if and only if

$$I_t = \int_0^t f(u) \left(\tilde{W}_u - \int_0^u \alpha(v)dX_v \right) du, \tag{4.30}$$

where $\alpha(t)$ is defined by

$$\alpha(t) = \frac{f(t)\Psi(t)}{\Psi'(t)}, \tag{4.31}$$

and $\Psi(t)$ is the solution of the Sturm-Liouville equation

$$\Psi''(t) = f^2(t)\Psi(t)$$

with initial conditions $\Psi(0) = 1$ and $\Psi'(0+) = 0$. In addition, solving the equation $X = W + I$, we get another representation of I as

$$I_t = \int_0^t \frac{f(s)}{\Psi'(s)} \left(\int_0^s \Psi'(u)d\tilde{W}_u - \int_0^s f(u)\Psi(u)dW_u \right) ds \tag{4.32}$$

(as in the equation (70) in Föllmer-Wu-Yor [**23**]).

Now let us investigate the existence of an equilibrium in this model. To simplify, we take here a simple example where the final price is given by $\tilde{W}_1$, i.e., $h(x,1) = x$. Since the price process (P_t) is $(\mathcal{F}_t^X)$-adapted and $E[\int_0^1 P_u^2 du] < \infty$, we have

$$E\left[\int_0^1 P_u dW_u\right] = E\left[\int_0^1 P_u dX_u\right] = 0.$$

Then the corresponding expected final wealth of the insider is given by

$$\begin{aligned} E[\Lambda_1(h,I)] &= E\left[\int_0^1 (P_1 - P_u)dI_u\right] = E\left[P_1 I_1 - \int_0^1 P_u d(I_u + W_u)\right] = E[\tilde{W}_1 I_1] \\ &= \int_0^1 \frac{f(s)}{\Psi'(s)} \int_0^s \Psi'(u)duds = \int_0^1 \alpha(u)du. \end{aligned}$$

Remark 4.6. Note that we do not need to compute the pricing rule explicitly here, since from the above equation we see that the insider's expected final profit is independent of the behavior of the price process between 0 and $1-$.

The following corollary provides an upper bound for the expected profit $\int_0^1 \alpha(u)du$.

Corollary 4.1. *Suppose that $f \in C^1(0,1) \cap \mathcal{A}(0,1)$ and α is given by (4.31). Then*

$$\int_0^1 \alpha(s)ds < 1.$$

Proof. Due to (4.30), we have, for all $0 \leq t \leq 1$,

$$\begin{aligned} E[(X_1 - \tilde{W}_1)^2] &= E[X_1^2] + E[\tilde{W}_1^2] - 2E[X_1\tilde{W}_1] \\ &= 2 - 2\int_0^1 \frac{f(s)}{\Psi'(s)} \int_0^s \Psi'(u)duds = 2 - 2\int_0^1 \alpha(s)ds, \end{aligned}$$

which is nonnegative. The equality holds if and only if $X_1 = \tilde{W}_1$. But as we have shown in Proposition 2.3, these two random variables cannot coincide. This leads to the desired result. □

Thus, $\int_0^1 \alpha(s)ds$, the expected final wealth of the insider, cannot be equal to 1. But how close can the insider come to the upper bound? Let us look at a simple example.

EXAMPLE 4.3. Let $f(t) = c$ with a constant $c \geq 0$. Then the solution of the associated Sturm-Liouville equation is given by

$$\Psi(t) = \frac{1}{2c}(e^{ct} - e^{-ct}).$$

This implies

$$\alpha(t) = \frac{e^{ct} - e^{-ct}}{e^{ct} + e^{-ct}}.$$

Hence, the expected final profit of the insider is

$$E[\Lambda_1] = \int_0^1 \alpha(t)dt = \frac{1}{c}\log\left(\frac{e^c + e^{-c}}{2}\right),$$

which is a strictly increasing function in c which starts at 0, and approaches 1 as $c \to \infty$.

This example shows that the supremum of $\int_0^1 \alpha(s)ds$ is equal to 1. But the supremum cannot be reached. Hence, we conclude that there exist no equilibrium if the insider observes an independent Brownian motion and is inconspicuous, even though the insider can come as close to the value 1 as he wants.

CHAPTER 5

Weak equilibrium and extended models

In Chapter 4 we have seen that an equilibrium in the sense of K. Back exists if the informed trader has either full information or a rather special kind of sequential information. In the present chapter, we introduce a modified notion of equilibrium. It is based on the idea that the pricing rule should minimize the expected combined profit of noise trading and insider trading. In addition we consider some extensions of the basic model: the insider information may jump at the final time, it may be delayed, and there may be several insiders with different degrees of information.

5.1. A modified notion of equilibrium

In the general context of Section 4.3, let us consider the expected final profit of an insider strategy $I \in \mathcal{I}_0$. In contrast to the last chapter, we now admit general pricing rules of the form

$$P_t = \begin{cases} h((X_u)_{u \le t}, t), & t < 1, \\ \\ h(S_1, 1), & t = 1, \end{cases} \tag{5.1}$$

where h is a nonanticipative functional on $C[0,1] \times [0,1]$ such that

$$E\left[\int_0^1 h^2((W_u)_{u \le t}, t) dt\right] < \infty; \tag{5.2}$$

see Definition 1.4. We assume that the process $(P_t)_{0 \le t \le 1}$ defined by (5.1) is a semimartingale whose paths are continuous on $[0,1)$.

From now on, we only consider insider strategies belonging to $\mathcal{I}_0$. As before, the cumulative order process induced by $I \in \mathcal{I}_0$ is given by

$$X_t = W_t + \int_0^t Y_u \, du \qquad (0 \le t \le 1).$$

Lemma 5.1. *Under the preceding assumptions on the pricing rule, the expected profit of a strategy $I \in \mathcal{I}_0$ is given by*

$$E[\Lambda_1(h, I)] = E[I_1 h(S_1, 1)] = E[X_1 h(S_1, 1)]. \tag{5.3}$$

In particular, it is independent of the special choice of the pricing rule $h(\cdot, t)$ for time $t < 1$.

PROOF. Using the same argument as in Section 4.1 we see that the final profit of an absolutely continuous strategy $I \in \mathcal{I}$ is given by

$$\begin{aligned}\Lambda_1(h,I) &= (P_1 - P_{1-})I_1 + \int_0^{1-} I_u dP_u \\ &= I_1 P_1 - \int_0^1 P_u dI_u,\end{aligned}$$

where we have used Itô's product rule in the last step. Since $(W_t)_{0\leq t\leq 1}$ is a Brownian motion with respect to the filtration$(\mathcal{F}_t^{W,S})$ and independent of S_1, we get

$$\begin{aligned}E[\Lambda_1(h,I)] &= E\left[(I_1+W_1)P_1 - \int_0^1 P_u d(I_u+W_u)\right] \\ &= E\left[X_1 h(S_1,1) - \int_0^1 P_u dX_u\right]; \qquad (5.4)\end{aligned}$$

here we use the fact that condition (5.2) implies

$$E\left[\int_0^1 P_u dW_u\right] = 0.$$

If I belongs to $\mathcal{I}_0$, then $(X_t)_{0\leq t\leq 1}$ is again a Wiener process. Using again (5.2) we see that equation (5.4) reduces to (5.3). □

In view of Lemma 5.1, we denote the expected final profit of the insider by $E[\Lambda_1(I)]$ instead of $E[\Lambda_1(h,I)]$. Let us introduce the following notion of profit maximization for the insider.

DEFINITION 5.1 (**Inconspicuous profit maximization**). An inconspicuous insider strategy $I^\star \in \mathcal{I}_0$ is called *optimal* if $I^\star$ maximizes the expected final profit $E[\Lambda_1(I)] = E[I_1 h(S_1,1)]$ of the insider for all $I \in \mathcal{I}_0$, i.e.,

$$E[\Lambda_1(I^\star)] = \max_{I\in\mathcal{I}_0} E[\Lambda_1(I)]. \qquad (5.5)$$

In the following discussion we will restrict ourselves to certain subclasses $\tilde{\mathcal{I}}_0$ of $\mathcal{I}_0$.

LEMMA 5.2. *Consider a subclass $\tilde{\mathcal{I}}_0$ of $\mathcal{I}_0$ consisting of strategies of the form*

$$I_t = \int_0^t \left(f(u)S_0 + \int_0^u F(u,v)dS_v + \int_0^u H(u,v)dX_v\right) du,$$

where $f \in L^2(0,1)\cap C(0,1)$ and F and H are Volterra kernels satisfying $F(t,u)\sqrt{V'(u)}$, $H(t,u) \in L^2((0,1)\times(0,1))$; see Section 2.2. A strategy $I^\star \in \tilde{\mathcal{I}}_0$ satisfies

$$E[\Lambda_1(I^\star)] = \max_{I\in\tilde{\mathcal{I}}_0} E[\Lambda_1(I)]$$

if and only if $I^\star$ minimizes the L^2-distance between $X_1 = W_1 + I_1$ and the final signal S_1 for all $I \in \tilde{\mathcal{I}}_0$, i.e.,

$$E[(X_1^\star - S_1)^2] = \min_{I\in\tilde{\mathcal{I}}_0} E[(X_1 - S_1)^2],$$

where $X_1^\star = W_1 + I_1^\star$*. In particular, optimality of* $I^\star$ *with respect to* $\tilde{\mathcal{I}}_0$ *does not depend on the special choice of the pricing rule for all time* $0 \le t \le 1$*. Furthermore, if* $X_1^\star = S_1$*, then* $I^\star$ *is an optimal inconspicuous insider strategy (in* $\mathcal{I}_0$*).*

PROOF. 1) First we recall that

$$0 \le E[Nh(N,1)] < \infty, \tag{5.6}$$

for all $N(0,1)$-distributed random variable N. In fact, since $h(\cdot,1)$ is strictly increasing, we see that

$$h(x,1) - h(-x,1) \ge 0, \qquad \text{for all } x \ge 0,$$

and this implies,

$$\begin{aligned} E[Nh(N,1)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x h(x,1) \exp\left(-\frac{x^2}{2}\right) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} (h(x,1) - h(-x,1))\, x \exp\left(-\frac{x^2}{2}\right) dx \ge 0. \end{aligned}$$

The second inequality in (5.6) follows from (4.2) and the Cauchy-Schwartz inequality.
2) Due to the special linear form of the strategies in $\tilde{\mathcal{I}}_0$, the distribution of (X_1, S_1) is Gaussian. Thus, the conditional expectation of X_1 given S_1 is of the form

$$E[X_1|S_1] = \frac{E[X_1 S_1]}{E[S_1^2]} S_1 = E[X_1 S_1] S_1.$$

Due to (5.3) we get

$$E[\Lambda_1(I)] = E[h(S_1,1)X_1] = E[h(S_1,1)E[X_1|S_1]] = E[S_1 h(S_1,1)]E[X_1 S_1]. \tag{5.7}$$

Moreover, we know that the final signal S_1 and the price function $h(\cdot,1)$ are already fixed before the trading begins. Hence, the insider can only influence the term $E[X_1 S_1]$. Furthermore, due to 1) and

$$\begin{aligned} E\left[(X_1 - S_1)^2\right] &= E\left[X_1^2\right] + E\left[S_1^2\right] - 2E\left[X_1 S_1\right] \\ &= 2\left(1 - E[X_1 S_1]\right), \end{aligned}$$

we conclude that

$$\begin{aligned} & E\left[(X_1^\star - S_1)^2\right] &&= \min_{I \in \tilde{\mathcal{I}}} E\left[(X_1 - S_1)^2\right] \\ \iff\; & E[S_1 X_1^\star] &&= \max_{I \in \tilde{\mathcal{I}}} E[S_1 X_1] \\ \iff\; & E[\Lambda_1(I^\star)] &&= \max_{I \in \tilde{\mathcal{I}}} E[\Lambda_1(I)]. \end{aligned}$$

3) If $X_1^\star = S_1$, then

$$\min_{I \in \mathcal{I}_0} E\left[(X_1 - S_1)^2\right] \le \min_{I \in \tilde{\mathcal{I}}_0} E\left[(X_1 - S_1)^2\right] = 0.$$

This implies

$$E[\Lambda_1(I^\star)] = \max_{I \in \mathcal{I}_0} E[\Lambda_1(I)],$$

i.e., $I^\star$ is an optimal inconspicuous strategy in $\mathcal{I}_0$. □

Let us now discuss the role of the market maker. Since the cumulative order process in the whole market is (X_t), the combined gain of the informed and uninformed traders is given by

$$\left((P_1 - P_{1-})X_1 + \int_0^{1-} X_u dP_u\right), \tag{5.8}$$

depending on the choice of the pricing rule in (5.1). We are going to characterize those pricing rules which generate a martingale with respect to the filtration $(\mathcal{F}_t^X)$ up to time $1-$, and which minimize expectation of the combined gain (5.8).

Definition 5.2 (**Weak Market Efficiency**). Given an inconspicuous strategy $I \in \mathcal{I}_0$, a pricing rule $h(\cdot, t)$ is called *rational with respect to I* if the following two conditions hold:
(i) The price process $P_t = h((X_u)_{u\leq t}, t)$ satisfies

$$P_t = E[P_{1-}|\mathcal{F}_t^X], \tag{5.9}$$

for all $t < 1$, and $\mathcal{L}(P_1) = \mathcal{L}(P_{1-})$. In other words, the price process is a martingale with respect to the information of the market maker $(\mathcal{F}_t^X)$ up to time $1-$, and the distribution of P_{1-} is the same as that of P_1.
(ii) The expected combined profit of the informed and uninformed traders is minimal, i.e.,

$$E\left[(h(S_1, 1) - h((X_u)_{u<1}, 1-))X_1 + \int_0^{1-} X_t dh((X_u)_{u\leq t}, u \leq t)\right] \stackrel{!}{=} \min, \tag{5.10}$$

over all pricing rules satisfying the assumptions preceding Lemma 5.1.

Remark 5.1. Suppose that the total profit on the market amounts to 0. Then the profit of the market maker is equal to the entire loss of the insider and the noise traders, i.e., at the terminal time 1 the market maker earns

$$-\left((P_1 - P_{1-})X_1 + \int_0^{1-} X_u dP_u\right).$$

Thus, the problem (5.10) may be viewed as a problem of profit maximization for the market maker. It means that the market maker has also a dealer function. This idea appears in Stoll [**55**] and also in the the bid-ask spread model of Copeland-Galai [**17**].

