Higher gap morasses

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Abstract

Velleman in [Velleman(1987)] proved the consistency of the existence of simpli fied gap 2 morasses (equivalent to the concrete morasses defined by Jensen) using a two stage forcing. We gave an essentially different proof of the same result and fill up some details from the Velleman's paper which were not clear. In fact the proof uses a slightly simpler and different definition of gap two simplified morasses and of the forcing conditions. We have eliminated the use of square-like sequences in the second stage, employing instead a "guessing" procedure for requirement. With these steps we hope to have laid the foundation for a future proof of gap n morasses in ZFC.

Keywords:
Logic, set theory, combinatorics, morasses

Zusammenfassung

Velleman im [Velleman(1987)] beweist die Konsistenz der Existenz vereinfachte Gap 2 Moraste (ein Begriff gleichwertig zu den ursprünglichen Morasten, geschafft von Jensen). Wir haben einen noch einfachen Begriff des Morastes in der Dissertation vorgeschlagen, Details aufgefüllt und wesentlich auch einen verschiedenen Beweis des Satzes erfunden und zwar in beide Stufe des Forcingverfahrens. Wir benötigen auch keine Squarefunktionereihenfolge (die ganz Kohärenzvoraussetzung fehlt aber ist linear und konfinal) sondern ein erratendes Verfahren für Sequenze, das nicht fest ist und nicht die ganze Kohärenzbedigung erfüllt wie bei Velleman. Wir hoffen, wir haben so eingelegt die Basis für einen zukunftigen Beweis des allgemeines Falls n in ZFC.

Schlagwörter:
Logik, Mengenlehre, kombinatorische Mathematik, Moraste

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Chapter 1

Introduction

In the 70's R. Jensen ([Jensen(1975)]) introduced the notion of gap 1 morass to solve initially the gap 2 principle of model theory, a generalization of the Lowenheim-Skolem theorem, namely given a structure of size κ^{++} with a first order predicate A of size κ (a (κ^{++}, κ)-structure) for κ regular cardinal (greater than ω) get a new (λ^{++}, λ)-structure for every regular cardinal $\lambda > \kappa$ elementary equivalent to the first one. These gap 1 morasses approximate a structure of size κ^{++} through strutures of size κ^{+} without increasing the size of the predicate A. Gap 1 morasses have also a great field of applications, they can be used to carry out constructions which could not be carry out in ZFC alone, namely combinatoric problems like existence of Kurepa trees, diamonds or squares notions in set theory (see [Velleman(1982)]), without mention the problemas in topology or algebra.

Succesfully Jensen managed to prove the existence of gap 1 morasses in L using strongly inner model features of L what is called now finestructure and additionally solved the gap 2 principle (in fact, morasses describe a special section of the L-hierarchy, we could say overkilling the gap 2 principle problem). So the existence of morasses is consistent with ZFC. Moreover they can be added by forcing to ZFC but its existence is not provable in ZFC alone.

The next step was to consider the gap-3 principle i.e. consider now (κ^{+++}, κ)-structures for κ regular but already the definition of gap 2 morasses introduced also by Jensen was quite intrincate, making very difficult already to prove alone its existence. So many who wanted to used the advantages of morasses and its applications left them to the very few morass "experts" and very little advanced was done since then.

Looking for a principle which explained why so many statements provable in L were also true in a forcing extension, for a special family of partial orders, Velleman found a forcing principle equivalent to the concrete morasses [Velleman(1982)]. In fact this forcing principle is a kind of Martin axiom, from which Velleman took the esence of the morass and hence deduce its simplified morass in [Velleman(1984a)] (of course a lot of work was done in this direction before (see [Kanamori(1982)] and [Shelah and Stanley(1982)]). So these simplified gap 1 morasses were equivalent to the original Jensen's definition but much simpler, more useful and clearer to understand the sort of constructions for which morasses could be applied to.

In [Velleman(1984b)] Velleman also proves that the existence of simplified gap 1 morass plus a weak form of square (a linear limit sequence of functions) is equivalent to a Martin's Axiomtype which allows to deduce many generalizations of several combinatorial principles known to follow from the existence of morasses. Simplified $(\kappa, 1)$ morasses with linear limits exist already in L for κ regular but not weakly compact [Donder(1985)].

Velleman in [Velleman(1987)] introduced also the simplified $(\kappa, 2)$ morass (for κ regular cardinal greater or equal than ω), using the notion of simplified gap 1 morass. He proves there that this gap 2 simplified morasses is consistence with ZFC using a two stage forcing. In the first step he added a (**neat**) simplified $(\kappa^+, 1)$ morass with *linear limits* and in the second stage the simplified gap 2 morass (in L Jensen's morasses implies the existence of neat morasses).

We gave an essentially different proof of the same result and fill up some details from the Velleman's paper ([Velleman(1987)]) which were not clear. In fact the proof uses a slightly simpler and different definition of gap two simplified morasses and of the forcing conditions in both stages. We have also eliminated the use of square-like sequences in the second stage, employing instead a "guessing" procedure for sequences which is not fixed and does not satisfy the full coherence requirement and used a two family notion (the identity and the "shift" function like in the first forcing stage) in the succesor steps of the second forcing stage instead of an infinite family of left branching embeddings.

Most of velleman's paper as this work are devoted to provide enough conditions to preserve cardinals in every forcing step (in fact we provide less of these Velleman's conditions). We have to garantize in deep the chain conditions and enougn clousure in the two forcingstages. The four so called amalgamation Lemmata garantize the compatibility of the conditions in the second forcing step (to find a counterexample to the antichain) and some of them to provide of course clousure. The first step ist quite easy to do so.

Let κ be a regular cardinal greater or equal than ω and M our ground model such that satisfies $2^{<\kappa} = \kappa$ and $2^{\kappa} = \kappa^+$. These are necessary conditions to prove κ^+-c.c. and κ^{++}-c.c. of $\mathbb{P}_0$ and of $\mathbb{P}_1$ respectively.

In the first step forcing $\mathbb{P}_1$ we added a simplified $(\kappa^+, 1)$ morass with gaps of size κ,i.e. only defined for $\kappa\rho \leq \kappa^+$. This morass will be fill up with the second forcing. Since the first step forcing or also here called *upper* forcing is quite simple we do not spend much time here, just note that there we added the linear limits functions or what we called *good sequences.* The rolle of these sequences is to garantize the κ^+-clousure of the *lower* forcing or second step forcing $\mathbb{P}_0$. Let G a $\mathbb{P}_1$-generic. Then in $M[G]$ there is simplified $(\kappa, 1)$ morass with jumps of size κ. $M[G]$ still satisfies $2^{,\kappa} = \kappa$ (a neccesary condition to prove in the next stage forcing clousure).

In $M[G]$ we define forcing $\mathbb{P}_0$. In the lower forcing $\mathbb{P}_0$ the conditions are small gap 2 morass segments plus an order preserving funtion F from the lenght of the morass segment to the top κ^+ (F depends on the condition), since the $rang(F)$ is not restricted to multiples of κ, we will add new levels to the simplified $(\kappa^+, 1)$ morass using an upward extension lemma, and the transition functions $\overline{d}_\zeta$, which are the functions which "connect" the level *lub* $F``\zeta$ with $F(\zeta)$ i.e. where the function F jumps (in Velleman they are denote like $F^\sharp(\zeta)$ and are unique, which is not our case), these connecting functions do not have to be part of these linear limit sequence but finite like in Velleman, they have to be in the range of F but finite (F is now an embedding between piecewise simplified morass).

Let H be a generic subset of $\mathbb{P}_0$, then $M[G][H]$ satifies there is a simplified $(\kappa, 2)$ morass.

With these simplifications, different and complete proof we hope to have laid the foundation for a future proof of gap n morasses in ZFC.

Chapter 2

Preliminaries

The following definitions and results are due to Jensen and can be found in [Jensen(1987)].

Definition 1 *Let $f : \varphi+1 \to \varphi'+1$ an order preserving funtion. f is a* **shift** *function of φ with* **split** *point $\sigma < \varphi$ iff $f \restriction \sigma = id \restriction \sigma$ and $f(\sigma + \delta) = \varphi + \delta$ where $\varphi' = \varphi + (\varphi - \sigma)$.*

Definition 2 *$M = \langle \boldsymbol{\varphi}, \boldsymbol{\mathcal{G}} \rangle$ is a* **gap 1 S-premorass of lenght** *μ (SPM) iff*

1) $\boldsymbol{\varphi} = \langle \varphi_\alpha \mid \alpha < \mu \rangle$ is a sequence of ordinals

2) $\boldsymbol{\mathcal{G}}$ is a set of triples $f = \langle \alpha, |f|, \alpha' \rangle$ such that $\alpha < \alpha' < \mu$ and $|f| : \varphi_\alpha + 1 \to \varphi_{\alpha'} + 1$ is order preserving.

Definition 3 *If $f = \langle \alpha, |f|, \alpha' \rangle$, we define $d(f) = \alpha$, $r(f) = \alpha'$. We also write: $rng(f) = rng(|f|)$, $dom(f) = dom(|f|)$ and $f \restriction X = |f| \restriction X$ and we set*

$$\mathcal{G}_{\alpha,\alpha'} = \{f \in \mathcal{G} \mid d(f) = \alpha,\ r(f) = \alpha'\}$$

for $\alpha < \alpha' < \mu$ and $\mathcal{G}_{\alpha\alpha} = \{id\}$

Definition 4 *Let $M = \langle \boldsymbol{\varphi}, \boldsymbol{\mathcal{G}} \rangle$ be a gap 1 SPM of lenght μ. M is a gap 1 S-morass segment (SMS) iff*

1) Either $\mathcal{G}_{\alpha,\alpha+1} = \{id\}$ and $\varphi_{\alpha+1} = \varphi_\alpha + 1$, or else $\mathcal{G}_{\alpha,\alpha+1} = \{id, f\}$, where f is a shift function with split point $\sigma < \varphi_\alpha$.

2) If $\alpha < \beta < \gamma < \mu$, then

$$\mathcal{G}_{\alpha,\gamma} = \{f \cdot g \mid f \in \mathcal{G}_{\beta\gamma},\ g \in \mathcal{G}_{\alpha\beta}\}$$

3) If $\lim(\alpha)$ *and* $g_i \in \mathcal{G}_{\beta_i\alpha}(i = 0,1)$*, then there are* h, γ, g'_i*,* $h \in \mathcal{G}_{\gamma\alpha}$*,* $g'_i \in \mathcal{G}_{\beta_i\gamma}$ $(i = 0,1)$ *sucht that* $g_i = hg'_i$ *(i=0,1).*

4) If $\lim(\alpha)$*, then* $\mathcal{G}_{\beta\alpha} \neq \emptyset$ *for* $\beta < \alpha$ *and*

$$\varphi_\alpha = \bigcup_{\beta<\alpha} \bigcup_{g\in\mathcal{G}_{\beta\alpha}} g``\varphi_\beta.$$

Definition 5 *M is called a **neat** gap 1 SMS of length* μ *if and only if M is a gap 1 SMS and* $\mathcal{G}_{\alpha,\alpha+1} = \{id, f\}$ *where f is a shift function for every* $\alpha < \mu$*. We say M splits at every* $\alpha < \mu$*.*

NOTE: The existence of neat SMS follows from the existence of SMS (see [Velleman(1984a)]).

Definition 6 *Let* $\kappa > \omega$ *be a regular cardinal. A* $(\kappa, 1)$ **S-morass (SM)** [1] *is an SMS of length* $\kappa + 1$ *such that* $\varphi_\kappa = \kappa^+$ *and* $\varphi_\alpha < \kappa$ *for* $\alpha < \kappa$*.*

NOTE: This notion of simplified morass differs inessentially from Velleman's in that we take our maps $f \in \mathcal{G}$ as being defined on $\varphi_\alpha + 1$ rather than on φ_α. It is easy to convert one of Velleman's morasses into one ours and conversely. We made the change in order to facilitate the later development of gap 2 morasses.

It follows trivially form the axioms that the sequence $\langle\varphi_\alpha \mid \alpha < \mu\rangle$ is monotone.

Some basic facts about SMS:

Lemma 7 *Let* $f, f' \in \mathcal{G}_{\beta,\alpha}$*,* $\nu, \nu' \leq \varphi_\beta$*. If* $lub f``\nu = lub f'``\nu'$*, then* $f \restriction \nu = f' \restriction \nu$*.*

Lemma 8 *Let* $f, f' \in \mathcal{G}_{\beta,\alpha}$ *such that* $f(\varphi_\beta) = f'(\varphi_\beta)$*. Then* $f = f'$*.*

Lemma 9 *Let* $f, f' \in \mathcal{G}_{\beta,\alpha}$ *such that* $f(\nu) = f'(\nu')$*, where* $\nu, \nu' < \varphi_\beta$ *or* $\nu = \nu' = \varphi_\beta$*. Then* $f \restriction (\nu + 1) = f' \restriction (\nu' + 1)$*.*

A gap one S-morass consists of ordinal levels connected by order preserving maps. The levels of a gap two S-morass will be gap one S-morass segments connected by structures preserving embeddings. We define now the notion of embedding precisely for S-morasses:

[1] The original Velleman's definition is the following: for κ regular cardinal (i.e. κ could be ω) $\mathcal{A} \subseteq P_\kappa(\kappa^+)$ is a $(\kappa, 1)$ simplified morass iff $\mathcal{A}$ is a well founded, homogeneous, locally small, directed locally, locally almost directed and which covers κ^+.

Definition 10 *Let $M = \langle \boldsymbol{\varphi}, \boldsymbol{\mathcal{G}} \rangle$, $M' = \langle \varphi', \mathcal{G}' \rangle$ be SMS of length μ, μ' respectively. F is a* **embedding** *of M into M' ($F : M \to M'$) iff*

1) $F : \mu \cup S \cup \mathcal{G} \to \mu' \cup S' \cup \mathcal{G}'$ *such that* $F``\mu \subset \mu'$, $F``S \subset S'$, $F``\mathcal{G} \subset \mathcal{G}'$.

2) *If* $\langle \alpha, \nu \rangle \in S$ *then* $F(\langle \alpha, \nu \rangle) = \langle F(\alpha), F_\alpha(\nu) \rangle$, *where* $F \restriction \mu : \mu \to \mu'$ *and* $F_\alpha : \varphi_\alpha + 1 \to \varphi'_{F(\alpha)} + 1$ *for* $\alpha < \mu$. *(Here* $S = \{\langle \alpha, \nu \rangle \mid \nu \leq \varphi_\alpha\}$, $S' = \{\langle \alpha, \nu \rangle \mid \nu \leq \varphi'_\alpha\}$*).*

3) $F``\mathcal{G}_{\alpha\beta} \subset \mathcal{G}'_{F(\alpha)F(\beta)}$

4) *If* $b \in \mathcal{G}_{\alpha\beta}$, *then* $F_\beta \cdot b = F(b) \cdot F_\alpha$ *and* $rng(F_\beta \cdot b) = rng(F_\beta) \bigcap rng(F(b))$.

5) *If* $b \in \mathcal{G}_{\alpha\beta}$, $a \in \mathcal{G}_{\beta\gamma}$, *then* $F(a \cdot b) = F(a) \cdot F(b)$.

6) *If* $\mathcal{G}_{\alpha,\alpha+1} = \{id\}$, *then* $\mathcal{G}'_{F(\alpha),F(\alpha)+1} = \{id\}$.

7) *If α has split point σ in M, then $F(\alpha)$ has split $F_\alpha(\sigma)$ in M'.*

We set $id_M =_{def}$ the identical imbedding. If $M = \langle \boldsymbol{\varphi}, \boldsymbol{\mathcal{G}} \rangle$ has length μ and $\mu' \leq \mu$, we set: $M \restriction \mu' = \langle \varphi \restriction \mu', \mathcal{G} \restriction \mu' \rangle$, where:

$$\begin{aligned} \varphi \restriction \mu' &= \langle \varphi_\alpha \mid \alpha < \mu' \rangle \\ \mathcal{G} \restriction \mu' &= \{ f \in \mathcal{G} \mid r(f) < \mu' \} \end{aligned}$$

We call M an **initial segment** of M' iff $M = M' \restriction length(M)$. If $F : M \to M'$ and $\overline{M}$ is an initial segment of M, we set: $F \restriction \overline{M} = F \cdot id_{\overline{M}}$

Definition 11 *Let M, M' be SMS of length $\theta + 1$, $\theta' + 1$ respectively, where M is an initial segment of M'. Let $\sigma < \theta$. $F : M \to M'$ is called a* **shift** *of M with* **split point** *σ iff*

1) $F \restriction (\theta + 1) : \theta + 1 \to \theta' + 1$ *is a shift with split point* σ

2) $F \restriction (M \restriction \sigma) = id_{M \restriction \sigma}$

3) $F_\sigma \restriction \varphi_\sigma = g \restriction \varphi_\sigma$ *for a* $g \in \mathcal{G}_{\sigma\theta}$ *and* $F_\sigma(\varphi_\sigma) = \varphi_\theta$

4) *If* $\sigma \leq \alpha < \beta \leq \theta$, *then*

$$F``\mathcal{G}_{\alpha\beta} = \mathcal{G}'_{F(\alpha)F(\beta)}.$$

Velleman shows that for $\sigma < \theta$, $b \in \mathcal{G}_{\sigma,\theta}$, there is exactly one shift f with split point σ such that $f_\sigma \restriction \varphi_\sigma = b \restriction \varphi_\sigma$. Its proof can be found in [Velleman(1987)] or [Jensen(1987)].

Lemma 12 (Upward extension of embeddings Lemma) *Let M be a gap 1 SMS of length θ. Let $F : \varphi_0 + 1 \to \varphi + 1$. Then there is exactly one pair f, M' such that $f : M \to M'$, length$(M') = \theta$, $f \upharpoonright \theta = id$, $f_0 = F$, $\varphi_0^{M'} = \varphi$ and $f``\mathcal{G}^M_{\alpha\beta} = \mathcal{G}^{M'}_{\alpha\beta}$ for $\alpha < \beta < \theta$.*

NOTE Using the upward extension of embeddings lemma 12 we can garantize the existence of shift given a M SMS of length θ and for every $b \in \mathcal{G}_{\sigma,\theta}$. This is a basic fact we will use in the construction of gap 2 S-moraress.

Definition 13 *By a gap two* **S-premorass** *(SPM) of length $\langle \lambda, \mu \rangle$ we mean a structure: $\boldsymbol{M} = \langle M_0, M_1 \rangle = \langle \langle \boldsymbol{\theta}, \boldsymbol{\mathcal{F}} \rangle, \langle \boldsymbol{\varphi}, \boldsymbol{\mathcal{G}} \rangle \rangle$ such that:*

1) M_1 is a gap 1 SMS of length μ.

2) $\boldsymbol{\theta} = \langle \theta_\alpha \mid \alpha < \lambda \rangle$ is an ordinal sequence

3)

$$\mu \geq \sup_{\alpha < \lambda} (\theta_\alpha + 1)$$

4) $\boldsymbol{\mathcal{F}}$ is a set of triples $f = \langle \alpha, |f|, \alpha' \rangle$ such that $\alpha < \alpha' < \lambda$ and

$$|f| : M_1 \upharpoonright (\theta_\alpha + 1) \to M_1 \upharpoonright (\theta_{\alpha'} + 1)$$

Definition 14 *Let $\boldsymbol{M} = \langle M_0, M_1 \rangle$ be a gap 2 SPM of length $\langle \lambda, \mu \rangle$. $\boldsymbol{M}$ is a* **gap 2 S-morass segment** *(SMS) iff the following five conditions hold:*

1) Either $\mathcal{F}_{\alpha,\alpha+1} = \{id\}$ and $\theta_{\alpha+1} = \theta_\alpha + 1$, or else $\mathcal{F}_{\alpha,\alpha+1} = \{id, f\}$, where f is a shift function with split point $\sigma < \theta_\alpha$.(In the later case σ is called the ***the split point of*** *α).*

2) If $\alpha < \beta < \gamma < \lambda$, then

$$\mathcal{F}_{\alpha,\gamma} = \{ f \cdot g \mid f \in \mathcal{F}_{\beta\gamma},\ g \in \mathcal{F}_{\alpha\beta} \}$$

3) If $\lim(\alpha)$ and $g_i \in \mathcal{F}_{\beta_i\alpha} (i = 0, 1)$, then there are h, γ, g'_i, $h \in \mathcal{F}_{\gamma\alpha}$, $g'_i \in \mathcal{F}_{\beta_i\gamma}$ $(i = 0, 1)$ sucht that $g_i = hg'_i$ (i=0,1).

4) If $\lim(\alpha)$, then $\mathcal{F}_{\beta\alpha} \neq \emptyset$ for $\beta < \alpha$ and

$$\theta_\alpha = \bigcup_{\beta < \alpha} \bigcup_{g \in \mathcal{F}_{\beta\alpha}} g``\theta_\beta.$$

5) If $\lim(\alpha)$, $\nu \leq \theta_\alpha$ *and*

$$\nu \in \bigcup_{\beta<\alpha} \bigcup_{g\in\mathcal{F}_{\beta\alpha}} rng(g)$$

then

(a)

$$\varphi_\nu + 1 = \bigcup_{\beta<\alpha} \bigcup_{\substack{g\in\mathcal{F}_{\beta\alpha}\\ g(\overline{\nu})=\nu}} rng(g_{\overline{\nu}})$$

(b)

$$\mathcal{G}_{\tau\nu} = \bigcup_{\beta<\alpha} \bigcup_{\substack{g\in\mathcal{F}_{\beta\alpha}\\ g(\overline{\nu},\overline{\tau})=\nu,\tau}} g``\mathcal{G}_{\overline{\tau}\overline{\nu}}$$

Definition 15 ***M*** *is called a* ***neat*** *gap 2 SMS of length* $\langle\lambda,\mu\rangle$ *if and only if* ***M*** *is a gap 2 SMS,* $\mathcal{G}_{\zeta,\zeta+1} = \{id, b\}$ *for every* $\zeta < \mu$ *and* $\mathcal{F}_{\alpha,\alpha+1} = \{id, f\}$ *where* b *is a split function for every* $\zeta < \mu$ *and* f *is a split function for every* $\alpha < \lambda$ *respectively.*

The following lemmas were proved by Jensen in [Jensen(1987)]

Lemma 16 *Let* $f \in \mathcal{F}_{\beta,\alpha}$, $\nu \leq \theta_\beta$, $\eta = lub f``\nu$. *There is* $\overline{f} \in \mathcal{F}_{\beta\alpha}$ *such that* $\overline{f} \restriction (M_1 \restriction \nu) = f \restriction (M_1 \restriction \nu)$, $\overline{f}(\nu) = \eta$ *and* $\overline{f}(\theta_\beta) \leq f(\theta_\beta)$. *Moreover, there is* $b \in \mathcal{G}_{\eta,f(\nu)}$ *such that* $b\overline{f}_\nu = f_\nu$.

