Non-Perturbative Renormalization of the B-Meson Axial Current

D I S S E R T A T I O N

zur Erlangung des akademischen Grades
doctor rerum naturalium
(Dr. rer. nat.)
im Fach Physik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät I
der Humboldt-Universität zu Berlin

von
Dipl.-Phys. Martin Kurth
geboren am 10.04.1971 in Berlin

Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. Dr. h. c. H. Meyer

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät I:
Prof. Dr. B. Ronacher

Gutachter:
1. Prof. Dr. Michael Müller-Preußker
2. Dr. Rainer Sommer
3. Prof. Dr. Fred Jegerlehner

eingereicht am: 22. Mai 2000
Abstract

The axial current of a light and a heavy quark is studied in the static approximation, with the aim of defining a non-perturbative renormalization scheme.

To keep lattice artifacts small, O(a) improvement in the static approximation is discussed in detail. It is explained how a finite size scheme can be used to avoid the necessity of accommodating a large energy range on a single lattice in the determination of the scale dependence of the renormalized static-light axial current. To that end, Schrödinger functional boundary conditions are imposed on the static quark field, and a renormalization condition is formulated. As a central object of the SF scheme, the “step scaling function”, connecting the renormalization constants at different scales, is introduced.

A large part of this thesis is dedicated to the expansion of suitable correlation functions to one loop order of perturbation theory. Using these expansions, the finite renormalization constants connecting the static-light axial current in the lattice MS scheme and the light-light axial current normalized by current algebra relations is calculated at one loop order. From this result, the relation of the renormalized static-light axial current in the SF scheme to the \( \overline{\text{MS}} \)-renormalized static-light axial current is derived. Using that relation, the static-light axial current’s two loop anomalous dimension in the SF scheme, which is needed for the calculation of the renormalization group invariant current, is calculated by conversion from the \( \overline{\text{MS}} \) scheme.

Further studies made in this thesis are the determination of discretization errors in the step scaling function at one loop order, and the calculation of an improvement coefficient for the static-light axial current at one loop order of perturbation theory.

Keywords:
lattice QCD, static approximation, renormalized axial current, O(a) improvement
Zusammenfassung

Diese Arbeit befaßt sich mit dem Problem der nichtperturbativen Renormierung des Axialstroms eines leichten und eines Bottom-Quarks. Solche nichtperturbativen Berechnungen können nur in der Gitter-QCD durchgeführt werden, d. h. die kontinuierliche Raumzeit wird durch ein vierdimensionales hyperkubisches Gitter ersetzt. Da einerseits die Kantenlänge des Gitters größer sein muß als typische physikalische Längenskalen des Problems, andererseits aber der durch die Gitterkonstante eingeführte Energie-Cutoff größer sein muß als die Masse des $b$-Quarks, sind für dieses Problem Gittergrößen erforderlich, für die die heutige Computerleistung nicht ausreicht. Es ist daher sinnvoll, das B-Meson in der statischen Näherung zu untersuchen, um dann zwischen dieser Näherung und leichten Quarkmassen interpolieren zu können.

Die Renormierung des Axialstroms in der statischen Näherung ist skalenabhängig. Um zu vermeiden, Rechnungen über einen großen Energiebereich hinweg auf einem einzigen Gitter durchführen zu müssen, wird als Renormierungsverfahren das SF-Schema vorgeschlagen, in dem die Renormierungsskala mit der inversen Kantenlänge des Raumzeitvolumens identifiziert wird. Das zentrale Objekt dieses Schemas ist die Step-Scaling-Funktion, die die Renormierungsconstanten bei verschiedenen Skalen miteinander in Beziehung setzt.

Ein wesentlicher Punkt dieser Arbeit ist die $O(a)$-Verbesserung, die die Diskretisierungsfehler reduziert. Nach einer Erklärung dieses Verfahrens für Eichfelder und leichte Quarks wird die statistische Approximation im Kontinuum und auf dem Gitter eingeführt, und die in der Gittertheorie erforderlichen $O(a)$-Verbesserungen werden diskutiert.

Für die eigentliche Renormierung werden Schrödinger-Funktional-Randbedingungen analog zum Fall leichter Quarks auch für die statistische Approximation eingeführt, und die durch diese Randbedingungen notwendige zusätzliche $O(a)$-Verbesserung diskutiert. Anschließend wird durch eine Renormierungsbedingung das SF-Schema für den statischen Axialstrom definiert.


Ein weiterer Untersuchungsgegenstand sind die Diskretisierungsfehler in der Step-Scaling-Funktion, die in der Einschleifennäherung berechnet wurden. Sie stellen sich ebenfalls als klein heraus. Abschließend wird der Einschleifenkoeffizient des $O(a)$-Verbesserungsterms für den statischen Axialstrom berechnet. Hierbei ergibt sich Übereinstimmung mit einem früheren Ergebnis anderer Autoren.

**Schlagwörter:**
Gitter-QCD, statische Approximation, renormierter Axialstrom, $O(a)$-Verbesserung
Acknowledgements

Above all, I would like to thank Rainer Sommer for directing my interest to the subject of this thesis, and for introducing me to the concepts of lattice gauge theory. His support has been very valuable throughout the three years I have been working with him.

I would also like to thank Michael Müller-Preußker (HU Berlin) and Fred Jegerlehner (DESY Zeuthen) for taking on the task of reporting on this thesis. Special thanks go to Martin Lüscher (DESY and CERN) for providing a set of FORTRAN programs, which I could use as a starting point for my own programming. Furthermore, I want to thank Stefan Sint (University of Rome Tor Vergata) for helpful correspondence on computer codes, and Peter Weisz (MPI Munich) for critical discussion of various aspects of my work. I am also grateful to Ulli Wolff (HU Berlin), both for discussion about my work, especially on the subject of local symmetries in the static approximation, and for organizing an informal seminar, which for unknown reasons is called “central committee”. Jochen Heitger (DESY Zeuthen) deserves my acknowledgement for helping me with my first attempts to use an APE 100 parallel computer.

Finally, I want to thank the Theory Group at DESY Zeuthen and the Computational Physics Group at the Humboldt University in Berlin for making the last three years a very enjoyable time.
Contents

1 Introduction 1

2 Renormalization strategy 4
  2.1 Problem 1: Heavy quarks on the lattice ...................... 4
  2.2 Problem 2: Scale dependent renormalization .................. 7

3 O(a) improved lattice QCD 10
  3.1 Lattice QCD .................................................. 10
  3.2 O(a) improvement ............................................. 12

4 The static approximation 15
  4.1 Static quarks in the continuum .............................. 15
  4.2 Static quarks on the lattice ................................ 16
  4.3 Symmetries in the static approximation ...................... 17
  4.4 O(a) improvement in the static approximation ............... 18

5 The Schrödinger functional 20
  5.1 The Schrödinger functional in the continuum .............. 20
    5.1.1 Gauge Fields ............................................. 20
    5.1.2 Relativistic quarks .................................... 21
    5.1.3 Static quarks ............................................. 22
    5.1.4 Correlation functions ................................... 25
  5.2 Discretization of the Schrödinger functional ............. 26
  5.3 O(a) improvement of the Schrödinger functional .......... 29
    5.3.1 Improved action ......................................... 29
    5.3.2 Improved correlation functions ......................... 31

6 Perturbation theory 32
  6.1 Integration over the quark fields .......................... 32
  6.2 Gauge fixing ................................................. 34
  6.3 Perturbative expansion of the correlation functions ....... 36
7 Renormalization
  7.1 General concepts ........................................ 42
  7.2 Renormalization in the lattice MS scheme .................. 44
  7.3 Matching to the \( \overline{\text{MS}} \) scheme .................. 47
  7.4 SF renormalization ........................................ 56
      7.4.1 The SF scheme and the step scaling function ........ 56
      7.4.2 Relation to the \( \overline{\text{MS}} \) scheme ............... 59
      7.4.3 The anomalous dimension of the static-light axial current
          at two loop order .................................. 59
      7.4.4 Cutoff effects in the SF scheme ....................... 61

8 \( \mathcal{O}(a) \) improvement revisited ...................... 63

9 Summary and Outlook ......................................... 65

A Notation ...................................................... 68
  A.1 Index conventions ........................................ 68
  A.2 Covariant derivatives in the Schrödinger
      functional ........................................... 68
  A.3 Lattice derivatives ....................................... 68
  A.4 Dirac matrices .......................................... 69
  A.5 Gauge group ............................................. 70
  A.6 Delta function .......................................... 70

B Perturbative expressions ...................................... 71
  B.1 The gluon propagator .................................... 71
  B.2 Tree level correlation functions ......................... 72
  B.3 Light quarks at one loop order .......................... 77
  B.4 Heavy quarks at one loop order .......................... 78
  B.5 Correlation functions at one loop order ................. 79
  B.6 Continuum limit of tree level correlation functions .... 82

C Extrapolation ................................................. 83

D Numerical results ............................................ 87

E The renormalized coupling in the SF scheme ............... 99
## List of Figures

5.1 The Schrödinger functional ............................... 21  
5.2 The correlation functions $f_A^{\text{stat}}$ and $f_1^{\text{stat}}$ ............................... 26  
6.1 One loop diagrams contributing to $f_A^{\text{stat}}$ ........................................ 39  
6.2 One loop diagrams contributing to $f_1^{\text{stat}}$ ........................................ 41  
7.1 The ratio $X_1$ in the lattice MS scheme ......................... 46  
7.2 Relation of different renormalization schemes ....................... 48  
7.3 The correlation functions $f_A$ and $f_1$ ......................... 49  
7.4 One loop diagrams contributing to $f_A$ .......................... 52  
7.5 One loop diagrams contributing to $f_1$ .......................... 53  
7.6 The ratio $Y_1$ ........................................ 55  
7.7 Extrapolation of $B_A^{\text{stat}}(z)$ ........................................ 57  
7.8 Discretization errors in the step scaling function .................... 62
List of Tables

B.1 The 3-point quark-gluon vertex ........................................ 79
B.2 The 4-point quark-gluon vertex ........................................ 80

D.1 The one loop coefficient of the critical quark mass ............... 87
D.2 $X_{1,\text{lat}}$ at one loop order for $\theta = 0.0$ and $\theta = 0.5$ ........ 88
D.3 $X_{1,\text{lat}}$ at one loop order for $\theta = 1.0$ .......................... 89
D.4 $Y_{1}$ at one loop order for $z = 1.0$ and $z = 1.5$ at $\theta = 0.5$ ...... 90
D.5 $Y_{1}$ at one loop order for $z = 2.0$ and $z = 2.5$ at $\theta = 0.5$ ...... 91
D.6 $Y_{1}$ at one loop order for $z = 3.0$ and $z = 3.5$ at $\theta = 0.5$ ...... 92
D.7 $Y_{1}$ at one loop order for $z = 4.0$ and $z = 4.5$ at $\theta = 0.5$ ...... 93
D.8 $Y_{1}$ at one loop order for $z = 5.0$ and $z = 6.0$ at $\theta = 0.5$ ...... 94
D.9 $Y_{1}$ at one loop order for $z = 8.0$ and $z = 12.0$ at $\theta = 0.5$ ...... 95
D.10 $Y_{1}$ at one loop order for $z = 16.0$ at $\theta = 0.5$ .................. 96
D.11 $B_{\text{stat}}^{\Lambda}$ at $\theta = 0.5$ ......................................... 97
D.12 The $O(a)$ part of $X_{\text{lat}}$ at one loop order ...................... 98
Chapter 1

Introduction

Since the discovery of the $\Upsilon$ resonance in 1977 [1] and the first observation of the B-meson in 1983 [2], the decays of “bottom” flavoured particles have attracted much attention both experimentally and theoretically. The B-meson, due to its large number of decay channels, offers a rich and interesting phenomenology. In the next few years, several B-meson experiments designed to study $CP$ violation, including BABAR$^1$ [3], HERA-B$^2$ [4], BELLE$^3$ [5], and LHCb$^4$ [6], will provide us with new insight into the fascinating world of heavy flavour physics. To be able to interpret the measured results, a quantitative theoretical understanding of the investigated properties is required.

The framework in which theoretical predictions are made is the Standard Model of elementary particles. The Standard Model is a local quantum field theory, with interactions originating from local gauge invariance under $SU(3)_c \times SU(2)_L \times U(1)_Y$ transformations. While the $SU(2) \times U(1)$ part of the Standard Model is the gauge theory of electroweak interactions [7, 8, 9] connected with weak isospin and hypercharge, the $SU(3)$ part of the gauge group describes the strong interaction; it involves particles carrying a quantum number called colour. These particles are called quarks [10, 11]. In the quark picture, a B-meson is composed of a $b$-quark and a light anti-quark ($u$, $\bar{d}$, or $s$), or of a $\bar{b}$ anti-quark and a light quark ($u$, $d$, or $s$).

The quark model and the $SU(3)$ gauge theory for strong interactions, called quantum chromodynamics (QCD) [12], together with the Feynman path integral method [13, 14], have been very successful in explaining phenomena connected with high energy processes, like scaling in deep inelastic scattering [15]. That QCD can be applied rather easily at high energy scales is due to the effect of asymptotic freedom, which is a general property of non-Abelian gauge theories [16, 17, 18]. Asymptotic freedom means that the gauge coupling becomes

---

$^1$Stanford Linear Accelerator Center SLAC, Stanford
$^2$Deutsches Elektronen-Synchrotron DESY, Hamburg
$^3$KEK High Energy Accelerator Research Organization, Tsukuba
$^4$European Laboratory for Particle Physics CERN, Geneva
small at high energies, which allows for a formal expansion of expectation values in powers of the coupling. Although the coefficients in such a series may be formally infinite, they can be made finite by regularization (introduction of an artificial momentum cutoff) and renormalization, i.e. subtraction of the infinities when removing the cutoff.

At low energies, the strong coupling constant is large, meaning that perturbation theory breaks down. For the calculation of low energy quantities, Wilson invented the concept of lattice QCD, which means the regularization of the euclidean theory by defining it on a four dimensional space-time lattice [19]. In this approach, expectation values can be evaluated non-perturbatively, which is usually done by Monte Carlo methods. Lattice gauge theory has by now become a mature method, although the accuracy of its prediction is limited by the accessible computing power. With today’s computers, satisfactory statistical errors can often only be achieved in the quenched approximation, setting the number of quark flavours \( N_f \) to zero, thus disregarding closed quark loops.

A quantity which is suitable for a lattice calculation, and which is interesting in the context of heavy quark physics, is the B-meson decay constant \( F_B \), connected with the axial current of one light and one \( b \)-quark,

\[
A_\mu(x) = \bar{\psi}_l(x) \gamma_\mu \gamma_5 \psi_b(x),
\]

by

\[
\langle 0 | (A_R)_0(0) | B \rangle = M_B F_B,
\]

where \( |B \rangle \) is a B-meson state with momentum zero, and \( M_B \) is the B-Meson mass. Here, the axial current has to be renormalized, and it is this renormalization to which this thesis aims to make a contribution.

The decay constant is not only interesting in itself, but it has a certain connection to the phenomenon of \( CP \)-violation. The quark fields \( \psi \) couple to the weak gauge bosons \( W^\pm_\mu \) through a term

\[
\bar{\psi}_U \gamma_\mu \frac{\gamma_5}{2} V_{UD} \psi_D W^\mu_+, \tag{1.3}
\]

where the indices \( U \) and \( D \) symbolize a sum over the up-type quarks and the down-type quarks respectively, and \( V_{UD} \) denotes the elements of the CKM matrix, which transforms weak interaction eigenstates into eigenstates of the quark mass matrix [20, 21]. The CKM matrix can be conveniently written in the Wolfenstein parameterization [22]:

\[
V = \left( \begin{array}{ccc}
1 - \frac{\lambda^2}{2} & \lambda & A \lambda^3 (\rho - i \eta) \\
-\lambda & 1 - \frac{\lambda^2}{2} - i A^2 \lambda^4 \eta & A \lambda^2 \\
A \lambda^3 (1 - \rho - i \eta) & -A \lambda^2 & 1
\end{array} \right). \tag{1.4}
\]

The parameters \( A \) and \( \lambda \) are known at rather good accuracy from several experiments [23, 24], while the still unknown parameters \( \rho \) and \( \eta \) have to be determined from B-meson decays. However, to connect experimental results to \( \rho \) and
\[ \frac{8}{3} B_B F_B^2 M_B^2 = \langle B_0 \mid (\bar{\psi}_d \gamma_\mu(1 - \gamma_5) \psi_b)(\bar{\psi}_d \gamma_\mu(1 - \gamma_5) \psi_b) \mid \bar{B}_0 \rangle \] (1.5)

between a \( B_0 \) and a \( \bar{B}_0 \) meson state is required. The constant \( B_B \) is not known very precisely (see e.g. [25]), but the non-perturbative knowledge of \( F_B \) can still yield rough approximative constraints on \( \rho \) and \( \eta \), using some estimate for \( B_B \). A precise determination of \( \rho \) and \( \eta \) is particularly interesting, since these parameters are the source of \( CP \) violation in the Standard Model, the measurement of which is the main goal of the new B-meson experiments.

This thesis is organized as follows: In Chapter 2, two problems arising in the lattice renormalization problem of the heavy quark axial current are described, and ideas for their solution are presented. Based on those ideas, a renormalization strategy is formulated. Chapter 3 describes the concepts of lattice QCD and \( O(a) \) improvement; in Chapter 4 the static approximation for heavy quarks is introduced. The Schrödinger functional, a certain way to choose boundary conditions, is discussed in Chapter 5, where also correlation functions are defined that are expanded at one loop order of perturbation theory in Chapter 6. Chapter 7, dealing with the renormalization of the static-light axial current in various renormalization schemes, is the heart of this thesis. In Chapter 8, the problem of \( O(a) \) improvement of the static-light axial current is revisited, and the calculation of a one loop improvement coefficient is presented. The thesis closes with a summary and an outlook to future developments in Chapter 9. Several appendices contain notational conventions and computational details as well as some tables with numerical results.
Chapter 2

Renormalization strategy

2.1 Problem 1: Heavy quarks on the lattice

Heavy quarks pose special difficulties for the discretization procedure, and, in general, require the use of effective theories (for a review see [25, 26, 27]). The main problem is that lattices have to be rather large to accommodate heavy quarks. On the one hand, the momentum cutoff given by the lattice spacing $a$ has to be considerably larger than the heavy (in our case the bottom) quark mass,

$$a^{-1} \gg m_b \approx 5 \text{GeV}, \quad (2.1)$$

to keep discretization errors controllable. On the other hand, only finite lattices can be used in numerical calculations, and the size of the chosen space-time volume (e.g. a box with size $L^4$) has to be larger than relevant physical length scales to keep finite size effects small. For the B-meson, this means [25, 26]

$$L > 1 - 1.5 \text{fm}. \quad (2.2)$$

Typical lattice sizes that can be used in Monte Carlo simulations on today’s computers lie in the range between $16^4$ and $48^4$ [28], which together with eq. (2.2) leads to a maximal cutoff of

$$a^{-1} \approx 2 - 4 \text{GeV}. \quad (2.3)$$

The condition (2.1) can thus not be satisfied in numerical simulations.

Several effective theories have been developed to avoid this problem and to calculate B-meson properties on the lattice at least in some approximation. One of the most widely used theories is non-relativistic QCD (NRQCD) [29, 30, 31, 32], where instead of the QCD Lagrangian one uses

$$\mathcal{L}_{\text{NRQCD}} = \phi^\dagger (D_0 - \frac{D^2}{2m_b} - \frac{\sigma B}{2m_b}) \phi, \quad (2.4)$$
where the covariant derivative is decomposed as $D_\mu = (D_0, \mathbf{D})$, $\phi$ is a two component spinor, $\mathbf{B}$ is the colour magnetic field strength, $\mathbf{\sigma}$ is the three-component vector composed of the Pauli matrices, and $m_b$ is the $b$-quark mass. NRQCD is being used in a large number of lattice calculations [28]. However, it has the disadvantage of being a non-renormalizable theory, which means that one cannot take the continuum limit, and it is not always clear how large the discretization errors are at the lattice spacings one has to keep.

A second way of dealing with heavy quarks on the lattice is the “Fermilab approach” [33, 34], which interpolates between the full QCD Lagrangian and a heavy quark effective Lagrangian. But here again it is hard to estimate how good the approximation is.

Another possibility is to calculate the decay constants $F_P$ of pseudoscalar mesons $|P\rangle$ with different auxiliary heavy quark masses $m_a$ that satisfy eq. (2.1), and then extrapolate to the physical $B$-meson mass. In practice, the calculation is done by computing some correlation function containing the axial current

$$A_\mu = \bar{\psi}_1 \gamma_\mu \gamma_5 \psi_a$$

(2.5)

of a light quark $\psi_1$ and a quark $\psi_a$ with mass $m_a$, for example

$$c_{AA}(x_0, m_a) = \int d^3x \langle A_0(x)(A_0)^\dagger(0) \rangle.$$  

(2.6)

After fixing a renormalization scheme for the coupling and the quark masses, a renormalization condition can be imposed on the axial current, meaning that a (finite) renormalization constant $Z_A$ is introduced,

$$(A_R)_\mu = Z_A A_\mu.$$  

(2.7)

Inserting this into $c_{AA}$, one gets a renormalized correlation function $(c_{AA})_R$, which for large time separations $x_0$ behaves like

$$(c_{AA})_R(x_0, m_a) \sim_{x_0 \to \infty} F_P^2 F_P e^{-M_P x_0},$$

(2.8)

where $M_P$ is the mass of the pseudoscalar meson. The factor $F_P \sqrt{M_P}$ can thus be extracted from the correlation function and then be extrapolated to $F_B \sqrt{M_B}$.

This method can be improved considerably by including a calculation in the static approximation, i. e. in the limit of an infinitely heavy $b$-quark. First, one introduces the static-light axial current

$$A_0^{\text{stat}} = \bar{\psi}_1 \gamma_0 \gamma_5 \psi_h,$$

(2.9)

with a light quark field $\bar{\psi}_1$ and a static quark field $\psi_h$. With this, the correlation function

$$c_{AA}^{\text{stat}}(x_0) = \int d^3x \langle A_0^{\text{stat}}(x)(A_0^{\text{stat}})^\dagger(0) \rangle$$

(2.10)
can be introduced. Then, a renormalized static-light axial current

\[
(A_{\text{R}}^{\text{stat}})_{0} = Z_{A}^{\text{stat}} A_{0}^{\text{stat}}
\]

(2.11)

and the corresponding function \((c_{AA}^{\text{stat}})_{R}\) can be defined, with a scale dependent renormalization constant \(Z_{A}^{\text{stat}}\). The static current obeys the renormalization group equation

\[
\mu \frac{\partial (A_{\text{R}}^{\text{stat}})_{0}}{\partial \mu} = \gamma(g_{R})(A_{\text{R}}^{\text{stat}})_{0},
\]

(2.12)

with an anomalous dimension

\[
\gamma(g_{R}) = -g_{R}^{2}(\gamma_{0} + \gamma_{1} g_{R}^{2} + \ldots),
\]

(2.13)

and a renormalized coupling \(g_{R}\), which satisfies the renormalization group equation

\[
\mu \frac{\partial g_{R}}{\partial \mu} = \beta(g_{R}),
\]

(2.14)

with the \(\beta\) function

\[
\beta(g_{R}) = -g_{R}^{3}(b_{0} + b_{1} g_{R}^{2} + \ldots).
\]

(2.15)

The finite part of \(Z_{A}^{\text{stat}}\) can be fixed by imposing the matching condition

\[
e^{-(m_{a} + \delta m)x_{0}}(c_{AA}^{\text{stat}})_{R}(x_{0}) = (c_{AA})_{R}(x_{0}, m_{a}) + O\left(\frac{1}{m_{a}}\right),
\]

(2.16)

where \(\delta m\) is a linearly divergent counterterm which has to be taken into account in the renormalization procedure, and it is understood that the static axial current is renormalized at \(\mu = m_{a}\). In this context, \(m_{a}\) means the renormalized quark mass \(m_{a}(\mu = m_{a})\), which has to be determined in a self-consistent way. Calculating the asymptotic behaviour of \((c_{AA}^{\text{stat}})_{R}\) for large time separations \(x_{0}\), one can define a function \(\Phi^{\text{stat}}(\mu)\) by

\[
(c_{AA}^{\text{stat}})_{R}(x_{0}, \mu) \sim_{x_{0} \to \infty} [\Phi^{\text{stat}}(\mu)]^{2} e^{-E x_{0}}.
\]

(2.17)

Evaluating eq. (2.17) at \(\mu = m_{a}\), and combining it with eq. (2.8) and the matching condition eq. (2.16), one can read off

\[
\frac{F_{P} \sqrt{M_{P}}}{\Phi^{\text{stat}}(m_{a})} = 1 + O\left(\frac{1}{m_{a}}\right).
\]

(2.18)

Here it is convenient to introduce the renormalization group invariant matrix element

\[
\Phi_{\text{RGI}}^{\text{stat}} = (2b_{0} g_{R}^{2})^{-\gamma_{0}/2b_{0}} \times \exp \left\{ - \int_{0}^{\varpi(\mu)} dg \left[ \frac{\gamma(g)}{\beta(g)} - \frac{\gamma_{0}}{b_{0} g} \right] \right\} \Phi^{\text{stat}}(\mu),
\]

(2.19)
and thus the renormalization group invariant current

\[
(A_{\text{RG1}})_{0} = (2b_{0}g_{R}^{2})^{-\gamma_{0}/2b_{0}} \times \exp \left\{ -\int_{0}^{m(\mu)} dg \left[ \frac{\gamma(g)}{\beta(g)} - \frac{\gamma_{0}}{b_{0}g} \right] \right\} (A_{\text{RG1}}^{\text{stat}})_{0}(\mu). \tag{2.20}
\]

There is no standard normalization of this current, the convention chosen here is the one corresponding to the Gasser-Leutwyler convention \cite{35} for renormalization group invariant quark masses. An important property of \(\Phi_{\text{RG1}}^{\text{stat}}\) and \((A_{\text{RG1}}^{\text{stat}})_{0}\) is that they do not depend on the renormalization scheme. To facilitate the conversion between different renormalization schemes, all running quantities should be quoted in terms of the corresponding renormalization group invariant. One should thus define

\[
\Phi(m_{a}) = \frac{\Phi_{\text{RG1}}^{\text{stat}}}{\Phi_{\text{stat}}(m_{a})} F_{P}\sqrt{m_{P}}. \tag{2.21}
\]

Inserting this into eq. (2.18), one gets

\[
\Phi(m_{a}) = \Phi_{\text{RG1}}^{\text{stat}} + O\left(\frac{1}{m_{a}}\right). \tag{2.22}
\]

If \(\Phi_{\text{RG1}}^{\text{stat}}\) is known, and one knows \(\Phi(\mu)\) in some renormalization scheme, one can interpolate between \(\Phi_{\text{RG1}}^{\text{stat}}\) and \(\Phi\) calculated at reasonably small quark masses \(m_{a}\). This makes it clear, that it is important to calculate the renormalized static axial current in some non-perturbative renormalization scheme, and to obtain the renormalization constant \(Z_{A,\text{RG1}}^{\text{stat}}\),

\[
(A_{\text{RG1}})_{0}^{\text{stat}} = Z_{A,\text{RG1}}^{\text{stat}}(A_{0}^{\text{stat}}), \tag{2.23}
\]

where \(A_{0}^{\text{stat}}\) means the bare lattice current.

### 2.2 Problem 2: Scale dependent renormalization

As mentioned above, the renormalization constant of the static-light axial current will be scale dependent. In lattice calculations, this can pose a problem. On the one hand, one wants to have \(L \gg 1\) fm to keep finite size effects small. On the other hand, one wants to have a non-perturbative renormalization up to scales of about \(10 - 100\) GeV. The reason for this is that the anomalous dimension is only known at two-loop order, and one has to apply a non-perturbative renormalization scheme up to perturbative scales, where it is safe to use the two-loop approximations of the \(\beta\) function and the anomalous dimension \(\gamma\) in the calculation of the renormalization group invariant current \((A_{\text{RG1}}^{\text{stat}})_{0}\) using Eq. (2.20). To keep discretization errors under control,

\[
a^{-1} \gg \mu \tag{2.24}
\]
is a necessary requirement. Together with the finite size constraint, this means

\[ \frac{L}{a} \gg 36. \]  
(2.25)

As explained above, this can hardly be achieved on the computers that are at our disposal at the moment.

In practical calculations, one can now try to be less strict in the requirements given above, and hope for a “window” of scales where the renormalization procedure works. A method along these lines has been introduced in [36].

A second possibility has been proposed in [37]. Here, one identifies the scale with the inverse box size,

\[ \mu = \frac{1}{L}, \]  
(2.26)

which means that a finite size effect is used as the observable on which renormalization conditions are imposed to define the renormalization scheme. This method has been successfully applied to the calculation of the running coupling \( \alpha_s \) [38] and renormalized quark masses [39] in the quenched approximation. For the renormalization of the static-light axial current we will adopt this scheme, following the strategy outlined in [40].

The calculation then proceeds with the following steps, assuming that the light quark mass has been set to zero, such that one does not have to bother with its renormalization:

1. Fix the boundary conditions. Here we will use Schrödinger functional boundary conditions [41, 42].

2. Define a renormalized coupling.

3. Choose a maximal renormalized coupling, corresponding to a maximal box size \( L_{\text{max}} \).

4. Calculate some low-energy observable (e.g. the potential of two static quarks) to get \( L_{\text{max}} \) in physical units.

5. Calculate the renormalized coupling up to the perturbative regime. This has been done in [38], and here it will be assumed that the renormalized coupling is known at all relevant scales.

6. Impose a renormalization condition on a suitable correlation function containing \( A_0^{\text{stat}} \), to define a renormalization constant \( Z_\Lambda^{\text{stat}}(L) \).

7. Calculate for a number of renormalized couplings \( g_R(L) \) the step scaling function

\[ \Sigma_{\Lambda}^{\text{stat}}(a/L, g_R) = \frac{Z_\Lambda^{\text{stat}}(2L)}{Z_\Lambda^{\text{stat}}(L)} \]  
(2.27)
on lattices with different lattice spacings $a$, and extrapolate it to its continuum limit $\sigma_{\text{stat}}^A(g_R)$.

