The Schrödinger functional for Gross-Neveu models

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Herr Dipl.-Phys. Björn Leder
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Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. Christoph Markschies

Dekan der
Mathematisch-Naturwissenschaftlichen Fakultät I:
Prof. Dr. Christian Limberg

Gutachter:

1. Dr. Rainer Sommer
2. Prof. Dr. Ulrich Wolff
3. Dr. Peter Weisz

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Abstract

Gross-Neveu type models with a finite number of fermion flavours are studied on a two-dimensional Euclidean space-time lattice. The models are asymptotically free and are invariant under a chiral symmetry. These similarities to QCD make them perfect benchmark systems for fermion actions used in large scale lattice QCD computations. The Schrödinger functional for the Gross-Neveu models is defined for both, Wilson and Ginsparg-Wilson fermions, and shown to be renormalisable in 1-loop lattice perturbation theory.

In two dimensions four fermion interactions of the Gross-Neveu models have dimensionless coupling constants. The symmetry properties of the four fermion interaction terms and the relations among them are discussed. For Wilson fermions chiral symmetry is explicitly broken and additional terms must be included in the action. Chiral symmetry is restored up to cut-off effects by tuning the bare mass and one of the couplings. The critical mass and the symmetry restoring coupling are computed to second order in lattice perturbation theory.

This result is used in the 1-loop computation of the renormalised couplings and the associated beta-functions. The renormalised couplings are defined in terms of suitable boundary-to-boundary correlation functions. In the computation the known first order coefficients of the beta-functions are reproduced. One of the couplings is found to have a vanishing beta-function. The calculation is repeated for the recently proposed Schrödinger functional with exact chiral symmetry, i.e. Ginsparg-Wilson fermions. The renormalisation pattern is found to be the same as in the Wilson case. Using the regularisation dependent finite part of the renormalised couplings, the ratio of the Lambda-parameters is computed.

Keywords:
Chiral Gross-Neveu model, Ginsparg-Wilson fermions, Schrödinger functional, Lattice perturbation theory

Zusammenfassung

In dieser Arbeit werden Gross-Neveu Modelle mit einer endlichen Anzahl von Fermiontypen auf einem zweidimensionalen Euklidischen Raumzeitgitter betrachtet. Modelle dieses Typs sind asymptotisch frei und invariant unter einer chiralen Symmetrie. Aufgrund dieser Gemeinsamkeiten mit QCD sind sie sehr gut geeignet als Testumgebungen für Fermionwirkungen die in großangelegten Gitter-QCD-Rechnungen benutzt werden. Das Schrödinger Funktional für die Gross-Neveu Modelle wird definiert für Wilson und Ginsparg-Wilson Fermionen. In 1-Schleifenstörungstheorie wird seine Renormierbarkeit gezeigt.

Die Vier-Fermionwechselwirkungen der Gross-Neveu Modelle habe dimensionslose Kopplungskonstanten in zwei Dimensionen. Die Symmetrieeigenschaften der Vier-Fermionwechselwirkungen und deren Beziehungen untereinander werden diskutiert. Im Fall von Wilson Fermionen ist die chirale Symmetrie explizit gebrochen und zusätzliche Terme müssen in die Wirkung aufgenommen werden. Die chirale Symmetrie wird durch das Einstellen der nackten Masse und einer der Kopplungen bis auf Cut-off-Effekte wiederhergestellt. Die kritische Masse und die symmetriewiederherstellende Kopplung werden bis zur zweiten Ordnung in Gitterstörungstheorie berechnet.

Dieses Resultat wird in der 1-Schleifenberechnung der renormierten Kopplungen und der zugehörigen Betafunktionen benutzt. Die renormierten Kopplungen werden definiert mit Hilfe von geeignete Rand-Rand-Korrelatoren. Die Rechnung reproduziert die bekannten führenden Koeffizienten der Betafunktionen. Eine der Kopplungen hat eine verschwindende Betafunktion. Die Rechnung wird mit dem vor kurzem vorgeschlagenen Schrödinger Funktional mit exakter chiraler Symmetrie, also Ginsparg Wilson Fermionen, wiederholt. Es werden die gleichen Divergenzen gefunden, wie im Fall von Wilson Fermionen. Unter Benutzung des regularisierungsabhängigen, endlichen Teils der renormierten Kopplungen werden die Verhältnisse der Lambda-Parameter bestimmt.

Schlagwörter:
Chirales Gross-Neveu Modell, Ginsparg-Wilson Fermionen, Schrödinger Funktional, Gitterstörungstheorie

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Chapter 1

Introduction

The theory of strong interactions, quantum chromodynamics (QCD), is a gauge theory with few parameters and is though assumed to describe many phenomenons, such as the mass spectrum of the hadrons and scattering processes involving quarks and gluons. Experimental data is available from the low energy regime of the light meson masses to the high energy regime of hadron-hadron scattering [1].

At high energies the fundamental degrees of freedom, the quarks and the gluons, are only weakly coupled. In such a situation a perturbative treatment, where the interactions of the quarks and gluons are small corrections, is justified. Indeed, perturbative calculations successfully describe the data, for example, of deep inelastic hadron-lepton scattering. At low energies the coupling becomes large and the interactions are not small but dominant. Therefore the perturbative approach is not applicable in this regime. In addition, the relevant degrees of freedom are no longer the fundamental quarks and gluons, but the lightest bound states (the light mesons, i.e. pions). Chiral perturbation theory has been developed to accommodate this. But as an effective theory it has to be matched to the experiments, thus losing the appeal of a first principle computation.

The lattice discretisation of quantum field theories is a powerful method that was first applied to QCD long ago by Wilson [2]. Since then lattice QCD has proved to allow for non-perturbative calculations from first principle and to connect the low and high energy regime (see [3] for an example). First of all the lattice regulates the theory, in which the inverse lattice spacing $1/a$ serves as a sharp momentum cut-off and thus renders the theory ultraviolet finite. In the infrared a non-vanishing mass or a finite volume scheme with specific boundary conditions like the Schrödinger functional (SF) provides a lower bound on the modes in the theory. (The merits of the SF are described in Chapter 4.) In this way a quantum field

theory on the lattice is mathematically well defined without reference to perturbation theory. Nevertheless perturbation theory can be used at this point. Being in general more complicated than similar studies in the continuum, lattice perturbation theory is needed, for example, to translate lattice results into the language of continuum renormalisation schemes like $\overline{\text{MS}}$ (minimal subtraction), that are mostly used by experimentalists. Furthermore, in the weak coupling regime, it serves as guidance and cross check for non-perturbative methods.

If the metric of the lattice theory is the Euclidean one, the path integral representation of quantum field theory is accessible to numerical evaluation via Monte Carlo simulations. Today this is the prominent approach to extract physics from lattice QCD. Since the days of Wilsons proposal there has been substantial progress in the understanding of lattice QCD as well as in the algorithms that are used to perform the simulations. Still, the by definition limited computational resources are the main obstacle to accomplish the goal of producing predictions without any compromise.

Along the way so called "toy models" were studied. The term refers to quantum field theories that are simpler but in some respects similar to QCD. They were used, for example, to test new methods [4] or to conjecture the phase structure of lattice QCD [5]. One reason to consider these simpler models is that there might be analytical tools at hand that allow one to solve the model exactly. Often another advantage is that Monte Carlo simulations of the toy model are much more cheaper. The knowledge and results gained in the simpler models can then be used to argue in the more complicated theory. Or assumptions and approximations used in lattice QCD without a chance to prove their validity there, can be applied and confronted with the exact result and/or high precision numerical data in the simpler theory.

For example, in order to make predictions about the real world, the lattice discretisation, as any regulator, has to be removed in the end. Numerical simulations of lattice QCD are only possible at finite lattice spacing a. In practice one computes the observable of interest for a number of lattice spacings and extrapolates to the continuum limit. If the measured points show a significant dependence on the lattice spacing one has to assume a functional form to perform the extrapolation. The only known prescription of the lattice artefacts goes back to the work of Symanzik [6–9]. His conclusions for the functional form of lattice artefacts in lattice field theories are based on an effective theory and perturbation theory. Using this form in the extrapolation step of lattice QCD computations introduces a possible source of systematic errors in a presumably first principle computation. In the two dimensional and asymptotically free non-linear sigma

models these aspects can be studied with high precision Monte Carlo simulations and analytic tools [10]. The message for lattice QCD is clear: try to avoid the extrapolation step. This can be achieved partly by eliminating the leading lattice artefacts by implementing Symanzik's improvement program [8, 9].

At the time of writing this thesis a number of collaborations of lattice physicists are simulating full lattice QCD with two or three light quarks and have presented first results [11–17]. The main difference between the approaches followed by these collaborations is in the used fermion action. Beside the already mentioned Wilson fermions, there are Wilson twisted mass, overlap, staggered fermions and the fixed point action. There are also differences in the gauge action and the various kinds of improvement applied, but here we concentrate on the fermionic part of the action (see [18] for an overview).

Some of these fermion actions are more theoretically sound than others. The hope is, based on universality arguments, that in the continuum limit, where the correlation length diverges, differences on the scale of the lattice spacing are unimportant. Clearly, a numerical test of this presumed agreement in the continuum limit is desirable. Due to the restricted numerical resources of today this is impossible in lattice QCD in the near future. In such a situation a two dimensional non-trivial fermionic quantum field theory could serve as a benchmark system. If the actions agree in the toy model, it would not be a proof for QCD, but some confidence would be gained. If they do not agree, it would be clear that there are serious problems and that most probably also lattice QCD simulations are affected.

In this work we study models of self-coupled fermions in two dimensional space-time. There are several theories of this kind in two dimensions referred to as Gross-Neveu [19] and Thirring models [20]. We consider here the first type. Among them the most similar to QCD is the chiral Gross-Neveu model (CGN) with N types (in the following referred to as flavours) of fermions. They are coupled through quartic interaction terms. Since in two dimensions the fermion fields have mass dimension $1/2$, the corresponding couplings are dimensionless. The CGN shares with QCD the features asymptotic freedom and a continuous chiral symmetry (in the massless theory). The continuum CGN has been studied in perturbation theory up to three loops [21–23]. Beyond perturbation theory the S-matrix and the particle spectrum are known to some extent [24–28]. Many properties of the model can be studied in the limit of infinite many flavours (large-N limit), which is also sensitive to the non-perturbative nature of the model. In this limit the model is asymptotically free and a fermion

mass is dynamically generated [19, 29].

On the lattice the model has been studied so far almost exclusively in the large-N limit [30, 31]. In this thesis we define the CGN with a finite number of fermion flavours on the lattice. For the fermion action we use Wilson's version [2] since it is the most rigorous and theoretically sound one. After the theory has been established in this way, it can be used to check other actions. As a first application we analyse a recently proposed Dirac operator [32] that is expected to be compatible with the Schrödinger functional boundary conditions and, at the same time, is a solution to the Ginsparg-Wilson relation [33] in the bulk of the lattice (up to exponentially decreasing tails). A Dirac operator that satisfies this relation has better chiral properties than standard Wilson fermions and is thus better suited in cases were chiral symmetry plays an important role. Since Ginsparg-Wilson fermions are very expensive in terms of computational costs, precise studies in two dimensions are very welcome. We define observables suitable for Monte Carlo simulations [34] and compute them in first order lattice perturbation theory. This is the first computation with Ginsparg-Wilson fermions in the Schrödinger functional beyond the free theory.

This thesis is organised as follows. In Chapter 2 we give a short review of the perturbative renormalisation and discretisation of quantum field theories. In Chapter 3 aspects of chiral symmetry on the lattice are addressed and the chiral properties of the Dirac operators used in this thesis are outlined. As indicated above we define the theory on a lattice with boundaries. The specific form of the boundary conditions and the implications of the presence of the boundaries on the renormalisability are discussed thoroughly in Chapter 4. The two dimensional fermionic model we utilise in this work is carefully defined in Chapter 5. In particular, we are concerned with the symmetries of the model, its lattice formulation and renormalisability. Since Wilson fermions explicitly break chiral symmetry, it has to be assured that the symmetry is recovered in the continuum limit. The employed strategy and the result are presented in Chapter 6. We use this result in Chapter 7 to define renormalised couplings. A next-to-leading order computation is then carried out for Wilson and Ginsparg-Wilson fermions. We draw conclusions and give an outlook in Chapter 8.

Chapter 2

Lattice perturbation theory

In this Chapter we introduce the basic concepts used in this thesis. Since we want to use lattice perturbation theory, we have to discuss perturbative renormalisation (Section 2.1). After these general remarks we list our notation and conventions for the lattice computation (Section 2.2). We close the Chapter with some remarks on the continuum limit and the analysis of lattice diagrams (Section 2.3).

2.1 Renormalisation

The bare couplings and masses that appear as parameters in the classical action of a quantum field theory are not the couplings and masses which are measured in experiments. Experimentalists rather gather data of cross sections and transition amplitudes. These quantities have to be computed in the theory that is supposed to describe the phenomenons. For each parameter in the action one input measurement is needed. Once all the parameters are fixed the theory can be used for predictions.

The crucial point of course is how many parameters are there in the action. The more parameter the less predicting power does the theory have. The number of terms in an action and thus the number of bare parameters is mainly restricted by symmetries and dimensional analysis.

Computing a cross section or transition amplitude yields a relation between an observable and the bare parameters of the theory. The observable itself may now be called coupling. In order to avoid confusion one calls it renormalised coupling since it is a redefinition of the bare coupling. In the same way all other couplings and masses may be redefined. The renormalised quantities may be regarded as the physical parameters of the theory because all observables can be expressed in terms of them.

In perturbation theory the necessity for renormalisation is encountered in the form of ultraviolet infinities when calculating loop corrections. To keep physical amplitudes finite these infinities have to be absorbed in a redefinition of the parameters and fields order by order in the perturbative expansion.

In order to handle the terms producing the infinities they first have to be rendered finite. In lattice perturbation theory the inverse lattice spacing $1/a$ provides an ultraviolet cut-off to the theory. This regularisation has to be removed before comparing with experiment. On the lattice this amounts to taking the continuum limit $a \to 0$. In this process the ultraviolet divergences show up and the renormalisation has to be implemented.

2.1.1 Mass independent renormalisation scheme

Let us consider a quantum field theory with one mass and one coupling constant that has been regularised on an infinite lattice, say QCD with N mass degenerated quarks. All information of the theory is contained in the n-point Green's functions. Any unrenormalised Green's function $\Gamma(p; g_0, m_0, 1/a)$ will then depend on the momenta of the external lines collectively labelled with p, on the bare mass m_0 and coupling constant g_0 and on the ultraviolet cut-off $1/a$.

QCD is a renormalisable quantum field theory. A renormalisation scheme is given through conditions that define the renormalised mass m_R, renormalised coupling g_R. Often also a wave function renormalisation factor Z_i for each type i of fields in the theory (i.e. in QCD one for the quark fields and one for the gluon fields) is introduced. This is not a necessity but convenient in the course of computations. Since we will consider massless field theory, with a renormalisation scale μ, a mass independent renormalisation scheme is needed to avoid infrared divergences [35]. The renormalisation conditions are then posed at the scale μ and vanishing renormalised mass. For the renormalised parameters one expects

$$Z_i = Z_i(g_0, a\mu)\,, \tag{2.1}$$

$$g_R = g_0\, Z_g(g_0, a\mu)\,, \tag{2.2}$$

$$m_R = m_q\, Z_m(g_0, a\mu)\,, \quad m_q = m_0 - m_c\,, \tag{2.3}$$

where m_c accounts for the additive mass renormalisation needed if chiral symmetry is broken by the regularisation (cf. Section 3.4). If the regularisation does not violate chiral invariance the renormalised mass vanishes at zero bare mass.

The renormalized coupling in lattice QCD, for example, may be defined through demanding the triple gluon vertex function to take its tree level value at momenta of order μ. Then the renormalised Green's functions have finite continuum limits. They are functions of the renormalised coupling and mass and are related to the bare ones as

$$\Gamma_R(p; g_R, m_R, \mu) = Z_\Gamma(g_0, a\mu)\Gamma(p; g_0, m_0, 1/a)\,, \tag{2.4}$$

where Z_Γ depends on the number and types of the external lines. (Note that there will also be some dependence on a gauge fixing parameter as in the continuum. However, this dependence disappears when physical quantities are computed and is not important for the aspects considered here. See [36] for a complete review of lattice perturbation theory.)

Eq. (2.4) really only holds in the continuum limit. At finite cut-off, that is finite lattice spacing a, perturbation theory states that the renormalised Green's functions are cut-off independent only up to terms of order a

$$\Gamma_R(p; g_R, m_R, \mu, a\mu) = \Gamma_R(p; g_R, m_R, \mu) + \mathrm{O}\left(a(\ln a)^k\right)\,, \tag{2.5}$$

at k-loop order [6]. These terms are called *scaling violations*. Since they are small near the continuum limit we suspend their discussion until Section 2.3 and neglect them in the following.

2.1.2 Renormalisation group equations

Since g_R and m_R depend on the renormalisation scale μ while Γ does not, differentiation on both sides of (2.4) yields the so called renormalisation group equations

$$\left\{\mu\frac{\partial}{\partial\mu} + \beta(g_R)\frac{\partial}{\partial g_R} + \tau(g_R)m_R\frac{\partial}{\partial m_R} - \gamma_\Gamma(g_R)\right\}\Gamma_R = 0\,, \tag{2.6}$$

where

$$\beta(g_R) \equiv \mu\frac{\partial}{\partial\mu}g_R(g_0, a\mu)\,, \tag{2.7}$$

$$\tau(g_R) \equiv \mu\frac{\partial}{\partial\mu}\ln Z_m(g_0, a\mu)\,, \tag{2.8}$$

$$\gamma_\Gamma(g_R) \equiv \mu\frac{\partial}{\partial\mu}\ln Z_\Gamma(g_0, a\mu)\,. \tag{2.9}$$

The renormalisation group functions β, τ and γ_Γ are the so called beta-function for the coupling and the anomalous dimensions of the mass and

the Green's function. (If two types of fields appear in Γ, say n_1 of type one and n_2 of type two, we have to take $Z_\Gamma = Z_1^{n_1} Z_2^{n_2}$. Then γ_Γ is the sum of the anomalous dimension of the two types $\gamma_\Gamma = n_1\gamma_1 + n_2\gamma_2$.) Note that the coefficient functions (2.7–2.9) must be independent of a because they appear in a differential equation of an cut-off independent quantity. Since they are dimensionless they must also be independent of μ. Thus they only depend on the renormalised coupling g_R.

The functions β, τ and γ_i can be calculated in perturbation theory as a power series in the renormalised coupling. In the case of QCD the beta-function

$$\beta(\alpha_s) = b_0\alpha_s^2 + b_1\alpha_s^3 + b_2\alpha_s^4 + \mathrm{O}(\alpha_s^5)\,, \tag{2.10}$$

is known to tree loops. The first two coefficients, for example, are [37–40]

$$b_0 = -\left(11 - \frac{2N}{3}\right) \quad b_1 = -\left(102 - \frac{38N}{3}\right)\,. \tag{2.11}$$

(Note that in many textbooks and publications another definition of the renormalised coupling of QCD is used. The relation to the one used here is $\alpha_s = g_R^2/4\pi$.)

2.1.3 Asymptotic freedom

From the shape of the beta-function the behaviour of the renormalised coupling at high energies may be deduced. The example above is characterised by a negative $\beta(g)$ for small $g \geq 0$ and leads to a vanishing renormalised coupling as $\mu \to \infty$. This behaviour is called asymptotic freedom. There are three other possible scenarios, we do not list them here but refer to the diverse textbooks on the topic [41–43]. To make the above statement more explicit and general, assume a beta-function that is negative for small positive g

$$\beta(g) \overset{g\to 0}{\to} -bg^n\,, \quad b > 0\,, \tag{2.12}$$

where g^n is the power of the coupling in front of the lowest-order divergent diagram contributing to $\beta(g)$ and therefore is always greater one. The renormalisation group equation is then

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}g(\mu) = -bg^n(\mu)\,. \tag{2.13}$$

Given the renormalised coupling at some scale μ the coupling at the energy E can be calculated by integrating this equation. The solution is

$$g(E) = g(\mu)\left[1 + (n-1)\,b\,\ln(E/\mu)\,(g(\mu))^{(n-1)}\right]^{-1/(n-1)}\,. \tag{2.14}$$

For $E \to \infty$ this solution becomes independent of $g(\mu)$

$$g(E) \overset{E\to\infty}{\to} \left[(n-1)\, b\, \ln(E/\mu)\right]^{-1/(n-1)} . \tag{2.15}$$

Thus starting from a value justifying the approximation (2.12) $g(E)$ always tends to zero for $E \to \infty$. On the other hand at small E the coupling may become large $g(E) > 1$. Thus perturbation theory becomes unreliably at small energies and non-perturbative methods are needed.

Along similar steps it can be shown that the effective dimensionality of operators and fields is given by dimensional analysis up to logarithmic corrections [41].

In this context it is worth mentioning that the first two coefficients of the beta-function are independent of how exactly the renormalised coupling is defined as long as for small bare coupling $g_R = g_0 + \mathrm{O}(g_0^2)$.[1] To see this assume two renormalised couplings g_A and g_B. Since there are no other dimensionless parameters g_A is a function only of g_B. We can expand the one in powers of the other

$$g_A(g_B) = g_B + c_1 g_B^2 + \mathrm{O}(g_B^3)\,, \tag{2.16}$$

or

$$g_B(g_A) = g_A - c_1 g_A^2 + \mathrm{O}(g_A^3)\,, \tag{2.17}$$

where the leading order coefficient is fixed by the condition that g_A and g_B at leading order are equal to the bare coupling. The two beta-functions can be related through

$$\beta_A(g_A) \overset{(2.7)}{=} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_A \overset{(2.16)}{=} \mu \frac{\partial g_B}{\partial \mu} \frac{\partial g_A}{\partial g_B} \overset{(2.7)}{=} \beta_B(g_B) \frac{\partial g_A}{\partial g_B}\,. \tag{2.18}$$

The beta-function has an expansion in the renormalised coupling

$$\beta_B(g_B) = b_0^B g_B^2 + b_1^B g_B^3 + \mathrm{O}(g_B^4)\,, \tag{2.19}$$

where the leading power is two but the argument holds for arbitrary leading power greater unity. In terms of g_A this becomes

$$\beta_B(g_A) = b_0^B g_A^2 + (b_1^B - 2c_1 b_0^B) g_A^3 + \mathrm{O}(g_A^4)\,, \tag{2.20}$$

and the derivative is

$$\frac{\partial g_A}{\partial g_B} = 1 + 2c_1 g_B + \mathrm{O}(g_B^2) = 1 + 2c_1 g_A + \mathrm{O}(g_A^2)\,. \tag{2.21}$$

[1] Note that notation might be misleading here. With g_R a renomalised coupling like α_s of QCD is meant. The more familiar renormlised coupling of QCD, that is called g_R, the beta-function woud start with a third power. See also the note under (2.11)

Now the right hand side of (2.18) can be evaluated in terms of g_A

$$\beta_A(g_A) = \left[b_0^B g_A^2 + (b_1^B - 2c_1 b_0^B) g_A^3 + \mathrm{O}(g_A^4)\right] \cdot \left[1 + 2c_1 g_A + \mathrm{O}(g_A^2)\right] , \tag{2.22}$$

$$= b_0^B g_A^2 + b_1^B g_A^3 + \mathrm{O}(g_A^4) , \tag{2.23}$$

proving that the first two coefficients are universal in the sense that they neither depend on the regularisation nor on the the renormalisation scheme. For the first coefficient this is a direct consequence of demanding the renormalised couplings to coincide at leading order (that is, at leading order they coincide with the unrenormalised coupling). The second order coefficients coincide because the expansion of beta-function starts at the next-to-leading order.

Finally we introduce the Λ-parameter

$$\Lambda = \mu (b_0 g_R^2)^{-b_1/(2b_0^2)} \mathrm{e}^{-1/(2b_0 g_R^2)} \cdot \exp\left\{ - \int_0^{g_R} \mathrm{d}g \, \left[\frac{1}{\beta(g)} + \frac{1}{b_0 g^3} + \frac{b_1}{b_0^2 g} \right] \right\} \tag{2.24}$$

In the massless theory the Λ-parameter is the only dimensionfull parameter. It is the standard solution of the renormalisation group equation for physical quantities

$$\left\{ \mu \frac{\partial}{\partial \mu} + \beta(g_R) \frac{\partial}{\partial g_R} \right\} P(\mu, g_R) = 0 , \tag{2.25}$$

which expresses the abitrariness of the refernce scale μ.

For each the two renormalised couplings g_A and g_B a Λ-parameter can be defined. Given the relation (2.16) between the couplings the ratio of the Λ-parameters is then a pure number [44]

$$\Lambda_A / \Lambda_B = \exp\left\{ \frac{1}{2b_0} \left(\frac{1}{g_B^2} - \frac{1}{g_A^2} \right) + \mathrm{O}(g_A^2) \right\} = \exp\left\{ \frac{c_1}{b_0} \right\} . \tag{2.26}$$

2.1.4 Multiple couplings

So far we considered theories with a single dimensionless coupling. It is not difficult to generalize the concepts to multiple such couplings [41]. There will be as many renormalised couplings g_l as bare couplings. The Green's functions depend on all these couplings and in (2.4) g_0 and g_R

may collectively refer to them. Then for each g_l there is a renormalisation group equation

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} g_l(\mu) = \beta_l(g(\mu)) \,, \tag{2.27}$$

with β_l depending in general on all the renormalised couplings g_l. In the case of one coupling the beta-function determines the asymptotic behaviour of this coupling. In the case of multiple couplings the beta-functions β_l determine the asymptotic trajectories in the space spanned by the couplings g_l. Clearly, there are many possibilities now. Let us concentrate on the prominent case of trajectories approaching a fixed point in g-space. A fixed point $g_(\mu) = g_l^*$ is defined through a mutual zero of the beta-functions

$$\beta_l(g*) = 0 \,. \tag{2.28}$$

Shifting $g_l \to g_l - g_l^*$ by the fixed point, Taylor-expanding $\beta_l(g - g^*)$ and ignoring terms $\mathrm{O}((g - g^*)^2)$ eq. (2.27) becomes

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} [g_l(\mu) - g_l^*] = \sum_k M_{lk} [g_k(\mu) - g_k^*] \,, \tag{2.29}$$

with the matrix M given by

$$M_{lk} = \left(\frac{\partial \beta_l(g)}{\partial g_k} \right)_{g=g^*} \,. \tag{2.30}$$

Suppose that the eigenvalues of this matrix are non-degenerate. Surely, that is not always true but it is the generic case. Then the eigenvectors v^m

$$\sum_k M_{lk} v_k^m = \lambda^m v_l^m \,, \tag{2.31}$$

form a complete set and can be used to express the solution to (2.29)

$$g_l(\mu) = g_l^* + \sum_m c_m v_l^m \mu^{\lambda_m} \,, \tag{2.32}$$

with coefficients c_m.