Proof: By induction on α.

CASE 1. $\alpha = 0$. Inmediately.

CASE 2. $\alpha = \alpha' + 1$. We observe that if the conclusion holds at β, α' and α', α, then it holds at β, α. Thus it suffices to prove it for $\beta = \alpha'$. But this follows by the special form of $\mathcal{F}_{\alpha',\alpha}$.

CASE 3. $\lim(\alpha)$. Pick $h \in \mathcal{F}_{\gamma,\alpha}$, $f_0 \in \mathcal{F}_{\beta,\gamma}$ such that $f = hf_0$ and $\eta \in rng(h)$ (this is possible by axioms (3) and (4) since $\eta \leq \theta_\alpha$, hence $\eta = g(\eta')$ for some $\beta' < \alpha$ and $g \in \mathcal{F}_{\beta',\alpha}$, but $\lim(\alpha)$, so there is $\gamma > \beta', \beta$ such that for some $h \in \mathcal{F}_{\gamma,\alpha}$ and $f_0 \in \mathcal{F}_{\beta,\gamma}$, $f_1 \in \mathcal{F}_{\beta',\gamma}$ and $g = hf_1$, $f = hf_0$, so $\eta \in rng(h)$ and $f = hf_0$). Let $h(\eta') = \eta$. Then $\eta' = lub f_0``\nu$ and there is $\overline{f}_0 \in \mathcal{F}_{\beta,\gamma}$ and $b_0 \in \mathcal{G}_{\eta',f_0(\nu)}$ such that $\overline{f}_0 \restriction \nu = f_0 \restriction \nu$, $\overline{f}_0(\nu) = \eta'$, $\overline{f}_0(\theta_\beta) \leq f_0(\theta_\beta)$ and $(f_0)_\nu = b_0(\overline{f}_0)_\nu$. Set $\overline{f} = h\overline{f}_0$ and $b = h(b_0)$.

$\square$

Lemma 17 *Let f, ν, η be as in lemma 16. There is $\overline{f} \in \mathcal{F}_{\beta\alpha}$ such that $\overline{f} \upharpoonright (M_1 \upharpoonright \nu) = f \upharpoonright (M_1 \upharpoonright \nu)$ and $\overline{f}(\nu + \delta) = \eta + \delta$ for $\nu + \delta \leq \theta_\beta$.*

Lemma 18 *Let $f \in \mathcal{F}_{\beta,\alpha}$, $\nu \leq \theta_\beta$ such that $f(\nu + \delta) = f(\nu) + \delta$ for all $\nu + \delta \leq \theta_\beta$. Then for all $\nu \leq \eta < \zeta \leq \theta_\beta$ we have*

$$\mathcal{G}_{f(\eta), f(\zeta)} = f``\mathcal{G}_{\eta,\zeta}.$$

Taking $\nu = 0$ in Lemma 17 we then get:

Lemma 19 $id_{\beta,\alpha} \in \mathcal{F}_{\beta,\alpha}$

Velleman [Velleman(1987)] also proves:

Lemma 20 *Let $f \in \mathcal{F}_{\beta,\alpha}$, $\nu \leq \theta_\beta$, $\eta = f(\nu) = lub\ f``\nu$. Let $\tau < \eta$, $b \in \mathcal{G}_{\tau,\eta}$. Then there is $\overline{\zeta} < \nu$ such that $\tau < \zeta = f(\overline{\zeta})$ and $b = f(\overline{b}) \cdot c$ where $\overline{b} \in \mathcal{G}_{\overline{\zeta},\nu}$, $c \in \mathcal{G}_{\tau,\zeta}$.*

Definition 21 *Let $\kappa > \omega$ be regular[2]. $\boldsymbol{M}$ is a $(\kappa, 2)$ S-morass (SM) iff $\boldsymbol{M}$ is a gap 2 SMS of length $\langle \kappa + 1, \kappa^+ + 1 \rangle$ with $\varphi_{\kappa^+} = \kappa^{++}$, $\varphi_\alpha < \kappa^+$ for $\alpha < \kappa^+$, $\theta_\kappa = \kappa^+$ and $\theta_\alpha < \kappa$ for $\alpha < \kappa$.*

[2]Velleman's original definition allows $\kappa = \omega$

Chapter 3

The first stage $\mathbb{P}_1$

We note however, that Lemma 16 was stated too weakly. A better statement is

Lemma 22 *Let* $f \in \mathcal{F}_{\beta,\alpha}$, $\nu \leq \theta_\beta$, $\eta = lub f``\nu$. *There is* $\overline{f} \in \mathcal{F}_{\beta\alpha}$ *such that* $\overline{f} \upharpoonright (M_1 \upharpoonright \nu) = f \upharpoonright (M_1 \upharpoonright \nu)$, $\overline{f}(\nu) = \eta$ *and* $\overline{f}(\theta_\beta) \leq f(\theta_\beta)$. *Moreover, there is* $b \in \mathcal{G}_{\eta,f(\nu)}$ *such that* $b\overline{f}(c) = f(c)$ *for all* $c \in \mathcal{G}_{\tau\nu}$, $\tau < \nu$ *and* $b\overline{f}_\nu = f_\nu$.

Note If $\xi < \varphi_\eta$ and suppose $\boldsymbol{M}$ is a neat gap 2 SMS, then by Lemma 20 $\xi \in rng(\overline{f}(c))$ for some $c \in \mathcal{G}_{\tau,\nu}$, $\tau < \nu$. Hence $b \upharpoonright \varphi_\eta$ is uniquely determined. Moreover, if $\varphi_\eta \in rng(c)$ for a $c \in \mathcal{G}_{\tau,\eta}$, $\tau < \eta$, thus $\varphi_\eta \in rng(\overline{f}(\overline{c}))$ for a $\overline{c} \in \mathcal{G}_{\overline{\tau},\nu}$, $\overline{\tau} < \nu$ by Lemma 20. Hence b is uniquely determined in this case. Similarly, b is uniquely determined if $\varphi_\eta \in rng(\overline{f}_\nu)$ since then $\overline{f}_\nu(\varphi_\nu) = \varphi_\eta$ and $b(\varphi_\eta) = b\overline{f}(\varphi_\nu) = f_\nu(\varphi_\nu)$. We could denote this b by $f^\sharp(\nu)$. If b is not determined, $b \upharpoonright \varphi_\nu$ is still determined and we could let $f^\sharp(\nu) =$ that b satisfying the above condition such that $b(\varphi_\nu) = lub\, b``\varphi_\nu$. However, we shall make little use of this notation.

Definition 23 *By a **generalized shift** we mean* $f : \overline{\varphi} + 1 \to \varphi + 1$ *such that for some* $\sigma < \overline{\varphi}$ *we have* $f \upharpoonright \sigma = id$, $f(\sigma + \xi) = \overline{\varphi} + \xi$ *for* $\sigma + \xi \leq \overline{\varphi}$. *(I.e. we drop the requirement:* $\varphi = lub f``\overline{\varphi} = f(\overline{\varphi})$*).*

Definition 24 *By a **generalized S-morass segment (GMS)** we mean a gap 1 SMS satisfying 1'), 2)- 4) of definition 4, where 1) is replaced by*

> *1') Either* $\mathcal{G}_{\alpha,\alpha+1} = \{id\}$ *(hence* $\varphi_{\alpha+1} \geq \varphi_\alpha$*) or else* $\mathcal{G}_{\alpha,\alpha+1} = \{id, b\}$, *where* b *is a generalized shift with split point* $\sigma < \varphi_\alpha$ *(hence* $\varphi_{\alpha+1} \geq \varphi_\alpha + (\varphi_\alpha - \sigma)$*).*

Definition 25 *If the length of M a gap 1 SMS is clear we write instead of* $\langle\langle\varphi_\zeta \mid \zeta \leq \mu\rangle, \langle\mathcal{G}_{\xi,\zeta} \mid \xi \leq \zeta \leq \mu\rangle\rangle$, $M = \langle \boldsymbol{\varphi}, \boldsymbol{\mathcal{G}}\rangle$.

Definition 26 *Let* $M' = \langle\langle\varphi_{\zeta'} \mid \zeta' < \mu'\rangle, \langle\mathcal{G}_{\zeta',\xi'} \mid \zeta' \leq \xi' < \mu'\rangle\rangle$ *be a 1 gap SMS of length* μ'*. By a* ***stretched gap 1 GMS*** *we mean* $M = \langle\langle\varphi_\zeta \mid \zeta \in I\rangle, \langle\mathcal{H}_{\zeta\xi} \mid \zeta \leq \xi; \zeta, \xi \in I\rangle\rangle$ *where* $I = I_M$ *is a set of ordinals and letting* $g : \mu' \to I$ *be order preserving such that* $I = g``\mu'$ *and* $M = g(M') = \langle\langle\varphi_{g(\zeta')} \mid \zeta' < \mu\rangle, \langle\mathcal{H}_{g(\xi'),g(\zeta')} \mid \xi' \leq \zeta' < \mu'\rangle\rangle$ *is a 1 gap GMS.*

Definition 27 $\mathbb{P}_1$ *= the upper part of our forcing (to add a* $(\kappa^+, 1)$*- stretched simplified morass) = the set of* $p = \langle\mu^p, M^p, B^p\rangle$ *such that*

1) $M^p = \langle\boldsymbol{\varphi}^p, \boldsymbol{\mathcal{H}}^p\rangle$ *is a stretched GMS with* $I_{M^p} = \{\kappa\rho \mid \rho \leq \overline{\mu}^p\}$ *for some* $\overline{\mu} = \overline{\mu}^p$*. We set:* $\mu^p = \kappa \cdot \overline{\mu}^p$, $I = \{\kappa\rho \mid \rho \leq \kappa^+\}$ *(Hence* $I_p =_{def} I_{M^p} = I \cap (\mu_p + 1)$*).*

2) $\varphi^p_0 = 1$, $\varphi^p_\xi < \kappa^+$, $|\mathcal{H}^p_{\xi,\zeta}| < \kappa^+$, $\varphi^p_{\xi+\kappa} = \varphi^p_\xi \cdot \kappa$ *for* $\zeta, \xi, \xi + \kappa \in I_p$*. Moreover,* $\mathcal{H}^p_{\xi,\xi+\kappa} = \{id\}$ *unless* $\xi = \kappa \cdot \rho + \kappa$*, in which case* $\mathcal{H}^p_{\xi,\xi+\kappa} = \{id, d\}$ *where* d *has a split point.*

3) $B^p : \varphi^p_{\mu^p} + 1 \to \kappa^{++}$ *is order preserving and* $rng(B^p)$ *is a union of intervals of the form* $[\kappa^+ \cdot \alpha, \kappa^+ \cdot \alpha + \nu)$ *for some* $\nu < \kappa^+$*.*

4) *Let* $\eta = \kappa \cdot \overline{\eta}$, $\lim(\overline{\eta})$ *and* $\overline{\eta} \leq \overline{\mu}^p$*. There is a sequence* $\langle\langle\eta_i, b_i\rangle \mid i < \tau\rangle$ $(\tau = \tau^p_\eta)$ *such that*

   (a) $\tau \leq \kappa$ *and* $\eta_i = \kappa \cdot \overline{\eta}_i$*, where* $\langle\overline{\eta}_i \mid i < \tau\rangle$ *is normal and* $\sup_i \overline{\eta}_i = \overline{\eta}$*.*

   (b) $b_i \in \mathcal{H}^p_{\eta_i,\eta}$ *such that* $b_{ij} = b_j^{-1} \cdot b_i \in \mathcal{H}^p_{\eta_i,\eta_j}$ *for* $i \leq j < \tau$*.*

   (c) *If* $b \in \mathcal{H}^p_{\xi,\eta}$, $\xi < \eta$*, there is* $i < \tau$ *such that* $b = b_i \cdot \overline{b}$ *for* $\overline{b} \in \mathcal{H}^p_{\xi,\eta_i}$*.*

   (d) *If* $\lim(\lambda)$ *and* $\lambda < \tau$, $\xi < \eta_\lambda$, $b \in \mathcal{H}^p_{\xi,\eta_\lambda}$*, then there is* $i < \lambda$, $\overline{b} \in \mathcal{H}^p_{\xi,\eta_i}$ *such that* $b = b_{i\lambda} \cdot \overline{b}$*.*

Note. We call any such $\langle\langle\eta_i, b_i\rangle \mid i < \tau\rangle$ a **good sequence** for η in $\mathbb{P}_1$.

Definition 28 *The partial order on* $\mathbb{P}_1$ *is defined by:* $p \leq q$ *if and only if*

1) $\mu^p \geq \mu^q$,

2) $M^q = M^p \upharpoonright \mu^q$,

3) $(B^p)^{-1} \cdot B^q \in \mathcal{G}^p_{\mu^q,\mu^p}$.

Lemma 29 *Let $p \in \mathbb{P}_1$. There is $p' \leq p$ such that $\mu^{p'} = \mu^p + \kappa$.*

Proof: It is enough to take $\mu^{p'} = \mu^p + \kappa$ and $B^{p'} \supseteq B^p$. Then $B^p = B^{p'} \cdot id_{\mu^p,\mu^{p'}}$.

□

We now show κ-strategy completeness: let Δ_i be dense open for $i < \delta \leq \kappa$ in $\mathbb{P}_1$ and $p_{i+1} \in \Delta_i$, find a condition p such that $p \leq p_i$ for all $i < \delta$ and $p \in \bigcap_{i<\delta} \Delta_i$.

Lemma 30 *Let Δ_i be strategically open sets in $\mathbb{P}_1$ $(i < \delta \leq \kappa)$ and $p_{i+1} \in \Delta_i$ such that $p_{i+1} \leq p_i$ $(i < \delta \leq \kappa)$. Then $\bigcap_{i<\delta}$ is dense i.e. for any $q \in \mathbb{P}_1$ there is $p \in \bigcap_{i<\delta} \Delta_i$ such that $p \leq p_i$ for all $i < \delta$.*

Proof: : We can choose $p \leq q$ such that $p \in \Delta_0$ and $p_{i+1} \in \Delta_i$ such that $p_{i+1} \leq p_i$ for all $i < \delta$ by the density of Δ_i. So if δ is a successor ordinal we are done. Let δ be a limit ordinal and suppose that $p_{i+1} \leq p_i$ and $p_{i+1} \in \Delta_i$ for all $i < \delta$. Set $D = \bigcup_{i<\delta} rng(B^{p_i})$. Let $B' : \varphi \to D$ be order preserving, where φ is the type order of D. Set $M^p = \bigcup_{i<\delta} M^i$. If $M^p = M^{p_i}$ for some $i < \delta$, then $p_i \leq p_j$ for all $j < \delta$ and we are done. Otherwise p is a GMS of length $\mu = lub_{i<\delta}\, \mu^{p_i}$. Set: $\varphi^p_\mu = \varphi$, $\varphi^p_i = \varphi^{p_j}_i$ for $i \leq j < \delta$. $\mathcal{H}^p_{\xi,\zeta} = \mathcal{H}^{p_j}_{\xi,\zeta}$ for $\xi, \zeta \in I \cap \mu_j$ $(I = \{\kappa\rho \mid \rho < \kappa^+\})$. We must define B^p and $\mathcal{H}^p_{\xi,\mu}$ for $\xi \in I \cap \mu$. Set $B^p = B'$, $B^p_{\xi,\mu} = (B^p)^{-1} \cdot B^{p_\xi}$, $\mathcal{G}^p_{\zeta\mu} = \{B^p_{\xi\mu} \cdot b \mid b \in \mathcal{G}^{p_\xi}_{\zeta\xi}$ for an $\zeta < \xi\}$.

The only difficult in verifying that p is a condition extending all previous ones is in verifying 4.d) in the definition 27 of condition of the first stage. But if we use the same **strategy** to choose p_λ for limit ordinals $\lambda < \delta$, then this will not be a problem. Then p is a condition, $p \leq p_i$ for $i < \delta$ and $p \in \bigcap_{i<\delta} \Delta_i$ (since Δ_i is closed for all $i < \delta$ and $p_{i+1} \in \Delta_i$).

□

(Note that $\langle\langle \eta_i, B^p_{\eta_i,\mu}\rangle \mid i < \tau\rangle$ is a good sequence for $\sup_{i<\tau} \eta_i = \mu$, $\langle \eta_i \mid i < \tau\rangle$ normal, $\eta_i \in I$.)

We need one further extension lemma for $\mathbb{P}_1$.

Lemma 31 *Let $\delta < \kappa^{++}$. There is $p' \leq p$ such that $\delta \in rng(B^{p'})$.*

Proof: Assume $\delta = \kappa^+\beta + \nu$ ($\nu < \kappa^+$), $\delta \in rng(B^p)$. By lemma 29 and 30 and the fact that $\varphi^p_\eta > \varphi^p_\tau \cdot \kappa$ for $\tau < \eta$, we can get $p' \leq p$ such that

$$\varphi^{p'}_{\mu^{p'}} - \xi > \nu \text{ for all } \xi < \mu^{p'}.$$

Suppose $\delta \notin rng(B^{p'})$, since otherwise nothing to prove.

CASE 1. $\kappa^+\beta \in rng(B^{p'})$.
Let $B^{p'}(\gamma) = \kappa^+\beta$. Then $B^{p'}(\gamma + \xi) = \kappa^+\beta + \xi$ for $\gamma + \xi < \varphi^{p'}_{\mu^{p'}}$. Hence $\delta = B^{p'}(\gamma + \nu)$. Contradiction.

CASE 2. $\kappa^+\beta \notin rng(B^{p'})$.
Extend p' to q with $\mu^q = \mu^{p'} + \kappa$, $\varphi^q_{\mu^q} = \varphi^{p'}_{\mu^{p'}} \cdot \kappa$. Let $\mathcal{G}^q_{\mu^{p'},\mu^q} = \{id, b\}$ where b has split point $\gamma = lub\ \{\xi \mid B^{p'}(\xi) < \kappa^+ \cdot \beta\}$. Then $B^{p'}(\gamma) > \kappa^+ \cdot \beta$ and we set $B^q \cdot b = B^{p'}$, $B^q(\gamma + \xi) = \kappa^+ \cdot \beta + \xi$. Then $\delta = B^q(\gamma + \mu)$.

□

NOTE Using "shift" to mean generalized shift, many of the definitions in the lemmas on SMS carry over to GMS - in particular

- The definition of "embedding" (definition 10).
- The definition of "$f : M \to M'$ is a shift with split point σ" (definition 11
- The upward extension of embeddings lemma (lemma 12).

Lemma 32 *If $2^\kappa = \kappa^+$, then $\mathbb{P}_1$ satisfies the κ^{++}-c.c.*

Proof: : Suppose not. Let A be a maximal antichain of size κ^{++}. Since $2^\kappa = \kappa^+$ we can suppose that every condition in A is identical (since $\mu^p = \kappa \cdot \overline{\mu}^p < \kappa^+$ and $|\mathcal{G}^p_{\xi,\zeta}| \leq \kappa$ for $\xi \leq \zeta < \mu^p$ we have control on these components of the condition p, there are at most κ^+, on the other hand to control the sequences $\langle \varphi^p_\zeta \mid \zeta < \mu^p \rangle$ we observe that since μ^p is a multiple of κ less than κ^+ and for $\zeta < \mu^p$, φ_ζ is multiple of κ less than κ^+, we have at most $(\kappa^+)^{<\kappa} = (2^\kappa)^{<\kappa} = \kappa^+$ such sequences) except for the function B^p. Define $\langle \nu_i \mid i \leq \kappa^+ \rangle$ as follows: $\nu_0 = \kappa^+$. For each $p \in \mathbb{P}_1$ such that $rng(B^p) \subset \nu_i$, pick $q_p \leq p$ such that $q_p \leq r \in A$ for some r. Pick $\nu_{i+1} \in (\nu_i, \kappa^{++})$ such that $rng(B^{q_p}) \subset \nu_{i+1}$ for all such p. For limit λ set $\nu_\lambda = \sup_{i<\lambda} \nu_i$. Let $\nu = \nu_{\kappa^+}$. Then $cf(\nu) = \kappa^+$ and hence ν is a multiple of κ^+ (since $\kappa^+ < \nu < \kappa^{++}$).

There is a $p \in A$ such that $rng(B^p) \nsubseteq \nu$, since otherwise $|A| \leq \kappa^+$. Let $\overline{\nu} < \varphi_{\mu^p}$ the first ordinal such that $B^p(\overline{\nu}) \geq \nu$. By the way of the range of each condition we have that $B^p \restriction \overline{\nu} \subset \nu$ and even more $lub B^p``\overline{\nu} < \nu$ where since $cf(\nu) = \kappa^+$ and $\overline{\nu} < \kappa^+$. Let p_1 identical to p and such that

$$\begin{aligned} B^{p_1} \restriction \overline{\nu} &= B^p \restriction \overline{\nu} \\ B^{p_1}(\overline{\nu}+\eta) &= lub\ B^p``\overline{\nu}+\eta \text{ for } \overline{\nu}+\eta \leq \varphi_{\mu^p}. \end{aligned}$$

So, in particular $rng(B^{p_1}) \subseteq \nu$. We can define a condition s such that $s \leq p, p_1$ as follows: $\mu^s = \mu^p + \kappa$, $\sigma^s_{\mu^p} = \overline{\nu}$ and

$$\begin{aligned} B^s \restriction \varphi_{\mu^p} &= B^{p_1}, \\ B^s(\varphi_{\mu^p}+\eta) &= B^p(\overline{\nu}+\eta) \text{ whenever } \overline{\nu}+\eta < \varphi_{\mu^p} \\ B^s(\varphi_{\mu^p}+\eta+\zeta) &= lub\ B^p``\overline{\nu}+\zeta \text{ if } \overline{\nu}+\eta = \varphi_{\mu^p} \\ &\quad \text{and } \varphi_{\mu^p}+\eta+\zeta < \varphi_{\mu^s} = \varphi_{\mu^p}\cdot\kappa. \end{aligned}$$

Then, since $rng(B^{p_1}) \subset \nu = \sup_{i<\kappa} \nu_i$, we have $rng(B^{p_1}) \subset \nu_i$ for an $i < \kappa^+$ and there is $q \leq p_1$ such that $q \leq r \in A$ for some r and $rng(B^q) \subseteq \nu_{i+1} \subseteq \nu$.