8. Determine the step scaling function in the whole relevant energy range by fitting the discrete values obtained in the last step with a suitable function.

9. Define the sequences

$$u_k = g_R(2^{-k}L_{\text{max}}), \quad k = 0, 1, 2, \ldots \quad (2.28)$$

and

$$v_k = \frac{Z_{\text{stat}}^A(2^{-k+1}L_{\text{max}})}{Z_{\text{stat}}^A(2L_{\text{max}})}, \quad k = 0, 1, 2, \ldots \quad (2.29)$$

10. Solve the recursion

$$v_0 = 1, \quad v_{k+1} = \frac{v_k}{\sigma_{\text{stat}}^A(u_k)} \quad (2.30)$$

up to $1/(2^{-k}L_{\text{max}}) \sim 10 - 100$ GeV.

11. Apply eq. (2.20) in perturbation theory to obtain the renormalization group invariant current.

Note that the lattice is only used in an intermediate step, to calculate the step scaling function. The calculation of the running parameters is then performed in the continuum.

In the next chapters, the quantities needed in this calculation will be defined precisely, and their properties will be investigated in perturbation theory.
Chapter 3

$O(a)$ improved lattice QCD

3.1 Lattice QCD

In the lattice approach, space-time is taken to be a 4-dimensional hyper-cubic lattice with lattice spacing $a$. A lattice quantum field theory is then set up by discretizing the action of the corresponding euclidean continuum theory. In this chapter, the basic concepts of lattice QCD are introduced. For a detailed overview see [43, 44, 45].

Gauge fields are included in the theory by introducing link variables $U(x, \mu)$ that are defined on the links between the lattice sites $x$ and their neighbours in $\mu$-direction, and that are matrices taken from the fundamental representation of the gauge group SU($N$), which in the case of QCD is SU(3). The closed loop consisting of four links is called a plaquette $p$, and the product of the link variables around a plaquette is denoted by

$$U(p) = U(x, \mu)U(x + a\mu, \nu)U(x + a\nu, \mu)^{-1}U(x, \nu)^{-1},$$

where $\mu \neq \nu$, and $\hat{\mu}$ is the unit vector in $\mu$-direction. For the action of the discretized gauge field, Wilson proposed

$$S_G[U] = \frac{1}{g^2} \sum_p \text{tr} \{1 - U(p)\},$$

where the sum runs over all oriented plaquettes.

The discretized light fermion fields are Grassmann fields $\psi(x)$, defined on the lattice sites $x$. They carry Dirac and colour indices, and transform in the fundamental representation of the gauge group. Introducing the discrete forward and backward covariant derivatives

$$\nabla_\mu \psi(x) = \frac{1}{a} [U(x, \mu)\psi(x + a\hat{\mu}) - \psi(x)]$$

and

$$\nabla^*_\mu \psi(x) = \frac{1}{a} [\psi(x) - U(x - a\hat{\mu}, \mu)^{-1}\psi(x - a\hat{\mu})],$$

...
a naïve discretized action can be written down,

$$S_{\text{1, naïve}} = a^4 \sum_x \bar{\psi}_1(x)(D + m_0)\psi_1(x), \quad (3.5)$$

with the naïve lattice Dirac operator

$$D = \frac{1}{2} \gamma_\mu(\nabla^*_\mu + \nabla_\mu). \quad (3.6)$$

However, the propagator derived from this action has not only one, but 16 poles, which survive the continuum limit. Different attempts to avoid this doubling problem have been undertaken. One of them is the staggered fermion approach by Kogut and Susskind [46, 47], where only one spin component of the quark field is defined at each lattice site, and remaining doublers are interpreted as different quark flavours. A second possibility, which will be adopted here, is the one proposed by Wilson [48]. Here, a term that breaks chiral symmetry is added to the naïve lattice Dirac operator, giving

$$D_W = D - \frac{a}{2}\nabla^*_\mu\nabla_\mu, \quad (3.7)$$

and the Wilson fermion action

$$S_I[U, \bar{\psi}_1, \psi_1] = a^4 \sum_x \bar{\psi}_1(x)(D_W + m_0)\psi_1(x). \quad (3.8)$$

The Wilson term removes the contribution of the fermion doublers to correlation functions in the continuum limit. The expectation value of an operator $\mathcal{O}$ can then be obtained through the path integral

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int D[U]D[\psi_1]D[\bar{\psi}_1]e^{-S_\text{cl}[U] - S_I[U, \bar{\psi}_1, \psi_1]}, \quad (3.9)$$

with the partition function

$$Z = \int D[U]D[\psi_1]D[\bar{\psi}_1]e^{-S_\text{cl}[U] - S_I[U, \bar{\psi}_1, \psi_1]}.$$  \quad (3.10)

In these expressions, $D[U]$ means

$$D[U] = \prod_{x, \mu} dU(x, \mu), \quad (3.11)$$

where $dU$ is the Haar measure of the gauge group SU(3), and the fermionic measures are

$$D[\psi_1] = \prod_x d\psi_1(x), \quad D[\bar{\psi}_1] = \prod_x d\bar{\psi}_1(x), \quad (3.12)$$

where $d\psi_1$ and $d\bar{\psi}_1$ denote the usual Grassmann algebra integration measures.
3.2 $O(a)$ improvement

The discretization of the gauge field and light quark actions is not unique. Irrelevant terms, i.e., terms that vanish in the continuum limit, can be added to the action. This freedom can be used to cancel $O(a)$ effects in correlation functions, producing a theory in which the continuum limit is reached at $O(a^2)$ rather than $O(a)$. According to Symanzik [49, 50], this is done by describing the lattice theory in terms of an effective continuum action, that contains the lattice spacing $a$ as an explicit parameter,

$$S_{\text{eff}} = S_0 + a S_1 + O(a^2).$$ (3.13)

In this expansion, $S_0$ is the standard continuum action, and $S_1 = \int d^4x L_1(x)$ has to be cancelled by adding counterterms to the lattice Lagrangian.

The pure gauge action does not generate any $O(a)$ effects, but the quark action needs improvement. The $O(a)$ counterterm that has to be added to the Lagrangian must be a dimension five operator that has the same space-time symmetries as the lattice Lagrangian. First of all, a basis of such operators has to be written down. In principle, all the operators contained in that basis can occur in the improvement term. However, if one is only interested in on-shell quantities, they are not completely independent, but connected by the equations of motion, that can be used to reduce the number of required operators [51, 52].

In the case of the light quark field, one then remains with three operators [52],

$$O_{\text{sw}} = \bar{\psi}_1 \sigma_{\mu \nu} F_{\mu \nu} \psi_1,$$ (3.14)

$$O_\xi = m_1 \text{tr}\{F_{\mu \nu} F_{\mu \nu}\},$$ (3.15)

and

$$O_m = m_1^2 \bar{\psi}_1 \psi_1,$$ (3.16)

where $m_1$ is the light quark mass, which will be given a precise meaning in Chapter 7. The operators $O_\xi$ and $O_m$ correspond to a shift in the bare coupling and the bare light quark mass. They will be treated in the context of renormalization in Chapter 7, and are disregarded until then. The operator $O_{\text{sw}}$ has been introduced by Sheikholeslami and Wohlert [53], and their improved light quark action is

$$S_{\text{impr}}[U, \bar{\psi}_1, \psi_1] = S_1[U, \bar{\psi}_1, \psi_1] + a^5 \sum_x c_{\text{sw}} \bar{\psi}_1(x) \frac{i}{4} \sigma_{\mu \nu} F_{\mu \nu}(x) \psi_1(x),$$ (3.17)

which means the addition of

$$\delta D_V = c_{\text{sw}} \frac{i}{4} \sigma_{\mu \nu} F_{\mu \nu}$$ (3.18)
to the Wilson-Dirac operator, where the “clover term”

\[ F_{\mu\nu}(x) = \frac{1}{8\alpha_s} \{ \mathcal{P}_{\mu\nu}(x) - \mathcal{P}_{\nu\mu}(x) \}, \quad (3.19) \]

\[ \mathcal{P}_{\mu\nu}(x) = U(x, \mu)U(x + a\hat{\mu}, \nu)U(x + a\hat{\nu}, \mu)^{-1}U(x, \nu)^{-1} \]

\[ + U(x, \nu)U(x - a\hat{\mu} + a\hat{\nu}, \mu)^{-1}U(x - a\hat{\mu}, \nu)^{-1}U(x - a\hat{\mu}, \mu) \]

\[ + U(x - a\hat{\mu}, \mu)^{-1}U(x - a\hat{\mu} - a\hat{\nu}, \nu)^{-1}U(x - a\hat{\mu} - a\hat{\nu}, \mu)U(x - a\hat{\nu}, \nu) \]

\[ + U(x - a\hat{\nu}, \nu)^{-1}U(x - a\hat{\nu}, \mu)U(x + a\hat{\mu} - a\hat{\nu}, \nu)U(x, \mu)^{-1}, \quad (3.20) \]

is a discretization of the field strength tensor \( F_{\mu\nu} \). The coefficient \( c_{sw} \) has been determined perturbatively by Wohlert [54]. At one-loop order, it is

\[ c_{sw} = 1 + 0.265907 g_0^2 + O(g_0^4), \quad (3.21) \]

which does not depend on the number \( N_f \) of dynamical quark flavours [55]. A non-perturbative value for \( c_{sw} \) is available in the quenched approximation [56],

\[ c_{sw} = \frac{1 - 0.656 g_0^2 - 0.152 g_0^4 - 0.054 g_0^6}{1 - 0.922 g_0^2}, \quad 0 \leq g_0 \leq 1, \quad (3.22) \]

as well as for \( N_f = 2 \) [57],

\[ c_{sw} = \frac{1 - 0.454 g_0^2 - 0.175 g_0^4 + 0.012 g_0^6 + 0.045 g_0^8}{1 - 0.720 g_0^2}. \quad (3.23) \]

Composite operators need separate \( O(\alpha) \) improvement. For example, the axial current of two light quarks \( \psi_1 \) and \( \psi_2 \),

\[ A(x) = \bar{\psi}_1(x)\gamma_\mu\gamma_5\psi_2(x), \quad (3.24) \]

has to be improved by adding dimension four operators with the same symmetry properties as the original lattice current. Again, one writes down a formal expansion of an effective continuum current in terms of the lattice spacing, and chooses a basis of operators with the required symmetries. As in the case of the action, the equations of motion can be used to reduce the number of operators that are needed [52]. A possible choice of remaining continuum operators is then

\[ (\delta A_\mu)_1 = \frac{1}{2}(m_1 + m_2)\bar{\psi}_1\gamma_\mu\gamma_5\psi_2 \quad (3.25) \]

and

\[ (\delta A_\mu)_2 = \partial_\mu P, \quad (3.26) \]

with the axial density

\[ P(x) = \bar{\psi}_1(x)\gamma_5\psi_2(x) \quad (3.27) \]
and the two light quark masses $m_1$ and $m_2$. The operator $(\delta A_\mu)_1$ will be dealt with in Chapter 7; for the moment we improve the lattice operator by discretizing the density $P$ and defining

$$\delta A_\mu = \frac{1}{2}(\partial^\mu + \partial^\mu_s)P,$$

with the forward and backward lattice derivatives $\partial_\mu$ and $\partial^\mu_s$ (see Appendix A.3), and the improved axial current is written as

$$(A_1)_\mu(x) = A_\mu(x) + ac_A \delta A_\mu.$$  \hspace{1cm} (3.29)

The coefficient $c_A$ is known both at one loop order of perturbation theory [55],

$$c_A = -0.00567(1)C_F g_0^2 + O(g_0^4), \quad C_F = \frac{4}{3},$$  \hspace{1cm} (3.30)

and non-perturbatively [56]

$$c_A = -0.00756 g_0^2 \times \frac{1 - 0.0748 g_0^2}{1 - 0.9777 g_0^2}, \quad 0 \leq g_0 \leq 1.$$  \hspace{1cm} (3.31)
Chapter 4

The static approximation

4.1 Static quarks in the continuum

To set up an effective theory for heavy quarks, an action for the heavy quark field has to be introduced. The usually chosen (euclidean) action defining the static theory is [58]

\[ S_h[A, \bar{\psi}_h, \psi_h] = \int d^4 x \bar{\psi}_h(x) D_0 \psi_h(x), \]  

(4.1)

where \( D_0 \) is the covariant derivative in the time direction. The heavy quark fields \( \psi_h \) and \( \bar{\psi}_h \) have Dirac and colour indices. They are written as 4-component spinors to keep the notation simple, but they satisfy the constraints

\[ P_+ \psi_h = \psi_h, \quad \bar{\psi}_h P_+ = \bar{\psi}_h, \quad P_+ = \frac{1}{2}(1 + \gamma_0), \]  

(4.2)

which means that the fields have only two degrees of freedom at each space-time point, and the other components are

\[ P_- \psi_h = \bar{\psi}_h P_- = 0, \quad P_- = \frac{1}{2}(1 - \gamma_0). \]  

(4.3)

This action can be obtained as the leading term in the \( 1/m \) expansion in Heavy Quark Effective Theory (for a review see [58]), or as the mass independent part of the NRQCD action.

The Lagrangian in eq. (4.1) produces a divergence, which is due to the static quark self energy [59]

\[ E_{\text{self}} = \frac{1}{a} \left\{ e_1 g_0^2 + O(g_0^4) \right\}, \quad e_1 = \frac{1}{12\pi^2} \times 19.95, \]  

(4.4)

and which has to be cancelled by a mass-like counterterm. To make the effect of this heavy quark mass renormalization explicit, the modified action

\[ S_h[A, \bar{\psi}_h, \psi_h] = \int d^4 x \bar{\psi}_h(x)(D_0 + \delta m) \psi_h(x) \]  

(4.5)
will be studied. The classical field equations derived from the action eq. (4.5) are

\[(D_0 + \delta m) \psi_{h,c}(x) = 0, \quad \bar{\psi}_{h,c}(x)(\overline{D}_0 - \delta m) = 0.\]

The heavy anti-quarks have to be treated separately, their action is

\[S_h[A, \bar{\psi}_R, \psi_R] = \int d^4x \bar{\psi}_R(x)(D_0 + \delta m)\psi_R(x),\]

with

\[P_- \bar{\psi}_R = \psi_R, \quad \bar{\psi}_R P_- = \bar{\psi}_R.\]

The action (4.5) leads to the heavy quark propagator

\[S_h(x, y) = \theta(x_0 - y_0)\delta(\mathbf{x} - \mathbf{y})e^{-\delta m(x_0 - y_0)}\mathcal{P}\exp\left\{\int_{y_0}^{x_0} dt A_0(t, \mathbf{x})\right\} P_+,\]

where \(\mathcal{P}\) symbolizes path-ordering. This propagator is local in space, in the sense that there is no propagation between points \(\mathbf{x}\) and \(\mathbf{y}\) with \(\mathbf{x} \neq \mathbf{y}\), meaning that the quark is “static” from an intuitive point of view.

### 4.2 Static quarks on the lattice

As in the case of light quarks, there are many possibilities to discretize the actions (4.1) and (4.5). Eichten and Hill propose [59, 60]

\[S_h[U, \bar{\psi}_h, \psi_h] = a^4 \sum_x \bar{\psi}_h(x) \nabla_0^* \psi_h(x).\]

Due to the heavy quark self energy, the linear divergence mentioned above appears, and has to be cancelled by a mass-like counterterm. There is no symmetry that can be used to determine it non-perturbatively, which is indeed a problem in the calculation of the \(b\) quark mass from the static lattice theory [61]. To derive the dependence of correlation functions on this counterterm analytically, we include it in the action,

\[S_h[U, \bar{\psi}_h, \psi_h] = a^4 \sum_x \bar{\psi}_h(x)(\nabla_0^* + \delta m)\psi_h(x).\]

The heavy quark propagator derived from the action (4.11) is

\[S_h(x, y) = \theta(x_0 - y_0)\delta(\mathbf{x} - \mathbf{y})(1 + a \delta m)^{-3/2} W(y, x)^{-1} P_+,\]

with the lattice theta and delta functions defined in Appendix A, and

\[W(x, x + R\hat{\mu}) = U(x, \mu)U(x + a\hat{\mu}, \mu) \times \ldots \times U(x + (R - a)\hat{\mu}, \mu) \quad \text{if} \quad R > 0,\]

\[W(x, x) = 1.\]
The propagator is only non-zero if \( x = y \), and \( W(y, x) \) in (4.12) is the time-like Wilson line connecting \( x \) and \( y \). This is due to the fact that the static quark action only connects the heavy fields at each lattice site with their only time-like neighbours but not with their space-like ones.

For completeness, the heavy anti-quark action is discretized as

\[
S_h[U, \bar{\psi}_h, \psi_h] = a^4 \sum_x \bar{\psi}_h(x)(\nabla_0 + \delta m)\psi_h(x).
\]

(4.15)

### 4.3 Symmetries in the static approximation

For the discussion of \( O(a) \) improvement, it is important to identify the symmetries of the static quark effective theory. The first thing to notice is that the static lattice action — in contrast to the full QCD action on the lattice — is not invariant under four-dimensional discrete rotations; only a three dimensional discrete euclidean symmetry is left, while the time direction is excluded from the symmetry.

On the other hand, the static approximation produces additional symmetries not known from full QCD:

1. Static quarks exhibit a heavy quark spin symmetry [62, 63, 64], which means that the action and the integration measure are invariant under the \( \text{SU}(2) \) transformations

\[
\psi_h \rightarrow V\psi_h, \quad \bar{\psi}_h \rightarrow \bar{\psi}_hV^{-1},
\]

(4.16)

with

\[
V = \exp\{-i\phi\epsilon_{ijk}\sigma_{jk}\}.
\]

(4.17)

The matrices \( \sigma_{jk} \) are defined in Appendix A.4

2. Invariance under rotations in flavour space when several heavy quark flavours are included [62, 63, 64]. As there is only one heavy quark flavour in our case, this symmetry will be irrelevant for the following discussion.

3. Invariance of the action and the integration measure under \( \text{U}(1) \) transformations that are local in space [65],

\[
\psi_h \rightarrow e^{i\eta(x)}\psi_h, \quad \bar{\psi}_h \rightarrow \bar{\psi}_he^{-i\eta(x)},
\]

(4.18)

where the parameter \( \eta(x) \) must be time independent. The static-light axial current transforms as

\[
A_0^\text{stat}(x) \rightarrow e^{i\eta(x)}A_0^\text{stat}(x).
\]

(4.19)
4.4 $O(a)$ improvement in the static approximation

With the knowledge of the relevant symmetries, the subject of $O(a)$ improvement in the static theory can now be addressed. As in the case of relativistic quarks, local dimension five operators with the same symmetry properties as the lattice heavy quark Lagrangian have to be found in an effective continuum formulation. Then they have to be discretized and added to the Lagrangian with appropriately chosen coefficients. A basis of possible dimension five operators is

\[
\begin{align*}
\mathcal{O}_1 &= \bar{\psi}_h \sigma_{0j} F_{0j} \psi_h, \\
\mathcal{O}_2 &= \bar{\psi}_h \sigma_{jk} F_{jk} \psi_h, \\
\mathcal{O}_3 &= \bar{\psi}_h D_0 D_0 \psi_h + \bar{\psi}_h \tilde{D}_0 \tilde{D}_0 \psi_h, \\
\mathcal{O}_4 &= \bar{\psi}_h D_j D_j \psi_h + \bar{\psi}_h \tilde{D}_j \tilde{D}_j \psi_h, \\
\mathcal{O}_5 &= m_1 (\bar{\psi}_h D_0 \psi_h - \bar{\psi}_h \tilde{D}_0 \psi_h), \\
\mathcal{O}_6 &= m_1 (\bar{\psi}_h D_j \psi_h - \bar{\psi}_h \tilde{D}_j \psi_h), \\
\mathcal{O}_7 &= m_1^2 \bar{\psi}_h \psi_h.
\end{align*}
\]  

(4.20)

Note that the time derivatives have to be treated separately from the space derivatives, as we have only three dimensional discrete euclidean invariance. The number of operators can again be reduced significantly. The operator $\mathcal{O}_1$ is zero, because of eq. (4.2) and

\[P_+ \gamma_j P_+ = 0.\]  

(4.21)

Furthermore, $\mathcal{O}_2$ cannot contribute, since it violates the heavy quark spin symmetry, eq. (4.16). Similarly, $\mathcal{O}_4$ and $\mathcal{O}_6$ can be dropped, since they are not invariant under the local $U(1)$ transformations in eq. (4.18). For on-shell calculations, the operators $\mathcal{O}_3$ and $\mathcal{O}_5$ are zero because of the equations of motion, which can be used in analogy to the relativistic case, following the detailed discussion in [52], which is not restricted to relativistic fields, but is valid for any set of composite operators that are local and gauge invariant. The quantity $\delta m$ does not appear here, since it is a counterterm and not a parameter of the theory.

We thus remain with $\mathcal{O}_7$, which is equivalent to a shift in the counterterm $\delta m$. It will be dealt with in Chapter 7; for the moment no $O(a)$ terms have to be added to the heavy quark action.

As a composite operator, the static-light axial current needs a separate discussion concerning $O(a)$ improvement. Here, a basis of possible dimension four operators with the appropriate space-time symmetries has to be found. A possible choice is

\[
\begin{align*}
(\delta A_0^{stat})_1 &= \bar{\psi}_1 \gamma_5 D_0 \psi_h, \\
(\delta A_0^{stat})_2 &= \bar{\psi}_1 \tilde{D}_0 \gamma_5 \psi_h.
\end{align*}
\]
\[(\delta A_{0}^{\text{stat}})_{3} = \bar{\psi}_{1} \gamma_{5} \gamma_{j} D_{j} \psi_{h}, \quad (4.22)\]
\[(\delta A_{0}^{\text{stat}})_{4} = \bar{\psi}_{1} \tilde{D}_{j} \gamma_{j} \gamma_{5} \psi_{h}, \quad (\delta A_{0}^{\text{stat}})_{5} = m_{1} \bar{\psi}_{1} \gamma_{0} \gamma_{5} \psi_{h}.\]

Now the heavy quark field equation can again be applied, which yields \((\delta A_{0}^{\text{stat}})_{1} = 0\). The operator \((\delta A_{0}^{\text{stat}})_{3}\) cannot give any contribution, since it does not transform correctly under local U(1) transformations, eq. (4.19). The remaining three operators are connected by the light quark field equation, such that one of them can be omitted. Here, \((\delta A_{0}^{\text{stat}})_{4}\) and \((\delta A_{0}^{\text{stat}})_{5}\) will be kept. The operator \((\delta A_{0}^{\text{stat}})_{5}\) will be discussed in the context of renormalization, Chapter 7, while \((\delta A_{0}^{\text{stat}})_{4}\) is discretized as

\[
\delta A_{0}^{\text{stat}}(x) = \bar{\psi}_{1}(x) \frac{1}{2} (\tilde{\nabla}_{j} + \tilde{\nabla}_{j}^{*}) \gamma_{j} \gamma_{5} \psi_{h}(x), \quad (4.23)
\]

and added to the unimproved lattice current with a coefficient \(c_{A}^{\text{stat}}\),

\[
(A_{1}^{\text{stat}})_{0}(x) = A_{0}^{\text{stat}}(x) + a c_{A}^{\text{stat}} \delta A_{0}^{\text{stat}}(x). \quad (4.24)
\]

A one loop value for \(c_{A}^{\text{stat}}\) has been calculated in [66], using NRQCD methods. The result is

\[
c_{A}^{\text{stat}} = g_{0}^{2} c_{A}^{\text{stat}(1)} + O(g_{0}^{4}), \quad c_{A}^{\text{stat}(1)} = -\frac{1}{4\pi} \times 1.00(1). \quad (4.25)
\]

In Chapter 8, a calculation of \(c_{A}^{\text{stat}(1)}\) from static approximation data will be presented.
Chapter 5
The Schrödinger functional

As described in Chapter 2, a finite size method is to be used in the non-perturbative renormalization procedure. This means that boundary conditions have to be fixed. One possibility to do this is the Schrödinger functional [41, 42], which is well-behaved in several respects. An important point to mention here is that the Schrödinger functional boundary conditions induce an energy gap in the spectrum of the Wilson-Dirac operator [42], such that quark masses can directly be set to zero in Monte Carlo simulations. In perturbation theory, this means that the Schrödinger functional is infrared finite, and no infrared regulator like an artificial gluon mass is needed. An introduction to the Schrödinger functional technique can be found in [67, 68].

First, the Schrödinger functional will be introduced in the continuum following [41] and [42], and then static quarks will be included. The discretization of the Schrödinger functional will happen in Section 5.2.

5.1 The Schrödinger functional in the continuum

The space-time topology of the Schrödinger functional is a finite cylinder, meaning a box with spatial size $L \times L \times L$ and time extent $T$, where the fields satisfy generalized periodic boundary conditions in the three space directions, and Dirichlet boundary conditions at the $x_0 = 0$ and $x_0 = T$ boundaries (see Fig. 5.1). In the following, all space integrals run from 0 to $L$, while all time integrals run from 0 to $T$.

5.1.1 Gauge Fields

The gauge field $A_k(x)$ is chosen to be periodic in space, i.e.

$$A_k(x + L\hat{k}) = A_k.$$  \hspace{1cm} (5.1)
At the \( x_0 = 0 \) and \( x_0 = T \) surfaces, boundary fields \( C(x) \) and \( C'(x) \) are introduced, such that
\[
A_k(x)|_{x_0=0} = C_k^0(x) \tag{5.2}
\]
and
\[
A_k(x)|_{x_0=T} = C_k'(x), \tag{5.3}
\]
where \( \Lambda \) denotes a gauge transformation by a periodic gauge function \( \Lambda(x) \),
\[
C_k^\Lambda(x) = \Lambda(x) C_k(x) \Lambda(x)^{-1} + \Lambda(x) \partial_k \Lambda(x)^{-1}, \tag{5.4}
\]
which can be used to project onto the subspace of gauge-invariant field configurations. The time components of the gauge field remain unconstrained. The action of the gauge field is
\[
S_G[A] = -\frac{1}{2g^2} \int dx_0 \int d^3x \text{tr}\{F_{\mu\nu} F^{\mu\nu}\}. \tag{5.5}
\]

### 5.1.2 Relativistic quarks

In the three space directions, the light quark fields are chosen to be periodic up to a phase \( \theta_k \),
\[
\psi_1(x + L\hat{k}) = e^{i\theta_k} \psi_1(x), \quad \bar{\psi}_1(x + L\hat{k}) = e^{-i\theta_k} \bar{\psi}_1(x). \tag{5.6}
\]
In the following, a somewhat different but completely equivalent notation will be used, namely the fermion fields will be taken to be periodic,

\[ \psi(x + L\hat{k}) = \psi(x), \quad \tilde{\psi}(x + L\hat{k}) = \tilde{\psi}(x), \]  

(5.7)
and the phases \( \theta_k \) are absorbed into the spatial components of the covariant derivatives, as given in Appendix A.2. An important point to note is that this change of notation does not affect correlation functions containing the static-light axial current, as it is equivalent to a substitution of \( \psi_h \) by \( e^{i\theta_k x_k / L} \psi_h \) in the current. But this does not matter, because of the local U(1) symmetry, eq. (4.18).

As the equation of motion derived from the light quark action is a first order differential equation, only half of the field components can be fixed at \( x_0 = 0 \) and \( x_0 = T \). Explicitly, boundary functions \( \rho_1(x), \bar{\rho}_1(x), \rho_1'(x), \) and \( \bar{\rho}_1'(x) \) are introduced, such that

\[ P_+ \psi_1(x)|_{x_0=0} = \rho_1(x), \quad P_- \psi_1(x)|_{x_0=T} = \rho_1'(x), \]  

(5.8)
and

\[ \tilde{\psi}_1(x) P_-|_{x_0=0} = \bar{\rho}_1(x), \quad \tilde{\psi}_1(x) P_+|_{x_0=T} = \bar{\rho}_1'(x). \]  

(5.9)

For consistency, the boundary functions must be such that the complementary components \( P_- \rho_1, \ldots, \bar{\rho}_1' P_- \) vanish, i.e.

\[ P_- \rho_1(x) P_+ = P_+ \rho_1'(x) = \bar{\rho}_1'(x) P_- = 0. \]  

(5.10)
The Schrödinger functional light quark action is [42]

\[
S[A, \bar{\psi}_1, \psi_1] = \int dx_0 \int d^3x \bar{\psi}_1(x) \left( \gamma_\mu D_\mu + m_0 \right) \psi_1(x) - \int d^3x \left[ \bar{\psi}_1(x) P_- \psi_1(x) \right]_{x_0=0} - \int d^3x \left[ \bar{\psi}_1(x) P_+ \psi_1(x) \right]_{x_0=T},
\]  

(5.11)
where the derivation of the boundary terms is analogous to the static quark case explained below.

### 5.1.3 Static quarks

In the static quark case, no spatial boundary conditions have to be specified, because the static quarks do not propagate in space, and thus cannot propagate across the spatial borders. However, to make the discussion of \( O(a) \) improvement analogous to the light quark case, periodic boundary conditions are chosen here,

\[ \psi_h(x + L\hat{k}) = \psi_h(x), \quad \tilde{\psi}_h(x + L\hat{k}) = \tilde{\psi}_h(x), \]  

(5.12)
but the results do not depend on these conditions.
At the boundaries in the time direction, boundary fields $\rho_h$ and $\rho_h'$ are defined as in the case of relativistic quarks, such that
\[
\psi_h(x)|_{x_0=0} = \rho_h(x), \quad \bar{\psi}_h(x)|_{x_0=T} = \bar{\rho}_h'(x).
\] (5.13)

As the heavy quark fields have only two components, no other boundary conditions are needed. Note also that no projectors are necessary here because of eq. (4.2).