Define the class $\mathcal{H}$ as the set of all pricing rules satisfying the first condition in Definition 5.2, i.e.,

$$\begin{aligned}\mathcal{H} \;=\; \{&P = (P_t)_{0\leq t<1} : P \text{ is a uniformly integrable martingale with respect to } (\mathcal{F}_t^X) \\ &\text{and } \mathcal{L}(P_1) = \mathcal{L}(P_{1-})\}.\end{aligned}$$

First let us look at a lemma.

LEMMA 5.3. *Let Z be an $N(0,1)$-distributed random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $h(\cdot)$ be a continuous increasing function. Then*

$$\max_N E\left[Zh(N)\right] = E\left[Zh(Z)\right],$$

where the maximum is taken over all random variables N on $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution $N(0,1)$. Furthermore, the maximum is attained if and only if $N = Z$, $\mathbb{P}$-a.s..

PROOF. Without loss of generality, we may assume $h(0) = 0$. Let N be an $N(0,1)$-distributed random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. Due to the Fubini Theorem we get

$$\begin{aligned} E[Zh(N)] &= E\left[Z\int_0^\infty 1_{(N\geq s)}dh(s)\right] - E\left[Z\int_{-\infty}^0 1_{(N\leq s)}dh(s)\right] \\ &= \int_0^\infty E\left[Z1_{(N\geq s)}\right]dh(s) - \int_{-\infty}^0 E\left[Z1_{(N\leq s)}\right]dh(s). \end{aligned}$$

Consider the maximum of $E[Z1_A]$ over all sets A with $\mathbb{P}(A) = 1 - \Phi(s)$, where $s \geq 0$ and Ψ is the distribution function of $N(0,1)$. Let $A^\star = \{Z \geq s\}$. Then for any set A with $\mathbb{P}(A) = 1 - \Phi(s)$ and $\mathbb{P}(A \setminus A^\star) > 0$, we obtain that

$$\begin{aligned} E[Z1_A] &= E[Z1_{A\setminus A^\star}] + E[Z1_{A\cap A^\star}] \\ &< s\,\mathbb{P}(A\setminus A^\star) + E[Z1_{A\cap A^\star}] = s\,\mathbb{P}(A^\star\setminus A) + E[Z1_{A\cap A^\star}] \\ &\leq E[Z1_{A^\star\setminus A}] + E[Z1_{A\cap A^\star}] = E[Z1_{A^\star}]. \end{aligned}$$

We see that the maximum is attained if and only if $A = A^\star = \{Z \geq s\}$. In the same way we see that the minimum of $E[Z1_A]$ over all sets A with $\mathbb{P}(A) = \Phi(s)$ and $s \leq 0$ is $E[Z1_{(Z\leq s)}]$. Hence,

$$E[Zh(N)] \leq \int_0^\infty E\left[Z1_{(Z\geq s)}\right]dh(s) - \int_{-\infty}^0 E\left[Z1_{(Z\leq s)}\right]dh(s) = E[Zh(Z)]$$

for all $N \sim N(0,1)$, and the equality holds if and only if $N = Z$, $\mathbb{P}$-a.s.. □

The following theorem gives the explicit form of a rational pricing rule in this class $\mathcal{H}$.

THEOREM 5.1. *Given an inconspicuous strategy $I \in \mathcal{I}_0$, there exists a unique pricing rule in the class $\mathcal{H}$ which is rational with respect to I in the sense of Definition 5.2. It is given by the space-time harmonic function $h(\cdot,\cdot)$ in (4.6), hence independent of the special choice of $I \in \mathcal{I}_0$. In particular, the resulting price P_t is only a function of the current cumulative orders X_t, not a functional of the past.*

PROOF. Let us fix $I \in \mathcal{I}_0$. Suppose $P \in \mathcal{H}$ with $P_{1-} = h(N,1)$, where N is an $N(0,1)$-distributed random variable. The expected combined final profit of the informed and uninformed traders is given by

$$E\left[(P_1 - P_{1-})X_1 + \int_0^{1-} X_u dP_u\right] = E\left[(P_1 - P_{1-})X_1\right], \tag{5.11}$$

since P is an $(\mathcal{F}_t^X)$-martingale up to $1-$ and X is a Brownian motion. Hence, the market maker wants to minimize

$$E\left[(h(S_1,1) - h(N,1))X_1\right] = E\left[h(S_1,1)X_1\right] - E\left[h(N,1)X_1\right],$$

over all $N(0,1)$-distributed random variables N. Since $P_1 = h(S_1,1)$ is fixed and $X_1 = W_1 + I_1$ is also fixed for the given insider strategy I, the market maker cannot influence the first term on the right-hand side. Hence, this optimization problem can be reformulated as follows: the market maker wants to find an $N(0,1)$-distributed random variable $N^\star$ such that

$$E\left[h(N^\star,1)X_1\right] = \max_{N\sim N(0,1)} E\left[h\left(N,1\right)X_1\right].$$

Due to Lemma 5.3 we see that $N^\star = X_1$. Thus, if the market maker determines the price (P_t) with $P_{1-} = h(X_1,1)$, he minimizes the expected combined terminal profit of the informed and uninformed traders. Moreover, from the martingale property and the strong Markov property, the price process should be of the form

$$P_t = E[P_{1-}|\mathcal{F}_t^X] = E[h(X_1,1)|\mathcal{F}_t^X] = E^{X_t}[h(X_{1-t},1)] = h(X_t,t), \tag{5.12}$$

i.e., h is the space-time harmonic pricing rule given by (4.6) above. □

Combining Definition 5.1 and Definition 5.2 we may define a modified equilibrium as follows.

DEFINITION 5.3 (**Weak Equilibrium**). Consider a pricing rule $h \in \mathcal{H}$ and an insider strategy $I^\star \in \mathcal{I}_0$. The pair $(h, I^\star)$ is called a *weak equilibrium* if h and $I^\star$ satisfy the conditions of inconspicuous profit maximization (Definition 5.1) and weak market efficiency (Definition 5.2). Furthermore, if $I^\star$ is an optimal inconspicuous insider strategy in some subclass $\tilde{\mathcal{I}}_0$, i.e.,

$$E[\Lambda_1(I^\star)] = \max_{I\in\tilde{\mathcal{I}}_0} E[\Lambda_1(I)],$$

then $(h, I^\star)$ is called a weak equilibrium in $\tilde{\mathcal{I}}_0$.

Now let us look at the relation between the notion of equilibrium in the sense of Definition 4.2 and of Definition 5.3.

THEOREM 5.2. *Let $I^\star \in \mathcal{I}_0$ and $W_1 + I_1^\star = S_1$. Then the pair $(h, I^\star)$, where h is given by* (4.6)*, is an equilibrium both in the sense of Definition 4.2 and in the sense of Definition 5.3.*

PROOF. 1) Due to Proposition 4.1 we see that $(h, I^\star)$ is an equilibrium in the sense of Definition 4.2.
2) From the proof of Theorem 5.1 we see that $P_{1-} = h(X_1, 1)$. Since $X_1 = S_1$, we get $P_{1-} = h(X_1, 1) = h(S_1, 1) = P_1$. This implies that $(P_t)_{0\le t\le 1}$ is a martingale with respect to $(\mathcal{F}_t^X)$, which coincides with the rational pricing rule in the sense of Definition 4.2. □

This theorem implies that the equilibrium defined in Definition 4.2 is equivalent to the one defined in Definition 5.3, provided that there is a process $I \in \mathcal{I}_0$ which yields $W_t + I_t \to S_1$ as $t \to 1$. In Kyle [**42**], Back [**7**] and Back-Pedersen [**11**] equilibrium is discussed only in this case. But if there is no insider in the market, a pricing rule which is rational in the sense of Definition 4.2 satisfies

$$P_t = E[P_1|\mathcal{F}_t^X] = E[h(S_1,1)|\mathcal{F}_t^W] = E[h(S_1,1)],$$

and this is of course unrealistic. In our weak equilibrium, the resulting rational pricing rule $h(x,t) = x$ due to Theorem 5.1. In the next three sections, we will introduce some other cases of insider information where the difference between these two notions of equilibrium will appear.

In the sequel, we consider two examples where a weak equilibrium can be computed explicitly.

EXAMPLE 5.1. As in Kyle [**42**], we suppose that the final price of the stock is given by $P_1 = h(S_1, 1) = m + \sigma S_1$, where $m \in \mathbb{R}$ and $\sigma > 0$. Thanks to Theorem 5.1 we know that the expected combined final profit of the informed and uninformed traders is minimal if the market maker determines the stock price at time t in the following form:

$$P_t = E[P_{1-}|\mathcal{F}_t^X] = E[h(X_1,1)|\mathcal{F}_t^X] = E[m + \sigma X_1|\mathcal{F}_t^X] = m + \sigma X_t.$$

Hence, the expected gain of the noise traders amounts to

$$E\left[(P_1 - P_{1-})W_1 + \int_0^{1-} W_u dP_u\right] = E\left[P_1 W_1\right] - E\left[\langle P, W\rangle_{1-}\right] = -\sigma.$$

The expected combined final profit of the informed and uninformed traders is given by

$$E[P_1 X_1] - E[\langle P, X\rangle_{1-}] = E[P_1 X_1] - \sigma.$$

(a) If there is no insider in the market,

$$E[P_1 X_1] = E[h(S_1,1)W_1] = 0.$$

Thus, the expected combined final profit of the informed and liquidity traders equal to $-\sigma$.

(b) If there is an informed trader with full information, we know from Theorem 5.2 that it is optimal for the insider to drive the cumulative demand (X_t) to the final point S_1, i.e., an optimal inconspicuous insider strategies is of the form

$$I_t^\star = \int_0^t \frac{S_1 - X_u}{1-u} du. \tag{5.13}$$

This implies that the expected final profit of the insider is equal to σ, hence the expectation of the informed and uninformed traders' combined final gain equals 0.

Example 5.2. Suppose that the final price of the stock is a log-normally distributed random variable with positive constants m and σ, i.e, $P_1 = \exp(m + \sigma S_1)$. Due to Theorem 5.1 we conclude that the *equilibrium* pricing rule is given by

$$P_t \;=\; E[P_{1-}|\mathcal{F}_t^X] = E[\exp(m+\sigma X_1)|\mathcal{F}_t^X] = \exp\left(m + \frac{1}{2}\sigma^2(1-t) + \sigma X_t\right),$$

which is a *geometric Brownian motion* relative to $(\mathcal{F}_t^X)$, i.e., P_t satisfies

$$dP_t = \sigma P_t \, dX_t,$$

with initial value $P_0 = \exp(m+\sigma^2/2)$. The expected combined loss of the uninformed traders is given by

$$\begin{aligned} -E\left[(P_1 - P_{1-})W_1 + \int_0^{1-} W_u dP_u\right] &= E[\langle P, W\rangle_{1-}] = E[\langle P, X\rangle_{1-}] \\ &= \sigma \exp\left(m + \frac{1}{2}\sigma^2\right). \end{aligned}$$

On the other hand, if the total profit in the market is equal to 0, the expected profit of the market maker is

$$E[\langle P, X\rangle_{1-}] - E[P_1 X_1] = \sigma \exp\left(m + \frac{1}{2}\sigma^2\right) - E[P_1 X_1],$$

which depends on the cumulative orders in the whole market at time 1. The associated expected profit of the informed trader is $E[P_1X_1]$. Let us consider two extreme cases:

(a) If there is no insider in the market,

$$E[P_1 X_1] = E[h(S_1, s)W_1] = 0.$$

We conclude that the expected final gain of the market maker is given by $\sigma \exp\left(m + \frac{1}{2}\sigma^2\right)$.

(b) If the informed trader has full information, the optimal inconspicuous insider strategy is given by (5.13), which implies $X_1 = S_1$. Hence, the maximal profit of the insider is given by

$$E[P_1 X_1] = E\left[S_1 \exp\left(m + \sigma S_1\right)\right] = \sigma \exp\left(m + \frac{1}{2}\sigma^2\right).$$

In this case, the expected profit of the market maker is equal to 0.

As the last two examples, we see that the expected loss of the noise trader is given by

$$-E\left[(P_1 - P_{1-})W_1 + \int_0^{1-} W_u dP_u\right] = E[\langle P, X\rangle_{1-}] = E[\langle h(X,\cdot), X\rangle_{1-}],$$

which does not depend on the insider strategy, i.e., even though there is no insider in the market, the noise traders will lose a certain amounts of wealth. But when the uninformed traders always lose their money, why should they trade? Milgrom-Strokey [**47**] propose a "no trade equilibrium", where the uninformed traders trade because of some particular exogenous reasons.

At the end of this section let us consider the model of insider trading with increasing extra information which we have introduced in Section 4.3. Suppose the insider information consists in observing a continuous centered square-integrable Gaussian martingale (S_t) with $var(S_1) = 1$.

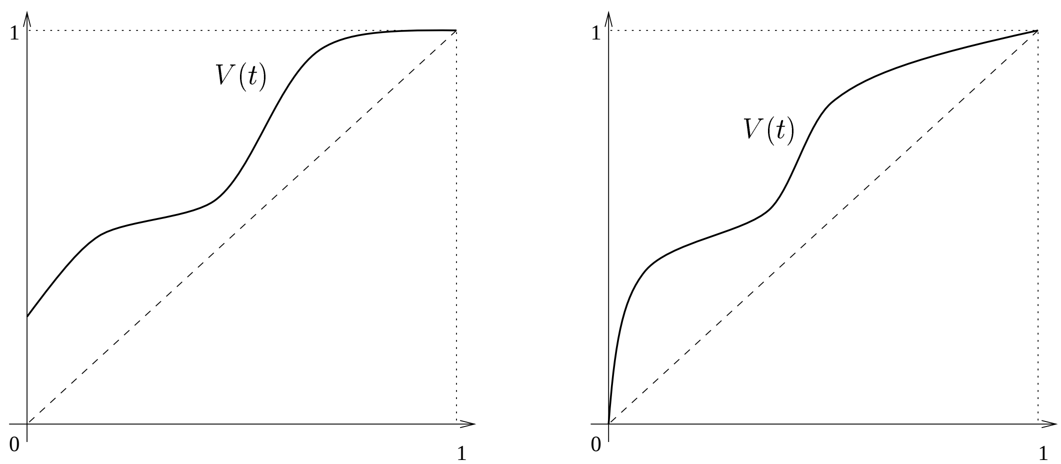

FIGURE 5.1

Case 1: The variance function of the insider extra information $V(t)$ satisfies the integrability conditions

$$\frac{1}{V(u) - u} \in \mathcal{A}(0,1) \cap L^1_{loc}([0,1));$$

see Figure 5.1. Due to the second assertion of Theorem 2.4 and Theorem 5.2, we conclude that the strategy

$$I^\star_t = \int_0^t \frac{S_u - X_u}{V(u) - u} du$$

is the unique optimal inconspicuous strategy for the insider in the class $\mathcal{I}_1$. Thus, in this case, there exists a weak equilibrium.