The qualitative behaviour for $\mu \to \infty$ is thus governed by the eigenvalues λ^m and the coefficients c_m. In particular, the fixed point is approached if and only if $c_m = 0$ for all $\lambda^m > 0$. A zero eigenvalue may be caused by a vanishing β_l, in which case the fixed point is reached for any value of this coupling in a region around the fixed point.

The eigenvectors of the negative eigenvalues define a subspace containing the trajectories attracted by the fixed point. Trajectories with support outside of this subspace may get very close to the fixed point but are eventually repelled.

We close this section by pointing out that the eigenvalues λ^m are invariant under a change of basis, that is going to a differently defined set of renormalised couplings $\tilde{g}_l$. The change $g \to \tilde{g}$ amounts to a similarity transformation of the matrix M and thus preserves its spectrum. This can be seen as follows. The new couplings will be functions of the gs and satisfy renormalisation group equations

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \tilde{g}_l(\mu) = \sum_m \frac{\partial \tilde{g}_l(g)}{\partial g_m} \beta_m(g) = \tilde{\beta}_l(\tilde{g}(\mu)) \,. \tag{2.33}$$

Thus β transforms as a contravariant vector in coupling space

$$\tilde{\beta}_l(\tilde{g}) = \sum_m \frac{\partial \tilde{g}_l(g)}{\partial g_m} \beta_m(g) \,. \tag{2.34}$$

Differentiating on both sides with respect to g_k we get

$$\sum_m \frac{\partial \tilde{\beta}_l(\tilde{g})}{\partial \tilde{g}_m} \frac{\partial \tilde{g}_m(g)}{\partial g_k} = \sum_m \frac{\partial^2 \tilde{g}_l}{\partial g_m \partial g_k} \beta_m(g) + \sum_m \frac{\partial \tilde{g}_l(g)}{\partial g_m} \frac{\partial \beta_m(g)}{g_k} \,. \tag{2.35}$$

At the fixed point g^* the first term on the right hand side vanishes and we are left with the matrix equation

$$\tilde{M}\, S = S\, M \,, \tag{2.36}$$

with

$$\tilde{M}_{lk} = \left(\frac{\partial \tilde{\beta}_l(\tilde{g})}{\partial \tilde{g}_k} \right)_{\tilde{g}=\tilde{g}(g^*)} \,, \quad S_{lk} = \left(\frac{\partial \tilde{g}_l(g)}{\partial g_k} \right)_{g=g^*} \,. \tag{2.37}$$

As long as S is invertible this is a similarity transformation and the eigenvalues of $\tilde{M}$ are those of M.

2.2 Discretisation

The regulator used in the computations of this thesis is the Euclidean lattice. If the (Minkowskian) time coordinate is Wick rotated to imaginary (Euclidean) time

$$x_0^E = i x_0^M \,, \tag{2.38}$$

the imaginary unit in front of the Minkowski-space action in the path integral of a quantum field theory becomes a minus sign

$$\int \mathrm{D}\, \mathrm{e}^{iS_M} \to \int \mathrm{D}\, \mathrm{e}^{-S_E}\,. \tag{2.39}$$

If the action is bounded from below (which is the case for physically relevant theories) the weight factor can be interpreted as a probability distribution for field configurations. Evidently there is a close connection between field theory and statistical physics if the weight is identified with the Boltzmann factor. The only subtle point at this stage is, one has to assure that the analytic continuation of the n-point Green's functions to imaginary time exists (see Section 1.3 in [42] and references therein). The path integral becomes a mathematically well defined object, that is a convergent multidimensional integral, by discretising Euclidean space-time on a finite lattice. This is the basis of non-perturbative Monte-Carlo techniques.

From now on we work in $D = d + 1$ dimensional space-time with Euclidean metric $\delta_{\mu\nu}$ and drop the superscript, that is x_0 refers to Euclidean time (Greek subscripts always run from 0 to d). The Euclidean Dirac matrices satisfy anti-commutation relations

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\,, \tag{2.40}$$

are all hermitian

$$\gamma_\mu^\dagger = \gamma_\mu\,, \tag{2.41}$$

and are related to their Minkowskian counterparts as

$$\gamma_0 = \gamma_0^M\,, \quad \gamma_i = -i\gamma_i^M\,. \tag{2.42}$$

We are here dealing with theories in two and four dimensions and the definition of the Euclidean γ_5 differs by a factor due to demanding hermiticity

$$D = 2:\ \gamma_5 = i\gamma_0\gamma_1\,, \quad D = 4:\ \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3\,. \tag{2.43}$$

Explicit representations of the Dirac matrices are given in the Appendix A.1.1.

Although the ultimate goal is to consider field theories on a lattice with boundaries we start here with the more common hypercubic lattice with either infinite extension or periodic boundary conditions. The sites of the lattice are labelled by $x_\mu = an_\mu$ with integer n_μ and lattice spacing a, which is the same in all directions. On a finite lattice the coordinates are restricted to $0 \leq x_\mu < L$ giving a total number of lattice sites $V/a^D = L^D/a^D$.

Continuum space-time integrals are on the lattice replaced by sums over all lattice sites

$$\int \mathrm{d}^D x \to \; a^D \sum_x . \tag{2.44}$$

The lattice spacing a is the minimal distance in the system and thus introduces an ultraviolet cut-off. The momenta can be restricted to the first Brillouin zone

$$-\frac{\pi}{a} < p_\mu \leq \frac{\pi}{a} . \tag{2.45}$$

On an infinite lattice continuum momentum integrals are cut-off

$$\int \frac{\mathrm{d}^D p}{(2\pi)^D} \; \to \; \int_{-\pi/a}^{+\pi/a} \frac{\mathrm{d}^D p}{(2\pi)^D} . \tag{2.46}$$

On a finite lattice the allowed momenta are a discrete set in the range (2.45). For periodic boundary conditions and integer n_μ the V/a^D allowed momenta are

$$p_\mu = \frac{2\pi n_\mu}{L} , \quad n_\mu = -L/2 < n_\mu \leq L/2 , \tag{2.47}$$

and the momentum integrals also become momentum sums

$$\int_{-\pi/a}^{+\pi/a} \frac{\mathrm{d}^D p}{(2\pi)^D} \; \to \; \frac{1}{V} \sum_p . \tag{2.48}$$

The discretisation of integrals was straightforward. However, continuum differential operators have infinitely many valid lattice representations. We introduce here the simplest possibilities which will serve as building blocks for more difficult choices. We consider lattice fields $\psi(x)$ defined at the sites of the lattice. On a finite lattice we have to specify boundary conditions. We choose general periodic boundary conditions

$$\psi(x + L\hat{\mu}) = \mathrm{e}^{ia\theta_\mu} \psi(x) , \quad -\pi < \theta_\mu \leq \pi , \tag{2.49}$$

parametrised by the phases θ_μ. The forward and backward finite difference operators are

$$\partial_\mu \psi(x) = \tfrac{1}{a} [\psi(x + a\hat{\mu}) - \psi(x)] , \tag{2.50}$$

$$\partial^*_\mu \psi(x) = \tfrac{1}{a} [\psi(x) - \psi(x - a\hat{\mu})] , \tag{2.51}$$

where $\hat{\mu}$ is a unit vector in μ-direction.

There is a different way of incorporating such general boundary conditions. One takes the lattice fields as periodic

$$\psi(x + L\hat{\mu}) = \psi(x)\,, \tag{2.52}$$

and defines the forward and backward finite difference operators as

$$\partial_\mu \psi(x) = \tfrac{1}{a}[\lambda_\mu \psi(x + a\hat{\mu}) - \psi(x)]\,, \tag{2.53}$$

$$\partial^*_\mu \psi(x) = \tfrac{1}{a}[\psi(x) - \lambda_\mu^{-1}\psi(x - a\hat{\mu})]\,. \tag{2.54}$$

The phase factors λ_μ depend on θ_μ

$$\lambda_\mu = \mathrm{e}^{ia\theta_\mu/L}\,, \quad -\pi < \theta_\mu \leq \pi\,. \tag{2.55}$$

The two notations are connected by an Abelian gauge transformation. A non-zero θ_μ introduces a "momentum" that is not restricted to the values of (2.47). The construction with the phase factors λ_μ in the finite differences is computationally easier and therefore adopted here. On infinite and periodic lattices the forward and backward finite difference operators obey

$$(\partial_\mu)^\dagger = -\partial^*_\mu\,. \tag{2.56}$$

Other important lattice operators are the anti-hermitian averaged finite difference operator

$$\widetilde{\partial}_\mu \psi(x) = \tfrac{1}{2}(\partial_\mu + \partial^*_\mu)\psi(x) = \tfrac{1}{2a}[\lambda_\mu \psi(x + a\hat{\mu}) - \lambda_\mu^{-1}\psi(x - a\hat{\mu})]\,, \tag{2.57}$$

and the hermitian lattice Laplace operator

$$\partial_\mu \partial^*_\mu \psi(x) = \partial_\mu \partial^*_\mu \psi(x) = \tfrac{1}{a^2}[\lambda_\mu \psi(x + a\hat{\mu}) + \lambda_\mu^{-1}\psi(x - a\hat{\mu}) - 2\psi(x)]\,. \tag{2.58}$$

As already mentioned there is some freedom in discretising differential operators and therefore the action of a given field theory. This is due to the smaller symmetry on the lattice. In particular Lorentz invariance is broken and an infinite number of (in this case irrelevant from the point of view of renormalisation) terms can appear in the lattice action. In this way infinitely many different lattice actions for the same continuum theory are possible. However, they are expected to be equal in the continuum limit, that is when the cut-off is removed. One says the different lattice actions fall into the same universality class characterised by the target continuum theory.

In numerical Monte-Carlo computations one is interested in lattice actions that balance between complexity (numerical cost) and the rate at which the continuum limit is approached (systematic error). Lattice perturbation theory is an essential tool to provide analytic understanding of these lattice artefacts.

2.3 Continuum limit and lattice artefacts

The renormalisation group function $\beta(g_R)$ describes the variation of the renormalised coupling g_R with the cut-off at fixed bare coupling

$$\beta(g_R) = \mu \frac{\partial g_R(g_0, a\mu)}{\partial \mu}\bigg|_{g_0} = a\mu \frac{\partial g_R(g_0, a\mu)}{\partial a\mu}\bigg|_{g_0} . \tag{2.59}$$

Now we can ask how the bare coupling has to be varied with the cut-off for fixed renormalised coupling. The lattice beta-function $\beta_{\mathrm{LAT}}(g_0)$ is defined through

$$a\frac{\mathrm{d}}{\mathrm{d}a} g_R = \left\{ a\frac{\partial}{\partial a} - \beta_{\mathrm{LAT}}(g_0)\frac{\partial}{\partial g_0} \right\} g_R(g_0, a\mu) = 0 . \tag{2.60}$$

$$\beta_{\mathrm{LAT}}(g_0) = -a\frac{\partial g_0(g_R, a\mu)}{\partial a}\bigg|_{g_R} . \tag{2.61}$$

Since

$$a\frac{\partial}{\partial a} g_R(g_0, a\mu) = \mu\frac{\partial}{\partial \mu} g_R(g_0, a\mu) = \beta(g_R) , \tag{2.62}$$

the two beta-function are related in the following way

$$\beta_{\mathrm{LAT}}(g_0)\frac{\partial g_R(g_0, a\mu)}{\partial g_0} = \beta(g_R) . \tag{2.63}$$

And because at lowest order in perturbation theory the both couplings are equal, $g_R = g_0 + \mathrm{O}(g_0^2)$, we know from Section 2.1.3, that the first two coefficients of the beta-functions are identical. In particular, this means that the bare coupling vanishes in the limit $a \to 0$.

As mentioned at the end of Section 2.1.1 renormalised lattice Green's functions have a finite continuum limit $a \to 0$ and differ from this limit by terms of order a

$$\Gamma_R(p; g_R, m_R, \mu, a\mu) = \Gamma_R(p; g_R, m_R, \mu) + \mathrm{O}\left(a(\ln a)^k\right) , \tag{2.64}$$

at k-loop order in perturbation theory [6]. It is widely believed that this behaviour also holds beyond perturbation theory and non-perturbative Monte Carlo data (naturally only available at finite a) is extrapolated to the continuum accordingly. Since perturbative computations are the only analytic tool to learn about the size of these *lattice artefacts* such computations are essential, especially if new methods are used. For example, the

amplitude of the lattice artefacts can be very different for different lattice actions.

In order to obtain numbers in lattice perturbation theory one almost always is forced to evaluate some lattice or momentum sums numerically. Consider a quantity $P(a/L)$ that is a sum of lattice diagrams at 2-loop order, where we suppress any dependence on the external lines. We supose that P is dimensionless. If it is not, it can be made so by appropriate factors of a. Futhermore we suppose that it is finite at $a/L = 0$, which always can be achieved by multiplication with appropriate factors of a/L. Then P has an expansion in a/L

$$P(a/L) = \sum_{n=0}^{\infty} \left[r_n + s_n \ln(a/L) + t_n \ln^2(a/L) \right] (a/L)^n . \tag{2.65}$$

The generalisation to abitrary loop order should be obvious.

In practice, $P(a/L)$ is computed at several values of a/L. In order to extract the coefficients of the expansion 2.65 we use the method described in Appendix D of Ref. [45]. In this way it is possible to reliably determine the systematic uncertainties that are inevitable involved in such a numerical analysis of lattice Feynman diagrams.

Chapter 3

Chiral symmetry on the lattice

The Euclidean Lagrangian of N free fermions is

$$\mathcal{L} = \overline{\psi}(x)\,(\gamma_\mu \partial_\mu + m)\,\psi(x)\,, \tag{3.1}$$

where the fermion fields carry suppressed Dirac and flavour indices. The flavour indices, labelling the N fermions, are contracted by a unit matrix in flavour space. For $m = 0$ the theory is invariant under a global $U(N) \times U(N)$ flavour symmetry which can be decomposed into $U(1)_V \times U(1)_A$ transformations

$$\psi \to \mathrm{e}^{i\omega_V + i\gamma_5\omega_A}\,\psi\,, \quad \overline{\psi} \to \overline{\psi}\,\mathrm{e}^{-i\omega_V + i\gamma_5\omega_A}\,, \tag{3.2}$$

acting equally on all flavours and chiral $SU(N) \times SU(N)$ transformations

$$\psi \to \mathrm{e}^{i\theta_V^a\lambda^a + i\gamma_5\theta_A^a\lambda^a}\,\psi\,, \quad \overline{\psi} \to \overline{\psi}\,\mathrm{e}^{-i\theta_V^a\lambda^a + i\gamma_5\theta_A^a\lambda^a}\,, \tag{3.3}$$

where the generators of the $SU(N)$ algebra λ^a act on the flavour indices. [1] The subscripts V and A refer to the associated Noether currents (see next section), which are of Lorentz vector and axial-vector type:

$$V_\mu = \overline{\psi}\gamma_\mu\psi\,, \quad A_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi\,, \tag{3.4}$$

and

$$V_\mu^a = \overline{\psi}\gamma_\mu\lambda^a\psi\,, \quad A_\mu^a = \overline{\psi}\gamma_\mu\gamma_5\lambda^a\psi\,. \tag{3.5}$$

We use the same letters for the $SU(N)$ singlet (3.4) and vector currents (3.5) but indicate the difference by an additional superscript for the vector ones.

[1] The generators λ_a are normalised to obey $\mathrm{Tr}\left\{\lambda^a\,\lambda^b\right\} = 2\delta_{ab}$. See Appendix A.1.2 for more details.

We can introduce a new set of generators

$$t_L^a = (1-\gamma_5)\lambda^a\,, \quad t_R^a = (1+\gamma_5)\lambda^a\,, \tag{3.6}$$

with commutation relations

$$[t_L^a, t_L^b] = 2if^{abc}t_L^c\,, \tag{3.7}$$
$$[t_R^a, t_R^b] = 2if^{abc}t_R^c\,, \tag{3.8}$$
$$[t_L^a, t_R^b] = 0\,. \tag{3.9}$$

The t_Ls and t_Rs obviously form two closed subalgebras. Thus, as indicated by the notation, chiral $SU(N) \times SU(N)$ is the direct sum of two $SU(N)$ subgroups acting independently on the left- and right-handed components of the fermion field

$$\psi_L = \tfrac{1}{2}(1-\gamma_5)\psi\,, \quad \psi_R = \tfrac{1}{2}(1+\gamma_5)\psi\,. \tag{3.10}$$

A general mass term $\overline{\psi}m\psi$ with $m = \mathrm{diag}(m_1, \ldots, m_N)$ in (3.1) breaks all of these symmetries but multiplication with an $U(1)_V$ phase. N mass degenerated fermions ($m \propto 1$) lift this to an $U(N)_V$ since also the generators λ^a remain unbroken.

In QCD chiral symmetry plays a key role in understanding the mass spectrum of the light mesons. The pions, for example, are seen as the Goldstone bosons [46, 47] associated with the spontaneously axial generators of $SU(2) \times SU(2)$ [48]. The QCD Lagrangian with only the two light u and d quarks has this symmetry in the massless limit, which is a good approximation because $m_{u,d}$ is much smaller than a typical hadron mass scale. Indeed isospin symmetry and quark number conservation, $U(1)_V$ and $SU(N)_V$ respectively, are experimentally confirmed to high precision [1]. The axial generators $\gamma_5\lambda^a$ are spontaneously broken in the quantum theory by a non-vanishing quark condensate leading to three massless Goldstone bosons: the pions. The small but non-zero mass of the pions is due to the fact that $SU(2) \times SU(2)$ is only an approximate symmetry.

So far, nothing has been said about the axial $U(1)_A$, i.e. continuous chiral phase transformations $\psi \to e^{i\omega_A\gamma_5}\psi$. In QCD this symmetry is also broken, but it must be in a different way since there is no light iso-singlet state expected from the Goldstone theorem. The effect can be traced back to the topological structure of the vacuum gauge field [49] and also explains the suppression of the electromagnetic decay of the neutral pion [43]. However, this chiral or axial *anomaly* is tightly connected to gauge symmetry. The two-dimensional fermion model considered in this thesis has no gauge symmetry and therefore the $U(1)_A$ is not anomalous.

As already stated above understanding the structure of the low energy regime of QCD needs understanding of chiral symmetry and how it is violated. Since the QCD coupling is large (cf. Section 2.1.3) at low energies non-perturbative methods such as numerical lattice QCD are needed. The discretisations used today handle chiral symmetry quite differently. We will discuss two different lattice Dirac operators and present their chiral properties. But before that we introduce the currents associated with the symmetries and derive operator identities from infinitesimal variable transformations in the path integral.

3.1 Continuum Ward identities

Here we derive operator identities, so called Ward identities. They are conveniently derived by variable transformations in the path integral for the expectation value of operators, where the new variables are connected to the old ones by infinitesimal *local* transformations

$$\psi \to \psi + \delta\psi\,, \qquad \overline{\psi} \to \overline{\psi} + \delta\overline{\psi}\,. \tag{3.11}$$

The expectation value of a local operator $\mathcal{O}$, that is a field composed of fermion fields and their derivatives evaluated at the same space-time point, is

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathrm{D}\overline{\psi}\mathrm{D}\psi\ \mathcal{O}\ \mathrm{e}^{-S}\,, \tag{3.12}$$

with the Euclidean action

$$S = \int \mathrm{d}^D x\ \mathcal{L} = \int \mathrm{d}^D x\ \overline{\psi}(x)\,(\gamma_\mu \partial_\mu + m)\,\psi(x)\,. \tag{3.13}$$

A linear change of the Grassmann valued fermion fields in (3.12) not only effects the operator and the action but also the integration measure by introducing a non-trivial Jacobian

$$\psi' = A\psi\,, \qquad \overline{\psi}' = \bar{A}\overline{\psi}\,, \tag{3.14}$$

$$\mathrm{D}\overline{\psi}\mathrm{D}\psi = \det A \det \bar{A} \cdot \mathrm{D}\overline{\psi}'\mathrm{D}\psi' = J \cdot \mathrm{D}\overline{\psi}'\mathrm{D}\psi'\,. \tag{3.15}$$

For simplicity we assume the infinitesimal transformations (3.11) can be represented by

$$A = 1 + \omega X\,, \quad \bar{A} = 1 + \omega \bar{X}\,, \tag{3.16}$$

where ω is an infinitesimal function of space-time evaluated at the point of the field and $X\,, \bar{X}$ are matrices acting on Dirac and flavour indices. Then

we can use $\det(1+\omega X) = 1 + \omega \operatorname{Tr}\{X\} + \mathrm{O}(\omega^2)$ to write $J = 1 + \delta J + \mathrm{O}(\omega^2)$.

Let δS and $\delta\mathcal{O}$ be the change in the action and the operator for a given change of fermion fields. Performing an infinitesimal change of variables in the path integral yields

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathrm{D}\overline{\psi}\mathrm{D}\psi \; \mathcal{O} \; \mathrm{e}^{-S}\,, \tag{3.17}$$

$$= \frac{1}{Z} \int \mathrm{D}\overline{\psi}\mathrm{D}\psi \; (1 + \delta J + \mathrm{O}(\omega^2)) \, (\mathcal{O} + \delta\mathcal{O}) \, \mathrm{e}^{-S}(1 - \delta S)\,, \tag{3.18}$$

$$= \langle \mathcal{O} \rangle + \langle \delta\mathcal{O} \rangle - \langle \mathcal{O}\, \delta S \rangle + \langle \mathcal{O}\, \delta J \rangle + \mathrm{O}(\omega^2)\,. \tag{3.19}$$

Thus we derived the general identity

$$\langle \delta\mathcal{O} \rangle - \langle \mathcal{O}\, \delta S \rangle + \langle \mathcal{O}\, \delta J \rangle = 0\,. \tag{3.20}$$

Let us restrict $\omega(x)$ to a region $\mathcal{R}$, that is $\omega(x) = 0$ for $x \notin \mathcal{R}$. If we choose an operator $\mathcal{O}_{\text{ext}}$ defined outside the region $\mathcal{R}$ where the variable transformation is performed then $\delta\mathcal{O}_{\text{ext}} = 0$. Likewise the change in the action is supported only in $\mathcal{R}$

$$\delta S = \int_{\mathcal{R}} \mathrm{d}^D x \; \delta\mathcal{L}\,. \tag{3.21}$$

Eq. (3.20) further simplifies to

$$\langle \mathcal{O}_{\text{ext}}\, \delta S \rangle = \langle \mathcal{O}_{\text{ext}}\, \delta J \rangle\,. \tag{3.22}$$

Now we promote the global transformations of (3.2) and (3.3) to local but infinitesimal transformations like

$$a) \quad u(1)_V\colon \quad \delta\psi = i\omega_V\psi\,, \qquad \delta\overline{\psi} = -i\omega_V\overline{\psi} \tag{3.23}$$

$$b) \quad u(1)_A\colon \quad \delta\psi = i\omega_A\gamma_5\psi\,, \qquad \delta\overline{\psi} = i\omega_A\overline{\psi}\gamma_5 \tag{3.24}$$

$$c) \quad su(N)_V\colon \quad \delta\psi = i\omega_V^a\lambda^a\psi\,, \qquad \delta\overline{\psi} = -i\omega_V^a\overline{\psi}\lambda^a \tag{3.25}$$

$$d) \quad su(N)_A\colon \quad \delta\psi = i\omega_A^a\lambda^a\gamma_5\psi\,, \qquad \delta\overline{\psi} = i\omega_A^a\overline{\psi}\gamma_5\lambda^a\,, \tag{3.26}$$

where the ωs are now infinitesimal functions of space-time evaluated at the point of the field.

For the $SU(N)$ vector transformations $c)$ and $d)$ the Jacobian is unity ($\delta J = 0$) because the $SU(N)$ generators are traceless. The change in the action is

$$\delta_V S = \int_{\mathcal{R}} \mathrm{d}^D x \; \omega_V^a \left[-\partial_\mu V_\mu^a(x) - \overline{\psi}(x)[\lambda^a, m]\psi(x) \right]\,, \tag{3.27}$$

and (3.28)

$$\delta_A S = \int_{\mathcal{R}} \mathrm{d}^D x \; \omega_A^a \left[-\partial_\mu A_\mu^a(x) + 2mP^a(x) \right]\,, \tag{3.29}$$

respectively. Then the identity (3.22) becomes

$$\left\langle \partial_\mu V^a_\mu(x)\, \mathcal{O}_{\text{ext}} \right\rangle = \left\langle \overline{\psi}(x)[m, \lambda^a]\psi(x)\, \mathcal{O}_{\text{ext}} \right\rangle \tag{3.30}$$

and

$$\left\langle \partial_\mu A^a_\mu(x)\, \mathcal{O}_{\text{ext}} \right\rangle = \left\langle \overline{\psi}(x)\gamma_5\{\lambda^a, m\}\psi(x)\, \mathcal{O}_{\text{ext}} \right\rangle . \tag{3.31}$$

In deriving these identities one has to make use of the arbitrariness of the exact definition of $\omega(x)$. For mass degenerate fermions one finds the $SU(N)$ vector current conservation

$$\left\langle \partial_\mu V^a_\mu(x)\, \mathcal{O}_{\text{ext}} \right\rangle = 0 \tag{3.32}$$

and the *partially conserved axial current* (PCAC)

$$\left\langle \partial_\mu A^a_\mu(x)\, \mathcal{O}_{\text{ext}} \right\rangle = 2m \left\langle P^a(x)\, \mathcal{O}_{\text{ext}} \right\rangle , \tag{3.33}$$

with the flavour vector pseudo-scalar density

$$P^a(x) = \overline{\psi}(x)\gamma_5\lambda^a\psi(x) . \tag{3.34}$$

Although we considered here free fermions without gauge interactions the calculation in QCD yields the same identities (see [49] for example).

For the $U(1)$ transformations $a)$ and $b)$ the change in the action is

$$\delta_V S = \int_{\mathcal{R}} \mathrm{d}^D x\; \omega_V \left[-\partial_\mu V_\mu(x) \right] , \tag{3.35}$$

and (3.36)

$$\delta_A S = \int_{\mathcal{R}} \mathrm{d}^D x\; \omega_A \left[-\partial_\mu A_\mu(x) + 2mP(x) \right] . \tag{3.37}$$

The corresponding operator identities are the fermion number conservation

$$\left\langle \partial_\mu V_\mu(x)\, \mathcal{O}_{\text{ext}} \right\rangle = 0 \tag{3.38}$$

and the singlet PCAC relation

$$\left\langle \partial_\mu A_\mu(x)\, \mathcal{O}_{\text{ext}} \right\rangle = 2m \left\langle P(x)\, \mathcal{O}_{\text{ext}} \right\rangle , \tag{3.39}$$

with the singlet pseudo-scalar density

$$P(x) = \overline{\psi}(x)\gamma_5\psi(x) . \tag{3.40}$$

As already mentioned in QCD the $U(1)_A$ symmetry is anomalous and $A_\mu(x)$ is not conserved in the quantum theory even for vanishing masses. Along the presented steps, also in QCD we would have arrived at (3.39), in contradiction to the anomaly. This is because the treatment here was very formal. That is, we performed variable transformations in an integral that is not well defined in first place. If QCD is regularised first (3.39) is changed and the anomaly is recovered. Using Ginsparg-Wilson fermions the anomaly can be computed easily [50].