Since s and q are not in A they are compatible, i.e. there is a $t \in \mathbb{P}_1$ such that $t \leq s, q$ and $p \neq r$ (since $rng(B^q) \subseteq \nu_{i+1} \subseteq \nu$, $rng(B^p) \nsubseteq \nu$ and $rng(B^s) \subseteq rng(B^s)$). Contradiction.

$\square$

Now let G be $\mathbb{P}_1$-generic. Define a GMS M by:

$$\begin{aligned} M \restriction \kappa^+ &= \bigcup_{p\in G} M^p \\ \varphi_{\kappa^+} &= \kappa^{++} = \bigcup_{p\in G} rng(B^p) \\ \mathcal{G}_{\eta,\kappa^+} &= \{B^p \cdot b \mid p \in G \text{ and } b \in \mathcal{G}^p_{\eta\mu^p}\} \end{aligned}$$

M is then a $\langle \kappa^+, \kappa^{++} \rangle$-stretched morass.

NOTE Inductively we can prove:
Let $\kappa\rho < \kappa^+$. There exist $\langle C_{\kappa\lambda}, H_{\kappa\lambda} \mid \lambda \leq \rho \text{ and } \lim(\lambda) \rangle$ such that

- $C_{\kappa\lambda} \subseteq \kappa\cdot\lambda$ is club in $\kappa\cdot\lambda$,
- type order$(C_{\kappa\lambda}) \leq \kappa$,
- $C_\eta = \eta \cap C_{\kappa\cdot\lambda}$ for η a limit point of $C_{\kappa\lambda}$

- $H_{\kappa\lambda} = \langle h_\eta \mid \eta \in C_{\kappa\lambda}\rangle$ such that
  1) $h_\eta \in \mathcal{G}_{\eta,\kappa\lambda}$.
  2) $h_{\eta,\eta'} = h_{\eta'}^{-1} \cdot h_\eta \in \mathcal{G}_{\eta\eta'}$ for $\eta \leq \eta' < \kappa$.
  3) and η a limit point of $C_{\kappa\lambda}$ and $H_\eta = \langle h^{\kappa\lambda}_{\eta,\eta'} \mid \eta' \in C_\eta\rangle$

Lemma 33 (Uniqueness Lemma) *Let M, M' be gap 1 GMS's of lenght $\theta+1$, $\theta'+1$ respectively, where θ, θ' are limit ordinals. Let $g : M \upharpoonright \theta \to M' \upharpoonright \theta'$ cofinally (i.e. $\sup g``\theta = \theta'$). There is at most one $f \supset g$ such that $f : M \to M'$ and :*

> $(*)$ *For every $\gamma < \theta'$, $c \in \mathcal{G}'_{\gamma\theta'}$, there is $\beta < \theta$ such that $\gamma \leq f(\beta)$ and $c = f(b) \cdot \overline{c}$ for a $b \in \mathcal{G}_{\beta,\theta}$, $\overline{c} \in \mathcal{G}'_{\gamma,f(\beta)}$.*

Let f_0, f_1 be two such completion of g. Set

$$\widetilde{\varphi} = \{\nu \leq \varphi'_{\theta'} \mid \nu \in rng(c) \text{ for some } c \in \mathcal{G}'_{\gamma\theta'} \text{ and } \gamma < \theta'\}$$

(1) Let $\nu \in \widetilde{\varphi}$. There is $b \in \mathcal{G}_{\beta,\theta}$ for some $\beta < \theta$ such that for some $\overline{\nu} \leq \varphi'_{g(\beta)}$ we have: $f_i(b)(\overline{\nu}) = \nu$ for $i = 0,1$ and $f_0(b) \upharpoonright (\overline{\nu}+1) = f_1(b) \upharpoonright (\overline{\nu}+1)$.

Proof (1). By $(*)$ there are b_i, β_i, ν_i such that $b_i \in \mathcal{G}_{\beta_i,\theta}$ ($\beta_i < \theta$), $\nu_i \leq \varphi'_{f_i(\beta_i)} = \varphi'_{g(\beta_i)}$ (since $f_i \supset g$ and $\beta_i < \theta$ for $i = 0,1$), $\nu_i < \varphi'_{g(\beta_i)}$ if $\nu < \varphi'_\theta$ ($i = 0,1$), and $f_i(b_i)(\nu_i) = \nu$. But there is a $b \in \mathcal{G}_{\beta,\theta}$ for a $\beta \geq \beta_0, \beta_1$ such that $b_i = b \cdot b'_i$, $b'_i \in \mathcal{G}_{\beta_i,\beta}$. Hence $f_i(b_i) = f_i(b) \cdot g(b'_i)$ and we can assume without lost of generality that $\beta_0 = \beta_1 = \beta$. Then $\nu_0, \nu_1 < \varphi_{g(\beta)}$ if $\nu < \varphi'_{\theta'}$; hence by lemma 9 $\nu_0 = \nu_1 = \overline{\nu}$ and $f_0(b) \upharpoonright (\overline{\nu}+1) = f_1(b) \upharpoonright (\overline{\nu}+1)$. If $\nu = \varphi'_{\theta'}$, then $\nu_0 = \nu_1 = \varphi'_{g(\beta)}$ and the same conclusion holds.

□

(2) $f_{0\theta} \upharpoonright \varphi_\theta + 1 = f_{1\theta} \upharpoonright \varphi_\theta + 1$

Proof. Let $\nu \leq \varphi_\theta$.

CASE 1. $f_{i\theta}(\nu) \in \widetilde{\varphi}$ for $i = 0$ or $i = 1$.

Then $f_{0,\theta}(\nu) = f_i(b)(\xi)$ for $i = 0,1$, where $b \in \mathcal{G}_{\beta,\theta}$, $f_0(b) \upharpoonright (\overline{\nu}+1) = f_1(b) \upharpoonright \overline{\nu}+1$. Hence $f_{0,\theta}(\nu) \in rng f_{0\theta} \cap rng(f_0(b)) = rng(f_{0\theta} \cdot b)$. Hence $\xi = g_\beta(\overline{\xi})$, $b(\overline{\xi}) = \nu$, and $f_{0\theta}(\nu) = f_0(b) g_\beta(\overline{\xi}) = f_1(b) g_\beta(\overline{\xi}) = (f_{1\theta} \cdot b)(\overline{\xi}) = f_{1\theta}(\nu)$.

□

CASE 2. $f_{i0}(\nu) \notin \widetilde{\varphi}$ for $i = 0, 1$. Then $f_{i\theta}(\nu) = \varphi_{\theta'}$ for $i = 0, 1$. Hence $\nu = \varphi_\theta$.

(3) $f_0(b) = f_1(b)$ for $b \in \mathcal{G}_{\beta\theta}$, $\beta < \theta$.

Proof. Set $\nu = f_0(b)(\varphi_{g(\beta)})$. By (1) there is b' such that $f_i(b')(\overline{\nu}) = \nu$ $(i = 0, 1)$, $b' \in \mathcal{G}_{\beta',\theta}$, $\beta \leq \beta' < \theta$ and $f_0(b') \restriction (\overline{\nu} + 1) = f_1(b') \restriction (\overline{\nu} + 1)$. We can assume without los of generality that $b = b' \cdot \overline{b}$ where $\overline{b} \in \mathcal{G}_{\beta,\beta'}$. (If not, find $b'' \in \mathcal{G}_{\beta''\theta}$, $\beta \leq \beta' \leq \beta'' < \theta$ such that $b = b''\overline{b}$ $(\overline{b} \in \mathcal{G}_{\beta,\beta''})$ and $b' = b'' \cdot \overline{b}'$ $(\overline{b}' \in \mathcal{G}_{\beta',\beta''})$). Then :

$$\begin{aligned} f_0(b) &= f_0(b')g(\overline{b}) \\ &= (f_0(b') \restriction (\overline{\nu} + 1)) \cdot g(\overline{b}) \\ &= (f_1(b') \restriction (\overline{\nu} + 1)) \cdot g(\overline{b}) \\ &= f_1(b') \cdot g(\overline{b}) \\ &= f_1(b) \end{aligned}$$

□

Definition 34 *If f is as in the uniqueness lemma, we call it the* ***good completion*** *of g.*

Definition 35 *Let $\langle S, \leq \rangle$ be a directed set. A* ***directed system*** *is a pair $\langle \langle M_i \mid i \in S \rangle, \langle f_{ij} \mid i \leq j \rangle \rangle$ where each M_i is a structure for a fixed language $\mathcal{L}$, and each $f_{ij} : M_i \to M_j$ is an elementary embedding such that $f_{ik} = f_{jk} \cdot f_{ij}$ for $i \leq j \leq k$ (f_{ii} is the identity). A* ***directed limit*** *of such a system is a structure M for $\mathcal{L}$ for which there are elmentary embeddings $f_i : M_i \to M$ for $i \in S$ with $f_i = f_j \cdot f_{ij}$ for $i \leq j$, such that: for each x in the domain of M, $x \in rng(f_i)$ for some $i \in S$.*

NOTE In our cases the structures M_i are quite simple, they are just ordinals φ with the usual order $\langle \varphi, \leq \rangle$, so the well foundness of the directed system is almost trivial

Chapter 4

The lower forcing $\mathbb{P}_0$

Let G be $\mathbb{P}_1$-generic. Let $M = \langle \boldsymbol{\varphi}, \boldsymbol{\mathcal{H}} \rangle = \langle \langle \varphi_{\kappa\rho} \mid \rho \leq \kappa^+ \rangle, \langle \mathcal{H}_{\kappa\rho',\kappa\rho} \mid \rho' \leq \rho \leq \kappa^+ \rangle \rangle$ be the (κ^+, κ^{++}) stretched morass given by G. We define forcing conditions $\mathbb{P}_0$. Each $p \in \mathbb{P}_0$ will have the form:

$$p = \langle \mathbf{M}^p, F^p, d^p, s^p \rangle,$$

where

(**C**1) $\boldsymbol{M}^p = \langle M_0^p, M_1^p \rangle = \langle \langle \boldsymbol{\theta}^p, \boldsymbol{\mathcal{F}}^p \rangle, \langle \boldsymbol{\varphi}^p, \boldsymbol{\mathcal{G}}^p \rangle \rangle$ is a gap 2 SMS with length $\lambda^p + 1$ respectively, $\mu^p + 1$ (with $\mu^p = \theta^p_{\lambda^p}$). Moreover $\theta^p_0 = \varphi^p_0 = 1$ and $\boldsymbol{M}^p$ is **neat** in the sense that M_i^p splits for $i = 0, 1$ at every $\alpha < \lambda^p + 1$ and $\zeta < \mu^p + 1$ respectively. We will write θ^p instead of $\theta^p_{\lambda^p}$ to simplify notation.

(**C**2) $F^p : \theta^p + 1 \to \kappa^+$ is order preserving and its range is a union of intervals $[\kappa\rho, \kappa\rho + \tau)$ $(\tau < \kappa)$. Moreover, if $\kappa\rho + \kappa \in rng(F^p)$, then $\kappa\rho \in rng(F^p)$ and $\kappa\rho + 1 \in rng(F^p)$. Moreover $F^p(0) = 0$. We define also $\overline{F}(\gamma) = lub F``\gamma$ and $\widetilde{F}(\gamma) = \overline{F}(\gamma)^*$ where $\delta^* =$ the smallest $\kappa\rho > \delta$.

$$D_p = \{\gamma \mid F^p(\gamma) = \kappa\rho \text{ for some } \rho\}$$

and

$$D_p^* = \{\gamma \mid F^p(\gamma) = \kappa\rho \text{ for a limit } \rho \text{ and } \overline{F}^p(\gamma) < F^p(\gamma)\} \subseteq D^p$$

(**C**3) $d^p = \langle d^p_\zeta \mid \zeta \in D_p \rangle$, where $d^p_\zeta \in \mathcal{H}_{\widetilde{F}^p(\zeta), F^p(\zeta)}$.

Note We observe that we do not define F^p at θ^p as κ^+, i.e. θ^p is not necesarily in D_p, this will simplify our proof in the case of the extension lemmas below and when we prove clousure for our last stage forcing.

In order that p be a condition there must exist a stretched GMS $\widetilde{M}_1^p = \langle\langle\widetilde{\varphi}_\zeta^p \mid \zeta \in I_p\rangle, \langle\widetilde{\mathcal{G}}_{\xi,\zeta}^p \mid \xi \leq \zeta, \xi,\zeta \in I_p\rangle\rangle$ on $I_p = rng(F^p) \cup rng(\overline{F}^p) \cup \{\kappa\rho \mid \rho \leq \kappa^+\}$ and an embedding $\widehat{F}^p : M_1^p \to \widetilde{M}_1^p$ with $\widehat{F}^p \restriction (\theta^p + 1) = F^p$, $\widehat{F}^p$ is defined as follows:

By induction on $\gamma \in D_p$ we simultaneosly define $\widetilde{M}_1^p \restriction (F^p(\gamma) + 1)$ and

$$\widehat{F}^p \restriction (\gamma + 1) : M_1^p \restriction (\gamma + 1) \to \widetilde{M}_1^p \restriction (F^p(\gamma) + 1),$$

where by $\widetilde{M}_1^p \restriction \delta$ it means the restriction of $\widetilde{M}_1^p$ to $I_p \restriction \delta =_{def} (I_p \cap \delta) \cup \{\kappa\rho \mid \rho < \kappa^+\}$. We also define an auxiliary embedding

$$\overline{F}^{p(\gamma)} : M_1^p \restriction (\gamma + 1) \to \widetilde{M}_1^p \restriction (\overline{F}^p(\gamma) + 1)$$

such that $\overline{F}^{p(\gamma)} \restriction (M_1^p \restriction \gamma) = \widehat{F}^p \restriction (M_1^p \restriction \gamma)$. In our notation we often ignore the distinction between $\widehat{F}^p$ and F^p, writing F_ζ^p for $\widehat{F}_\zeta^p$ and $F^p(b)$ for $\widehat{F}^p(b)$ where $b \in \mathcal{G}^p$. Similarly we write $\overline{F}_\zeta^p$, $\overline{F}^p(b)$ for $\overline{F}_\zeta^{p(\gamma)}$, $\overline{F}^{p(\gamma)}(b)$, when γ is clear from the context. (Then $\overline{F}_\zeta^p = F_\zeta^p$ for $\zeta < \gamma$ and $\overline{F}^p(b) = F^p(b)$ for $b \in \mathcal{G}_{\xi\zeta}^p$ with $\zeta < \gamma$.)

We assume $\widetilde{M}_1^p \restriction (F^p(\delta) + 1)$, $\widehat{F}^p \restriction (M_1^p \restriction \delta + 1)$, $\overline{F}^{p(\delta)}$ to be defined for $\delta \in D_p \cap \gamma$ (hence if γ is a limit point of D_p, $\widetilde{M}_1^p \restriction \overline{F}^p(\gamma)$, $\widehat{F}^p \restriction (M_1^p \restriction \gamma)$ are defined). In the course of the construction we shall to impose additional requirements to (C4) and (C5), whithout which p cannot be a condition (we use them to prove that F^p is well defined in $\boldsymbol{\mathcal{G}}^p$). We shall inductively verify for $\gamma \in D_p$:

(**I**1) $\mathcal{H}_{\eta,\tau} \subset \widetilde{\mathcal{G}}_{\eta,\tau}^p$ for $\eta < \tau \leq F^p(\gamma)$ such that η, τ are multiple of κ.

(**I**2) Let $b \in \mathcal{G}_{\eta,\tau}^p$ for $\eta < \tau \leq \gamma$ such that $b(\varphi_\eta) < \varphi_\tau$ in M_1^p. Then $F^p(b)(\widetilde{\varphi}_{F^p(\eta)}) < \widetilde{\varphi}_{F^p(\tau)}$.

(**I**3) Let $b \in \mathcal{G}_{\eta,\tau}^p$ for $\eta < \tau \leq \gamma$ such that $b(\varphi_\eta) < \varphi_\tau$ in M_1^p. Then $\overline{F}^p(b)(\widetilde{\varphi}_{F^p(\eta)}) < \widetilde{\varphi}_{\overline{F}^p(\tau)}$.

CASE 1. $\gamma = \min D$. Then $\gamma = 0$ since $F(0) = 0$. Then $\widehat{F}^p \restriction 1 : M_1^p \restriction 1 \to \widetilde{M}_1^p \restriction 1$ is the identity.

CASE 2. γ immediately succeeds δ in D_p. Then $\gamma = \delta + \xi$ where $\xi \geq 1$ and $\overline{F}^p(\gamma) = F^p(\delta) + \xi$. $F_\delta^p : \varphi_\delta^p + 1 \to \widetilde{\varphi}_{F^p(\delta)}^p + 1$ is given. By the extension of embedding lemma (lemma 12) we have:

- a stretched gap 1 SMS M' on $I' = [F^p(\delta), \overline{F}^p(\gamma)]$.

- An $F' : M_1^p \restriction [\delta,\gamma] \to M'$ such that $F'_\delta = F^p_\delta$, $F'(\delta+\eta) = F^p(\delta)+\eta$, and
$$\mathcal{G}'_{F(\delta)+\nu,F(\delta)+\eta} = F' \text{``} \mathcal{G}_{\delta+\nu,\delta+\eta}.$$
for $\delta+\nu \le \delta+\eta \le \gamma$. These are unique. We then set

- $\widetilde{\varphi}^p_{F^p(\delta)+\nu} = \varphi'_{F^p(\delta)+\nu}$ for $F^p(\delta)+\nu \le \overline{F}^p(\gamma)$.
- $\widetilde{\mathcal{G}}^p_{\eta,\tau} = \mathcal{G}'_{\eta,\tau}$ for $F^p(\delta) \le \eta \le \tau \le \overline{F}^p(\gamma)$.
- $\widetilde{\mathcal{G}}^p_{\nu,\tau}$ = the set of $c \cdot b$ with $c \in \mathcal{G}'_{F^p(\delta),\tau}$, $b \in \mathcal{G}^p_{\nu,F^p(\delta)}$ for $\nu < F^p(\delta) < \tau < \overline{F}^p(\gamma)$.
- $F^p_{\delta+\nu} = F'_{\delta+\nu}$ for $\delta+\nu < \gamma$.
- $\overline{F}^p_\gamma = F'_\gamma$.
- $F^p(b) = F'(b)$ for $b \in \mathcal{G}^p_{\eta,\nu}$. $\delta \le \eta \le \nu < \delta+\xi = \gamma$.
- $\overline{F}^p(b) = F'(b)$ for $b \in \mathcal{G}_{\eta,\delta+\xi}$, $\delta \le \eta \le \delta+\xi = \gamma$.

Now let $b \in \mathcal{G}_{\tau,\eta}$, $\tau < \delta < \eta \le \delta+\xi = \gamma$. We set:

- $F^p(b) = F^p(d) \cdot F^p(c)$ for $b = d \cdot c$, where $d \in \mathcal{G}^p_{\delta,\eta}$, $c \in \mathcal{G}^p_{\tau,\delta}$ and $\eta < \delta+\xi$.
- $\overline{F}^p(b) = F^p(d) \cdot \overline{F}^p(c)$, where d, c as above and $\eta = \delta+\xi = \gamma$.

We must show that these definitions are independent of the choice of d, c. Let $b = d_0 \cdot c_0 = d_1 \cdot c_1$ where d_0, c_0 and d_1, c_1 are as above. We use the

Lemma 36 (Fact) *Let $a : \varphi_\tau + 1 \to \varphi_\nu + 1$ in M_1 and $a(\varphi_\tau) = \varphi_\nu$. Then $\varphi_\nu = lub\ a\text{``}\varphi_\tau$.*

This is a general fact about gap 1 **neat** SMS and is proven by induction on ν.

CASE 1. $c_i(\varphi_\tau) = \varphi_\delta$ for $i = 0, 1$.
Then $c_0 = c_0 \restriction (\varphi_\tau + 1) = c_1 \restriction (\varphi_\tau + 1) = c_1$. Since $d_i(\varphi_\delta) = b(\varphi_\tau)$ for $i = 0, 1$, we conclude $d_0 = d_0 \restriction (\varphi_\delta + 1) = d_1 \restriction (\varphi_\delta + 1) = d_1$.

□

CASE 2. $c_i(\varphi_\tau) < \varphi_\delta$ for $i = 0, 1$.

Let $v_i = c_i(\varphi_\tau)$. Then $d_i(v_i) = b(\varphi_\tau)$ for $i = 0, 1$. Hence $v_0 = v_1 = v$ and $c_0 = c_1 = c$. By (I2) $F^p(c)(\widetilde{\varphi}_{F(\tau)}) = \widetilde{v} < \widetilde{\varphi}^p_\delta$. Since $F^p(d_i)(\widetilde{v}) = F^p(b)(\widetilde{\varphi}^p_{F^p(\tau)})$ for $i = 0, 1$ we have $F^p(d_0) \restriction \widetilde{v}+1 = F^p(d_1) \restriction \widetilde{v}+1$, where $rng(F^p(c)) \subseteq \widetilde{v}+1$. Hence $F^p(d_0) \cdot F^p(c) = F^p(d_1) \cdot F^p(c)$.