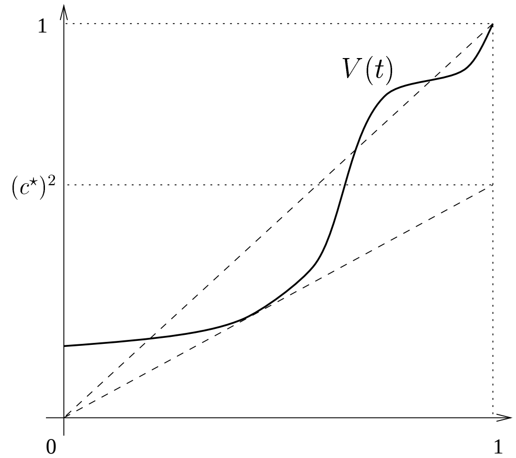

FIGURE 5.2

Case 2: Suppose the variance function of S satisfies

$$\frac{1}{V(u) - c^2 u} \in \mathcal{A}(0,1) \cap L^1([0,1)), \tag{5.14}$$

for some positive constant $c < 1$, but it does not hold for $c = 1$ (See Figure 5.2. The case $c = 1$ is concluded in case 1). Define $\tilde{\mathcal{I}}_1$ as the class of all $I \in \mathcal{I}_1$ such that the process $X = W + I$ solves the linear stochastic differential equation

$$dX_t = dW_t + \frac{cS_t - c^2 X_t}{V(t) - c^2 t} dt, \tag{5.15}$$

for some c satisfying (5.14). It follows from the first assertion in Theorem 2.4 that for all $I \in \tilde{\mathcal{I}}_1$, $W + I$ is a Brownian motion with respect to its own filtration. Furthermore, given a process $I \in \tilde{\mathcal{I}}_1$, we see that the solution of (5.15) is given by

$$\begin{aligned} X_t &= \int_0^t \exp\left(-\int_u^t \frac{c^2}{V(v) - c^2 v} dv\right) dW_u \\ &\quad + \int_0^t \exp\left(-\int_u^t \frac{c^2}{V(v) - c^2 v} dv\right) \frac{cS_u}{V(u) - c^2 u} du. \end{aligned}$$

In fact, we only have to consider the case where $P_1 = S_1$ to get an optimal inconspicuous strategy in the general case; see Lemma 5.2. Hence, when the insider follows the strategy I, the resulting expected final profit amounts to

$$\begin{aligned} E[\Lambda_1(I)] &= E[P_1 I_1] = E[S_1 X_1] = \int_0^1 \exp\left(-\int_u^1 \frac{c^2}{V(v) - c^2 v} dv\right) \frac{cV(u)}{V(u) - c^2 u} du \\ &= c. \end{aligned}$$

Define

$$c^\star := \sup\left\{ c > 0 : \frac{1}{V(u) - c^2 u} \in \mathcal{A}(0,1) \cap L^1(0,t) \text{ for all } t < 1 \right\},$$

whose square is the largest slope of the linear function under the curve $V(t)$ and passing through the origin. Using a similar argument as in the proof of Lemma 2.5, we conclude that (5.14) does not hold for $c^\star$. This implies

$$c^\star = \sup_{I \in \tilde{\mathcal{I}}_1} E[\Lambda_1(I)],$$

but a maximum on the right-hand side is not attained. Thus, there is no equilibrium in $\tilde{\mathcal{I}}_1$ in this case.

Case 3: Suppose S does not satisfy the above two cases, i.e.,

$$\lim_{t \to 0} \left(\frac{V(t)}{t} \right) < \infty.$$

Figure 5.3 shows two typical examples. In the special case where S is a Wiener process $\tilde{W}$, we see from Subsection 4.3.2 that there exists no insider strategy $I^\star$ such that

$$E[\Lambda_1(I^\star)] = \max_{I \in \mathcal{I}_0} E[\Lambda_1(I)].$$

This implies that there exists no equilibrium in this case. In the case where the variance of the insider information $V(t)$ is t^p for some $p \geq 1$, (5.14) does not hold for all $c > 0$. It follows from Theorem 2.4 that there exists no inconspicuous insider strategy of the form

$$dX_t = dW_t + (f(t)S_t + g(t)X_t)\, dt,$$

with initial value $X_0 = 0$, where f and g are nonzero continuous functions satisfy $\int_0^t f^2(u) u^p du < \infty$, for all $t < 1$ and $g \in \mathcal{A}(0,1)$. However, it does not imply that the insider should not trade at all. He could start to trade a little later. In Section 5.3 we will discuss this further.

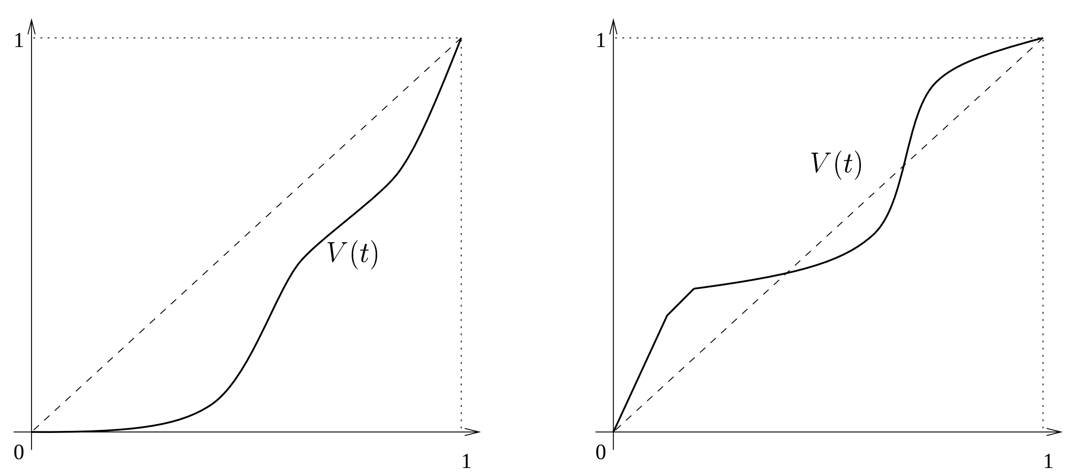

FIGURE 5.3

5.2. Noisy information

In this section we consider the case where the insider information is obtained by observing the process

$$S_t = \begin{cases} S_1 + cN, & 0 \le t < 1, \\ S_1, & t = 1, \end{cases}$$

where S_1 and N are independent with standard normal distribution $N(0,1)$, and where c is a nonnegative constant. In contrast to the previous chapter, the Gaussian process (S_t) is now constant during the interval $[0,1)$ and then jumps to the final value S_1. Thus, at time $t < 1$ the insider observes the final signal corrupted by some noise. We call c the *size of the noise*. In Karatzas-Pikovsky [**39**] the authors discuss this kind of insider information, but in the context of a small investor model where the insider is a price taker. In our model prices are affected in this context by the insider strategy. We want to examine the existence of weak equilibrium. According to Theorem 5.1 a rational pricing rule exists. Hence we have only to find an optimal inconspicuous strategy for the informed trader.

As before we denote by $(X_t)_{0\le t\le 1}$ the cumulative orders in the market. Suppose that the insider follows a strategy given by a linear transformation of $S_1 + cN$ and X. More precisely, we introduce the class $\mathcal{I}^c$ of all strategies $I = (I_t)_{0\le t\le 1} \in \mathcal{I}_0$ such that the process $X = W + I$ solves the equation

$$X_t = W_t + \int_0^t (f(u)(S_1 + cN) + g(u)X_u)du, \tag{5.16}$$

for some $f \in C^1(0,1) \cap L^2_{loc}([0,1))$ and $g \in C^1(0,1) \cap \mathcal{A}(0,1)$. The requirement $I \in \mathcal{I}_0$ means that X is again a Brownian motion. The following proposition gives a characterization of these strategies. In other words, we characterize Brownian motions in the class of processes X which are of the form (5.16).

PROPOSITION 5.1. *A process X satisfying* (5.16) *is a Brownian motion with respect to its natural filtration if and only if f and g are of the following form*

$$\begin{aligned} f(t) &= \frac{M}{M^2(1+c^2) - t}, \\ g(t) &= \frac{-1}{M^2(1+c^2) - t}, \end{aligned} \tag{5.17}$$

where $M^2(1+c^2) \ge 1$. In other words, X is a solution of

$$dX_t = dW_t + \frac{M(S_1 + cN) - X_t}{M^2(1+c^2) - t}dt, \tag{5.18}$$

with initial value $X_0 = 0$.

PROOF. We use a similar technique as in Chapter 2. If a solution of X (5.16) is a Wiener process, then due to the characterization in Theorem 2.2 the functions f and g must satisfy the relation

$$f(s)f(t) = (sg(s) - 1)g(t),$$

for almost all $s \leq t$. This implies $f(t) = -Mg(t)$ for some constant M. Substituting this result again into the above equation, we get (5.17). Conversely, the solution of the process X satisfying (5.18) is of the form

$$X_t = (M^2(1+c^2) - t)\int_0^t \frac{dW_u}{M^2(1+c^2) - u} + \frac{t}{M(1+c^2)}(S_1 + cN). \tag{5.19}$$

Hence for $s \leq t$, we have

$$\begin{aligned} E[X_s X_t] &= (M^2(1+c^2) - s)(M^2(1+c^2) - t)\int_0^s \frac{du}{(M^2(1+c^2) - u)^2} + \frac{st(1+c^2)}{M^2(1+c^2)^2} \\ &= s. \end{aligned}$$

This means that X is a Brownian motion. □

From this proposition we know that all processes in the class $\mathcal{I}^c$ are of the form

$$\begin{aligned} I_t &= \int_0^t \frac{M(S_1 + cN) - X_u}{M^2(1+c^2) - u} du \\ &= \frac{t}{M(1+c^2)}(S_1 + cN) - (M^2(1+c^2) - t)\int_0^t \frac{W_u}{(M^2(1+c^2) - u)^2} du. \end{aligned}$$

Using this strategy, the expected final wealth of the insider is given by

$$E[\Lambda_1(I)] = E[P_1 I_1] = E[P_1 X_1] = \frac{E[h(S_1,1)S_1]}{M(1+c^2)}.$$

It follows from $M^2(1+c^2) \geq 1$ and $E[h(S_1,1)S_1] \geq 0$ that

$$E[\Lambda_1(I)] \leq \frac{E[h(S_1,1)S_1]}{\sqrt{1+c^2}},$$

for all $I \in \mathcal{I}^c$. The maximum is attained at $M = 1/\sqrt{1+c^2}$, and the corresponding strategy of the informed trader is given by

$$I_t^\star = \int_0^t \frac{(1+c^2)^{-\frac{1}{2}}(S_1 + cN) - X_u}{1-u} du. \tag{5.20}$$

EXAMPLE 5.3. Suppose the final price of the stock P_1 is normally distributed, i.e., $h(S_1, 1) = m + \sigma S_1$ with constants $\sigma > 0$ and $m \in \mathbb{R}$. Then the final profit of the insider is given by

$$E[\Lambda_1] = \frac{E[P_1 S_1]}{\sqrt{1+c^2}} = \frac{\sigma}{\sqrt{1+c^2}}.$$

The associated optimal inconspicuous strategy in $\mathcal{I}^c$ is of the form (5.20), which converges to $(S_1 + cN)/\sqrt{1+c^2}$ as $t \to 1$. Hence, we see that

$$E[(X_1^\star - S_1)^2] = \min_{I \in \mathcal{I}^c} E[(X_1 - S_1)^2] \neq 0.$$

This means that no process I in $\mathcal{I}^c$ yields $W_1+I_1=S_1$. Furthermore, due to Example 5.1 we know that the price process is of the form $P_t = m+\sigma X_t$. Hence, the expected profit of the uninformed traders is $-\sigma$, as in Example 5.1. The expected combined profit of the informed and uninformed traders amounts to $-\sigma(1-1/\sqrt{1+c^2})$.

For the case $c=0$, we see that the extra information is S_1, the final signal. The corresponding optimal strategy is given by

$$I_t^\star = \int_0^t \frac{S_1 - X_u}{1-u} du.$$

This coincides with the results in Kyle [**42**] and Back [**7**] (see also Section 4.2).

EXAMPLE 5.4. Suppose that the final price is of the form $P_1 = \exp(m+\sigma S_1)$ with $\sigma > 0$ and $m \in \mathbb{R}$. Then, due to Lemma 5.2 and the above example, the strategy $I^\star$ defined in (5.20) is an optimal inconspicuous strategy in $\mathcal{I}^c$ and the optimal final profit of the insider is given by

$$E[\Lambda(I^\star)] = \frac{1}{\sqrt{1+c^2}}\,\sigma \exp\left(m + \frac{1}{2}\sigma^2\right).$$

The expected gain of the uninformed traders is $-\sigma\exp(m+\sigma^2/2)$.

REMARK 5.2. Let us consider the equilibrium in the sense of Definition 4.2 for this case. We have shown in Example 5.3 that no process in $\mathcal{I}^c$ converges to $S_1 - W_1$ and that the optimal strategy $I^\star$ in $\mathcal{I}^c$ is given by (5.20). If the final price P_1 is equal to $m+\sigma S_1$, then the rational price process in the sense of Back's Definition 4.2 should be of the form

$$P_t^B = E[h(S_1,1)|\mathcal{F}_t^X] = E[m+\sigma S_1|\mathcal{F}_t^X] = m + \frac{\sigma}{\sqrt{1+c^2}} X_t,$$

i.e., the rational pricing rule is given by

$$h^B(x,t) = m + \frac{\sigma\, t}{\sqrt{1+c^2}},$$

for $t<1$. Hence, $(h^B, I^\star)$ is an equilibrium in the sense of Definition 4.2. Furthermore, we see that this equilibrium depends on the insider information and $P_t^B \neq P_t$ if $c \neq 0$. In this equilibrium the expected final wealth of the insider is $\sigma/\sqrt{1+c^2}$, as in Example 5.3. But the expected combined profit of the informed and uninformed traders amounts to

$$\begin{aligned} E\left[(P_1 - P_{1-})X_1 - \int_0^{1-} X_u dP_u\right] &= E\left[\left((m+\sigma S_1) - \left(m + \frac{\sigma}{\sqrt{1+c^2}} X_1\right)\right) X_1\right] \\ &= \sigma\left(E[S_1 X_1] - \frac{1}{\sqrt{1+c^2}}\right) = 0, \end{aligned}$$

which is not equal to the result in Example 5.3.