3.2 Lattice Ward identities

The steps that led us to the continuum PCAC relations can be applied on the lattice [51]. The expressions for lattice currents are then multilocal, in the sense that they involve more then one lattice site, and additional terms can appear that are allowed by the lattice symmetries. However, the difference between multilocal and local[2] becomes irrelevant in the continuum limit. Futhermore the continuum PCAC relations are part of the definition of the theory and thus, at finite lattice spacing and for properly renormalised operators they hold up to cutoff effects.

In perturbation theory on expects terms of $\mathrm{O}(a\ln(a)^l)$ in l-loop order. In the case of massless fermions, for example, eq. (3.33) implies the lattice version

$$\langle \widetilde{\partial}_\mu \left(A^a_\mu(x) \right)_{\mathrm{R}} \, (\mathcal{O}_{\mathrm{ext}})_{\mathrm{R}} \rangle = \mathrm{O}(a\ln(a)^l)\,. \tag{3.41}$$

On the lattice we also use the local currents and densities. Thus the continuum definitions (3.5), (3.4), (3.34), (3.40) transfer to lattice. Although the axial-vector and vector currents are conserved in the massless theory they receive a finite renormalistion on the lattice

$$\left(A^a_\mu(x) \right)_{\mathrm{R}} = Z_A A^a_\mu(x) \quad \text{and} \quad \left(A_\mu(x) \right)_{\mathrm{R}} = Z_A A_\mu(x)\,, \tag{3.42}$$

$$\left(V^a_\mu(x) \right)_{\mathrm{R}} = Z_V V^a_\mu(x) \quad \text{and} \quad \left(V_\mu(x) \right)_{\mathrm{R}} = Z_V V_\mu(x)\,, \tag{3.43}$$

with

$$Z_I = 1 + Z_I^{(1)}\, g^2 + \ldots\,. \tag{3.44}$$

3.3 Nielsen-Ninomiya theorem

Before discussing the two lattice Dirac operators used in this thesis we briefly describe the broader context of lattice fermions and chiral symmetry. As we will see chiral symmetry in the lattice regularisation is tightly connected to other desirable properties of the operator $D = \gamma_\mu D_\mu$ in the massless action

$$S = a^D \sum_x \overline{\psi}(x)\, D\psi(x)\,. \tag{3.45}$$

[2]Here locality refers to operators composed of fields that are all taken at the same space-time point. It may not be confused with the locality of a differential operator like the Dirac operator.

The kernel $D(x-y)$ and the Fourier transform $\widetilde{D}(p)$ of the Dirac operator are defined through

$$D\psi(x) = a^D \sum_y D(x-y)\psi(y)\,, \quad D(x-y) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^D p}{(2\pi)^D}\, \mathrm{e}^{ip(x-y)} \widetilde{D}(p)\,. \tag{3.46}$$

To give an example consider the simplest lattice Dirac operator one could think of

$$D_{\text{naive}} = \tfrac{1}{2}\{\gamma_\mu(\partial_\mu^* + \partial_\mu)\}\,, \tag{3.47}$$

that is the averaged finite difference (2.57) contracted with the gamma-matrices. Indeed, D_{naive} anti-commutes with γ_5 and thus (3.45) would be invariant under the chiral transformations (3.24) and (3.26). But it turns out that this Dirac operator leads to a proliferation of fermion species. The operator in Fourier space

$$\widetilde{D}_{\text{naive}}(p) = \tfrac{1}{a}\sum_\mu \sin(ap_\mu)\,, \quad -\pi/a < p_\mu \leq \pi/a\,, \tag{3.48}$$

has 4 and 16 zeros in two and four dimensions respectively. Thus $\widetilde{D}^{-1}_{\text{naive}}$ is not only propagating a single physical fermion (the zero at $p_\mu = 0$), but also the so called *doubler modes*.

This is not an accident. The appearance of the doublers is in accordance with the Nielsen-Ninomiya theorem [52–55]. It can be stated in the form:[3]

Theorem 1 *Any massless lattice Dirac operator cannot satisfy the following four properties simultaneously:*

a) $aD(x)$ *is local in the sense that it is bounded by* $C\mathrm{e}^{-\gamma|x-y|/a}$

b) $\widetilde{D}(p) = i\gamma_\mu p_\mu + \mathrm{O}(ap^2)$ *for* $|p| \ll \pi/a$

c) $\widetilde{D}(p)$ *is invertible at all non-zero momenta*

d) $\gamma_5 D - D\gamma_5 = 0$

In the bound in a) C and $\gamma > 0$ are constants that do not depend on a. With respect to the universality of the continuum limit a Dirac operator with such exponentially small tails is certainly as good as an ultra-local operator with only nearest neighbour interactions. The property b) is the right continuum behaviour at small momentum. Doublers are excluded

[3] The theorem was originally proven for fermions on two- and four-dimensional lattices [54].

by c) and chiral symmetry in the form of (3.24) and (3.26) is guaranteed by d).

Any fermion discretisation in two or four dimensions has to abandon at least one of these desirable properties. In the example given in this section one finds a bunch of doubler modes, i.e. c) is violated. In the free theory this may not be a problem, but in the interacting theory they would contribute in the loop corrections.

In the following two sections we discuss the Wilson and the Ginsparg-Wilson fermions, both abandoning d), but in a very different way and with very different consequences.

3.4 Wilson fermions

In the view of the last section Wilson's approach [56] to remove the doublers is to sacrifice chiral symmetry by adding an irrelevant term to (3.47). The Wilson-Dirac operator is given by[4]

$$D_{\mathrm{W}} = \tfrac{1}{2}\{\gamma_\mu(\partial^*_\mu + \partial_\mu) - a\partial^*_\mu\partial_\mu\}\,, \tag{3.49}$$

and corresponding massive Dirac operator is simply

$$D_m = D_{\mathrm{W}} + m_0\,. \tag{3.50}$$

Looking at the Fourier transform

$$\widetilde{D}_{\mathrm{W}}(p) = \tfrac{1}{a}\sum_\mu\{\gamma_\mu \sin(ap_\mu) + \tfrac{2}{a}\sin^2(ap_\mu/2)\}\,, \tag{3.51}$$

we see that the doublers receive a mass of the order of the cut-off and hence are strongly suppressed at finite a and eventually disappear in the continuum limit.

The price to pay is that the lattice Laplace operator $a\partial^*_\mu\partial_\mu$ explicitly breaks chiral symmetry at finite lattice spacing. As a consequence a vanishing bare mass does not automatically imply a conserved axial current as in the continuum (3.33). But this operator identity has to hold for the renormalised quantities also on the lattice up to scaling violations

$$\left\langle\left[\widetilde{\partial}_\mu\left(A^a_\mu(x)\right)_{\mathrm{R}} - 2m_R\left(P(x)\right)_{\mathrm{R}}\right]\left(\mathcal{O}_{\mathrm{ext}}\right)_{\mathrm{R}}\right\rangle = \mathrm{O}(a\ln(a)^l)\,. \tag{3.52}$$

[4]Often there is an additional parameter $0 < r \leq 1$ multiplying the lattice Laplace operator: $-ar\partial^*_\mu\partial_\mu$. Throughout this thesis we set $r = 1$.

In fact, together with (2.3)

$$m_R = m_q Z_m(g_0, a\mu)\,, \quad m_q = m_0 - m_c\,, \tag{3.53}$$

eq. (3.52) serves as definition of the *critical mass* m_c [51], that is, the value of the bare mass at which the axial current is conserved up to scaling violations

$$m_0 = m_c(g) \quad \text{such that} \quad \langle \widetilde{\partial}_\mu \left(A^a_\mu(x) \right)_{\mathrm{R}} \, (\mathcal{O}_{\mathrm{ext}})_{\mathrm{R}} \rangle = \mathrm{O}(a \ln(a)^l)\,. \tag{3.54}$$

In the two-dimensional Gross-Neveu model the global chiral $U(1)$ symmetry is not anomalous and thus a critical mass for the conservation of the singlet axial current can be defined

$$m_0 = m_c(g) \quad \text{such that} \quad \langle \widetilde{\partial}_\mu \left(A_\mu(x) \right)_{\mathrm{R}} \, (\mathcal{O}_{\mathrm{ext}})_{\mathrm{R}} \rangle = \mathrm{O}(a \ln(a)^l)\,. \tag{3.55}$$

In this thesis we employ the latter definition. As indicated the critical mass can be computed in perturbation theory as a power series in the coupling constant

$$m_c(g) = m_c^{(0)} + m_c^{(1)} g^2 + m_c^{(2)} g^4 + \mathrm{O}(g^6)\,. \tag{3.56}$$

Having mass dimension one, m_c will cancel the linear divergence spoiling the PCAC relation ((3.33) or (3.39)) due to the explicitly broken chiral symmetry. However, there may also be a term of order $\mathrm{O}(1)$. In QCD such a term is not present because there is no chiral symmetry breaking operator of mass dimension four. Therefore it is enough to tune m_c. But in two dimensions there are chiral symmetry breaking four fermion interactions and one has to tune a dimensionless parameter in addition to m_c (see Section 5.3.1).

3.5 Ginsparg-Wilson fermions

From the Nielsen-Ninomiya theorem (page 25) one may conclude that it is not possible to construct a meaningful lattice theory of fermions with chiral symmetry. However, the way out was discovered shortly after the original paper by Nielsen and Ninomiya. Studying block-spin renormalisation in lattice QCD, Ginsparg and Wilson [33] found the relation[5]

$$\gamma_5 D + D \gamma_5 = a D \gamma_5 D\,, \tag{3.57}$$

[5] In Ref. [33] the authors actually derive a more general relation, but (3.57) is the most spread version and is used solely in this thesis.

for the lattice Dirac operator. But it was not recognised at the time and remained unappreciated until Hasenfratz [57] "rediscovered" it in the context of the perfect action. Indeed, the Dirac operator of the perfect action obeys the Ginsparg-Wilson relation (3.57).

In the following Neuberger [58] found an explicit solution to (3.57) and the dimensional reduced Dirac operator of the domain wall fermion was recognised to obey the Ginsparg-Wilson relation [59].

Before turning to the solution presented by Neuberger we point out that the importance of (3.57) is due to the fact that it allows for an exact chiral symmetry [50]. Consider the infinitesimal transformation of the fermion fields[6]

$$\psi \to \psi + \epsilon\gamma_5(1-aD)\psi\,, \quad \overline{\psi} \to \overline{\psi} + \epsilon\overline{\psi}\gamma_5\,. \tag{3.58}$$

This is an exact symmetry of the action (3.45)

$$\delta S = a^D \sum_x \epsilon\overline{\psi}(x)\,(D\gamma_5(1-aD)+\gamma_5 D)\,\psi(x) \overset{(3.57)}{=} 0\,. \tag{3.59}$$

In the continuum limit (3.58) become the familiar chiral phase transformations (3.24). Chiral flavour transformations are defined analogously by replacing $\epsilon\gamma_5 \to \epsilon^a\gamma_5\lambda^a$ in (3.58).

As a consequence on the lattice one gets (3.41), and similar for the singlet axial current, for free. Of course there is a exactly conserved current corresponding to the symmetry (3.58), but it is complicated in structure and not ultra-local.

The Neuberger-Dirac operator [58] is given by the expression

$$D = \frac{1}{\bar{a}}\left\{1 - A\,(A^\dagger A)^{-1/2}\right\}\,, \quad \bar{a} = \frac{a}{1+s}\,, \tag{3.60}$$

$$A = 1 + s - aD_{\mathrm{W}}\,, \tag{3.61}$$

with the Wilson Dirac operator D_{W} as introduced in Section 3.49 and a parameter $0 \le s \le 1/2$ that allows for some optimisation. For any A obeying $A\gamma_5 = \gamma_5 A^\dagger$ this operator satisfies (3.57) with a replaced by $\bar{a}$. That is certainly the case for D_{W}. More properties of the operator (3.60) are discussed in [60]. The best choice for the massive Dirac operator, for example, is

$$D_m = (1 - \tfrac{1}{2}\bar{a}m)\,D + m\,. \tag{3.62}$$

[6] Note that the fermion fields can be transformed independently in Euclidean space.

Chapter 4

The Schrödinger functional

Nonperturbative lattice QCD is set up to deliver predictions from first principles without uncontrolled approximations, mostly in the form of MC simulations. One ingredient on the way from numbers to physics is renormalisation. In order to stay on the road of first principles, also the renormalisation has to be done non-perturbatively. Here the Schrödinger functional (SF) has proved to be a powerful framework.

The Schrödinger functional in a quantum field theory is the transition amplitude between field configurations at time zero and some later time. In an Euclidean prescription it can be represented by a functional integral

$$Z = \int \mathrm{e}^{-S}, \tag{4.1}$$

where the fields are defined on a space-time manifold with boundaries and obey Dirichlet boundary conditions.

There exists a reasonable amount of literature about the Schrödinger functional in lattice QCD and its merits. One of the motivations to study the SF was, that it provides a infrared cut-off $\mathrm{O}(1/T)$ to the theory [61], where $T \propto L$ is the time extension of the lattice. Since it is a finite size regularisation scheme, the size of the system L is a natural scale in the theory. By employing finite size recursion techniques, it is thus possible to connect the low energy regime of the light mesons with the high energy regime of perturbative QCD. In this way the running of the strong coupling and the fundamental parameters of QCD can be computed [3, 11, 62]. We can not give a thorough introduction to these topics here. Instead we refer the reader to the Les Houches lectures by Lüscher [49], which give an overview about the techniques. In the present chapter we discuss the renormalisability of the SF and on its lattice representation.

In the continuum formulation of the Schrödinger functional the fermion fields are defined in the time interval $x_0 = [0, T]$. At the boundaries they are subject to the conditions

$$P_+\psi(x) = \overline{\psi}(x)P_- = 0 \quad \text{at} \quad x_0 = 0\,, \tag{4.2}$$

$$P_-\psi(x) = \overline{\psi}(x)P_+ = 0 \quad \text{at} \quad x_0 = T\,. \tag{4.3}$$

The projectors occurring in these equations are defined as

$$P_\pm = \tfrac{1}{2}(1 \pm \gamma_0)\,. \tag{4.4}$$

Note that these boundary conditions are invariant under space rotations, parity, time reflections and charge conjugation, but not under chiral transformations (3.2) and (3.3).

In this chapter we concentrate on bilinear fermionic actions

$$S_\mathrm{F} = \int_0^T \mathrm{d}\, x_0 \int \mathrm{d}^d\, \mathbf{x}\; \overline{\psi}(x)\, D\, \psi(x)\,, \tag{4.5}$$

$$D = \gamma_\mu \partial_\mu + \mathcal{A} + m_0\,, \tag{4.6}$$

where $d = 1$ or $d = 3$ and $\mathcal{A}(x)$ is the sum of bosonic fields mediating the interactions. In the case of QCD obviously $\mathcal{A} = \gamma_\mu A_\mu(x)$, with the gauge field $A_\mu(x)$. In the case of the GN the four fermion interaction terms can be replaced by bosonic auxiliary fields in the functional integral. The fermion action is then also of the form (4.5) (cf. Section C.1). The gauge field or the auxiliary fields will play a spectator role most of the time. However, desirable properties like renormalisability and locality (in the lattice formulation) of the theory may depend on their details.

In the standard formulation of the SF inhomogeneous boundary conditions are adopted and the boundary values are used as sources for the fermion fields at the boundary [61]. With the homogeneous boundary conditions of Eqs. (4.2, 4.3) the boundary fermion fields can be defined through the non-zero Dirac components

$$\zeta(\mathbf{x}) = P_-\psi(x)\,, \quad \overline{\zeta}(\mathbf{x}) = \overline{\psi}(x)P_+ \quad \text{at} \quad x_0 = 0\,, \tag{4.7}$$

$$\zeta'(\mathbf{x}) = P_+\psi(x)\,, \quad \overline{\zeta}'(\mathbf{x}) = \overline{\psi}(x)P_- \quad \text{at} \quad x_0 = T\,. \tag{4.8}$$

This is the convention used in [32].

The boundary conditions of the SF may cause additional divergences. In quantum field theory one has learned to deal with divergences. They are absorbed into a "normalisation" of the fields and parameter of the theory. If one gets away with a redefinition of a finite number of parameters,

the theory is called renormalisable. It is then not obvious, that the theory with the boundaries stays finite.

Symanzik argues in [63] (see also [64] for an introduction) that the SF of any renormalisable quantum field theory can be rendered finite by adding a finite number of boundary counterterms. These are local polynomials in the fields and their derivatives, integrated over the boundary. They are restricted by the symmetries of the theory and have mass dimension d or less.

The argument is based on explicit a calculation in scalar ϕ^4 theory, where one has to add two new boundary counterterms ϕ^2 and $\phi\partial_0\phi$. The expectation could be shown to hold up to 2-loop of perturbation theory in SU(N) Yang-Mills theory [65, 66] and in QCD [45, 67]. In the former case the symmetries forbid any boundary counterterms and in the later they can be absorbed in a multiplicative renormalisation of the quark fields at the boundary . Although a proof is still missing, there is little doubt that the SF of QCD is renormalisable, given the success of the method in non-perturbative computations [68].

Discretising the Schrödinger functional of a given quantum field theory involves then two questions. Boundary conditions like Eqs. (4.2, 4.3) make only sense when imposed on smooth functions, but the lattice is discrete by definition and the lattice fields are only defined at the sites of the lattice. So the question arises how the continuum boundary conditions are represented on the lattice. Furthermore the lattice breaks some continuum symmetries, like continuous rotations, translations and chiral transformations. This may give rise not only to new terms in the bulk of the lattice, but also to additional terms at the boundaries.

In Section 4.1 we address the first question with an heuristic argument to make plausible, that the Schrödinger functional boundary conditions arise naturally in the continuum limit and need no special adjustments of the lattice action. Then in Section 4.2 we get explicit and present formulae for the well studied lattice SF with Wilson fermions. We present a rather recent proposal [32] for an overlap Dirac operator in the SF in Section 4.3. There is another proposal for a SF with Ginsparg-Wilson fermions by Taniguchi [69]. There the boundary conditions are realised through an orbifold projection. However, there are some technical difficulties, i.e the fermion determinant has a phase and fermions masses can not be introduced straightforwardly. Therefore we consider only the proposal of Ref. [32]. At the end of this Chapter we come back to the question of boundary counter terms.

4.1 Lattices with boundaries

Discretising a field theory involves some freedom in the definition of the lattice action. For example the differential operator ∂_μ can be replaced by the forward finite difference $a\partial_\mu\psi(x) = \psi(x+a\mu) - \psi(x)$ or by the averaged finite difference $a\widetilde{\partial}_\mu\psi(x) = \psi(x+a\mu) - \psi(x-a\mu)$. That both choices lead to the same continuum theory is assured by the local nature of the resulting actions. In the continuum limit the relevant length scale of the theory becomes much larger than the lattice spacing a and the microscopic (length scale a) details become irrelevant. One says the two actions belong to the same universality class.

These classes are characterised by global properties like dimensionality and symmetries. In statistical mechanics universal behaviour occurs in the vicinity of the critical lines and there the concept of universality has been extended to systems with boundaries (see [70] for a review). This means for the discretisation of field theories that boundary conditions imposed in the continuum theory, together with the dimensionality and the symmetries of the theory, define an universality class. In general requiring locality, symmetries and power-counting reduce the number of possible boundary conditions and therefore the number of universality classes.

The line of argument followed here is borrowed from Section 3 in Ref. [32] and we also start with the simple example of a free scalar field.

4.1.1 Free scalar field

We consider a free scalar field in the half-space $x_0 \leq 0$. We do not specify the boundary condition at $x_0 = 0$, but rather explore where we are lead to in the continuum limit starting from different lattice actions. The lattice fields $\phi(x)$ are defined at the sites of a hypercubic lattice with spacing a. Only the fields at $x_0 = a, 2a, 3a, \ldots$ are dynamical degrees of freedom and are integrated over in the functional integral. A possible lattice action reads

$$S = a^4 \sum_{x_0 \geq a} \sum_{\mathbf{x}} \tfrac{1}{2}\{\partial_\mu\phi(x)\partial_\mu\phi(x) + m^2\phi(x)^2\}\,, \tag{4.9}$$

where ∂_μ is the forward difference operator in μ direction. Note that the action depends only on the dynamical degrees of freedom.

In order to calculate the propagator we write Eq. (4.9) as a quadratic form in ϕ

$$S = a^4 \sum_{x_0 \geq a} \sum_{\mathbf{x}} \tfrac{1}{2}\phi(x)\{-\partial^*_\mu\partial_\mu + m^2 + P\}\phi(x) \tag{4.10}$$

with the backward difference operator ∂^*_μ. Replacing the symmetric nearest neighbour interaction of Eq. (4.9) by the one in Eq. (4.10) produces some mismatched terms near the boundary. This is cured by the inclusion of the boundary term

$$P = -\frac{1}{a}\,\delta_{x_0,a}\,\partial^*_0\,. \tag{4.11}$$

The defining equation of the propagator $G(x,y)$ follows directly from the action (4.10)

$$(-\partial^*_\mu\partial_\mu + m^2 + P)\,G(x,y) = a^{-4}\delta_{x,y}\,, \quad x_0\,, y_0 \geq a\,. \tag{4.12}$$

Due to translation invariance in the space directions the propagator can be calculated in a time-momentum representation

$$G(x,y) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\mathrm{e}^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}\,\widetilde{G}(x_0,y_0,\mathbf{p})\,. \tag{4.13}$$

This is done by first determining the eigenfunctions of $-\partial^*_\mu\partial_\mu + m^2$ that are annihilated by P

$$\mathrm{e}^{i\mathbf{p}\mathbf{x}}\,\cos\left((x_0-\tfrac{a}{2})p_0\right)\,. \tag{4.14}$$

It is then easy to write down the propagator

$$\widetilde{G}(x_0,y_0,\mathbf{p}) = \int_0^{2\pi/a}\frac{\mathrm{d}p_0}{\pi}\,\frac{1}{m^2+\hat{p}^2}\,\cos\left((x_0-\tfrac{a}{2})p_0\right)\cos\left((y_0-\tfrac{a}{2})p_0\right)\,. \tag{4.15}$$

In the continuum limit the integral can be done and the result is

$$G_{\mathrm{cont}}(x,y) = \int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\frac{\mathrm{e}^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}}{2\epsilon(\mathbf{p})}\,\left(\mathrm{e}^{-\epsilon(\mathbf{p})|x_0-y_0|}+\mathrm{e}^{-\epsilon(\mathbf{p})(x_0+y_0)}\right)\,, \tag{4.16}$$

$$\epsilon(\mathbf{p}) = \sqrt{m^2+\hat{\mathbf{p}}^2}\,. \tag{4.17}$$

Examining the behaviour at the boundary one infers that the propagator satisfies Neumann boundary conditions in the continuum limit

$$\partial_0\,G_{\mathrm{cont}}(x,y)\big|_{x_0=0} = 0\,. \tag{4.18}$$

Now we slightly modify the action by adding a boundary term

$$S \to S + a^3\sum_{\mathbf{x}}\frac{c}{2a}\,\phi(x)^2\Big|_{x_0=a}\,. \tag{4.19}$$

Note that the powers of a are such that $c > 0$ is dimensionless. The propagator is still given by (4.12) but now P has an additional term

$$P = \frac{1}{a}\,\delta_{x_0,a}\,\left(-\partial_0^* + \frac{c}{a}\right)\,. \tag{4.20}$$

The eigenfunctions that are annihilated by P are a bit more complicated

$$\mathrm{e}^{i\mathbf{px}}\,\sin\left((x_0-a)p_0+\varphi(p_0)\right)\,, \tag{4.21}$$

$$\varphi(p_0) = \arctan\left(\frac{\sin(p_0)}{\cos(p_0)+c-1}\right)\,. \tag{4.22}$$

The propagator is similar to (4.15) but constructed with the eigenfunctions (4.21). Again the integral can be performed in the continuum limit and the result is

$$G^c_{\mathrm{cont}}(x,y) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\frac{\mathrm{e}^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}}{2\epsilon(\mathbf{p})}\,\left(\mathrm{e}^{-\epsilon(\mathbf{p})|x_0-y_0|} - \mathrm{e}^{-\epsilon(\mathbf{p})(x_0+y_0)}\right)\,. \tag{4.23}$$

This propagator satisfies Dirichlet boundary conditions

$$G^c_{\mathrm{cont}}(x,y)\big|_{x_0=0} = 0\,. \tag{4.24}$$

A small change in the action led to a totally different class of boundary conditions. This can be made a bit more transparent by looking at the field equation

$$\left\langle \eta(x)\phi(y)\right\rangle = a^{-4}\delta_{x,y}\,,\quad \eta(x) = \frac{\delta S}{\delta\phi(x)}\,. \tag{4.25}$$

From the action in the form (4.19) we derive

$$\eta(x) = \left\{-\partial_\mu^*\partial_\mu + m^2\right\}\phi(x)\,,\quad x_0 > a\,. \tag{4.26}$$

Thus in the bulk of the lattice we find the Klein-Gordon equation. But at the boundary $x_0 = a$ we find

$$\eta(x) = \frac{c}{a^2}\,\phi(x) - \frac{1}{a}\,\partial_0\,\phi(x) + \left\{-\partial_k^*\partial_k + m^2\right\}\phi(x)\,. \tag{4.27}$$

The negative powers of a in front of the first two terms are due to the fact that they are supported only at the boundary. Taking the limit $a \to 0$ the first term dominates for all $c > 0$. Thus in the continuum limit the field equation at the boundary implies Dirichlet boundary conditions at $x_0 = 0$. In the somehow special case $c = 0$ the second term dominates and the field equation implies Neumann boundary conditions.

4.1.2 The SF universality classes

In the example of the last section no boundary conditions were imposed onto the lattice fields. The boundary conditions were encoded in the lattice action and emerged when we took the limit $a \to 0$. But we found two distinct classes of boundary conditions. One that is generic, in the sense that it is found for a wide range of actions (all $c > 0$). And one that is sensitive to small perturbations of the action.