□

CASE 3. The above fail. Let e.g. $c_0(\varphi_\tau) = \nu < \varphi_\eta$ and $c_1(\varphi_\tau) = \varphi_\eta$. Pick $\xi > \nu$, $\xi < \varphi_\eta$ such that $\xi \in rng(c_1)$. This is possible by the above fact. Let $\xi = c_1(\alpha)$ and set $\overline{\xi} = c_0(\alpha)$. Then $\overline{\xi} < \nu < \xi$, but $d_0(\overline{\xi}) = d_1(\xi) = b(\alpha)$. Contradiction!

□

This shows that $F^p(b)$ is uniquely defined for $b \in \mathcal{G}_{\tau,\eta}$, $\eta < \delta + \xi = \gamma$. The uniqueness of $\overline{F}^p(b)$ is proven similarly using (I3).

We have constructed $\widehat{F}^p \restriction (M_1 \restriction \gamma)$ and $\overline{F}^{p(\gamma)}$. We must still, however, verify (I2) for $\tau < \gamma$ and (I3) for $\tau \leq \gamma$. We verify (I2). Let τ be the least counterexample. Then $\eta < \tau$. Let $b \in \mathcal{G}^p_{\eta,\tau}$ be a counterexample.

CASE 1. There is a $\rho < \tau$ such that $\eta < \rho$ and $c(\varphi_\eta) < \varphi_\rho$ for a pair c, d such that $c : \varphi_\eta \to \varphi_\rho$, $d : \varphi_\rho \to \varphi_\tau$ in M^p_1 and $b = d \cdot c$. Then $\nu =_{def} F^p(c)(\widetilde{\varphi}^p_{F^p(\eta)}) < \widetilde{\varphi}^p_{F^p(\rho)}$, by the minimality of η. But then $F^p(b)(\widetilde{\varphi}^p_{F^p(\eta)}) = F^p(d)(F^p(c)(\widetilde{\varphi}^p_{F^p(\eta)})) = F^p(d)(\nu) < F^p(d)(\widetilde{\varphi}^p_{F^p(\rho)}) \leq \widetilde{\varphi}^p_{F^p(\tau)}$. Contradiction.

CASE 2. Case 1 fails. Then $\tau = \rho + 1$, $b = d \cdot c$, where $c : \varphi_\eta \to \varphi_\rho$, $d : \varphi_\rho \to \varphi_\tau$ in M^p_1 and $c(\varphi_\eta) = \varphi_\rho$. Let $\mathcal{G}^p_{\rho,\tau} = \{id, a\}$ where a has split point σ. Then $d = id_{\rho,\tau}$ and $b(\varphi_\eta) = \varphi_\rho < \varphi_\tau$. Hence $\widetilde{\mathcal{G}}^p_{F^p(\rho),F^p(\tau)} = \{id, F(a)\}$, where $id_{F^p(\rho),F^p(\eta)} = F^p(id_{\rho,\eta})$ and $F^p(a)$ has split point $F^p_\rho(\sigma)$. Let $\nu' = F^p(c)(\widetilde{\varphi}^p_{F^p(\eta)})$. Then $\nu' \leq \widetilde{\varphi}^p_{F^p(\rho)}$ and $F^p(b)(\widetilde{\varphi}^p_{F^p(\eta)}) = id_{\rho,\eta}(\nu') < \widetilde{\varphi}^p_{F^p(\tau)} = F^p(a)(\widetilde{\varphi}^p_{F^p(\rho)})$. Contradiction.

□

The verification of (I3) is similar.

We now define $\widetilde{\mathcal{G}}^p_{\eta,\tau}$ for $\widetilde{F}^p(\gamma) = F^p(\delta) + \kappa \leq \tau$, τ multiple of κ, $\eta \in (I_p \cap (\overline{F}^p(\gamma) + 1))$ or η multiple of κ. If $\eta \leq \overline{F}^p(\gamma)$, set:

$$\widetilde{\mathcal{G}}^p_{\eta,\tau} = \{b \cdot id_{\overline{F}^p(\gamma),\widetilde{F}^p(\gamma)} \cdot c \mid c \in \widetilde{\mathcal{G}}^p_{\eta,\overline{F}^p(\gamma)},\ b \in \mathcal{H}_{\widetilde{F}^p(\gamma),\tau}\}.$$

Otherwise:

$$\widetilde{\mathcal{G}}^p_{\eta,\tau} = \mathcal{H}_{\eta,\tau}.$$

NOTE. If $F^p(\delta) = \eta + \kappa$, $\mathcal{H}_{F^p(\delta),\widetilde{F}^p(\gamma)} = \{id, d\}$, where d has a split point σ. We know, in this case, that $F^p(\delta) + 1 \in I_p$, but we need a further requirement to ensure that (I1) holds:

(**C**4) Let $F^p(\delta) = \eta + \kappa$. Then $\mathcal{G}^p_{\delta,\delta+1} = \{id, d'\}$, where d' has split point σ' and

$$\mathcal{H}_{F^p(\delta),F^p(\delta)+\kappa} = \{id, d\},$$

where d has split point $\sigma = F^p_\delta(\sigma')$ $(= \sigma_{F^p(\delta)}!)$.

This ensures that $d \in \widetilde{\mathcal{G}}^p_{F^p(\delta),F^p(\delta)+\kappa}$, since $d = id_{F^p(\delta)+1,F^p(\delta)+\kappa} \cdot F^p(d')$. It remains only to define F^p_γ and $F^p(b)$ for $b : \varphi_\xi \to \varphi_\gamma$ in M^p_1. Recall that $d^p_\gamma \in \mathcal{H}_{\widetilde{F}^p(\gamma),F^p(\gamma)}$, where in this case $\widetilde{F}^p(\gamma) = F^p(\delta) + \kappa$. Set

$$\overline{d}^p_\gamma = d^p_\gamma \cdot id_{\overline{F}^p(\gamma),\widetilde{F}^p(\gamma)}.$$

(Hence $\overline{d}^p_\gamma \in \widetilde{\mathcal{G}}^p_{\overline{F}^p(\gamma),F^p(\gamma)}$). Set:

$$\begin{aligned} F^p_\gamma &= \overline{d}^p_\gamma \cdot \overline{F}^p_\gamma \\ F^p(b) &= \overline{d}^p_\gamma \cdot \overline{F}^p(b) \text{ for } b \in \mathcal{G}^p_{\tau,\gamma}, \tau < \gamma \end{aligned}$$

This completes the definition of $\widetilde{M}^p_1 \restriction (F^p(\gamma) + 1)$, $\widehat{F}^p \restriction (M^p_1 \restriction \gamma + 1)$ in case 2. (The verification of I2, I3 for $\tau = \gamma$ is trivial by the way we defined $\widetilde{\mathcal{G}}^p_{F^p(\eta)F^p(\gamma)}$ and in a similar way if $\lim(\gamma)$ since in this case $\overline{F}^p(\gamma)$ is a multiple of κ and $\mathcal{H}_{\overline{F}^p(\gamma),\overline{F}^p(\gamma)+\kappa} = \{id\}$).

NOTE. $\overline{d}^p_\gamma = id_{\overline{F}^p(\gamma),F^p(\gamma)}$ if $\widetilde{F}^p(\gamma) = F^p(\gamma)$. This holds in particular if $F^p(\gamma) = F^p(\delta) + \kappa$.

CASE 3. $\lim(\gamma)$ for $\gamma \in D_p$. Then $\overline{F}^p(\gamma)$ is a multiple of κ and $\widehat{F}^p \restriction (M^p_1 \restriction \gamma) : M^p_1 \restriction \gamma \to \widetilde{M}^p_1 \restriction \overline{F}^p(\gamma)$ cofinally. We require

(**C**5) $\widehat{F}^p \restriction (M^p_1 \restriction \gamma)$ has a good completion for $\lim(\gamma)$ in D_p.

We let $\overline{F}^{p(\gamma)}$ be this completion. We then define $\widetilde{\mathcal{G}}^p_{\eta,\tau}$ for $\overline{F}^p(\gamma) \le \tau$, τ multiple of κ by

$$\widetilde{\mathcal{G}}^p_{\eta,\tau} = \{b \cdot c \mid c \in \widetilde{\mathcal{G}}^p_{\eta,\overline{F}(\gamma)} \text{ and } b \in \mathcal{H}_{\overline{F}^p(\gamma),\tau}\}$$

for $\eta < \overline{F}^p(\gamma) < \tau$. Otherwise set

$$\widetilde{\mathcal{G}}^p_{\eta,\tau} = \mathcal{H}_{\eta,\tau} \text{ for } \overline{F}^p(\gamma) \leq \eta.$$

Note that $\overline{F}^p(\gamma) + \kappa = \widetilde{F}^p(\gamma)$ and $\overline{d}^p_\gamma = d^p_\gamma \cdot id_{\overline{F}^p(\gamma),\widetilde{F}^p(\gamma)}$. We set

$$\begin{aligned} F^p_\gamma &= \overline{d}^p_\gamma \cdot \overline{F}^p_\gamma \\ F^p(b) &= \overline{d}^p_\gamma \cdot \overline{F}^p(b) \text{ for } b \in \mathcal{G}^p_{\tau,\gamma}, \text{ and } \tau < \gamma. \end{aligned}$$

This completes the definition of $\widetilde{M}^p_1 \restriction (\overline{F}^p(\gamma)+1)$, $\widehat{F}^p \restriction (M^p_1 \restriction \gamma + 1)$ and $\overline{F}^{p(\gamma)}$ for $\gamma \in D_p$.

We note that these are always defined as long as (C1)- (C5) hold. Finally, if $\gamma = \max D_p < \theta^p$, then $F^p(\gamma+\xi) = F^p(\gamma)+\xi$ and we define $\widetilde{M}^p_1$, $\widehat{F}^p = \widehat{F}^p \restriction (\theta^p+1)$ just as $\widetilde{M}^p_1 \restriction F^p(\gamma)$, $\widehat{F}^p \restriction (\gamma)$ were defined in the case that γ immediately succeedds δ in D_p (here too we need (C4) if $F^p(\gamma) = \nu + \kappa$).

It remains to specify s^p:

(**C**6) $s^p = \langle s^p_\zeta \mid \zeta \in D^*_p \rangle$ and $s^p_\zeta = \langle \langle s^p_{\zeta,i}, \delta^p_{\zeta,i} \rangle \mid i < \tau^p_\zeta \rangle$ where:

- $\tau^p_\zeta \leq \kappa$.
- $s^p_{\zeta,i} \in \mathcal{H}_{\delta^p_{\zeta,i},F^p(\zeta)}$ where $\langle \delta_{\zeta,i} \mid i < \tau^p_\zeta \rangle$ is a normal function converging to $F^p(\zeta)$.
- $\delta^p_{\zeta,0} \leq \overline{F}^p(\zeta) < \delta^p_{\zeta,0} + \kappa$ and $rng(s_{\zeta,0}) \subset rng(d_\zeta)$.
- Let $\xi < F^p(\zeta)$, ξ a multiple of κ, $b \in \mathcal{H}_{\xi,F^p(\zeta)}$. There is i such that $b = s^p_{\zeta,i} \cdot \overline{b}$ for a $\overline{b} \in \mathcal{H}_{\xi,\delta^p_{\zeta,i}}$.
- Either $\delta^p_{\zeta,i+1} = \kappa\rho$ for a limit ρ or else $\delta^p_{\zeta,i+1} = \delta^p_{\zeta,i} + \kappa$.

We now define the partial ordering of $\mathbb{P}_0$:

Definition 37 *Let $p, q \in \mathbb{P}_0$. $p \leq q$ if and only if the following hold:*

- *$\theta^p \geq \theta^q$, $\mu^p \geq \mu^q$, $M^p_0 \restriction \theta^q + 1 = M^q_0$ and $M^p_1 \restriction \mu^q + 1 = M^q_1$.*
- *$rng(F^q) \subset rng(F^p)$ and*

$$F^{qp} = (F^p)^{-1} \cdot F^q \in \mathcal{F}^p_{\lambda^q,\lambda^p}$$

- *If $F^{q,p}(\overline{\nu}) = \nu$ (hence $\widetilde{F}^q(\overline{\nu}) \leq \widetilde{F}^p(\nu)$), then*

$$d^{qp}_{\overline{\nu}} = (d^p_\nu)^{-1} \cdot d^q_{\overline{\nu}} \in \mathcal{H}_{\widetilde{F}^q(\overline{\nu}),\widetilde{F}^p(\nu)}$$

- *For all but finitely many* $\overline{\nu} \in D_q$*: If* $F^{qp}(\overline{\nu}) = \nu$ *and* $\overline{F}^p(\nu) > \overline{F}^q(\overline{\nu})$*, then* $\overline{d}^q_{\overline{\nu}} = F^p(d)$ *for some* $d \in \mathcal{F}^p$*.*

- *Let* $F^{qp}(\overline{\nu}) = \nu \in D^*_p$*. Let*

$$\delta^q_{\overline{\nu},i} \leq \overline{F}^p(\nu) < \delta^q_{\overline{\nu},i+1}$$

then $\tau^p_\nu = \tau^q_{\overline{\nu}} - i$*,* $s^p_{\nu,j} = s^q_{\overline{\nu},i+j}$ *for some* $i > 0$*, all* $j \geq 0$ *and* $\overline{F}^p(\nu) < \delta^q_{\overline{\nu},i} + \kappa$*.*

NOTE By the definition of $\leq_{\mathbb{P}_0}$ if we write f for F^{qp}, since $rng(F^q) \subseteq rng(F^p)$: for $\xi, \zeta \in rng(\overline{F}^q) \cup rng(F^q)$

- $\widetilde{\varphi}^q_\zeta = \widetilde{\varphi}^p_\zeta$
- $\widetilde{\mathcal{G}}^q_{\xi\zeta} \subseteq \widetilde{\mathcal{G}}^p_{\xi\zeta}$
- for $\zeta \leq \mu^q$ $(F^q)_\zeta = (F^p)_{f(\zeta)} \cdot f_\zeta$.

Chapter 5

Properties of the lower forcing $\mathbb{P}_0$

Definition 38 *Let* $p \in \mathbb{P}_0$, $\zeta \in D_p$, $p' = \langle \boldsymbol{M}^p, F', d', s' \rangle$ *is the* ***reduction*** *of* p *at* ζ *if and only if:*

1) $F' \restriction \zeta = F^p \restriction \zeta$, $F'(\zeta + \eta) = \overline{F}^p(\zeta) + \eta$ *for* $\zeta + \eta \leq \theta^p$.

2) $s'_\nu = s_\nu$ *for* $\nu < \zeta$, *otherwise undefined.*

3) $d'_\nu = d_\nu$ *for* $\nu < \zeta$, *otherwise undefined.*

It is easily proved that p has exactly one reduction at $\zeta \in D_p$.

Definition 39 $\mathbb{P}_\zeta$ *= the set of* $p \in \mathbb{P}_0$ *such that* $F^p(\overline{\zeta}) = \zeta$ *where* $\overline{\zeta} = max D_p$.

Hence $F^p(\overline{\zeta} + \nu) = \zeta + \nu$ for $\overline{\zeta} + \nu \leq \theta^p$ and if p' is the reduction of p at $\overline{\tau} \in D_p$, then $p' \in \mathbb{P}_{(\eta)}$ for an $\eta \leq \overline{F}^p(\overline{\tau})$.

Before proceeding we need some preliminary definitions:

Definition 40 *Let* M *be a gap 1 GMS of length* θ*. Let* $I = [\alpha, \beta)$ *be an interval in* θ *(i.e.* $\alpha < \beta \leq \theta$*).* $M \restriction I$ *is the stretched GMS on* I *defined by:* $\varphi_\gamma^{M \restriction I} = \varphi_\gamma^M$, $\mathcal{G}_{\gamma,\delta}^{M \restriction I} = \mathcal{G}_{\gamma,\delta}^M$ *for* $\alpha \leq \gamma \leq \delta < \beta$.

Definition 41 *Let* $f : M \to M'$*, where* M*,*M' *are gap 1 GMS. Let* $I = [\alpha, \beta)$ *be an interval in length*(M)*.* $f \restriction M : M \restriction I \to M'$ *is defined by:* $f \restriction I = f'$ *where* $f'(\gamma) = f(\gamma)$*,* $f'_\gamma = f_\gamma$ *for* $\gamma \in I$ *and* $f'(b) = f(b)$ *for* $b \in \mathcal{G}_{\gamma,\delta}^M$*,* $\alpha \leq \gamma \leq \delta < \beta$.

Definition 42 *Let M, M' be stretched gap 1 GMS on $I = [\alpha, \beta)$, $I' = [\alpha', \beta')$ respectively. Let $f : M \to M'$. f, M' are determined by α', φ, M, b if and only if:*

- $\beta' - \alpha' = \beta - \alpha$ *and* $f(\alpha + \eta) = \alpha' + \eta$ *for* $\eta < \beta - \alpha$.
- $\varphi = \varphi^{M'}_{\alpha'}$, $b = f_\alpha : \varphi_\alpha + 1 \to \varphi + 1$.
- $\mathcal{G}^{M'}_{f(\gamma)f(\delta)} = f``\mathcal{G}^{M}_{\gamma,\delta}$ *for* $\alpha \leq \gamma \leq \delta < \beta$.

By the extension of embedding lemma 12 if $\alpha' \in On$ and $b : \varphi_\alpha \to \varphi$ is order preserving, there is exactly one pair $\langle f, M' \rangle$ determined by α', φ, M, b. It is easily seen that:

FACT: If $f : M \to M'$ is determined by α', φ, M, b and $f' : M' \to M''$ is determined by $\alpha'', \varphi', M', b'$ then $f' \cdot f : M \to M''$ is determined by $\alpha'', \varphi', M, b' \cdot b$.

By the remarks at the end of Chapter 3, we also have:

FACT : If $\boldsymbol{M} = \langle M_0, M_1 \rangle$ is a gap 2 neat SMS and $f \in \mathcal{F}_{\mu,\delta}$, $\alpha < \beta \leq \varphi_\mu + 1$ such that $f(\alpha + \eta) = f(\alpha) + \eta$ for $\eta < \beta - \alpha$, then $f \upharpoonright [\alpha, \beta) : M_1 \upharpoonright [\alpha, \beta) \to M_1 \upharpoonright [f(\alpha), \widetilde{\beta})$ is determined by $f(\alpha)$, $\varphi_{f(\alpha)}$, $M_1 \upharpoonright [\alpha, \beta)$, f_α (where $\widetilde{\beta} = f(\alpha) + (\beta - \alpha)$).

Lemma 43 (First amalgamation lemma) *Let $p \in \mathbb{P}_{(\zeta)}$, $\mu \in D_p$ and $\overline{F}^p(\mu) < F^p(\mu)$. Let p' be the reduction of p at μ. Let $p' \in \mathbb{P}_{(\tau)}$ and $q \in \mathbb{P}_{(\tau)}$ such that $q \leq p'$. There is an $r \in \mathbb{P}_{(\zeta)}$ such that $r \leq q, p$.*

NOTE: The lemma 43 can be improved to:

Lemma 44 *Let $p \in \mathbb{P}_{(\zeta)}$, $\mu \in D_p$, $\overline{F}^p(\mu) < F^p(\mu)$. Let p' be the reduction of p at μ. Let $p' \in \mathbb{P}_{(\tau)}$ and let $q \in \mathbb{P}_{(\tau)}$ such that $q \leq p'$. There is $r \in \mathbb{P}_{(\zeta)}$ such that $r \leq q, p$ and for $F^p(\mu) = F^r(\mu')$, $\overline{F}^r(\mu') \in [\tau, \tau + \kappa)$ and $d^r_{\mu'} = d^p_\mu$.*

We first devise a trivial extension $q' \leq q$ such that $\theta^{q'} > \theta^q$ and $F^{q'} \upharpoonright (\theta^q + 1) = F^q \upharpoonright (\theta^q + 1)$ (For this take $\lambda^{q'} = \lambda^q + 1$, $\mathcal{F}_{\lambda^q, \lambda^{q'}} = \{id, f\}$ with split point 0 and $f_0 = id$. Then set $F^{q'} \upharpoonright (\theta^q + \eta) = F^q(\theta^q) + \eta$ for $\theta^q + \eta \leq \theta^{q'} = \theta^q \cdot 2$. Set $s^{q'} = s^q$, $d^{q'} = d^q$). We now define:

- $\boldsymbol{M}^r$ = the extension of $\boldsymbol{M}^{q'}$ with $\lambda^r = \lambda^{q'}+1 = \lambda^q+2$, $\mathcal{F}_{\lambda^{q'}\lambda^r} = \{id, g\}$ with split point $\widetilde{\mu} = F^{p',q'}(\mu)$ and $g_{\widetilde{\mu}} = id_{\widetilde{\mu},\theta^{q'}}$ (in fact $\widetilde{\mu} = F^{p'q}(\mu)$ since $F^{q'} \upharpoonright (\theta^q + 1) = F^q$).

- $F^r \upharpoonright \theta^{q'} = F^{q'} \upharpoonright \theta^{q'}$, $F^r(\theta^{q'} + \eta) = F^p(\mu + \eta)$ for $\mu + \eta \leq \theta^p$; $F^r(\theta^{q'} + (\theta^p - \mu) + \eta) = F^p(\theta^p) + \eta$ for $\theta^{q'} + (\theta^p - \mu) + \eta \leq \theta^{q'} + \theta^{q'} - \widetilde{\mu} = \theta^r$.

- $s^r_\zeta = s^{q'}_\zeta$, $d^r_\zeta = d^{q'}_\zeta$ for $\zeta < \theta^{q'}$.

- Let $1 \leq \eta$, $\mu+\eta \leq \theta^p$. We set: $d^r_{\theta^{q'}+\eta} = d^p_{\mu+\eta}$ (noting that $\overline{F}^r(\theta^{q'}+\eta) = \overline{F}^p(\mu+\eta)$, $F^r(\theta^{q'}+\eta) = F^p(\mu+\eta)$).