5.3. Delayed information

So far we have discussed the case where the insider gets his extra information at the initial time 0. In the present section we deal with the case where the insider gets his information at time $t_0 > 0$. Can he invest in such a way that the cumulative demand in the market is again a Brownian motion? If so, which strategy yields the maximal expected profit?

As before we suppose that the extra information is given by a continuous centered square-integrable Gaussian martingale $(S_t)_{0 \le t \le 1}$ with $var(S_1) = 1$. The informed trader starts to observe this process only at time t_0. In order not to be discovered, the informed trader has to invest in such a way that the cumulative demand in the market X is again a Brownian motion.

Consider the stochastic differential equation

$$dX_t = \begin{cases} dW_t, & t \le t_0, \\ \\ dW_t + (f(t)S_t + g(t)W_{t_0} + h(t)X_t)\,dt, & t_0 < t \le 1, \end{cases} \tag{5.21}$$

with $X_0 = 0$, $X_{t_0} = W_{t_0}$ and continuously differentiable functions f, g, h satisfying $\int_{t_0}^t f^2(u)V(u)du < \infty$, for all $t \in (t_0, 1)$, $g \in L^1_{loc}([t_0, 1))$, $h \in \mathcal{A}(t_0, 1)$.

In the following proposition we characterize those cases where the process given by (5.21) is again a Wiener process.

Proposition 5.2. *A process X satisfying* (5.21) *is a Brownian motion if and only if the functions f, g and h are of the form:*

$$\begin{aligned} f(t) &= \frac{c}{c^2V(t) + t_0 - t}, \\ g(t) &= \frac{1}{c^2V(t) + t_0 - t}, \\ h(t) &= \frac{-1}{c^2V(t) + t_0 - t}, \end{aligned} \tag{5.22}$$

where c is a nonzero constant satisfying

$$\frac{1}{c^2V(u) + t_0 - u} \in \mathcal{A}(t_0, 1) \cap L^1_{loc}([t_0, 1)). \tag{5.23}$$

Thus, the process X satisfying

$$dX_t = \begin{cases} dW_t, & \text{for } t < t_0, \\ \\ dW_t + \dfrac{cS_t + W_{t_0} - X_t}{c^2V(t) + t_0 - t}dt, & \text{for } t \ge t_0, \end{cases} \tag{5.24}$$

is a Brownian motion with respect to its own filtration.

Proof. 1) If the process X is a Brownian motion, the covariance function satisfies $E[X_sX_t] = s$ for all $0 \le s \le t \le 1$. For $t_0 \le s \le t$, applying Theorem 2.1 and

Theorem 2.2, we see that f, g and h must satisfy

$$h(t)(sh(s)-1)=V(s)f(s)f(t)+t_0 g(s)g(t). \tag{5.25}$$

For $s\le t_0\le t$, we have due to (5.21)

$$X_t=W_t+\int_{t_0}^t\left(f(u)S_u+g(u)W_{t_0}+h(u)X_u\right)du.$$

Multiplying both sides by X_s and taking expectation, we see that

$$\begin{aligned} s &= E[X_sX_t]=E[X_sW_t]+\int_{t_0}^t\left(g(u)E[X_sW_{t_0}]+h(u)E[X_sX_u]\right)du \\ &= s+s\int_{t_0}^t(g(u)+h(u))du. \end{aligned}$$

This yields $g(t)+h(t)=0$. Substituting this result in (5.25), we see that the associated solution is given by (5.22).
2) The solution of (5.24) is given by

$$X_t=\begin{cases} W_t, & \text{for } t\le t_0,\\ W_{t_0}+\dfrac{1}{G(t)}\displaystyle\int_{t_0}^t G(u)dW_u+\dfrac{c}{G(t)}\displaystyle\int_{t_0}^t\frac{G(u)S_u}{c^2V(u)+t_0-u}du, & \text{for } t>t_0, \end{cases} \tag{5.26}$$

with a deterministic function

$$G(t):=\exp\left(\int_{t_0}^t\frac{1}{c^2V(u)+t_0-u}du\right).$$

This implies $E[X_sX_t]=s$ after some calculation. □

Proposition 5.2 shows that possible inconspicuous insider strategies are of the form

$$\begin{aligned} I_t &:= \begin{cases} 0, & \text{for } t<t_0\\ \displaystyle\int_{t_0}^t\frac{cS_u+W_{t_0}-X_u}{c^2V(u)+t_0-u}du, & \text{for } t\ge t_0 \end{cases}\\ &= \begin{cases} 0, & \text{for } t<t_0\\ \displaystyle\int_{t_0}^t\left(\frac{G(u)}{G(t)}-1\right)dW_u+\frac{c}{G(t)}\int_{t_0}^t\frac{G(u)S_u}{c^2V(u)+t_0-u}du, & \text{for } t\ge t_0, \end{cases} \end{aligned} \tag{5.27}$$

for some c satisfying (5.23). We denote the set of such strategies by $\mathcal{I}(t_0)$. In particular, we have $\mathcal{I}(0)=\tilde{\mathcal{I}}_1$. Note that $\mathcal{I}(s)\cap\mathcal{I}(t)=\{0\}$, if $s\ne t$.

Let us consider the condition (5.23). Using a similar argument as in Section 2.3 we see that a necessary condition for the constant c to satisfy (5.23) is given by

$$c^2\ge\sup_{t_0\le t\le 1}\left(\frac{t-t_0}{V(t)}\right), \tag{5.28}$$

i.e.,

$$\frac{1}{c^2} \leq \inf_{t_0 \leq t \leq 1} \left(\frac{V(t)}{t - t_0} \right).$$

This means, $1/c^2$ is the positive slope of an affine linear function under $V(t)$ which passes $t = t_0$ and does not intersect $V(t)$ on $[t_0, 1)$. Hence, we have to calculate the lower bound of c by (5.28) and check whether all constants c larger than this lower bound satisfy condition (5.23).

Sometimes, the insider is forced to delay his investment if he wants to remain inconspicuous. For example, if the insider information yields $\lim_{t\to 0}(V(t)/t) < \infty$, he cannot apply linear strategies of the form

$$I_t = \int_0^t (f(u)S_u + g(u)X_u)du,$$

where f and g are continuous and satisfy some integrability conditions; see Theorem 2.4 above. But when he starts his trading later, he can use linear inconspicuous strategies of the form (5.27). However, we have to find the optimal time where the insider starts his trading. We need three steps to compute this optimal starting time.

a) For fixed $t_1 > t_0$, let

$$k = \sup_{t_1 \leq t \leq 1} \left(\frac{t - t_1}{V(t)} \right)^{\frac{1}{2}}.$$

Furthermore, we have to check whether k satisfies (5.23).

b) If the insider starts his trading at time t_1, the supremum of his expected profit is given by

$$\sup_{I \in \mathcal{I}(t_1)} E[\Lambda_1(I)] = \frac{1 - t_1}{k}.$$

If k satisfies (5.23), there exists an optimal inconspicuous insider strategy in $\mathcal{I}(t_1)$.

c) Let

$$t^\star := \arg\max_{t \geq t_0} \sup_{I \in \mathcal{I}(t)} E[\Lambda_1(I)].$$

Then $t^\star$ is the optimal time for the insider to start his trading.

In the next example, we will calculate this time explicitly.

Example 5.5. Consider Example 4.1 with $\alpha = \sigma = 1$, i.e., the final price of the stock is $P_1 = S_1$. The price process P_t is equal to X_t, and the expected profit of the insider is given by

$$E[\Lambda_1(I)] = E[S_1 X_1] = \frac{c}{G(1)} \int_{t_0}^1 \frac{G(v)V(v)}{c^2 V(v) + t_0 - v} dv = \frac{1 - t_0}{c},$$

where we use (5.26) and integration by parts. In order to obtain the maximal expected profit of the insider in the class $\mathcal{I}(t_0)$, we have to find a minimal positive constant c satisfying (5.28). Consider the following special cases:

(1) Full information: Due to (5.28), we see that

$$c^2 \geq \sup_{t_0 \leq t \leq 1} \left(\frac{t - t_0}{V(t)} \right) = 1 - t_0.$$

Furthermore, we see that for $(c^\star)^2 = 1 - t_0$,

$$\frac{1}{(c^\star)^2 V(u) + t_0 - u} = \frac{1}{1-u} \in \mathcal{A}(t_0, 1) \cap L^1_{loc}([t_0, 1)).$$

This implies that $c^\star$ is valid for (5.23). Thus, the maximal value of $E[\Lambda_1(I)]$ is $\sqrt{1-t_0}$ with $c^\star = \sqrt{1-t_0}$. Therefore, the optimal strategy of the insider in this class of strategies $\mathcal{I}(t_0)$ is of the form:

$$I^\star_t = \begin{cases} 0, & \text{for } t < t_0, \\ \displaystyle\int_{t_0}^t \frac{\sqrt{1-t_0} S_1 + W_{t_0} - X_u}{1-u} du, & \text{for } t \geq t_0. \end{cases} \tag{5.29}$$

The expected profit of the noise traders is -1. If $t_0 = 0$, the result coincides with the one in Section 4.2.

(2) Sequential information $(\tilde{W}_t)_{t_0 \leq t \leq 1}$ $(t_0 > 0)$: Since $var(\tilde{W}_t) = t$, we have

$$c^2 \geq \sup_{t_0 \leq t \leq 1} \left(\frac{t - t_0}{var(\tilde{W}_t)} \right) = \sup_{t_0 \leq t \leq 1} \left(\frac{t - t_0}{t} \right) = 1 - t_0.$$

We see that for $(c^\star)^2 = 1 - t_0$,

$$\frac{1}{(c^\star)^2 V(u) + t_0 - u} = \frac{1}{t_0(1-u)} \in \mathcal{A}(t_0, 1) \cap L^1_{loc}([t_0, 1)),$$

provided $t_0 \neq 0$. Therefore, if $t_0 \neq 0$, the optimal strategy in $\mathcal{I}(t_0)$ is of the form

$$I^\star_t = \begin{cases} 0, & \text{for } t < t_0, \\ \displaystyle\int_{t_0}^t \frac{\sqrt{1-t_0}\, \tilde{W}_u + W_{t_0} - X_u}{t_0(1-u)} du, & \text{for } t \geq t_0, \end{cases}$$

and the corresponding maximal profit is $\sqrt{1-t_0}$. In fact, for $t_0 \to 0$, we get the expected final profit of the insider converges to 1. Hence, there exists a sequence of insider strategies whose expected profit converges to the supremum of $E[\Lambda(I)]$. Nevertheless, if $t_0 = 0$, there exists no inconspicuous insider strategy in $\bigcup_{0 \leq t \leq 1} \mathcal{I}(t)$. This coincides with the results in Section 4.3.2, i.e., there exists no optimal inconspicuous strategy for the insider, but he can come arbitrarily close to the maximal value.

(3) Sequential information with $V(t) = ((t - t_0)/(1 - t_0))^p$ for $0 < p < 1$: Since

$$\sup_{t_0 \leq t \leq 1} \left(\frac{t - t_0}{V(t)} \right) = \sup_{t_0 \leq t \leq 1} (t - t_0)^{1-p} (1 - t_0)^p = 1 - t_0,$$

(5.23) holds for $c^2 = 1 - t_0$. The maximal insider profit in $\mathcal{I}(t_0)$ is $\sqrt{1-t_0}$ and the corresponding strategy is given by

$$I_t^\star = \begin{cases} 0, & \text{for } t < t_0, \\ \displaystyle\int_{t_0}^t \frac{cS_u + W_{t_0} - X_u}{(u-t_0)[(u-t_0)^{p-1}(1-t_0)^{p-1} - 1]} du, & \text{for } t \geq t_0. \end{cases}$$

The case $p = 0$ is just like the case (1) above, i.e., $V(1) = 1 = E[S_1^2]$.

(4) Partial information with $V(t) = ((t-t_0)/(1-t_0))^p$ for $p > 1$: Since

$$\sup_{t_0 \leq t \leq 1} \left(\frac{t - t_0}{V(t)} \right) = \frac{(1-t_0)^p}{(t-t_0)^{p-1}} = \infty,$$

we have $c = \infty$. This means that if the insider does not want to be discovered, he cannot use strategies in the class $\tilde{\mathcal{I}}_1$ except $I \equiv 0$. Nevertheless, it does not mean that the insider should not invest. If he submits orders after time $t_1 > t_0$, he could still get a positive profit without being discovered.

In the following we want to calculate the optimal time for the insider to begin his investment. Let

$$\begin{aligned} k^2 &= \sup_{t_1 \leq t \leq 1} \left(\frac{t-t_1}{V(t)} \right) = \sup_{t_1 \leq t \leq 1} \frac{(t-t_1)(1-t_0)^p}{(t-t_0)^p} \\ &= \begin{cases} \left(\dfrac{p-1}{t_1 - t_0} \right)^{p-1} \left(\dfrac{1-t_0}{p} \right)^p, & \text{if } t_1 \in (t_0, 1 - \dfrac{1-t_0}{p}), \\ 1 - t_1, & \text{if } t_1 \in [1 - \dfrac{1-t_0}{p}, 1]. \end{cases} \end{aligned}$$

We can check that if $t_1 \in (t_0, 1 - (1-t_0)/p)$, the constant k does not satisfy (5.23). On the other hand, if $t_1 \in [1 - (1-t_0)/p, 1]$, (5.23) holds for k. Therefore, if the insider starts to trade at time t_1, the supremum of his expected gain for strategies in $\mathcal{I}(t_1)$ is given by

$$\begin{aligned} \sup_{I \in \mathcal{I}(t_1)} E[\Lambda_1(I)] &= \frac{1-t_1}{k} \\ &= \begin{cases} \sqrt{\dfrac{p^p(1-t_1)(t_1-t_0)^{(p-1)}}{(p-1)^{(p-1)}(1-t_0)^p}}, & \text{if } t_1 \in (t_0, 1 - \dfrac{1-t_0}{p}), \\ \sqrt{1-t_1}, & \text{if } t_1 \in [1 - \dfrac{1-t_0}{p}, 1]. \end{cases} \end{aligned}$$

Since for $t_1 \in [1 - (1-t_0)/p, 1]$, k satisfies (5.23). This implies that $\sqrt{1-t_1}$ is not only supremum, but also maximum of the expected final profit. In order to calculate

$$t^\star = \arg \max_{t \geq t_0} \sup_{I \in \mathcal{I}(t)} E[\Lambda_1(I)]$$

and the corresponding expected profit, we have to consider two different cases.

i) If $1 < p \leq 2$, the optimal time for the insider to start trading is $t^\star = 1-(1-t_0)/p$. He can use the strategy

$$I_t^\star = \begin{cases} 0, & \text{for } t < t^\star, \\ \displaystyle\int_{t^\star}^{t} \frac{\sqrt{p(1-t_0)}\, S_u + W_{t^\star} - X_u}{(u-t_0)^p(1-t_0)^{1-p} + p(1-u) - (1-t_0)}\, du, & \text{for } t \geq t^\star. \end{cases}$$

to attain his maximal profit $\sqrt{(1-t_0)/p}$.

ii) If $p > 2$, we get that

$$\arg\max_{t \geq t_0} \sup_{I \in \mathcal{I}(t)} E[\Lambda_1(I)] = \frac{1+t_0}{2},$$

i.e., the optimal time for the insider to start his trading is $(1+t_0)/2$. But the insider can only come arbitrarily close to the supremum of his expected profit

$$\sqrt{\frac{p^p}{2^{(p+1)}(p-1)^{(p-1)}}(1-t_0)}.$$

In particular, we may consider the case where $t_0 = 0$, i.e., the insider information consists in observing a continuous centered Gaussian martingale S with $E[S_t^2] = t^p$ starting at time 0. Since $\inf_{0 \leq t \leq 1}(V(t)/t) = 0$, we conclude from the above discussion that the insider should not trade immediately. Explicitly, we get the following results.