Clearly a free scalar field is a trivial example. In the case of interacting theories or more complicated actions the analysis of the field equations will not be so transparent. But boundary conditions that respect locality will always be of the form

$$\mathcal{O}(x)\big|_{x_0=0} = 0\,, \tag{4.28}$$

where $\mathcal{O}(x)$ is a linear combination of local fields and their derivatives with the appropriate symmetry properties.

Inspired by the above example it is plausible that the generic boundary conditions, that are stable under perturbations of the lattice action, are those imposed on the fields with the lowest dimension. All other possible boundary conditions will require some tuning of the lattice action, unless there are symmetries that protect them.

We want to apply this argument now to the case of fermions in two (GN) and four dimensions (QCD). In both cases the fermions are represented on the lattice by Dirac spinors $\psi(x)$ at each lattice site. Both theories are asymptotically free, assuring that the scaling dimension of local fields is equal to their engineering dimension (cf. Section 2.1). Thus the fields of lowest dimension are the fermion fields themselves and the generic boundary conditions are of the from

$$B\,\psi(x)\big|_{x_0=0} = 0\,, \tag{4.29}$$

where B is a constant matrix with Dirac, flavour and, in the case of QCD, colour indices. Boundary conditions at a later time $x_0 = T$ are then linked to those at $x_0 = 0$ by time reflection symmetry and the conditions for the anti-fermion field are given by charge conjugation symmetry.

The possible boundary conditions are further restricted by the lattice symmetries. If the lattice theory is invariant under gauge and flavour transformations, cubic rotations and parity, so should be the boundary conditions. And finally the matrix B cannot have full rank since the Dirac equation is a first order differential equation and the two conditions at

$x_0 = 0$ and $x_0 = T$ would imply a vanishing fermion propagator. Up to a constant factor there are only two matrices satisfying all these properties

$$B = P_- \quad \text{and} \quad B = P_+ \,. \tag{4.30}$$

The homogeneous Dirichlet boundary conditions of the Schrödinger functional are therefore the generic ones and do not need any fine tuning or particular adjustment of the lattice action. There are two universality classes of lattice theories which differ by the sign in the boundary condition

$$P_\pm \, \psi(x)\big|_{x_0=0} = 0 \,. \tag{4.31}$$

Since the two are connected by a finite chiral transformation, the difference matters only for non-vanishing fermion masses. In this case the sign can be determined by inspection of the free propagator.

4.2 Free Wilson fermions

In the continuum the Schrödinger functional can be represented by a functional integral

$$Z = \int \mathrm{D}\psi \mathrm{D}\overline{\psi} \, \mathrm{e}^{-S_\mathrm{F}[\overline{\psi},\psi]} \,, \tag{4.32}$$

where the fields obey Dirichlet boundary conditions (4.2–4.3) and the action may be given by (4.5). For simplicity we omit any interaction (gauge or four fermion) and introduce a $d+1$ dimensional lattice with spacing a and label the sites by integer multiples $x_\mu/a \in \mathsf{Z}\,,\ \mu = 0,\ldots,d$. The space directions are taken to be periodic with length L

$$\psi(x + L\hat{k}) = \psi(x) \,, \tag{4.33}$$

where $\hat{k}$ is an unit length vector in direction $k = 1,\ldots,d$. The fermion fields at times $x_0 = a, 2a, \ldots, T-a$ are the dynamical degrees of freedom (the fields that are integrated over in the functional integral). It is convenient to assume that the fermion fields are defined at all other values of x_0 as well, but that they are zero there. The free lattice action takes then the familiar form

$$S_0 = a^D \sum_x \overline{\psi}(x) \, D_m \, \psi(x) \,, \tag{4.34}$$

where D_m is some discretisation of the massive Dirac operator.

4.2.1 Dirac operator and propagator

In the present section we consider

$$D_m = D_{\mathrm{W}} + m_0\,, \tag{4.35}$$

where the Wilson Dirac operator is defined as (cf. Section 3.4)

$$D_{\mathrm{W}} = \tfrac{1}{2}\{\gamma_\mu(\partial^*_\mu + \partial_\mu) - a\partial^*_\mu\partial_\mu\}\,, \tag{4.36}$$

in the range $0 < x_0 < T$. At all other times the target field $\chi = D_m\psi$ is set to zero[1]

$$D_m\psi(x)\Big|_{x_0\le 0} = 0 = D_m\psi(x)\Big|_{x_0\ge T}\,. \tag{4.37}$$

Note that we set $r = 1$ and that we introduce factors $\mathrm{e}^{ia\theta_k/L}$ in the spacial lattice difference operators (which is equivalent with periodic boundary conditions with a phase, see Section 2.2). For $\theta = 0$ this is the Dirac operator introduced in [61].

Thus the Dirac operator can be considered as a linear mapping in the space of fermion fields that vanish at the boundaries. The propagator $S(x,y)$ is defined through

$$\Big(\tfrac{1}{2}\{\gamma_\mu(\partial^*_\mu + \partial_\mu) - a\partial^*_\mu\partial_\mu\} + m_0\Big)\; S(x,y) = \frac{1}{a^{d+1}}\,\delta_{x,y}\,, \quad 0 < x, y < T\,, \tag{4.38}$$

with boundary values

$$P_+\,S(x,y)\Big|_{x_0=0} = P_-\,S(x,y)\Big|_{x_0=T} = 0\,. \tag{4.39}$$

Since the operator $\gamma_5 D_m$ is hermitian the propagator has the property

$$S(x,y)^\dagger = \gamma_5\,S(y,x)\,\gamma_5\,. \tag{4.40}$$

In the free theory it is possible to derive an explicit expression for the propagator in a time-momentum representation [71]. An elegant form is

$$S(x,y) = (D^\dagger_{\mathrm{W}} + m_0)\,G(x,y)\,, \quad 0 < x_0, y_0 < T\,, \tag{4.41}$$

[1] The Dirac operator maps the space of fermion fields that are defined at all x_0, but are zero at $x_0 < a$ and $x_0 > T - a$, into itself.

where $G(x,y)$ is defined through

$$\begin{aligned} G(x,y) &= L^{-d} \sum_{\mathbf{p}} \Big\{ -2i \mathring{p}_0^+ A(\mathbf{p}^+) R(p^+) \Big\}^{-1} \, \mathrm{e}^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \\ &\times \Big\{ (M(p^+) - i \mathring{p}_0^+) \, \mathrm{e}^{-\omega(\mathbf{p})|x_0-y_0|} + (M(p^+) + i \mathring{p}_0^+) \, \mathrm{e}^{-\omega(\mathbf{p})(2T-|x_0-y_0|)} \\ &- (M(p^+) + i\gamma_0 \mathring{p}_0^+) \, \mathrm{e}^{-\omega(\mathbf{p})(x_0+y_0)} - (M(p^+) - i\gamma_0 \mathring{p}_0^+) \, \mathrm{e}^{-\omega(\mathbf{p})(2T-x_0-y_0)} \Big\} . \end{aligned} \tag{4.42}$$

All undefined functions and notations in this expression are introduced in Appendix A.2. In particular the momenta p_μ are given by eq. A.19 and A.34.

4.3 Ginsparg-Wilson fermions

The Wilson lattice Dirac operator violates chiral symmetry explicitly and leads to computational difficulties such as additive mass and multiplicative current renormalisation. Therefore it is desirable to have a lattice operator with better chiral properties. In Section 3.3 we learned that it is not possible to formulate a lattice theory of fermions with a continuum like chiral symmetry. The way out in Section 3.5 was to ease the restriction to an exact continuum like chiral symmetry in favour of a lattice chiral symmetry and a lattice Dirac operator given as the solution to the Ginsparg-Wilson relation

$$\gamma_5 D + D \gamma_5 = a D \gamma_5 D . \tag{4.43}$$

In the limit $a \to 0$ this relation becomes the known anticommutation relation and the lattice chiral transformations become the continuum ones.

In the presence of the boundaries the same approach immediately leads to inconsistencies. As stated in the introduction in the continuum the SF boundary conditions break chiral symmetry. This can be seen by considering the solution to the massless Dirac equation

$$D \psi(x) = 0 , \tag{4.44}$$

which has to obey the boundary conditions (4.2–4.3). Note that the massless Dirac operator anti-commutes with γ_5. Therefore, given the propagator $D S(x,y) = \delta(x-y)$ the sum $\gamma_5 S(x,y) + S(x,y)\gamma_5$ is a solution of the Dirac equation for all y. And since the propagator obeys the boundary conditions (4.2–4.3) this solution can also be obtain from the boundary

values at $x_0 = 0$ and $x_0 = T$ (actually the non-zero components there)

$$\gamma_5 S(x,y) + S(x,y)\gamma_5 = \int_{x_0=0} \mathrm{d}^d z\, S(x,z)\gamma_5 S(z,y) + \int_{x_0=T} \mathrm{d}^d z\, S(x,z)\gamma_5 S(z,y)\,. \quad (4.45)$$

Thus the continuum propagator anti-commutes with γ_5 up to boundary terms. In QCD with more than one massless quark this leads to a non-singlet chiral Ward identity with a unit mass term at the boundaries [32].

Now consider a lattice Dirac operator satisfying (4.43) in the presence of the boundaries. Then the lattice propagator anti-commutes with γ_5

$$\gamma_5 S(x,y) + S(x,y)\gamma_5 = \frac{1}{a^d}\,\delta_{x,y}\gamma_5\,, \quad (4.46)$$

for any finite separation $x - y$ and any finite a. Thus this theory can not have the right continuum limit, i.e. the one where the propagator obeys (4.45).

4.3.1 Modified Neuberger-Dirac operator

As explained in Section 4.1.2 the SF boundary conditions form a universality class and need no special adjustment of the lattice action. This means in particular that there may be many possible lattice operators leading to the right continuum limit. In Ref. [32] it is proposed to allow for an additional term on the right hand side of Eq. (4.43)

$$\gamma_5 D + D\gamma_5 = a\, D\gamma_5 D + \Delta_B\,, \quad (4.47)$$

where Δ_B is supported in the vicinity of the boundaries and decays exponentionally with the distance to them.

The operator introduced in the same reference

$$D_{\mathrm{N}} = \frac{1}{\bar{a}}\left\{1 - \tfrac{1}{2}(U + U^{\sim})\right\}\,, \quad (4.48)$$

$$U = A\left(A^\dagger A + caP\right)^{-1/2}\,, \quad U^{\sim} = \gamma_5 U^\dagger \gamma_5\,, \quad \bar{a} = \frac{a}{1+s}\,, \quad (4.49)$$

is a modification of the Neuberger-Dirac operator (3.60). Here A is essentially the Wilson-Dirac operator in the presence of the boundaries (4.36–4.37)

$$A = 1 + s - aD_{\mathrm{W}}\,. \quad (4.50)$$

The parameters s and c can be used to optimise numerical computations. The modification is due to the boundary operator

$$P\psi(x) = \frac{1}{a}\left\{\delta_{x_0,a}P_-\psi(x)\big|_{x_0=a} + \delta_{x_0,T-a}P_+\psi(x)\big|_{x_0=T-a}\right\}. \tag{4.51}$$

The construction is such that D_{N} inherits the transformation properties of the Wilson-Dirac operator under cubic rotations, parity, time-reflections and charge conjugation. And having the combination $U + U^{\sim}$ renders $\gamma_5 D_{\mathrm{N}}$ hermitian. Although we will be mainly concerned with massless fermions the extension to massive fermions is simple [60] and given by

$$D_m = (1 - \tfrac{1}{2}\bar{a}m_0)D_{\mathrm{N}} + m_0\,. \tag{4.52}$$

In Section 4.3.3 we will show that the operator (4.48) indeed obeys (4.47) with a replaced by $\bar{a}$.

4.3.2 Free theory

For our perturbative computation we need the Dirac operator and the fermion propagator in the free theory (no gauge field, no four fermion interactions). To obtain an explicit expression for the free propagator like in the Wilson case might be possible but is much more difficult. Instead it is computed numerically using well established methods [72].

Nevertheless, the operator under the square root in Eq. (4.49) can be worked out in the time-momentum representation and it is reassuring that for the interesting range of the parameters s and c its eigenvalues are positive. This can be seen as follows. In the free theory the operator under the square root explicitly reads

$$A^\dagger A + caP = (1+s)^2 + sa^2\sum_\mu \partial^*_\mu\partial_\mu + \tfrac{1}{2}a^4\sum_{\mu<\nu}\partial^*_\mu\partial_\mu\partial^*_\nu\partial_\nu + (c-1)aP\,, \tag{4.53}$$

and acts on the fermion fields in the presence of the boundaries (cf. Section 4.2.1). It is hermitian and therefore has real eigenvalues. Due to translation invariance in space the spatial eigenfunctions are plane waves

$$w(x) = \frac{1}{L^d}\sum_{\mathbf{p}} \mathrm{e}^{i\mathbf{p}\mathbf{x}}\, w_{\mathbf{p}}(x_0)\,, \tag{4.54}$$

with spatial momenta p_k in the range (A.33). In the following we restrict ourself to a definite momentum $\mathbf{p}$. Then the Dirac components of $w_{\mathbf{p}}(x_0)$

are ordinary functions of x_0. Therefore we drop the subscript in the eigenvalue equation

$$\Big(-q\,a^2\partial_0^*\partial_0 + m + (c-1)aP\Big)\; w(x_0) = \lambda\, w(x_0)\,, \tag{4.55}$$

where q and m are short hand for

$$q = \tfrac{a^2}{2}\hat{\mathbf{p}}^2 - s\,, \quad m = (1+s)^2 - sa^2\hat{\mathbf{p}}^2 + \tfrac{a^4}{2}\sum_{k<l}\hat{p}_k^2\hat{p}_l^2\,. \tag{4.56}$$

Because of the projectors in P, eqs. (4.55) are two coupled equations for the plus and minus components in $w(x_0) = P_+\, w_+(x_0) + P_-\, w_-(x_0)$. In the special case $c = 1$ they have to be identical and one easily finds the solutions

$$\sin(p_0 x_0)\,, \quad p_0 = \frac{n\pi}{T}\,, \quad n = 1, 2, \ldots, T/a - 1\,, \tag{4.57}$$

and the correspondent eigenvalues

$$\lambda = (1+s)^2 - sa^2\hat{p}^2 + \tfrac{a^4}{2}\sum_{\mu<\nu}\hat{p}_\mu^2\hat{p}_\nu^2 = q\,\hat{p}_0^2 + m\,. \tag{4.58}$$

For arbitrary c the solutions to (4.55) are

$$P_-\sin\big(p_0 x_0 + b\big) + P_+\sin\big(p_0(T - x_0) + b\big)\,, \tag{4.59}$$

$$b = -\arctan\left(\frac{\sin(ap_0)}{\frac{q}{c-1} + \cos(ap_0)}\right)\,, \tag{4.60}$$

and the allowed values of p_0 are given by the solutions of the equation

$$\tan(p_0 T) = \frac{\sin(ap_0)}{\frac{q}{c-1} + \cos(ap_0)}\,. \tag{4.61}$$

The solutions of this equation are all real for non-negative $q/(c-1)$. Since the eigenfunctions are odd functions of p_0 it is sufficient to stick to the $T/a - 1$ solutions in the interval $[0, \pi/a)$. In any case the eigenvalues (4.58) are bounded from below by $(1 - |s|)^2$.

For $q/(c-1) < 0$ one finds a pure imaginary solution $p_0^* = ik$, $k > 0$. This solution enters the eigenvalue (4.58) through the product $q\,\hat{p}_0^2$ and might lead to zero modes and negative eigenvalues. However, for $c \geq 1$ imaginary solutions exist only for $q < 0$ and thus the product $q\,(\hat{p}_0^*)^2$ is

positive and the eigenvalues (4.58) are as well bounded from below by $(1-|s|)^2$.

Since the operator under the square root is bounded from below we can adopt the argument in [73], using expansion in Legendre polynomials, to conclude that in the free theory the locality of the Dirac operator D_{N} is guaranteed for all $|s| < 1$ and $c \geq 1$. The eigenfunctions (4.59) can be orthonormalised and used to write down an analytical expression for the kernel $D_{\mathrm{N}}(x,y)$ of the Dirac operator. But the evaluation of $D_{\mathrm{N}}(x,y)$ in this way would be very expensive, since it involves a sum over momenta p_0 which in turn are determined for each set of parameter values by the roots of (4.61).

Even more desirable would be an explicit expression for the propagator. With the eigenfunctions of the operator under the square root at hand one would hope to find the eigenfunctions of the hermitian operator $D_{\mathrm{N}}D_{\mathrm{N}}^{\dagger}$ and write the propagator as

$$S(x,y) = \left[D_{\mathrm{N}}^{\dagger} \frac{1}{D_{\mathrm{N}} D_{\mathrm{N}}^{\dagger}} \right] (x,y)\,. \tag{4.62}$$

But the eigenvalue problem of $D_{\mathrm{N}}D_{\mathrm{N}}^{\dagger}$ is much more complicated than the one treated above and we do not attempt to solve it here. Instead the propagator is computed numerically using established methods. This includes a polynomial approximation of the inverse square root in the Dirac operator and solving the linear equation

$$D_{\mathrm{N}}\psi = \eta\,, \tag{4.63}$$

for ψ with the approximate Dirac operator and appropriate sources η. The precision can be controlled throughout all steps of the computation [72]. In this way the whole propagator can be evaluated with a given accuracy. In practice we also Fourier transform to the space component to obtain the time-momentum representation (see Section 5.5.2).

4.3.3 Chiral properties

As a check, it was tested numerically whether the operator (4.48) is a solution to

$$\gamma_5\, D + D\, \gamma_5 = \bar{a}\, D\gamma_5 D + \Delta_B\,, \tag{4.64}$$

with

$$||a\Delta_B(x,y)|| \approx \mathrm{e}^{-\kappa\cdot\tau/a} + \mathrm{e}^{-\kappa\cdot(T-\tau)/a}\,, \quad \kappa > 0\,, \tag{4.65}$$

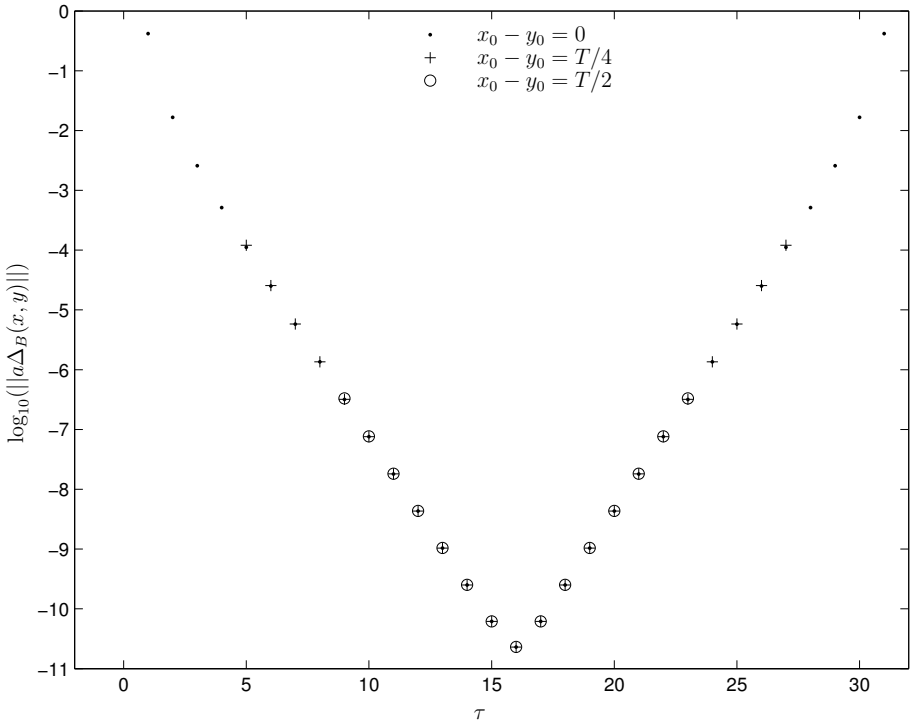


FIGURE 4.1: The deviation from the Ginsparg-Wilson relation $a\Delta_B$ in Eq. (4.64) is localised at the boundaries with tails that decrease exponentially with the distance from the boundaries. The plot is for a 16×32 lattice, $\theta = 0.5$ and $x_1 = y_1 = L/2$.

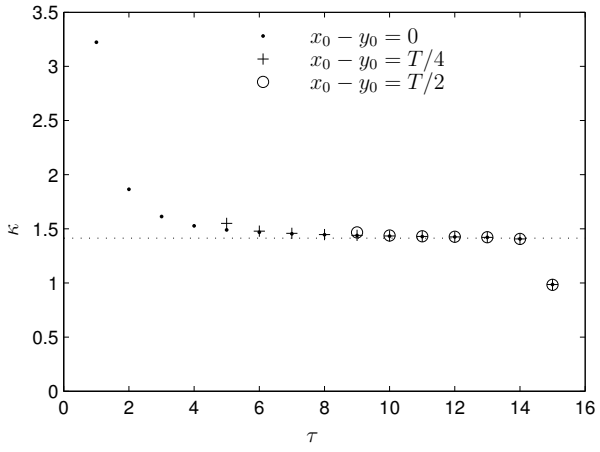


FIGURE 4.2: "Effective mass" plot for $a\Delta_B(x,y)$. For large distances from the boundary $a\Delta_B(x,y)$ is compatible with (4.65) with $\kappa \approx \sqrt{2}$ (dotted line). The plot is for the same parameters as Fig. 4.1

for large distances τ and $T-\tau$ from the boundaries. For simplicity we define τ as the distance from the boundary at $x_0 = 0$ along the time direction as

$$\tau = \frac{y_0 + x_0}{2}\,. \tag{4.66}$$

In Fig. 4.1 $a\Delta_B(x,y)$ is plotted for a 16×32 lattice and $\theta = 0.5$. The deviations from the Ginsparg-Wilson relation decay exponentially and from the "effective mass" plot Fig. 4.2 the rate seems to approach $\kappa = \sqrt{2}$ for $\tau \to \infty$. It should be possible to compute $\Delta_B(x,y)$ analytically to some extent, but we did not attempt to.

4.4 The generating functional

We want to compute expectation values of polynomials $\mathcal{O}$ in the fermion and anti-fermion bulk and boundary fields. A possible lattice representation of the boundary fields (4.7–4.8) is [2]

$$\zeta(\mathbf{x}) = P_-\psi(x)\Big|_{x_0=a}\,, \quad \overline{\zeta}(\mathbf{x}) = \overline{\psi}(x)\Big|_{x_0=a}P_+\,, \tag{4.67}$$

$$\zeta'(\mathbf{x}) = P_+\psi(x)\Big|_{x_0=T-a}\,, \quad \overline{\zeta}'(\mathbf{x}) = \overline{\psi}(x)\Big|_{x_0=T-a}P_-\,. \tag{4.68}$$

With this choice it is enough to introduce sources for the fermion and anti-fermion fields $\eta(x)$ and $\overline{\eta}(x)$ in the interior of the lattice $0 < x_0 < T$. The generating functional is then

$$Z[\overline{\eta},\eta] = \int \mathrm{D}\psi\mathrm{D}\overline{\psi}\,\exp\left\{-S_0[\overline{\psi},\psi] + a^{d+1}\sum_x [\overline{\psi}(x)\eta(x) + \overline{\eta}(x)\psi(x)]\right\}. \tag{4.69}$$

As said before only the fields at $0 < x_0 < T$ are integrated over in this functional integral. We can perform this integration and obtain

$$\ln Z[\overline{\eta},\eta] = a^{2(d+1)}\sum_{x,y}\overline{\eta}(x)\,S(x,y)\,\eta(y) + \text{const.}\,. \tag{4.70}$$

Thus the generating functional is an exponential of a quadratic expression in the sources. Replacing the fields in $\mathcal{O}$ by functional derivatives

$$\psi(x) \to \frac{\delta}{\delta\overline{\eta}(x)}\,, \qquad \overline{\psi}(x) \to -\frac{\delta}{\delta\eta(x)}\,, \tag{4.71}$$

[2] In QCD one has to include link variables into this definition of the boundary fields.

we may write the expectation value as

$$\langle \mathcal{O} \rangle = \left\{ \frac{1}{Z} \mathcal{O} Z \right\}_{\overline{\eta},\eta=0} . \tag{4.72}$$

They are given as the sum of all Wick contractions. Below we list the basic contractions.

$$[\psi(x)\overline{\psi}(y)] = S(x,y) , \tag{4.73}$$
$$[\psi(x)\overline{\zeta}(\mathbf{y})] = S(x;a,y_1)P_+ , \tag{4.74}$$
$$[\psi(x)\overline{\zeta}'(\mathbf{y})] = S(x;T-a,y_1)P_- , \tag{4.75}$$
$$[\zeta(\mathbf{x})\overline{\psi}(y)] = P_- S(a,x_1;y) , \tag{4.76}$$
$$[\zeta'(\mathbf{x})\overline{\psi}(y)] = P_+ S(T-a,x_1;y) , \tag{4.77}$$
$$[\zeta(\mathbf{x})\overline{\zeta}'(\mathbf{y})] = P_- S(a,x_1;T-a,y_1)P_- , \tag{4.78}$$
$$[\zeta'(\mathbf{x})\overline{\zeta}(\mathbf{y})] = P_+ S(T-a,x_1;a,y_1)P_+ , \tag{4.79}$$
$$[\zeta(\mathbf{x})\overline{\zeta}(\mathbf{y})] = P_- S(a,x_1;a,y_1)P_+ , \tag{4.80}$$
$$[\zeta'(\mathbf{x})\overline{\zeta}'(\mathbf{y})] = P_+ S(T-a,x_1;T-a,y_1)P_- . \tag{4.81}$$

4.5 Boundary counter terms

The boundary conditions of the SF may give rise to new counter terms defined on the boundary, in the sense that the associated bare coefficients are needed to absorb infinities on the way to the continuum limit. Relevant operators or composite fields living at the boundary are those which have dimension d or less. Thus objects like $\overline{\psi}\Gamma\psi$ with $\Gamma = \{1, \gamma_0, \gamma_1, \gamma_5\}$ can appear.

The identity and γ_0 can be written as $P_+ + P_-$ and $P_+ - P_-$ respectively. Hence the corresponding terms are proportional to the boundary fields (4.7,4.8) in the continuum or (4.67,4.68) on the lattice. The terms with γ_1 and γ_5 violate parity and are therefore not present.

Thus it is enough in the course of renormalisation to introduce a renormalisation factor Z_ζ for all boundary fields

$$\zeta(\mathbf{x}) \to Z_\zeta \, \zeta(\mathbf{x}) , \quad \overline{\zeta}(\mathbf{x}) \to Z_\zeta \, \overline{\zeta}(\mathbf{x}) , \tag{4.82}$$

and equivalently for $\zeta', \overline{\zeta}'$.