- Let $p' \in \mathbb{P}_{(\tau)}$. Then $q' \in \mathbb{P}_{(\tau)}$. If $\tau < \overline{F}^p(\mu)$, then $\widetilde{F}^r(\theta^{q'}) = \widetilde{F}^p(\mu)$ and we set: $d^r_{\theta^{q'}} = d^p_\mu$. If $\theta^{q'} \in D^*_r$, then $\mu \in D^*_p$ and we set: $s^r_{\theta^{q'}} = s^p_\mu$.

- If $\tau = \overline{F}^p(\mu)$, then μ is a limit point of D_p (otherwise $F^p(\mu) = \kappa\rho + \kappa$ and $\kappa\rho + 1 \in rng(F^p)$; hence $\kappa\rho = \tau < \overline{F}^p(\mu)$). Hence

$$\tau = \sup_{\gamma \in D_p \cap \mu} F^p(\gamma)$$

  is a limit in $\{\kappa\rho \mid \rho \leq \kappa^+\}$. Hence $\mathcal{H}^p_{\tau,\tau+\kappa} = \{id\}$, where $\widetilde{F}^r(\theta^{q'}) = \tau + \kappa = \widetilde{F}^p(\mu)$. Set $d^r_{\theta^{q'}} = d^p_\mu$. If $\theta^{q'} \in D^*_r$, then $\mu \in D^*_p$ and we set $s^r_{\theta^{q'}} = s^p_\mu$. [Note that in both cases, if $\mu \in D^*_p$, then $\delta^p_{\mu,0} = \delta^r_{\theta^{q'},0} = \tau$.]

This defines r. We first prove:

Claim 1: $r \in \mathbb{P}_{(\zeta)}$.

(C1)-(C3) and (C6) are trivial. By induction on $\gamma \in D_r$ we prove the existence of $\widetilde{M}^r \upharpoonright (F^r(\gamma)+1)$, $F^r \upharpoonright (\gamma+1)$, $\overline{F}^r(\gamma)$ and verify (C4) at $\delta < \gamma$ and (C5) at γ.

Recall that $q \in \mathbb{P}_{(\tau)}$, $\tau \leq F^{p'}(\mu)$ (hence $q' \in \mathbb{P}_{(\tau)}$). Let $F^q(\overline{\tau}) = \tau$, then $F^{q'}(\overline{\tau}) = \tau$ and hence $F^r(\overline{\tau}) = \tau$ since $F^r \upharpoonright \theta^{q'} = F^{q'} \upharpoonright \theta^{q'}$ and $\overline{\tau} \leq \theta^q < \theta^{q'}$. For $\gamma \leq \overline{\tau}$ we then have $F^r \upharpoonright \gamma + 1 = F^q \upharpoonright \gamma + 1$. Hence $\widetilde{M}^r_1 \upharpoonright (F^r(\gamma)+1) = \widetilde{M}^q_1 \upharpoonright F^q(\gamma)+1$, $F^r \upharpoonright (\gamma+1) = F^q \upharpoonright (\gamma+1)$ and $\overline{F}^{r(\gamma)} = \overline{F}^{q(\gamma)}$ are given. Now let $\theta = \theta^{q'}$. Then θ is the immediate successor of $\overline{\tau}$ in D_r and $F^r(\theta) = F^p(\mu)$, $\overline{F}^r(\theta) = \overline{F}^{q'}(\theta)$. Thus $\overline{F}^{r(\theta)} = \overline{F}^{q'(\theta)} = \widehat{F}^{q'(\theta)}$ is given. Note that if $f = \overline{F}^{r(\theta)} \upharpoonright [\overline{\tau}, \theta]$, then

$$f : M^{q'}_1 \upharpoonright [\overline{\tau}, \theta] \to \widetilde{M}^{q'}_1 \upharpoonright [\tau, \tau + (\theta - \overline{\tau})]$$

is determined by

$$\tau,\ \widetilde{\varphi}^q_\tau,\ M^{q'}_1 \upharpoonright [\overline{\tau}, \theta],\ F^q_{\overline{\tau}}.$$

We then have $F^r \upharpoonright [\overline{\tau}, \theta) = \overline{F}^{r(\theta)} \upharpoonright [\overline{\tau}, \theta)$ and we define $\widetilde{\mathcal{G}}^r_{\eta\gamma}$ for $\eta \leq \gamma \in I_r \cap (\theta + 1)$ as usual. Finally, since $d^r_\theta \in \mathcal{H}_{\widetilde{F}^r(\theta), F(\theta)}$, we again set

$$\overline{d}^r_\theta = d^r_\theta \cdot id_{\overline{F}^r(\theta)\widetilde{F}^r(\theta)}$$

and

$$\begin{aligned} F^r_\theta &= \overline{d}^r_\theta \cdot \overline{F}^r_\theta, \\ \widehat{F}^r(b) &= \overline{d}^r \cdot \overline{F}^r(b). \end{aligned}$$

This defines $\widehat{F}^r \upharpoonright \theta+1$. The verification of (C4), (C5) for $\gamma \in D_q \cap (\theta+1)$ is trivial, since then $\gamma \in D_q$ and q is a condition. C5 is not appliable to θ, since θ immediately succeeds $\overline{\tau}$ in D_r. We do, however, have (C4) for $\delta = \overline{\tau}$ if $\tau = \eta + \kappa$, since q is a condition.

We now define $\widehat{F}^r \upharpoonright \gamma + 1$, $\widetilde{M}^r_1 \upharpoonright F^r(\gamma) + 1$, $\overline{F}^{r(\gamma)}$ for $\gamma \in D_r$ such that $\gamma > \theta$. At the same time we shall verify (C4) for $\delta < \gamma$ and (C5) at γ. We first note the following facts:

1) Let $f = F^{p'q} \upharpoonright [\mu, \theta^p)$, $\widetilde{\theta} = f(\theta^p)$, $\widetilde{\mu} = f(\mu)$. Then $f, M^q_1 \upharpoonright [\widetilde{\mu}, \widetilde{\theta}]$ are determined by:
$$\widetilde{\mu},\ \varphi^q_{\widetilde{\mu}}, M^p_1 \upharpoonright [\mu, \theta^p],\ f_\mu.$$

2) $M^q_1 \upharpoonright [\widetilde{\mu}, \widetilde{\theta}] = M^{q'}_1 \upharpoonright [\widetilde{\mu}, \widetilde{\theta}] = M^r_1 \upharpoonright [\widetilde{\mu}, \widetilde{\theta}]$. Let $\mathcal{F}^r_{\lambda^{q'}, \lambda^r} = \{id, g\}$ and let $\theta = \theta^{q'} = g(\widetilde{\mu})$. Then $g \upharpoonright [\widetilde{\mu}, \theta]$, $M^r \upharpoonright [\theta, g(\theta)]$ are determined by: θ, $\varphi^{q'}_\theta$, $M^{q'}_1 \upharpoonright [\widetilde{\mu}, \theta]$, $g_{\widetilde{\mu}} = id$ (in particular, if $\theta^* = g(\widetilde{\theta})$, then $g \upharpoonright [\widetilde{\mu}, \widetilde{\theta}]$, $M^r_1 \upharpoonright [\theta, \theta^*]$ are determined by θ, $\varphi^{q'}_\theta$, $M^q_1 \upharpoonright [\widetilde{\mu}, \widetilde{\theta}]$, $g_{\widetilde{\mu}} = id$). It follows easily.

3) $g \cdot f \upharpoonright [\mu, \theta^p]$, $M^r_1 \upharpoonright [\theta, \theta^*]$ are determined by: θ, $\varphi^{q'}_\theta$, $M^p_1 \upharpoonright [\mu, \theta^p]$, $g_{\widetilde{\mu}} f_\mu = f_\mu$.
Note that:

4) $F^q_{\widetilde{\mu}} \circ f_\mu = F^{p'}_\mu = \overline{F}^p_\mu$.

Using these facts we prove by induction on $\gamma \in D_p \setminus \mu$:

a) $\widetilde{M}^r_1 \upharpoonright \overline{F}^p(\gamma) + 1$ exists.

b) $\widetilde{M}^r_1 \upharpoonright F^p(\gamma) + 1$ exists.

c) $\widetilde{M}^r_1 \upharpoonright [F^p(\mu), \overline{F}^p(\gamma)] = \widetilde{M}^p_1 \upharpoonright [F^p(\mu), \overline{F}^p(\gamma)]$.

d) $\widetilde{M}_1^r \restriction [F^p(\mu), F^p(\gamma)] = \widetilde{M}_1^p \restriction [F^p(\mu), F^p(\gamma)]$.

e) $\overline{F}^{r(gf(\gamma))} \cdot g \cdot f \restriction [\mu,\gamma] = \overline{F}^{p(\gamma)} \restriction [\mu,\gamma]$ (i.e. $\overline{F}^r_{gf(\eta)} = \overline{F}^p_\eta$, $\overline{F}^r(gf(b)) = \overline{F}^p(b)$ for $b \in \mathcal{G}^p_{\eta\zeta}$ $(\mu \leq \eta \leq \zeta \leq \gamma)$ and $\overline{F}^r gf(\eta) = F^p(\eta)$ for $\mu \leq \eta \leq \gamma$).

f) $\widehat{F}^r \cdot g \cdot f \restriction [\mu,\gamma] = \widehat{F}^p \restriction [\mu,\gamma]$.

g) Let γ immediate succeeds $\delta \geq \mu$ in D_p. (Hence $gf(\gamma)$ immediate succeeds $gf(\delta)$ in D_r) Them (C4) holds at $gf(\delta)$ with F^r.

h) Let γ be a limit point of D_p, $\gamma > \mu$. (Hence $gf(\gamma)$ is a limit point of D_r since $D_r \cap [\theta,\theta^*] = gf``D_p \cap [\mu,\theta_1^p]$ where $\theta^* = gf(\theta^p)$). Then (C5) holds at γ with F^r.

Proof:

CASE 1. $\gamma = \mu$. Trivial since $\widehat{F}^r \restriction (\mu+1) = \widehat{F}^{q'} \restriction (\mu+1) = \widehat{F}^q \restriction (\mu+1)$ and q is a condition.

CASE 2. $\gamma > \mu$ immediate succeeds δ in D_p. Then $gf(\gamma)$ immediate succeeds $gf(\delta) \geq \theta$ in D_r. We can then define $\widetilde{M}_1^r \restriction \overline{F}^r(gf(\gamma))$, $\overline{F}^{r(gf(\gamma))}$ in the usual way with: $\overline{F}^{r(gf(\gamma))} \restriction [gf(\delta), gf(\gamma)]$, $\widetilde{M}_1^r \restriction [F^r(gf(\delta)), \overline{F}^r(gf(\gamma))]$ are determined by :

$$F^r(gf(\delta)),\ \widetilde{\varphi}^r_{F^r(gf(\delta))},\ M_1^r \restriction [gf(\delta), gf(\gamma)],$$

and $F^r_{gf(\delta)}$. Since $F^r_{gf(\delta)}(gf)_\delta = F^p_\delta$ and $F^r gf(\delta) = F^p(\delta)$ by (f) (induction hypothesis), and $\widetilde{\varphi}^r_{F^r gf(\delta)} = \widetilde{\varphi}^p_{F^p(\delta)}$ by (d), it follows that $\overline{F}^{p(\gamma)} \restriction [\delta,\gamma]$, $\widetilde{M}_1^r \restriction [F^p(\delta), \overline{F}^p(\gamma)]$ are determined by: $F^p(\delta)$, $\widetilde{\varphi}^p_{F^p(\delta)}$, $M_1^p \restriction [\delta,\gamma]$, F^p_δ. Hence $\widetilde{M}_1^r \restriction [F^p(\mu), \overline{F}^p(\gamma)] = \widetilde{M}_1^p \restriction [F^p(\mu), \overline{F}^p(\gamma)]$ and $\overline{F}^{r(gf(\gamma))} \cdot g \cdot f \restriction [\mu,\gamma] = \overline{F}^{p(\gamma)} \restriction [\mu,\gamma]$. (a), (c), (e) follows easily. If $F^p(\delta) = \eta + \kappa$, we must prove (g). This, however, follows easily from the fact that (C4) holds at δ with F^p.

We have: $\widetilde{F}^r(gf(\gamma)) = \widetilde{F}^p(\gamma) = F^p(\delta) + \kappa$ by (e). But $d^r_{gf(\gamma)} = d^p_\gamma \in \mathcal{H}^p_{\widetilde{F}^p(\gamma), F^p(\gamma)}$. Hence $\overline{d}^r_{gf(\gamma)} = d^p_\gamma \cdot id_{\overline{F}^p(\gamma), \widetilde{F}^p(\gamma)} = \overline{d}^p_\gamma \in \widetilde{\mathcal{G}}^r_{\overline{F}^r(gf(\gamma)), \widetilde{F}^r(gf(\gamma))}$ since by definition $F^r(gf(\gamma)) = F^p(\gamma)$.

We then define $F^r_{gf(\gamma)} = \overline{d}^r_{gf(\gamma)} \cdot \overline{F}^r_{gf(\gamma)}$ (hence $F^r_{gf(\gamma)} \cdot (gf)_\gamma = \overline{d}^p_\gamma \cdot \overline{F}^r_{gf(\gamma)} \cdot (gf)_\gamma = \overline{d}^p_\gamma \cdot \overline{F}^p_\gamma = F^p_\gamma$). For $b \in \mathcal{G}^r_{\eta, gf(\gamma)}$, $\eta < gf(\gamma)$, we set: $F^r(b) = \overline{d}^p_\gamma \cdot \overline{F}^r(b)$. In particular if $\widetilde{\mu} = \eta$, then $\eta = gf(\eta')$ for an $\eta' < \gamma$ and $b \in \mathcal{G}^r_{\eta, gf(\gamma)} = gf''\mathcal{G}^p_{\eta',\gamma}$. Hence if $b = gf(b')$, we have $F^r(b) = F^p(b')$. This defines $\widetilde{M}_1^r \restriction F^r(gf(\gamma)) + 1$, $F^r \restriction gf(\gamma) + 1$. It is trivial that (b), (d), (f) hold at γ.

□

CASE 3. γ is a limit point in $D_p \setminus \mu$. We are then given $\widehat{F}^r \restriction gf(\gamma)$, since $gf(\gamma)$ is a limit point in $D_r \setminus \theta$. We are also given $\widetilde{M}_1^r \restriction F^r(gf(\gamma)) = \widetilde{M}_1^r \restriction F^p(\gamma)$. We must find a good completion of

$$\widehat{F}^r \restriction gf(\gamma) : M_1^r \restriction gf(\gamma) \to \widetilde{M}_1^r \restriction F^p(\gamma)$$

with respect to $\widetilde{M}_1^r \restriction \overline{F}^p(\gamma) + 1$ (where $\widetilde{\mathcal{G}}^r_{\tau,\overline{F}^p(\gamma)}$ is the set of $d \cdot c$ such that $c \in \mathcal{G}^r_{\tau,\eta}$ for an $\eta < \overline{F}^p(\gamma)$ and η is a multiple of κ and $d \in \widetilde{\mathcal{G}}^p_{\eta,\overline{F}^p(\gamma)}$). We define such a completion F' setting:

1) $F' \restriction gf(\gamma) = \widehat{F}^r \restriction gf(\gamma)$

2) Let $\zeta < \varphi^r_{gf(\gamma)}$. Then $\zeta = b(\overline{\zeta})$ for a $b \in \mathcal{G}^r_{gf(\eta),gf(\gamma)}$, where $\eta \geq \mu$. Hence $b = gf(\overline{b})$, where $\overline{b} \in \mathcal{G}^p_{\eta,\gamma}$. Set:

$$F'_{gf(\gamma)}(\zeta) = F^p(\overline{b})(F^r_{gf(\eta)}(\overline{\zeta}))$$

(it is easily established that this definition is independent of the choice of $\overline{b}$)

3) Let $b \in \mathcal{G}_{\eta,gf(\gamma)}$. Then $b = d \cdot c$ where $c \in \mathcal{G}_{\eta,gf(\rho)}$, $d \in \mathcal{G}^r_{gf(\rho),gf(\gamma)}$ and $gf(\rho) \geq \theta$. But then $d = gf(\overline{d})$ for a $\overline{d} \in \mathcal{G}^p_{\rho,\gamma}$. Set

$$F'(b) = \widehat{F}^p(\overline{d}) \cdot F^r(c).$$

(Again, the independence of the choice of d, c is easily established using the fact that γ is a limit point of D_p.)

Using the fact that $\overline{F}^{p(\gamma)}$ is a good completion of

$$F^p \restriction \gamma : M_1^p \restriction \gamma \to \widetilde{M}_1^p \restriction \overline{F}^p(\gamma),$$

it follows easily that F' is the (unique) good completion of $F^r gf(\gamma)$. We set $\overline{F}^{r(gf(\gamma))} = F'$. This completes Case 3.

□

Hence we have define $\widetilde{M}_1^r \restriction \zeta + 1$, $\widehat{F}^r \restriction gf(\overline{\zeta}) + 1$, where $p \in \mathbb{P}_{(\zeta)}$ and $F^p(\overline{\zeta}) = \zeta$. We complete the definition of $\widehat{F}^r$ as usual with:

$$\widehat{F}^r \restriction [gf(\overline{\zeta}), (\theta^r)], \ \widetilde{M}_1^r \restriction [\zeta, F^r(\theta^r)]$$

is determined by:

$$\zeta,\ \widetilde{\varphi}_\zeta,\ M_1^r \restriction [gf(\overline{\zeta}), \theta^r],\ F^r_{gf(\overline{\zeta})}.$$

Note that if $\theta^* = gf(\theta^p)$, then $\widehat{F}^r \restriction [gf(\overline{\zeta}), \theta^*]$, $\widetilde{M}_1^r \restriction [\zeta, F^r(\theta^*)]$ is determined by:

$$\zeta,\ \widetilde{\varphi}_\zeta,\ M_1^r \restriction [gf(\zeta), \theta^*],\ F^r_{gf(\overline{\zeta})}.$$

Hence $\widehat{F}^r \cdot gf = \widehat{F}^p$. We use this to establish (C4) for $\delta = gf(\overline{\zeta})$ in the case that ζ is a successor multiple of κ. It follows that $r \in \mathbb{P}_{(\zeta)}$.

□

Claim 2. $r \leq p$.
Proof: . By the above, $(\widehat{F}^r)^{-1} \cdot \widehat{F}^p = gf \in \mathcal{F}^r_{\lambda^{p'}\lambda^r}$. The remainig conditions in the definition of $\leq$ are easily verified.

□

Claim 3. $r \leq q$.
Proof: . $(\widehat{F}^r)^{-1} \cdot \widehat{F}^q = id_{\lambda^q,\lambda^r} \in \mathcal{F}^r_{\lambda^q\lambda^r}$.

□

Lemma 45 (Second amalgamation lemma) *Let $p \in \mathbb{P}_{(\zeta)}$. Let $\mu \in D^*_p$, let p' the reduction of p at μ. Let $q \leq p'$ such that $q \in \mathbb{P}_{(\delta^p_{\mu,i})}$ and there is $b = F^q(\overline{b})$ such that $\overline{d}^p_\mu = s^p_{\mu,i} \cdot b$. Then there is $r \in \mathbb{P}_{(\zeta)}$ such that $r \leq p, q$.*

Then $p' \in \mathbb{P}_{(\delta^p_{\mu,0})}$. Assume $i > 0$, since otherwise nothing to prove, by the first amalgamation lemma 43. As before, find a trivial $q' \leq q$ such that $\theta^{q'} > \theta^q$, $\widehat{F}^{q,q'} = id_{\lambda^q,\lambda^{q'}}$ (where $\lambda^{q'} = \lambda^q + 1$) and $q' \in \mathbb{P}_{(\delta)}$, where $\delta = \delta^p_{\mu,i}$. We now define r:

- $\boldsymbol{M}^r = \langle M_0^r, M_1^r\rangle$ extends $\boldsymbol{M}^{q'}$ with $\lambda^r = \lambda^{q'}+1$, $\mathcal{F}_{\lambda^{q'},\lambda^r} = \{id, g\}$, where g has split point $\widetilde{\mu} = F^{p'q}(\mu) = F^{p'q'}(\mu)$ and $g_{\widetilde{\mu}}$ is defined as follows: Let b as above. Then $\overline{d}^p_\mu = s^p_{\mu,i} \cdot b = d \cdot c \cdot b$, where $d \in \mathcal{H}_{\delta+\kappa,F^p(\mu)}$, $rng(d) \subset rng(s^p_{\mu,i+1})$, and $c \in \mathcal{H}_{\delta,\delta+\kappa}$. For c we have one of the two cases:

  CASE 1. $c = id_{\delta,\delta+\kappa}$ (δ not a succesor multiple of κ).

CASE 2. $\delta = \eta + \kappa$, $\widetilde{\mathcal{G}}^q_{\delta,\delta+1} = \{id, e\}$, where e has split point σ and $c = id_{\delta+1,\delta+\kappa} \cdot e$. In this case $F^q(\overline{\delta}+1) = F^{q'}(\overline{\delta}+1) = \delta + 1$ and $\mathcal{G}^q_{\overline{\delta},\overline{\delta}+1} = \{id, \overline{e}\}$, where $\overline{e}$ has split point $\overline{\sigma}$ and $F^q(\overline{\sigma}) = F^{q'}(\overline{\sigma}) = \sigma$ (i.e $F^q(\overline{e}) = e$).

Set $b' = c' \cdot \overline{b}$, where $c' = id_{\overline{\delta},\theta^{q'}}$ in case 1 and $c' = id_{\overline{\delta}+1,\theta^{q'}} \cdot \overline{e}$ in case 2. We set: $g_{\widetilde{\mu}} = b' \in \mathcal{G}_{\widetilde{\mu},\theta^{q'}}$.