Corresponding to (5.11), a boundary term has to be added to the action:
\[
S_h[A, \bar{\psi}_h, \psi_h] = \int dx_0 \int d^3x \bar{\psi}_h(x) (D_0 + \delta m) \psi_h(x) - \int d^3x \left[ \bar{\psi}_h(x) \psi_h(x) \right]_{x_0=0} - \int d^3x \left[ \bar{\psi}_h(x) R(x) \psi_h(x) \right]_{x_0=T}.
\] (5.14)

This can be seen by the requirement that the action eq. (5.14) defines a sensible classical field theory, i.e. that the action as a functional on the space of smooth ($C^\infty$) functions has stationary points that lie within that space. This means that one has to apply a variation principle to look for these stationary points $\psi_{hcl}$ and $\bar{\psi}_{hcl}$, and demand that they are smooth functions. To that end, we follow the discussion for the relativistic quark case in [42], and write a general ansatz for the action,
\[
S_h[A, \bar{\psi}_h, \psi_h] = \int dx_0 \int d^3x \bar{\psi}_h(x) (D_0 + \delta m) \psi_h(x)
+ \int d^3x \left[ \bar{\psi}_h(x) R_0(x) \psi_h(x) \right]_{x_0=0}
+ \int d^3x \left[ \bar{\psi}_h(x) R_T(x) \psi_h(x) \right]_{x_0=T}.
\] (5.15)

As the heavy quark fields satisfy eq. (4.2), we can assume
\[
R_0 = P_+ R_0 P_+, \quad R_T = P_+ R_T P_+
\] (5.16)

without any loss of generality. We can then write down the variations
\[
\delta S_h(\bar{v}) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ S_h[A, \bar{\psi}_h + \epsilon \bar{v}, \psi_h] - S_h[A, \bar{\psi}_h, \psi_h] \right\},
\] (5.17)
\[
\delta S_h(v) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ S_h[A, \bar{\psi}_h, \psi_h + \epsilon v] - S_h[A, \bar{\psi}_h, \psi_h] \right\},
\] (5.18)

where $v$ and $\bar{v}$ are smooth functions with
\[
P_+ v = v, \quad \bar{v} P_+ = \bar{v}, \quad P_- v = \bar{v} P_- = 0,
\] (5.19)

which satisfy
\[
v(x)|_{x_0=0} = 0
\] (5.20)

and
\[
\bar{v}(x)|_{x_0=T} = 0,
\] (5.21)
to keep the Schrödinger functional boundary conditions intact. Calculating the first variation explicitly, one gets

$$
\tilde{\delta}S_h(\bar{v}) = \int dx_0 \int d^3x \bar{v}(x) \{ D_0 + \delta m + \delta(x_0)R_0(x) \} \psi_h(x),
$$
\begin{equation}
(5.22)
\end{equation}

where eq. (5.21) has been used. Using eq. (5.20), the second variation gives

$$
\delta S_h(v) = \int dx_0 \int d^3x \tilde{\psi}_h(x) \{ D_0 + \delta m + \delta(x_0 - T)R_T(x) \} v(x),
$$
\begin{equation}
(5.23)
\end{equation}

and, after partial integration,

$$
\delta S_h(v) = -\int dx_0 \int d^3x \tilde{\psi}_h(x) \left\{ \tilde{D}_0 - \delta m - \delta(x_0 - T)[R_T(x) + 1] \right\} v(x).
$$
\begin{equation}
(5.24)
\end{equation}

Now we demand that smooth functions $\psi_{h,\text{cl}}$ and $\tilde{\psi}_{h,\text{cl}}$ exist that are solutions of

$$
\tilde{\delta}S_h(\bar{v}) = 0, \quad \delta S_h(v) = 0.
$$
\begin{equation}
(5.25)
\end{equation}

From eq. (5.22) one can see that $\psi_{h,\text{cl}}$ has to satisfy

$$
(D_0 + \delta m) \psi_{h,\text{cl}}(x) = 0
$$
\begin{equation}
(5.26)
\end{equation}

for $0 < x_0 < T$. As the function $\psi_h$ is demanded to be smooth, this field equation must also hold at the boundaries, which implies

$$
P_+ R_0(x) P_+ = 0.
$$
\begin{equation}
(5.27)
\end{equation}

Analogously,

$$
P_+ [R_T(x) + 1] P_+ = 0
$$
\begin{equation}
(5.28)
\end{equation}

follows from eq. (5.24). In the light of eq. (5.16) this finally gives

$$
R_0(x) = 0, \quad R_T(x) = -1,
$$
\begin{equation}
(5.29)
\end{equation}

yielding the heavy quark action (5.14).

The complete action is then

$$
S[A, \bar{\psi}, \psi, \bar{\psi}_h, \psi_h] = S_{\text{cl}}[A] + S_1[A, \bar{\psi}, \psi] + S_h[A, \bar{\psi}_h, \psi_h],
$$
\begin{equation}
(5.30)
\end{equation}

and the Schrödinger functional is defined as the partition function, which is a functional of the boundary fields,

$$
Z[C', \bar{\rho}_1', \rho_1', \bar{\rho}_h', \bar{\rho}, \rho_1, \rho_h] = \int D[A] \int D[\bar{\psi}] D[\psi] D[\bar{\psi}_h] D[\psi_h] e^{-S[A, \bar{\psi}, \psi, \bar{\psi}_h, \psi_h]},
$$
\begin{equation}
(5.31)
\end{equation}

where the integration measures

$$
D[A] = \prod_{x, \mu} dA_\mu(x),
$$
\begin{equation}
(5.32)
\end{equation}
\[ D[\psi_1] = \prod_x d\psi_1(x) , \quad D[\bar{\psi}_1] = \prod_x d\bar{\psi}_1(x) , \]  
and
\[ D[\psi_h] = \prod_x d\psi_h(x) , \quad D[\bar{\psi}_h] = \prod_x d\bar{\psi}_h(x) \]
are only formally defined. The integral over the gauge transformations \( \Lambda \) in eq. (5.31) ensures gauge invariance of the Schrödinger functional \([41]\).

In order to have simple renormalization properties, expectation values of operators will be defined for vanishing boundary gauge fields \( C_k(x) = C'_k(x) = 0 \) in this thesis, as in \([52]\), and are given by the path integral expression
\[
\langle \mathcal{O} \rangle = \left\{ \frac{1}{Z} \int D[A] D[\psi_1] D[\bar{\psi}_1] D[\psi_h] D[\bar{\psi}_h] \right. \\
\left. \mathcal{O} e^{-S[A, \bar{\psi}_1, \psi_1, \bar{\psi}_h, \psi_h]} \right|_{\bar{\rho}_1' = \bar{\rho}_1' = \bar{\rho}_h = \rho_1 = \rho_h = 0} .
\]
Here, the operator \( \mathcal{O} \) can, apart from the gauge field and the quark and anti-quark fields, contain the light quark “boundary fields”
\[
\zeta_1(x) = \frac{\delta}{\delta \rho_1(x)} , \quad \bar{\zeta}_1(x) = -\frac{\delta}{\delta \rho_1(x)} , \\
\zeta_1'(x) = \frac{\delta}{\delta \bar{\rho}_1'(x)} , \quad \bar{\zeta}_1'(x) = -\frac{\delta}{\delta \bar{\rho}_1'(x)},
\]
and the corresponding heavy quark fields
\[
\zeta_h'(x) = \frac{\delta}{\delta \rho_h'(x)} , \quad \bar{\zeta}_h(x) = -\frac{\delta}{\delta \rho_h(x)} .
\]

### 5.1.4 Correlation functions

To renormalize the static-light axial current, a correlation function containing that current has to be used. A suitable choice is
\[
f_A^{\text{stat}}(x_0) = -\frac{1}{2} \int d^3y \, d^3z \, \langle A_0^{\text{stat}}(x) \bar{\zeta}_h(y) \gamma_5 \zeta_1(z) \rangle .
\]
As we are only interested in the renormalization of the current here, and not of the light and heavy quark wave functions, a ratio in which the wave function renormalization constants cancel should be defined. To that end, we introduce a second correlation function
\[
f_1^{\text{stat}} = -\frac{1}{2L^6} \int d^3u \, d^3v \, d^3y \, d^3z \, \langle \bar{\zeta}_1'(u) \gamma_5 \bar{\zeta}_h'(v) \bar{\zeta}_h(y) \gamma_5 \zeta_1(z) \rangle ,
\]
and form the ratio
\[
X = \frac{f_A^{\text{stat}}(T/2)}{\sqrt{f_1^{\text{stat}}}} ,
\]
where indeed the wave function renormalizations cancel. In Figure 5.2, a schematic view of the two correlation functions is shown.
5.2 Discretization of the Schrödinger functional

To derive finite quantities from the Schrödinger functional, a regularization has to be chosen, which is the lattice regularization in our case. However, it has to be stressed that the Schrödinger functional is not restricted to the lattice; perturbative studies can also be performed in dimensional regularization, for example [41, 69].

In the discretized version of the Schrödinger functional, all fields are treated as in the continuum concerning their periodicity in the three space directions, i.e. they are chosen to be periodic, with angles $\theta_k$ included in the discretized covariant derivatives (see Appendix A.3).

At the boundaries $x_0 = 0$ and $x_0 = T$, the gauge field satisfies the Dirichlet conditions
\[ U(x, k)|_{x_0=0} = W_k(x) \]  \hfill (5.41)
and
\[ U(x, k)|_{x_0=T} = W'_k(x), \]  \hfill (5.42)
where the boundary fields $W$ and $W'$ are obtained from smooth continuum gauge fields $C$ and $C'$ by
\[ W_k(x) = \mathcal{P} \exp \left\{ a \int_0^1 dt \, C_k(x + a\hat{k} - ta\hat{k}) \right\}, \]  \hfill (5.43)
\[ W'_k(x) = \mathcal{P} \exp \left\{ a \int_0^1 dt \, C'_k(x + a\hat{k} - ta\hat{k}) \right\}. \]  \hfill (5.44)
The symbol \( P \) denotes path ordering of the exponential, such that the fields at larger values of \( t \) precede those at smaller \( t \). As in the continuum, the time components of the gauge field remain unconstrained.

The gauge field action can be written as

\[
S_G[U] = \frac{1}{g^2} \sum_p w(p) \text{tr} \{1 - U(p)\},
\]

where \( U(p) \) as in eq. (3.1) is the product of the link variables around the plaquette \( p \), and the sum runs over all oriented plaquettes that have only corners with time components \( x_0 \) in the range \( 0 \leq x_0 \leq T \). The weight \( w(p) \) is \( \frac{1}{2} \) for the spatial plaquettes at \( x_0 = 0 \) and \( x_0 = T \), while it is 1 for all other plaquettes.

For the light quark fields, the boundary conditions

\[
P_+ \phi_1(x)|_{x_0=0} = \rho_1(x), \quad P_- \phi_1(x)|_{x_0=T} = \rho_1'(x)
\]

and

\[
\bar{\phi}_1(x) P_-|_{x_0=0} = \bar{\rho}_1(x), \quad \bar{\phi}_1(x) P_+|_{x_0=T} = \bar{\rho}_1'(x)
\]

are fixed as in the continuum. Extending the fields to \( x_0 < 0 \) and \( x_0 > T \) by

\[
\phi_1(x) = \bar{\phi}_1(x) = 0 \quad \text{if} \quad x_0 < 0 \quad \text{or} \quad x_0 > T,
\]

and

\[
P_- \phi_1(x)|_{x_0=0} = P_+ \phi_1(x)|_{x_0=T} = 0,
\]

\[
\bar{\phi}_1(x)|_{x_0=0} P_+ = \bar{\phi}_1(x)|_{x_0=T} P_- = 0,
\]

the light quark action can be written in the simple form

\[
S_l[U, \bar{\psi}_1, \psi_1] = a^4 \sum_x \bar{\psi}_1(x) (D_W + m_0) \psi_1(x),
\]

where the sum runs over all \( x \) with spatial components \( x_k \) in the interval \( 0 \leq x_k < L \).

The heavy quarks again satisfy the conditions

\[
\phi_h(x)|_{x_0=0} = \rho_h(x), \quad \bar{\phi}_h(x)|_{x_0=T} = \bar{\rho}_h'(x).
\]

Here, it is also convenient to extend the fields by

\[
\phi_h(x) = \bar{\phi}_h(x) = 0 \quad \text{if} \quad x_0 < 0 \quad \text{or} \quad x_0 > T,
\]

and

\[
\bar{\phi}_h(x)|_{x_0=0} = 0.
\]

The heavy quark action is then

\[
S_h[U, \bar{\psi}_h, \psi_h] = a^4 \sum_x \bar{\psi}_h(x) (\nabla_0^* + \delta m) \psi_h(x) - a^3 \sum_x \bar{\psi}_h(x) \psi_h(x)|_{x_0=T}.
\]
In the case $\delta m = 0$, which below will be shown to be interesting in our case, one can also set
\[ \psi_h(x)|_{x_0=T} = 0, \tag{5.55} \]
and then write the action as
\[ S_h[U, \tilde{\psi}_h, \psi_h]|_{\delta m=0} = a^4 \sum_x \tilde{\psi}_h(x) \nabla^2 \psi_h(x). \tag{5.56} \]
The complete action is
\[ S[U, \tilde{\psi}_h, \psi_h, \tilde{\psi}_h, \psi_h] = S_0[U] + S_1[U, \tilde{\psi}_1, \psi_1] + S_h[U, \tilde{\psi}_h, \psi_h], \tag{5.57} \]
and the Schrödinger functional is defined as
\[ Z[W', \rho_1', \rho_1', \rho_h', W, \rho_1, \rho_1, \rho_h] = \int D[U] D[\psi_1] D[\tilde{\psi}_1] D[\psi_h] D[\tilde{\psi}_h] e^{-S[U, \tilde{\psi}_h, \psi_h, \tilde{\psi}_h, \psi_h]}, \tag{5.58} \]
where the integration measures are defined as in eq. (3.11) and eq. (3.12), and
\[ D[\psi_h] = \prod_x d\psi_h(x), \quad D[\tilde{\psi}_h] = \prod_x d\tilde{\psi}_h(x). \tag{5.59} \]
Expectation values
\[ \langle \mathcal{O} \rangle = \left\{ \frac{1}{Z} \int D[U] D[\psi_1] D[\tilde{\psi}_1] D[\psi_h] D[\tilde{\psi}_h] \mathcal{O} e^{-S[U, \tilde{\psi}_h, \psi_h, \tilde{\psi}_h, \psi_h]} \right\} \bigg|_{\tilde{\rho}_1' = \rho_1' = \tilde{\rho}_h' = \rho_1 = \rho_h = 0} \tag{5.60} \]
are taken at $\tilde{W}_k(x) = \tilde{W}_k'(x) = 1$ in this thesis, and the operator $\mathcal{O}$ can again contain boundary derivatives analogous to eq. (5.36) and (5.37).

Now, a discretized version of the correlation functions can be written down,
\[ f_{\Delta}^{\text{stat}}(x_0) = -a^6 \sum_{y,z} \frac{1}{2} \left\langle A_\Delta^{\text{stat}}(x) \tilde{\zeta}_h(y) \gamma_5 \zeta_1(z) \right\rangle \tag{5.61} \]
and
\[ f_1^{\text{stat}} = -\frac{a^{12}}{L^6} \sum_{u,v,y,z} \frac{1}{2} \left\langle \tilde{\zeta}_1'(u) \gamma_5 \zeta_h' (v) \tilde{\zeta}_h(y) \gamma_5 \zeta_1(z) \right\rangle \tag{5.62} \]
Again, the ratio
\[ X = \frac{f_{\Delta}^{\text{stat}}(T/2)}{\sqrt{f_1^{\text{stat}}}} \tag{5.63} \]
can be formed, in which the wave function renormalization constants cancel. Indeed, this ratio has a second advantage. The dependence of the correlation
functions $f_0^{\text{stat}}$ and $f_1^{\text{stat}}$ on the mass counterterm $\delta m$ can be extracted analytically from the heavy quark action. The results are

$$f_0^{\text{stat}}(x_0) \propto (1 + a \delta m)^{-x_0/a} \quad (5.64)$$

and

$$f_1^{\text{stat}} \propto (1 + a \delta m)^{-T/a}. \quad (5.65)$$

This means that the $\delta m$-dependent terms cancel in the ratio $X$, and our calculation of $X$ can be performed at $\delta m = 0$, without picking up any heavy quark mass renormalization.

## 5.3 O(a) improvement of the Schrödinger functional

As the Schrödinger functional boundary conditions might (and in fact do) induce $O(a)$ effects, the improvement problem has to be reconsidered, taking the boundary effects into account.

### 5.3.1 Improved action

To achieve $O(a)$ improvement of the gauge field action, an $O(a)$ counterterm $\delta S_{G, b}$ has to be added,

$$S_{G, \text{impr}}[U] = S_G[U] + \delta S_{G, b}[U], \quad (5.66)$$

which is a sum over operators of local fields at the boundaries and has the same three-dimensional discrete euclidean symmetries as the original action. Using these symmetries and the equations of motion as in Sect. 3.2 to reduce the number of relevant operators, one remains with [41]

$$\delta S_{G, b} = \frac{1}{2g_0^2} (c_s - 1) \sum_{p_b} \text{tr} \left\{ 1 - U(p_b) \right\}$$

$$+ \frac{1}{g_0^2} (c_t - 1) \sum_{p_t} \text{tr} \left\{ 1 - U(p_t) \right\}, \quad (5.67)$$

where $p_b$ denotes the oriented space-like plaquettes at the boundaries $x_0 = 0$ and $x_0 = T$, and $p_t$ means the corresponding time-like plaquettes. Both $c_s$ and $c_t$ are only known perturbatively. The value of $c_s$ will not be needed in our context; $c_t$ is known at two loop order,

$$c_t = 1 + c_t^{(1)} g_0^2 + c_t^{(2)} g_0^4 + O(g_0^6), \quad (5.68)$$

with [41, 70]

$$c_t^{(1)} = -0.08900(5) + 0.0191410(1) \times N_f \quad (5.69)$$
and \[71, 72\]

\[
c_i^{(2)} = -0.0294(3) + 0.002(1) \times N_f + 0.0000(1) \times N_f^2. \tag{5.70}
\]

Counterterms also have to be added to the light quark action,

\[
S_{1, \text{impr}}[U, \bar{\psi}, \psi] = S_1[U, \bar{\psi}, \psi] + \delta S_{1, v}[U, \bar{\psi}, \psi] + \delta S_{1, b}[U, \bar{\psi}, \psi]. \tag{5.71}
\]

The volume counterterm \(\delta S_{1, v}[U, \bar{\psi}, \psi]\) is the Sheikholeslami-Wohlert term already known from Section 3.2. For the boundary counterterms, again a basis of operators with the appropriate symmetries has to be found in a formal continuum theory. The number of necessary counterterms is reduced by application of the equations of motion, and the remaining operators are discretized. A possible choice is \[52\]

\[
\delta S_{1, b}[U, \bar{\psi}, \psi] = a^4 \sum_x \left\{ (\bar{c}_a - 1) \left[ \bar{\mathcal{O}}_a(x) + \mathcal{O}'_a(x) \right] + (\bar{c}_t - 1) \left[ \bar{\mathcal{O}}_t(x) - \mathcal{O}'_t(x) \right] \right\} \tag{5.72}
\]

with the operators

\[
\bar{\mathcal{O}}_a(x) = \frac{1}{2} \bar{\rho}_1(x) \gamma_k (\nabla_k + \nabla_k) \rho_1(x), \tag{5.73}
\]

\[
\bar{\mathcal{O}}_t(x) = \frac{1}{2} \bar{\rho}_1'(x) \gamma_k (\nabla_k^* + \nabla_k) \rho_1'(x), \tag{5.74}
\]

\[
\mathcal{O}_t(x) = \left\{ \bar{\psi}_1(y) P_+ \nabla_0^* \psi_1(y) + \bar{\psi}_1(y) \nabla_0 P_- \psi_1(y) \right\} \bigg|_{y=(a, x)}, \tag{5.75}
\]

\[
\mathcal{O}_t'(x) = \left\{ \bar{\psi}_1(y) P_- \nabla_0 \psi_1(y) + \bar{\psi}_1(y) \nabla_0 P_+ \psi_1(y) \right\} \bigg|_{y=(T-a, x)}. \tag{5.76}
\]

The operators \(\bar{\mathcal{O}}_t\) and \(\mathcal{O}_t\) can be accounted for by adding \(\delta D_b\) with

\[
\delta D_b \psi_1(x) = (\bar{c}_t - 1) \left\{ \delta(x_0 - a) \left[ \psi_1(x) - U(x - a \hat{0}, 0)^{-1} P_+ \psi_1(x - a \hat{0}) \right] \right. \right. \right.
\]

\[
\left. \left. + \delta(x_0 - (T - a)) \left[ \psi_1(x) - U(x, 0) P_- \psi_1(x + a \hat{0}) \right] \right\} \right\} \tag{5.77}
\]

to the lattice Dirac operator. Here again, the coefficients are only known perturbatively. The \(\bar{c}_a\) term will not play a rôle in our computation; the one loop expansion of \(\bar{c}_t\) is \[55, 73\]

\[
\bar{c}_t = 1 - 0.01346(1) C_F g_0^2 + O(g_0^4), \quad C_F = \frac{4}{3}. \tag{5.78}
\]

Again, a possible operator

\[
\mathcal{O}_m = m_1 \bar{\psi}_1 \psi_1 \tag{5.79}
\]

has been disregarded, and its discussion is postponed to Chapter 7.

A priori, one should expect that also some boundary terms could be necessary for the heavy quark action,

\[
S_{h, \text{impr}}[U, \bar{\psi}_h, \psi_h] = S_h[U, \bar{\psi}_h, \psi_h] + \delta S_{h, b}[U, \bar{\psi}_h, \psi_h]. \tag{5.80}
\]
This means the addition of dimension four operators to the heavy quark Lagrangian at the boundaries. A basis of such operators, that have the same three dimensional euclidean symmetry properties as the original heavy quark action, consists of

\[
\begin{align*}
\hat{\mathcal{O}}_1 &= \bar{\psi}_h D_0 \psi_h, \\
\hat{\mathcal{O}}_2 &= \bar{\psi}_h \hat{D}_0 \psi_h, \\
\hat{\mathcal{O}}_3 &= \bar{\psi}_h \gamma_j D_j \psi_h - \bar{\psi}_h \hat{D}_j \gamma_j \psi_h, \\
\hat{\mathcal{O}}_4 &= m_1 \bar{\psi}_h \psi_h.
\end{align*}
\] (5.81)

The operators \( \hat{\mathcal{O}}_1 \) and \( \hat{\mathcal{O}}_2 \) vanish in on-shell quantities because of the equations of motion for the heavy quark field. The third operator \( \hat{\mathcal{O}}_3 \) is zero because of eq. (4.21). The only remaining operator is \( \hat{\mathcal{O}}_4 \), which is equivalent to a rescaling of the heavy quark boundary fields. It will be neglected for the moment and reconsidered in Chapter 7. Apart from this, the Schrödinger functional boundary conditions do not produce any \( O(a) \) effects concerning the static quark field.

5.3.2 Improved correlation functions

Using the improved actions for the light and the static quark fields, including the bulk and boundary \( O(a) \) counterterms, the correlation functions \( f_{\delta A}^{\text{stat}} \) and \( f_{\delta A}^{\text{stat}} \) can be recalculated. In the following chapters it is always understood that correlation functions are calculated using the improved action, unless stated otherwise.

To achieve complete \( O(a) \) improvement, the \( \delta A_0^{\text{stat}} \) term from eq. (4.23) has to be included. To do this, we define

\[
f_{\delta A}^{\text{stat}}(x_0) = -a^2 \sum_{y, z} \frac{1}{2} \left\langle \delta A_0^{\text{stat}}(x) \bar{\zeta}_h(y) \gamma_5 \zeta_1(z) \right\rangle,
\] (5.82)

and

\[
f_{\delta A}^{\text{stat}}(x_0) = f_{A}^{\text{stat}}(x_0) + a c_{A}^{\text{stat}} f_{\delta A}^{\text{stat}}(x_0).
\] (5.83)

Then the improved ratio

\[
X_1(g_0 / L, a) = \frac{f_{A}^{\text{stat}}(T/2)}{\sqrt{f_{\delta A}^{\text{stat}}}}
\] (5.84)

can be defined, which will be used in our renormalization procedure.
Chapter 6

Perturbation theory

The perturbative expansion of our correlation functions will follow [55] very closely, with the difference that static quarks will be included in our case.

6.1 Integration over the quark fields

The first step in our perturbative expansion is to write expectation values of operators $O$ as

$$\langle O \rangle = \langle [O]_F \rangle_G, \quad (6.1)$$

where $\langle \ldots \rangle_F$ means the expectation value obtained with the fermionic action

$$S_F[U, \bar{\psi}_1, \psi_1, \bar{\psi}_h, \psi_h] = S_f[U, \bar{\psi}_1, \psi_1] + S_h[U, \bar{\psi}_h, \psi_h] \quad (6.2)$$

for a fixed gauge field configuration $U$, and $\langle \ldots \rangle_G$ is an expectation value calculated with the effective gauge field action

$$S_{\text{eff}}[U] = S_G[U] - \text{tr} \left\{ \ln (D_W + \delta D_V + \delta D_b + m_0) \right\}. \quad (6.3)$$

With our choice of boundary values it is easy to see that the static fermion matrix does not give any contribution to $S_{\text{eff}}$, reflecting the fact that there are no closed static quark loops. Introducing source fields $\eta_h$, $\bar{\eta}_h$, $\eta_h$, and $\bar{\eta}_h$ for the quark fields, the generating functional

$$\mathcal{Z}_F[\tilde{\rho}_1', \rho_1'; \tilde{\rho}_1, \rho_1; \tilde{\rho}_h', \rho_h; U] =$$

$$\int D[\psi_1] D[\bar{\psi}_1] D[\psi_h] D[\bar{\psi}_h] \exp \left\{ -S_F[U, \bar{\psi}_1, \psi_1, \bar{\psi}_h, \psi_h] \right\}$$

$$+ a^4 \sum_x \left[ \bar{\psi}_1(x) \eta_1(x) + \bar{\eta}_1(x) \psi_1(x) \right]$$

$$+ a^4 \sum_x \left[ \bar{\psi}_h(x) \eta_h(x) + \bar{\eta}_h(x) \psi_h(x) \right] \quad (6.4)$$
can be written down. After substituting
\[
\psi_1(x) \longrightarrow \frac{\delta}{\delta \tilde{\eta}_1(x)}, \quad \tilde{\psi}_1(x) \longrightarrow -\frac{\delta}{\delta \eta_1(x)} \tag{6.5}
\]
and
\[
\psi_h(x) \longrightarrow \frac{\delta}{\delta \tilde{\eta}_h(x)}, \quad \tilde{\psi}_h(x) \longrightarrow -\frac{\delta}{\delta \eta_h(x)} \tag{6.6}
\]
in the operator $\mathcal{O}$, the fermionic expectation value can be written as
\[
[\mathcal{O}]_F = \left\{ \frac{1}{Z_F} \mathcal{O} Z_F \right\}_{\rho_1' = \ldots = \eta_h = 0}. \tag{6.7}
\]
Now the quark fields are written in terms of the fluctuations around their classical values,
\[
\psi_1(x) = \psi_{1,\text{cl}}(x) + \chi_1(x), \quad \tilde{\psi}_1(x) = \tilde{\psi}_{1,\text{cl}}(x) + \tilde{\chi}_1(x), \tag{6.8}
\]
and
\[
\psi_h(x) = \psi_{h,\text{cl}}(x) + \chi_h(x), \quad \tilde{\psi}_h(x) = \tilde{\psi}_{h,\text{cl}}(x) + \tilde{\chi}_h(x), \tag{6.9}
\]
where the fluctuation fields must vanish at $x_0 = 0$ and $x_0 = T$ in order to satisfy the Schrödinger functional boundary conditions. Using this fact at the boundaries, and applying the light and heavy quark field equations in the bulk, one gets
\[
S_F[U, \tilde{\psi}_1, \psi_1, \tilde{\psi}_h, \psi_h] = S_F[U, \tilde{\psi}_{1,\text{cl}}, \psi_{1,\text{cl}}, \tilde{\psi}_{h,\text{cl}}, \psi_{h,\text{cl}}] + S_F[U, \tilde{\chi}_1, \chi_1, \tilde{\chi}_h, \chi_h]. \tag{6.10}
\]
Inserting this into the generating functional and changing the integration variables from the $\psi$ fields to the $\chi$ fields, we get
\[
\ln Z_F = \ln Z_F|_{\rho_1' = \ldots = \eta_h = 0} - S_F[U, \tilde{\psi}_{1,\text{cl}}, \psi_{1,\text{cl}}, \tilde{\psi}_{h,\text{cl}}, \psi_{h,\text{cl}}]
+ a^4 \sum_{x,y} [\tilde{\eta}_1(x) \eta_1(y)] S_1(x,y) \eta_1(x) + a^4 \sum_{x} \left[ \tilde{\eta}_1(x) \psi_{1,\text{cl}}(x) + \tilde{\psi}_{1,\text{cl}}(x) \eta_1(x) \right]
+ a^8 \sum_{x,y} [\tilde{\eta}_h(x) \eta_h(y)] S_h(x,y) \eta_h(x)
+ a^4 \sum_{x} \left[ \tilde{\eta}_h(x) \psi_{h,\text{cl}}(x) + \tilde{\psi}_{h,\text{cl}}(x) \eta_h(x) \right] \tag{6.11}
\]
with the light quark propagator $S_1(x,y)$ and the heavy quark propagator $S_h(x,y)$. Using the field equations for the classical quark fields in the bulk of our space-time box, we can write
\[
S_F[U, \tilde{\psi}_{1,\text{cl}}, \psi_{1,\text{cl}}, \tilde{\psi}_{h,\text{cl}}, \psi_{h,\text{cl}}] =
\sum_{x} \left\{ \frac{1}{2} a \tilde{c}_{\chi} \left[ \tilde{\rho}_1(x) \rho_1(x) + \rho_1'(x) \rho_1'(x) \right] - a \left[ \tilde{\rho}_1(x) U(x, 0) \psi_{1,\text{cl}}(x) \right]_{x_0 = a} + \rho_1'(x) U(x, 0) \psi_{1,\text{cl}}(x) \right\} \tag{6.12}
\]

\[
- \sum_{x} \left\{ \frac{1}{2} a \tilde{c}_{\chi} \left[ \tilde{\rho}_h(x) \rho_h(x) + \rho_h'(x) \rho_h'(x) \right] - a \left[ \tilde{\rho}_h(x) U(x, 0) \psi_{h,\text{cl}}(x) \right]_{x_0 = T - a} \right\}
- \sum_{x} \left\{ \frac{1}{2} a \tilde{c}_{\chi} \left[ \tilde{\rho}_h(x) \rho_h(x) + \rho_h'(x) \rho_h'(x) \right] - a \left[ \tilde{\rho}_h(x) U(x, 0) \psi_{h,\text{cl}}(x) \right]_{x_0 = T - a} \right\}
\]
Expectation values can be computed by differentiating eq. (6.11) with respect to the source fields. Some results, which will be used later, are:

\[
\begin{align*}
[\zeta_1(x) \tilde{\psi}_1(y)]_F &= \frac{\delta \tilde{\psi}_1(y)}{\delta \rho_1(x)} \\
&= \tilde{c}_1 P_- U(x - a\hat{0}, 0) S_1(x, y) |_{x_0 = a}, \\
[\tilde{\psi}_1(x) \tilde{\zeta}_1(y)]_F &= \frac{\delta \tilde{\psi}_1(x)}{\delta \rho_1(y)} \\
&= \tilde{c}_1 S_1(x, y) U(y - a\hat{0}, 0)^{-1} P_+ |_{y_0 = a}, \\
[\zeta_1'(x) \tilde{\zeta}_1(y)]_F &= \tilde{c}_1^2 P_+ U(x, 0)^{-1} S_1(x, y) U(y - a\hat{0})^{-1} P_+ |_{x_0 = T - a, y_0 = a}, \\
[\tilde{\zeta}_1(x) \tilde{\zeta}_1'(y)]_F &= \tilde{c}_1^2 P_- U(x - a\hat{0}, 0) S_1(x, y) U(y, 0) P_- |_{x_0 = a, y_0 = T - a}, \\
[\psi_h(x) \tilde{\zeta}_h(y)]_F &= \frac{\delta \psi_h(x)}{\delta \rho_h(y)} \\
&= S_h(x, y) U(y - a\hat{0}, 0)^{-1} |_{y_0 = a}, \\
[\zeta_h'(x) \tilde{\zeta}_h(y)]_F &= U(x, 0)^{-1} S_h(x, y) U(y - a\hat{0})^{-1} |_{x_0 = T - a, y_0 = a}.
\end{align*}
\]