	$1 < p < 2$	$p \geq 2$
optimal time to start trading	$1 - 1/p$	$1/2$
maximal expected final profit	$\sqrt{1/p}$ (maximum)	$\sqrt{\frac{p^p}{2^{(p+1)}(p-1)^{(p-1)}}}$ (supremum)
Existence of equilibrium	yes	no

For the case $1 < p < 2$, the optimal insider strategy in $\bigcup_{0 \leq t \leq 1} \mathcal{I}(t)$ is given by

$$I_t^\star = \begin{cases} 0, & \text{for } t < 1 - \dfrac{1}{p}, \\ \displaystyle\int_{1-1/p}^{t} \frac{\sqrt{p}\, S_u + W_{1-1/p} - X_u}{u^p + p(1-u) - 1}\, du, & \text{for } t \geq 1 - \dfrac{1}{p}. \end{cases}$$

(5) Partial information with $V(t) = \frac{1}{4}(2t^2 + t + 1)$.

i) $t_0 > 0.2$: The function $(t-t_0)/V(t)$ is increasing on the interval $(t_0, 1)$. This implies

$$c^2 \geq \sup_{t_0 \leq t \leq 1} \left(\frac{t-t_0}{V(t)}\right) = \frac{1-t_0}{V(1)} = 1 - t_0.$$

For $c=\sqrt{1-t_0}$, the condition (5.23) holds. Therefore, the optimal strategy in $\mathcal{I}(t_0)$ is of the form

$$I_t^\star = \begin{cases} 0, & \text{for } t < t_0 \\ \displaystyle\int_{t_0}^t \frac{4(\sqrt{1-t_0}S_u + W_{t_0} - X_u)}{(1-u)(1+3t_0-2u(1-t_0))}du, & \text{for } t \geq t_0. \end{cases}$$

The maximal profit of the insider is given by $\sqrt{1-t_0}$.

ii) $t_0 \leq 0.2$: The function $(t-t_0)/V(t)$ attains its maximum on $[t_0,1]$ at $\tilde{t} = t_0 + \sqrt{t_0^2 + \frac{1}{2}t_0 + \frac{1}{2}}$. Thus,

$$c^2 \geq \sup_{t_0 \leq t \leq 1} \left(\frac{t-t_0}{V(t)}\right) = \frac{\tilde{t}-t_0}{V(\tilde{t})} = \frac{16}{7}\left(\sqrt{t_0^2 + \frac{t_0}{2} + \frac{1}{2}} - t_0 - \frac{1}{4}\right) =: \tilde{c}^2.$$

Hence, the optimal insider strategy in $\mathcal{I}(t_0)$ is of the form

$$I_t^\star = \begin{cases} 0, & \text{for } t < t_0, \\ \displaystyle\int_{t_0}^t \frac{4(|\tilde{c}|S_u + W_{t_0} - X_u)}{\tilde{c}^2(2u^2+u+1)+4t_0-4u}du, & \text{for } t \geq t_0. \end{cases}$$

The corresponding profit is given by $(1-t_0)\sqrt{\sqrt{t_0^2+\frac{t_0}{2}+\frac{1}{2}}+t_0+\frac{1}{4}}$ and it is easy to check that this value is strictly smaller than $\sqrt{1-t_0}$.

From the discussion of case (4), we see that the insider may not be able to use linear strategies from the beginning of the trading, since otherwise he would be discovered. However, we can also provide an example where the insider can get some positive profit, provided that he invests from time 0 on. But if he delays his orders and starts at some positive time, he could get more gain. For instance, let

$$V(t) = \begin{cases} \dfrac{7}{6}t - \dfrac{1}{6}, & \text{if } \dfrac{1}{4} \leq t < 1, \\ \dfrac{1}{4}t + \dfrac{1}{16}, & \text{if } t < \dfrac{1}{4}. \end{cases}$$

We see that $\sup_{0\leq t\leq 1}(V(t)/t)$ is equal to $1/2$. The supremum of the insider's profit in $\tilde{\mathcal{I}}_1$ is $\sqrt{2}/2 \approx 0.707107$. But this supremum will not be attained. Consequently, *there is no weak equilibrium in* $\tilde{\mathcal{I}}_1$. However, if the insider starts to trade a little later, he could get more profit. Since

$$\arg\max_{0\leq t\leq 1} \frac{1-t}{\sup_{t\leq s\leq 1}\left(\dfrac{s-t}{V(s)}\right)^{1/2}} = \frac{1}{7},$$

and since the associated expected profit amounts to $(6/7)^{3/2} \approx 0.79356$, we see that in this case the insider obtains more profit if he begins to trade first at time

1/7 instead of trading from the beginning (see Figure 5.4). Moreover, this value is maximal. It means *there exists a weak equilibrium in* $\bigcup_{0\le t\le 1}\mathcal{I}(t)$.

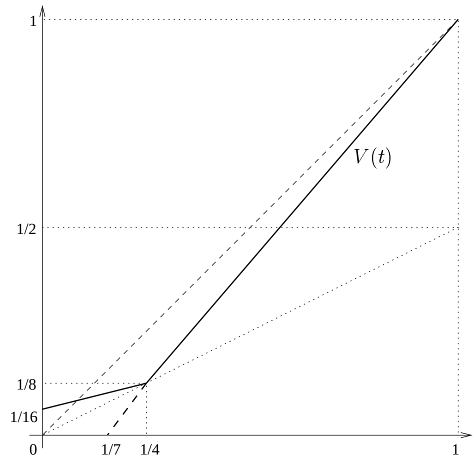

FIGURE 5.4

EXAMPLE 5.6. Suppose that the final price of the stock P_1 is of the form

$$P_1 = \exp(m + \sigma S_1),$$

with constants $m \in \mathbb{R}$ and $\sigma > 0$. Thanks to (5.7) we see that the final profit of the insider with the strategy (5.27) is given by

$$E[\Lambda_1(I)] = E[\exp(m+\sigma S_1)X_1] = \frac{\sigma(1-t_0)}{c}\exp\left(m+\frac{\sigma^2}{2}\right).$$

We have to find a minimal positive constant c satisfying (5.28). The rest of the discussion is just as in the above example.

REMARK 5.3. Now let us look at the equilibrium in the sense of Definition 4.2 in the model of delayed information. Consider the case (1) in Example 5.5, i.e., the case where the insider knows the final signal S_1 from the time t_0 on. We know that the optimal inconspicuous strategy is of the form (5.29). Using this optimal strategy, the cumulative order process (X_t) converges to $\sqrt{1-t_0}S_1 + W_{t_0}$ as $t \to 1$. If (P_t^B) is a rational price process in the sense of Back's Definition 4.2, it must satisfy the martingale property for $t \in [0,1]$. Thus,

$$\begin{aligned} P_t^B &= E[S_1|\mathcal{F}_t^X] = E\left[\frac{1}{\sqrt{1-t_0}}(X_t - W_{t_0})\middle|\,\mathcal{F}_t^X\right] \\ &= \frac{1}{\sqrt{1-t_0}}X_t - \frac{1}{\sqrt{1-t_0}}E[W_{t_0}|\mathcal{F}_t^X] = \begin{cases} 0, & t \le t_0, \\ \dfrac{1}{\sqrt{1-t_0}}(X_t - X_{t_0}), & t > t_0. \end{cases} \end{aligned}$$

For $t \ge t_0$, P_t^B is not of the form $h(X_t, t)$. Hence, we conclude that there is no equilibrium in the sense of Definition 4.2 if we consider only insider strategies in $\mathcal{I}(t_0)$,.

5.4. A model with two insiders

Up to now we have discussed the situation where there is one insider in the market. In the present section we want to consider the case of two insiders with different degrees of information. Such model has been discussed in, for example, Back-Cao-Willard [**10**].

Suppose the final stock price is given by $P_1 = h(N_1, N_2)$, where h is a smooth function on $\mathbb{R}^2$ and N_1, N_2 are two independent $N(0,1)$-distributed random variables. We classify the agents in the market into four groups: market maker, noise traders, partially informed trader and fully informed trader. The roles of the market maker and the noise traders are as before. The cumulative order of the noise trader is a Brownian motion W which is independent of N_1 and N_2. Both insiders can observe the cumulative orders in the market. The partially informed trader knows N_1 at time 0; he is not aware of the existence of the other signal N_2 and the other insider, and he assumes that the final price will be a function of N_1. The discussion in Section 4.2 suggests that his strategy will be given by

$$I_t^{(1)} = \int_0^t \frac{N_1 - X_u}{1-u} du, \tag{5.30}$$

where X is the cumulative demand in the market. The fully informed traders is aware of the presence of the partially informed trader, and he observes the signals N_1 and N_2. We want to consider the optimal strategy of the fully informed trader under this condition.

For the fully informed trader, not to be discovered by the market maker is his main purpose. Hence, we define the class $\mathcal{I}^{(2)}$ of all strategies $(I_t^{(2)})$ of the fully informed trader of the form

$$I_t^{(2)} = \int_0^t (f(u)N_1 + g(u)N_2 + k(u)X_u)du, \tag{5.31}$$

where $f, g \in C^1(0,1) \cap L^2[0,1)$, $h \in C^1(0,1) \cap \mathcal{A}(0,1)$, and the process $W + I^{(1)} + I^{(2)}$ is again a Wiener process. In the next proposition we want to characterize Brownian motions of the form

$$\begin{aligned} dX_t &= dW_t + dI_t^{(1)} + dI_t^{(2)} \\ &= dW_t + \left(\left(f(t) + \frac{1}{1-t} \right) N_1 + g(t)N_2 + \left(k(t) - \frac{1}{1-t} \right) X_t \right) dt \\ &= dW_t + (F(t)N_1 + G(t)N_2 + K(t)X_t)\, dt. \end{aligned} \tag{5.32}$$

PROPOSITION 5.3. *The process X satisfying* (5.32) *is a Brownian motion if and only if the functions F, G, K are of the forms*

$$\begin{aligned} F(t) &= \frac{k}{k^2(1+c^2)-t}, \\ G(t) &= \frac{ck}{k^2(1+c^2)-t}, \\ K(t) &= \frac{-1}{k^2(1+c^2)-t}, \end{aligned} \tag{5.33}$$

where $k^2(1+c^2) \geq 1$.

PROOF. 1) Assume that X is a Brownian motion and the process $(\xi_t)_{0\leq t\leq 1}$ is defined by

$$\xi_t = X_t - \int_0^t K(u)X_u du \tag{5.34}$$

$$= W_t + \left(\int_0^t F(u)du\right)N_1 + \left(\int_0^t G(u)du\right)N_2. \tag{5.35}$$

The covariance of ξ in (5.34) is given by

$$\begin{aligned} E[\xi_s\xi_t] &= s - 2\int_0^s uK(u)du - s\int_s^t K(u)du + 2\int_0^s\int_0^u K(u)K(v)vdvdu \\ &\quad + \left(\int_0^s uK(u)du\right)\left(\int_s^t K(u)du\right). \end{aligned}$$

The covariance of ξ in (5.35) is given by

$$E[\xi_s\xi_t] = s + \left(\int_0^s F(u)du\right)\left(\int_0^t F(u)du\right) + \left(\int_0^s G(u)du\right)\left(\int_0^t G(u)du\right).$$

These two values should be identical. Differentiating with respect to s and t, we know that if the process X is a Brownian motion, F, G and K must satisfy the equation

$$K(t)(sK(s)-1) = F(s)F(t) + G(s)G(t),$$

for $s \leq t$. Solving this equation we get (5.33).
2) The solution of (5.32) with (5.33) is given by

$$X_t = (k^2(1+c^2)-t)\int_0^t \frac{dW_u}{k^2(1+c^2)-u} + \frac{t}{k(1+c^2)}(N_1 + cN_2). \tag{5.36}$$

Hence,

$$\begin{aligned} E[X_sX_t] &= (k^2(1+c^2)-s)(k^2(1+c^2)-t)\int_0^s \frac{du}{(k^2(1+c^2)-u)^2} + \frac{st}{k^2(1+c^2)} \\ &= \frac{(k^2(1+c^2)-t)s}{k^2(1+c^2)} + \frac{st}{k^2(1+c^2)} = s. \end{aligned}$$

This means that X is a Brownian motion. □

From this proposition we see that a strategy $I^{(2)}$ of the form (5.31) in $\mathcal{I}^{(2)}$ if and only if

$$\begin{aligned} f(t) &= \frac{k}{k^2(1+c^2)-t} - \frac{1}{1-t}, \\ g(t) &= \frac{ck}{k^2(1+c^2)-t}, \\ k(t) &= \frac{-1}{k^2(1+c^2)-t} + \frac{1}{1-t}, \end{aligned}$$

where $k^2(1+c^2) \geq 1$. Applying (5.36) we rewrite $I^{(2)}$ as

$$\begin{aligned} I_t^{(2)} &= B(W,t) + \int_0^t \left(f(u) + k(u)\frac{u}{k(1+c^2)} \right) du \cdot N_1 \\ &\quad + \int_0^t \left(g(u) + k(u)\frac{cu}{k(1+c^2)} \right) du \cdot N_2 \\ &= B(W,t) + \left(\left(1 - \frac{1}{k(1+c^2)} \right) N_1 - \frac{cN_2}{k(1+c^2)} \right) \log(1-t), \end{aligned} \tag{5.37}$$

where

$$B(W,t) := \int_0^t \left(\frac{k^2(1+c^2)-u}{1-u} - 1 \right) \int_0^u \frac{dW_v}{k^2(1+c^2)-v} du$$

Let us look at an example.