Chapter 5

Self-coupled fermions in two dimensions

5.1 Four fermion operators

In two dimensions fermion fields have mass dimension $1/2$ and a local four fermion operator

$$\overline{\psi}_{\alpha i}\overline{\psi}_{\beta j}\psi_{\gamma k}\psi_{\delta l}\,, \tag{5.1}$$

has a dimensionless coupling in the action. From the point of view of dimensional analysis such a interaction term is renormalisable. This means that if such a four fermion interaction is added to a theory that is renormalisable, it stays so. All new divergences can be absorbed into a redefinition of the four fermion coupling.

In the literature one finds several two-dimensional fermion models with different symmetry and interaction content [19–21]. This is because there are a lot of possible ways to contract the indices of four fermion fields. But in two dimensions there are also several relations between the possible contractions. In the following we discuss this in some detail.

The most general four fermion operator is an arbitrary contraction of the Dirac (Greek letters $\alpha, \beta, \ldots$) and flavour (Latin letters $i, j, \ldots$) indices in Eq. (5.1). Since we want a theory that is as similar to QCD as it can be in two dimensions, it certainly should be Lorentz invariant, even under parity and, in the massless case, have an $U(N)$ flavour symmetry. [1]

Lorentz invariance strongly constrains the possible contractions of the Dirac indices. Since Eq. (5.1) must be a Lorentz scalar it must be a product of two scalars, two pseudo-scalars, two vectors or two axial-vectors.

[1] This is meant in the continuum. At the end of the day we are interested in continuum QCD.

Consider the case of two scalars[2]

$$\overline{\psi}_{\alpha i}\psi_{\alpha j}\overline{\psi}_{\beta k}\psi_{\beta l}\,. \tag{5.2}$$

Invariance under $U(N)$ transformations

$$\psi_{\alpha i} \to U_{im}\,\psi_{\alpha m}\,, \quad \overline{\psi}_{\alpha j} \to \overline{\psi}_{\alpha n}\,U^*_{jn}\,, \tag{5.3}$$

$$\text{where} \quad U^\dagger U = UU^\dagger = 1\,, \tag{5.4}$$

allows for the following two flavour contractions

$$\overline{\psi}_{\alpha i}\psi_{\alpha i}\overline{\psi}_{\beta j}\psi_{\beta j}\,, \quad \overline{\psi}_{\alpha i}\psi_{\alpha j}\overline{\psi}_{\beta j}\psi_{\beta i}\,. \tag{5.5}$$

The second contraction can be expanded in a basis of $N \times N$ matrices (Appendix A.3.2)

$$\overline{\psi}_i\psi_j\overline{\psi}_j\psi_i = \frac{1}{N}(\overline{\psi}\psi)^2 + \frac{1}{2}\sum_a (\overline{\psi}\lambda^a\,\psi)^2\,, \tag{5.6}$$

where we suppressed all subscripts on the left hand side. The first term is proportional to the first one in (5.5) and the sum is over the generators λ^a of $SU(N)$, which act on the flavour indices of $\overline{\psi}$ and ψ.

Therefore the most general four fermion operator consistent with the symmetries can be expanded in a basis of eight different contractions. One half of them are products of flavour singlet bilinear operators

$$\begin{aligned} O_{SS} &= (\overline{\psi}\psi)^2\,, \\ O_{PP} &= (\overline{\psi}\gamma_5\psi)^2\,, \\ O_{VV} &= \textstyle\sum_\mu (\overline{\psi}\gamma_\mu\psi)^2\,, \\ O_{AA} &= \textstyle\sum_\mu (\overline{\psi}\gamma_\mu\gamma_5\psi)^2\,, \end{aligned} \tag{5.7}$$

and the other half are products of flavour vector bilinear opeartors

$$\begin{aligned} O'_{SS} &= \textstyle\sum_a (\overline{\psi}\lambda^a\psi)^2\,, \\ O'_{PP} &= \textstyle\sum_a (\overline{\psi}\gamma_5\lambda^a\psi)^2\,, \\ O'_{VV} &= \textstyle\sum_{\mu,a} (\overline{\psi}\gamma_\mu\lambda^a\psi)^2\,, \\ O'_{AA} &= \textstyle\sum_{\mu,a} (\overline{\psi}\gamma_\mu\gamma_5\lambda^a\psi)^2\,. \end{aligned} \tag{5.8}$$

But in two dimensions these operators are not independent. Suppose there is only a single fermion ($N = 1$). From the above list one would expect four different terms (no flavour vector operators for a single fermion).

[2]Repeated indices are summed over if not indicated otherwise.

But there are only four independent field components at each space-time point. Thus there is only one local four fermion operator.

For $N > 1$ the number of independent field components is no restriction. Nevertheless there are relations among the operators above that reduce the number of really independent ones to three. Because of a peculiarity of the γ-matrices in two dimensions ($\gamma_\mu\gamma_5 = i\epsilon_{\mu\nu}\gamma_\nu$) there is no difference between vector and axial-vector and thus

$$O_{AA} = -O_{VV}\,, \quad O'_{AA} = -O'_{VV}\,. \tag{5.9}$$

More dependencies are due to Fierz identities. Fierz identities connect products of Dirac bilinear forms by rearranging the order of the Dirac spinors (see Appendix A.3.1). For the two flavour contractions of (5.5), but with general Dirac structure, we find

$$(\overline{\psi}_i\Gamma\psi_j)\,(\overline{\psi}_j\Gamma\psi_i) = -\frac{1}{4}\sum_I\,\mathrm{tr}(\Gamma_I\,\Gamma\,\Gamma_I\,\Gamma)(\overline{\psi}_i\Gamma_I\psi_i)\,(\overline{\psi}_j\Gamma_I\psi_j)\,, \tag{5.10}$$

where $\Gamma, \Gamma_I \in \{\Gamma_S = 1\,, \Gamma_P = \gamma_5\,, \Gamma_V = \gamma_0\,, \gamma_1\}$. Using also (5.6) this yields three identities relating the flavour-singlet and the flavour-vector operators

$$\begin{aligned} O'_{SS} &= -(1+2/N)\,O_{SS} - O_{PP} - O_{VV} && (5.11)\\ O'_{PP} &= -O_{SS} - (1+2/N)\,O_{PP} + O_{VV} && (5.12)\\ O'_{VV} &= -2\,O_{SS} + 2\,O_{PP} - 2/N\,O_{VV}\,. && (5.13) \end{aligned}$$

The number of independent operators has been reduced from eight to three. One possible choice would be

$$O_{SS}\,, O_{PP}\,, O_{VV}\,. \tag{5.14}$$

But any other combination of three operators from (5.7) and (5.8) is equally good.

5.2 Chiral symmetry

The eight operators of (5.7) and (5.8), of which only a set of three is independent, are invariant under $U(N)$ transformations (5.3) by construction. Since massless QCD at the classical level has a global chiral $U(N)$ symmetry we investigate here the transformation properties of these operators under the infinitesimal transformations (3.24) and (3.26). In Section 3.1

these were introduced as local transformations. But since they will act on operators that are local and contain no derivatives of fields the outcome is also applicable for the global case.

For the above mentioned symmetry transformations the change $\delta\mathcal{O}$ of the operator $\mathcal{O}$ linear in the infinitesimal parameter ω is

$$\mathcal{O} \to \mathcal{O} + \delta\mathcal{O} + \mathrm{O}(\omega^2)\,. \tag{5.15}$$

If $\delta\mathcal{O}$ vanishes the operator is invariant under finite transformations that can be build up from the infinitesimal ones.

Axial $U(1)$ transformation Consider first the axial $U(1)$ transformation. Because of the γ-matrices in O_{VV} and O'_{VV} they are invariant like the kinetic term.

$$\delta O_{VV} = 0\,, \quad \delta O'_{VV} = 0\,. \tag{5.16}$$

But δO_{SS} and δO_{PP} do not vanish. Using the projectors $P_{R,L} = \frac{1}{2}(1 \pm \gamma_5)$ we write

$$O_{SS} = (\overline{\psi}\,(P_R + P_L)\,\psi)^2\,, \quad O_{PP} = (\overline{\psi}\,(P_R - P_L)\,\psi)^2\,. \tag{5.17}$$

Now it is easy to see that

$$\begin{aligned} \delta O_{SS} &= 2i\omega_A(\overline{\psi}\,(P_R - P_L)\,\psi)\,(\overline{\psi}\,(P_R + P_L)\,\psi)\,, \\ \delta O_{PP} &= 2i\omega_A(\overline{\psi}\,(P_R + P_L)\,\psi)\,(\overline{\psi}\,(P_R - P_L)\,\psi)\,. \end{aligned}$$

Therefore the difference of the two operators is invariant

$$\delta(O_{SS} - O_{PP}) = 0\,, \tag{5.18}$$

and similar for the primed operators

$$\delta(O'_{SS} - O'_{PP}) = 0\,. \tag{5.19}$$

Axial $SU(N)$ transformation The axial $SU(N)$ transformation (3.26) affects Dirac and flavour indices. However, O_{VV} has a trivial flavour structure and it is easy to see that it is invariant

$$\delta O_{VV} = 0\,. \tag{5.20}$$

Computing $\delta O'_{VV}$ the commutator $[\lambda^a, \lambda^b] = 2if^{abc}$ with the structure constants f^{abc} appear (cf. Appendix A.3.2)

$$\begin{aligned} \delta O'_{VV} &= 2i\omega^a_A(\overline{\psi}\,\gamma_5\lambda^a\gamma_\mu\lambda^b\,\psi + \overline{\psi}\,\gamma_\mu\lambda^b\lambda^a\gamma_5\,\psi)\,(\overline{\psi}\,\gamma_\mu\lambda^b\,\psi) \\ &= -2i\omega^a_A(\overline{\psi}\,\gamma_\mu\gamma_5[\lambda^a, \lambda^b]\,\psi)\,(\overline{\psi}\,\gamma_\mu\lambda^b\,\psi) \\ &= 4\omega^a_A f^{abc}(\overline{\psi}\,\gamma_\mu\gamma_5\lambda^c\,\psi)\,(\overline{\psi}\,\gamma_\mu\lambda^b\,\psi) \\ &= 4i\omega^a_A f^{abc}\big\{(\overline{\psi}\,\gamma_1\lambda^c\,\psi)\,(\overline{\psi}\,\gamma_0\lambda^b\,\psi) - (\overline{\psi}\,\gamma_0\lambda^c\,\psi)\,(\overline{\psi}\,\gamma_1\lambda^b\,\psi)\big\}\,. \end{aligned}$$

In the last step we used the definition of γ_5 and the Clifford algebra (cf. Appendix A.1.1). Since the structure constants are totally anti-symmetric, the whole expression is symmetric in the indices b and c and hence does not vanish for general ω^a_A

$$\delta O'_{VV} \neq 0\,. \tag{5.21}$$

The result for O_{VV} and O'_{VV} can be used together with (5.11–5.13) to infer

$$\delta(O_{SS} - O_{PP}) \neq 0\,, \quad \text{and} \quad \delta(O'_{SS} - O'_{PP}) \neq 0\,. \tag{5.22}$$

5.3 Lattice chiral Gross-Neveu model

The continuum chiral Gross-Neveu model in Euclidean space-time is given by the action

$$S^c_{\mathrm{CGN}} = \int \mathrm{d}^2x \, \left\{ \overline{\psi}\, \gamma_\mu \partial_\mu \, \psi - \tfrac{1}{2} g^2 (O_{SS} - O_{PP}) - \tfrac{1}{2} g_V^2 O_{VV} \right\} . \tag{5.23}$$

From the analysis of the last two sections we know that this is the most general action with chiral $U(1) \times U(1)$ and $SU(N)$ flavour symmetry. But we also learned that this form is not the only possibility. Using the identity (5.13) the action (5.23) may equally be written as

$$S^c_{\mathrm{CGN}} = \int \mathrm{d}^2x \, \left\{ \overline{\psi}\, \gamma_\mu \partial_\mu \, \psi + \tfrac{1}{4} g^2 O'_{VV} - \tfrac{1}{2} \delta_V^2 O_{VV} \right\} , \tag{5.24}$$

$$\text{with} \quad \delta_V^2 = g_V^2 - g^2/N\,. \tag{5.25}$$

The coupling δ_V^2 is believed to have an exactly vanishing beta-function. This is connected to the decoupling of the $U(1)$ part of correlation functions that is derived by formal manipulations of the continuum path integral [74, 75]. The vanishing of the beta-function has been proved up to two-loop perturbation theory [23].

Passing now over to the lattice inevitably breaks part of the continuum symmetries. Generally this leads to more possibilities for the mixing of operators under renormalisation and to additonal parameters in the action. For example, an operator that is multiplicative renormalisable in the continuum may lose this property on the lattice.[3] Being lattice artefacts[4] one has to assure that these effects disappear in the continuum limit. As

[3] This also occurs in other regularisation schemes. In the two-loop computation of [23] using dimensional regularization so called "evanescent operators" appear at an intermediate state of the calculation.

[4] This is not true if there is an anomaly that breaks the symmetry.

we will see, this may happen automatically, but in some cases additional parameters in the action have to be tuned.

At first the theory is set up on a hypercubic lattice with periodic boundary conditions. The introduction of boundaries as in the Schrödinger functional involves additional problems that are addressed in a separate section (Section 5.5).

In the action we have to include terms that are forbidden by the continuum symmetries, but are allowed by the less restricting lattice symmetries. By dimensional analysis such operators with mass dimension n will have a factor a^{n-D} in front (where $D = d + 1$ is the dimension of space-time and should not be confused with the Dirac operator). Operators with $n > D$ can safely be neglected since they automatically disappear for $a \to 0$.[5] But the operators with mass dimension $n \leq D$ are needed and lead to an additional renormalisation.

Obviously (Euclidean) Lorentz symmetry $O(2)$ is broken on the lattice. The symmetry group of the hypercubic lattice, the hypercubic group, is a subgroup of $O(2)$ and contains rotations by $\pi/2$ and reflections. Luckily, the operators allowed by the hypercubic group are the ones already present in (5.23) or (5.24) and operators with mass dimension $n > D$. So, ignoring all other symmetries for a moment, the effects of the broken Lorentz symmetry in the action automatically disappear in the continuum limit and we may use

$$S_{\mathrm{CGN}} = a^2 \sum_x \left\{ \overline{\psi}\, D\, \psi - \tfrac{1}{2} g^2 (O_{SS} - O_{PP}) - \tfrac{1}{2} g_V^2 O_{VV} \right\} , \tag{5.26}$$

with the lattice Dirac operator D as the lattice action. The fermion fields $\overline{\psi}(x)$ and $\psi(x)$ are defined at the sites of a hypercubic lattice with periodic boundary conditions and lattice spacing a (c.f. Section 2.2). Note that there may be mixing amongst operators that cannot be neglected and that thus has to be taken into account when calculating expectation values of such operators.

Depending on the exact definition of D there are further broken or modified symmetries that might lead to a modification of (5.26). The well studied Wilson-Dirac operator explicitly breaks chiral symmetry and leads to an additive mass renormalisation. For staggered fermions the broken flavour symmetry causes new mixings. The lattice chiral symmetry associated with Ginsparg-Wilson fermions forbids additional operators of mass dimension $n \leq D$ in the action. In the next two sections we discuss Wilson and Ginsparg-Wilson fermions.

[5] The $n = D + 1$ operators may be considered if one wants to achieve $\mathrm{O}(a)$ improvement.

5.3.1 Wilson fermions

As we have seen in Section 3.4 the Wilson-Dirac operator explicitly breaks chiral symmetry and a fine tuning of the bare mass is needed to restore chiral symmetry. The action should therefore contain all terms allowed by the remaining $U(N)$ flavour symmetry. In Section 5.1 a basis for $U(N)$ invariant four fermion operators was established. Due to dependencies among the diverse operators it is enough to choose as set of three operators out of (5.7) and (5.8). In analogy to (5.26) one possibility is

$$S_{\mathrm{CGN,W}} = a^2 \sum_x \Big\{ \overline{\psi}\,(D_{\mathrm{W}} + m_0)\,\psi \\ - \tfrac{1}{2}g^2(O_{SS} - O_{PP}) - \tfrac{1}{2}\delta_P^2 O_{PP} - \tfrac{1}{2}g_V^2 O_{VV} \Big\}, \quad (5.27)$$

where we added a chiral symmetry breaking mass term and the coupling of O_{SS} and O_{PP} are no longer related.

There are many different choices one could make. For example, one can invert eqs. (5.11)-(5.13) to obtain

$$S_{\mathrm{CGN,W}} = a^2 \sum_x \Big\{ \overline{\psi}\,(D_{\mathrm{W}} + m_0)\,\psi \\ - \tfrac{1}{2}g'^2(O'_{SS} - O'_{PP}) - \tfrac{1}{2}\delta'^2_P O'_{PP} - \tfrac{1}{2}g'^2_V O'_{VV} \Big\}, \quad (5.28)$$

where the primed couplings are related to the unprimed ones

$$g'^2 = \frac{N^2}{4(N^2-1)} \Big\{ (2/N)\, g^2 - \delta_P^2 - 2g_V^2 \Big\}, \quad (5.29)$$

$$\delta'^2_P = \frac{N^2}{4(N^2-1)} \Big\{ 2(1/N - 1)\, \delta_P^2 \Big\}, \quad (5.30)$$

$$g'^2_V = \frac{N^2}{4(N^2-1)} \Big\{ -2g^2 + \delta_P^2 + 2/N\, g_V^2 \Big\}. \quad (5.31)$$

The traceless λ-matrices in the four fermion operators in (5.28) can help to reduce the number of non-zero diagrams in perturbative calculations. (This will be utilized in Section 6) Yet another version is obtained by decomposing the interaction as in (5.24)

$$S_{\mathrm{CGN,W}} = a^2 \sum_x \Big\{ \overline{\psi}\,(D_{\mathrm{W}} + m_0)\,\psi + \tfrac{1}{4}g^2 O'_{VV} - \tfrac{1}{2}\delta_P^2 O_{PP} - \tfrac{1}{2}\delta_V^2 O_{VV} \Big\}, \quad (5.32)$$

$$\text{with} \quad \delta_V^2 = g_V^2 - g^2/N\,. \tag{5.33}$$

As pointed out in Section 3.4 the bare mass m_0 has to be tuned to a non-zero value in order to restore chiral symmetry. In the same way δ_P^2 has to be tuned since the value $\delta_P^2 = 0$ is not distinguished by a greater symmetry. A whole section (Section 6) is devoted to the restoration of chiral symmetry.

5.3.2 Ginsparg-Wilson fermions

A Dirac operator satisfying the Ginsparg-Wilson relation (3.57) implicates the lattice chiral symmetry (3.58). This symmetry becomes the familiar chiral symmetry in the continuum limit and forbids a mass term. However, at finite lattice spacing the invariant four fermion operators are different from the continuum ones.

The transformations (3.58) may be written as

$$\psi \to \psi + \epsilon\hat{\gamma}_5\psi\,, \quad \overline{\psi} \to \overline{\psi} + \epsilon\overline{\psi}\gamma_5\,, \tag{5.34}$$

$$\text{with} \quad \hat{\gamma}_5 = \gamma_5(1 - aD) \quad \text{and} \quad \hat{\gamma}_5^2 = 1\,. \tag{5.35}$$

This looks like the usual chiral transformations except for the $\hat{\gamma}_5$. Using (3.58) one proofs the identity

$$(1 - \tfrac{a}{2}D)\hat{\gamma}_5 = \gamma_5(1 - \tfrac{a}{2}D)\,. \tag{5.36}$$

Which means that $(1 - \frac{a}{2}D)\psi$ transforms under (5.34) like ψ under (3.58)

$$(1 - \tfrac{a}{2}D)\psi \overset{(5.34)}{\to} (1 - \tfrac{a}{2}D)\psi + \epsilon\gamma_5(1 - \tfrac{a}{2}D)\psi\,. \tag{5.37}$$

Therefore the chirally invariant operators of Section 5.2 are invariant under the lattice symmetry after replacing $\psi \to (1 - \frac{a}{2}D)\psi$. In particular, if we define

$$\begin{aligned} \hat{O}_{SS} &= (\overline{\psi}(1 - \tfrac{a}{2}D)\psi)^2\,, \\ \hat{O}_{PP} &= (\overline{\psi}\gamma_5(1 - \tfrac{a}{2}D)\psi)^2\,, \\ \hat{O}_{VV} &= \textstyle\sum_\mu (\overline{\psi}\gamma_\mu(1 - \tfrac{a}{2}D)\psi)^2\,, \end{aligned} \tag{5.38}$$

then the operators invariant under 5.2 are

$$\delta\hat{O}_{VV} = 0\,, \quad \delta(\hat{O}_{SS} - \hat{O}_{PP}) = 0\,. \tag{5.39}$$

The treatment and result in the case of the operators which are products of flavour vector bilinear operators (5.8) is equivalent. Note that the operators (5.38) are not ultra-local any more. Nevertheless the identities (5.9)

and (5.11)-(5.13) hold also for these operators. But in the derivation in Section 5.1 one has to replaces $\psi \to \hat{\psi} = (1 - \frac{a}{2}D)\psi$.

Now we gathered all prerequisites to write down the lattice action for Ginsparg-Wilson fermions in the form

$$S_{\mathrm{CGN,GW}} = a^2 \sum_x \left\{ \overline{\psi}\, D\, \psi - \tfrac{1}{2} g^2 (\hat{O}_{SS} - \hat{O}_{PP}) - \tfrac{1}{2} g_V^2 \hat{O}_{VV} \right\}, \tag{5.40}$$

or

$$S_{\mathrm{CGN,GW}} = a^2 \sum_x \left\{ \overline{\psi}\, D\, \psi + \tfrac{1}{4} g^2 \hat{O}'_{VV} - \tfrac{1}{2} \delta_V^2 \hat{O}_{VV} \right\}, \tag{5.41}$$

$$\text{with} \quad \delta_V^2 = g_V^2 - g^2/N\,, \tag{5.42}$$

where in both cases

$$\gamma_5 D + D\gamma_5 = a D \gamma_5 D\,. \tag{5.43}$$

5.4 The discrete Gross-Neveu model

Note that for $g_V^2 = 0$ and $\delta_P^2 = g^2$ (5.27) becomes the action of the discrete Gross-Neveu model

$$S_{\mathrm{DGN}} = a^2 \sum_x \left\{ \overline{\psi}\, (D + m_0)\, \psi - \tfrac{1}{2} g^2 O_{SS} \right\}. \tag{5.44}$$

The model is invariant under finite chiral transformations $\psi \to \psi\gamma_5$, $\overline{\psi} \to -\gamma_5\overline{\psi}$ and under a hidden $\mathrm{O}(2N)$ symmetry. This invariance becomes evident in the Majorana representation [76]. Since its beta-function is well known in perturbation theory [77], we will use this model as a cross-check of our calculation.

5.5 Schrödinger functional of the CGN model

The Schrödinger functional of free Wilson and Ginsparg-Wilson fermions in two dimensions reads (cf. Section 4)

$$Z_0 = \int \mathrm{D}\psi \mathrm{D}\overline{\psi}\, \exp\left\{ -a^2 \sum_x \overline{\psi}(x)\, D\, \psi(x) \right\}, \tag{5.45}$$

where the Dirac operator D in presence of the boundaries is given by the left hand side of (4.35) and (4.48) respectively. The subscript of Z_0 is to indicate that it refers to the free theory. Remember that only the fermion

fields at times $x_0 = a, 2a, \ldots, T - a$ are integrated over in the functional integral and that they obey Dirichlet boundary conditions (4.2–4.3). It is convenient to assume, as we do in (5.45), that the fermion fields are defined at all other values of x_0 as well, but that they are zero there.

Since the four fermion operators have mass dimension two, they are irrelevant in the discussion of the naturalness of the Schrödinger functional boundary conditions (c.f. Section 4.1.2) and the needed boundary counter terms (Section 4.5). Thus with the fermion fields defined as above the Schrödinger functional of the chiral Gross-Neveu model is

$$Z = \int \mathrm{D}\psi\mathrm{D}\overline{\psi}\, \exp\left\{ -a^2 \sum_x \left[\overline{\psi}(x)\, D\, \psi(x) + \tfrac{1}{2} \sum_I c_I\, O_I \right] \right\}, \qquad (5.46)$$

where the sum is over the four fermion operators multiplied by the corresponding coupling constant. This notation covers all the possible choices of operators presented in the Section 5.3.

5.5.1 Generating functional

In Section 4.4 we outlined how to compute expectation values of polynomials $\mathcal{O}$ in the fermion and anti-fermion bulk and boundary fields. A definition of the lattice boundary fields is given by (4.67) and (4.68). The generating functional for the interacting theory is then

$$Z[\overline{\eta}, \eta] = \int \mathrm{D}\psi\mathrm{D}\overline{\psi}\, \exp\left\{ -a^2 \sum_x \left[\overline{\psi}\, D\, \psi - \tfrac{1}{2} \sum_I c_I\, O_I - (\overline{\psi}\eta + \overline{\eta}\psi) \right] \right\}. \qquad (5.47)$$

Replacing in $\mathcal{O}$ the field operators by functional derivatives as defined in (4.71), the expectation value can be written in a compact form

$$\langle \mathcal{O} \rangle = \left\{ \frac{1}{Z}\, \mathcal{O}\, Z[\overline{\eta}, \eta] \right\}_{\overline{\eta}, \eta = 0}. \qquad (5.48)$$

For our perturbative expansion it is convenient to rewrite (5.48). If we also replace the fields in the four fermion operators O_I by functional derivatives the generating functional becomes

$$Z[\overline{\eta}, \eta] = \mathrm{e}^{\frac{a^2}{2} \sum_x c_I O_I}\, Z_0[\overline{\eta}, \eta]\,. \qquad (5.49)$$

where $Z_0[\overline{\eta}, \eta]$ is the generating functional of the free theory (cf. Section 4.4). After integration over the fermion fields it is given by

$$Z_0[\overline{\eta}, \eta] = \exp\left\{ a^4 \sum_{x,y} \overline{\eta}(x)\, S(x,y)\, \eta(y) \right\}, \qquad (5.50)$$

where $S(x,y)$ is the propagator associated with the Dirac operator in (5.47)

$$D\,S(x,y) = \frac{1}{a^2}\,\delta_{x,y}\,, \quad 0 < x,y < T\,, \tag{5.51}$$

with boundary values

$$P_+\,S(x,y)\Big|_{x_0=0} = P_-\,S(x,y)\Big|_{x_0=T} = 0\,. \tag{5.52}$$

Finally we use (5.49) in (5.48) to write

$$\langle \mathcal{O} \rangle = \left\{ \frac{1}{Z}\,\mathcal{O}\,\mathrm{e}^{\frac{a^2}{2}\sum_x c_I\,O_I}\,Z_0[\overline{\eta},\eta] \right\}_{\overline{\eta},\eta=0}\,, \tag{5.53}$$

Expanded in powers of the couplings the generating functional (5.49) is the sum of all vacuum diagrams, that is, diagrams with no external lines. Clearly the factor $1/Z$ in (5.53) cancels all diagrams in the expansion of the expectation value, that contain vacuum diagrams as subdiagrams. Therefore we write the expansion of the expectation value as

$$\langle \mathcal{O} \rangle = \langle \mathcal{O} \rangle_0 + \tfrac{a^2}{2}\sum_x c_I\,\langle \mathcal{O}\,O_I(x) \rangle_0 + \tfrac{a^4}{8}\sum_{x,y} c_I\,c_J\,\langle \mathcal{O}\,O_I(x)\,O_J(y) \rangle_0 + \dots\,, \tag{5.54}$$

where $\langle \mathcal{O}\,X \rangle_0$, with X a product of four fermion operators or unity, is defined as

$$\langle \mathcal{O}\,X \rangle_0 = \left\{ \frac{1}{Z_0}\,\mathcal{O}\,X\,Z_0[\overline{\eta},\eta] \right\}_{\overline{\eta},\eta=0} - \begin{array}{l}\text{contractions with}\\ \text{vacuum subdiagrams}\end{array}\,. \tag{5.55}$$

The terms in the expansion (5.54) are given by the sum of all Wick contractions of the fields in the operator $\mathcal{O}$ and the increasing number of four fermion operator insertions, that do not contain vacuum diagrams as subdiagrams. The basic contractions are given by (4.73)-(4.81). The contractions are most conveniently represented by Feynman diagrams. The rules for the corresponding expressions are given in the next section.