### 6.2 Gauge fixing

To perform a perturbative calculation, i.e., an expansion around the classical vacuum configuration, a gauge has to be fixed. Here, the gauge fixing procedure of [41, 55] is used. First of all, we fix the boundary conditions

\[ W(x, k) = W'(x, k) = 1. \]

(6.19)

These boundary values are not affected by all gauge functions \( \Lambda(x) \) which are constant at \( x_0 = 0 \) and \( x_0 = T \). The group of these transformations is called \( \hat{G} \). Now the subgroup of gauge transformations that act trivially on the classical vacuum configuration \( U(x, \mu) = 1 \) has to be divided out. This subgroup is the group of constant gauge functions \( \Lambda(x) \), which is isomorphic to SU(3). We thus define

\[ G = \hat{G} / SU(3) \]

(6.20)

as the group that has to be fixed. This group is identified with the set of gauge functions \( \Lambda \in \hat{G} \) that satisfy

\[ \Lambda(x)|_{x_0 = T} = 1. \]

(6.21)

Then the Lie algebra \( \mathcal{L} \) of \( G \) can be understood as the set of fields \( \omega(x) \) with values in the Lie algebra of SU(3), which satisfy

\[ \omega(x)|_{x_0 = 0} = \text{constant}, \quad \omega(x)|_{x_0 = T} = 0. \]

(6.22)

Now we can introduce vector fields \( q_\mu(x) \) with components in the Lie algebra of SU(3), which are defined for all links carrying link variables, i.e., for \( 0 \leq x_0 \leq T \).
if $\mu = 1, 2, 3$, and for $0 \leq x_0 < T$ if $\mu = 0$, and which satisfy the boundary conditions

$$q_k(x)|_{x_0=0} = q_k(x)|_{x_0=T} = 0.$$  

(6.23)

The space of all such vector fields is denoted by $\mathcal{H}$. The vector fields can be used to parameterize an expansion of the gauge configurations around the classical vacuum $U(x, \mu) = 1$ by writing

$$U(x, \mu) = \exp \{g_0aq_\mu(x)\},$$  

(6.24)

with

$$q_\mu(x) = q_\mu^a(x)T^a,$$  

(6.25)

where $T^a$ are the generators of the group SU(3), see Appendix A.5 Now, a gauge fixing function

$$F : \mathcal{H} \mapsto \mathcal{L}$$  

(6.26)

has to be defined. To this end, we introduce a mapping

$$d^* : \mathcal{H} \mapsto \mathcal{L}$$  

(6.27)

by demanding that the relation

$$- \sum_x \text{tr}\{d^*q(x)\omega(x)\} = \sum_{x,\mu} \text{tr}\{q_\mu(x)\partial_\mu\omega(x)\}$$  

(6.28)

holds for all $q \in \mathcal{H}$ and $\omega \in \mathcal{L}$, with the lattice partial derivative $\partial_\mu$ (see Appendix A.3). Calculating this explicitly gives

$$d^*q(x) = \begin{cases} 
(a^2/L^3) \sum_y q_0(y)|_{y_0=0} & \text{if } x_0 = 0, \\
\partial_\mu q_\mu(x) & \text{if } 0 < x_0 < T, \\
0 & \text{if } x_0 = T.
\end{cases}$$  

(6.29)

Now, the gauge fixing part of the action can be chosen as

$$S_{GF[q]} = -\frac{1}{2} \sum_x \text{tr}\{d^*q(x)d^*q(x)\}.$$  

(6.30)

This choice of action defines the *Feynman gauge*. The total action to be considered is then

$$S_{tot}[q, \bar{c}, c] = S_{eff}[U] + S_{GF[q]} + S_{ghost[q, \bar{c}, c]},$$  

(6.31)

with a ghost field action $S_{ghost}$, which is of no interest here, since it contributes to our correlation functions only at higher orders than one loop.

Expanding $S_{tot}$ in the coupling, we can write [55]

$$S_{tot}[q] = - \sum_{x,\mu} \text{tr}\{q_\mu(x)\Delta_1 q_\mu(x)\} + O(g_0),$$  

(6.32)
with
\[ \Delta_1 q_k(x) = - \partial^*_\mu \partial_\mu q_k(x), \quad 0 < x_0 < T, \] (6.33)
and
\[ \Delta_1 q_0(x) = \begin{cases} 
- \partial^*_k \partial_k q_0(x) - a^{-1} \partial_0 q_0(x) & \text{if } x_0 = 0, \\
+ (a/L^3) \sum_y q_0(y) |_{y_0=0} & \text{if } 0 < x_0 < T - a, \\
- \partial^*_\mu \partial_\mu q_0(x) & \text{if } x_0 = T - a.
\end{cases} \] (6.34)

With this action, the gluon propagator
\[ \langle q^a_\mu(x) q^b_\nu(y) \rangle_G = \delta^{ab} D_{\mu\nu}(x, y) + O(g_0) \] (6.35)
can be calculated. The result can be found in Appendix B.1.

### 6.3 Perturbative expansion of the correlation functions

Applying Wick’s theorem [74] to our correlation functions yields
\[ f_\Lambda^{\text{Stat}}(x_0) = a^6 \sum_{y,z} \frac{1}{2} \left\langle \text{tr} \left\{ \left[ \zeta_1(z) \bar{\psi}_1(x) \right] F \gamma_5 \gamma_5 [\psi_h(x) \bar{\xi}_h(y)] F \gamma_5 \right\} \right\rangle_G, \] (6.36)
\[ f_1^{\text{Stat}} = \frac{a^{12}}{L^6} \sum_{u,v,y,z} \frac{1}{2} \left\langle \text{tr} \left\{ \left[ \zeta_1(z) \bar{\xi}_1(v) \right] F \gamma_5 [\psi_h'(v) \bar{\xi}_h(y)] F \gamma_5 \right\} \right\rangle_G. \] (6.37)

Noting that
\[ \gamma_5 S_1(x, y) \gamma_5 = S_1(y, x)^\dagger, \quad \gamma_5 S_h(x, y) \gamma_5 = S_h(y, x)^\dagger, \] (6.38)
one can use eq. (6.13) and (6.16) to derive
\[ \gamma_5 [\zeta_1(z) \bar{\psi}_1(x)] F \gamma_5 = \{ [\psi_1(x) \bar{\xi}_1(z)] F \}^\dagger, \] (6.39)
\[ \gamma_5 [\zeta_1(z) \bar{\xi}_1(v)] F \gamma_5 = \{ [\xi_1'(v) \bar{\xi}_1(z)] F \}^\dagger. \] (6.40)

With this and eq. (6.13) and (6.17), the correlation functions assume the form
\[ f_\Lambda^{\text{Stat}}(x_0) = -a^6 \sum_{y,z} \frac{1}{2} \left\langle \text{tr} \left\{ \partial_\mu \bar{\psi}_1(x) \right\} \right\rangle \left\langle \text{tr} \left\{ \gamma^0 \frac{\delta \psi_h(x)}{\delta \rho_1(y)} \right\} \right\rangle_G, \] (6.41)
\[ f_1^{\text{Stat}} = \frac{a^{12}}{L^6} \sum_{u,v,y,z} \frac{1}{2} \left\langle \text{tr} \left\{ \left[ \zeta_1'(u) \bar{\xi}_1(z) \right] F \right\} \left\langle \left[ \xi_1'(v) \bar{\xi}_1(y) \right] F \right\} \right\rangle_G. \] (6.42)

It is now convenient to introduce the matrices
\[ H_1(x) = a^2 \sum_y \frac{\delta \psi_1(x)}{\delta \rho_1(y)} \] (6.43)
and
\[ K_1 = \hat{c}_1 \frac{a^3}{L^3} \sum_x P_+ U(x, 0)^{-1} H_1(x) |_{x_0 = \tau - a} \]  (6.44)

following [55], and the corresponding static quark matrices
\[ H_h(x) = a^3 \sum_y \frac{\delta \psi_h \alpha_1(x)}{\delta \rho_h(y)} \]  (6.45)

and
\[ K_h = \frac{a^3}{L^3} \sum_x P_+ U(x, 0)^{-1} H_h(x) |_{x_0 = \tau - a} \]  (6.46)

With these, the correlation functions can be written as
\[ f_{A}^{stat}(x_0) = -\frac{1}{2} \langle tr \left\{ H_1(x)^\dagger \gamma_0 H_h(x) \right\} \rangle_G \]  (6.47)

and
\[ f_{1}^{stat} = \frac{1}{2} \langle tr \left\{ K_1^+ K_h \right\} \rangle_G \]  (6.48)

Now, the correlation functions can be expanded in the coupling \( g_0 \),
\[ f_{A}^{stat}(x_0) = f_{A}^{stat(0)}(x_0) + g_0^2 f_{A}^{stat(1)}(x_0) + O(g_0^4), \]  (6.49)
\[ f_{1}^{stat} = f_{1}^{stat(0)} + g_0^2 f_{1}^{stat(1)} + O(g_0^4), \]  (6.50)

by expanding the matrices
\[ H_1(x) = H_1^{(0)}(x) + g_0 H_1^{(1)}(x) + g_0^2 H_1^{(2)}(x) + O(g_0^3), \]  (6.51)
\[ H_h(x) = H_h^{(0)}(x) + g_0 H_h^{(1)}(x) + g_0^2 H_h^{(2)}(x) + O(g_0^3), \]  (6.52)
\[ K_1 = K_1^{(0)} + g_0 K_1^{(1)} + g_0^2 K_1^{(2)} + O(g_0^3), \]  (6.53)
\[ K_h = K_h^{(0)} + g_0 K_h^{(1)} + g_0^2 K_h^{(2)} + O(g_0^3). \]  (6.54)

The tree level formulae
\[ f_{A}^{stat(0)}(x_0) = -\frac{3}{2} \text{tr} \left\{ H_1^{(0)}(x)^\dagger H_h^{(0)}(x) \right\}, \]  (6.55)
\[ f_{1}^{stat(0)} = \frac{3}{2} \text{tr} \left\{ K_1^{(0)} K_h \right\}, \]  (6.56)

where the traces now run only over the Dirac indices, are written out explicitly in Appendix B.2.

The matrices \( H_1^{(2)}(x) \) and \( H_h^{(2)}(x) \) can be decomposed,
\[ H_1^{(2)}(x) = H_1^{(2)}(x)_{1} + H_1^{(2)}(x)_{2} + m_0^{(1)} \frac{\partial}{\partial m_0} H_1^{(0)}(x), \]  (6.57)
\[ H_h^{(2)}(x) = H_h^{(2)}(x)_{1} + H_h^{(2)}(x)_{2}, \]  (6.58)
where it is understood that all terms are evaluated at the tree level value $m_0(0)$ of the light quark mass. The matrix $H_1^{(2)}(x)_b$ is the term arising from the boundary $O(a)$ improvement for the light quark field, $H_1^{(2)}(x)_1$ and $H_1^{(2)}(x)_1$ contain a vertex with one gluon leg, and $H_1^{(2)}(x)_2$ and $H_1^{(2)}(x)_2$ contain a vertex with two gluon legs. The exact form of all these matrices is given in Appendices B.3 and B.4.

With these matrices, the one loop coefficient of the correlation function $f_{A}^{\text{stat}}(x_0)$ can be decomposed into

$$f_{A}^{\text{stat}}(x_0) = f_{A}^{\text{stat}}(x_0)_b + f_{A}^{\text{stat}}(x_0)_1a + f_{A}^{\text{stat}}(x_0)_1b + f_{A}^{\text{stat}}(x_0)_2a + f_{A}^{\text{stat}}(x_0)_2b + f_{A}^{\text{stat}}(x_0)_3 + m_0^{(1)} \frac{\partial}{\partial m_0} f_{A}^{\text{stat}}(x_0).$$

(6.59)

The $O(a)$ boundary improvement term

$$f_{A}^{\text{stat}}(x_0)_b = -\frac{1}{2} \text{tr} \left\{ H_1^{(2)}(x)_b^\Gamma_0 H_h^{(0)}(x) \right\}$$

(6.60)

can be obtained analytically from the explicit form of $H_1^{(2)}(x)_b$, and also the one-loop mass correction term can be calculated explicitly. The other terms, which have to be calculated numerically, are

$$f_{A}^{\text{stat}}(x_0)_1a = -\frac{1}{2} \left\langle \text{tr} \left\{ H_1^{(2)}(x)_1^\Gamma_0 H_h^{(0)}(x) \right\} \right\rangle \tilde{\mathcal{C}},$$

(6.61)

$$f_{A}^{\text{stat}}(x_0)_1b = -\frac{1}{2} \left\langle \text{tr} \left\{ H_1^{(2)}(x)_1^\Gamma_0 H_h^{(0)}(x) \right\} \right\rangle \tilde{\mathcal{C}},$$

(6.62)

$$f_{A}^{\text{stat}}(x_0)_2a = -\frac{1}{2} \left\langle \text{tr} \left\{ H_1^{(2)}(x)_2^\Gamma_0 H_h^{(0)}(x) \right\} \right\rangle \tilde{\mathcal{C}},$$

(6.63)

$$f_{A}^{\text{stat}}(x_0)_2b = -\frac{1}{2} \left\langle \text{tr} \left\{ H_1^{(2)}(x)_2^\Gamma_0 H_h^{(0)}(x) \right\} \right\rangle \tilde{\mathcal{C}},$$

(6.64)

$$f_{A}^{\text{stat}}(x_0)_3 = -\frac{1}{2} \left\langle \text{tr} \left\{ H_1^{(1)}(x)^\Gamma_0 H_h^{(1)}(x) \right\} \right\rangle \tilde{\mathcal{C}},$$

(6.65)

where $\langle \ldots \rangle \tilde{\mathcal{C}}$ means the gauge field average using $S_{\text{tot}}$ at $g_0 = 0$. They can be interpreted in terms of the Feynman diagrams shown in Figure 6.1 for $x_0 = T/2$.

The $O(a)$ improvement term for the static axial current is expanded as

$$a c_{\Lambda}^{\text{stat}} f_{\delta A}^{\text{stat}}(x_0).$$

(6.66)

The coefficients in the expansion of $K_1$ and $K_h$ allow the decomposition

$$K_1^{(1)} = K_{1,1}^{(1)} + K_{1,2}^{(1)},$$

(6.67)

$$K_h^{(1)} = K_{h,1}^{(1)} + K_{h,2}^{(1)},$$

(6.68)

$$K_1^{(2)} = m_0^{(1)} \frac{\partial}{\partial m_0} K_{1}^{(0)} + K_{1,1}^{(2)} + K_{1,2}^{(2)} + K_{1,3}^{(2)} + K_{1,4}^{(2)},$$

(6.69)

$$K_h^{(2)} = K_{h,1}^{(2)} + K_{h,2}^{(2)} + K_{h,3}^{(2)} + K_{h,4}^{(2)},$$

(6.70)
Figure 6.1: One loop diagrams contributing to $f_\text{stat}(T/2)$.

with

$$K_{1,1}^{(1)} = \frac{a^3}{L^3} \sum_x P_+ H_1^{(1)}(x) |_{x_0 = T/2} - a,$$  
(6.71)

$$K_{h,1}^{(1)} = \frac{a^3}{L^3} \sum_x H_h^{(1)}(x) |_{x_0 = T/2} - a,$$  
(6.72)

$$K_{1,2}^{(1)} = -\frac{a^4}{L^3} \sum_x P_+ q_0^a(x) T^a H_1^{(0)}(x) |_{x_0 = T/2} - a,$$  
(6.73)

$$K_{h,2}^{(1)} = -\frac{a^4}{L^3} \sum_x q_0^a(x) T^a H_h^{(0)}(x) |_{x_0 = T/2} - a,$$  
(6.74)

$$K_{1,1}^{(2)} = \frac{a^3}{L^3} \sum_x P_+ \left\{ c_1^{(1)} H_1^{(0)}(x) + H_1^{(2)}(x) \right\} |_{x_0 = T/2} - a,$$  
(6.75)

$$K_{1,1}^{(2)} = \frac{a^3}{L^3} \sum_x P_+ H_1^{(2)}(x) |_{x_0 = T/2} - a,$$  
(6.76)

$$K_{h,1}^{(2)} = \frac{a^3}{L^3} \sum_x H_h^{(2)}(x) |_{x_0 = T/2} - a,$$  
(6.77)

$$K_{1,2}^{(2)} = \frac{a^3}{L^3} \sum_x P_+ H_1^{(2)}(x) |_{x_0 = T/2} - a,$$  
(6.78)

$$K_{h,2}^{(2)} = \frac{a^3}{L^3} \sum_x H_h^{(2)}(x) |_{x_0 = T/2} - a,$$  
(6.79)

$$K_{1,3}^{(2)} = -\frac{a^4}{L^3} \sum_x P_+ q_0^a(x) T^a H_1^{(1)}(x) |_{x_0 = T/2} - a,$$  
(6.80)
\[ K_{h,3}^{(2)} = -\frac{a^4}{3} \sum_x q_0^a(x) T^a H_h^{(1)}(x)|_{x_0 = T - \alpha}, \]  
\[ K_{1,4}^{(2)} = \frac{a^5}{2L^3} \sum_x P_+ q_0^a(x) q_0^b(x) T^a T^b H_1^{(0)}(x)|_{x_0 = T - \alpha}, \]  
\[ K_{h,4}^{(2)} = \frac{a^5}{2L^3} \sum_x q_0^a(x) q_0^b(x) T^a T^b H_h^{(0)}(x)|_{x_0 = T - \alpha}. \]  

This decomposition allows us to write the correlation function as

\[ f_1^{\text{stat}(1)} = (f_1^{\text{stat}(1)})_b + (f_1^{\text{stat}(1)})_{1a} + (f_1^{\text{stat}(1)})_{1b} + (f_1^{\text{stat}(1)})_{2a} + (f_1^{\text{stat}(1)})_{2b} \]
\[ + (f_1^{\text{stat}(1)})_3 + (f_1^{\text{stat}(1)})_{4a} + (f_1^{\text{stat}(1)})_{4b} + (f_1^{\text{stat}(1)})_{5a} + (f_1^{\text{stat}(1)})_{5b} \]
\[ + (f_1^{\text{stat}(1)})_{6a} + (f_1^{\text{stat}(1)})_{6b} + (f_1^{\text{stat}(1)})_{7a} + (f_1^{\text{stat}(1)})_{7b} \]
\[ + m_0^{(1)} \frac{\partial}{\partial m_0} f_1^{\text{stat}(0)}, \]

with

\[ (f_1^{\text{stat}(1)})_b = \frac{1}{2} \left\langle \text{tr} \left\{ K_1^{(2)} \dagger \right\} \right\rangle, \]
\[ (f_1^{\text{stat}(1)})_{1a} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{11}^{(2)} \dagger \right\} \right\rangle, \]
\[ (f_1^{\text{stat}(1)})_{1b} = \frac{1}{2} \left\langle \text{tr} \left\{ K_1^{(2)} \dagger \right\} \right\rangle, \]
\[ (f_1^{\text{stat}(1)})_{2a} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{12}^{(2)} \dagger \right\} \right\rangle, \]
\[ (f_1^{\text{stat}(1)})_{2b} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{12}^{(2)} \dagger \right\} \right\rangle, \]
\[ (f_1^{\text{stat}(1)})_3 = \frac{1}{2} \left\langle \text{tr} \left\{ K_{13}^{(1)} \dagger \right\} \right\rangle, \]
\[ (f_1^{\text{stat}(1)})_{4a} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{13}^{(1)} \dagger \right\} \right\rangle, \]
\[ (f_1^{\text{stat}(1)})_{4b} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{13}^{(1)} \dagger \right\} \right\rangle, \]
\[ (f_1^{\text{stat}(1)})_{5a} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{14}^{(2)} \dagger \right\} \right\rangle, \]
\[ (f_1^{\text{stat}(1)})_{5b} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{14}^{(2)} \dagger \right\} \right\rangle, \]
\[ (f_1^{\text{stat}(1)})_{6a} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{14}^{(2)} \dagger \right\} \right\rangle, \]
\[ (f_1^{\text{stat}(1)})_{6b} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{14}^{(2)} \dagger \right\} \right\rangle, \]
\[ (f_1^{\text{stat}(1)})_{7a} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{14}^{(2)} \dagger \right\} \right\rangle, \]
\[ (f_1^{\text{stat}(1)})_{7b} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{14}^{(2)} \dagger \right\} \right\rangle. \]
Figure 6.2: One loop diagrams contributing to $f_1^{\text{stat}}$. The dotted lines denote the links from $T - a$ to $T$.

Here, again, the boundary term and the one loop mass term can be calculated analytically. The other terms are connected to the Feynman diagrams in Figure 6.2.

With the results for all these diagrams, one can calculate the one loop expansion of the ratio $X_1$,

$$X_1(g_0, L/a) = X_1^{(0)}(L/a) + g_0^2 X_1^{(1)}(L/a) + O(g_0^4),$$  \hspace{1cm} (6.98)

with

$$X_1^{(0)}(L/a) = \frac{f_1^{\text{stat}(0)}(T/2)}{\sqrt{f_1^{\text{stat}(0)}}}$$  \hspace{1cm} (6.99)

and

$$X_1^{(1)}(L/a) = \frac{1}{f_1^{\text{stat}(0)}} \left\{ f_A^{\text{stat}(1)}(T/2) \sqrt{f_1^{\text{stat}(0)}} - \frac{f_A^{\text{stat}(0)}(T/2)}{2 \sqrt{f_1^{\text{stat}(0)}}} f_1^{\text{stat}(1)} 
+ a c_A^{\text{stat}(1)} f_\delta^{\text{stat}(0)}(T/2) \sqrt{f_1^{\text{stat}(0)}} \right\}.$$  \hspace{1cm} (6.100)
Chapter 7

Renormalization

7.1 General concepts

As one encounters infinities in many quantities when taking the continuum limit, renormalization is required. It is known from [75] that lattice QCD with relativistic quarks is renormalizable at any order of perturbation theory. In the static approximation, the situation is not completely clear. In this case, renormalizability has not been established rigorously, although many perturbative calculations have been done in the static effective theory [58, 59, 60, 76, 77, 78], and none of them indicated a problem with renormalizability. However, some care in the renormalization process is appropriate. In this chapter it is assumed that renormalization works “as usual”, meaning that infinities are removed by adding local counterterms that can be identified by power counting arguments. Here we will stick to mass independent renormalization schemes, in which the renormalization constants only depend on the coupling, but not on the light quark mass.

First of all, the light quark mass requires an additive renormalization, due to the Wilson term in the relativistic quark action, thus defining the subtracted light quark mass

\[ m_q = m_0 - m_c, \]  

(7.1)

where \( m_c \) is called the critical mass of the light quark. The subtracted mass \( m_q \) is then renormalized multiplicatively. In the definition of the renormalized quark mass, one has to remember the \( O(a) \) counterterm derived from the operator \( \mathcal{O}_m \) in eq. (3.16), which is now added with an appropriate coefficient \( b_m \),

\[ m_R = Z_m (1 + \alpha b_m m_q) m_q, \]  

(7.2)

where the renormalization constant \( Z_m \) depends only on the coupling and the renormalization scale. The coefficient \( b_m \) is known at one loop order of perturbation theory,

\[ b_m = b_m^{(0)} + b_m^{(1)} g_0^2 + O(g_0^4), \]  

(7.3)
with [73]
\[
b_m^{(0)} = -\frac{1}{2}, \quad b_m^{(1)} = -0.07217(2) \times C_F, \quad C_F = \frac{4}{3}, \tag{7.4}
\]

Different definitions of the critical quark mass are possible, differing by \( \mathcal{O}(a^2) \) terms. Usually, one chooses correlation functions containing the light-light axial current and density, and then demands that the PCAC mass defined by those correlation functions is zero. In perturbation theory, such a critical mass can be expanded in terms of the coupling,
\[
m_c = m_c^{(0)} + m_c^{(1)} g_0^2 + \mathcal{O}(g_0^4), \tag{7.5}
\]
and the coefficients in this perturbation series can be extrapolated to the continuum limit at each order. Then one gets
\[
m_c^{(0)} = 0, \tag{7.6}
\]
and the continuum limit of the one loop coefficient is [54, 55, 79]
\[
m_c^{(1)} = -0.2025565(1) \times C_F. \tag{7.7}
\]
It is this continuum limit which will be used as the one-loop critical quark mass in the following calculations.

Also the coupling is renormalized multiplicatively,
\[
g_R = Z_g (1 + ab_g m_q) g_0, \tag{7.8}
\]
where the \( \mathcal{O}(a) \) improvement term connected with the operator \( \mathcal{O}_g \) in eq. (3.15) has been included. The coefficient \( b_g \) can again be expanded in the bare coupling,
\[
b_g = b_g^{(0)} + b_g^{(1)} g_0^2 + \mathcal{O}(g_0^4). \tag{7.9}
\]
For the purpose of our one loop calculation, it is only important to know that
\[
b_g^{(0)} = 0, \tag{7.10}
\]
which means
\[
g_R^2 = g_0^2 + \mathcal{O}(g_0^4). \tag{7.11}
\]

The light quark boundary fields are also renormalized multiplicatively, taking the mass dependent \( \mathcal{O}(a) \) terms from the operator \( \mathcal{O}_m \), eq. (5.79), into account,
\[
\zeta_{1,R} = Z_l (1 + ab_l m_q) \zeta_l, \tag{7.12}
\]
\[
\tilde{\zeta}_{1,R} = Z_l (1 + ab_l m_q) \tilde{\zeta}_l, \tag{7.13}
\]
\[
\zeta_{1,R} = Z_l (1 + ab_l m_q) \zeta_1', \tag{7.14}
\]
\[
\tilde{\zeta}_{1,R} = Z_l (1 + ab_l m_q) \tilde{\zeta}_1'. \tag{7.15}
\]
The heavy quark fields are treated similarly, including the $O(a)$ contribution produced by the operator $\tilde{O}_4$ in eq. (5.81),

\[
\begin{align*}
\tilde{\zeta}_{h,R} &= Z_h (1 + ab_h m_q) \tilde{\zeta}_{h}, \\
\tilde{\zeta}_{h,R}' &= Z_h (1 + ab_h m_q) \tilde{\zeta}_{h}'.
\end{align*}
\] (7.16) (7.17)

All the boundary field renormalization constants and their mass dependent $O(a)$ improvement terms cancel in the ratio $X_1$.

The static-light axial current is renormalized by

\[
(A_R^{\text{stat}})_0 = Z_A^{\text{stat}} (1 + ab_A^{\text{stat}} m_q) (A_T^{\text{stat}})_0,
\] (7.18)

where $b_A^{\text{stat}}$, originating from $(\delta A_0^{\text{stat}})_5$ in eq. (4.22), is not known up to now.