EXAMPLE 5.7. Suppose that the final price of the stock is given by $P_1 = S_1 = AN_1 + \sqrt{1-A^2}N_2$ with $0 \leq A \leq 1$. This implies $P_1 \sim N(0,1)$. From Example 5.1 we know that the pricing rule of the stock is $h(x,t) = x$, i.e., $(P_t)_{0 \leq t < 1} = (X_t)_{0 \leq t < 1}$ and this implies

$$\langle P, W \rangle_{1-} = \langle P, X \rangle_{1-} = 1.$$

Hence, the expected final profit of the noise traders is given by

$$\begin{aligned} E\left[(P_1 - P_{1-})W_1 + \int_0^{1-} W_u dP_u \right] &= E\left[W_1 P_1 - \int_0^1 P_u dW_u - \langle P, W \rangle_{1-} \right] \\ &= E\left[W_1(AN_1 + \sqrt{1-A^2}N_2) - 1 \right] = -1, \end{aligned}$$

which is independent of the strategies of both insiders. Now let us compute the expected profit of the insiders. Due to Itô's product rule, we get that the expected profit of the partially informed trader is given by

$$E\left[(P_1 - P_{1-})I_1^{(1)} + \int_0^{1-} I_u^{(1)} dP_u \right] = E\left[P_1 I_1^{(1)} - \int_0^1 P_u dI_u^{(1)} \right]$$

and that the expected profit of the fully informed trader is of the form

$$E\left[(P_1 - P_{1-})I_1^{(2)} + \int_0^{1-} I_u^{(2)} dP_u \right] = E\left[P_1 I_1^{(2)} - \int_0^1 P_u dI_u^{(2)} \right].$$

Furthermore, if the fully informed trader applies a strategy of the form (5.37), the cumulative order X is a Brownian motion satisfying (5.36). This implies that

$$
\begin{aligned}
& E\left[P_1 I_t^{(1)} - \int_0^t P_u dI_u^{(1)}\right] = E\left[S_1 I_t^{(1)} - \int_0^t X_u dI_u^{(1)}\right] \\
= & \int_0^t \frac{E[S_1 N_1] - E[S_1 X_u]}{1-u} du - \int_0^t \frac{E[N_1 X_1] - E[X_u^2]}{1-u} du \\
= & \int_0^t \frac{1}{1-u}\left(A - \frac{u}{k(1+c^2)}\left(A + c\sqrt{1-A^2}\right) - \frac{u}{k(1+c^2)} + u\right) du \\
= & \left(\left(\frac{1}{k(1+c^2)} - 1\right)(A+1) + \frac{c\sqrt{1-A^2}}{k(1+c^2)}\right)\log(1-t) \\
& \qquad + \left(\frac{A + c\sqrt{1-A^2}}{k(1+c^2)} + \left(\frac{1}{k(1+c^2)} - 1\right)\right) t.
\end{aligned}
$$

As $t \to 1$, this value converges to the expected profit of the partially informed trader, and this is dependent on the parameters c and k of the strategies of the fully informed trader. In addition, using a similar method we get that the expected profit of the fully informed trader amounts to

$$
\begin{aligned}
& E\left[P_1 I_1^{(2)} - \int_0^1 P_u dI_u^{(2)}\right] \\
= & \lim_{t\to 1} E\left[P_1 I_t^{(2)} - \int_0^t P_u dI_u^{(2)}\right] = \lim_{t\to 1} E\left[S_1 I_t^{(2)} - \int_0^t X_u dI_u^{(2)}\right] \\
= & \lim_{t\to 1}\left(\left(1 - \frac{1}{k(1+c^2)}\right)(A+1) - \frac{c\sqrt{1-A^2}}{k(1+c^2)}\right)\log(1-t) - \left(1 - \frac{1}{k(1+c^2)}\right) t.
\end{aligned}
$$

If the fully informed trader uses a strategy of the form (5.37) with

$$
0 < c < \sqrt{\frac{1-A}{1+A}} \qquad \text{and} \qquad \frac{1}{\sqrt{1+c^2}} < k < \frac{1}{1+c^2}\left(1 + c\sqrt{\frac{1-A}{1+A}}\right),
$$

then his expected final profit is equal to ∞, but at the same time that of the partially informed trader equals $-\infty$.

So far we have assumed that the partially informed trader is not aware of the existence of the fully informed trader. But, since the partially informed trader can observe the cumulative demand in the market, he also knows the process $W + I^{(2)}$. Hence, if the fully informed trader does not want to be discovered by the partially informed, he has to drive the process $W + I^{(2)}$ to be again a Brownian motion. However, we can prove that a process $I^{(2)}$ given by (5.31) turns both $W + I^{(1)} + I^{(2)}$

and $W + I^{(2)}$ into Brownian motions if and only if f, g and k are of the form

$$\begin{aligned} f(t) &= \frac{k}{k-t} - \frac{1}{1-t}, \\ g(t) &= \frac{\sqrt{k(k-1)}}{k-t}, \\ k(t) &= -\frac{1}{k-t} + \frac{1}{1-t}, \end{aligned}$$

with some constant $k > 1$. This implies that the expected profit of the fully informed trader amounts to

$$\lim_{t\to 1} \sqrt{\frac{(k-1)(1-A^2)}{k}} \log\left(\frac{1}{1-t}\right) = \infty,$$

provided $k > 1$ and $A < 1$. Hence, if the fully informed trader applies the strategy

$$\begin{aligned} & I_t^{(2)} \\ = & \int_0^t \left(\left(\frac{k}{k-u} - \frac{1}{1-u} \right) N_1 + \frac{\sqrt{k(k-1)}}{k-u} N_2 + \left(\frac{1}{1-u} - \frac{1}{k-u} \right) X_u \right) du \\ = & \left(k \log\left(\frac{k}{k-t}\right) + \log(1-t) \right) N_1 + \sqrt{k(k-1)} \log\left(\frac{k}{k-t}\right) N_2 \\ & + \int_0^t \left(\frac{1}{1-u} - \frac{1}{k-u} \right) X_u du, \end{aligned} \tag{5.38}$$

with $k > 1$, then the fully informed trader can get an infinite profit and the expected loss of the partially informed trader is also infinite. Moreover, we see that, if the fully informed trader follows the strategy (5.38), then the combined cumulative demand of the noise and fully informed traders is a Brownian motion independent of the random variable N_1. This means that the partially informed trader cannot discover the fully informed trader if the latter follows the strategy (5.38).

CHAPTER 6

Insider trading and information costs

In the last two chapters, we have studied pricing rules and insider strategies in some models where the insider's extra information is simply given. In the paper of Grossman-Stiglitz [**30**], the authors consider a model where the insider has to pay certain costs to obtain this extra information. In this chapter, our main purpose is to introduce information costs in our model and to analyze the optimal strategy of the insider in such a setting.

As in Chapter 4 and 5, we assume that all market participants are risk neutral. The uninformed traders have no extra information and their cumulative order process W is a Brownian motion. There is a market maker who determines the price of the stock according to the cumulative demand in the market. Furthermore, there is an investor who may obtain some additional information. But in contrast to the models we have discussed so far, we now consider the case where this agent has to pay some costs to get this information.

6.1. Noisy information

Suppose the full information is S_1, a standard normal random variable. The insider can buy at time $t = 0$ the noisy information $S_1 + cN$ at costs $\alpha(c) \geq 0$, where N is an $N(0,1)$ distributed random variable independent of S_1 and $c \geq 0$ is the size of the noise. If $c = 0$, the insider buys the exact information about the final stock price. If the size of the noise increases, the costs will decrease. Hence, we assume that the cost function $\alpha(c)$ is non-increasing in c.

For a given value of c, it has been shown in Section 5.2 that the inconspicuous insider strategy

$$I_t^\star = \int_0^t \frac{(1+c^2)^{-\frac{1}{2}}(S_1 + cN) - X_u}{1-u} du, \tag{6.1}$$

yields the maximal expected profit

$$E[\Lambda_1(I^\star)] = \frac{E[h(S_1,1)X_1]}{\sqrt{1+c^2}}.$$

Hence, after paying the information costs the expected final net profit of the insider amounts to

$$E[\Lambda_1(I^\star) - \alpha(c)] = \frac{E[h(S_1,1)X_1]}{\sqrt{1+c^2}} - \alpha(c).$$

Based on this quantity as a function of c, the insider will decide how much information to buy.

Let us look at an example.

EXAMPLE 6.1. Suppose the terminal stock price is given by S_1. Then the final profit of the insider is given by

$$E[\Lambda_1] = \frac{1}{\sqrt{1+c^2}} - \alpha(c),$$

and the associated optimal strategy $I^\star$ given by (6.1). Should the investor become an insider? If yes, what kind of information should he buy? Let us now look at some different cost functions $\alpha(c)$ and the corresponding optimal constants $c^\star$.

(i) $\alpha \equiv 0$. In other words, the insider candidate does not need to pay any costs to be an insider. The optimal expected final wealth is given by $1/\sqrt{1+c^2}$. Obviously, it is always profitable for the investor to get this information regardless of the size of the noise. Figure 6.1 shows the expected profit of insider with noisy information $S_1 + cN$. We see that this value is maximal if he gets the exact information ($c = 0$) at time $t = 0$. The corresponding optimal strategy is given by

$$I_t^\star = \int_0^t \frac{\tilde{W}_1 - X_u}{1-u} du.$$

This strategy coincides with the results in the papers of Kyle [**42**] and Back [**7**]; see section 4.2.

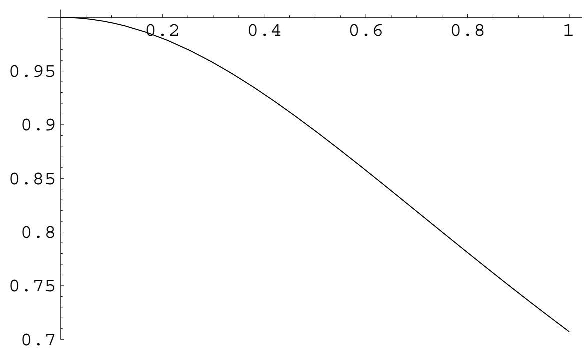

FIGURE 6.1. the case (i): $\alpha(c) \equiv 0$

(ii) $\alpha(c) = 1/(1+c^2)$. The insider has to pay $1/(1+c^2)$ to obtain the information $S_1 + cN$. Figure 6.2 shows that it is not optimal for the insider to buy the information without noise, since the full information is too expensive. In fact, we see that if the insider buys the exact information, his profit is exactly equal to 0. However, it is always profitable for the insider candidate to buy the information if $c > 0$. The optimal $c^\star$ is $\sqrt{3}$ and the corresponding profit is $1/4$. This implies that the optimal information for the insider is of the form $S_1 + \sqrt{3}N$ and the corresponding optimal strategy for the insider is given by

$$I_t^\star = \int_0^t \frac{S_1 + \sqrt{3}N - 2X_u}{2(1-u)} du.$$

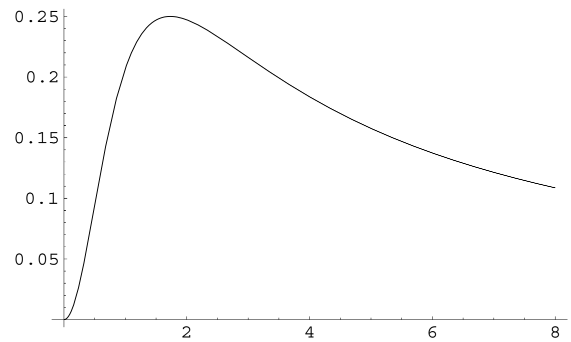

FIGURE 6.2. the case (ii): $\alpha(c) = \dfrac{1}{1+c^2}$

(iii) $\alpha(c) = 1/(4c)$. Figure 6.3 shows that if the investor buys the exact information, his profit will be $-\infty$. If the size of the noise c is larger than $1/\sqrt{15}$, the informed trader earns a positive profit. Thus, if $c < 1/\sqrt{15}$, it is not profitable for the insider candidate to buy the information. The optimal value of c is given by $c^\star = 0.811149\cdots$, and the corresponding maximal gain is $0.468422\cdots$.

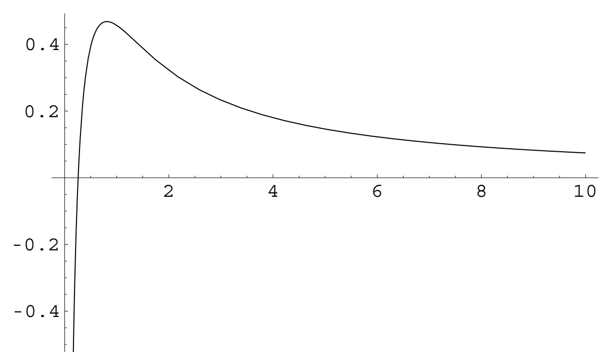

FIGURE 6.3. the case (iii): $\alpha(c) = \dfrac{1}{4c}$

This example shows that the decision of the investor to become an insider or not depends on the costs of the information. It may be profitable to buy some noisy information, and it may be better not to buy any information.

6.2. Increasing information

Suppose the investor can choose one of several different processes (S_t) of increasing extra information. Which process should he buy to maximize his expected profit?

Suppose that the insider can get increasing information by observing a continuous centered square-integrable Gaussian martingale $(S_t)_{0\le t\le 1}$ with $S_1 \sim N(0,1)$. Denote by $V_S(t)$ the variance function of S_t. Let us consider the L^2-distance between S_t and S_1 which is given by

$$E[(S_t - S_1)^2] = 1 - 2E[S_1 S_t] + E[S_t^2] = 1 - E[S_t^2] = 1 - V_S(t).$$

The smaller this distance is, the more valuable this information is. Thus, we introduce a cost function of the form

$$\int_0^1 \beta(V_S(u))du,$$

where $\beta : [0,1] \longrightarrow \mathbb{R}$ is continuous and nondecreasing in t. For example, the insider has to pay

$$\int_0^1 \beta(V_S(1))du = \beta(1)$$

for obtaining the process $S_t \equiv S_1$. If (S_t) is a Wiener process, the information cost is equal to $\int_0^1 \beta(u)du$. Especially, if $\beta(t) = t$ for all t, then information cost of (S_t) is equal to the area under the curve $V(t)$ (see the area A in Figure 6.4).