5.5.2 Feynman rules

Because translation invariance is broken in the time direction the Feynman rules in the Schrödinger functional are given in a half Fourier transformed space. That is, the half transformed Propagator $\widetilde{S}(x_0,y_0,p_1)$ is defined through

$$S(x,y) = \frac{1}{L}\sum_{p_1} \mathrm{e}^{ip_1(x_1-y_1)}\,\widetilde{S}(x_0,y_0,p_1)\,. \tag{5.56}$$

The rules are listed in Fig. 5.1. In the case of Wilson fermions the propagators from the boundary to the interior $H(x_0, p_1)$, $H'(x_0, p_1)$ and from boundary to boundary $K(p_1)$, $K'(p_1)$ can be evaluated to some extent (see Appendix B.1.1).

Statistical factors Because the four fermion interactions consist of a product of two identical bilinear operators, each insertion of a four fermion interaction can be connected to the rest of a diagram in two ways that lead to the same final contribution. In other words there are two different contractions giving the same diagram. This factor is always cancelled by the $1/2$ that comes with each insertion (c.f. Eq. (5.55)).

The factor $1/n!$ from the Taylor expansion needs some more words. At the nth order of this expansion it multiplies terms with n four fermion interactions. Their general strucure is

$$\left(\sum_{I=1}^{F} c_I\, O_I\right)^n = \sum_{m_1+m_2+\cdots+m_F=n} \frac{n!}{m_1!\, m_2! \ldots m_F!}\, O_1^{m_1}\, O_2^{m_2} \ldots O_F^{m_F}\,, \quad (5.57)$$

where F is the number of different four fermion operators in the action ($F = 3$ for Wilson, $F = 2$ for Ginsparg-Wilson fermions). The number of equal contractions for a term of this sum due to interchange of vertices is $m_1!\, m_2! \ldots m_F!$. Thus, taking also into account the overall factor $1/n!$ from the Taylor expansion of the exponential function (see (5.55)) we have $(m_1!\, m_2! \ldots m_F!)/n!$ which is exactly cancelled by the factor in Eq. (5.57). Therefore none of these factors appears in the Feynman rules.

Sign The sign of a diagram is determined by the number of traces. Each trace comes with a minus sign. This gives the overall sign

$$(-1)^{\#\text{traces}}\,.$$

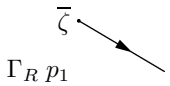
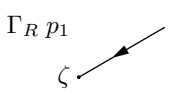
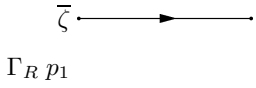
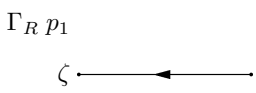
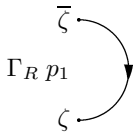
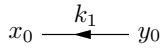
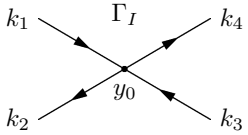
	$H(x_0,p_1)=\widetilde{S}(x_0,a,p_1)P_+$
	$\Gamma_R\gamma_5\,H^\dagger(x_0,p_1)\,\gamma_5=\Gamma_R\,P_-\widetilde{S}(a,x_0,p_1)$
	$\Gamma_{R'}\,\delta_{p_1p_1'}K(p_1)=\Gamma_{R'}\,\delta_{p_1p_1'}\,P_+\widetilde{S}(T-a,a,p_1)P_+$
	$\Gamma_R\gamma_5\,\delta_{p_1p_1'}K^\dagger(p_1)\,\gamma_5=\Gamma_R\,\delta_{p_1p_1'}\,P_-\widetilde{S}(a,T-a,p_1)P_-$
	$\Gamma_R\,L(p_1)=\Gamma_R\,P_-\widetilde{S}(a,a,p_1)P_+$
	$\sum_{k_1}\ \widetilde{S}(x_0,y_0,k_1)$
	$\sum_{y_0,I}\ \frac{a}{L}\,\delta(k_1-k_2-k_3+k_4)$
	Γ_I, for example $\gamma_5\lambda^a$
	Γ_X, current/density insertion, e.g. $\gamma_0\gamma_5$
	same vertex but separate Dirac and flavour traces

FIGURE 5.1: Feynman rules for the Schrödinger functional of the chiral Gross-Neveu model.

Chapter 6

Chiral symmetry restoration

In this Section we use the chiral Ward identity to restore chiral symmetry in the Wilson discretisation at finite lattice spacing (up to $\mathrm{O}(a)$). We show here that this can be achieved by a perturbative computation of the critical mass m_c and a symmetric $\delta_{P,s}$. The value of the third coupling g_V^2 has not to be tuned.

On the lattice, using the Wilson discretisation, chiral symmetry is broken and nothing in general prevents m_0 to take a finite value or $\delta_P^2 \neq 0$. However, as we pointed out in Section 3.4, the renormalised axial current $\left(A_\mu\right)_{\mathrm{R}}$ of properly defined fields is expect to obey

$$\left\langle \left(\mathcal{O}\right)_{\mathrm{R}} \widetilde{\partial}_\mu \left(A_\mu\right)_{\mathrm{R}} (x) \right\rangle = \mathrm{O}(a)\,. \tag{6.1}$$

This condition can be used to fix the bare parameters in perturbation theory [51, 71] (as well as in numerical simulations [34]). In the following section we discuss the strategy and the result in detail.

6.1 Correlation functions

We define correlation functions in the Schödinger functional (SF) set up in order to utilise (6.1). The correlation functions

$$f_X(x_0) = -\frac{a^2}{2N} \sum_{y_1,z_1} \left\langle\, \overline{\psi}(x)\, \Gamma_X\, \psi(x)\, \overline{\zeta}(y_1)\, \gamma_5\, \zeta(z_1) \,\right\rangle, \tag{6.2}$$

$$\Gamma_A = \gamma_0\gamma_5\,, \quad \Gamma_P = \gamma_5\,, \tag{6.3}$$

to be considered are correlators of a zero momentum pseudo scalar boundary state built from the boundary fields ζ, $\overline{\zeta}$ (eq. (4.67)) and insertions of

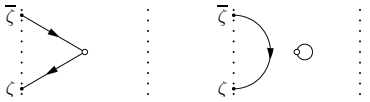


FIGURE 6.1: Tree level diagrams for f_X.

the time component of the axial current (f_A) and the pseudo scalar density (f_P) respectively. The vacuum expectation value $\langle\cdot\rangle$ has been defined in Section 5.5.1.

As discussed in Section 5.3.1 there is some ambiguity in the operators of the four fermion interaction. Here we use the following form of the action

$$S_{\mathrm{CGN,W}} = a^2 \sum_x \Big\{ \overline{\psi}\,(D_{\mathrm{W}} + m_0)\,\psi \\ - \tfrac{1}{2} g'^2 (O'_{SS} - O'_{PP}) - \tfrac{1}{2}\delta'^2_P O'_{PP} - \tfrac{1}{2} g'^2_V O'_{VV} \Big\}\,, \quad (6.4)$$

for the actual computation. Using (6.4) in the perturbative expansion the computation gets simplified, because a number of contractions vanish due to vanishing flavour traces. Results obtained for the primed couplings are then translated back into the more common unprimed couplings of (5.27) using (5.29)-(5.31).

For small couplings the expectation value in (6.2) can be expanded as indicated in (5.54). Then the right hand side of (6.2) is a sum of an increasing number of insertions of the interactions but the expectation value taken with the free action S_0 only and all contractions with vacuum subdiagrams subtracted (cf. (5.55))

$$f_X(x_0) = f_X^{(0)}(x_0) + \sum_I c_I\, f_{X,I}^{(1)}(x_0) + \sum_{I,J} c_I c_J\, f_{X,IJ}^{(2)}(x_0) + \mathrm{O}(c^3)\,, \quad (6.5)$$

where

$$c_I = \{g'^2, \delta'^2_P, g'^2_V\} \quad \text{and} \quad O_I = \{O'_{SS} - O'_{PP}, O'_{PP}, O'_{VV}\}\,. \quad (6.6)$$

The tree level amplitude

$$f_X^{(0)}(x_0) = -\frac{a^2}{2N} \sum_{y_1,z_1} \left\langle\, \overline{\psi}(x)\,\Gamma_X\,\psi(x)\;\overline{\zeta}(y_1)\,\gamma_5\,\zeta(z_1)\,\right\rangle_0\,, \quad (6.7)$$

is the sum of the two diagrams sketched in Fig. 6.1. In these diagrams the dotted lines represent the time slices $x_0 = 0$ and $x_0 = T$. Plain lines are

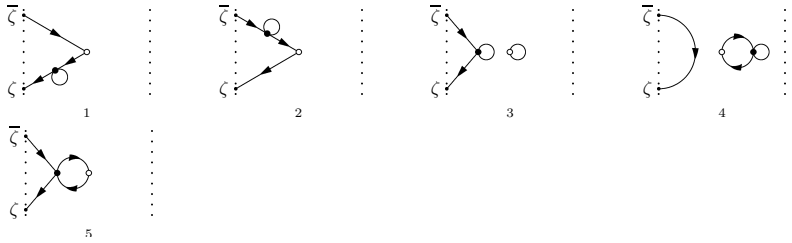


FIGURE 6.2: First order diagrams for f_X.

used for the fermion propagator and the small open circle in the middle symbolises the insertion of a current or density operator. Using the explicit form of the porpagator (4.42) the tree level amplitudes can be calculated analytically to some extent. The somewhat lengthy expressions are listed in Appendix B.3.

The first order amplitudes

$$f^{(1)}_{X,I}(x_0) = -\frac{a^4}{4N} \sum_{y_1,z_1,u} \left\langle\, \overline{\psi}(x)\, \Gamma_X\, \psi(x)\; \overline{\zeta}(y_1)\, \gamma_5\, \zeta(z_1)\; O'_I(u) \,\right\rangle_0 , \tag{6.8}$$

are sums of five diagrams. Due to the γ_5-hermiticity of the Wilson-Dirac operator, and thus the fermion propagator, the diagrams 1 and 2 in Fig. 6.2 are equal. In these diagrams the small filled circles (dots) represent the insertion of a four fermion interaction. Since the free propagator is diagonal in flavour space the flavour traces factorise from the rest of the computation. Differences in the resulting factors originate from different orders of the λ-matrices in the four fermion interaction and the number of separate traces. Only those combinations of the λ-matrices that can be reduced to the identity give a non-zero contribution. In the case of the first order diagrams there is only one possible order

$$\lambda^a \lambda^a = \frac{2(N^2-1)}{N}\, \mathbb{1} . \tag{6.9}$$

The number of separate traces is one (diagrams 1,2,5) and two (diagrams 3,4). Together with the factor $1/N$ in the definition of the correlation function the first order amplitudes can be organised in powers of N^p with $p = 0, 1$ and an overall factor (6.9).

The second order amplitudes

$$f^{(2)}_{X,IJ}(x_0) = -\frac{a^6}{8N} \sum_{y_1,z_1,u,v} \left\langle\, \overline{\psi}(x)\, \Gamma_X\, \psi(x)\; \overline{\zeta}(y_1)\, \gamma_5\, \zeta(z_1)\; O'_I(u)\; O'_J(v) \,\right\rangle_0 , \tag{6.10}$$

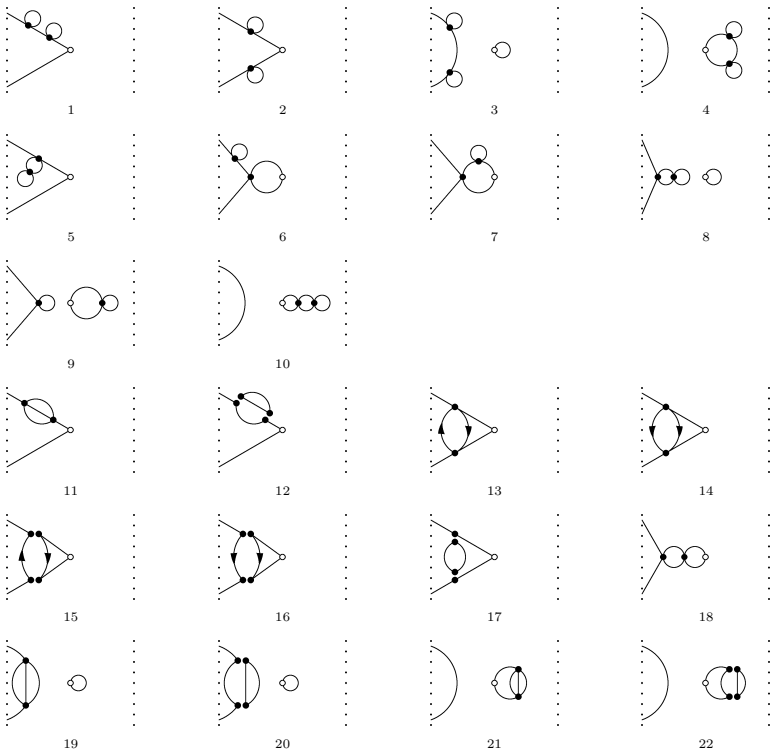


FIGURE 6.3: Second order diagrams for f_X.

are sums of the 22 diagrams depicted in Fig. 6.3. The diagrams 1, 5-7, 11 and 12 have to be counted twice since when reflected at an horizontal line they give diagrams with the same numerical value but a different contraction (just like the first order diagrams 1 and 2 are equal). There are now four λ-matrices to combine producing factors

$$\lambda^a\lambda^a\lambda^b\lambda^b = \frac{4(N^2-1)^2}{N^2}\,\mathbb{1}, \tag{6.11}$$

$$\lambda^a\lambda^b\lambda^a\lambda^b = -\frac{4(N^2-1)}{N^2}\,\mathbb{1}, \tag{6.12}$$

$$\lambda^a\lambda^b \otimes \lambda^a\lambda^b = \frac{4(N^2-1)}{N^2}\,\mathbb{1}\otimes\mathbb{1}. \tag{6.13}$$

The number of separate traces reaches from one to three. Together with the factor $1/N$ in the definition of the correlation function the second order amplitudes can be organised in powers of N^p with $p = 0,1,2,3$ and an overall factor $4(N^2-1)/N^2$.

6.2 Strategy and result

We work with the Schrödinger functional of the Gross-Neveu model at fixed ratio $T/L = 2$. The phase $\theta \equiv \theta_1$ characterising the spatial boundary conditions (cf. Section 2.2) is a free parameter of this regularisation. The Ward identities are independent of θ. Hence it provides a probe for the critical mass in the sense that (6.1) must hold for all θ.

We define the renormalised correlation function $(f_A(x_0))_{\mathrm{R}}$ and its time derivative $h_A(\theta, x_0/L)$ as dimensionless quantities

$$(f_A(x_0))_{\mathrm{R}} = Z_A\, Z_\zeta^2\, f_A(x_0) \quad \text{and} \quad h_A(\theta, x_0/L) = L\widetilde{\partial}_0\ (f_A(x_0))_{\mathrm{R}}\ , \tag{6.14}$$

where we introduced normalisation factors for the axial current and the boundary fields

$$Z_A = 1 + c_I\, Z_{A,I}^{(1)} + \mathrm{O}(c^2)\,, \quad Z_\zeta = 1 + c_I\, Z_{\zeta,I}^{(1)} + \mathrm{O}(c^2)\,. \tag{6.15}$$

In general, because $\partial_\mu A_\mu(x)$ can mix with $\frac{1}{a}\,P(x)$ we expect for h_A an expansion in powers of a/L starting with a linear divergence

$$h_A(\theta, x_0/L) = A_{-1}(\theta, x_0/L)\, L/a + A_0(\theta, x_0/L) + \mathrm{O}(a/L)\,. \tag{6.16}$$

Eq. (6.1) enforces the coefficients of the divergence and the finite part to vanish [1]

$$A_{-1}(\theta, x_0/L) = A_0(\theta, x_0/L) = 0\,, \quad \text{for all} \quad \theta\,, x_0/L\,, \tag{6.17}$$

$$\text{at} \quad am_0 = am_c \quad \text{and} \quad \delta'^2_P = \delta'^2_{P,s}\,. \tag{6.18}$$

Thus we have a two-dimensional parameter space spanned by θ and x_0/L for which these coefficients must vanish.

For free fermions (6.1) is satisfied for $am_0 = am_c = 0$. In the interacting theory m_c can be expanded

$$am_c = am^{(1)}_{c,I}\, c_I + am^{(2)}_{c,IJ}\, c_I c_J + \mathrm{O}(c^3)\,. \tag{6.19}$$

As we will see also δ'^2_P is constrained by (6.1) and is given in terms of the other two couplings.

The dimensionless h_A just as well posses an expansion in powers of the couplings

$$h_A(\theta, x_0/L) = h^{(0)}_A + \sum_I c_I\, h^{(1)}_{A,I} + \sum_{I,J} c_I c_J\, h^{(2)}_{A,IJ} + \mathrm{O}(c^3)\,. \tag{6.20}$$

Now the coefficients of this expansion, sums of lattice diagrams, may be expanded in powers of a/L. As already indicated $h^{(0)}_A = h_0$ is at least $\mathrm{O}(a^3/L^3)$ and therefore $am^{(0)}_c = 0$. In the next two sections we compute the divergent and finite part of $h^{(1)}_{A,I}$ and $h^{(2)}_{A,IJ}$. Although there will be some analytic arguments to simplify the expressions, the final evaluation of the contributing diagrams is performed numerically as described in Section 2.3.

6.2.1 First order

Using expansions (6.5) and (6.19) in (6.20) yields for the first order term

$$\sum_I c_I\, h^{(1)}_{A,I} = \sum_I c_I \left\{ h_{1,I} + am^{(1)}_{c,I}\, h_2 + \left(Z^{(1)}_{A,I} + 2Z^{(1)}_{\zeta,I} \right) h_0 \right\}, \tag{6.21}$$

with

$$h_0 = L\widetilde{\partial}_0 f^{(0)}_A(x_0)\,, \quad h_{1,I} = L\widetilde{\partial}_0 f^{(1)}_{A,I}(x_0)\,, \quad h_2 = \frac{\partial}{\partial am_0} h_0\,, \tag{6.22}$$

[1] Be careful to not confuse the coefficient $A_0(\theta, x_0/L)$ with the time component of the axial vector current $A_0(x)$.

all defined at $am_0 = 0$. The tree level amplitude h_0 vanishes identically for $\theta = 0$ and is $\mathrm{O}(a^3/L^3)$ for $\theta \neq 0$. Its derivative h_2 with respect to am_0 though diverges linearly with L/a. This implies at first order

$$\sum_I c_I \left\{ h_{1,I} + am_{c,I}^{(1)}\, h_2 \right\} = \mathrm{O}(a/L)\,. \tag{6.23}$$

Linear divergence The first order amplitudes $h_{1,I} = \sum_{d=1}^{5} h_{1,I}^d$ are the sums of the diagrams depicted in Fig. 6.2. Only diagrams one to four contribute to the linear divergence due to the contact term induced by the bubble contraction (cf. Appendix B.1.3). The contribution of diagram five $h_{1,I}^5$ is $\mathrm{O}(1)$ and thus is important for enforcing $A_0(\theta, x_0/L) = 0$.

For the linearly divergent diagrams one to four we find (with the help of (B.24))

$$\sum_{d=1}^{4} h_{1,I}^d = -\frac{a^3}{2N}\,\frac{2(N^2-1)}{N}\,L\widetilde{\partial}_0 \sum_{y_1,z_1,u} \times \\ \left\langle\, A_0(x)\,\overline{\zeta}(y_1)\,\gamma_5\,\zeta(z_1)\,\overline{\psi}(u)\,\Big(B(u_0) + F_I(u)\,a^2/L^2\,\theta_1 + \ldots\Big)\,\psi(u)\,\right\rangle_0 . \tag{6.24}$$

The factor $2(N^2-1)/N$ originates from the sum over the λ-matrices (cf. Eq. (6.9) and comment before). For fixing the critical mass we need only the leading term in (6.24). Using (B.21) and (B.25) we find

$$\sum_{d=1}^{4} h_{1,I}^d = -\frac{2(N^2-1)}{N}\,B_1\,h_2 + \mathrm{O}(1)\,. \tag{6.25}$$

Thus the contribution of these diagrams is proportional to h_2 which also multiplies $am_{c,I}^{(1)}$ in (6.21). Abbreviating $am_c^{(1)} = \sum_I am_{c,I}^{(1)}\,c_I$ the linearly divergent part at this order is given by

$$A_{-1}(\theta, x_0/L) = \left(am_c^{(1)} - \left(\delta'^2_P + 2g'^2_V\right)\,\frac{2(N^2-1)}{N}\,B_1 \right)\,C_{-1}(N,\theta,x_0/L)\,. \tag{6.26}$$

Where $C_{-1}(N,\theta,x_0/L)$ is the coefficients of the linear divergences in h_2

$$h_2 = C_{-1}(N,\theta,x_0/L)\,\tfrac{L}{a} + \mathrm{O}(1)\,, \tag{6.27}$$

$$\text{and} \quad C_{-1}(N,\theta,x_0/L) = C_{-1,0}(\theta,x_0/L) + N\,C_{-1,1}(\theta,x_0/L)\,. \tag{6.28}$$

The linear divergences in h_2 arise from the mass derivative of the two tree level diagrams in Fig. 6.1. Diagram two receives an additional factor of N because its evaluation involves two separate flavour traces (it has two closed fermion loops). These amplitudes can be determined numerically. The right hand side of (6.26) vanishes for all $\theta, x_0/L$ only if the critical mass is

$$am_c^{(1)} = \frac{2(N^2-1)}{N} B_1 \left(\delta'^2_P + 2g'^2_V\right) . \tag{6.29}$$

Finite part Summing up all $\mathrm{O}(1)$ contributions of the five diagrams and organising it in powers of N we find that the terms proportional to g'^2 and g'^2_V cancel. Explicitly we have

$$A_0(\theta, x_0/L) = \delta'^2_P \frac{2(N^2-1)}{N} C_0(N, \theta, x_0/L) , \tag{6.30}$$

with

$$C_0(N, \theta, x_0/L) = C_{0,0}(\theta, x_0/L) + N\, C_{0,1}(\theta, x_0/L) . \tag{6.31}$$

The right hand side of (6.30) has to vanish for all θ, x_0/L. Therefore chiral symmetry is restored for

$$\delta'^2_{P,s} = 0 , \tag{6.32}$$

at first order of perturbation theory.

Summary Translating Eqs. (6.29) and (6.32) back to the unprimed couplings using (5.29)-(5.31) yields

$$am_c^{(1)} = -0.3849001 \times \left(2Ng^2 - \delta_P^2 - 2g_V^2\right) , \tag{6.33}$$

$$\delta^2_{P,s} = \mathrm{O}(g^4) . \tag{6.34}$$

At this order we find no constraint on the vector coupling g_V^2. Or phrased in another way, the vector coupling has not to be tuned in order to restore chiral symmetry.

6.2.2 Second order

The relevant terms for the second order term in (6.20) are

$$\sum_{I,J} c_I c_J\, h^{(2)}_{A,IJ} =$$

$$\sum_{I,J} c_I c_J \left\{ am^{(2)}_{c,IJ}\, h_2 + h_{1,IJ} + am^{(1)}_{c,I} h_{3,J} + am^{(1)}_{c,I} am^{(1)}_{c,J}\, h_4 + \mathrm{O}(a/L) \right\} , \tag{6.35}$$

with

$$h_{1,IJ} = L\widetilde{\partial}_0 f^{(2)}_{A,IJ}(x_0)\,, \quad h_{3,I} = \frac{\partial}{\partial a m_0} h_{1,I}\,, \quad h_4 = \frac{1}{2}\frac{\partial^2}{\partial a m_0^2} h_0\,. \tag{6.36}$$

It is understood that the first order critical mass is set to the value found above and the constraint $\delta^2_{P,s} = \mathrm{O}(g^4)$ is employed. Also terms like

$$(Z^{(2)}_{A,IJ} + 2Z^{(2)}_{\zeta,IJ})\, h_0 \quad \text{and} \quad (Z^{(1)}_{A,I} + 2Z^{(1)}_{\zeta,I})\, h^{(1)}_{A,J}\,,$$

that are at least $\mathrm{O}(a/L)$, are suppressed in (6.35).

Divergences Recall the reduction of bubble diagrams like the diagrams 1-10 in Fig. 6.3 to insertions of the scalar density and the relation between the mass derivative of a expectation value and the insertion of the scalar density outlined in Appendix B.1.3. After a little thought it is clear that the derivatives $h_{3,J}$ and h_4 cancel all the power divergences in the second order amplitudes $h_{1,IJ}$ that are due to bubbles, i.e. the contribution of diagrams 1-10. That this is indeed the case was checked numerically. The only other linearly divergent diagrams are number 11, 12 and 21, 22 in the same Figure. In order to keep the equations clear we define

$$a m_c^{(2)} = \sum_{IJ} a m^{(2)}_{c,IJ}\, c_I c_J\,, \tag{6.37}$$

and the linear divergent part of the summed up diagrams 11, 12, 21 and 22

$$\sum_{d=11,12,21,22} h^d_{1,IJ} = C^{IJ}_{-1}(N,\theta,x_0/L)\,\tfrac{L}{a} + \mathrm{O}(1)\,, \tag{6.38}$$

$$\text{with} \quad C^{IJ}_{-1}(N,\theta,x_0/L) = \sum_{n=0}^{2} N^n\, C^{IJ}_{-1,n}(\theta,x_0/L)\,. \tag{6.39}$$

The coefficient of the linear divergence at second order is then

$$A_{-1}(\theta,x_0/L) = a m_c^{(2)}\, C_{-1} + \frac{4(N^2-1)}{N^2} \sum_{I,J} c_I c_J\, C^{IJ}_{-1}\,, \tag{6.40}$$

where $C_{-1} = C_{-1}(N,\theta,x_0/L)$ is defined in (6.27).