The renormalization of the light-light axial current is analogous to the static case, with the counterterm $(\delta A_\mu)_1$ from eq. (3.25),

\[
(A_R)_\mu = Z_A \left( 1 + b_A \frac{1}{2} (m_{q,1} + m_{q,2}) \right) (A_1)_\mu,
\] (7.19)

where $m_{q,1}$ and $m_{q,2}$ are the subtracted bare masses of the light quarks, and $b_A$ is known at one loop order of perturbation theory,

\[
b_A = b_A^{(0)} + b_A^{(1)} g_0^2 + O(g_0^4),
\] (7.20)

with [73]

\[
b_A^{(0)} = 1, \quad b_A^{(1)} = 0.11414(4) \times C_F.
\] (7.21)

In the discussion of renormalization schemes in this chapter, the subtracted light quark mass is always set to zero, meaning

\[
m_0 = m_c,
\] (7.22)

such that the $b_A^{\text{stat}}$ term does not have to be taken into account. The ratio of time and spatial extent of the space-time volume is chosen to be 1,

\[
T = L.
\] (7.23)

Furthermore, the angles $\theta_k$ are chosen to be identical,

\[
\theta \equiv \theta_1 = \theta_2 = \theta_3.
\] (7.24)

### 7.2 Renormalization in the lattice MS scheme

The first perturbative renormalization scheme to be introduced here is the lattice minimal subtraction (MS) scheme. The coefficients in the perturbative expansion
of couplings, quark masses, and the static-light axial current are expected to diverge logarithmically in the continuum limit. The lattice MS scheme is "minimal" in the sense that it removes only these logarithms and nothing else. Explicitly, the lattice MS scheme is defined by the requirement that, at each order of perturbation theory, the renormalization constants are polynomials in \( \ln(a \mu) \), where \( a \) is the lattice spacing, and \( \mu \) is the renormalization scale.

For our purposes, only the tree level value of the renormalized coupling is needed, i.e.

\[
g^2_{\text{lat}} = g^2_0 + O(g^4_0). \tag{7.25}
\]

Although it is not needed for the calculation of \( X_1 \) at \( m_0 = m_c \), the light quark mass renormalization constant is given here for later use. It is

\[
Z_{m, \text{lat}} = 1 - d_0 \ln(a \mu) g^2_{\text{lat}} + O(g^4_{\text{lat}}), \tag{7.26}
\]

with the scheme independent one-loop anomalous dimension

\[
d_0 = \frac{1}{2\pi^2}. \tag{7.27}
\]

By eq. (7.2), the constant \( Z_{m, \text{lat}} \) defines the lattice MS mass \( m_{\text{lat}} \).

Similarly, the static-light axial current is renormalized by

\[
Z_{A, \text{lat}}^{\text{stat}} = 1 - \gamma_0 \ln(a \mu) g^2_{\text{lat}} + O(g^4_{\text{lat}}), \tag{7.28}
\]

where \([80, 81]\)

\[
\gamma_0 = -\frac{1}{4\pi^2} \tag{7.29}
\]

is the scheme-independent one loop anomalous dimension of the current. This defines the renormalized current (\( A_{\text{lat}}^{\text{stat}} \)) and the lattice MS-renormalized ratio \( X_{1, \text{lat}} \),

\[
X_{1, \text{lat}}(L/a) = X_1^{(0)}(L/a) + X_{1, \text{lat}}^{(1)}(L/a) g^2_{\text{lat}} + O(g^4_{\text{lat}}), \tag{7.30}
\]

with

\[
X_{1, \text{lat}}^{(1)}(L/a) = X_1^{(1)}(L/a) - \gamma_0 \ln(a \mu) X_1^{(0)}(L/a). \tag{7.31}
\]

To check whether the assumptions on renormalizability are justified, the ratio \( X_{1, \text{lat}} \) has to be obtained numerically. This has been done on an IBM SP2 and a Linux PC, using a set of FORTRAN programs. As an example, the formulae used in the programs for diagrams 1a and 3 are given in Appendix B.5. In each run of the program, \( X_1^{(0)} \) and \( X_{1, \text{lat}}^{(1)} \) were calculated for fixed \( \theta \) and for a set of lattice sizes \( (L/a) = 4, 6, \ldots, 48 \), which took about one day on the PC. Results are given in tables D.2 and D.3, and they are graphically shown in Figure 7.1 for the set of parameters mentioned in Section 7.1, at \( \mu = 1/L \). The results depend on the angle theta; the choice here is \( \theta = 0.0, \theta = 0.5, \) and \( \theta = 1.0 \). The dotted lines in the plots are fits that have been produced as described in Appendix C. One can see that \( X_{1, \text{lat}}^{(1)} \) does indeed have a continuum limit. Note that the label on the horizontal axis is \((a/L)^2\); the curves as well as a detailed analysis of the numerical data indicate that \( O(a) \) improvement has been successful.
Figure 7.1: The one loop coefficient of the lattice MS-renormalized ratio $X_{\text{Lat}}$ at different values of $\theta$. The data points are for $L/a = 8, 10, \ldots, 48$. The dotted lines are fits to the data, where the filled symbols have been included in the fits, while the open symbols were disregarded.
7.3 Matching to the \( \overline{\text{MS}} \) scheme

The lattice MS scheme is related to other renormalization schemes by finite renormalization. This means that in any renormalization scheme the renormalized coupling, light quark mass, and static-light axial current can be written as

\[
\begin{align*}
  g_R^2 &= (\chi_g)_{\text{R, lat}} g_{\text{lat}}^2, \\
  m_R &= (\chi_m)_{\text{R, lat}} m_{\text{lat}}, \\
  (A_{\text{A, lat}}^{\text{stat}}) &= (\chi_A)_{\text{R, lat}} (A_{\text{lat}}^{\text{stat}}),
\end{align*}
\]

(7.32) (7.33) (7.34)

up to lattice artifacts. If one chooses a mass independent renormalization scheme, the finite renormalization constants \((\chi_g)_{\text{R, lat}}, (\chi_m)_{\text{R, lat}}, \text{ and } (\chi_A)_{\text{R, lat}}\) depend only on the coupling.

A very popular perturbative renormalization scheme for QCD calculations is the modified minimal subtraction (\( \overline{\text{MS}} \)) scheme [82]. As many interesting quantities have been calculated in the \( \overline{\text{MS}} \) scheme, like the static axial current’s two-loop anomalous dimension, which plays an important rôle in the calculation of the renormalization group invariant current, it is desirable to know the relation between the renormalized currents in the \( \overline{\text{MS}} \) and the lattice MS schemes. As in the general case above, this relation is written as

\[
(A_{\text{A, lat}}^{\text{stat}}) = (\chi_A)_{\overline{\text{MS, lat}}} (A_{\text{lat}}^{\text{stat}}),
\]

(7.35)

with

\[
(\chi_A)_{\overline{\text{MS, lat}}} = 1 + (\chi_A)_{\overline{\text{MS, lat}}}^{(1)} g_{\overline{\text{MS}}}^2 + O(g_{\overline{\text{MS}}}^4)
\]

(7.36)

and it is the aim to compute the one loop coefficient \((\chi_A)_{\overline{\text{MS, lat}}}^{(1)}\). To achieve this, we will introduce an intermediate scheme, which uses the axial current of two relativistic quarks \(\psi_1\) and \(\psi_2\),

\[
A_0(x) = \bar{\psi}_1(x) \gamma_0 \gamma_5 \psi_2(x),
\]

(7.37)

with bare quark masses \(m_{0,1}\) and \(m_{0,2}\). The renormalized mass of the first quark will again be set to zero, i. e.

\[
m_{0,1} = m_c.
\]

(7.38)

The coupling and the second quark mass are renormalized by the \( \overline{\text{MS}} \) prescription, while one imposes a renormalization condition on the axial current by demanding that in the case in which the second quark mass also vanishes, the renormalized axial isovector current

\[
(A_{\text{A, lat}}^{\text{stat}}) = Z_{A_{\text{A, lat}}} \bar{\psi}_1(x) \frac{1}{2} \tau^a \gamma_0 \gamma_5 \psi_2(x)
\]

(7.39)

satisfies the euclidean current algebra explained in [83], where \(\tau^a\) is a Pauli matrix acting on the flavour indices of the quark field. The (finite) renormalization
constant $Z_{A,CA}$ is then used to renormalize the current in eq. (7.37), giving a
renormalized current $(A_{CA})_0$. The current in this renormalization scheme is rel-
ted to the static current in the other two schemes by

$$(A_{CA})_0 = (\chi_A^{stat})_{CA, lat} (A_{lat})_0,$$

$$(A_{MS})_0 = (\chi_A^{stat})_{MS, CA} (A_{CA})_0,$$

as shown in Figure 7.3. Then, $(\chi_A^{stat})_{MS, lat}$ can be obtained as

$$(\chi_A^{stat})_{MS, lat} = (\chi_A^{stat})_{MS, CA} (\chi_A^{stat})_{CA, lat}. $$

The renormalization constant $(\chi_A^{stat})_{MS, CA}$ is known at one loop order [59],

$$(\chi_A^{stat})_{MS, CA} = 1 + (\chi_A^{stat})_{MS, CA}^{(1)} g_{MS}^2 + O(g_{MS}^4),$$

$$(\chi_A^{stat})_{MS, CA}^{(1)} = -\gamma_0 \ln \left( \frac{\mu}{m_{2, MS}} \right) + \frac{1}{6\pi^2},$$

where $m_{2, MS}$ is the $\overline{MS}$ mass of the second quark. As the one-loop anomalous
dimension is scheme independent, $(\chi_A^{stat})_{CA, lat}$ is expected to contain the same
logarithm as $(\chi_A^{stat})_{MS, CA}$, and an unknown finite part,

$$(\chi_A^{stat})_{CA, lat} = 1 + (\chi_A^{stat})_{CA, lat}^{(1)} g_{MS}^2 + O(g_{MS}^4),$$

Figure 7.2: The relation between the static axial current in the lattice MS scheme, the static
axial current in the $\overline{MS}$ scheme, and the axial current renormalized by current algebra relations.
Figure 7.3: Schematic drawing of the correlation functions \( f_A \) (left) and \( f_1 \) (right). The dot in the middle of the left diagram symbolizes the axial current at \( x_0 \).

\[
(\chi_A^{\text{stat}})^{(1)}_{\text{CA, lat}} = \gamma_0 \ln \left( \frac{\mu}{m_{2, \text{MS}}} \right) + B_A^{\text{stat}}. 
\]  

This finite constant \( B_A^{\text{stat}} \) is now the only missing part in the relation of the lattice MS scheme to the MS scheme.

For the calculation of \( B_A^{\text{stat}} \), one has to compute a correlation function containing the light-light axial current \( A_0 \). We choose to do this in the lattice regularization, and use the function

\[
f_A(x_0) = -a^6 \sum_{y,z} \frac{1}{2} \langle A_0(x) \overline{\zeta}_2(y) \gamma_5 \zeta_1(z) \rangle, 
\]  

which is schematically shown in Figure 7.3.

To include the \( O(a) \) improvement term from eq. (3.28), we also introduce the correlation function

\[
f_P(x_0) = -a^6 \sum_{y,z} \frac{1}{2} \langle P(x) \overline{\zeta}_2(y) \gamma_5 \zeta_1(z) \rangle, 
\]  

with the axial density \( P \) defined by discretization of eq. (3.27), and define

\[
f_{\delta A}(x_0) = \frac{1}{2} (\partial^*_0 + \partial_0) f_P(x_0). 
\]  

Taking the mass dependent \( O(a) \) improvement eq. (3.25) into account, and using \( m_{0,1} = m_c \), we introduce the improved correlation function

\[
f_{A,1}(x_0) = (1 + a \frac{1}{2} b_A m_{q,2}) \left\{ f_A(x_0) + a c_A f_{\delta A}(x_0) \right\}. 
\]
As in the static case, it is desirable to have ratios of correlation functions in which the wave function renormalization constants cancel. To that end, a further function

\[ f_1 = -\frac{a^{12}}{L^6} \sum_{u,v,y,z} \frac{1}{2} \left\langle \zeta_1'(u) \gamma_5 \zeta_2'(v) \zeta_2(y) \gamma_5 \zeta_1(z) \right\rangle \]  

(7.51)
is introduced, which is also schematically shown in Figure 7.3. We now set

\[ \mu = \frac{1}{L}, \]  

(7.52)
and introduce the abbreviation

\[ z \equiv L m_{2,\overline{\text{MS}}}. \]  

(7.53)
Then, the ratio

\[ Y_1(g_0, z, L) = \frac{f_{\lambda,1}(L/2)}{\sqrt{f_1}} \]  

(7.54)
can be defined, where indeed the wave function renormalization constants and the \( b_1 \) terms cancel. To get a matching of our perturbative renormalization schemes, the ratio \( Y_1 \) has to be expanded in terms of the coupling,

\[ Y_1(g_0, z, L) = Y^{(0)}_1(z, L) + Y^{(1)}_1(z, L) g_0^2 + O(g_0^4). \]  

(7.55)
To achieve this, we first have to establish the relation between the bare quark mass on the lattice, and the \( \overline{\text{MS}} \) mass \( z \), which means that we have to know the conversion factor \( (\chi_m)_{\overline{\text{MS}}, \text{lat}} \):

\[ z = L (\chi_m)_{\overline{\text{MS}}, \text{lat}} m_{\text{lat}}. \]  

(7.56)
This factor is expanded perturbatively,

\[ (\chi_m)_{\overline{\text{MS}}, \text{lat}} = 1 + (\chi_m)_{\overline{\text{MS}}, \text{lat}}^{(1)} g_0^2 + O(g_0^4), \]  

(7.57)
and the one loop coefficient is \[79\]

\[ (\chi_m)_{\overline{\text{MS}}, \text{lat}}^{(1)} = 0.122282 \times C_F. \]  

(7.58)
Inserting this into eq. (7.2), and using eq. (7.26) and eq. (7.56), we get

\[ m_{0,2} = m_{0,2}^{(0)} + m_{0,2}^{(1)} g_0^2 + O(g_0^4), \]  

(7.59)
with

\[ a m_{0,2}^{(0)} = 1 - \sqrt{1 - 2 \frac{a}{L} z}, \]  

(7.60)

\[ a m_{0,2}^{(1)} = a m_c^{(1)} + \frac{(d_0 \ln(\frac{a}{L}) - (\chi_m)_{\overline{\text{MS}}, \text{lat}}^{(1)} \frac{a}{L} z + 2b_1^{(1)}(\frac{a}{L} z - a m_{0,2}^{(0)}))}{\sqrt{1 - 2 \frac{a}{L} z}}. \]  

(7.61)
Now, the correlation functions have to be expanded at one loop order. For \( f_A \),
this is done by calculating the matrix \( H_1 \) from Section 6.3 with the bare quark masses \( m_{0,1} \) and \( m_{0,2} \), giving two matrices \( H_1 \) and \( H_2 \). Up to one loop order, \( f_A \) is
\[
f_A(x_0) = f_A^{(0)}(x_0) + f_A^{(1)}(x_0) g_0^2 + O(g_0^4),
\]
(7.62)
where \( f_A^{(0)} \) is given in Appendix B.2, and \( f_A^{(1)} \) can be written as
\[
f_A^{(1)}(x_0) = f_A^{(1)}(x_0)_{ba} + f_A^{(1)}(x_0)_{bb} + f_A^{(1)}(x_0)_{1a} + f_A^{(1)}(x_0)_{1b}
+ f_A^{(1)}(x_0)_{2a} + f_A^{(1)}(x_0)_{2b} + f_A^{(1)}(x_0)_{3}
+ m_{0,1} \frac{\partial}{\partial m_{0,1}} f_A^{(0)}(x_0) + m_{0,2} \frac{\partial}{\partial m_{0,2}} f_A^{(0)}(x_0),
\]
(7.63)
with the boundary terms
\[
f_A^{(1)}(x_0)_{ba} = -\frac{1}{2} \text{tr} \left\{ H_1^{(2)}(x) \gamma_0 H_2^{(0)}(x) \right\},
\]
(7.64)
\[
f_A^{(1)}(x_0)_{bb} = -\frac{1}{2} \text{tr} \left\{ H_1^{(0)}(x) \gamma_0 H_2^{(2)}(x) \right\},
\]
(7.65)
and the loop terms
\[
f_A^{(1)}(x_0)_{1a} = -\frac{1}{2} \left\langle \text{tr} \left\{ H_1^{(2)}(x) \gamma_0 H_2^{(0)}(x) \right\} \right\rangle_G,
\]
(7.66)
\[
f_A^{(1)}(x_0)_{1b} = -\frac{1}{2} \left\langle \text{tr} \left\{ H_1^{(0)}(x) \gamma_0 H_2^{(2)}(x) \right\} \right\rangle_G,
\]
(7.67)
\[
f_A^{(1)}(x_0)_{2a} = -\frac{1}{2} \left\langle \text{tr} \left\{ H_1^{(2)}(x) \gamma_0 H_2^{(0)}(x) \right\} \right\rangle_G,
\]
(7.68)
\[
f_A^{(1)}(x_0)_{2b} = -\frac{1}{2} \left\langle \text{tr} \left\{ H_1^{(0)}(x) \gamma_0 H_2^{(2)}(x) \right\} \right\rangle_G,
\]
(7.69)
\[
f_A^{(1)}(x_0)_{3} = -\frac{1}{2} \left\langle \text{tr} \left\{ H_1^{(1)}(x) \gamma_0 H_2^{(1)}(x) \right\} \right\rangle_G.
\]
(7.70)
These can be interpreted in terms of the Feynman diagrams shown in Figure 7.4.

The correlation function \( f_1 \) is also expanded,
\[
f_1 = f_1^{(0)} + f_1^{(1)} g_0^2 + O(g_0^4),
\]
(7.71)
using the matrices \( K_1 \) and \( K_2 \), calculated by inserting \( m_{0,1} \) and \( m_{0,2} \) into \( K_1 \) (see Section 6.3). The tree level function \( f_1^{(0)} \) is given in Appendix B.2, the one loop coefficient can again be decomposed,
\[
f_1^{(1)} = (f_1^{(1)})_{ba} + (f_1^{(1)})_{bb} + (f_1^{(1)})_{1a} + (f_1^{(1)})_{1b} + (f_1^{(1)})_{2a} + (f_1^{(1)})_{2b}
+ (f_1^{(1)})_{3} + (f_1^{(1)})_{4} + (f_1^{(1)})_{5a} + (f_1^{(1)})_{5b}
+ (f_1^{(1)})_{6a} + (f_1^{(1)})_{6b} + (f_1^{(1)})_{7a} + (f_1^{(1)})_{7b}
+ m_{0,1} \frac{\partial}{\partial m_{0,1}} f_1^{(0)} + m_{0,2} \frac{\partial}{\partial m_{0,2}} f_1^{(0)},
\]
(7.72)
Figure 7.4: One loop diagrams contributing to $f_A(T/2)$.

where

$$
(f_1^{(1)})_{ba} = \frac{1}{2} \text{tr} \left\{ K_{1,b}^{(2)} K_{2}^{(0)} \right\},
$$

$$
(f_1^{(1)})_{bb} = \frac{1}{2} \text{tr} \left\{ K_{1}^{(0)} K_{2,b}^{(2)} \right\}
$$

are the boundary counterterms, and

$$
(f_1^{(1)})_{1a} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{1,1}^{(2)} K_{2}^{(0)} \right\} \right\rangle_{\tilde{G}},
$$

$$
(f_1^{(1)})_{1b} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{1}^{(0)} K_{2,1}^{(2)} \right\} \right\rangle_{\tilde{G}},
$$

$$
(f_1^{(1)})_{2a} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{1,2}^{(2)} K_{2}^{(0)} \right\} \right\rangle_{\tilde{G}},
$$

$$
(f_1^{(1)})_{2b} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{1}^{(0)} K_{2,2}^{(2)} \right\} \right\rangle_{\tilde{G}},
$$

$$
(f_1^{(1)})_{3} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{1,1}^{(1)} K_{2,1}^{(1)} \right\} \right\rangle_{\tilde{G}},
$$

$$
(f_1^{(1)})_{4} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{1,2}^{(1)} K_{2,2}^{(1)} \right\} \right\rangle_{\tilde{G}},
$$

$$
(f_1^{(1)})_{5a} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{1,3}^{(2)} K_{2}^{(0)} \right\} \right\rangle_{\tilde{G}},
$$

$$
(f_1^{(1)})_{5b} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{1}^{(0)} K_{2,3}^{(2)} \right\} \right\rangle_{\tilde{G}},
$$

$$
(f_1^{(1)})_{6a} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{1,4}^{(2)} K_{2}^{(0)} \right\} \right\rangle_{\tilde{G}},
$$

$$
(f_1^{(1)})_{6b} = \frac{1}{2} \left\langle \text{tr} \left\{ K_{1}^{(0)} K_{2,4}^{(2)} \right\} \right\rangle_{\tilde{G}}.
$$
Figure 7.5: One loop diagrams contributing to $f_1$. The dotted lines denote the links from $T - a$ to $T$.

$$(f^{(1)}_{1})_{0b} = \frac{1}{2} \left\langle \text{tr} \left\{ K^{(0)}_{1} K^{(2)}_{2} \right\} \right\rangle \hat{\mathcal{G}}, \quad (7.84)$$

$$(f^{(1)}_{1})_{7a} = \frac{1}{2} \left\langle \text{tr} \left\{ K^{(1)}_{1,1} K^{(1)}_{2,2} \right\} \right\rangle \hat{\mathcal{G}}, \quad (7.85)$$

$$(f^{(1)}_{1})_{7b} = \frac{1}{2} \left\langle \text{tr} \left\{ K^{(1)}_{1,2} K^{(1)}_{2,1} \right\} \right\rangle \hat{\mathcal{G}} \quad (7.86)$$

are connected to the Feynman diagrams in Figure 7.5.

With these expressions, the expansion coefficients of $Y_1$ can now be calculated,

$$Y^{(0)}_1(z, L/a) = \left(1 + a \frac{1}{2} b^{(0)}_A m^{(0)}_{q,2} \frac{f^{(0)}_A(L/2)}{\sqrt{f^{(0)}_1}}\right)$$

$$Y^{(1)}_1(z, L/a) = \frac{1}{f^{(0)}_1} \left\{ f^{(1)}_A(L/2) \sqrt{f^{(0)}_1} - \frac{f^{(0)}_A(L/2)}{2 \sqrt{f^{(0)}_1}} f^{(1)}_1 \right. \right.$$  

$$\left. + \frac{1}{2} (f^{(1)}_A m^{(0)}_{q,2} + b^{(0)}_A m^{(1)}_{q,2} f^{(0)}_A(L/2) \sqrt{f^{(0)}_1} \right) + a c^{(1)}_A f^{(0)}_A(L/2) \sqrt{f^{(0)}_1}, \quad (7.88)$$
While $Y_i^{(0)}$ can be obtained analytically from the formulae in Appendix B.2, $Y_i^{(1)}$ has to be calculated numerically. This has again been done with a set of FORTRAN programs on an IBM SP2 and on a PC. In each run of the program, the correlation functions were calculated at fixed $\theta$ and for a set of lattice sizes $(L/a) = (L/a)_{\text{min}} \ldots, 48$, where $(L/a)_{\text{min}} = 2z + 2$ for integer $z$ values and $(L/a)_{\text{min}} = 2z + 1$ for half-integer $z$ values. For $z = 12$ and $z = 16$, the computations were extended up to $(L/a) = 64$ and $(L/a) = 80$, respectively. Computing times ranged from a bit less than two days for the $z < 12$ runs, up to more than three weeks for the $z = 16$ run. Results for $\theta = 0.5$ and for various values of $z$ are listed in tables D.4–D.10. As in the case of $\lambda_{\text{lat}}$, the data have been extrapolated to the continuum limit using the fitting procedure explained in Appendix C. As an example, data and fits for $z = 2$, $z = 5$, and $z = 8$ are shown in Figure 7.6. The large deviation of the $z = 8$ data from the fit for smaller lattices reflects the large discretization errors that are connected with large quark masses.

The renormalized ratio $Y_{1,\text{CA}}$ can be written as

$$Y_{1,\text{CA}} = Z_{A,\text{CA}} Y_i,$$  \hspace{1cm} (7.89)

where $Z_{A,\text{CA}}$ is known both non-perturbatively [83], and at one loop order of perturbation theory [79],

$$Z_{A,\text{CA}} = 1 + Z_{A,\text{CA}}^{(1)} \alpha_s + O(\alpha_s^4),$$  \hspace{1cm} (7.90)

$$Z_{A,\text{CA}}^{(1)} = -0.0873435 \times C_F.$$  \hspace{1cm} (7.91)

From eq. (7.40), we get

$$Y_{1,\text{CA}} = \left(x_{\text{A}}^{\text{stat}}\right)_{\text{CA,lat}} X_{1,\text{lat}}.$$  \hspace{1cm} (7.92)

Expanding this to one loop order, we get

$$Y_i^{(0)}(z, L/a) = X_i^{(0)}(L/a) + O \left(\frac{a}{L}\right)^2 + O \left(\frac{1}{z}\right),$$ \hspace{1cm} (7.93)

$$Y_i^{(1)}(z, L/a) + Z_{A,\text{CA}} Y_i^{(0)}(z, L/a) = X_i^{(1)}_{\text{lat}}(L/a) + \left\{ -\gamma_0 \ln(z) + B_{A,\text{lat}}^{\text{stat}} \right\} X_i^{(0)}(L/a) + O \left(\frac{a}{L}\right)^2 + O \left(\frac{1}{z}\right).$$ \hspace{1cm} (7.94)

Solving eq. (7.94) for $B_{A,\text{lat}}^{\text{stat}}$ and inserting eq. (7.93) gives

$$B_{A, \text{lat}}^{\text{stat}} = Z_{A,\text{CA}}^{(1)} + \gamma_0 \ln(z) + \frac{1}{X_i^{(0)}} \left\{ Y_i^{(1)} - X_i^{(1)}_{\text{lat}} \right\} + O \left(\frac{a}{L}\right)^2 + O \left(\frac{1}{z}\right).$$ \hspace{1cm} (7.95)
Figure 7.6: The one loop coefficient of the ratio $Y_1$ at $\theta = 0.5$ and at different values of $z$. The data points are for $L/a = 10, 12, \ldots, 48$. The dashed lines are fits to the data, where the filled symbols have been included in the fits, while the open symbols were disregarded.
Denoting the continuum limit of the right hand side by $\hat{B}_A^{\text{stat}}(z)$,

$$ Z_{A,CA}^{(1)} + \gamma_0 \ln(z) + \frac{1}{X_1^{(0)}} \left\{ Y_1^{(1)} - X_{l,\text{lat}}^{(1)} \right\} = \hat{B}_A^{\text{stat}}(z) + O\left( \left( \frac{a}{L} \right)^2 \right), \quad (7.96) $$

we can write

$$ B_A^{\text{stat}} = \hat{B}_A^{\text{stat}}(z) + O \left( \frac{1}{z} \right), \quad (7.97) $$

which means that $B_A^{\text{stat}}$ has to be obtained by extrapolation of $\hat{B}_A^{\text{stat}}(z)$ to $1/z = 0$. This has again been done by a fit to the numerical data, as explained in Appendix C. Figure 7.7 shows the data and the fit, where the two points with the largest values of $z$ ($z = 12$ and $z = 16$) have not been included in the fit because of their large errors from the extrapolation to $a/L = 0$. However, they agree with the fit within their errors.

The extrapolation to $1/z = 0$ yields

$$ B_A^{\text{stat}} = -0.137(1), \quad (7.98) $$

which finally gives

$$ (\chi_A^{\text{stat}})^{\text{(1)}}_{\text{MS, lat}} = \frac{1}{6\pi^2} - 0.137(1). \quad (7.99) $$

The quantity $B_A^{\text{stat}}$ has previously been calculated by Borrelli and Pittori for a "rotated" axial current [78], and using a different method. However, it is possible to extract a result for the local current from the numerical data they quote, which is

$$ B_A^{\text{stat}} = -0.140. \quad (7.100) $$

It is not entirely clear how large their errors are. They only claim that their numerically calculated constants have errors of "less than five per cent", which would lead to an overall error of about $\Delta B_A^{\text{stat, BP}} \sim 0.015$. Even if their error is considerably smaller, the results (7.98) and (7.100) are in good agreement.

A similar calculation has been done by Hernández and Hill [84], but they also use rotated operators, and it is not possible to get a result for the local current from the numbers they quote. Calculations of the quantity corresponding to $B_A^{\text{stat}}$ in the theory without $\text{O}(a)$ improvement can be found in [59, 76].

### 7.4 SF renormalization

#### 7.4.1 The SF scheme and the step scaling function

A significant disadvantage of the lattice MS scheme and the $\overline{\text{MS}}$ scheme is that they are only defined in perturbation theory. To be able to perform a non-perturbative renormalization, a further scheme has to be introduced. One possible choice is the SF scheme [38, 39, 41, 70].
Figure 7.7: The function $\hat{B}_A^{\text{stat}}$. The dotted curve is a fit including the filled symbols. Its value at $1/z = 0$ is $B_A^{\text{stat}}$. 
To set up the scheme, one first has to define a renormalized coupling $g_{\text{SF}}(L)$, where the renormalization scale is $\mu = 1/L$, as explained in Section 2.2. This has been introduced in [41], a short definition is given in Appendix E. Then, a prescription for the critical mass has to be chosen. The definition used in the previous chapter can only be applied in perturbation theory; an appropriate choice for non-perturbative purposes would be to define $m_c$ through the correlation function $f_{A,1}$ and the PCAC relation

$$\left\{ \frac{1}{2}(\partial_0^2 + \partial_0) f_A(x_0) + a c_A \delta_0^2 \partial_0 f_P(x_0) \right\}_{m_0,1 = m_0, 2 = m_c} = 0. \quad (7.101)$$

Then, a renormalization condition has to be imposed on a correlation function containing the static-light axial current. Our choice is again the ratio $X_1$, and we define the SF-renormalized ratio $X_{1,\text{SF}}$ by demanding that

$$X_{1,\text{SF}}(L/a) = X_1^{(0)}(L/a). \quad (7.102)$$

This defines a renormalization constant for the static axial current,

$$(A_{\text{SF}}^{\text{stat}})_0 = Z_{A,\text{SF}}^{\text{stat}} A_0^{\text{stat}}, \quad (7.103)$$

with

$$Z_{A,\text{SF}}^{\text{stat}}(g_0, L/a) = \frac{X_1^{(0)}(L/a)}{X_1(g_0, L/a)}. \quad (7.104)$$

With this renormalization constant, the step scaling function $\Sigma_{A}^{\text{stat}}$ introduced in Section 2.2 can be defined. To do this for a given renormalized coupling $u$, one first has to adjust the bare coupling $g_0$ such that

$$g_{\text{SF}}^2(L) = u, \quad (7.105)$$

and then one defines

$$\Sigma_{A}^{\text{stat}} \left( u, \frac{a}{L} \right) = \frac{Z_{A,\text{SF}}^{\text{stat}}(g_0, 2L/a)}{Z_{A,\text{SF}}^{\text{stat}}(g_0, L/a)}. \quad (7.106)$$

This function can be extrapolated to its continuum limit $\sigma_{A}^{\text{stat}}$,

$$\Sigma_{A}^{\text{stat}} \left( u, \frac{a}{L} \right) = \sigma_{A}^{\text{stat}}(u) + O \left( \frac{a}{L} \right). \quad (7.107)$$

If $O(a)$ improvement is applied rigorously, the continuum limit should be reached at $O((a/L)^2)$. However, one has to be careful, because the improvement coefficients $c_A^{\text{stat}}, c_I$ and $\tilde{c}_I$ are only known perturbatively. This means that, in non-perturbative calculations, one has to be prepared for $O(a)$ effects.
7.4.2 Relation to the $\overline{\text{MS}}$ scheme

As the $\overline{\text{MS}}$ scheme is one of the most widely used schemes in perturbation theory, it is interesting to know the relation between the $\overline{\text{MS}}$ scheme and the SF scheme. Here, this relation is established at one loop order.