This defines $\boldsymbol{M}^r$. We define further:

- $F^r \upharpoonright \theta^{q'} = F^{q'} \upharpoonright \theta^{q'}$, $F^r(\theta^{q'} + \eta) = F^p(\mu + \eta)$ for $\mu + \eta \le \theta^p$ and otherwise $F^r(\theta^{q'} + (\theta^p - \mu) + \eta) = F^p(\theta^p) + \eta$.
- $s^r_\nu = s^{q'}_\nu$, $d^r_\nu = d^{q'}_\nu$ for $\nu < \theta^{q'}$.
- Let $1 \le \eta$, $\mu + \eta \le \theta^p$. We set

$$\begin{aligned} d^r_{\theta^{q'}+\eta} &= d^p_{\mu+\eta}, \\ s^r_{\theta^{q'}+\eta} &= s^p_{\mu+\eta}. \end{aligned}$$

We now define $d^r_{\theta^{q'}}$, $s^r_{\theta^{q'}}$ by

- $s^r_{\theta^{q'},j} = s^p_{\mu,i+j}$ where i is the i such that $\delta = \delta^p_{\mu,i}$.
- $d^r_{\theta^{q'}} = d$, where $d, c, \overline{c}, c'$ are as above (in the definition of $\boldsymbol{M}^r$).

This defines r.

Let $\theta = \theta^{q'}$. We define $\widehat{F}^r \upharpoonright (\theta+1)$ by $\widehat{F}^r \upharpoonright \theta = \widehat{F}^{q'} \upharpoonright \theta$; $F^r(\theta) = F^p(\mu)$; $F^r_\theta = \overline{d} \cdot \overline{F}^r_\theta = \overline{d} \cdot F^{q'}_\theta$, where $\overline{d} = \overline{d}^r_\theta = d \cdot id_{\overline{F}^{q'}(\theta),\delta+\kappa}$ and $d = d^r_\theta$. For $a \in \mathcal{G}^r_{\tau,\theta}$ we set: $\widehat{F}^r(a) = \overline{d} \cdot F^{q'}(a)$. This gives:

1) Take again $f = F^{p'q}$, then

$$\begin{aligned} F^r_\theta \cdot (gf)_\mu &= F^r_\theta \cdot b' \cdot f_\mu \\ &= d \cdot id_{F^{q'}(\theta),\delta+\kappa} \cdot F^{q'}_\theta \cdot b' \cdot f_\mu \\ &= d \cdot id_{F^{q'}(\theta),\delta+\kappa} \cdot F^{q'}_\theta \cdot c' \cdot \overline{b} \cdot f_\mu, \end{aligned}$$

now, if $c' = id_{\overline{\delta},\theta}$ (case 1) and since $F^{q'}_\theta \cdot c' = F^{q'}(id_{\overline{\delta},\theta}) \cdot F^{q'}_{\overline{\delta}} = id_{\delta,F^{q'}(\theta)} \cdot F^{q'}_{\overline{\delta}}$. So in case 1, we have

$$\begin{aligned} F^r_\theta \cdot (gf)_\mu &= d \cdot id_{F^{q'}(\theta),\delta+\kappa} \cdot id_{\delta,F^{q'}(\theta)} \cdot F^{q'}_{\overline{\delta}} \cdot \overline{b} \cdot f_\mu \\ &= d \cdot id_{\delta,\delta+\kappa} \cdot F^{q'}(\overline{b}) \cdot F^{q'}_{\widetilde{\mu}} \cdot f_\mu \\ &= d \cdot c \cdot b \cdot F^{q'}_{\widetilde{\mu}} \cdot f_\mu, \end{aligned}$$

similarly if $c' = id_{\overline{\delta}+1,\theta^{q'}} \cdot \overline{e}$ (case 2), since in this case

$$\begin{aligned}
F^r_\theta \cdot (gf)_\mu &= d \cdot id_{F^{q'}(\theta),\delta+\kappa} \cdot id_{\delta+1,F^{q'}(\theta)} \cdot F^{q'}_{\overline{\delta}+1} \cdot \overline{e} \cdot \overline{b} \cdot f_\mu \\
&= d \cdot id_{\delta+1,\delta+\kappa} \cdot F^{q'}(\overline{e}) \cdot F^{q'}_{\overline{\delta}} \cdot \overline{b} \cdot f_\mu \\
&= d \cdot id_{\delta+1,\delta+\kappa} \cdot e \cdot F^{q'}(\overline{b}) \cdot F^{q'}_{\widetilde{\mu}} \cdot f_\mu \\
&= d \cdot id_{\delta+1,\delta+\kappa} \cdot e \cdot b \cdot F^{q'}_{\widetilde{\mu}} \cdot f_\mu \\
&= d \cdot c \cdot b \cdot F^{q'}_{\widetilde{\mu}} \cdot f_\mu .
\end{aligned}$$

Hence in any case

$$\begin{aligned}
F^r_\theta \cdot (gf)_\mu &= \overline{d}^p_\mu \cdot F^{p'}_\mu \text{ (where } F^{q'}(\widetilde{\mu}) = \mu) \\
&= \overline{d}^p_\mu \cdot \overline{F}^p_\mu \\
&= F^p_\mu
\end{aligned}$$

2) Let $a \in \mathcal{G}^p_{\tau,\mu}$. Then

$$\begin{aligned}
\widehat{F}^r(gf(a)) &= \overline{d} \cdot \widehat{F}^{q'}(gf(a)) \\
&= \overline{d} \cdot \widehat{F}^{q'}(b' \cdot f(a)) \text{ since } g \text{ is a shift embedd.} \\
&= d \cdot c \cdot b \cdot \widehat{F}^{q'}(f(a)) \text{ similar as above} \\
&= \overline{d}^p_\mu \cdot \widehat{F}^{p'}(a) \\
&= \overline{d}^p_\mu \cdot \overline{F}^p(a) \\
&= \widehat{F}^p(a).
\end{aligned}$$

The rest of the proof is a virtual repetition of lemma 43.

□

Definition 46 *$p \leq_\zeta q$ if and only if ($p \leq q$ and $p, q \in \mathbb{P}_{(\zeta)}$).*

Lemma 47 (Extension lemma) *Let $p \in \mathbb{P}_{(\zeta)}$, $F^p(\overline{\zeta}) = \zeta$.*

a) Let $\delta < \zeta + \kappa$. There is a $q \leq_\zeta p$ such that $\delta \in rng(F^q)$.

b) Let $\nu \in rng(F^p)$, $\delta < \widetilde{\varphi}^p_\nu$. There is $q \leq_{(\zeta)} p$ such that $\delta \in rng(F^q_{\nu'})$ where $F^q(\nu') = \nu$.

c) Let $\nu, \tau \in rng(F^p)$, $b \in \widetilde{\mathcal{G}}^p_{\nu\tau}$. There is $q \leq_{(\zeta)} p$ such that $b = F^q(b')$ for some b'.

Proof: Induction on ζ, using the amalgamation lemma.

We first proof (**a**). Suppose not. Let δ be the least counterexample (for a $p \in \mathbb{P}_{(\zeta)}$).

CASE 1. $\delta \geq \zeta$. Then $\delta > F^p(\theta^p)$. By repeated split we can extend M_1^p to an M_1' such that $\theta^{M'} > \delta$. There is then an obvious $p' \leq p$, $p' \in \mathbb{P}_{(\zeta)}$ such that $F^{p'} \upharpoonright \theta^p + 1 = F^p$, $F^{p'}(\theta^p + \eta) = F^p(\theta^p) + \eta$. Hence $\delta \in rng(F^{p'})$.

□

CASE 2. $\delta < F^p(\tau) + \kappa$ for a $\tau < \overline{\zeta}$. Let $\xi =$ the least $\xi \in D_p$ such that $\xi > \tau$. (then $\xi \leq \overline{\zeta}$, where $F^p(\overline{\zeta}) = \zeta$). Let p' be the reduct of p at ξ. Then $p' \in \mathbb{P}_{(\gamma)}$, where $\gamma \leq F^p(\tau) < \gamma + \kappa$. Hence $\delta < \gamma + \kappa$ and there is $q \leq_\gamma p'$ such that $\delta \in rng(F^q)$. Using the first amalgamation Lemma, let $r \leq q, p$ such that $r \in \mathbb{P}_{(\zeta)}$. Then $\delta \in rng(F^r)$.

□

CASE 3. The above fail. Then $\overline{F}^p(\overline{\zeta}) + \kappa le \delta < \zeta$ and $\zeta = \kappa\rho$ for a limit ρ, since $\kappa\overline{\rho} + \kappa \in rng(F^p)$ implies that $\kappa\overline{\rho} \in rng(F^p)$. We can assume without lost of generality that $\delta = \kappa\rho'$ for some ρ', since otherwise $\kappa\overline{\rho} < \delta < \kappa\overline{\rho} + \kappa$ for some $\overline{\rho}$ and by the minimality of δ there is $p' \leq_\zeta p$ such that $\kappa\overline{\rho} \in rng(F^{p'})$ and Case 2 would apply to p'. Clearly $\delta > \overline{F}^p(\overline{\zeta})$. Consider $s^p_{\overline{\zeta}} = \langle s^p_{\overline{\zeta},i} \mid i < \tau_{\overline{\zeta}} \rangle$.

CASE 3.1 $\delta^p_{\overline{\zeta},i} = \kappa\rho$ for a limit ρ, where $\delta \leq \delta^p_{\overline{\zeta},i}$. Let p' the reduction of p at $\overline{\zeta}$. Clearly, there is $p'' = \langle \boldsymbol{M}^p, F'', d'', s'' \rangle$ such that $F'' \upharpoonright \overline{\zeta} = F^p \upharpoonright \overline{\zeta}$, $F''(\overline{\zeta} + \eta) = \delta^p_{\overline{\zeta},i} + \eta$, $d''_\nu = d^p_\nu$, $s''_\nu = s^p_\nu$ for $\nu < \overline{\zeta}$. Then $p'' \in \mathbb{P}_{(\delta^p_{\overline{\zeta},i})}$ and p' is the reduction of p'' at $\overline{\zeta}$ and since $\overline{F}^p(\overline{\zeta}) < \overline{F}^p(\overline{\zeta}) + \kappa \leq \delta < \zeta$ then $\overline{\zeta} \in D^*_{p''}$. Hence by the first amalgamation Lemma there is $q \in \mathbb{P}_{(\delta^p_{\overline{\zeta},i})}$ such that $q \leq p', p''$. By induction hypothesis (**c**) there is $q' \leq_{\delta^p_{\overline{\zeta},i}} q$ such that $b \in rng(F^{q'})$, where $\overline{d}^p_{\overline{\zeta}} = s^p_{\overline{\zeta},i} \cdot b$. Hence there is $r \in \mathbb{P}_{(\zeta)}$ such that $r \leq q', p$ by the second amalgamation lemma. But then $\delta^p_{\overline{\zeta},i} \in rng(F^r)$and $\delta^p_{\overline{\zeta},i} \geq \delta$. Hence Case 2 applies at r. Contradiction.

□

CASE 3.2 Case 3.1 fails.

Let $\delta^p_{\overline{\zeta},i} < \delta \leq \delta^p_{\overline{\zeta},i+1} = \delta^p_{\overline{\zeta},i} + \kappa$ (if $\delta \leq \delta^p_{\overline{\zeta},j}$ then j must be a succesor since $\lim(j)$ implies $\delta^p_{\overline{\zeta},j}$ a multiple of κ limit). By the minimality of δ there is $p' \leq_\zeta p$ such that $\delta^p_{\overline{\zeta},i} + 1 \in rng(F^{p'})$. Hence $\delta = \delta^p_{\overline{\zeta},i+1}$, since otherwise Case 2 would apply to p'. Clearly $F^{p'}$"$\overline{\zeta} \subseteq \delta$, since otherwise Case 2 would apply to p'. Hence $\delta^{p'}_{\overline{\zeta},0} = \delta^p_{\overline{\zeta},i}$ and $\delta^{p'}_{\overline{\zeta},j} = \delta^p_{\overline{\zeta},i+j} = \delta^{p'}_{\overline{\zeta},i} + \kappa \cdot j$ for $j < \omega$, where $\zeta = \delta^p_{\overline{\zeta},i} + \kappa\omega$. Hence we may assume without lost of generality:

$$\delta^p_{\overline{\zeta},0} < \delta = \delta^p_{\overline{\zeta},1},$$

$\delta^p_{\overline{\zeta},i} = \delta^p_{\overline{\zeta},0} + \kappa \cdot i$ and $\delta^p_{\overline{\zeta},0} + 1 \in rng(F^p)$. We obtain a contradiction by constructing $r \leq_\zeta p$ such that $\delta \in rng(F^r)$. Let p' be the reduction of p at $\overline{\zeta}$. Then $p' \in \mathbb{P}_{(\gamma)}$ where $\gamma = \delta^p_{\overline{\zeta},0}$. Let $\mathcal{H}_{\gamma,\delta} = \{id, f\}$, where f has split point σ. Note that $\widetilde{\varphi}_\delta = \widetilde{\varphi}_\gamma \cdot \kappa$. Let $\sigma < \varphi_\gamma \cdot \alpha$ for an $\alpha < \kappa$.

Pick $p'' \leq_\gamma p'$ such that $\gamma + \alpha \in rng(F'')$. Thus, since $\widetilde{\varphi}_\gamma - \beta = \widetilde{\varphi}_\gamma$ for $\beta < \widetilde{\varphi}_\gamma$ and since we split α many times, we have $\widetilde{\varphi}^{p''}_{\gamma+\alpha} \geq \widetilde{\varphi}_\gamma \cdot \alpha$. Hence $\sigma < \widetilde{\varphi}^{p''}_{\gamma+\alpha}$ and by induction hypothesis (**b**) there is $p_0 \leq_\gamma p''$ such that $\sigma \in rng(F^{p_0}_{\overline{\gamma}+\alpha})$, where $F^{p_0}(\overline{\gamma}) = \gamma$. Let $F^{p_0}_{\overline{\gamma}+\alpha}(\overline{\sigma}) = \sigma$. Then $\overline{\sigma} < \varphi_{\theta^{p_0}}$. Since $M^{p_0}_1$ is a gap **neat** 1 SMS, there is $b \in \mathcal{G}^{p_0}_{0,\theta^{p_0}}$ such that $\overline{\sigma} \in rng(b \restriction \varphi_0)$. But $\varphi_0 = 1$. Hence $b(0) = \overline{\sigma}$. Now split $M^{p_0}_0$ at 0, getting $\boldsymbol{M}' = \langle M'_0, M'_1 \rangle$ with $\lambda' = \lambda^{p_0} + 1$, $\mathcal{F}^{M'}_{\lambda,\lambda'} = \{id, g\}$ using $g_0 : 1 \to \varphi^{p_0}_\theta$ as b. Since $\mathcal{G}^{p_0}_{01} = \{id, h\}$ with split point 0, we have $\mathcal{G}^{M'}_{\theta^{p_0},\theta^{p_0}+1} = \{id, h'\}$, where $h' = g(h)$ has split point $g_0(0) = \overline{\sigma}$. Form $q \leq p''$ by setting:

$\boldsymbol{M}^q = \boldsymbol{M}'$, $F^q \restriction \theta^{p_0} = F^{p_0} \restriction \theta^{p_0}$, $F^q(\theta^{p_0} + \eta) = \delta + \eta$ for $\theta^{p_0} + \eta < \theta^q$. (Hence $D_q = D_p \cup \{\theta^{p_0}\}$). Set $d^q_\nu = d^{p_0}_\nu$, $s^q_\nu = s^{p_0}_\nu$ for $\nu < \theta^{p_0}$. Since $\delta = \gamma + \kappa = F^q(\theta^{p_0})$, we have $d^q_{\theta^{p_0}} \in \mathcal{F}_{\delta,\delta}$; hence $d^q_{\theta^{p_0}} = id$ and $\overline{d}^q_{\theta^{p_0}} = id_{\overline{F}^q(\theta^{p_0}),\delta}$. In order to show that q is a condition we must verify (C4) at δ. We have:

$$\begin{aligned} F^{p_0}_{\theta^{p_0}}(\overline{\sigma}) &= F^{p_0}_{\theta^{p_0}}(id_{\overline{\gamma}+\alpha,\theta^{p_0}}(\overline{\sigma})) \\ &= F^{p_0}_{\overline{\gamma}+\alpha}(\overline{\sigma}) = \sigma \text{ (since } F^{p_0}(id_{\overline{\gamma}+\alpha,\overline{\gamma}+\beta}) = id_{\gamma+\alpha,\gamma+\beta}). \end{aligned}$$

Hence

$$\begin{aligned} F^q_{\theta^{p_0}}(\overline{\sigma}) &= \overline{d}^q_{\theta^{p_0}} \cdot \overline{F}^q_{\theta^{p_0}}(\overline{\sigma}) \\ &= \overline{d}^q_{\theta^{p_0}} \cdot F^{p_0}_{\theta^{p_0}}(\overline{\sigma}) \\ &= \overline{d}^q_{\theta^{p_0}}(\sigma) \\ &= \sigma, \end{aligned}$$

where $\overline{\sigma}$ is the split point of h' and σ is the split point of f. (Hence $\mathcal{G}^q_{\theta^{p_0},\theta^{p_0}+1} = \{id, h'\}$ and $\mathcal{H}_{\gamma,\delta} = \{id, f\}$, where $F^q(\theta^{p_0}) = \delta$.) Thus $q \in \mathbb{P}_{(\delta)}$,

$q \le p'$. Let $d^p_\mu = s^p_{\mu,i+1} \cdot b$. By induction hypothesis there is $q' \le_\delta q$ such that $b = F^{q'}(\bar{b})$ for some b. Hence by the first amalgamation lemma (lemma 43) there is $r \in \mathbb{P}_{(\zeta)}$ such that $r \le q', p$. Hence $\delta \in rng(F^r)$. Contradiction.

□

Proof: (**c**)

Let $F^p(\overline{\nu}, \overline{\tau}) = (\nu, \tau)$. We prove the statement by induction on τ. So suppose the statement true for every $\eta < \tau$.

CASE 1. $\overline{F}^p(\overline{\tau}) = F^p(\overline{\tau})$

CASE 1.1 $\overline{\tau} = \eta + 1$. Then $F^p(\overline{\tau}) = F^p(\eta) + 1$ and we can write $b = e \cdot d$ for some $d \in \widetilde{\mathcal{G}}^p_{\nu, F^p(\eta)}$ and $e \in \widetilde{\mathcal{G}}^p_{F^p(\eta), F^p(\eta)+1} = \{id, c\}$ where c is a shift function. By induction hypothesis there is $r \le p$ such that $d = F^r(d')$ and it is also true that $e = F^p(e')$ for some $e' \in \mathcal{G}^p_{\eta,\eta+1}$ (by lemma 18), so $b = F^r(e') \cdot F^r(d') = F^r(e' \cdot d')$.

CASE 1.2. $\lim(\overline{\tau})$. So $F^p : M^p_1 \restriction \overline{\tau} \to \widetilde{M}^p_1 \restriction \tau$ is cofinal and we can consider its good completion $F^p : M^p_1 \restriction (\overline{\tau} + 1) \to \widetilde{M}^p_1 \restriction (\tau + 1)$, hence there is $\eta < \overline{\tau}$ such that $\nu \le F^p(\eta)$ and $b = F^p(b') \cdot e$ for a $e \in \mathcal{G}^p_{\nu, F^p(\eta)}$ and $b' \in \mathcal{G}^p_{\eta\overline{\tau}}$. By induction hypothesis there is $q \le p$ such that $e = F^q(e')$ and we are done. Note that the case $F^p(\overline{\tau}) = \overline{F}^p(\overline{\tau}) = \kappa\rho$ is included here since it must be $\lim(\overline{\tau})$ (and $\lim(\rho)$!) but not the case $\overline{F}^p(\overline{\tau}) < F^p(\overline{\tau}) = \kappa\rho$.

CASE 2. $\overline{F}^p(\overline{\tau}) < F^p(\overline{\tau}) = \tau$. Hence $\tau = \kappa\rho$. Let p' be the reduct of p at $\overline{\tau}$ so $p' \in \mathbb{P}_{(\zeta')}$ for some $\zeta' \le \tau' = \overline{F}^p(\overline{\tau}) < \tau \le \zeta$.

CASE 2.1. There is $b' \in \mathcal{G}^p_{\nu,\tau'}$ such that $b = \overline{d}^p_{\overline{\tau}} \cdot b'$ (if $\tau = \kappa(\rho+1)$ where are in this case since $\overline{d}^p_{\overline{\tau}} = id_{\overline{F}^p(\overline{\tau}),\tau}$ and $\mathcal{G}^p_{\tau',\tau} = \{id_{\tau',\tau}\}$).