With these information costs we conclude that the expected final net profit of the informed trader is given by

$$E\left[\Lambda_1(I) - \int_0^1 \beta(V_S(u))du\right] = E\left[h(S_1,1)I_1 - \int_0^1 \beta(V_S(u))du\right].$$

Let us first consider the following simple example.

Example 6.2. Suppose the final price P_1 is given by a Gaussian random variable $S_1 \sim N(0,1)$. Furthermore, suppose $\beta(u)$ is given by βu with $\beta \ge 0$. Then the expected final profit of this information is given by

$$E[\bar{\Lambda}_1] := E\left[\Lambda_1(I) - \int_0^1 \beta(V_S(u))du\right] = E[S_1 I_1] - \beta\int_0^1 V_S(u)du.$$

First we recall some results from Section 5.1 and use them to compute the expected net profit of this information.

Case 1: If S satisfies the integrability conditions

$$\frac{1}{V_S(u) - u} \in \mathcal{A}(0,1) \cap L^1_{loc}([0,1)), \tag{6.2}$$

then using the strategy

$$I_t^\star = \int_0^t \frac{S_u - X_u}{V_S(u) - u}du,$$

the expected gross profit is 1; see Example 4.1. Hence, the expected net profit of the insider amounts to

$$E[\bar{\Lambda}_1] = 1 - \beta\int_0^1 V_S(u)du.$$

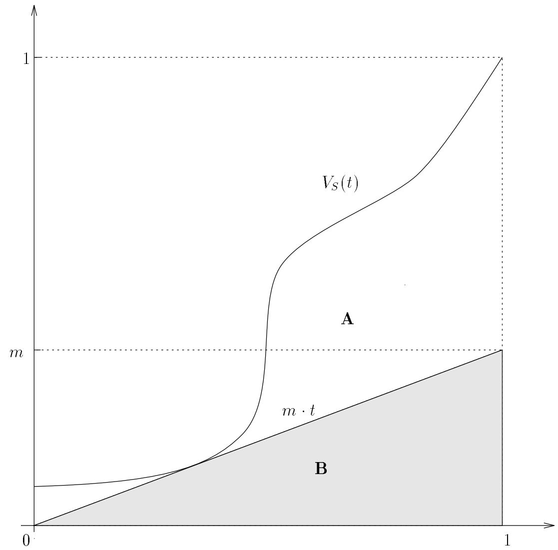

FIGURE 6.4

Since $V_S(t)$ satisfies (6.2), using the third assertion in Lemma 2.5 we conclude that $V_S(t) > t$ for all $t \in (0, 1)$. This results in

$$E[\bar{\Lambda}_1] < 1 - \beta \int_0^1 u du = 1 - \frac{\beta}{2}.$$

Case 2: If S is a Brownian motion $\tilde{W}$, the information cost amounts to

$$\int_0^1 \beta(V_S(u))du = \int_0^1 \beta u du = \frac{\beta}{2}.$$

Due to the discussion in Subsection 4.3.2, we see that the insider can use this information to realize an expected profit as close to $1 - \beta/2$ as he wishes.

Combining Case 1 and Case 2, we see that if $\beta > 1$, the information as in the above two cases is not valuable. If $\beta < 1$, it is profitable to buy the information S which forms a Brownian motion.

Case 3: Suppose $(S_t)_{0 \le t \le 1}$ is a continuous Gaussian martingale satisfying

$$\frac{1}{V_S(u) - c^2 u} \in \mathcal{A}(0, 1) \cap L^1_{loc}([0, 1)) \tag{6.3}$$

for some positive constant $c < 1$. But S does not satisfy (6.2). Let

$$c^\star := \sup \left\{ c > 0 : \frac{1}{V(u) - c^2 u} \in \mathcal{A}(0, 1) \cap L^1(0, t) \text{ for all } t < 1 \right\}.$$

Thus, supremum of $E[\Lambda_1(I)]$ amounts to $c^\star$. This implies that the expected gain $E[\bar{\Lambda}_1]$ is bounded above by $c^\star - \beta \int_0^1 V_S(u)du$. Furthermore, from the discussion in Section 2.3 we see that

$$c^2 \le m := \inf_{0 \le t \le 1} \left(\frac{V_S(t)}{t} \right) \le 1$$

for all c satisfying (6.3). This implies $c^\star \le \sqrt{m}$. We can interprete m as the smallest slope of the linear function passing the origin 0 which intersects the curve $V_S(u)$ on the interval $(0,1]$; see Figure 6.4. We see that the integral $\int_0^1 V_S(u)du$ is the area A under the curve $V_S(u)$, and this is larger than the area B which is $m/2$. Hence,

$$E[\bar{\Lambda}_1] < c^\star - \beta\,\frac{m}{2} \le \sqrt{m} - \frac{\beta\, m}{2} \le \begin{cases} 1 - \dfrac{\beta}{2}, & \text{if } \beta \le 1, \\[2ex] \dfrac{1}{2\beta}\left(> 1 - \dfrac{\beta}{2}\right), & \text{if } \beta > 1. \end{cases}$$

If $\beta \le 1$, the discussion is as in Cases 1 and 2. But if $\beta > 1$, it is advantageous for the insider to buy a process S whose variance function is, for example, of the form

$$V_S(t) = \begin{cases} \delta + \dfrac{1}{\beta^2}\, t, & \text{if } 0 \le t \le 1-\delta, \\[2ex] 1 - \left(1 - \dfrac{1}{\beta^2}\right)\left(\dfrac{1}{\delta} - 1\right)(1-t), & \text{if } 1-\delta < t \le 1, \end{cases} \tag{6.4}$$

for some $\delta > 0$ small enough.

Case 4: Suppose the variance function of S satisfies

$$\lim_{t\to 0}\left(\frac{V(t)}{t}\right) < \infty.$$

Then if the insider does not want to be discovered, he cannot use linear strategies of the form

$$I_t = \int_0^t (f(u)S_u + g(u)X_u)du,$$

where f and g are continuous functions satisfying some integrability conditions; see Theorem 2.4. However, he may start to trade at some later time t_0, but he has to pay the full information costs, since he obtains the information from time 0 on. Suppose the insider uses a delayed strategy of the form (5.27) where the constant c satisfies

$$\frac{1}{c^2} \le m := \inf_{t_0 \le t \le 1}\left(\frac{V(t)}{t - t_0}\right);$$

see Section 5.3. Then the expected net profit of this information is given by

$$\begin{aligned} E[\bar{\Lambda}_1] &< \sqrt{m}(1-t_0) - \beta\int_0^1 V_S(u)du \le \sqrt{m}(1-t_0) - \beta\frac{m(1-t_0)}{2} \\ &= (1-t_0)\left(\sqrt{m} - \frac{\beta m}{2}\right) \le \begin{cases} (1-t_0)\left(1 - \dfrac{\beta}{2}\right), & \text{if } \beta \le 1, \\[2ex] \dfrac{1-t_0}{2\beta}\left(< \dfrac{1}{2\beta}\right), & \text{if } \beta > 1. \end{cases} \end{aligned}$$

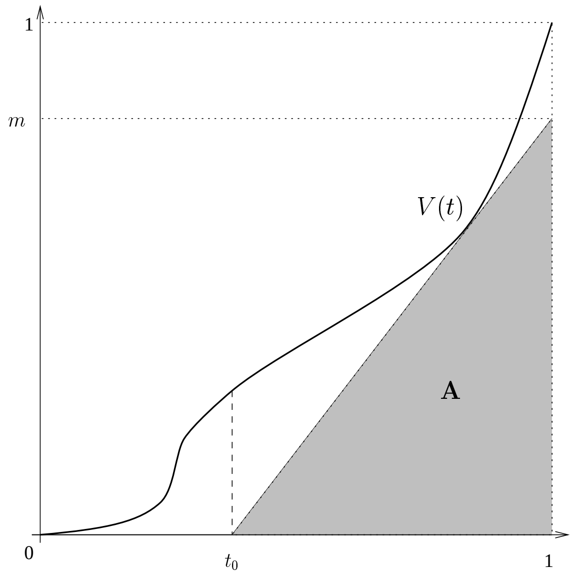

FIGURE 6.5

From the above discussion we conclude that it is profitable for the insider to buy information whose variance function is linear with slope $1/\beta^2 \wedge 1$ on the interval $[0, 1-\delta]$ for some small $\delta > 0$; for example, of the form (6.4).

6.3. Delayed information

In this section we look at the case where the insider can decide at which time he buys his extra (full or partial) information (S_u). When he buys the information at time $t > 0$, he pays no information cost during the time interval $[0, t]$. Some natural questions arise: At which point of time is it optimal for the investor to buy the information? Can he achieve his maximal expected gain while remaining inconspicuous?

Let (I_s) be the process defined by

$$I_s := \begin{cases} 0, & \text{for } s < t, \\ \displaystyle\int_t^s \frac{cS_u + W_t - X_u}{c^2 V(u) + t - u} du, & \text{for } s \geq t, \end{cases} \tag{6.5}$$

with

$$\frac{1}{c^2 V(u) + t - u} \in \mathcal{A}(t, 1) \cap L^1_{loc}([t, 1)).$$

From the discussion in Section 5.3 we see that if the insider starts to trade at time $t \in (0, 1)$ and follows the strategy I, he will not be discovered during the whole trading interval. Due to the discussion in the last section we see that the information cost from time t to 1 amounts to $\int_t^1 \beta(V_S(u))du$. Hence, the expected final profit is given by

$$E\left[\Lambda_1(I) - \int_t^1 \beta(V_S(u))du\right] = E\left[h(S_1,1)X_1 - \int_t^1 \beta(V_S(u))du\right].$$

Let us look at a simple example.

EXAMPLE 6.3. Suppose the final price of the stock is given by $P_1 = S_1$. Suppose the insider decides to purchase the information (S_u) first at time $t \geq 0$ and uses the strategy given by (6.5). Then his profit is given by

$$G(t) = E\left[\Lambda_1(I) - \int_t^1 \beta(V_S(u))du\right] = \frac{1-t}{c} - \int_t^1 \beta(V_S(u))du,$$

where β is the cost function. Let us consider the following cases:

(1) $\beta(u) \equiv 0$, i.e., no information costs. Then the discussion is just the same as in Example 5.5. The insider can reach his maximal profit if he gets the extra information from the beginning, but in this case the cumulative demand in the market is not necessarily a Brownian motion. This depends on the structure of the information process (S_u) the insider desires to buy. If (S_u) is a standard Brownian motion, he cannot use strategies of the form (6.5) if he does not want to be discovered. In the case $S_t \equiv S_1$ he can achieve the maximum while remaining inconspicuous.

(2) $\beta(t) = \beta \geq 0$. Since

$$\int_t^1 \beta(V_S(u))du = \int_t^1 \beta du = \beta(1-t),$$

the expected profit of the insider is given by

$$G(t) = \sqrt{1-t} - \beta(1-t).$$

If $\beta \leq 1/2$, it is profitable for the insider to buy the information at time 0. But whether he can reach his maximal profit without being discovered depends again on the structure of (S_u).

For the case $\beta > 1/2$, if the informed trader wants to reach his maximal profit, he should not buy the information at the beginning of the trading. It is easy to check that in this case the optimal point of time for the insider to buy the information is $1 - 1/(4\beta^2)$. In the following we look at the optimal time for the informed trader to buy the information for different values of the constant β.

(i) $\beta = 0.8$, i.e., the profit of the insider is given by

$$G(t) = \sqrt{1-t} - 0.8(1-t).$$

Figure 6.6 shows that $G(t)$ does not attain its maximum at $t = 0$, but at time $t = 39/64$. Thus, in this case it is not optimal for the insider to buy the information too early.

(ii) $\beta = 1$, i.e., the profit of the insider is given by

$$G(t) = \sqrt{1-t} - (1-t).$$

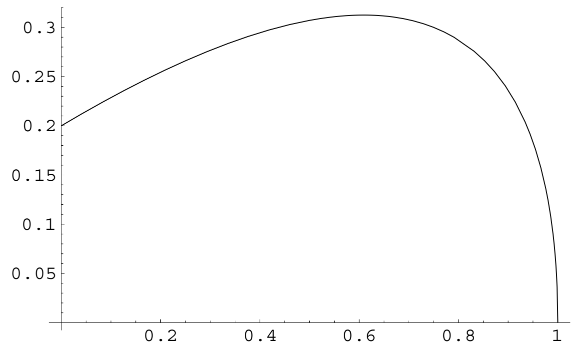

FIGURE 6.6. the case $\beta = 0.8$

In this case the optimal time to buy the information is at $t = 3/4$. But whenever the informed trader buys the information, he will get a non-negative profit; see Figure 6.7. Hence, it is always worth for the insider to buy the information.

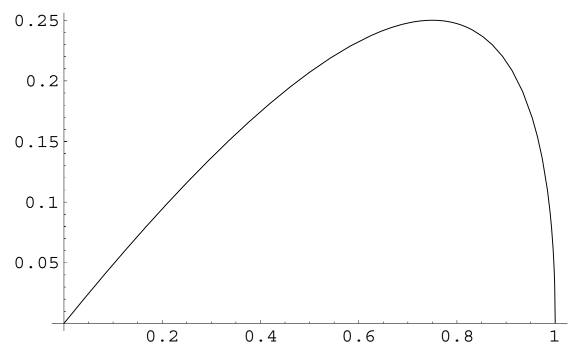

FIGURE 6.7. the case $\beta = 1$

(iii) $\beta = 1.2$. From Figure 6.8, we see that the insider should not buy his information too early, otherwise he will get a negative profit. More precisely, he should not buy the information at time $t < 11/36$. After this time, he will get a positive profit. The optimal time to buy the information is $t = 119/144$.

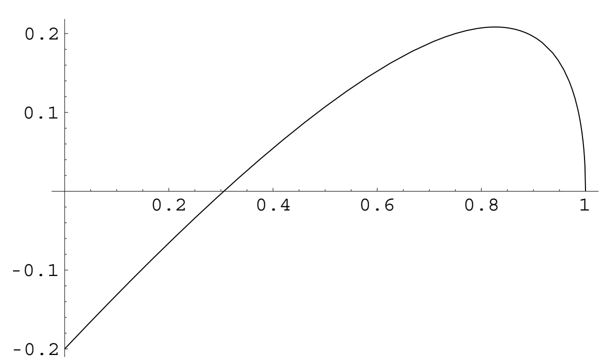

FIGURE 6.8. the case $\beta = 1.2$

(3) $\beta(u) = \beta u$ with a constant β. Then the information cost is given by

$$\int_t^1 \beta(V_S(u))du = \beta \int_t^1 V_S(u)du.$$

Hence, the profit of the insider is given by

$$G(t) = \sqrt{1-t} - \beta \int_t^1 V_S(u)du.$$

(a) If $S_u = S_1$, i.e., $V_S(u) = 1$ for all u, then the expected gain is of the form

$$G(t) = \sqrt{1-t} - \beta(1-t).$$

The rest of the discussion is the same as in the case (2).
(b) If (S_u) is a Brownian motion, then

$$G(t) = \sqrt{1-t} - \frac{1}{2}\beta(1-t^2).$$

(i) $\beta = 1$. If the investor wants to get the full information from time t, he has to pay $\frac{1}{2}(1-t^2)$. From Figure 6.9, we see that it is optimal for him to buy the information from the beginning of the trading. But as we have shown, there are no such strategies which remain inconspicuous.