To this end, we write again

$$ (A_{\text{SF}}^{\text{stat}})_0 = (\chi_{\text{SF,} \overline{\text{MS}}}^{\text{stat}})(A_{\text{SF}}^{\text{stat}})_0, \quad (7.108) $$

and expand

$$ (\chi_{\text{SF,} \overline{\text{MS}}}^{\text{stat}}) = 1 + (\chi_{\text{SF,} \overline{\text{MS}}}^{\text{stat}})^{(1)} g_{\overline{\text{MS}}}^2 + O(g_{\overline{\text{MS}}}^4). \quad (7.109) $$

To calculate $(\chi_{\text{SF,} \overline{\text{MS}}}^{\text{stat}})^{(1)}$, we first relate the SF scheme to the lattice MS scheme,

$$ (A_{\text{SF}}^{\text{stat}})_0 = (\chi_{\text{SF, lat}}^{\text{stat}}) (A_{\text{lat}}^{\text{stat}})_0 $$

$$ = (\chi_{\text{SF, lat}}^{\text{stat}}) (\chi_{\text{lat,} \overline{\text{MS}}}^{\text{stat}})^{-1} (A_{\text{lat}}^{\text{stat}})_0. \quad (7.110) $$

This means that

$$ (\chi_{\text{SF,} \overline{\text{MS}}}^{\text{stat}}) = (\chi_{\text{SF, lat}}^{\text{stat}}) (\chi_{\text{lat,} \overline{\text{MS}}}^{\text{stat}})^{-1}. \quad (7.111) $$

As $(\chi_{\text{lat,} \overline{\text{MS}}}^{\text{stat}})$ is known at one loop order from the calculation in Section 7.3, only $(\chi_{\text{SF, lat}}^{\text{stat}})$ remains to be calculated. This is done by writing

$$ (\chi_{\text{SF, lat}}^{\text{stat}}) X_{1, \text{lat}}(g_0, L/a) = X_{1, \text{SF}}(g_0, L/a) $$

$$ = X_{1(0)}. \quad (7.112) $$

Expanding eq. (7.112) up to order $g_0^2$, and solving for $(\chi_{\text{SF, lat}}^{\text{stat}})^{(1)}$, we get

$$ (\chi_{\text{SF, lat}}^{\text{stat}})^{(1)} = -\frac{X_{1(0)}^{(1)}}{X_{1(0)}^{(0)}} (L/a), \quad (7.113) $$

where it is understood that the right hand side is extrapolated to the continuum limit. Performing this extrapolation with our data from tables D.2 and D.3, and inserting $(\chi_{\text{lat,} \overline{\text{MS}}}^{\text{stat}})^{(1)}$ from eq. (7.99) into eq. (7.111), we finally get

$$ (\chi_{\text{SF,} \overline{\text{MS}}}^{\text{stat}})^{(1)}(\theta = 0.0) = 0.056(1) \quad (7.114) $$

$$ (\chi_{\text{SF,} \overline{\text{MS}}}^{\text{stat}})^{(1)}(\theta = 0.5) = 0.036(1) \quad (7.115) $$

$$ (\chi_{\text{SF,} \overline{\text{MS}}}^{\text{stat}})^{(1)}(\theta = 1.0) = 0.016(1). \quad (7.116) $$

7.4.3 The anomalous dimension of the static-light axial current at two loop order

As indicated earlier, knowledge of the anomalous dimension at two loop order is required for a safe extrapolation to the renormalization group invariant current.
In any renormalization scheme, the static-light axial current’s anomalous dimension $\gamma$ is defined by the renormalization group equation (2.12), with the coupling satisfying eq. (2.14). As stated in Section 2.2, the anomalous dimension and the beta function can be expanded as

\[
\gamma(g_R) = -g_R^2(\gamma_0 + \gamma_1 g_R^2 + \ldots),
\]
\[
\beta(g_R) = -g_R^3(b_0 + b_1 g_R^2 + \ldots).
\]

(7.117)  
(7.118)

The one loop anomalous dimension

\[
\gamma_0 = -\frac{1}{4\pi^2}
\]

and the coefficients

\[
b_0 = \frac{1}{(4\pi)^2} \left\{ 11 - \frac{2}{3} N_f \right\}
\]

(7.120)

and

\[
b_1 = \frac{1}{(4\pi)^4} \left\{ 102 - \frac{38}{3} N_f \right\},
\]

(7.121)

with $N_f$ relativistic quark flavours, are scheme independent, while the two loop anomalous dimension $\gamma_1$ has to be calculated separately in each renormalization scheme. To compute it in the SF scheme, we will follow the concepts of [85], where the two loop anomalous dimension of quark masses has been calculated, and convert $\gamma_1$ from the \underline{MS} scheme, where it is known to be [86, 87, 88]

\[
\gamma_1^{\underline{MS}} = -\frac{1}{576\pi^4} \left\{ \frac{127}{2} + 28\zeta(2) - 5N_f \right\},
\]

(7.122)

with the Riemann zeta function $\zeta$. For the calculation, we need the conversion factor relating the couplings in both schemes,

\[
g_{SF}^2 = (\chi_\xi)_{SF,\underline{MS}}^2 = \left( (\chi_\xi)_{SF,\underline{MS}} \right)^2 = 1 + (\chi_\xi)_{SF,\underline{MS}}^{(1)} g_{MS}^2 + O(g_{MS}^4),
\]

(7.123)

where for our choice of the SF-coupling, the one loop coefficient is

\[
(\chi_\xi)_{SF,\underline{MS}}^{(1)} \mu=1/L = -\frac{1}{4\pi}(c_{1,0} + c_{1,1} N_f),
\]

(7.124)

with [70]

\[
c_{1,0} = 1.25563(4), \quad c_{1,1} = 0.039863(2).
\]

(7.125)

Inserting these and the conversion factor $(\chi^{\text{stat}}_A)_{SF,\underline{MS}}$ into eq. (2.12) and eq. (2.14), and expanding to order $g_{MS}^4$, one gets

\[
\gamma_1^{SF} = \gamma_1^{\underline{MS}} + 2b_0(\chi^{\text{stat}}_A)^{(1)}_{SF,\underline{MS}} - \gamma_0(\chi_\xi)^{(1)}_{SF,\underline{MS}}.
\]

(7.126)
Using the results for \( \chi_{A}^{\text{stat}}_{\text{SF,MS}}^{(1)} \) from the previous section, we finally obtain

\[
\gamma_{1}^{\text{SF}}(\theta = 0.0) = \frac{1}{(4\pi)^{2}} \{ 0.52(2) - 0.0733(13)N_{f} \} 
\]

\[
\gamma_{1}^{\text{SF}}(\theta = 0.5) = \frac{1}{(4\pi)^{2}} \{ 0.08(2) - 0.0466(13)N_{f} \} 
\]

\[
\gamma_{1}^{\text{SF}}(\theta = 1.0) = \frac{1}{(4\pi)^{2}} \{ -0.36(2) - 0.0199(13)N_{f} \} ,
\]

showing that the anomalous dimensions are reasonably small, which is a prerequisite for a safe extrapolation to the renormalization group invariant operator using the two loop formula.

### 7.4.4 Cutoff effects in the SF scheme

As the step scaling function is the central object of our renormalization strategy, it is interesting to know how large its discretization errors are. As a measure for the lattice artifacts,

\[
\delta(u, a/L) = \frac{\Sigma_{A}^{\text{sat}}(u, a/L) - \sigma_{A}^{\text{stat}}(u)}{\sigma_{A}^{\text{stat}}(u)}
\]

(7.130)

is introduced. With our numerical data, we are able to evaluate this at one loop order, i.e. we can calculate the one loop coefficient in the expansion

\[
\delta(u, a/L) = \delta^{(1)}(a/L)u + O(u^{2}).
\]

(7.131)

From eq. (7.104) and eq. (7.106) we get

\[
\Sigma_{A}^{\text{sat}}(u, a/L) = 1 + \left\{ \frac{X_{1,\text{lat}}^{(1)}(g_{0}, L/a)}{X_{1}^{(0)}(L/a)} - \frac{X_{1,\text{lat}}^{(1)}(g_{0}, 2L/a)}{X_{1}^{(0)}(2L/a)} + \gamma_{0}\ln(2) \right\} u + O(u^{2}),
\]

(7.132)

which can be extrapolated to the continuum,

\[
\sigma_{A}^{\text{stat}}(u) = 1 + \gamma_{0}\ln(2)u + O(u^{2}),
\]

(7.133)

giving finally

\[
\delta^{(1)}(a/L) = \frac{X_{1,\text{lat}}^{(1)}(g_{0}, L/a)}{X_{1}^{(0)}(L/a)} - \frac{X_{1,\text{lat}}^{(1)}(g_{0}, 2L/a)}{X_{1}^{(0)}(2L/a)}.
\]

(7.134)

In Figure 7.8, \( \delta^{(1)} \) is shown, using two different definitions of the critical mass. In the left plot, the extrapolated one loop mass is used as in all the perturbative calculations; in the right plot, the critical mass has been defined by eq. (7.101) at \( \theta = 0.0 \), which is the definition which will be used in a non-perturbative calculation [89]. In this case, one still gets \( m_{c}^{(0)} = 0 \), but \( m_{c}^{(1)} \) takes the values given in Table D.1. However, it can be seen that the two definitions do not yield very different results. In both cases, the discretization errors are rather moderate.
Figure 7.8: Discretization errors in the step scaling functions. In the left plot, the one loop coefficient of the critical light quark mass has been extrapolated to the continuum, in the right plot it has been calculated at finite lattice spacing using the PCAC relation.
Chapter 8

\[ O(a) \] improvement revisited

For the \( O(a) \) improvement of the static-light axial current, the \( c_A^{\text{stat}(1)} \) value (4.25) obtained by Morningstar and Shigemitsu has been used up to this point. But we are also able to calculate this coefficient using our numerical data. To that end, we use the lattice MS-renormalized ratio \( X_{1, \text{lat}} \), and write its one loop coefficient as

\[
X_{1, \text{lat}}(L/a) = X_{\text{lat}}(L/a)_{\xi_t=1} + X_{b}^{(1)}(L/a) + c_A^{\text{stat}(1)} X_{\delta A}^{(0)}(L/a),
\]  

(8.1)

with

\[
X_{b}^{(1)}(L/a) = \frac{1}{f_1^{\text{stat}(0)}} \left\{ f_A^{\text{stat}(1)}(L/a) \sqrt{f_1^{\text{stat}(0)}} - f_A^{\text{stat}(0)}(L/a) \left( f_1^{\text{stat}(1)} \right)_b \right\},
\]  

(8.2)

and

\[
X_{\delta A}^{(0)}(L/a) = a f_A^{\text{stat}(0)}(L/a) f_1^{\text{stat}(0)},
\]  

(8.3)

where \( X_{\text{lat}}^{(1)}(L/a)_{\xi_t=1} \) is the lattice MS-renormalized ratio without boundary and current improvement terms. Now, the \( O(a/L) \) part of eq. (8.1) has to be determined. As \( X_b \) and \( X_{\delta A} \) are \( O(a) \) counterterms, their continuum limit is zero, and we have

\[
X_{b}^{(1)} \left( \frac{L}{a} \right) \propto \frac{a}{L} + O \left( \left( \frac{a}{L} \right)^2 \right), \quad X_{\delta A}^{(0)} \left( \frac{L}{a} \right) \propto \frac{a}{L} + O \left( \left( \frac{a}{L} \right)^2 \right).
\]  

(8.4)

To calculate the \( O(a/L) \) part of \( X_{\text{lat}}^{(1)}(L/a)_{\xi_t=1} \), we introduce the symmetric derivative

\[
\frac{1}{2} (\partial + \partial^*) X_{\text{lat}}^{(1)} \left( \frac{L}{a} \right) = \frac{1}{2a} \left\{ X_{\text{lat}}^{(1)} \left( \frac{L + a}{a} \right) - X_{\text{lat}}^{(1)} \left( \frac{L - a}{a} \right) \right\}.
\]  

(8.5)

In a formal continuum theory containing \( (a/L) \) as an explicit parameter, one would write the \( O(a/L) \) coefficient as

\[
\frac{d}{d(a/L)} X_{\text{lat}}^{(1)}(L/a)_{\xi_t=1, a/L=0} = \lim_{a/L \to 0} \left( \frac{d X_{\text{lat}}^{(1)}(L/a)_{\xi_t=1}}{d(L/a)} \right) \frac{d(L/a)}{d(a/L)}
\]  

63
\[
= \lim_{a/L \to 0} \left\{ -\frac{L^2}{a^2} \frac{dX^{(1)}_{\text{lat}}(L/a)|_{\tilde{c}_i=1}}{d(L/a)} \right\}.
\]

Discretizing
\[
\frac{dX^{(1)}_{\text{lat}}(L/a)|_{\tilde{c}_i=1}}{d(L/a)} = \frac{1}{2} \left\{ X^{(1)}_{\text{lat}} \left( \frac{L}{a} + 1 \right)|_{\tilde{c}_i=1} - X^{(1)}_{\text{lat}} \left( \frac{L}{a} - 1 \right)|_{\tilde{c}_i=1} \right\},
\]
and comparing with eq. (8.5), the \(O(a/L)\) part of \(X^{(1)}_{\text{lat}}(L/a)|_{\tilde{c}_i=1}\) can be written as
\[
\lim_{a/L \to 0} \left\{ -\frac{L^2}{2a} (\partial + \partial^*) \ X^{(1)}_{\text{lat}} \left( \frac{L}{a} \right)|_{\tilde{c}_i=1} \right\}.
\]

Putting all together, we can extract the \(O(a/L)\) part of eq. (8.1). The desired \(O(a)\) improvement is then achieved by demanding that this \(O(a/L)\) part is zero in the continuum limit,
\[
\lim_{a/L \to 0} \left\{ -\frac{L^2}{2a} (\partial + \partial^*) \ X^{(1)}_{\text{lat}} \left( \frac{L}{a} \right)|_{\tilde{c}_i=1} + \frac{L}{a} X^{(1)}_{\text{b}} \left( \frac{L}{a} \right) + c^{\text{stat}(1)}_{A} \frac{L}{a} X^{(0)}_{\delta A} \left( \frac{L}{a} \right) \right\} = 0.
\]

Solving for \(c^{\text{stat}(1)}_{A}\) gives
\[
c^{\text{stat}(1)}_{A} = -\frac{\lim_{a/L \to 0} \frac{L^2}{2a} (\partial + \partial^*) \ X^{(1)}_{\text{lat}} \left( \frac{L}{a} \right)|_{\tilde{c}_i=1} - \lim_{a/L \to 0} \frac{L}{a} X^{(1)}_{\text{b}} \left( \frac{L}{a} \right)}{\lim_{a/L \to 0} \frac{L}{a} X^{(0)}_{\delta A} \left( \frac{L}{a} \right)},
\]
where \((L/a) X^{(1)}_{\text{b}}\) and \((L/a) X^{(0)}_{\delta A}\) can be extrapolated to the continuum analytically, using the formulae given in Appendix B.6. For \(\frac{L^2}{2a} (\partial + \partial^*) X^{(1)}_{\text{lat}} \left( \frac{L}{a} \right)|_{\tilde{c}_i=1}\), data have to be obtained numerically at finite lattice spacing. For \(\theta = 0.5\) and \(\theta = 1.0\), the data can be found in table Table D.12. They have to be extrapolated to the continuum limit (see Appendix C), which gives
\[
c^{\text{stat}(1)}_{A} (\theta = 0.5) = -\frac{1}{4\pi} \times 1.01(7),
\]
\[
c^{\text{stat}(1)}_{A} (\theta = 1.0) = -\frac{1}{4\pi} \times 1.01(5).
\]

This is in good agreement with the Morningstar-Shigemitsu result (4.25), but has considerably larger errors.
Chapter 9
Summary and Outlook

As fascinating physical phenomena, like CP violation in the standard model, are connected with the axial current of a light and a heavy quark, this quantity is an interesting object for non-perturbative studies. Because the mass of the $b$ quark is larger than cutoffs that can be reached in Monte Carlo computations, the $B$-meson cannot be studied directly, but one can get physical results by interpolation between light quark masses and the static approximation. In this approximation, a scale dependent renormalization is required for the axial current.

In this respect, several fundamental questions have been addressed:

- It has been explained how difficulties in scale dependent renormalizations in the lattice regularized theory can be avoided by using a finite size scheme, and an outline of the renormalization strategy has been given.

- The formulation of the static effective theory on the lattice has been explained, and $O(a)$ improvement in that theory has been discussed in a systematic way.

- Static quarks have been included in the Schrödinger functional. An appropriate action has been defined, followed by a detailed discussion of $O(a)$ improvement including Schrödinger functional boundary conditions.

- Suitable correlation functions have been introduced, in which wave function renormalization constants and the heavy quark mass counterterm cancel.

After the general discussion of the static approximation and the proposed renormalization procedure, some aspects of this method have been studied numerically in perturbation theory.

- The correlation functions have been expanded to one loop order of perturbation theory, and have been evaluated numerically.

- It has been shown that the correlation functions can be made finite by the lattice MS renormalization prescription. Furthermore, the analysis of the data reveals that $O(a)$ improvement has been successful.
• The finite renormalization constant connecting the renormalized static axial current in the lattice MS scheme with that in the $\overline{\text{MS}}$ scheme has been calculated at one loop order, confirming a result known from [78], where it had been obtained from a completely different quantity.

• The SF scheme, which can also be used non-perturbatively, has been introduced. One of its basic ingredients, the step scaling function, has been defined.

• The relation of the SF scheme to the $\overline{\text{MS}}$ scheme has been computed at one loop order of perturbation theory.

• The static-light axial current's two loop anomalous dimension in the SF scheme has been calculated by conversion from the $\overline{\text{MS}}$ scheme. The results are reasonably small, inducing the hope that in non-perturbative calculations one can reach scales where it is safe to apply the two loop approximation to extrapolate to the renormalization group invariant current.

• Discretization errors in the step scaling function have been shown to be moderate at one loop order.

• Finally, the one loop $O(a)$ improvement coefficient for the static-light axial current has been calculated, giving a result which is in good agreement with the one obtained in [66] by NRQCD methods.

It has to be stressed that the calculations listed above are based on the assumption that the static theory can be renormalized by adding local counterterms which can be found by power counting methods. In contrast to the relativistic case, this assumption has not been proven, but the numerical results mentioned above do not indicate any problem with renormalizability. The good agreement with results for renormalization constants and improvement coefficients obtained by other authors using completely different methods is particularly convincing.

Interesting projects for the near future could be:

• A non-perturbative calculation of the step scaling function in the SF scheme. This is actually in progress [89].

• The calculation of the B-meson decay constant $F_B$. Instead of a current-current correlator, one could try to get the matrix element $\langle 0 | A_\mu^\text{stat} | B \rangle$ directly from the Schrödinger functional with large time extent $T \approx 3 \text{ fm}$, following [90, 91]. In the case of the pion decay constant, this method has produced more accurate results than the standard procedure. At the moment, this does not seem to be the case in the static approximation, but there is some hope that this problem can be solved.
• The complete determination of the CKM matrix, connected with the phenomenon of CP violation. While knowledge of $F_B$ can already yield approximate constraints, it would clearly be desirable to have a non-perturbative renormalization of the four-fermion operator connected with $B$-$\bar{B}$ mixing in the static approximation. As the Schrödinger functional seems to work fine with the static approximation concerning the axial current, one could hope that the SF scheme can also be used in the renormalization of the four fermion operator. However, this operator mixes with three other operators under renormalization [92], creating a situation to which the Schrödinger functional has not yet been applied. This means that such a project might not only yield important physical results, but could also be interesting from a conceptual point of view.
Appendix A

Notation

A.1 Index conventions

In this thesis, letters $\mu, \nu, \ldots$ taken from the middle of the Greek alphabet are Lorentz indices and run from 0 to 3, while the components of spatial vectors are labelled by Latin indices $k, l, \ldots$ running from 1 to 3. The $N^2 - 1$ generators of the gauge group SU($N$) are labelled by indices $a, b, \ldots$ taken from the beginning of the Latin alphabet.

A.2 Covariant derivatives in the Schrödinger functional

In the Schrödinger functional, the angles $\theta_k$ introduced in Chapter 5 are included in the covariant derivatives,

$$D_\mu = \partial_\mu + g_0 A_\mu + i \frac{\theta_\mu}{L}, \quad \text{(A.1)}$$

with $\theta_0 = 0$. The derivatives act to the left as

$$\mathcal{D}_\mu = \mathcal{F}\partial_\mu - g_0 A_\mu - i \frac{\theta_\mu}{L}. \quad \text{(A.2)}$$

A.3 Lattice derivatives

Denoting the unit vector in $\mu$-direction by $\hat{\mu}$, the forward and backward derivatives on the lattice are defined by

$$\partial_\mu \psi(x) = \frac{1}{a} \{ \psi(x + a\hat{\mu}) - \psi(x) \} \quad \text{(A.3)}$$

and

$$\partial^*_\mu \psi(x) = \frac{1}{a} \{ \psi(x) - \psi(x - a\hat{\mu}) \}. \quad \text{(A.4)}$$
They act to the left as
\[ \bar{\psi}(x) \frac{\partial}{\partial \mu} = \frac{1}{a} \left\{ \bar{\psi}(x + a\hat{\mu}) - \bar{\psi}(x) \right\}, \quad (A.5) \]
\[ \bar{\psi}(x) \frac{\partial^*}{\partial \mu} = \frac{1}{a} \left\{ \bar{\psi}(x) - \bar{\psi}(x - a\hat{\mu}) \right\}. \quad (A.6) \]

With the phases \( \theta_\mu \) introduced in chapter 5, we define the factors
\[ \lambda_\mu = \exp \left\{ ia \frac{\theta_\mu}{L} \right\}, \quad \theta_0 = 0, \quad -\pi < \theta_k \leq \pi, \quad (A.7) \]
and the gauge covariant derivatives in the Schrödinger functional
\[ \nabla_\mu \psi(x) = \frac{1}{a} \left\{ \lambda_\mu U(x, \mu) \psi(x + a\hat{\mu}) - \psi(x) \right\}, \quad (A.8) \]
\[ \nabla^*_\mu \psi(x) = \frac{1}{a} \left\{ \psi(x) - \lambda_\mu^{-1} U(x - a\hat{\mu}, \mu)^{-1} \psi(x - a\hat{\mu}) \right\}. \quad (A.9) \]

They act to the left as
\[ \bar{\psi}(x) \nabla_\mu = \frac{1}{a} \left\{ \bar{\psi}(x + a\hat{\mu}) U(x, \mu)^{-1} \lambda_\mu^{-1} - \bar{\psi}(x) \right\}, \quad (A.10) \]
\[ \bar{\psi}(x) \nabla^*_\mu = \frac{1}{a} \left\{ \bar{\psi}(x) - \bar{\psi}(x - a\hat{\mu}) U(x - a\hat{\mu}, \mu) \lambda_\mu \right\}. \quad (A.11) \]

### A.4 Dirac matrices

For the Dirac matrices, the chiral representation is used, i.e.
\[ \gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ e_\mu^\dagger & 0 \end{pmatrix}, \quad (A.12) \]
where the \( 2 \times 2 \) matrices \( e_\mu \) are
\[ e_0 = -1, \quad e_k = -i \sigma_k, \quad (A.13) \]
with the Pauli matrices \( \sigma_k \). In this representation, the Dirac matrices are hermitian, and satisfy
\[ \{ \gamma_\mu, \gamma_\nu \} = 2 \delta_{\mu\nu}. \quad (A.14) \]

The matrix \( \gamma_5 \) is defined as \( \gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \), which gives
\[ \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (A.15) \]

The hermitian \( \sigma \)-matrices are defined as
\[ \sigma_{\mu\nu} = \frac{i}{2} \{ \gamma_\mu, \gamma_\nu \}. \quad (A.16) \]
A.5 Gauge group

The vector fields $q$ have values in the Lie algebra $\text{su}(3)$, and can be decomposed as

$$q_\mu(x) = q_\mu^a(x)T^a,$$  \hspace{1cm} (A.17)

where $T^a$ are the generators of the group $\text{SU}(3)$. They satisfy

$$(T^a)^\dagger = -T^a,$$  \hspace{1cm} (A.18)

and are normalized to

$$\text{tr}\{T^aT^b\} = -\frac{1}{2}\delta^{ab}. \hspace{1cm} (A.19)$$

A.6 Delta function

On the lattice, it is convenient to normalize the $\delta$ function as

$$\delta(x_\mu) = a^{-1}\delta_0 x_\mu,$$  \hspace{1cm} (A.20)

and to define

$$\delta(x) = \prod_{\mu=0}^3 \delta(x_\mu), \quad \delta(\mathbf{x}) = \prod_{k=1}^3 \delta(x_k). \hspace{1cm} (A.21)$$

The Heavyside function $\theta(x_\mu)$ is defined as

$$\theta(x_\mu) = \begin{cases} 1 & \text{if } x_\mu \geq 0 \\ 0 & \text{otherwise.} \end{cases} \hspace{1cm} (A.22)$$
Appendix B

Perturbative expressions

In this appendix, explicit formulae for expressions occurring in the perturbative expansion of our correlation functions are given. All formulae related to gluons or light quarks are taken from [55] and [93].