Setting the right hand side of (6.40) zero leads to

$$a m_c^{(2)} = -\frac{4(N^2-1)}{N^2} \sum_{I,J} c_I c_J\, \frac{C^{IJ}_{-1}(N,\theta,x_0/L)}{C_{-1}(N,\theta,x_0/L)}\,. \tag{6.41}$$

When one evaluates the ratios in the last expression numerically, one finds that the θ and x_0/L-dependence of the numerator is cancelled by the denominator. In other words $C^{IJ}_{-1}(N,\theta,x_0/L)$ factorises

$$C^{IJ}_{-1}(N,\theta,x_0/L) = (D^{IJ}_{-1,0} + D^{IJ}_{-1,1}\,N)\cdot C_{-1}(N,\theta,x_0/L)\,. \tag{6.42}$$

where $D^{IJ}_{-1,n}$ are computable constants. Organising the terms in powers of N we find at this order

$$\begin{aligned} am_c^{(2)} = &-\frac{4(N^2-1)}{N^2} \\ &\times\left\{(D_1+N\,D_2)\,(g'^4+g'^4_V)+2(D_2+N\,D_1)\,g'^2\,g'^2_V\right\} \end{aligned} \tag{6.43}$$

with

$$D_1 = 0.01195(1)\,,\quad D_2 = 0.22870(1)\,. \tag{6.44}$$

As mentioned the first order result $\delta'_{P,s} = \mathrm{O}(g'^4)$ must be used here in order to cancel all divergences.

Finite part The first order result allows for a second order contribution. Thus we have to include the first order finite part (6.30) at second order

$$A_0(\theta,x_0/L) = \delta'_{P,s}\,\frac{2(N^2-1)}{N}\,C_0 + \frac{4(N^2-1)}{N^2}\sum_{I,J} c_I c_J\,C_0^{IJ}\,. \tag{6.45}$$

The coefficients

$$C_0^{IJ}(N,\theta,x_0/L) = \sum_{n=0}^{3} N^n\,C^{IJ}_{0,n}(\theta,x_0/L)\,, \tag{6.46}$$

receive contribution from all the 21 diagrams of Fig. 6.3, which all have a factor of $4(N^2-1)/N^2$ in common. $C_0 = C_0(N,\theta,x_0/L)$ was introduced in (6.30) and is the finite part of the first order diagrams up to a common factor of $2(N^2-1)/N$. Setting (6.45) zero yields

$$\delta'_{P,s} = \frac{2}{N}\sum_{I,J} c_I c_J\,\frac{C_0^{IJ}(N,\theta,x_0/L)}{C_0(N,\theta,x_0/L)}\,. \tag{6.47}$$

Again we find that the θ and x_0/L-dependence of the numerator is cancelled by the one of the denominator, which means that chiral symmetry

is really restored. Note that in the sum on the left hand side many terms vanish because of $\delta'_{P,s} = \mathrm{O}(g^4)$. Organised in powers of N the result is

$$\delta'_{P,s} = \frac{2D_3}{N}\left[N\,g'^4 + \left(1+N-N^2\right)\,g'^4_V + 2\,g'^2\,g'^2_V\right] + \mathrm{O}(g'^6)\,, \quad (6.48)$$

with

$$D_3 = 0.6192(1)\,. \quad (6.49)$$

Summary Translating Eqs. (6.43) and (6.48) back to the more common couplings g^2, δ_P^2, g_V^2 using (5.29)-(5.31) yields

$$am_c^{(2)} = (D_1 - N\,D_2)\,(g^4 + g_V^4) + 2(D_2 - N\,D_1)\,g^2\,g_V^2 + \mathrm{O}(g^6)\,, \quad (6.50)$$

$$\delta_{P,s} = D_3\,\left[N\,g^4 - 2\,g^2\,g_V^2 - g_V^4\right] + \mathrm{O}(g^6)\,. \quad (6.51)$$

6.3 Conclusion

We demanded the chiral Ward identity (6.1) to hold up to $\mathrm{O}(a)$ on the lattice for the renormalised operators. The computation is carried out in second order perturbation theory in the Schrödinger functional of the Gross-Neveu model with Wilson fermions. The result is that the bare mass has to diverge in order to cancel a linear divergence and that the chiral symmetry breaking coupling δ_P^2, although zero at first order, has to take a finite value at second order.

Explicitly the result is

$$am_c = am_c^{(1)} + am_c^{(2)} + \mathrm{O}(g^6)\,. \quad (6.52)$$

with

$$am_c^{(1)} = -B_1 \times \left(2Ng^2 - \delta_P^2 - 2g_V^2\right)\,, \quad (6.53)$$

$$am_c^{(2)} = (D_1 - N\,D_2)\,(g^4 + g_V^4) + 2(D_2 - N\,D_1)\,g^2\,g_V^2 + \mathrm{O}(g^6)\,, \quad (6.54)$$

and

$$\delta_{P,s}^2 = D_3\,\left(N\,g^4 - 2\,g^2\,g_V^2 - g_V^4\right) + \mathrm{O}(g^6)\,. \quad (6.55)$$

We find no constraint on the vector coupling g_V^2. Or phrased in another way, the vector coupling has not to be tuned in order to restore chiral symmetry.

In order to compare with the large N result of [30] we set and rescale

$$g^2 = g_S^2/N\,, \quad \delta_P^2 = (g_P^2 + g_S^2)/N \quad \text{and} \quad g_V^2 \to g_V^2/N\,, \tag{6.56}$$

and take the $N \to \infty$ limit in (6.52) and (6.55)

$$am_c^{N\to\infty} = -0.7698002 \times g_S^2 + \mathrm{O}(g^6)\,, \tag{6.57}$$

$$\left[g_P^2/g_S^2\right]_s^{N\to\infty} = 1 - 0.6192(1) \times g_S^2 + \mathrm{O}(g^4)\,. \tag{6.58}$$

For am_c this is the complete large N result. The authors neglect the vector four fermion interaction from the beginning, but their result is not changed if it is taken into account. In the large N limit chiral symmetry is restored for $1/g_P^2 = 1/g_S^2 + 0.619$ which is reproduced by our result at second order.

Chapter 7

Renormalised coupling

If the time extension T of the Schrödinger functional of the massless chiral Gross-Neveu model is fixed to a multiple of the spatial extension L, say $T = 2L$, then all dimensionfull quantities will depend on L. Keeping the volume finite usually leads to systematic errors. But here the finite volume is utilised to probe the theory. That is renormalised quantities are defined at the scale $\mu = 1/L$. Therefore by definition there are no finite size effects in the Schrödinger functional.

In view of a Monte-Carlo simulation of the model we aim here at the definition of renormalised couplings at zero renormalised mass in terms of renormalised correlation functions of the boundary fields (4.67), (4.68). For each coupling g_I^2 in the action a combination F_I of such correlation functions defines the corresponding renormalised coupling $\tilde{g}_I^2$ such that it is equal to the bare coupling at leading order of perturbation theory up to corrections of order a

$$\tilde{g}_I^2 = F_I \quad \text{such that} \quad \tilde{g}_I^2 = g_I^2 + \mathrm{O}(ag^2) + \mathrm{O}(g^4)\,. \tag{7.1}$$

In the continuum limit of asymptotically free theories this ensures $\tilde{g}_I^2 \stackrel{a\to 0}{=} g_I^2 + \mathrm{O}(g^4)$.[1] In turn, as we have seen in Section 2.1.3, such a definition of the renormalised coupling ensures the universality of the first two coefficients of the corresponding beta-function in the single coupling case. For multiple couplings the situation is more complicated (see Section 2.1.4). Nevertheless condition (7.1) is a possible choice also in that case.

In the next Section the used correlation functions are discussed. In Section 7.2 they are formally expanded in powers of the couplings exploiting the cancellation of some diagrams. The results for Wilson and

[1] Throughout the thesis $\mathrm{O}((g^2)^n)$ means all terms $\mathrm{O}((g_{I_1}^2)^{n_1}\dots(g_{I_m}^2)^{n_m})$, $\sum_{i=1}^m n_i = n$.

Ginsparg-Wilson fermions are presented in Sections 7.3 and 7.4 respectively. In the last Section (7.5) we use the results to compute the ratio of the Λ-parameters.

7.1 Correlation functions

The two- and four-point correlators of the boundary fields are defined as

$$f_2 = -\frac{a^2}{NL} \sum_{u_1 z_1} \left\langle \overline{\zeta}(u_1)\, \zeta'(z_1) \right\rangle , \tag{7.2}$$

and

$$f_4 = -\frac{a^4}{2(N^2-1)L^2} \sum_{u_1 v_1 y_1 z_1} \left\langle \overline{\zeta}'(u_1)\, \gamma_5 \lambda^a \zeta'(v_1)\; \overline{\zeta}(y_1)\, \gamma_5\, \lambda^a \zeta(z_1) \right\rangle . \tag{7.3}$$

We sum over the spatial direction to project on to zero momentum. The powers of the lattice spacing a are chosen to render the correlation functions dimensionless. The remaining factors are to normalise the tree level amplitudes.

The matrices Γ_B and Γ'_B contract the Dirac indices of the boundary fields. In two dimensions such a matrix can be expanded in the identity, γ_μ and γ_5. But only terms with Γ_B and Γ'_B equal to γ_1 and/or γ_5 have a non-vanishing contribution because of the projectors $P_\pm$ at the boundary of the Schrödinger functional (apparent for example in the basic Wick contractions at the end of Section 4.4 or the Feynman rules in Fig. 5.1). For the same reason the contribution of the possible combinations of the two are all equal up to a factor of i and the overall sign ($P_\pm \gamma_5 P_\mp = \pm i P_\pm \gamma_1 P_\mp$). Therefore in the computation of f_4 we use for the Dirac structure on the boundaries $\Gamma'_B = \Gamma_B = \gamma_5$.

For the flavour structure we choose flavour vector bilinears. In combination with four fermion interactions of the flavour scalar type (5.7) this choice simplifies the perturbative computation. This is because diagrams which do not connect the two boundaries involve the flavour trace $\mathrm{tr}_f\{\lambda^a\}$ and hence vanish.

The renormalised two- and four-point correlation functions are

$$(f_2)_{\mathrm{R}} = Z_\zeta^2 f_2 \quad \text{and} \quad (f_4)_{\mathrm{R}} = Z_\zeta^4 f_4 \tag{7.4}$$

The normalisation factor for the boundary fields was already introduced in (6.15) in Section 6.2.

In the ratio of appropriate powers of the renormalised quantities the unknown wave function normalisation is cancelled. Then it is enough to evaluate the unrenormalised correlators at zero renormalised mass

$$R(\theta) = \frac{(f_4)_{\mathrm{R}}}{(f_2)^2_{\mathrm{R}}} - 1 = \frac{f_4}{(f_2)^2} - 1 \quad \text{at} \quad m_{\mathrm{R}} = 0\,. \tag{7.5}$$

For later convenience we subtract the free theory value at zero mass. As indicated the ratio $R = R(\theta)$ depends on the phase $\theta \equiv \theta_1$ parametrising the spatial boundary conditions. This dependence will allow us to find combinations $R(\theta) + b \cdot R(\theta')$ to define the renormalised couplings as in (7.1).

7.2 Perturbative expansion

In order to expand R_g in powers of the couplings g_I^2 we need the expansion of f_4 and f_2. We abbreviate $c_I = g_I^2$ and indicate that opposite to (6.6) here c_I refers to g^2, δ_P^2 and g_V^2.

$$f_4 = f_4^{(0)} + c_I\, f_{4,I}^{(1)} + c_I c_J\, f_{4,IJ}^{(2)} + \mathrm{O}(c^3)\,, \tag{7.6}$$

and $(f_2)^2$

$$(f_2)^2 = \left(f_2^{(0)}\right)^2 + c_I\, 2 f_2^{(0)} f_{2,I}^{(1)} + c_I c_J \left\{ 2 f_2^{(0)} f_{2,IJ}^{(2)} + f_{2,I}^{(1)} f_{2,J}^{(1)} \right\} + \mathrm{O}(c^3)\,. \tag{7.7}$$

Multiplying (7.6) with the inverse of (7.7) yields

$$\begin{aligned} \frac{f_4}{(f_2)^2} &= 1 + \frac{c_I}{\left(f_2^{(0)}\right)^2} \left\{ f_{4,I}^{(1)} - 2 f_{2,I}^{(1)} f_2^{(0)} \right\} \\ &\quad + \frac{c_I c_J}{\left(f_2^{(0)}\right)^2} \left\{ f_{4,IJ}^{(2)} - 2 f_{2,IJ}^{(2)} f_2^{(0)} + 3 f_{2,I}^{(1)} f_{2,J}^{(1)} - 2 \frac{f_{2,I}^{(1)} f_{4,J}^{(1)}}{f_2^{(0)}} \right\} + \mathrm{O}(c^3)\,. \end{aligned} \tag{7.8}$$

The first and second order diagrams of f_2 and f_4 are listed in Figs. 7.1 and 7.2. Half of the diagrams of f_4 at first and second order can be shown to be identical to products of f_2 diagrams. To see this, consider a diagram $f_{4,i}^{(n)}$ at order n in the expansion of f_4, where the subscript i labels the diagram. Then we define:

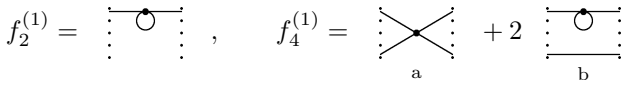


FIGURE 7.1: First order diagrams of f_2 and f_4. The bubble diagrams stand for the sum of the connected (one trace) disconnected (two traces) diagram.

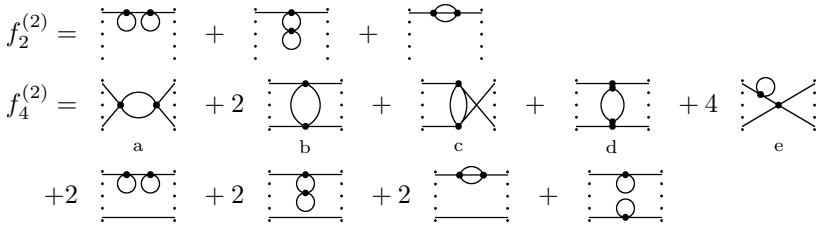


FIGURE 7.2: Second order diagrams of f_2 and f_4. The bubble diagrams stand for the sum of the connected (one trace) disconnected (two traces) diagram.

Definition 1 *If a diagram $f_{4,i}^{(n)}$ at order n in the expansion of f_4 can be cut horizontally without cutting through a fermion line or a vertex, then it is called reducible.*

In Appendix B.2.4 we prove the following Lemma:

Lemma 1 *If a diagram $f_{4,i}^{(n)}$ at order n in the expansion of f_4 is reducible, then $f_{4,i}^{(n)}$ can be written as a product $f_{2,j}^{(r)} \cdot f_{2,k}^{(s)}$ of two diagrams appearing in the expansion of f_2, with $n = r + s$. The amputated part of $f_{2,j}^{(r)}$ and $f_{2,k}^{(s)}$ is equal to the amputated part of the upper and the lower half of $f_{4,i}^{(n)}$, respectively.*

Diagrammatically this is depicted in Fig. 7.3. As a consequence all products of f_2 diagrams in the expansion (7.8) are cancelled by corresponding

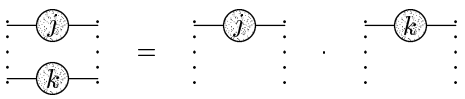


FIGURE 7.3: Diagrammatical representation of Lemma 1.

f_4 diagrams. Only f_4 diagrams that are not reducible are left

$$\frac{f_4}{(f_2)^2} = 1 + \frac{c_I}{f_4^{(0)}}\, f^{(1)}_{4,I,\mathrm{a}} + \frac{c_I c_J}{f_4^{(0)}} \left\{ f^{(2)}_{4,IJ,\mathrm{a}} + 2f^{(2)}_{4,IJ,\mathrm{b}} + f^{(2)}_{4,IJ,\mathrm{c}} + f^{(2)}_{4,IJ,\mathrm{d}} + 4f^{(2)}_{4,IJ,\mathrm{e}} - 2\frac{f^{(1)}_{2,I}}{f_2^{(0)}}\, f^{(1)}_{4,J,\mathrm{a}} \right\} + \mathrm{O}(c^3)\,. \quad (7.9)$$

From this result one easily reads off the expansion of R

$$R(\theta) = R^{(1)}(\theta) + R^{(2)}(\theta) + \mathrm{O}(c^3)\,, \quad (7.10)$$

with

$$R^{(1)}(\theta) = c_I\, R^{(1)}_I(\theta) \quad \text{and} \quad R^{(2)}(\theta) = c_I c_J\, R^{(2)}_{IJ}(\theta)\,. \quad (7.11)$$

The first and second order terms are

$$R^{(1)}_I(\theta) = \frac{1}{f_4^{(0)}}\, f^{(1)}_{4,I,\mathrm{a}}\,, \quad (7.12)$$

$$R^{(2)}_{IJ}(\theta) = \frac{1}{f_4^{(0)}} \left\{ f^{(2)}_{4,IJ,\mathrm{a}} + 2f^{(2)}_{4,IJ,\mathrm{b}} + f^{(2)}_{4,IJ,\mathrm{c}} + f^{(2)}_{4,IJ,\mathrm{d}} + 4f^{(2)}_{4,IJ,\mathrm{e}} - 2\frac{f^{(1)}_{2,I}}{f_2^{(0)}}\, f^{(1)}_{4,J,\mathrm{a}} \right\}. \quad (7.13)$$

We now proceed with the explicit computation for Wilson fermions.

7.3 Wilson fermions

The lattice action of the chiral Gross-Neveu model with Wilson fermions we use here is given by (5.27)

$$S_{\mathrm{CGN,W}} = a^2 \sum_x \left\{ \overline{\psi}\,(D_{\mathrm{W}} + m_0)\,\psi - \tfrac{1}{2}g^2(O_{SS} - O_{PP}) - \tfrac{1}{2}\delta_P^2 O_{PP} - \tfrac{1}{2}g_V^2 O_{VV} \right\}. \quad (7.14)$$

The renormalised mass vanishes if the bare mass is set to its critical value and the coupling of the pseudo-scalar interaction to its symmetric value

$$m_0 = m_c \quad \text{and} \quad \delta_P^2 = \delta_{P,s}^2 \,. \tag{7.15}$$

Since Lemma 1 holds for any m_0, it is also true for $m_0 = m_c$. In order to incorporate the expansion of m_c in the expansion of R, two new terms have to be added at second order, corresponding to the derivative with respect to am_0 of the first order term in (7.9) ($\partial_m = \partial/\partial am_0$)

$$R_I^{(1)}(\theta) = \frac{1}{f_4^{(0)}} f_{4,I,\mathrm{a}}^{(1)} \,, \tag{7.16}$$

$$R_{IJ}^{(2)}(\theta) = \frac{1}{f_4^{(0)}} \Bigg\{ f_{4,IJ,\mathrm{a}}^{(2)} + 2 f_{4,IJ,\mathrm{b}}^{(2)} + f_{4,IJ,\mathrm{c}}^{(2)} + f_{4,IJ,\mathrm{d}}^{(2)} + 4 f_{4,IJ,\mathrm{e}}^{(2)} + a m_{c,I}^{(1)} \partial_m f_{4,J,\mathrm{a}}^{(1)} - 2 \frac{f_{2,I}^{(1)} + a m_{c,I}^{(1)} \partial_m f_2^{(0)}}{f_2^{(0)}} f_{4,J,\mathrm{a}}^{(1)} \Bigg\} \,, \tag{7.17}$$

where all diagrams are evaluated at $m_0 = 0$. We do not set $\delta_P^2 = \delta_{P,s}^2$ from the beginning, but keep it as a free parameter and only set it to its symmetric value at the end of the computation. This will allow us to compare the general result with computations in the discrete Gross-Neveu model (cf. Section 5.4).

In the case of $R_I^{(1)}(\theta)$ we have to evaluate only one diagram which involves a sum over the time coordinate of the four fermion interaction. This sum can be computed analytically to some extent (cf. Appendix B.2.2), i.e. the continuum limit can be extracted. Using (B.35) and (B.27) we find

$$R^{(1)}(\theta) = \frac{T}{2L\, C(\theta)} \left\{ 2\, g^2 + \delta_P^2 \left(A(\theta) - 1 \right) - 2\, g_V^2\, B(\theta) \right\} + \mathrm{O}(a) \,. \tag{7.18}$$

with

$$A(\theta) = \frac{L}{2\theta T} \sinh(2\theta T/L) \overset{\theta \to 0}{\to} 1 \tag{7.19}$$

$$B(\theta) = \cosh(2\theta T/L) \overset{\theta \to 0}{\to} 1 \tag{7.20}$$

$$C(\theta) = \cosh^2(\theta T/L) \overset{\theta \to 0}{\to} 1 \,. \tag{7.21}$$

In the case of $R_{IJ}^{(2)}(\theta)$ there are five non-reducible diagrams involving one momentum loop. The diagrams are evaluated numerically for a range

of lattice sizes and several θ values and extract the finite and logarithmic divergent terms (cf. Section 2.3 and Appendix B.2.3). Using Eqs. (B.39-B.43) we find

$$R^{(2)}(\theta) = \frac{T}{2L\,C(\theta)}\,\frac{\ln(a/L)}{2\pi}\,\Big\{ -4\,g^4\,(N-B(\theta)) + 2\,\delta_P^4\,(N-1)\,(A(\theta)-1) \\ -4\,g^2\,\delta_P^2\,\left((N-1)A(\theta)+B(\theta)-N\right) + 8\,g_V^2\,\delta_P^2\,A(\theta)\Big\} + \ldots\,, \quad (7.22)$$

where the dots indicate the finite part that has been suppressed here.

7.3.1 Discrete Gross-Neveu model

Our general four fermion interaction theory also contains the well studied Gross-Neveu model [19]. We call it here the discrete Gross-Neveu model for its discrete chiral symmetry $\psi \to \gamma_5\psi$, $\overline{\psi} \to -\overline{\psi}\gamma_5$ and in order to discriminate it from the chiral Gross-Neveu model. The discrete Gross-Neveu model has only the scalar four fermion interaction $g^2/2\,(\overline{\psi}\psi)^2$ (cf. Section 5.4). In our notation this amounts to setting

$$\delta_P^2 = g^2 \qquad g_V^2 = 0\,. \quad (7.23)$$

This model possesses a $\mathrm{O}(2N)$ symmetry that allows no other interaction term. This strictly holds only for $\theta = 0\,,\ \pi$. For all other values the boundary conditions break this symmetry. However, the local Ward identities associated with this symmetry will still hold and therefore the ultra-violet divergences are expected to remain unchanged.

The results of the last section can be used to calculate the one-loop beta-function, which then can be compared to previous results.

With the settings (7.23), eqs. (7.18) and (7.22) simplify to

$$R^{(1)}_{\mathrm{dgn}}(\theta) = \frac{T}{2L\,C(\theta)}\,\left(A(\theta)+1+\mathrm{O}(a)\right)\,g^2\,, \quad (7.24)$$

$$R^{(2)}_{\mathrm{dgn}}(\theta) = -\frac{T}{2L\,C(\theta)}\,\frac{\ln(a/L)}{\pi}\,(N-1)\,(A(\theta)+1)\,g^2 + \ldots\,. \quad (7.25)$$

A renormalised coupling can be defined by

$$\tilde{g}^2_{\mathrm{dgn}} \equiv F_{\mathrm{dgn}} = \frac{2L\,C(\theta)}{T(A(\theta)+1)}\,R_{\mathrm{dgn}}(\theta)\,, \quad (7.26)$$

with the expansion

$$\tilde{g}^2_{\mathrm{dgn}} \overset{a\to 0}{=} g^2 - g^4\,\left\{\frac{N-1}{\pi}\,\ln(a/L) - c_{\mathrm{dgn}}\right\} + \mathrm{O}(g^6)\,, \quad (7.27)$$

with $c_{\text{dgn}} = c^0_{\text{dgn}} + c^1_{\text{dgn}}\, N$. The corresponding beta-function (2.7) is ($\mu = 1/L$)

$$\beta(\tilde{g}^2_{\text{dgn}}) = -\frac{N-1}{\pi}\,\tilde{g}^4_{\text{dgn}} + \mathrm{O}(\tilde{g}^6_{\text{dgn}})\,. \tag{7.28}$$

This result is in accordance with previous continuum calculations [19, 23, 77, 78] and recent lattice calculations [76].