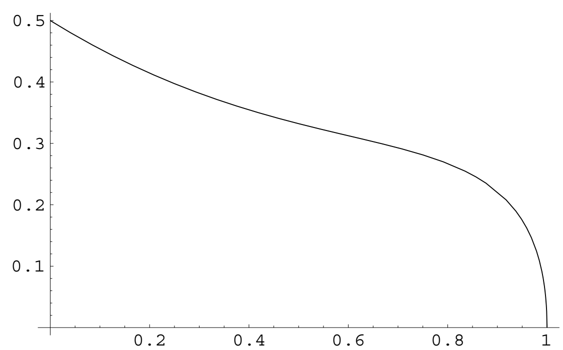

FIGURE 6.9. the case $\beta = 1$

(ii) $\beta = 1.8$. From Figure 6.10 we see that whenever the insider buys the information, he gets a positive profit, and at time $t = 0.927319\cdots$, he can reach his maximal profit.
(iii) $\beta = 2$. See Figure 6.11. The optimal time to buy the information is $t = 0.905997\cdots$. In contrast to the last case, the expected profit of the insider will be negative if he buys the information too early.

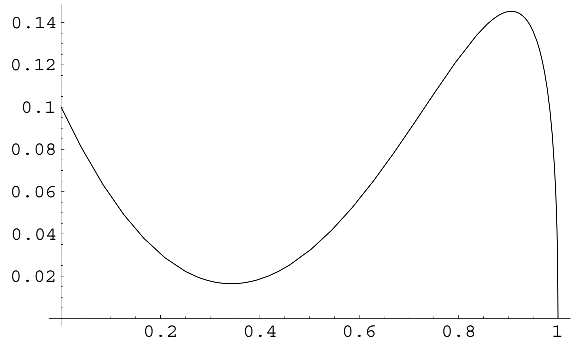

FIGURE 6.10. the case $\beta = 1.8$

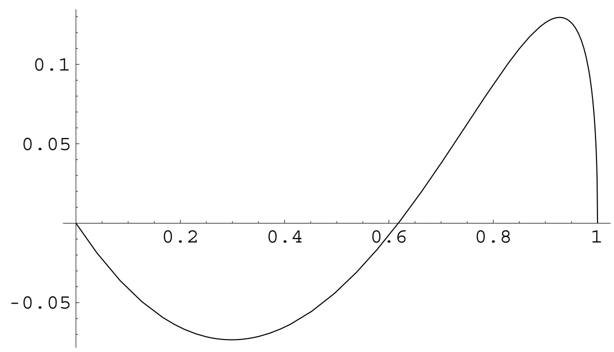

FIGURE 6.11. the case $\beta = 2$

Bibliography

[1] A. Admati and P. Pfleiderer. A theory of intraday patterns: volume and price variability. *Review of Financial Studies*, 1:3–40, 1988.

[2] A. Admati and P. Pfleiderer. Divide and conquer: a theory of intraday and day-of-the-week mean effects. *Review of Financial Studies*, 2:189–224, 1989.

[3] J. Amendinger. *Initial enlargement of filtrations and additional information in financial markets*. PhD thesis, Technische Universität Berlin, 1998.

[4] J. Amendinger. Martingale representation theorems for initially enlarged filtrations. 1998. Working paper.

[5] J. Amendinger, P. Imkeller, and M. Schweizer. Additional logarithmic utility of an insider. *Stochastic Processes and Their Applications*, 75:271–294, 1998.

[6] L. Bachelier. Théorie de la spéculation. *Annales Scientifiques de l'École Normale Supérieure*, 17:21–86, 1990.

[7] K. Back. Insider trading in continuous time. *Review of Financial Studies*, 5:387–409, 1992.

[8] K. Back. Asymmetric information and options. *Review of Financial Studies*, 6:435–472, 1993.

[9] K. Back. Continuous trading with asymmetric information and imperfect competition. In M. H. A. Davis, D. Duffie, W. H. Fleming, and S. E. Shreve, editors, *IMA, Mathematical finance*, pages 1–12. Spring, 1995.

[10] K. Back, H. H. Cao, and G. A. Willard. Imperfect competition among informed traders. *Journal of Finance*, 1998. forthcoming.

[11] K. Back and H. Pedersen. Long-lived information and intraday patterns. *Journal of Financial Market*, 1:385–402, 1998.

[12] W. Bagehot. The only game in town. *Financial Analysts Journal*, 27:12–14, 1971.

[13] B. Biais and J. C. Rochet. Risk sharing, adverse selection and market structure. In B. Biais, T. Björk, and J. Cvitanic, editors, *Financial mathematics*, volume 1656 of *Lecture Note in Mathematics*, pages 1–51. Springer, 1997.

[14] Y. Chiu. From an example of Lévy's. In J. Azema, P. A. Meyer, and M. Yor, editors, *Séminaire de Probabilités XXIX*, volume 1613 of *Lecture Notes in Mathematics*, pages 162–165. Springer, 1995.

[15] K.-H. Cho and N. El Karoui. Continuous auction equilibrium with asymmetric information: risk neutral vs. risk aversion; convergence. 1998. Working paper.

[16] K.-H. Cho and N. El Karoui. Single auction equilibrium with insider trading: non-linear equilibria and risk aversion. 1998. Working paper.

[17] T. Copeland and D. Galai. Information effects and the bid-ask spread. *Journal of Finance*, 38:1457–1469, 1983.

[18] C. Corduneanu. *Integral equations and applications*. Cambridge University Press, 1991.

[19] J. Cvitanic and J. Ma. Hedging options for a large investor and forward-backward SDE's. *Annals of Applied Probability*, 6:370–398, 1996.

[20] M. H. A. Davis. *Linear estimation and stochastic control*. Chapman and Hall, New York, 1977.

[21] P. Deheuvels. Invariance of Wiener processes and of Brownian bridges by integral transformations and applications. *Stochastic Processes and their Applications*, 13:311–318, 1982.

[22] O. Easley and M. O'Hara. Price, trade size and information in securities markets. *Journal of Financial Economics*, 19:69–90, 1987.

[23] H. Föllmer, C.-T. Wu, and M. Yor. Canonical decomposition of linear transformations of two independent Brownian motions motivated by models of insider trading. *Stochastic Processes and their Applications*, 1998. To appear.

[24] H. Föllmer, C.-T. Wu, and M. Yor. On weak Brownian motions of arbitrary order. 1999. Laboratoire de Probabilités et Modéles aléatoires, Université Paris VI & VII. Working paper.

[25] R. Frey. Perfect option hedging for a large trader. *Finance and Stochastics*, 2:115–141, 1998.

[26] L. Glosten and P. Milgrom. Bid, ask and transaction prices in a specialist market with heterogeneously informed traders. *Journal of Financial Economics*, 13:71–100, 1985.

[27] I. C. Gohberg and M. G. Krein. *Theory and Applications of Volterra operators in Hilbert space*, volume 24 of *Translations of Mathematical Monographs*. American Mathematical Society, 1970.

[28] A. Grorud and M. Pontier. Insider trading in a continuous time market model. *International Journal of Theoretical and Applied Finance*, 1:331–347, 1998.

[29] S. J. Grossman. *The informational role of prices*. MIT Press, Cambridge, MA, 1989.

[30] S. J. Grossman and J. Stiglitz. On the impossibility of informationally efficient markets. *Amer. Economic Review*, 70:393–408, 1980.

[31] Y. Hibino, M. Hitsuda, and H. Muraoka. Construction of noncanonical representations of a Brownian motion. *Hiroshima Math. J.*, 27:439–448, 1997.

[32] T. Hida and M. Hitsuda. *Gaussian processes*. American Mathematical Society, 1993.

[33] M. Hitsuda. Representation of Gaussian processes equivalent to Wiener process. *Osaka J. Math.*, 5:299–312, 1968.

[34] R. Jarrow. Market manipulation, bubbles, corners and short squeezes. *Journal of Financial and Quantitative Analysis*, 27(3):311–336, 1992.

[35] R. Jarrow. Derivative securities markets, market manipulation and option pricing theory. *Journal of Financial and Quantitative Analysis*, 29:241–261, 1994.

[36] Th. Jeulin and M. Yor. Inégalités de Hardy, semimartingales, et faux-amis. In C. Dellacherie, P. A. Meyer, and M. Weil, editors, *Séminaire de Probabilités XIII1977/78*, volume 721 of *Lecture Notes in Mathematics*, pages 332–359. Springer, 1979.

[37] Th. Jeulin and M. Yor. Filtration des ponts browniens et équations differentielles linéaires. In J. Azema, P. A. Meyer, and M. Yor, editors, *Séminaire de Probabilitiés XXIV 1988/89*, volume 1426 of *Lecture Notes in Mathematics*, pages 227–265. Springer, 1990.

[38] G. Kallianpur. *Stochastic filtering theory*. Springer, New York, 1980.

[39] I. Karatzas and I. Pikovsky. Anticipative portfolio optimization. *Advances in Applied Probability*, 28(4):1095–1122, 1996.

[40] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*. Springer, New York, second edition, 1991.

[41] A. S. Kyle. Market structure, information, futures markets, and price formulation in international agricultural trade: Advance readings in price formation, market structure and price instability. Westview press, Boulder and London, 1984.

[42] A. S. Kyle. Continuous auctions and insider trading. *Econometrica*, 53:1315–1335, 1985.

[43] A. S. Kyle. Informed speculation with imperfect competition. *Review of Economic Studies*, 56:317–355, 1989.

[44] P. Lévy. Sur une classe de courbes de l'espace de Hilbert et sur une équation intégrale non linéaire. *Ann. Sci. Ecol. Norm. Sup.*, 73(3):121–156, 1956.

[45] P. Lévy. Fonctions aléatoires à corrélations linéaires. *Illinois J. Math*, 1:217–258, 1957.

[46] R. Sh. Liptser and A. N. Shiryaev. *Statistics of random processes, Volume I: General theory.* Springer, New York, 1977.

[47] P. Milgrom and Strokey N. Information, trade, and common knowledge. *Journal of Economic Theory*, 26:17–27, 1982.

[48] J. Neveu. *Processus aléatoires gaussiens.* Les presses de l'Université de Montréal, 1968.

[49] M. O'Hara. *Market microstructure theory.* Blackwell, 1995.

[50] I. Pikovsky. Market equilibrium with differential information. *Finance and Stochastics*, 1997. forthcoming.

[51] Ph. Protter. *Stochastic integration and differential equations. A new approach*, volume 21 of *Applications of Mathematics.* Springer, 1992.

[52] D. Revuz and M. Yor. *Continuous martingales and Brownian motion.* Springer, 1991.

[53] J. C. Rochet and J. C. Vila. Insider trading without normality. *Review of Economic Studies*, 61:131–152, 1994.

[54] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes and martingales, Volume 2: Itô calculus.* Wiley, 1987.

[55] H. Stoll. The supply of dealer services in securities markets. *Journal of Finance*, 33:1133–1151, 1978.

[56] C.-T. Wu and M. Yor. Linear transformations of two independent Brownian motions and orthogonal decompositions of Brwonian filtrations. 1999. In preparation.

[57] M. Yor. *Some aspects of Brownian motion, Part I: some special functionals.* Birkhäuser, Basel, 1992.

[58] K. Yosida. *Lectures on differential and integral equations.* Dover Pubns, 1991.

Lebenslauf

1. 2. 1969	geboren in Keelung, Taiwan.
9. 1975 – 6. 1981	Besuch der Keelung municipal Chung-Shiao elementary school.
9. 1981 – 6. 1984	Besuch der Keelung municipal Cheng-Bin junior high school.
9. 1984 – 6. 1987	Besuch der Taipei municipal Cheng-Kung senior high school.
9. 1987 – 6. 1991	Bachelor of Science der Falkultät für Mathematik, staatliche Tsing-Hua Universität, Hsinchu, Taiwan.
9. 1987 – 6. 1993	Verschiedene Stipendien für begabte Studenten.
7. 1990 – 6. 1993	Tutor am Institute of Mathematics der staatlichen Tsing-Hua Universität, Hsinchu, Taiwan. Betreuung von Übung in Analysis, Algebra, lineare Algebra, diskrete Mathematik, Differentialgleichung und komplexe Analysis.
9. 1991 – 6. 1993	Master of Science des Instituts für Mathematik der staatlichen Tsing-Hua Universität, Hsinchu, Taiwan. Thema der Master Arbeit: "Almost everywhere convergence of the Walsh-Fourier series".
9. 1991 – 6. 1993	Stipendiat des National Science Council, Taiwan.
7. 1993 – 6. 1994	Forschungsassistent am Institut für Mathematik, Academia Sinica, Taipei, Taiwan.
7. 1994 – 9. 1994	Besuch eines Deutschkurses des Goethe Instituts Bonn.
10. 1994 – 2. 1995	Besuch eines Deutschkurses an der Universität Bielefeld.
4. 1995 – 1. 1996	Studium an der Universität Bielefeld.
2. 1996 –	Studium an der Humboldt-Universität zu Berlin.
1. 1997 –	Stipendiat des Berliner Graduiertenkolleges "Stochastische Prozesse und Probabilistische Analysis".

Veröffentlichungen

1. (mit C.-P. Chen) *Double Walsh series with coefficients of bounded variation of higher order. Transactions of AMS* **350** (1998), 395-417.

2. (mit H. Föllmer und M. Yor) *Canonical decomposition of linear transformations of two independent Brownian motions motivated by models of insider trading.* To appear in: *Stochastic Processes and Their Applications.*

3. (mit H. Föllmer und M. Yor) *On weak Brownian motions of arbitrary order.* Laboratoire de Probabilités et Modéles aléatoires, Université Paris VI & VII. Working paper.

4. (mit M. Yor) *Linear transformations of two independent Brownian motions and orthogonal decompositions of Brownian filtrations.* In Vorbereitung.

Erklärung

Hiermit versichere ich, daß ich die vorliegende Dissertation selbständig und ohne unerlaubte Hilfe angefertigt habe.