B.1 The gluon propagator

Following [55], the gluon propagator eq. (6.35) is written as

$$D_{\mu\nu}(x, y) = \frac{1}{L^3} \sum_p e^{i p (x-y)} d_{\mu\nu}(x_0, y_0; p),$$  \hspace{1cm} (B.1)

where the sum is over all momenta $p$ with

$$p_k = \frac{2\pi n_k}{L},$$  \hspace{1cm} (B.2)

with whole numbers $n_k$ and

$$-\frac{\pi}{a} < p_k \leq \frac{\pi}{a}.$$  \hspace{1cm} (B.3)

Now, a gluon “energy” $\epsilon$ is introduced by

$$\cosh(\epsilon a) = 1 + \frac{1}{2} a^2 \hat{p}^2, \quad \hat{p}_k = \frac{2}{a} \sinh(ap_k).$$  \hspace{1cm} (B.4)

With these definitions, the gluon propagator can be written down explicitly. For $p \neq 0$, this gives

$$d_{00}(x_0, y_0; p) = \frac{a}{\sinh(\epsilon a) \sinh(\epsilon T)} \times \begin{cases} \cosh[(\epsilon T - x_0) - \frac{1}{2} \epsilon a] \cosh(\epsilon y_0 + \frac{1}{2} \epsilon a) & \text{if } x_0 \geq y_0, \quad (B.5) \\ \cosh(\epsilon x_0 + \frac{1}{2} \epsilon a) \cosh[(\epsilon T - y_0) - \frac{1}{2} \epsilon a] & \text{if } x_0 < y_0, \quad (B.5) \end{cases}$$
and
\[ d_{jk}(x_0, y_0; p) = \delta_{jk} \frac{a}{\sinh(\epsilon a) \sinh(\epsilon T)} \times \begin{cases} \sinh[\epsilon(T - x_0)] \sinh(\epsilon y_0) & \text{if } x_0 \geq y_0, \\ \sinh(\epsilon x_0) \sinh[\epsilon(T - y_0)] & \text{if } x_0 < y_0. \end{cases} \tag{B.6} \]

In the case \( p = 0 \) one obtains
\[ d_{00}(x_0, y_0; 0) = \begin{cases} y_0 + a & \text{if } x_0 \geq y_0, \\ x_0 + a & \text{if } x_0 < y_0, \end{cases} \tag{B.7} \]

and
\[ d_{jk} = \delta_{jk} \frac{1}{T} \times \begin{cases} (T - x_0)y_0 & \text{if } x_0 \geq y_0, \\ x_0(T - y_0) & \text{if } x_0 < y_0. \end{cases} \tag{B.8} \]

### B.2 Tree level correlation functions

To write down the correlation functions at tree level, some definitions are needed. First of all, the lattice momenta
\[ p^+_\mu = p_\mu + \frac{\theta_\mu}{L}, \quad \theta_0 = 0 \tag{B.9} \]

are introduced, where
\[ p_k = \frac{2\pi n_k}{L}, \tag{B.10} \]

with whole numbers \( n_k \) and
\[ -\frac{\pi}{a} < p_k \leq \frac{\pi}{a}. \tag{B.11} \]

Furthermore, we define for any momentum \( q_\mu \)
\[ \hat{q}_\mu = \frac{1}{a} \sin(a q_\mu), \tag{B.12} \]

and
\[ \hat{q}_\mu = \frac{2}{a} \sin(a q_\mu/2). \tag{B.13} \]

With these, an effective mass
\[ M(q) = m_0 + \frac{1}{2} a \hat{q}^2 \tag{B.14} \]

for the light quark field is defined. Writing any solution of the free lattice Dirac equation
\[ \left\{ \gamma_\mu \frac{1}{2} (\nabla_\mu + \nabla_\mu) - \frac{a}{2} \nabla_\mu \nabla_\mu + m_0 \right\} \psi^{(0)}_{1,\alpha}(x) = 0 \tag{B.15} \]
as
\[ \psi_{1, \text{cl}}^{(0)}(x) = u_i e^{ip x}, \] (B.16)
and looking for positive energy solutions
\[ \text{Im } p_0 \geq 0, \] (B.17)
one can deduce that
\[ p_0 = p_0^+ = i \omega(p^+) \mod 2\pi/a, \] (B.18)
with \( \omega(q) \) defined by
\[ \sinh \left[ \frac{a}{2} \omega(q) \right] = \frac{a}{2} \left\{ \frac{q^2 + (m_0 + \frac{1}{2} a q^2)^2}{1 + a(m_0 + \frac{1}{2} a q^2)} \right\}^{\frac{1}{2}}. \] (B.19)
To simplify the notation, one introduces
\[ A(q) = 1 + a(m_0 + \frac{1}{2} a q^2) \] (B.20)
and
\[ R(q) = M(q) \left\{ 1 - e^{-2\omega(q)R} \right\} - i q^0 \left\{ 1 + e^{-2\omega(q)R} \right\}. \] (B.21)
Having this, one can calculate the tree level propagator \( S_1^{(0)} \), defined by
\[ (D_W + m_0)S_1^{(0)}(x, y)|_{q_0=0} = \delta(x - y) \] (B.22)
and the boundary conditions
\[ S_1^{(0)}(x, y)|_{x_0=0} = S_1^{(0)}(x, y)|_{y_0=0} = S_1^{(0)}(x, y)|_{x_0=T} = S_1^{(0)}(x, y)|_{y_0=T} = 0. \] (B.23)
Then, one can also obtain its Fourier transform
\[ \tilde{S}_1^{(0)}(x_0, y_0; p) = a^3 \sum_x e^{-ip(x-y)} S_1^{(0)}(x, y). \] (B.24)
If \( m_0 = 0 \) and \( p^+ = 0 \), the propagator is
\[ \tilde{S}_1^{(0)}(x_0, y_0; p) = \begin{cases} P_+ & \text{if } x_0 > y_0, \\ P_- & \text{if } x_0 < y_0, \\ 1 & \text{if } x_0 = y_0. \end{cases} \] (B.25)
In all other cases, one can deduce from the form given in [55] that for \( x_0 > y_0 \)
\[ \tilde{S}_1^{(0)}(x_0, y_0; p) = - \left\{ 2i p_0^0 A(p^+) R(p^+) \right\}^{-1}. \]
\[ \times \left\{ (M(p^+) - i\gamma_{\mu} p_{\mu}^+(M(p^+) - i p_0^+)e^{-\omega(p^+)(x_0 - y_0)} \\
+ (M(p^+) + i\gamma_0 p_0^+ - i\gamma_k p_k^+)(M(p^+) + i p_0^+)e^{-\omega(p^+)(2T - x_0 + y_0)} \\
- \left[ M^2(p^+) + (p_0^+)^2 - i\gamma_k M(p^+) p_k^+ - \gamma_0\gamma_k p_0^+ p_k^+ \right] e^{-\omega(p^+)(x_0 + y_0)} \\
- \left[ M^2(p^+) + (p_0^+)^2 \\
i\gamma_k M(p^+) p_k^+ + \gamma_0\gamma_k p_0^+ p_k^+ \right] e^{-\omega(p^+)(2T - x_0 - y_0)} \right\}. \] (B.26)

For \( x_0 < y_0 \), one obtains

\[ \hat{S}_1^{(0)}(x_0, y_0; \mathbf{p}) = -\left\{ 2i p_0^+ A(p^+) R(p^+) \right\}^{-1} \]
\[ \times \left\{ (M(p^+) + i\gamma_0 p_0^+ - i\gamma_k p_k^+)(M(p^+) - i p_0^+)e^{-\omega(p^+)(y_0 - x_0)} \\
+ (M(p^+) - i\gamma_{\mu} p_{\mu}^+(M(p^+) + i p_0^+)e^{-\omega(p^+)(2T - y_0 + x_0)} \\
- \left[ M^2(p^+) + (p_0^+)^2 - i\gamma_k M(p^+) p_k^+ - \gamma_0\gamma_k p_0^+ p_k^+ \right] e^{-\omega(p^+)(y_0 + x_0)} \\
- \left[ M^2(p^+) + (p_0^+)^2 \\
i\gamma_k M(p^+) p_k^+ + \gamma_0\gamma_k p_0^+ p_k^+ \right] e^{-\omega(p^+)(2T - y_0 - x_0)} \right\}, \] (B.27)

while in the case \( x_0 = y_0 \) one gets

\[ \hat{S}_1^{(0)}(x_0, y_0; \mathbf{p}) = \frac{1}{A(p^+) P_-} - \left\{ 2i p_0^+ A(p^+) R(p^+) \right\}^{-1} \]
\[ \times \left\{ (M(p^+) - i\gamma_{\mu} p_{\mu}^+(M(p^+) - i p_0^+) \\
+ (M(p^+) + i\gamma_0 p_0^+ - i\gamma_k p_k^+)(M(p^+) + i p_0^+)e^{-\omega(p^+)(2T)} \\
- \left[ M^2(p^+) + (p_0^+)^2 - i\gamma_k M(p^+) p_k^+ - \gamma_0\gamma_k p_0^+ p_k^+ \right] e^{-\omega(p^+)(2x_0)} \\
- \left[ M^2(p^+) + (p_0^+)^2 \\
i\gamma_k M(p^+) p_k^+ + \gamma_0\gamma_k p_0^+ p_k^+ \right] e^{-\omega(p^+)(2T - 2x_0)} \right\}. \] (B.28)
For $a \leq x_0 \leq T$, a solution of the free Wilson-Dirac equation can be written as
\[
\psi^{(0)}_1(x) = a^3 \sum_y \{ S^{(0)}_1(x, y) \rho_1(y) \big|_{y_0=a} + S^{(0)}_1(x, y) \rho_1'(y) \big|_{y_0=T-a} \}, \quad (B.29)
\]
which yields
\[
H^{(0)}_1(x) = \frac{1}{R(p^+)} \left\{ (M(p^+) - i p^0 - i \gamma_k p^+_k) e^{-\omega(p^+)} x_0 \\
- (M(p^+) + i p^0 - i \gamma_k p^+_k) e^{-\omega(p^+)(2T-x_0)} \right\} P_+, \quad (B.30)
\]
with $p = 0$. As $H^{(0)}_1$ does not depend on $x$, a function
\[
\chi_1(x_0) = H^{(0)}_1(x) \quad (B.31)
\]
can be introduced. In the static quark case, setting $\delta m = 0$, one gets from eq. (4.12)
\[
\tilde{S}^{(0)}_h(x_0, y_0; p) = P_+, \quad (B.32)
\]
and
\[
\chi_1(x_0) = H^{(0)}_1(x) = P_. \quad (B.33)
\]
Putting all together, we obtain
\[
f^{\text{stat}(0)}_h(x_0) = - \frac{3}{2} \text{tr} \left\{ \chi_1(x_0)^d P_+ \right\}
\]
\[
= - \frac{3}{R(p^+)} \left\{ (M(p^+) - i p^0) e^{-\omega(p^+)} x_0 \\
- (M(p^+) + i p^0) e^{-\omega(p^+)(2T-x_0)} \right\} P_+ \quad (B.34)
\]
and
\[
f^{(0)}_h(x_0) = - \frac{3}{2} \text{tr} \left\{ \chi_1(x_0)^d \gamma_0 \chi_2(x_0) \right\}
\]
\[
= - \frac{3}{R_1((p^+_1)_{1})R_2((p^+_2)_{2})} \left\{ \left[ (M_1((p^+_1)_{1}) - i (p^0_{1})_{1}) e^{-\omega_1(p^+)} x_0 \\
- (M_1((p^+_1)_{1}) + i (p^0_{1})_{1}) e^{-\omega_1(p^+)(2T-x_0)} \right] \right. \\
\times \left. \left[ (M_2((p^+_2)_{2}) - i (p^0_{2})_{2}) e^{-\omega_2(p^+)} x_0 \\
- (M_2((p^+_2)_{2}) + i (p^0_{2})_{2}) e^{-\omega_2(p^+)(2T-x_0)} \right] \right. \\
\times \left. \left[ (p^0_{1})_{2} (p^0_{2})_{2} \left[ e^{-\omega_1(p^+)} x_0 - e^{-\omega_1(p^+)(2T-x_0)} \right] \right] \right), \quad (B.35)
\]
where the traces run only over Dirac indices, the indices 1 and 2 label the two relativistic quarks with bare masses $m_{0,1}$ and $m_{0,2}$, $p = 0$, and $(p_0^+)^j$ means $p_0^n$ calculated with the mass $m_{0,j}$. For the $O(a)$ improvement of $f_\lambda$, one needs the correlation function $f_P$ at tree level. It is

$$
\begin{align*}
  f_P^{(0)}(x_0) &= \frac{3}{2} \text{tr} \left\{ \chi_1(x_0) \dagger \chi_2(x_0) \right\} \\
  &= \frac{3}{R_1((p^+)_1)R_2((p^+)_2)} \left\{ \left[ (M_1((p^+)_1) - i(p_0^+)_1)e^{-\omega_1(p^+)_x_0} \\
  &\quad - (M_1((p^+)_1) + i(p_0^+)_1)e^{-\omega_1(p^+)_x_0} \right] \\
  &\quad \times \left[ (M_2((p^+)_2) - i(p_0^+)_2)e^{-\omega_2(p^+)x_0} \\
  &\quad - (M_2((p^+)_2) + i(p_0^+)_2)e^{-\omega_2(p^+)x_0} \right] \\
  &\quad + (p_0^+)_1(p_0^+)_2 \left[ e^{-\omega_1(p^+)x_0} - e^{-\omega_1(p^+)}(T_x_0) \right] \\
  &\quad \times \left[ e^{-\omega_2(p^+)x_0} - e^{-\omega_2(p^+)}(2T_x_0) \right] \right\} .
\end{align*}
$$

(B.36)

The tree level results for the $K$ matrices are

$$
K_1^{(0)} = P_+ \chi_1(T - a)
$$

(B.37)

and

$$
K_h^{(0)} = P_+,
$$

(B.38)

giving

$$
\begin{align*}
  f_1^{\text{stat}(0)} &= \frac{3}{2} \text{tr} \left\{ \chi_1(T - a) \dagger P_+ \right\} \\
  &= \frac{3}{R(p^+)} \left\{ (M(p^+) - i(p_0^+)^+e^{-\omega(p^+)(T - a)} \\
  &\quad - (M(p^+) + i(p_0^+)^+e^{-\omega(p^+)(T + a)} \right\}
\end{align*}
$$

(B.39)

and

$$
\begin{align*}
  f_1^{(0)} &= \frac{3}{2} \text{tr} \left\{ \chi_1(T - a) \dagger P_+ \chi_2(T - a) \right\} \\
  &= - \frac{3}{R_1((p^+)_1)R_2((p^+)_2)} \left\{ \left[ (M_1((p^+)_1) - i(p_0^+)_1)e^{-\omega_1(p^+)(T - a)} \\
  &\quad - (M_1((p^+)_1) + i(p_0^+)_1)e^{-\omega_1(p^+)(T + a)} \right]
\end{align*}
$$
\[
\times \left[ (M_2((p^+)_2) - i(p^+_0)_2)e^{-\omega_2(p^+)(T-a)} - (M_2((p^+)_2) + i(p^+_0)_2)e^{-\omega_2(p^+)(T+a)} \right], \\
\text{(B.40)}
\]

where, again, \( p = 0 \).

For the \( O(a) \) improvement of the static-light axial current, \( f_{\delta A}^{\text{stat}} \) has to be known at tree level. It is easy to see that

\[
f_{\delta A}^{\text{stat}(0)}(x_0) = -\frac{3(p^+)^2}{R(p^+)} \left\{ e^{-\omega(p^+)x_0} - e^{-\omega(p^+)(2T-x_0)} \right\}, \\
\text{(B.41)}
\]

with \( p = 0 \).

### B.3 Light quarks at one loop order

For the one loop calculation, the light quark matrices \( H_1 \) introduced in Chapter 6 are needed. The term for the one loop mass correction is

\[
\frac{\partial}{\partial m_0} H_1^{(0)}(x) = -a \sum_{y_0=a}^{T-a} \tilde{S}_1^{(0)}(x_0, y_0; 0) \chi_1(y_0), \\
\text{(B.42)}
\]

which can be obtained by differentiating eq. (B.15) with respect to \( m_0 \). The boundary counterterm is

\[
H_1^{(2)}(x)_b = -\tilde{c}_1^{(1)} \left\{ \tilde{S}_1^{(0)}(x_0, a; 0) \left[ \chi_1(a) - P_+ \right] - \tilde{S}_1^{(0)}(x_0, T-a; 0) \chi_1(T-a) \right\}. \\
\text{(B.43)}
\]

The matrices containing vertices are [93]

\[
H_1^{(1)}(x) = -\frac{a}{L^3} \sum_k \sum_{y_0=a}^{T-a} e^{i k \cdot x} \tilde{S}_1^{(0)}(x_0, y_0; k) \\
\times \sum_{i=1}^{16} q_{\mu(i)}(u_0 + at(i); k) T^\mu V^{(1)}_i(k, 0) \chi_1(u_0 + as(i)), \\
\text{(B.44)}
\]

\[
H_1^{(2)}(x)_1 = \frac{a^2}{L^3} \sum_{k, q} \sum_{y_0, y_0=a}^{T-a} e^{i(k+q) \cdot x} \tilde{S}_1^{(0)}(x_0, y_0; k+q) \\
\times \sum_{i=1}^{16} q_{\mu(i)}^a(u_0 + at(i); k) T^\mu V^{(1)}_i(k, q) \tilde{S}_1^{(0)}(u_0 + as(i), y_0; q) \\
\times \sum_{j=1}^{16} q_{\mu(j)}^a(v_0 + at(j); q) T^{\mu} V^{(1)}_j(q, 0) \chi_1(v_0 + as(j)), \\
\text{(B.45)}
\]
and

\[ H^{(2)}_1(x) = -\frac{a}{L^3} \sum_{k, q} \sum_{u_0=a}^{T-a} e^{i(k+q)x} \tilde{\zeta}_1^{(0)}(x_0, u_0; k + q) \]

\[ \times \sum_{i=1}^{5} \tilde{q}_\mu(i)(u_0 + at(i); k) \tilde{q}_\rho(i)(u_0 + at(i); q) \]

\[ \times T^a T^b \tilde{V}_i^{(2)}(k, q, 0) \chi(0 + as(i)), \quad (B.46) \]

where the quantities \( \mu(i), s(i), t(i) \) connected with the three point vertex \( V_i^{(1)} \) and the four point vertex \( V_i^{(2)} \) are given in Table B.1 and Table B.2, and the Fourier transform of the gauge field fluctuation is defined by

\[ q_0^a(x) = \frac{1}{L^3} \sum_p e^{ipx} \tilde{\xi}_0^a(x_0, p), \quad (B.47) \]

\[ q_k^a(x) = \frac{1}{L^3} \sum_p e^{ipx} e^{iqp_k} \tilde{\xi}_k^a(x_0, p). \quad (B.48) \]

### B.4 Heavy quarks at one loop order

The classical equation of motion for the heavy quark, without \( \delta m \) term,

\[ \nabla_0^* \psi_{h, cl}(x) = 0, \quad (B.49) \]

is solved by

\[ \psi_{h, cl}(x) = U(x - a0\hat{0}, 0)^{-1} \cdots U(x - x_0\hat{0}, 0)^{-1} \rho_h(x), \quad (B.50) \]

which leads to

\[ H_h(x) = U(x - a\hat{0}, 0)^{-1} \cdots U(x - x_0\hat{0}, 0)^{-1} P_+. \quad (B.51) \]

Expanding this in powers of the bare coupling, and inserting eq. (B.47), we get

\[ H_h^{(0)}(x) = P_+, \quad (B.52) \]

\[ H_h^{(1)}(x) = -\frac{a}{L^3} \sum_k \sum_{u_0=a}^{T-a} e^{ikx} \tilde{\xi}_h^{(0)}(x_0, u_0; k) \tilde{\xi}_0^a(u_0 - a; k) T^a P_+, \quad (B.53) \]

and

\[ H_h^{(2)}(x) = H_h^{(1)}(x) + H_h^{(1)}(x) 2, \quad (B.54) \]
Table B.1: Table from [33]. Data describing the three-point quark-gluon vertex, with \( r_j = \frac{1}{2} k_j + q_j + \theta_j / L \). The terms for \( i = 6 \ldots 16 \) originate from the Sheikholeslami-Wohlert term in the action.

\[
H^{(2)}_h(x)_1 = \frac{a^2}{L^6} \sum_{k,q} \sum_{u_0=a}^{T-a} \sum_{v_0=a}^{T-a} e^{i(k+q)x} \tilde{S}_h^{(0)}(x_0,u_0; k+q)\bar{q}_0^a(u_0-a;k)T^q(B.55)
\times \tilde{q}_0^a(v_0-a;q)T^bP_+ \tag{B.56}
\]

and

\[
H^{(2)}_h(x)_1 = \frac{a^2}{2L^6} \sum_{k,q} \sum_{u_0=a}^{T-a} e^{i(k+q)x} \tilde{S}_h^{(0)}(x_0,u_0; k+q)\bar{q}_0^a(u_0-a;k)T^a(B.57)
\times \tilde{q}_0^a(u_0-a;q)T^bP_+. \tag{B.58}
\]

B.5 Correlation functions at one loop order

In the numerical calculations, the correlation functions have been used in a form where one works in momentum space in space-like directions, while the time
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
i & \(\mu(i)\) & \(t(i)\) & \(s(i)\) & \(\frac{2}{a}V_{i}^{(2)}(k,q,p)\) \\
\hline
1 & 0 & 0 & 1 & \(-\frac{1}{2}(1 - \gamma_0)\) \\
2 & 0 & -1 & -1 & \(-\frac{1}{2}(1 + \gamma_0)\) \\
3 & 1 & 0 & 0 & \(i\sin(ar_1)\gamma_1 - \cos(ar_1)\) \\
4 & 2 & 0 & 0 & \(i\sin(ar_2)\gamma_2 - \cos(ar_2)\) \\
5 & 3 & 0 & 0 & \(i\sin(ar_3)\gamma_3 - \cos(ar_3)\) \\
\hline
\end{tabular}
\caption{Table B.2: Table from [93]. Data describing the four-point quark-gluon vertex, with \(r_j = \frac{1}{2}(k_j + q_j) + p_j + \theta_j/L\).}
\end{table}

coordinate \(x_0\) has been kept. This form can be obtained directly from the perturbative expansion of the matrices \(H_1, H_n, K_1,\) and \(K_n\). The one loop expressions for the correlation functions \(f_{A}^{stat}(1), f_{A}^{stat}(1), f_{A},\) and \(f_{1}\) can be calculated by using

\[
\langle \bar{q}_{i}(x_0;k)\bar{q}_{j}(y_0;q) \rangle = L^3 \delta_{ij} \delta_{q,-k} d_{\mu\nu}(x_0, y_0; k). \tag{B.59}
\]

As examples, the resulting formulae for the diagrams 1a and the diagrams 3 are given below.

\[
f_{A}^{stat(1)}(x_0)_{1a} = 2a^2 \sum_{k} \sum_{u_0, v_0=a, i,j=1}^{T-a} d_{\mu(i)\mu(j)}(u_0 + at(i), v_0 + at(j); k)
\times \text{tr} \left\{ \chi_0(v_0 + a s(j)) \dagger V_{j}^{(1)}(-k, 0) \dagger 
\times \tilde{S}_{1}^{(0)}(u_0 + a s(i), v_0; -k) \dagger V_{i}^{(1)}(k, -k) \dagger 
\times \tilde{S}_{1}^{(0)}(x_0, u_0; 0) \dagger P_{+} \right\}, \tag{B.60}
\]

\[
f_{A}^{(1)}(x_0)_{1a} = 2a^2 \sum_{k} \sum_{u_0, v_0=a, i,j=1}^{T-a} d_{\mu(i)\mu(j)}(u_0 + at(i), v_0 + at(j); k)
\times \text{tr} \left\{ \chi_0(v_0 + a s(j)) \dagger V_{j}^{(1)}(-k, 0) \dagger 
\times \tilde{S}_{1}^{(0)}(u_0 + a s(i), v_0; -k) \dagger V_{i}^{(1)}(k, -k) \dagger 
\times \tilde{S}_{1}^{(0)}(x_0, u_0; 0) \dagger \gamma_0 \chi_2(x_0) \right\}, \tag{B.61}
\]

\[
(f_{1}^{stat(1)})_{1a} = -2a^2 \sum_{k} \sum_{u_0, v_0=a, i,j=1}^{T-a} d_{\mu(i)\mu(j)}(u_0 + at(i), v_0 + at(j); k)
\times \text{tr} \left\{ \chi_0(v_0 + a s(j)) \dagger V_{j}^{(1)}(-k, 0) \dagger 
\times \tilde{S}_{1}^{(0)}(u_0 + a s(i), v_0; -k) \dagger V_{i}^{(1)}(k, -k) \dagger 
\times \tilde{S}_{1}^{(0)}(T - a, u_0; 0) \dagger P_{+} \right\}, \tag{B.62}
\]
\[(f^{(1)})_1 = -2 \frac{a^2}{L^3} \sum_{k} \sum_{u_0, v_0 = a}^{T-a} \sum_{i,j=1}^{16} d_{\mu(i), \mu(j)}(u_0 + at(i), v_0 + at(j); k) \]
\[
\times tr \left\{ \chi_1(v_0 + as(j)) V_j^{(1)}(-k, 0) \right\} \\
\times \tilde{S}_1^{(0)}(u_0 + as(i), v_0; -k) V_i^{(1)}(k, -k) \tilde{S}_2^{(0)}(T - a, u_0; 0) \right\}, \]
\[
(f^{\text{stat}})^{(1)}_3(x_0) = -2 \frac{a^2}{L^3} \sum_{k} \sum_{u_0, v_0 = a}^{T-a} \sum_{i=1}^{16} d_{\mu(i), 0}(u_0 - 1, v_0 + at(i), k) \]
\[
\times tr \left\{ \chi_1(v_0 + as(i)) V_i^{(1)}(k, 0) \right\} \\
\times \tilde{S}_1^{(0)}(x_0, v_0, k) \right\}, \]
\[
(f^{\text{stat}})^{(1)}_3(x_0) = -2 \frac{a^2}{L^3} \sum_{k} \sum_{u_0, v_0 = a}^{T-a} \sum_{i=1}^{16} d_{\mu(i), 0}(u_0 - 1, v_0 + at(i), k) \]
\[
\times tr \left\{ \chi_1(v_0 + as(i)) V_i^{(1)}(k, 0) \right\} \\
\times \tilde{S}_2^{(0)}(x_0, u_0; k) \right\}, \]
\[
(f_1)_3 = 2 \frac{a^2}{L^3} \sum_{k} \sum_{u_0, v_0 = a}^{T-a} \sum_{i,j=1}^{16} d_{\mu(i), \mu(j)}(u_0 + at(j), v_0 + at(i), k) \]
\[
\times tr \left\{ \chi_1(v_0 + as(i)) V_i^{(1)}(k, 0) \right\} \\
\times \tilde{S}_1^{(0)}(T - a, u_0; 0) \right\}, \]

with the notation explained in Section B.1–B.4. These formulae have been programed in FORTRAN, with fields, propagators and vertices treated as 4 × 4 arrays. This is not the most efficient way of programming the quantities above. Identifying their Dirac structure first, and then calculating with the corresponding coefficients, would have had the effect of speeding up the code. However, the direct matrix multiplication has been fast enough for our purposes.
B.6 Continuum limit of tree level correlation functions

For the determination of $c_A^{\text{stat}(1)}$ in Chapter 8, the continuum limits of certain tree-level functions are needed. These can be obtained from the continuum limits

\begin{align*}
L\omega(p^+) &\longrightarrow \omega_c = \sqrt{z_c^2 + \theta_k \theta_k}, \quad \text{(B.68)} \\
LM(p^+) &\longrightarrow z_c, \quad \text{(B.69)} \\
A(p^+) &\longrightarrow 1, \quad \text{(B.70)} \\
LR(p^+) &\longrightarrow R_c = z_c \left(1 - e^{-2\omega_c}\right) + L\omega_c \left(1 + e^{-2\omega_c}\right), \quad \text{(B.71)}
\end{align*}

for $p = 0$, where $z_c$ is the continuum limit of $Lm_0^{(0)}$. Inserting these into eq. (B.34), (B.39), (B.41) and (B.43), and setting $T = L$ and $z = 0$, one gets the required limits

\begin{align*}
f_A^{\text{stat}(0)}(L/2) &\longrightarrow -\frac{3\omega_c}{R_c} \left( e^{-\omega_c / 2} + e^{-3\omega_c / 2} \right), \quad \text{(B.72)} \\
f_1^{\text{stat}(0)} &\longrightarrow \frac{6\omega_c}{R_c} e^{-\omega_c}, \quad \text{(B.73)} \\
L^2 f_A^{\text{stat}(1)}(L/2)_b &\longrightarrow -\tilde{c}_1^{(1)} \frac{18\omega_c}{R_c^2} \left( e^{-\omega_c / 2} + e^{-3\omega_c / 2} \right) \left(1 - e^{-2\omega_c}\right) + 2 \left( e^{-\omega_c / 2} - e^{-3\omega_c / 2} \right) e^{-\omega_c}, \quad \text{(B.74)} \\
L(f_1^{\text{stat}(1)})_b &\longrightarrow \tilde{c}_1^{(1)} \frac{72\omega_c}{R_c^2} \left(1 - e^{-2\omega_c}\right) e^{-\omega_c}, \quad \text{(B.75)} \\
f_\delta^{\text{stat}(0)}(L/2) &\longrightarrow -\frac{3\theta_k \theta_k}{R_c} \left( e^{\omega_c / 2} - e^{-3\omega_c / 2} \right). \quad \text{(B.76)}
\end{align*}
Appendix C

Extrapolation

To extrapolate the one loop data to the continuum limit, the fitting procedure from [72] is used, which is a generalization of the blocking method described in [94].

From the numerical computations, we have a set of data $F(L/a)$, with rounding errors $\delta_{F(L/a)}$ due to the limited precision of floating point operations in computer calculations. The data in this thesis have been obtained using double precision (8 byte) real numbers. For some lattice sizes, the rounding errors were determined by comparison with quadruple precision results. These were then used to estimate the errors for other lattice sizes.

One assumes that the one loop coefficients allow an asymptotic expansion in terms of functions $f_k, k = 1, 2 \ldots$. Using the first $n_f$ of these functions, one can write

$$F(L/a) = \sum_{k=1}^{n_f} \alpha_k(L/a)f_k(L/a) + R(L/a), \quad (C.1)$$

where $R(L/a)$ vanishes in the limit $L/a \to \infty$.

Assuming that we have $n$ data points $F((L/a)_j)$ with

$$(L/a)_1 < (L/a)_2 < \ldots (L/a)_n \quad (C.2)$$

and

$$n \geq n_f, \quad (C.3)$$

we can write eq. (C.1) in matrix notation,

$$F = f\alpha + R, \quad (C.4)$$

where $f$ is an $n \times n_f$ matrix, and it is the aim to determine the $n_f$ dimensional column vector $\alpha$. This is done by minimizing the quadratic form

$$\chi^2 = (F - f\alpha)^\top W^2(F - f\alpha), \quad (C.5)$$
where \( W \) is an \( n \times n \) diagonal matrix, which can be used to give different weights to different data points. Minimization of eq. (C.5) yields

\[
 f^\top W^2 f \alpha = f^\top W^2 F.  
\]  
(C.6)

We assume that the functions \( f_k \) are chosen such that the columns of \( Wf \) are linearly independent. Then they span an \( n_f \) dimensional subspace of our \( n \) dimensional space. Denoting the projector onto this subspace by \( P \), eq. (C.6) can be rewritten,

\[
 Wf \alpha = PWF.  
\]  
(C.7)

To extract \( \alpha \) from this equation, a singular value decomposition [95] of \( Wf \) is applied, which means that \( Wf \) is decomposed as

\[
 Wf = USV^\top.  
\]  
(C.8)

In this decomposition, \( S \) and \( V \) are \( n_f \times n_f \) matrices, where \( S \) is diagonal and \( V \) is orthonormal, and the \( n \times n_f \) matrix \( U \) is column-orthonormal with

\[
 U^\top U = 1, \quad UU^\top = P.  
\]  
(C.9)

Having these matrices, we can solve for \( \alpha \),

\[
 \alpha = VS^{-1}U^\top W F.  
\]  
(C.10)

The errors of \( F(L/a) \) are treated like statistical errors, i.e. they are propagated quadratically,

\[
 \delta_{\alpha_k}^2 = \sum_{(L/a)} (VS^{-1}U^\top W)_{k,(L/a)}^2 \delta_{F(L/a)}^2.  
\]  
(C.11)

This is a bit questionable, as the rounding effects are not really statistical errors. However, there does not seem to be a better solution.

The second error source is the truncation of the fit function after the first \( n_f \) terms. To estimate the size of this systematic error, the amount by which the coefficient \( \alpha_k \) changes when including one of the next two functions in the fit was calculated, and the maximum of these two used as the systematic error. This is again not a very convincing method, but about all one can do, and the resulting errors appear to be rather sensible.