Since $\zeta' < \zeta$ by induction hypothesis there is $q \in \mathbb{P}_{(\zeta')}$ such that $q \le p'$ and $b' \in rng(F^q)$. Let $r \le q, p$ as in Lemma 43, so if $F^r(\tau_0) = F^p(\overline{\tau})$, then $\zeta' \le \tau' \le \tau_1 < \zeta + \kappa$ where $\tau_1 = \overline{F}^r(\tau_0)$. Let $b' = F^q(b_0) = F^r \cdot F^{qr}(b_0)$. So $b' \in rng(F^r)$. Take $b_1 = F^{qr}(b_0) \in \mathcal{G}^r_{F^{pr}(\overline{\nu}), F^{p'r}(\overline{\tau})}$. Since $r \le q$ and $q \in \mathbb{P}_{(\zeta')}$, there is $\zeta_1 < \tau_0$ such that $F^r(\zeta_1) = \zeta'$ and there is no $\gamma \in [\zeta_1, \tau_0)$ such that $\gamma \in D_r$ (if there were such a γ then $\zeta' < F^r(\gamma) = \kappa\nu < \overline{F}^r(\tau_0) < \zeta' + \kappa$ which is impossible). So $F^r(\zeta_1 + \eta) = \overline{F}^r(\zeta_1 + \eta) = \zeta' + \eta$ if $\zeta_1 + \eta < \tau_0$. Now since $F^{p'}(\overline{\tau}) = \overline{F}^p(\overline{\tau}) < F^p(\overline{\tau}) = \tau = F^r(\tau_0)$,then $F^{p'r}(\overline{\tau}) < \tau_0$. But since $\zeta' = F^r(\zeta_1) \le F^{p'}(\overline{\tau})$ we have also that $\zeta_1 \le F^{p'r}(\overline{\tau})$. So $F^r(F^{p'r}(\overline{\tau})) = \overline{F}^r(F^{p'r}(\overline{\tau}))$. Hence $\overline{F}^r(F^{p'r}(\overline{\tau})) = \tau'$ and $\overline{F}^r(id_{F^{p'r}(\overline{\tau}),\tau_0}) = id_{\tau',\tau_1}$. We define

$$b^* = id_{F^{p'r}(\overline{\tau}),\tau_0} \cdot b_1,$$

so

$$\begin{aligned}\overline{F}^r(b^*) &= \overline{F}^r(id_{F^{p'r}(\overline{\tau}),\tau_0}\cdot b_1)\\ &= \overline{F}^r(id_{F^{p'r}(\overline{\tau}),\tau_0})\cdot \overline{F}^r(b_1)\\ &= id_{\tau',\tau_1}\cdot \overline{F}^r(b_1)\\ &= id_{\tau',\tau_1}\cdot b'\end{aligned}$$

But then

$$\begin{aligned}F^r(b^*) &= \overline{d}^r_{\tau_0}\cdot \overline{F}^r(b^*)\\ &= \overline{d}^r_{\tau_0}\cdot id_{\tau',\tau_1}\cdot \overline{F}^r(b_1)\\ &= \overline{d}^r_{\tau_0}\cdot b'\\ &= b.\end{aligned}$$

CASE 2.2. $\tau=\kappa\rho$ for $\lim(\rho)$ and for all $b'\in\mathcal{G}^p_{\nu,\tau'}$ $(b\neq\overline{d}^p_{\overline{\tau}}\cdot b')$. We use in this the sequence $s^p_{\overline{\tau}}=\langle s^p_{\overline{\tau},i}\mid i<\sigma^p_{\overline{\tau}}\rangle$. By the properties of the sequence there is $i<\sigma^p_{\overline{\tau}}$ such that $b=s^p_{\overline{\tau},i}\cdot b'$ for some $b'\in\widetilde{\mathcal{G}}^p_{\nu,\delta_{\overline{\tau},i}}$. By (a) there is $q\leq_{(\zeta)}p$ such that $\delta^p_{\overline{\tau},i}\in rng(F^q)$. Let $\delta^p_{\overline{\tau},i}=F^q(\delta)$. We have now two cases. Let $F^q(\tau')=\tau$.

CASE 2.2.1 $\overline{F}^q(\tau')=F^q(\tau')=\tau$. As Case 1.

CASE 2.2.2 $\overline{F}^q(\tau')<F^q(\tau')=\kappa\rho$ for $\lim(\rho)$. Since $q\leq p$ for some $j\geq i$, $\delta^p_{\overline{\tau},j}\leq\overline{F}^q(\tau')<\delta^p_{\overline{\tau},j+1}$ (since $\delta^p_{\overline{\tau},i}\in rng(F^q)$ then $\overline{F}^q(\overline{\tau}')\geq\delta^p_{\overline{\tau},i}$). But $\delta^p_{\overline{\tau},j}=\delta^q_{\overline{\tau}',0}$ and $\delta^p_{\overline{\tau},j+1}=\delta^p_{\overline{\tau},j}+\kappa$. Hence $\delta^p_{\overline{\tau},j}\in rng(\overline{F}^q)$. On the other hand since $j\geq i$ we are allowed to write $b=s^p_{\overline{\tau},j}\cdot b_1$ ($s_{\overline{\tau},i}=s_{\overline{\tau},j}\cdot s_{\overline{\tau},ij}$) and since $s^q_{\overline{\tau}',0}=s^p_{\overline{\tau},j}$ and $rng(s^q_{\tau',0})\subset rng(d^q_{\tau'})=rng(\overline{d}^q_{\tau'})$ we can write $s^p_{\overline{\tau},j}=\overline{d}^q_{\tau'}\cdot c$ for some $c\in\mathcal{G}^q_{\delta_{\overline{\tau},j},\overline{F}^q(\tau')}$. So $b=\overline{d}^q_{\tau'}\cdot c\cdot b_1$ and we can proceed as before.

□

Proof: (b)

By induction on ν. If $\nu=0$ then $\varphi^p_\nu=1$ so $\delta=0$, but $F^p(0)=0$ implies that $F^p_0=id$ so $\delta\in rng(F^p_\nu)$.

CASE 1. $\nu>0$ and $\nu\neq\kappa(\rho+1)$ (ν can be $\kappa\rho$ for limit ρ). Then we have that $\varphi^p_\nu=\bigcup\{b''\varphi^p_\eta\mid\eta<\nu,\ b\in\widetilde{\mathcal{G}}^p_{\eta,\nu}\}$ (if $\nu=\kappa\rho$ for limit ρ by the property (d)(iv) of the upper forcing $\mathbb{P}_1$ satisfies it or by the basic extension lemma 12 the new segment of morasses is also neat). So $\delta=b(\tau)$ for some $b\in\widetilde{\mathcal{G}}^p_{\eta,\nu}$ for some $\eta<\nu$ and $\tau<\widetilde{\varphi}^p_\eta$. Now we get a $q\leq_{(\zeta)}p$ such that $\eta\in rng(F^q)$ (by (a)) and such that $b\in\widetilde{\mathcal{G}}^q_{\eta,\nu}$ By induction hypothesis and

(c) there is $r \leq_{(\zeta)} q$ such that $\tau = F^r_{\eta'}(\tau')$ for some $\tau' < \widetilde{\varphi}^r_{\eta'}$ such that $F^r(\eta') = \eta$ and $b = F^r(\bar{b})$ where $\bar{b} \in \mathcal{G}^r_{\eta',\nu'}$ such that $F^r(\eta', \nu') = (\eta, \nu)$. So $\delta = b(\tau) = F^r(\bar{b})(F^r_{\eta'}(\tau')) = F^r_{\nu'} \cdot \bar{b}(\tau')$. So $\delta \in rng(F^r_{\nu'})$ and $F^r(\nu') = \nu$.

CASE 2. $\nu = \kappa(\rho+1)$ Since $\delta < \widetilde{\varphi}_{\kappa(\rho+1)} = \widetilde{\varphi}_{\kappa\rho} \cdot \kappa$ then there is $\eta < \kappa$ such that $\delta < \varphi^p_{\kappa\rho} \cdot \eta$. Now since $\kappa(\rho+1) \in rng(F^p)$, we have also that $\kappa\rho \in rng(F^p)$. Let $F^p(\zeta') = \kappa(\rho+1)$.

We build first a condition $p' \leq_{(\zeta)} p$ such that $\lambda^{p'} = \lambda^p + \eta + 1$ (this is possible since $\eta < \kappa$) and then using the reduct of p' at ζ' and the first extension lemma (lemma 43) we get a condition r such that $\kappa\rho+\eta \in rng(F^r)$ (hence $\overline{F}^r(\zeta') \geq \kappa\rho + \eta$ where $F^q(\zeta') = \kappa(\rho+1)$) and $\varphi^r_{\overline{F}^r(\zeta')} \geq \varphi^r_{\kappa\rho+\eta} \geq \varphi^p_{\kappa\rho} \cdot \eta > \delta$ so $\delta = id(\delta)$ where $id \in \mathcal{G}^r_{\overline{F}(\zeta'),\kappa(\rho+1)}$ and we can proceed as before.

We define p' by induction and such that $\lambda^{p'} = \lambda^p + \eta + 1$. We add new levels to p one at a time. Suppose $\lambda^p + \tau$ has been chosen for some $\tau \leq \eta$. To add level $\lambda^p + \tau + 1$ we use in this case the basic extension lemma 12 with $\mathcal{F}_{\lambda+\tau,\lambda+\tau+1} = \{id, b\}$ and b is a split function such that $b(\zeta) = \theta_{\lambda^p+\tau} + \zeta$ and $b_0 = id$. So $\sigma_{\theta_{\lambda^p+\tau}} = b_0(\sigma_0) = b_0(0) = 0$ and hence $\varphi_{\theta_{\lambda^p+\tau}} = \tau \cdot \varphi_{\theta_{\lambda^p}}$.

If τ is a limit ordinal we take the union, i.e.

$$\theta_{\lambda^p+\tau} = \sup\{\theta_{\lambda^p+i} \mid i < \tau\},$$

$$\varphi_{\theta_{\lambda^p+\tau}} = \sup\{\varphi_\eta \mid \eta < \theta_{\lambda^p+\tau}\},$$

$$\mathcal{G}_{\xi,\theta_{\lambda^p+\tau}} = \bigcup\{\mathcal{G}_{\xi,i} \mid \xi < i < \theta_{\lambda^p+\tau}\},$$

and

$$\mathcal{F}_{\alpha,\lambda^p+\tau} = \bigcup\{\mathcal{F}_{\alpha,\beta} \mid \alpha < \beta < \lambda^p + \tau\}$$

We have still to define $F^{p'}$. $F^{p'} \restriction \theta^p = F^p$, $F^{p'}(\theta^p + \tau) = \overline{F}^p(\theta^p) + \tau$ for $\theta^p + \tau \leq \theta^{p'}$. Let q be the reduct of p' at ζ' where $F^{p'}(\zeta') = F^p(\zeta') = \kappa(\rho+1)$, so $F^q \restriction \zeta' = F^{p'} \restriction \zeta'$ and $F^q(\zeta' + i) = \overline{F}^{p'}(\zeta') + i$ for $\zeta' + i < \theta^{p'}$. Since $\kappa\rho \in rng(F^p) \subset rng(F^{p'})$, there is $\xi \leq \theta^p$ such that $F^p(\xi) = F^{p'}(\xi) = \kappa\rho$, so for $\xi + \tau \leq \theta^q = \theta^{p'}$ in fact $F^q(\xi+\tau) = \kappa\rho+\tau$. Note that $\xi + \eta \leq \theta^p + \eta < \theta^{p'}$ so $\kappa\rho + \eta \in rng(F^q)$. Also note that if $\xi + \tau < \theta^q$, then $F^q_{\xi,\xi+\tau}(id) = id$, $F^q_\xi(0) = 0$, and $id(0) = 0$. From the definition of embedding it follows that $F^q_{\xi+\tau}(0) = 0$. Thus if $\sigma_{\xi+\tau} = 0$, then $\sigma_{\kappa\rho+\tau} = 0$. Since there is an increasing η- sequence of ordinals τ such that $\sigma_{\zeta+\tau} = 0$, it follows that $\varphi^q_{\overline{F}^q(\theta^q)} \geq \varphi_{\kappa\rho}\eta$. Using lemma 43 there is $r \in \mathbb{P}_\xi$ such that $r \leq q, p'$. But then $\delta = id(\delta)$ where $id \in \mathcal{F}^r_{\overline{F}(\zeta'),\kappa(\rho+1)}$ and we can proceed as before.

□

Lemma 48 (Third amalgamation Lemma) *Let $p \in \mathbb{P}_{(\zeta)}$, $\mu \in D_p$ and $\overline{F}^p(\mu) < F^p(\mu)$. Let p' be the reduction of p at μ. Let $q \leq p'$ such that $F^q(\theta^q) < F^p(\mu) = \eta$ where $p' \in \mathbb{P}_{(\tau)}$ (so $\overline{F}^p(\mu) < F^p(\mu)$ and $\tau \leq \overline{F}^p(\mu)$). There is $r \in \mathbb{P}_{(\zeta)}$ such that $r \leq p, q$.*

Proof: : If $\eta = \kappa(\rho+1)$ then $\kappa\rho \in rng(F^p)$ and $q, p \in \mathbb{P}_{(\kappa\rho)}$. Hence the first amalgamation lemma applies. Now let η be a limit multiple of κ. Then $\mu \in D^*_p$. Let $\delta = \delta^p_{\mu,i} > F^q(\theta^q)$. There is p_1 such that $\boldsymbol{M}^{p_1} = \boldsymbol{M}^p$, $F^{p_1} \restriction \mu = F^p \restriction \mu$, $F^{p_1}(\mu+\nu) = \delta+\nu$. Moreover p' is the reduction of p_1 at μ. Hence there is $q' \in \mathbb{P}_{(\delta)}$ such that $q' \leq p_1, q$. Let $\overline{d}^p_\mu = s^p_{\mu,i} \cdot b$ and pick $q' \leq q$, $q' \in \mathbb{P}_{(\delta)}$ such that $b = F^{q'}(\overline{b})$ for some $\overline{b}$. Then $q' \leq q \leq p'$ and hence q', p satisfy the assumption of the second amalgamation Lemma 45. Hence there is $r \in \mathbb{P}_{(\zeta)}$ such that $r \leq q', p$.

□

Lemma 49 *If $2^{<\kappa} = \kappa$, then $\mathbb{P}_0$ satisfies the κ^+-c.c.*

Proof: : Suppose not. Let A be a maximal antichain of size κ^+, since $2^{<\kappa} = \kappa$ we can suppose that every $p \in A$ is identical except for the function F^p (since $\lambda^p < \kappa$ and $|\mathcal{F}^p_{\alpha,\beta}| \leq \kappa$ for $\alpha \leq \beta < \lambda^p$ we have control on these components of the condition p, there are at most κ, on the other hand since $\langle \theta_\alpha \mid \alpha < \lambda^p \rangle)$ and λ^p is less than κ, we have at most $\kappa^{<\kappa} = 2^{<\kappa} = \kappa$ such sequences). Define $\langle \nu_i \mid i \leq \kappa \rangle$ as follows: $\nu_0 = \kappa$. For each $p \in \mathbb{P}_0$ such that $rng(F^p) \subset \nu_i$, pick $q_p \leq p$ such that $q_p \leq r \in A$ for some r. Pick $\nu_{i+1} \in (\nu_i, \kappa^+)$ such that $rng(F^{q_p}) \subset \nu_{i+1}$ for all such p. For limit λ set $\nu_\lambda = \sup_{i<\lambda} \nu_i$. Let $\nu = \nu_\kappa$. Then $cf(\nu) = \kappa$, hence ν is a multiple of κ. There is a $p \in A$ such that $rng(F^p) \nsubseteq \nu$, since otherwise $|A| \leq \kappa$. By the extension lemma 47 there is $p' \leq p$ such that $\nu \in rng(F^{p'})$. Hence $\overline{F}^{p'}(\overline{\nu}) < \nu$ where $F^{p'}(\overline{\nu}) = \nu$, since $cf(\nu) = \kappa$. But then $\overline{\nu} \in D_{p'}$. Let p_1 be the reduction of p' at $\overline{\nu}$. Then, since $rng(F^{p_1}) \subset \nu = \sup_{i<\kappa} \nu_i$, we have $rng(F^{p_1}) \subset \nu_i$ for an $i < \kappa$ and there is $q \leq p_1$ such that $q \leq r \in A$ for some r and $rng(F^q) \subseteq \nu_{i+1} \subseteq \nu$. By the third amalgamation lemma there is $s \leq q, p$. But then $r|p$ since $p \in A$ and $q \leq r \in A$ such that $r \neq p$. Contradiction.

□

Chapter 6

The Statement

The task now is to prove that no cardinal are collapsed in the second stage forcing. Velleman has proved that this stage has the κ^+-c.c., so every cardinal $\geq \kappa^+$ is not collapsed. We do it proving like Velleman that the entire forcing iteration is (κ, ∞)-distributive. So a condition p consists of pairs $\langle p_1, p_2 \rangle$, where p_1 is a condition for the first stage and $p_1 \Vdash$ "$\check{p}_2$ is a condition for $\mathbb{P}_1$". Since the first stage is κ^+-closed, we may extend p_1 to completely determine p_2 in the ground model, so we may suppose all components of p_2 are determined. We may also assume that μ^{p_1} is a limit multiple of κ, $\mu^{p_1} = \kappa \cdot \mu^{p_1}$, $cf(\mu^{p_1}) = \kappa$ and $\mu^{p_1} \geq \overline{F}^{p_2}(\theta^{p_2})$ (for all conditions).

Let p be a forcing condition, $\sigma < \kappa$, and for each $\alpha < \sigma$, D_α is a dense open set. We will construct a descending sequence $\langle p_\alpha \mid \alpha \leq \sigma \rangle$ such that $p_0 = p$ and $p_{\alpha+1} \in D_\alpha$ for all $\alpha < \sigma$. Clearly then $p_\sigma < p$ and $p_\sigma \in \bigcap\{D_\alpha \mid \alpha < \sigma\}$ as required. As usual, when discussing the components of p_α we we use a superscript α instead of p_α.

6.1 Succesor case

Let $p_0 = p$. Now suppose p_α has been chosen and we wish to choose $p_{\alpha+1}$. First choose $q \leq p_\alpha$ such that $q \in D_\alpha$ and $q \in \mathbb{P}_{(\zeta)}$. Recall that there may be finitely many $\nu_\alpha \in D^*_{p_\alpha}$ such that $\overline{d}^\alpha_{\nu_\alpha} \notin rng(F^q)$. Since there is $0 < i_\alpha < \tau$ such that $\overline{d}^\alpha_{\nu_\alpha} = s^p_{\nu_\alpha, i_\alpha} \cdot e$ for some $e \in \mathcal{G}_{\overline{F}^\alpha(\nu_\alpha)\delta^p_{\nu_\alpha i_\alpha}}$, choose $p_{\alpha+1} \leq q$ such that $\overline{F}^{\alpha+1}(\nu_{\alpha+1}) > \delta^\alpha_{\nu_\alpha, i_\alpha}$ (using the extension lemma 47 a), where $F^{\alpha+1}(\nu_{\alpha+1}) = F^\alpha(\nu_\alpha) = \kappa\rho$ for some $\nu_{\alpha+1} \leq \theta^{\alpha+1}$, such that $\mu^\alpha < \overline{F}^{\alpha+1}(\theta^{\alpha+1})$ and $\mu^\alpha \in rng(F^{\alpha+1})$ (if $\mu^\alpha < \zeta + \kappa$ apply lemma 47 a), if not apply the following fact:

Fact For $\mu = \kappa\overline{\rho} > \zeta + \kappa$ for $\overline{\rho}$ limit ordinal there is an $r \leq q$ such that $\mu \in rng(F^r)$.

Proof: We define r as follows: $\lambda^r = \lambda^q + 1$, let F^r be

$$\begin{aligned} F^r \restriction (\theta^q + 1) &= F^q \\ F^r(\theta^q + 1 + \nu) &= \mu^\alpha + \nu \text{ for } \nu < \theta^q. \end{aligned}$$

We split M_0^q at 0 to get M_0^r, let $\theta = \theta^q$, define $\mathcal{F}_{\lambda^q,\lambda^r} = \{id, h\}$, where $h(\nu) = \theta + \nu$, so $\theta^r = 2 \cdot \theta^q$, we take $h_0 = id_{0\theta}$ and $d^q_{\theta+1} \in \mathcal{G}_{\zeta+\kappa,\mu^r}$ as id. If we call $\mu' = \theta + 1$ then $r \in \mathbb{P}_{(\mu)}$ and since $\overline{F}^r(\mu') < F^r(\mu')$ then $\mu' \in D^*_r$. So we must still define $s_{\mu'} = s^r_{\mu'}$. Let $\langle b_i \mid i < \kappa\rangle$ be a good sequence for μ in M_1^p, $b_i : \varphi_{\gamma_i} \longrightarrow \varphi_{\mu_1}$, $\langle \gamma_i \mid i < \kappa\rangle$ is normal (without lost of generality γ_i a multiple of κ) and

$$\varphi_\mu = \bigcup_{i<\kappa} b_i``\varphi_{\gamma_i}.$$

If $b \in \mathcal{G}_{\eta,\mu}$ for $\eta < \mu$, then $b = b_i \cdot c$ for an $i < \kappa$, $c \in \mathcal{G}_{\eta,\gamma_i}$. Define

$$d^r_{\mu'} = id_{\zeta+\kappa,\mu} = b_i \cdot c$$

for an $i < \kappa$. Set

$$\begin{aligned} \delta^r_0 &= \delta^r_{\mu'0} = \zeta \\ \delta^r_{j+1} &= \gamma_{i+j} \\ s^r_{\mu',0} &= id_{\zeta,\mu_1} \\ s^r_{\mu',j+1} &= b_{i+j} \quad (j > 0)). \end{aligned}$$

□

Take $p_{\alpha+1} = r$. Observe that for $i \leq \alpha$, $\mu^i \in rng(F^{\alpha+1})$. Additionally, according to the following cases $p_{\alpha+1}$ is such that:

CASE A.1. If for all $\nu_\alpha \in D^*_{p_\alpha}$ and such that $F^{\alpha,q}(\nu_\alpha) =_{def} \nu \in D^*_q$ and $\overline{F}^q(\nu) > \overline{F}^\alpha(\nu_\alpha)$, then $\overline{d}^\alpha_{\nu_\alpha} = F^q(d)$ for some $d \in \mathcal{G}^q$. Then take $p_{\alpha+1} = q$. We observe here that in this case if

$$\delta^\alpha_{\nu_\alpha,i} \leq \overline{F}^{\alpha+1}(\nu_{\alpha+1}) < \delta^\alpha_{\nu_\alpha,i+1},$$

where $\nu = \nu_{\alpha+1}$ and $i = i_\alpha \geq 0$

CASE A.2. If there are finite $\nu^k_\alpha \in D^*_{p_\alpha}$ for $k \leq j_\alpha$ such that $F^{p_\alpha q}(\nu^k_\alpha) = \nu$ and $\overline{F}^q(\nu) > \overline{F}^\alpha(\nu^k_\alpha)$ for $0 \leq k \leq j_\alpha$ but $\overline{d}^\alpha_{\nu^k_\alpha} \notin rng(F^q)$. Using the extension lemma 47 finite times we choose $p_{\alpha+1} \leq q$ such that for all $0 \leq k \leq j_\alpha$, $\overline{d}^{\alpha}_{\nu^k_\alpha} \in rng(F^{\alpha+1})$.

NOTE In any case we have got a $\alpha+1$-sequence i_k for $k \leq \alpha$ such that

$$\delta^0_{\nu_0,l_0} \leq \overline{F}^1(\nu_1) < \delta^0_{\nu_0,l_1} \leq \cdots \leq \delta^0_{\nu_0,l_k} \leq \overline{F}^{k+1}(\nu_{k+1}) < \delta^0_{\nu_0,l_{k+1}} \leq \cdots$$

where $\nu_k \in D^*_{p_k}$ is such that $F^{k,k+1}(\nu_\kappa) = \nu_{k+1}$. Hence $\delta^{\alpha+1}_{\nu_{\alpha+1},0} = \delta^0_{\nu_0,l_\alpha}$ for some $l_\alpha < \tau^0_{\nu_0}$.