As a further check on the consistency of the method, all fits have been calculated for several values of \( n \), ranging from the minimal value \( n_f \), using the \( n_f \) largest available lattice sizes, up to the complete set of data we have. As long as \( (L/a) \) is not too small in the data points included in the fit, the coefficients turn out to be rather stable under this variation of \( n \). The process of estimating errors is explained in detail below, on the example of the \( 1/z \) extrapolation to get \( B_A^{sat} \), eq. (7.97).
In detail, the following fits have been performed, with the choice $W = 1$. The lattice MS-renormalized ratio $X_{1,\text{lat}}^{(1)}$ from tables D.2 and D.3 was fitted with a function of the form

$$f_0 + f_1 \left( \frac{a}{L} \right)^2 + f_2 \left( \frac{a}{L} \right)^2 \ln(a/L) + f_3 \left( \frac{a}{L} \right)^3 + f_4 \left( \frac{a}{L} \right)^3 \ln(a/L),$$  \hspace{1cm} (C.12)

where $f_0$ is the continuum limit one wants to calculate. For the estimation of the systematic error, $(a/L)^4$ and $(a/L)^4 \ln(a/L)$ were included. To check whether $X_{1,\text{lat}}^{(1)}$ is really $O(a)$ improved, a fit which also contained $(a/L)$ and $(a/L) \ln(a/L)$ has been applied. In the $\theta = 0.5$ case, for example, this gave 0.03(9) for the $(a/L)$ coefficient and $-0.007(18)$ for the $(a/L) \ln(a/L)$ coefficient. These have to be compared with the $O((a/L)^2)$ terms, $f_1 = 0.9(3)$ and $f_2 = -0.20(6)$. This shows that the $(a/L)$ terms are small, and within their errors compatible with zero, thus confirming that $O(a)$ improvement does work, which already has been the first impression from the inspection of Figure 7.1.

In the extrapolation of $Y_1^{(1)}$, a fit of the same form as for $X_{1,\text{lat}}^{(1)}$ was used up to $z = 5$. For $z > 5$, fits of this form yield increasingly large errors, and for those $z$ values a fit of the form

$$f_0 + f_1 \left( \frac{a}{L} \right)^2 + f_2 \left( \frac{a}{L} \right)^2 \ln(a/L)$$  \hspace{1cm} (C.13)

was used. For the determination of the systematic error, the next two terms $(\frac{a}{L})^3$ and $(\frac{a}{L})^3 \ln(a/L)$ were included. Some examples concerning fits for $Y_1^{(1)}$ are shown in Figure 7.6.

The fitting procedure described above has also been used for the extrapolation of $B_A^{\text{stat}}$ to $1/z = 0$. Here the data were fitted with

$$f_0 + f_1 \frac{1}{z} + f_2 \frac{1}{z} \ln(z),$$  \hspace{1cm} (C.14)

and the error was estimated using $(1/z)^2$ and $(1/z)^2 \ln(z)$. The fit shows that the $(1/z) \ln(z)$ term plays a dominating rôle. In detail, the fit is performed with the data from Table D.11 including the points with $2.5 \leq z \leq 8.0$, giving a value of

$$(B_A^{\text{stat}})_0 = -0.13716(52),$$  \hspace{1cm} (C.15)

where the error is the one obtained by quadratic error propagation from the errors of the single data points. Then, the fit is repeated, but this time, a term $f_3 (1/z)^2$ is added to the fit function (C.14). This fit leads to the result

$$(B_A^{\text{stat}})_1 = -0.13726(192).$$  \hspace{1cm} (C.16)

We keep the difference of $(B_A^{\text{stat}})_1$ and $(B_A^{\text{stat}})_0$ in mind,

$$(\Delta B_A^{\text{stat}})_1 = 0.00010.$$  \hspace{1cm} (C.17)
Then we repeat the fit, this time adding $f_4(1/z)^2 \ln(z)$ to the function (C.14), which yields

$$ (B_\text{A}^{\text{stat}})_{2} = -0.13719(127). $$  \hspace{1cm} (C.18)

Again, we calculate the difference of $(B_\text{A}^{\text{stat}})_{2}$ and $(B_\text{A}^{\text{stat}})_0$ and obtain

$$ (\Delta B_\text{A}^{\text{stat}})_{2} = 0.0003. $$  \hspace{1cm} (C.19)

The maximum of $(\Delta B_\text{A}^{\text{stat}})_{1}$ and $(\Delta B_\text{A}^{\text{stat}})_{2}$, which is $(\Delta B_\text{A}^{\text{stat}})_{1}$ is taken to be the systematic error, and we write

$$ B_\text{A}^{\text{stat}} = -0.1372(5)(1), $$  \hspace{1cm} (C.20)

where the first error is the propagated one, the second error is the systematic one. However, a variation of the number $n$ of included data points shows that this estimate might be a bit too optimistic. Excluding the point $z = 2.5$ from the fit, we get

$$ B_\text{A}^{\text{stat}} = -0.1371(7)(13), $$  \hspace{1cm} (C.21)

and including $z = 2$ gives

$$ B_\text{A}^{\text{stat}} = -0.1377(4)(18). $$  \hspace{1cm} (C.22)

Repeating the calculation for all possible $n$ shows that the third digit is stable, the fourth digit is not. I finally decided to use a more conservative estimate than (C.20),

$$ B_\text{A}^{\text{stat}} = -0.137(1). $$  \hspace{1cm} (C.23)

In the fits for $X_{1,\text{lat}}$ and $Y_{1}$, this complication did not occur, and the estimated systematic error was consistent with the results obtained by variation of $n$.

For the calculation of the static axial currents $O(a)$ improvement term, $\frac{1}{2}(\partial + \partial')X_{\text{lat}}^{(1)}(L/a)|_{a=1}$ has been fitted with

$$ f_0 + f_1 \left( \frac{a}{L} \right) + f_2 \left( \frac{a}{L} \right) \ln(a/L), $$  \hspace{1cm} (C.24)

and $(a/L)^2$ and $(a/L)^2 \ln(a/L)$ were used to determine the systematic error.
Appendix D

Numerical results

<table>
<thead>
<tr>
<th>$L/a$</th>
<th>$m^{(1)}_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>$-0.274824120653$</td>
</tr>
<tr>
<td>6</td>
<td>$-0.270958239932$</td>
</tr>
<tr>
<td>8</td>
<td>$-0.270212641443$</td>
</tr>
<tr>
<td>10</td>
<td>$-0.270103882639$</td>
</tr>
<tr>
<td>12</td>
<td>$-0.270082783452$</td>
</tr>
<tr>
<td>14</td>
<td>$-0.270077422224$</td>
</tr>
<tr>
<td>16</td>
<td>$-0.270075781321$</td>
</tr>
</tbody>
</table>

Table D.1: The one loop coefficient of the critical quark mass $m_c$ calculated from eq. (7.101) at $\theta = 0.0$. The errors are smaller than the last given digit.
| $L/a$ | $X_{1,\text{lat}}^{(1)}(L/a)|_{\theta=0.0}$ | $X_{1,\text{lat}}^{(1)}(L/a)|_{\theta=0.5}$ |
|-------|---------------------------------|----------------------------------|
| 4     | $-0.0689208524812999(4)$        | $-0.1149433534610605(7)$        |
| 6     | $-0.094739776327626$            | $-0.125049528337982$            |
| 8     | $-0.10229450172845(1)$          | $-0.13211380614649(1)$          |
| 10    | $-0.10545788153831$             | $-0.1337427349878$              |
| 12    | $-0.10705997283293(8)$          | $-0.13450844932370(5)$          |
| 14    | $-0.1079798825075$              | $-0.13490847555555$             |
| 16    | $-0.1085575317511(5)$           | $-0.135140278122(1)$            |
| 18    | $-0.1089448340286$              | $-0.1352851188882$              |
| 20    | $-0.109217620780(2)$            | $-0.1353809450545$              |
| 22    | $-0.109417209915$               | $-0.1354471509440$              |
| 24    | $-0.109567736884$               | $-0.135494457696$               |
| 26    | $-0.109684105785$               | $-0.135529177256$               |
| 28    | $-0.10977594270$                | $-0.135555220124$               |
| 30    | $-0.10984969816$                | $-0.135575111120$               |
| 32    | $-0.10990892995$                | $-0.1359053658$                 |
| 34    | $-0.10999590030$                | $-0.1360265514$                 |
| 36    | $-0.1100010036$                 | $-0.13561228225$                |
| 38    | $-0.1100360377$                 | $-0.13562000385$                |
| 40    | $-0.1100658808$                 | $-0.1356262487$                 |
| 42    | $-0.1100915102$                 | $-0.1356313355$                 |
| 44    | $-0.110136835$                  | $-0.1356355047$                 |
| 46    | $-0.1101329954$                 | $-0.1356389399$                 |
| 48    | $-0.1101499180(2)$              | $-0.1356417830(2)$              |

Table D.2: The one loop coefficient of $X_{1,\text{lat}}$ for $\theta = 0.0$ and $\theta = 0.5$, calculated at double precision. Where an error is given, it has been determined by comparison with the quadruple precision result.
| $L/a$ | $X_{I, \text{lat}}^{(1)} (L/a)|_{\theta=1.0}$ |
|-------|------------------------------------------|
| 4     | $-0.1444256774452487(1)$                |
| 6     | $-0.1504774828123731$                   |
| 8     | $-0.151311015610704(4)$                |
| 10    | $-0.151014840631847$                   |
| 12    | $-0.15053624553571(1)$                |
| 14    | $-0.15010230033606$                   |
| 16    | $-0.1497502382763(1)$                   |
| 18    | $-0.1494716172749$                   |
| 20    | $-0.1492507429573(5)$                   |
| 22    | $-0.1490737923469$                   |
| 24    | $-0.148930173449$                   |
| 26    | $-0.148812071979$                   |
| 28    | $-0.148713753727$                   |
| 30    | $-0.148630981187$                   |
| 32    | $-0.14856058588$                   |
| 34    | $-0.14850016711$                   |
| 36    | $-0.14844788156$                   |
| 38    | $-0.14840229572$                   |
| 40    | $-0.14836228142$                   |
| 42    | $-0.14832694085$                   |
| 44    | $-0.14829555218$                   |
| 46    | $-0.14826752954$                   |
| 48    | $-0.14824239323(6)$                   |

Table D.3: The one loop coefficient of $X_{I, \text{lat}}$ for $\theta = 1.0$, calculated at double precision. Where an error is given, it has been determined by comparison with the quadruple precision result.
| $L/a$ | $Y_1^{(1)}(L/a)_{|z=1.0}$ | $Y_1^{(1)}(L/a)_{|z=1.5}$ |
|-------|-----------------------------|-----------------------------|
| 4     | $-0.034799450182243(4)$     | $-0.046726542279477(1)$     |
| 6     | $-0.05480691134963$         | $-0.064482508279915$        |
| 8     | $-0.06256988654455(3)$      | $-0.07153366402190(2)$      |
| 10    | $-0.06637419380832$         | $-0.07503739743745$         |
| 12    | $-0.06850667918060(2)$      | $-0.07702580717532(4)$      |
| 14    | $-0.0698174034465$          | $-0.0782612731973$          |
| 16    | $-0.0706792278524(5)$       | $-0.0790812489264(4)$       |
| 18    | $-0.071275689810$           | $-0.0796533793141$          |
| 20    | $-0.071705412778$           | $-0.080068526471$           |
| 22    | $-0.072025166551$           | $-0.080379400924$           |
| 24    | $-0.072269484061$           | $-0.080618291063$           |
| 26    | $-0.07246034290$            | $-0.08080587570$            |
| 28    | $-0.07261226689$            | $-0.0809558954$             |
| 30    | $-0.07273516469$            | $-0.08107785497$            |
| 32    | $-0.07283598335$            | $-0.08117817487$            |
| 34    | $-0.07291971004$            | $-0.08126185597$            |
| 36    | $-0.0729899998$             | $-0.0813323515$             |
| 38    | $-0.0730495806$             | $-0.0813923013$             |
| 40    | $-0.0731005219$             | $-0.0814437150$             |
| 42    | $-0.0731444165$             | $-0.0814881444$             |
| 44    | $-0.073185064$              | $-0.0815268037$             |
| 46    | $-0.0732157717$             | $-0.0815606536$             |
| 48    | $-0.0732449939$             | $-0.0815904626$             |

Table D.4: The one loop coefficient of $Y_1$ for $z = 1.0$ and $z = 1.5$ at $\theta = 0.5$, calculated at double precision. Where an error is given, it has been determined by comparison with the quadruple precision result.
| \(L/a\) | \(Y_1^{(1)}(L/a)\)\(|_{z=2.0}\) | \(Y_1^{(1)}(L/a)\)\(|_{z=2.5}\) |
|---|---|---|
| 6  | -0.07553623375670 | -0.086393020146079 |
| 8  | -0.08151028318403(1) | -0.091525082394953(8) |
| 10 | -0.08458623081636 | -0.09408708524314 |
| 12 | -0.08637681185566(2) | -0.09563345987252(4) |
| 14 | -0.0875112560049 | -0.0966421621022 |
| 16 | -0.0882761259022(7) | -0.0973380782442(6) |
| 18 | -0.088816864568 | -0.097839360230 |
| 20 | -0.089213666548(3) | -0.098213004967(3) |
| 22 | -0.089513724985 | -0.098499348738 |
| 24 | -0.089746304950 | -0.098723885295 |
| 26 | -0.08993035038 | -0.09890338694 |
| 28 | -0.09007857161 | -0.09904926659 |
| 30 | -0.09019975953 | -0.09916951603 |
| 32 | -0.09030015464 | -0.09926987140 |
| 34 | -0.09038428824 | -0.09935453921 |
| 36 | -0.0904555163 | -0.0994266629 |
| 38 | -0.0905163682 | -0.0994886312 |
| 40 | -0.0905687800 | -0.0995422863 |
| 42 | -0.0906142547 | -0.0995890681 |
| 44 | -0.0906539734 | -0.0996301155 |
| 46 | -0.0906888752 | -0.0996663395 |
| 48 | -0.0907197143 | -0.0996984759 |

Table D.5: The one loop coefficient of \(Y_1\) for \(z = 2.0\) and \(z = 2.5\) at \(\theta = 0.5\), calculated at double precision. Where an error is given, it has been determined by comparison with the quadruple precision result.
| $L/a$ | $Y_1^{(1)}(L/a)|_{z=3.0}$ | $Y_1^{(1)}(L/a)|_{z=3.5}$ |
|-------|--------------------------|--------------------------|
| 8     | -0.100939176319015(5)    | -0.108301927893913(8)   |
| 10    | -0.10308960144557        | -0.11134170256066       |
| 12    | -0.10437324428024(6)     | -0.1124299071542(3)     |
| 14    | -0.1052391485069         | -0.11315386179243       |
| 16    | -0.1058549181752(5)      | -0.1136844572075(3)     |
| 18    | -0.1063096402219         | -0.1140880494711        |
| 20    | -0.106655607057(3)       | -0.114402978776(2)      |
| 22    | -0.106925343115          | -0.114653771652         |
| 24    | -0.107139984952          | -0.114856933526         |
| 26    | -0.10731377072           | -0.11502394736          |
| 28    | -0.10745658650           | -0.11516301324          |
| 30    | -0.10757547702           | -0.11528011760          |
| 32    | -0.10767557662           | -0.11537971562          |
| 34    | -0.10776070175           | -0.11546517973          |
| 36    | -0.1078337391            | -0.1155391018           |
| 38    | -0.1078969060            | -0.1156035013           |
| 40    | -0.1079519295            | -0.1156599708           |
| 42    | -0.1080001717            | -0.1157097808           |
| 44    | -0.1080427188            | -0.1157539550           |
| 46    | -0.1080804462            | -0.1157933259           |
| 48    | -0.1081140658            | -0.1158285764           |

Table D.6: The one loop coefficient of $Y_1$ for $z = 3.0$ and $z = 3.5$ at $\theta = 0.5$, calculated at double precision. Where an error is given, it has been determined by comparison with the quadruple precision result.
| $L/a$ | $Y^{(1)}_1(L/a)|_{z=4.0}$ | $Y^{(1)}_1(L/a)|_{z=4.5}$ |
|-------|-----------------------------|-----------------------------|
| 10    | -0.11845797720597           | -0.1225923233833           |
| 12    | -0.1197285721389(1)        | -0.12615269408065(4)       |
| 14    | -0.1203586028376           | -0.1268605744892           |
| 16    | -0.1208115019015(7)        | -0.1272699667877(6)        |
| 18    | -0.1211640903287           | -0.1275803934167           |
| 20    | -0.121446719199(2)         | -0.127833049472(2)         |
| 22    | -0.121677271361            | -0.128043767466            |
| 24    | -0.121867913790(7)         | -0.128221736087(8)         |
| 26    | -0.122027395254            | -0.12837341178             |
| 28    | -0.12216218551             | -0.12850367795(2)          |
| 30    | -0.12227716025             | -0.12861633972             |
| 32    | -0.12237605012             | -0.12871440674             |
| 34    | -0.12246174780             | -0.12880028329             |
| 36    | -0.1225365232             | -0.1288759059              |
| 38    | -0.1226021771             | -0.1289428463              |
| 40    | -0.1226601530             | -0.1290023889              |
| 42    | -0.1227116183             | -0.1290555909              |
| 44    | -0.1227575257             | -0.1291033281              |
| 46    | -0.1227986589             | -0.1291463303              |
| 48    | -0.1228356671             | -0.1291852098              |

Table D.7: The one loop coefficient of $Y_1$ for $z = 4.0$ and $z = 4.5$ at $\theta = 0.5$, calculated at double precision. Where an error is given, it has been determined by comparison with the quadruple precision result.
| $L/a$ | $Y_1^{(1)}(L/a)|_{z=5.0}$ | $Y_1^{(1)}(L/a)|_{z=6.0}$ |
|-------|--------------------------|--------------------------|
| 12    | -0.131252103580717(2)    | -0.1409824342089         |
| 14    | -0.13263728584142        | -0.1428661785797(3)     |
| 16    | -0.133095728079(6)       | -0.1433883403059        |
| 18    | -0.1333936211493         | -0.143658256402(2)      |
| 20    | -0.133624914496(2)       | -0.143850642084         |
| 22    | -0.133819002104          | -0.144008281430(8)      |
| 24    | -0.133985543249(8)       | -0.14414474710          |
| 26    | -0.13412989840           | -0.14426550789(2)       |
| 28    | -0.13425582514(2)        | -0.14437340829          |
| 30    | -0.13436624644           | -0.14447030673          |
| 32    | -0.13446352982           | -0.14455761633          |
| 34    | -0.13454962353           | -0.1446365000           |
| 36    | -0.1346261428            | -0.1447079495           |
| 38    | -0.1346944329            | -0.1447728229           |
| 40    | -0.1347556190            | -0.1448318670           |
| 42    | -0.1348106456            | -0.1448857334           |
| 44    | -0.1348603091            | -0.1449349913           |
| 46    | -0.1349052833            | -0.1449801383           |

Table D.8: The one loop coefficient of $Y_1$ for $z = 5.0$ and $z = 6.0$ at $\theta = 0.5$, calculated at double precision. Where an error is given, it has been determined by comparison with the quadruple precision result.
| $L/a$ | $Y_1^{(1)}(L/a)|_{z=8.0}$ | $Y_1^{(1)}(L/a)|_{z=12.0}$ |
|-------|-----------------------------|-----------------------------|
| 18    | -0.1547482175608            |                             |
| 20    | -0.158020230955(2)          | -0.170396740023             |
| 22    | -0.158954201907             | -0.17669557845(1)          |
| 24    | -0.159372775121(6)          | -0.17879414394             |
| 26    | -0.159619711304             | -0.17979334032             |
| 28    | -0.15979460424(2)           | -0.18036719192             |
| 30    | -0.16005079012              | -0.1807408706              |
| 32    | -0.16015398356              | -0.1810076697              |
| 34    | -0.1602465322               | -0.1812118180              |
| 36    | -0.160305381                | -0.1813763759              |
| 38    | -0.1604073525               | -0.1815142771              |
| 40    | -0.1604779293               | -0.1816332205              |
| 42    | -0.1605429955               | -0.1817380289              |
| 44    | -0.1606031377               | -0.1818318635              |
| 46    | -0.1606588472               | -0.181916883               |
| 48    | -0.1607050228               | -0.181994619               |
| 50    | -0.1607588472               | -0.182066196               |
| 52    | -0.1608031377               | -0.182132465               |
| 54    | -0.1608429955               | -0.182194091               |
| 56    | -0.1608828322               | -0.182251605               |
| 58    | -0.1609226683               | -0.182305442               |

Table D.9: The one loop coefficient of $Y_1$ for $z = 8.0$ and $z = 12.0$ at $\theta = 0.5$, calculated at double precision. Where an error is given, it has been determined by comparison with the quadruple precision result.
| $L/a$ | $Y_i^{(1)}(L/a)|_{z=16.0}$ |
|-------|---------------------------|
| 34    | -0.17852969527            |
| 36    | -0.1877057352             |
| 38    | -0.1910206343             |
| 40    | -0.1926899299             |
| 42    | -0.1936802091             |
| 44    | -0.1943320419             |
| 46    | -0.1947941950             |
| 48    | -0.1951409161             |
| 50    | -0.1954128468             |
| 52    | -0.195633817              |
| 54    | -0.195818575              |
| 56    | -0.195976659              |
| 58    | -0.196114471              |
| 60    | -0.196236450              |
| 62    | -0.196345760              |
| 64    | -0.196444718              |
| 66    | -0.196535057              |
| 68    | -0.196618102              |
| 70    | -0.196694885              |
| 72    | -0.19676623               |
| 74    | -0.19683279               |
| 76    | -0.19689511               |
| 78    | -0.19695364               |
| 80    | -0.19700876               |

Table D.10: The one loop coefficient of $Y_i$ for $z = 16.0$ at $\theta = 0.5$, calculated at double precision.
<table>
<thead>
<tr>
<th>$z$</th>
<th>$\hat{B}_A^{\text{stat}}(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>$-0.143566(1)$</td>
</tr>
<tr>
<td>1.5</td>
<td>$-0.150940(1)$</td>
</tr>
<tr>
<td>2.0</td>
<td>$-0.154210(1)$</td>
</tr>
<tr>
<td>2.5</td>
<td>$-0.155499(1)$</td>
</tr>
<tr>
<td>3.0</td>
<td>$-0.155796(1)$</td>
</tr>
<tr>
<td>3.5</td>
<td>$-0.155595(1)$</td>
</tr>
<tr>
<td>4.0</td>
<td>$-0.155152(1)$</td>
</tr>
<tr>
<td>4.5</td>
<td>$-0.154603(2)$</td>
</tr>
<tr>
<td>5.0</td>
<td>$-0.154018(6)$</td>
</tr>
<tr>
<td>6.0</td>
<td>$-0.15288(2)$</td>
</tr>
<tr>
<td>8.0</td>
<td>$-0.1510(1)$</td>
</tr>
<tr>
<td>12.0</td>
<td>$-0.1484(4)$</td>
</tr>
<tr>
<td>16.0</td>
<td>$-0.1468(6)$</td>
</tr>
</tbody>
</table>

Table D.11: $\hat{B}_A^{\text{stat}}$ at $\theta = 0.5$. The errors have been determined from the $a/L \to 0$ extrapolation as explained in Appendix C.
\[
L/a \quad \frac{L^2}{\alpha} (\partial^{\prime} + \partial) X^{(1)}_{1, \text{lat}} (L/a) |_{\tilde{c}_i=1, \theta=0.5} \quad \frac{L^2}{\alpha} (\partial^{\prime} + \partial) X^{(1)}_{1, \text{lat}} (L/a) |_{\tilde{c}_i=1, \theta=1.0}
\]

| $L/a$ | $\frac{L^2}{\alpha} (\partial^{\prime} + \partial) X^{(1)}_{1, \text{lat}} (L/a) |_{\tilde{c}_i=1, \theta=0.5}$ | $\frac{L^2}{\alpha} (\partial^{\prime} + \partial) X^{(1)}_{1, \text{lat}} (L/a) |_{\tilde{c}_i=1, \theta=1.0}$ |
|-------|-------------------------------------------------|-------------------------------------------------|
| 5     | -0.1045003413087                                 | 0.0652091607382                                 |
| 7     | -0.0400431529453                                 | 0.129409479246                                  |
| 9     | -0.006575748785                                 | 0.165233011236                                  |
| 11    | 0.013544685919                                   | 0.183838106777                                  |
| 13    | 0.02580031427                                   | 0.192469416927                                  |
| 15    | 0.03352854355                                   | 0.19597196491                                  |
| 17    | 0.03867716607                                   | 0.19699684365                                  |
| 19    | 0.0423117590                                   | 0.1968611090                                  |
| 21    | 0.0450105579                                   | 0.1961964482                                  |
| 23    | 0.0470966807                                   | 0.1953042820                                  |
| 25    | 0.048759688                                   | 0.194329553                                   |
| 27    | 0.050117081                                   | 0.193342560                                   |
| 29    | 0.051245649                                   | 0.192377318                                   |
| 31    | 0.052197955                                   | 0.191449774                                   |
| 33    | 0.05301138                                   | 0.19056667                                   |
| 35    | 0.05371337                                   | 0.18972997                                   |
| 37    | 0.0543246                                   | 0.1889392                                   |
| 39    | 0.0548608                                   | 0.1881925                                   |
| 41    | 0.0553346                                   | 0.1874876                                   |
| 43    | 0.0557556                                   | 0.1868218                                   |
| 45    | 0.0561319                                   | 0.1861925                                   |
| 47    | 0.0564698                                   | 0.1855971                                   |

Table D.12: The one loop coefficient of $\frac{L^2}{\alpha} (\partial^{\prime} + \partial) X_{\text{lat}}$ for the determination of the improvement coefficient $c_A^{\text{stat}(1)}$. Values are for $\tilde{c}_i = 1$ at $\theta = 0.5$ and $\theta = 1.0$, calculated at double precision. Estimating the accuracy from tables D.2 and D.3, the errors are expected to be of the order of the last given digit.
Appendix E

The renormalized coupling in the SF scheme

As renormalized coupling, a quantity is suitable if it is finite on removal of the regulator, and depends on exactly one scale. In the SF scheme, the coupling $g_{SF}$ is defined using the Schrödinger functional including a gauge field and light quark fields, with non-zero boundary conditions for the gauge field [41]. One assumes that one has a solution $B_0(x)$ of the classical equations of motion for the gauge field, and solutions $\psi_{1,cl}$ and $\bar{\psi}_{1,cl}$ of the Dirac equation, that satisfy the boundary conditions in eq. (5.2), (5.3), (5.8), and (5.9). Furthermore, one assumes that the action satisfies

$$S[A, \bar{\psi}_{1,cl}, \psi_{1,cl}] > S[B, \bar{\psi}_{1,cl}, \psi_{1,cl}],$$

(E.1)

as long as $A$ is not a gauge transform of the classical solution $B$. One can then introduce the effective action

$$\Gamma[B, \bar{\psi}_{1,cl}, \psi_{1,cl}] = -\ln Z[C', \bar{\rho}_1', \rho_1'; C, \bar{\rho}_1, \rho_1],$$

(E.2)

and its perturbative expansion,

$$\Gamma[B, \bar{\psi}_{1,cl}, \psi_{1,cl}] = \frac{1}{g_0^2} \Gamma_0[B, \bar{\psi}_{1,cl}, \psi_{1,cl}] + \Gamma_1[B, \bar{\psi}_{1,cl}, \psi_{1,cl}] + g_0^2 \Gamma_2[B, \bar{\psi}_{1,cl}, \psi_{1,cl}] + O(g_0^4).$$

(E.3)

One then chooses the specific boundary conditions

$$C_k = \frac{i}{L} \left( \begin{array}{ccc} \phi_1 & 0 & 0 \\ 0 & \phi_2 & 0 \\ 0 & 0 & \phi_3 \end{array} \right), \quad C'_k = \frac{i}{L} \left( \begin{array}{ccc} \phi'_1 & 0 & 0 \\ 0 & \phi'_2 & 0 \\ 0 & 0 & \phi'_3 \end{array} \right),$$

(E.4)

where $k = 1, 2, 3$, and the matrix entries $\phi_1 \ldots \phi'_3$ only depend on one single parameter $\eta$,

$$\phi_1 = \eta - \frac{\pi}{3}, \quad \phi'_1 = -\phi_1 - \frac{4\pi}{3},$$

$$\phi_2 = -\frac{1}{2} \eta, \quad \phi'_2 = -\phi_3 + \frac{2\pi}{3},$$

$$\phi_3 = -\frac{1}{2} \eta + \frac{\pi}{3}, \quad \phi'_3 = -\phi_2 + \frac{2\pi}{3}. \quad (E.5)$$

99
After introducing

$$\Gamma'[B, \bar{\psi}_{1,\text{cl}}, \psi_{1,\text{cl}}] = \frac{\partial}{\partial \eta} \Gamma[B, \bar{\psi}_{1,\text{cl}}, \psi_{1,\text{cl}}]|_{\eta=0},$$

(E.6)

one finally defines the renormalized coupling by

$$g_{SF}^2 (L) = \frac{\Gamma'[B, \bar{\psi}_{1,\text{cl}}, \psi_{1,\text{cl}}]}{\Gamma'[B, \bar{\psi}_{1,\text{cl}}, \psi_{1,\text{cl}}]}.$$

(E.7)

Note that this definition is not restricted to a particular regularization. It is a non-perturbative definition suitable for Monte Carlo treatment, but it can as well be expanded in the coupling and then be dealt with in dimensional regularization [69].
Bibliography


[34] A.S. Kronfeld, Application of heavy-quark effective theory to lattice QCD. I: Power corrections, hep-lat/0002008.


[93] M. Lüscher, Improved axial current at one-loop order of perturbation theory, internal notes of the ALPHA Collaboration.


Lebenslauf

Name: Martin Kurth
geboren am 10.04.1971 in Berlin
Nationalität: Deutsch

6/1991 Abitur an der Lessing-Oberschule (Gymnasium) in Berlin-Wedding
10/1991 - 01/1997 Studium an der Technischen Universität Berlin in der Fachrichtung Physik
seit 05/1997 Wissenschaftlicher Mitarbeiter beim Deutschen Elektronensynchrotron DESY, Zeuthen

Publikationen

Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfaßt zu haben und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Martin Kurth
22. Mai 2000