Now since $rng(s^{\alpha+1}_{\nu_{\alpha+1},0}) \subseteq rng(d^{\alpha+1}_{\nu_{\alpha+1}}) = rng(\overline{d}^{\alpha+1}_{\nu_{\alpha+1}})$, we can suppose that $s^{\alpha}_{\nu_\alpha,i_\alpha} = s^{\alpha+1}_{\nu_{\alpha+1},0} = s^0_{\nu,l_\alpha} = F^{\alpha+2}(\hat{s}_{\alpha+1})$ for some $\hat{s}_{\alpha+1} \in \mathcal{G}^{\alpha+2}$ (since $s^{\alpha+1}_{\nu_{\alpha+1},0} = \overline{d}^{\alpha+1}_{\nu_{\alpha+1}} \cdot c$ and

$$c \in \widetilde{\mathcal{G}}^{\alpha+1}_{\delta^{\alpha+1}_{\nu_{\alpha+1},0},\overline{F}^{\alpha+1}(\nu_{\alpha+1})},$$

and $\delta^{\alpha+1}_{\nu_{\alpha+1},0} \in rng(F^{\alpha+1})$, we get that $c, \overline{d}^{\alpha+1}_{\nu_{\alpha+1}} \in rng(F^{\alpha+2})$, so $s^{\alpha+1}_{\nu_{\alpha+1},0} = s^0_{\nu,l_\alpha} = F^{\alpha+2}(\hat{s}_\alpha)$). In a similar way since $\mu^i \in rng(F^{\alpha+1})$ for $i \leq \alpha$, we can suppose that $B_{ij} = (B^j)^{-1} \cdot B^i : \varphi_{\mu^i} \to \varphi_{\mu^j}$ is in $rng(F^{\alpha+1})$, i.e. there are function b_{ij} such that $B_{ij} = F^{\alpha+1}(b_{ij})$ for $i \leq j \leq \alpha$.

6.2 Limit case

Suppose $\lim(\alpha)$ and we have already chosen p_β for all $\beta < \alpha$. It is enough to considere the case $\alpha = \omega$.

Suppose p_i has been chosen for all $i < \omega$. We have now to define p^ω, i.e the level λ^ω, F^ω and $\mathcal{F}^\omega_{\alpha\lambda^\omega}$ for $\alpha < \lambda^\omega$.

We let $\lambda^\omega = \sup\{\lambda^i \mid i < \omega\}$. Now from the descending sequence of conditions $\langle p_i \mid i < \omega\rangle$ we have the sequence of functions $\langle F^i : \theta^i + 1 \to \kappa^+ + 1 \mid i < \omega\rangle$ such that if $i < j$ then $F^j = F^i \circ f$ for some $f \in \mathcal{F}^i_{\lambda_j,\lambda_i}$ which we will call F_{ij} (actually f is $(F^j)^{-1} \cdot F^i : \theta^i + 1 \to \theta^j + 1$), we consider the related directed system $\langle \theta^i, F_{ij} \mid i \leq j < \omega\rangle$ and its direct limit

$$\langle \theta^\omega, F_{i\omega} \mid \theta^i + 1 \to \theta^\omega\rangle = \text{ dir } \lim\langle \theta^i + 1, F_{ij} : i < j < \omega\rangle$$

where $\theta^\omega = \bigcup\{F_{i\omega} “\theta^i + 1 : i < \omega\} = \sup\{F_{i\omega}(\theta^i) \mid i < \omega\}$.

6.2.1 $\eta < \theta^\omega$

Define from this limit a function $F^\omega : \theta^\omega + 1 \to \kappa^+$: for $\eta < \theta^\omega$ then there is $i \in \omega$ such that $\eta \in rang(F_{i\omega})$ so $\eta = F_{i\omega}(\xi)$ for some $\xi < \theta^i$. Define

$$F^\omega(\eta) =_{def} F^i(\xi) < \kappa^+,$$

and we define:

$$F^\omega(\theta^\omega) =_{def} \overline{F}^\omega(\theta^\omega).$$

It is not difficult to prove that it is indeep a function. By definition it holds $F^i = F^\omega \cdot F_{i\omega}$ i.e. for every $i \in \omega$, F^ω extends F^i.

In a similar way we are able to define F^ω_η for $\eta < \theta^\omega$ since $\eta = F_{i\omega}(\eta')$ for some $i < \omega$ and some $\eta' \leq \theta^i$ and using the expression

$$F^i_{\eta'} = F^\omega_\eta \cdot (F_{i\omega})_{\eta'}$$

More exactly, since $\theta^\omega = \bigcup\{F_{i\omega}``\theta^i + 1 : i < \omega\}$ we define for $\eta < \theta^\omega$

$$X_\eta =_{def} \{\langle i, \eta'\rangle : F_{i\omega}(\eta') = \eta\}$$

and

$$\begin{aligned} \eta'_i &= \text{that } \eta' \text{ such that } \langle i, \eta'\rangle \in X_\eta \\ &\quad \text{for } \exists\eta'\langle i, \eta'\rangle \in X_\eta \\ \overline{X}_\eta &= \{i : \eta'_i \text{ exists }\} \end{aligned}$$

i.e. $\overline{X}_\eta$ is the set of $i < \omega$ such that $F^i(\eta'_i) = F^\omega(\eta)$. We can check easily that $\langle\langle\varphi^i_{\eta'_i}, \langle F^{ij}_{\eta'_i}\rangle \mid i \leq j < \omega, i, j \in \overline{X}_\eta\rangle$ is a directed system, so now we considere its direct limit $\langle\widetilde{\varphi}_\eta, (F_{i\omega})_{\eta'_i} \mid \varphi_{\eta'_i} \to \widetilde{\varphi}_\eta\rangle$ where

$$\widetilde{\varphi}_\eta = \bigcup\{(F_{i\omega})_{\eta'_i}``\varphi_{\eta'_i} \mid i \in \overline{X}_\eta\},$$

and define then $\varphi^\omega_\eta =_{def} \widetilde{\varphi}_\eta$ for $\eta < \theta^\omega$.

We define also $\mathcal{G}^\omega_{\xi\zeta}$ for $\xi \leq \zeta < \theta^\omega$, if $b_i \in \mathcal{G}^i_{\xi_i,\zeta_i}$ and such that $F_{i\omega}(\xi_i, \zeta_i) = (\xi, \zeta)$ then

$$\mathcal{G}^\omega_{\xi\zeta} =_{def} \{(F_{i\omega})_{\zeta_i} \cdot b_i \cdot (F_{i\omega})^{-1}_{\xi_i} : i \in \overline{X}_\zeta \text{ and } b_i \in \mathcal{G}^i_{\xi_i\zeta_i}\},$$

and if we define $F_{i\omega}(b_i) =_{def} (F_{i\omega})_{\zeta_i} \cdot b_i \cdot (F_{i\omega})^{-1}_{\xi_i}$ for $b_i \in \mathcal{G}^i_{\xi_i\zeta_i}$ then for $b \in \mathcal{G}^\omega_{\xi\zeta}$, $F^\omega(b) = F^i(b_i)$ for $i \in \overline{X}_\zeta$ i.e. $F_{i\omega}(\zeta_i) = \zeta$ since

$$\begin{aligned} F^\omega(b) &= F^\omega((F_{i\omega})_{\zeta_i} \cdot b_i \cdot (F_{i\omega})^{-1}_{\xi_i}) \\ &= F^\omega(F_{i\omega}(b_i) \cdot (F_{i\omega})_{\xi_i} \cdot (F_{i\omega})^{-1}_{\xi_i}) \\ &= F^\omega(F_{i\omega}(b_i)) \\ &= F^i(b_i). \end{aligned}$$

Now we define for $\eta < \theta^\omega$, $F^\omega_\eta : \varphi^\omega_\eta \to \varphi^\omega_{F^\omega(\eta)}$, if $\nu < \varphi^\omega_\eta$ there is $i < \omega$ such that $\eta = F_{i\omega}(\eta')$, $\nu = (F_{i\omega})_{\eta'}(\overline{\nu})$ and $\overline{\nu} < \theta^i$ then

$$\begin{aligned} F^\omega_\eta(\nu) &= F^\omega_\eta((F_{i\omega})_{\eta'}(\overline{\nu})) \\ &= F^i_{\eta'}(\overline{\nu}). \end{aligned}$$

and we have now to define

$$\overline{d}^\omega_\nu$$

and

$$\overline{F}^\omega(b)$$

for $b \in \mathcal{G}^\omega_{\xi,\nu}$ for $\nu \in (F^\omega)^{-1}(\{\kappa \cdot \rho \mid \rho < \kappa^+\})$:

CASE 1. $F^\omega(\nu) = \kappa \cdot (\rho + 1)$.
Since $\widetilde{\mathcal{G}}^\omega_{\overline{F}^\omega(\nu), F^\omega(\nu)} = \{id\}$, $\overline{d}^\omega_\nu = id_{\gamma, \overline{F}^\omega(\nu)}$ and define $\overline{F}^\omega(b) = F^\omega(b)$.

CASE 2. $F^\omega(\nu) = \kappa\rho$ for $lim(\rho)$.
If $\overline{F}^\omega(\nu) = F^\omega(\nu)$ then $\overline{d}^\omega_\nu = id_{\overline{F}^\omega(\nu), \kappa\rho}$ and define $\overline{F}^\omega(b) = F^\omega(b)$.

CASE 3. $\overline{d}^\omega_\nu$ for $\nu \in D^*_{p^\omega}$.
Let $\nu < \theta^\omega$ and let $\gamma = \overline{F}^\omega(\nu)$.

Lemma 50 *Then* $\gamma = \sup\{\overline{F}^i(\nu_i) \mid i < \omega,\ F^i(\nu_i) = F^\omega(\nu)\}$.

It is clear that $\sup\{\overline{F}^i(\nu_i) \mid i < \omega,\ F^i(\nu_i) = F^\omega(\zeta)\} \leq \gamma$. Let now $\delta < \gamma$ then there is $\xi < \nu$, $i < \omega$ such that $\delta \leq F^\omega(\xi) = F^i(\xi_i) < F^\omega(\nu) = F^i(\nu_i)$ for $\xi < \nu_i$) so $\delta < \overline{F}^i(\nu_i)$ and hence $\gamma \leq \sup\{\overline{F}^i(\nu_i) \mid i < \omega,\ F^i(\nu_i) = F^\omega(\zeta)\}$.

□

and hence writing p for p_0, $\delta_{\nu_0,\xi} = \delta^0_{\nu_0,\xi}$ and $\tau^p_{\nu_0} = \tau_{\nu_0}$.

Lemma 51 *If* $\gamma < F^\omega(\nu)$*, then* $\gamma = \delta^p_{\nu_0,\eta}$ *for some limit ordinal* $\eta < \tau^p_{\nu_0}$.

By the way we chose the sequence p_i and the normality of the function $\{\delta^p_{\nu,i} \mid i < \tau^p_\nu\}$ where $F^p(\nu_0) = \kappa\rho = F^\omega(\nu)$, take $\eta = \sup\{l_i \mid i < \omega\} < \tau^p_{\nu_0}$ hence $\delta^p_{\nu_0\eta} = \sup\{\delta^p_{\nu_0,l_i} \mid i < \omega\}$ (recall that the function is normal).

□

By the way we chose the succesor extension $p_{\alpha+1}$ and the above lemma we have that $\overline{F}^{\omega}(\nu) = \delta^0_{\nu_0,\eta} =_{def} \delta^{\omega}_{\nu,0}$. We define in this case $\overline{d}^{\omega}_{\nu} = s^0_{\nu_0,\eta} =_{def} s^{\omega}_{\nu,0}$.

Let $F^{\omega}(b) = F^i(b_i)$ for some $i < \omega$, since $\overline{d}^i_{\nu_i} = s^0_{\nu_0,j} \cdot c$ for some $j < \eta$ and some $c \in \widetilde{\mathcal{G}}_{\overline{F}^i(\nu_i),\delta^0_{\nu_0,j}}$, we define

$$\overline{F}^{\omega}(b) = (s^{\omega}_{\nu,0})^{-1} \cdot (s^0_{\nu_0,j}) \cdot c \cdot \overline{F}^i(b_i),$$

then

$$\begin{aligned} F^{\omega}(b) &= \overline{d}^{\omega}_{\nu} \cdot \overline{F}^{\omega}(b) \\ &= s^{\omega}_{\nu,0} \cdot (s^{\omega}_{\nu,0})^{-1} \cdot (s^0_{\nu_0,j}) \cdot c \cdot \overline{F}^i(b_i) \\ &= (s^0_{\nu_0,j}) \cdot c \cdot \overline{F}^i(b_i), \\ &= \overline{d}^i_{\nu_i} \cdot \overline{F}^i(b_i) \\ &= F^i(b_i). \end{aligned}$$

Similarly we can define $\overline{F}^{\omega}_{\zeta}$ in such a way that

$$F^{\omega}_{\zeta} = \overline{d}^{\omega}_{\zeta} \cdot \overline{F}^{\omega}_{\zeta},$$

and the rest we define it as follows for $\nu \in D^*_{p^{\omega}}$,

$$\begin{aligned} \tau^{\omega}_{\nu} &= \tau_{\nu_0} - \eta \\ \delta^{\omega}_{\nu,j} &= \delta^0_{\nu_0,\eta+j} \\ s^{\omega}_{\nu} &= s^0_{\nu_0,\eta}. \end{aligned}$$

Note that $s_{ij} =_{def} (s^j_{\nu_j,0})^{-1} \cdot s^i_{\nu_i,0}$ for $i < j < \omega$ is a directed sytem which directed system is the same $s_{i\eta}$ (i.e $s^0_{\nu_0,\eta}$) and moreover since $s^i_{\nu_i,0} = F^{i+1}(\hat{s}_i) = F^{\omega}(F_{i+1,\omega}(\hat{s}_i))$ for some $\hat{s}_i \in \mathcal{G}^{i+1}$, if we call $F_{i+1,\omega}(\hat{s}_i) = \tau_i$ we have that $s_{ij} = F^{\omega}(\tau_{ij})$ and $\langle \tau_i \mid i < \omega \rangle$ is also a directed system such that its directed system $\tau_{i\eta}$ satisfies $F^{\omega}(\tau_{i\eta}) = s_{i\eta}$.

6.2.2 $\eta = \theta^\omega$

We have first to define φ_{θ^ω}.

Since

$$\cdots \leq \mu^i = F^{i+1}(\zeta^{i+1}) < \overline{F}^{i+1}(\theta^{i+1}) \leq \mu^{i+1} = F^{i+2}(\zeta^{i+2}) < \overline{F}^{i+2}(\theta^{i+2}) \leq$$

we can suppose that $B_{ij} = F^{j+1}(b_{ij}) = F^\omega(F_{j+1\,\omega}(b_{ij}))$ where $b_{ij} : \varphi_{F_{ij+1}(\zeta^i)} \to \varphi_{F_{jj+1}(\zeta^j)}$, let $e_{ij} =_{def} F_{j+1\omega}(b_{ij})$, then $e_{ij} \in \mathcal{G}^\omega_{F_{i\omega}(\zeta^i)F_{j\omega}(\zeta^j)}$. Let $\langle \widetilde{\varphi}_\omega, e_{i\omega}\rangle$ be the direct limit of $\langle e_{ij}, \varphi_{F_{i\omega}(\zeta_i)} \mid i < \omega\rangle$ and $\langle \hat{\varphi}_\omega, B_{i\omega} \mid i < \omega\rangle$ the direct limit of $\langle \varphi_{\zeta_i}, B_{ij} \mid i \leq j < \omega\rangle$ (all of them are directed systems), we define

$$\begin{aligned} \varphi_{\theta^\omega} &=_{def} \widetilde{\varphi}_\omega \\ \varphi_{\mu^\omega} &=_{def} \hat{\varphi}_\omega \end{aligned}$$

since $\theta^\omega = \sup\{F_{i\omega}(\theta^i) \mid i < \omega\} = \sup\{F_{i\omega}(\zeta^i) \mid i < \omega\}$ and $e_{ij} : \varphi_{F_{i\omega}(\zeta^i)} \to \varphi_{F_{j\omega}(\zeta^j)}$. And since $B_{i\omega} = F^\omega(e_{i\omega})$, define $F^\omega_{\theta^\omega}$ using the relation:

$$F^\omega_{\theta^\omega} \cdot e_{i\omega} = B_{i\omega} \cdot F^\omega_{F_{i\omega}(\zeta^i)}$$

Define $\mu^\omega = \sup\{\mu^i \mid i < \omega\}$. So $F^\omega(\theta^\omega) = \overline{F}^\omega(\theta^\omega) = \mu^\omega$.

By the way we have also defined an embedding $F_{i\omega}$, we let $F_{i\omega} \in \mathcal{F}^\omega_{\lambda^i\lambda^\omega}$. In general for $\gamma < \lambda^\omega$ we let

$$\mathcal{F}^\omega_{\gamma\lambda^\omega} = \{f \cdot g : \exists i < \omega(\gamma < \lambda^i, g \in \mathcal{F}^i_{\gamma\lambda^i}, \text{ and } F^i = F^\omega \cdot f)\}.$$

and for $\zeta < \theta^\omega$:

$$\mathcal{G}^\omega_{\zeta,\theta^\omega} = \{e_{i\omega} \cdot \bar{b} \mid \bar{b} \in \mathcal{G}_{\zeta,F_{i\omega}(\zeta^i)} \text{ for some } i < \omega \text{ such that } \zeta < F_{i\omega}(\zeta^i)\}$$

Since $F^\omega(\theta^\omega) = \overline{F}^\omega(\theta^\omega) = \mu^\omega$ we define $\overline{d}^\omega_{\theta^\omega} = id$.

Let H be $\mathbb{P}_0$-generic over $M[G]$. In $M[G][H]$, define a simplified $(\kappa, 2)$-morass $\boldsymbol{M}$ by:

$$\begin{aligned} \boldsymbol{M} \restriction \kappa &= \bigcup_{p \in H} \boldsymbol{M}^p \\ \theta_\kappa &= \kappa^+ = \bigcup_{p\in H} rng(F^p) \\ \mathcal{F}_{\alpha,\kappa} &= \{F^p \cdot f \mid \exists p \in H,\ \lambda^p > \alpha \text{ and } f \in \mathcal{F}^p_{\eta\lambda^p}\} \text{ for } \alpha < \kappa. \end{aligned}$$

We have to check that for every $i < \omega$ there is a unique embedding f such that $F^i = F^\omega \cdot f$. We put $f \in \mathcal{F}^\omega_{\lambda^i,\lambda^\omega}$.

We have also to check that p^ω is a condition extending p_i for all $i < \omega$. For all $i < \omega$ we check that:

1) $\theta^\omega > \theta^i$, $\mu^\omega > \mu^i$, $M_0^\omega \restriction \theta^i + 1 = M_0^i$ and $M_1^\omega \restriction \mu^i + 1 = M_1^i$.

   - $\lambda^\omega > \lambda^i$. Since we define $\lambda^\omega = \sup\{\lambda^i : i < \omega\}$. $\theta^\omega > \theta^i$ and $\mu^\omega > \mu^i$
   - $\theta^i_\alpha = \theta^\omega_\alpha$ for $\alpha \leq \lambda^i$. Trivial
   - $\varphi^\omega_\zeta = \varphi^i_\zeta$ for $\zeta \leq \theta^i$.
     If $\zeta \leq \theta^i$ then $\zeta = F_{j\omega}(\zeta_j)$ for some $\zeta_j \leq \theta^j$, but also $\zeta = g(\nu)$ for some $g \in \mathcal{F}_{\lambda_l \lambda_i}$ and we can stablish an order preserving function between φ^ω_ζ and φ^i_ζ. Indeep, since
     $$\varphi^i_\zeta = \bigcup\{b``\varphi^i_\xi : \xi < \zeta, b \in \mathcal{G}^i_{\xi\zeta}\}$$
     and
     $$\varphi^\omega_\zeta = \bigcup\{(F_{j\omega})_{\zeta_j}``\varphi^j_{\zeta_j} : F_{j\omega}(\zeta_j) = \zeta\},$$
     we can stablish a bijection using the properties of neatness and the way we built each extension.
   - $\mathcal{G}^\omega_{\xi\zeta} = \mathcal{G}^i_{\xi\zeta}$ for $\xi < \zeta \leq \theta^i$.
     By above, if $b \in \mathcal{G}^\omega_{\xi\zeta}$ then $b : \varphi^\omega_\xi \to \varphi^\omega_\zeta$, but $\varphi^\omega_\xi = \varphi^i_\xi$ and $\varphi^\omega_\zeta = \varphi^i_\zeta$ so $b \in \mathcal{G}^i_{\xi,\zeta}$. Similar in the other way, using the fact that $\zeta = F_{j\omega}(\zeta_j)$ so $F_{j\omega} \in \mathcal{F}^i_{ji}$.
   - $\mathcal{F}^\omega_{\alpha\beta} = \mathcal{F}^i_{\alpha\beta}$ for $\alpha < \beta \leq \lambda^i$.

2) We defined $\mathcal{F}^\omega_{\lambda^i\lambda^\omega}$ such that $F^i = F^\omega \cdot f$, so $rng(F^i) \subset rng(F^\omega)$.

3) For every $i < \omega$ and for every $\nu_i \in D_{p_i}$: if $F_{i\omega}(\nu_i) = \nu$ and $\overline{F}^\omega(\nu) > \overline{F}^i(\nu_i)$, then $\overline{d}^i_{\nu_i} = F^\omega(d)$ for some $d \in \mathcal{F}^\omega$ (by the way we chose the extension of each condition).

So we have proved that for $i < \omega$, $p_\omega \leq p_i$.

Lemma 52 *In $M[G][H]$ there is a simplified $(\kappa, 2)$-morass.*

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Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfaßt und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.
Franqui Cárdenas

31. August 2004