7.3.2 Chiral Gross-Neveu model

Now that we have confidence in our expansion, we set the pseudo-scalar coupling to its symmetric value $\delta^2_P = \delta^2_{P,s}$ to ensure continuous chiral symmetry at finite lattice spacing (up to $\mathrm{O}(a)$, cf. Chapter 6)

$$\delta^2_{P,s} = D_3\,\left(N\,g^4 - 2\,g^2\,g_V^2 - g_V^4\right) + \mathrm{O}(g^6)\,. \tag{7.29}$$

Since the correction to zero is of order g^4, this enters as a finite term in $R^{(2)}(\theta)$. Explicitly, in the chirally symmetric case Eqs. (7.18) and (7.22) become

$$R^{(1)}_{\text{cgn}}(\theta) = \frac{T}{L\,C(\theta)}\,\left\{g^2 - g_V^2\,B(\theta)\right\} + \mathrm{O}(a)\,, \tag{7.30}$$

$$R^{(2)}_{\text{cgn}}(\theta) = -\frac{T}{L\,C(\theta)}\,\frac{\ln(a/L)}{\pi}\,(N - B(\theta))\,g^4 + \dots\,, \tag{7.31}$$

where we suppressed the finite part. Thus renormalised couplings may be defined through

$$\tilde{g}^2 \equiv F_g = \frac{L}{T(B(\theta)-1)}\,\left(B(\theta)\,R_{\text{cgn}}(0) - C(\theta)\,R_{\text{cgn}}(\theta)\right)\,, \tag{7.32}$$

and

$$\tilde{g}^2_V \equiv F_V = \frac{L}{T(B(\theta)-1)}\,\left(R_{\text{cgn}}(0) - C(\theta)\,R_{\text{cgn}}(\theta)\right)\,. \tag{7.33}$$

They obey the condition (7.1) as can be seen from their expansions

$$\tilde{g}^2 \overset{a\to 0}{=} g^2 - g^4\,\frac{N}{\pi}\,\ln(a/L) + c_g + \mathrm{O}(g^6)\,, \tag{7.34}$$

and

$$\tilde{g}^2_V \overset{a\to 0}{=} g_V^2 - g^4\,\frac{1}{\pi}\,\ln(a/L) + c_V + \mathrm{O}(g^6)\,, \tag{7.35}$$

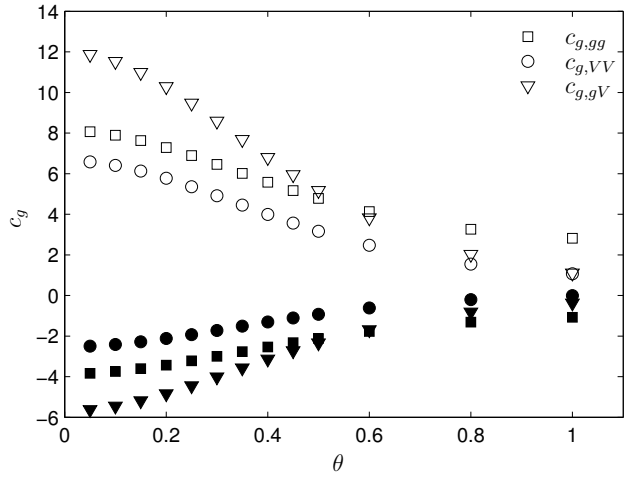


FIGURE 7.4: Dependence of the finite part of $\tilde{g}^2$ on θ at order g^4.

where the finite parts at second order are given by

$$c_g = c_{g,gg}\, g^4 + c_{g,VV}\, g_V^4 + c_{g,Vg}\, g_V^2 g^2 - \frac{D_3 D(\theta)}{2}\left(N\, g^4 - 2\, g^2\, g_V^2 - g_V^4\right), \tag{7.36}$$

and

$$c_V = c_{V,gg}\, g^4 + c_{V,VV}\, g_V^4 + c_{V,Vg}\, g_V^2 g^2 - \frac{D_3 D(\theta)}{2}\left(N\, g^4 - 2\, g^2\, g_V^2 - g_V^4\right), \tag{7.37}$$

where $D(\theta)$ is the ratio

$$D(\theta) = \frac{A(\theta) - 1}{B(\theta) - 1}\,. \tag{7.38}$$

The coefficients $c_{I,JK} = c^0_{I,JK} + c^1_{I,JK}\, N$ show a rather mild dependence on θ. They have been determined numerically for several values of θ in the interval $[0, 1]$. The result is shown in Fig. 7.4 and Fig. 7.5, where open and filled symbols refer to $c^0_{I,JK}$ and $c^1_{I,JK}$ respectively. The last term in (7.36) and (7.37) originates from $\delta_{P,s}\ \delta_P^2$ (7.29), which has to be included at this order.

It is straightforward to derive the associated beta-functions ($\mu = 1/L$)

$$\beta_g(\tilde{g}^2) = -\tilde{g}^4\, \frac{N}{\pi} + \mathrm{O}(\tilde{g}^6)\,, \tag{7.39}$$

and

$$\beta_V(\tilde{g}^2) = -\tilde{g}^4\, \frac{1}{\pi} + \mathrm{O}(\tilde{g}^6)\,. \tag{7.40}$$

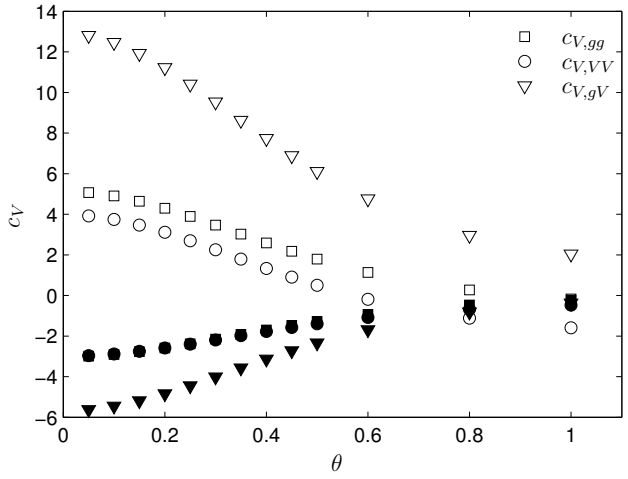


FIGURE 7.5: Dependence of the finite part of $\tilde{g}_V^2$ on θ at order g^4.

Thus, from (7.39) we see that the coupling g^2 is asymptotically free and from (7.35) it is obvious that the coupling g_V^2 receives an additive renormalisation. The one-loop beta-functions derived here agree with the ones derived in the $\overline{\text{MS}}$ scheme in Ref. [23].

As we have seen in Section 5.3 there is some freedom in the choice of four fermion interactions due to the identities (5.11)-(5.13). In particular, it is possible to find a combination of terms were one of the two beta-functions vanishes. This is also indicated by the formal continuum argumentation in [74, 75]. Indeed if we use (5.13) to rewrite (7.14) as in (5.32), the new coupling δ_V^2 is related to the original ones by (5.33)

$$\delta_V^2 = g_V^2 - g^2/N\,. \tag{7.41}$$

Then (7.32) remains unchanged. But the renormalised coupling associated with δ_V^2 is given by

$$\tilde{\delta}_V^2 = \tilde{g}_V^2 - \tilde{g}^2/N = \delta_V^2 + c_V - c_g/N + \mathrm{O}(g^6)\,. \tag{7.42}$$

There is no logarithmic divergence and therefore the corresponding beta-function vanishes at this order

$$\beta_\delta(\tilde{\delta}_V^2) = \mathrm{O}(\tilde{g}^6)\,. \tag{7.43}$$

7.4 Ginsparg-Wilson fermions

The free modified Neuberger Dirac operator D_{N} defined in Section 4.3.1 obeys

$$\gamma_5 D_{\mathrm{N}} + D_{\mathrm{N}} \gamma_5 = a D_{\mathrm{N}} \gamma_5 D_{\mathrm{N}} + \Delta_B \,, \tag{7.44}$$

The parameters c and s are set to 1 and 0, respectively, in the following. The term Δ_B is supported in the vicinity of the boundaries and decays exponentionally with the distance to them (cf. Fig 4.1). In particular, the rate of the decay is constant if the distance is measured in lattice units. Then transformation (5.34) is a symmetry of the action and the associated Ward identities are expected to hold in the interior (well separated from the boundaries) of lattice with small corrections which vanish in the continuum limit. Therefore we can use an action with two four fermion interactions (cf. Section 5.3.2)

$$S_{\mathrm{CGN,GW}} = a^2 \sum_x \left\{ \overline{\psi} D_{\mathrm{N}} \psi - \tfrac{1}{2} g^2 (\hat{O}_{SS} - \hat{O}_{PP}) - \tfrac{1}{2} g_V^2 \hat{O}_{VV} \right\} , \tag{7.45}$$

to compute the correlation functions (7.2) and (7.3) in the Schrödinger functional with Ginsparg-Wilson fermions. The operators $\hat{O}_I$ differ from the operators O_I by the substition $\psi \to \hat{\psi} = (1 - \frac{a}{2} D)\psi$.

The free propagator entering in the perturbative expansion is computed numerically as indicated in Section 4.3.2 for lattice sizes $L/a = 4\,,5\,,\ldots\,,48$ and $\theta = 0\,,0.1\,,0.5\,,1$.

7.4.1 Discrete Gross-Neveu model

As in the case of Wilson fermions we check if the beta-function of the discrete Gross-Neveu model is correctly reproduced. The renormalised coupling was defined in (7.26). At finite lattice spacing one expects

$$\tilde{g}^2_{\mathrm{dgn}} = g^2 \, k_{\mathrm{dgn}}(a/L) - g^4 \left\{ \frac{N-1}{\pi} \, \ln(a/L) - c_{\mathrm{dgn}}(a/L) \right\} + \mathrm{O}(g^6) \,, \tag{7.46}$$

where $c_{\mathrm{dgn}}(a/L) = c^0_{\mathrm{dgn}}(a/L) + c^1_{\mathrm{dgn}}(a/L)\, N$. The coefficient of the leading order term $k_{\mathrm{dgn}}(a/L)$ is expected to be unity in the continuum limit. In Fig. 7.6 its lattice spacing dependence is plotted for $\theta = 0\,,0.1\,,0.5\,,1$. In any case it has the right continuum limit (the systematic error of the extrapolation is $\mathrm{O}(10^{-5})$). In Fig 7.7 the finite part at second order is plotted for three θ-values. These numbers are obtained by subtracting the expected logarithmic divergence from the second order diagrams (we also

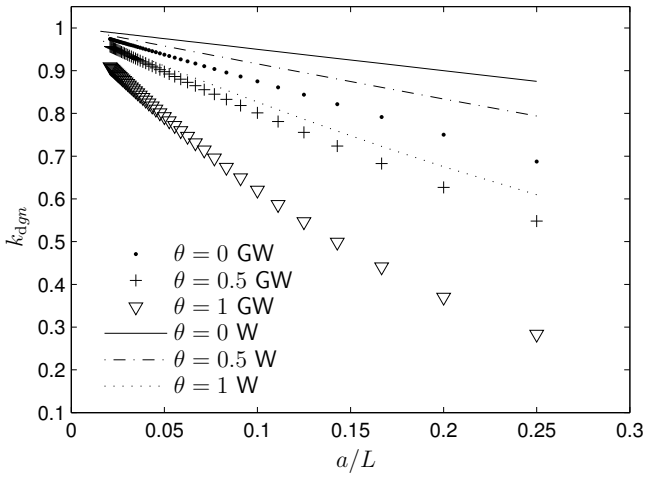


FIGURE 7.6: Cut-off dependence of the coefficient of the leading order term in the expansion of the renormalised coupling of the discrete Gross-Neveu model. It is unity in the continuum limit, as it should be. For comparison we plot Ginsparg-Wilson and Wilson fermions.

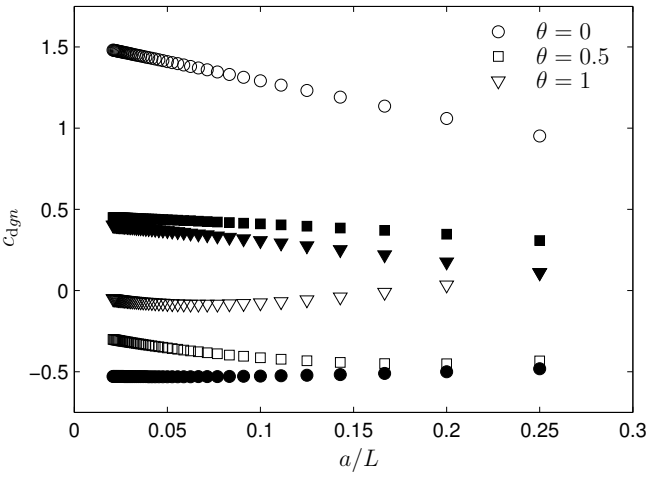


FIGURE 7.7: Subtracting the expected logarithmic divergence (as indicated in (7.46)) from the second order diagrams one obtains the finite part of $\tilde{g}^2_{\rm dgn}$ at order g^4. We plot here the result for three different values of θ. Open and filled symbols refer to $c^0_{\rm dgn}(a/L)$ and $c^1_{\rm dgn}(a/L)$ respectively.

determined the coefficient of the logarithmic divergence directly with results compatible with $-(N-1)/\pi$ and a systematic error of $\mathrm{O}(10^{-5})$). Note that the cut-off effects of the Ginsparg-Wilson fermions are roughly twice as large as for the Wilson fermions.

Thus the coupling of the lattice theory with the modified Neuberger operator (4.48) can be renormalised in the same way as with Wilson fermions. We now proceed with the chiral Gross-Neveu model.

7.4.2 Chiral Gross-Neveu model

The renormalised couplings for the two interaction terms are defined in the same way as in the Wilson case, i.e. by implementing the θ-dependence of the leading order term in the expansion of $R(\theta)$

$$\tilde{g}^2 \equiv F_g = \frac{L}{T(B(\theta)-1)} \left(B(\theta)\, R_{\mathrm{cgn,gw}}(0) - C(\theta)\, R_{\mathrm{cgn,gw}}(\theta) \right) , \qquad (7.47)$$

and

$$\tilde{g}_V \equiv F_V = \frac{L}{T(B(\theta)-1)} \left(R_{\mathrm{cgn,gw}}(0) - C(\theta)\, R_{\mathrm{cgn,gw}}(\theta) \right) . \qquad (7.48)$$

The difference is that no additive mass renormalisation is needed ($am_0 = 0$) and that the pseudo-scalar coupling vanishes exactly ($\delta_P^2 = 0$).

Since in the case of the modified Neuberger operator we have no analytic handle on the leading order, we first check whether the definitions above satisfy (7.1). To this end we expand

$$\tilde{g}^2 = g^2\, k_{g,g}(a/L) + g_V^2\, k_{g,V}(a/L) + \mathrm{O}(g^4) , \qquad (7.49)$$

and

$$\tilde{g}_V^2 = g^2\, k_{V,g}(a/L) + g_V^2\, k_{V,V}(a/L) + \mathrm{O}(g^4) . \qquad (7.50)$$

The coefficients were computed for lattice sizes $L/a = 4, 5, \ldots, 48$. Fig. 7.8 shows the coefficients of $\tilde{g}^2$ and Fig. 7.9 the coefficients of $\tilde{g}_V^2$. Their extrapolations to the continuum limit have the expected values within the systematic errors. For example, in the case $\theta = 0.5$ the result is

$$k_{g,g} = 0.999995(17) \quad k_{g,V} = 0.000012(27) , \qquad (7.51)$$

and

$$k_{V,g} = -0.000005(18) \quad k_{V,V} = 1.000011(28) . \qquad (7.52)$$

The results are similar for $\theta = 0.1, 1$. Again the cut-off effects of Ginsparg-Wilson fermions exceed the ones of Wilson fermions.

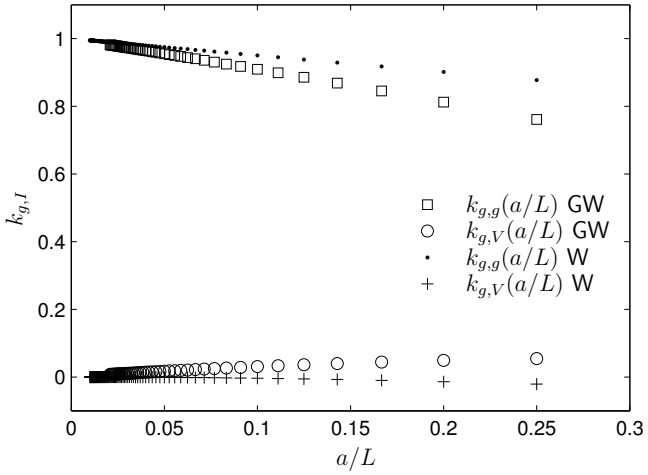


FIGURE 7.8: Cut-off dependence of the coefficients of the leading order terms in the expansion of the renormalised coupling $\tilde{g}^2$ of the chiral Gross-Neveu model for $\theta = 0.5$. For comparison we plot Ginsparg-Wilson and Wilson fermions.

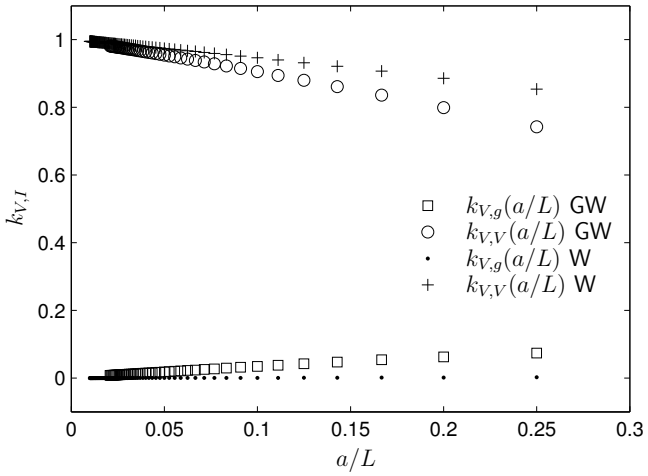


FIGURE 7.9: Cut-off dependence of the coefficients of the leading order terms in the expansion of the renormalised coupling $\tilde{g}_V^2$ of the chiral Gross-Neveu model for $\theta = 0.5$. For comparison we plot Ginsparg-Wilson and Wilson fermions.

At next to leading order we expect to find the logarithmic divergences with coefficients $-N/\pi$ and $-1/\pi$, respectively, for $\tilde{g}^2$ and $\tilde{g}_V^2$. As in the case of the discrete Gross-Neveu model we first determined the coefficient of the divergence explicitly. After we were convinced that it has the right value, we subtracted it from the sum of second order diagrams. The resulting amplitudes should have a finite or vanishing continuum limit. Expanding the renormalised couplings to second order

$$\tilde{g}^2 = g^2\, k_{g,g}(a/L) + g_V^2\, k_{g,V}(a/L) - g^4\, \frac{N}{\pi}\, \ln(a/L) + c_g(a/L) + \mathrm{O}(g^6)\,, \tag{7.53}$$

and

$$\tilde{g}_V^2 = g^2\, k_{V,g}(a/L) + g_V^2\, k_{V,V}(a/L) - g^4\, \frac{1}{\pi}\, \ln(a/L) + c_V(a/L) + \mathrm{O}(g^6)\,, \tag{7.54}$$

these amplitudes are $c_g(a/L)$ and $c_V(a/L)$. They can be sorted by powers of the bare couplings

$$c_g = c_{g,gg}\, g^4 + c_{g,VV}\, g_V^4 + c_{g,Vg}\, g_V^2 g^2\,, \tag{7.55}$$

and

$$c_V = c_{V,gg}\, g^4 + c_{V,VV}\, g_V^4 + c_{V,Vg}\, g_V^2 g^2\,, \tag{7.56}$$

and powers of N

$$c_{I,JK} = c_{I,JK}^0 + c_{I,JK}^1\, N\,. \tag{7.57}$$

In the continuum limit we do not expect to find the same values for these coefficients as in the Wilson case, because the finite part of the renormalised coupling is scheme dependent. However, as indicated above they should have a finite or vanishing continuum limit. Any uncancelled divergence would mean that additional terms are needed in the action to absorb them. But as can be seen from Figs. 7.10 and 7.11 there is no such divergence, all coefficients have a well defined continuum limit. The dependence of the values of the extrapolation on θ is depicted in Figs. 7.12 and 7.13. Although the amplitudes are different, the functional dependence is, as it should be, identical to the Wilson case.

Since we find the same divergences the beta-functions associated with the renormalised couplings are the same as in the Wilson case. This, of course, comes as no suprise. If the theory with the modified Neuberger operator in the action is renormalisable and describes the same continuum theory, all coefficients of the beta-function must be equal since it is universal as a whole.

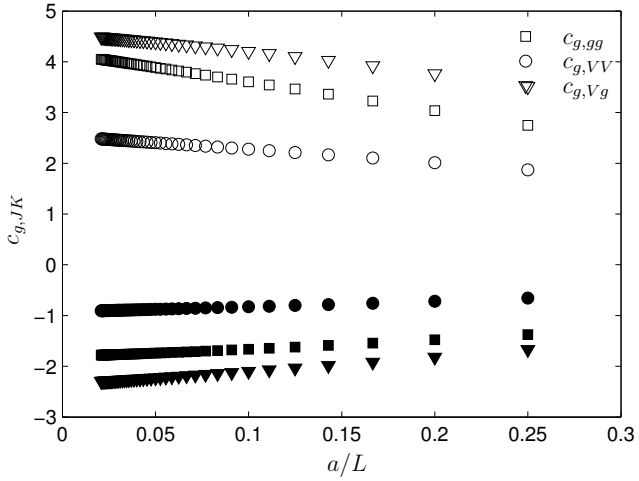


FIGURE 7.10: Cut-off dependence of the coefficients of the second order terms in the expansion of the renormalised coupling $\tilde{g}^2$ of the chiral Gross-Neveu model for $\theta = 0.1$. Open and filled symbols refer to $c^0_{I,JK}(a/L)$ and $c^1_{I,JK}(a/L)$ respectively.

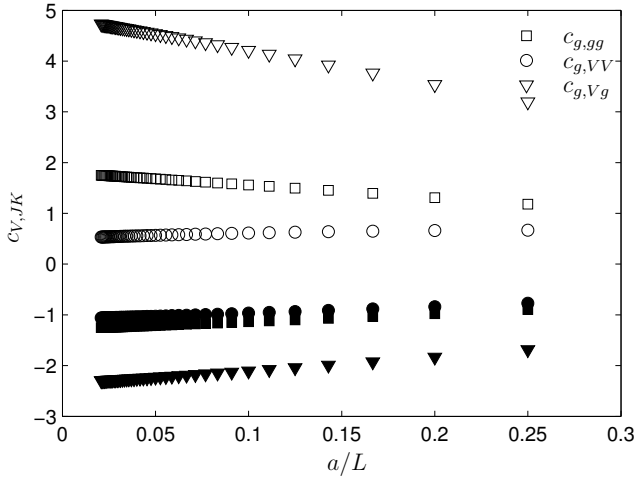


FIGURE 7.11: Cut-off dependence of the coefficients of the second order terms in the expansion of the renormalised coupling $\tilde{g}^2_V$ of the chiral Gross-Neveu model for $\theta = 0.1$. Open and filled symbols refer to $c^0_{I,JK}(a/L)$ and $c^1_{I,JK}(a/L)$ respectively.

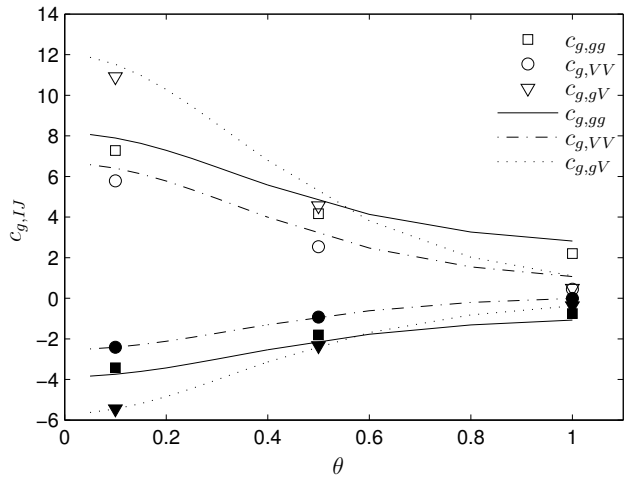


FIGURE 7.12: Dependence of the continuum extrapolation of the finite part of $\tilde{g}^2$ on θ at order g^4. The symbols are Ginsparg-Wilson and curves are Wilson fermions. Open and filled symbols refer to $c^0_{I,JK}$ and $c^1_{I,JK}$ respectively.

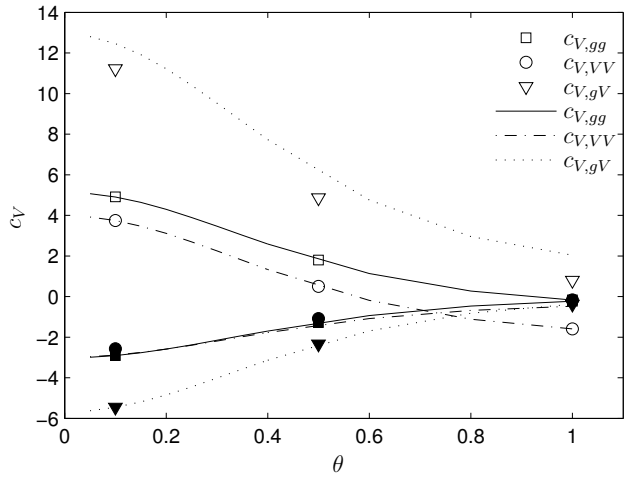


FIGURE 7.13: Dependence of the continuum extrapolation of the finite part of $\tilde{g}^2_V$ on θ at order g^4. The symbols are Ginsparg-Wilson and curves are Wilson fermions. Open and filled symbols refer to $c^0_{I,JK}$ and $c^1_{I,JK}$ respectively.

7.5 Summary and ratio of Lambda parameters

In the last two Sections we calculated the renormalised couplings and beta-functions of the discrete and chiral Gross-Neveu model in 1-loop perturbation theory for Wilson and Ginsparg-Wilson fermions in the Schrödinger functional. Both models have a asymptotically free coupling. In the chiral Gross-Neveu model there is also a coupling that does not renormalise, i.e. its beta-function vanishes. The first coefficient of the beta-function is universal and we correctly reproduce the known values

$$\text{DGN:} \quad b_0 = -\frac{N-1}{\pi} \qquad \text{CGN:} \quad b_0 = -\frac{N}{\pi}\,. \tag{7.58}$$

A nontrivial test of the equality of the theory with Wilson and Ginsparg-Wilson fermions is now the computation of the ratio of the lattice Λ-parameters to show its independence of the angle θ.

Discrete Gross-Neveu model: The continuum limit of the renormalised coupling of the discrete Gross-Neveu model calculated with Wilson fermions

$$\tilde{g}^2 \stackrel{a\to 0}{=} g_{\mathrm{w}}^2 + g_{\mathrm{w}}^4\ \{b_0\ \ln(a/L) + c^{\mathrm{w}}\} + \mathrm{O}(g_{\mathrm{w}}^6)\,, \tag{7.59}$$

and with Ginsparg-Wilson fermions

$$\tilde{g}^2 \stackrel{a\to 0}{=} g_{\mathrm{gw}}^2 + g_{\mathrm{gw}}^4\ \{b_0\ \ln(a/L) + c^{\mathrm{gw}}\} + \mathrm{O}(g_{\mathrm{gw}}^6)\,, \tag{7.60}$$

must be equal. This allows to relate the bare couplings, e.g.

$$g_{\mathrm{gw}}^2 = g_{\mathrm{w}}^2 + a_1\, g_{\mathrm{w}}^4 + \mathrm{O}(g_{\mathrm{w}}^6)\,, \quad a_1 = c^{\mathrm{w}} - c^{\mathrm{gw}}\,. \tag{7.61}$$

The finite parts $c_{\mathrm{w}}^{\mathrm{dgn}}$ and $c_{\mathrm{gw}}^{\mathrm{dgn}}$ are functions of θ. Since it is a probe to the theory like external momenta, the relation between the bare parameters (7.61) can not depend on it. Therefore the difference a_1 has to be independent of θ. This is equal to saying that the ratio of the lattice Λ-parameters

$$\Lambda_{\mathrm{LAT,\,gw}}/\Lambda_{\mathrm{LAT,\,w}} = \exp\left\{\frac{a_1}{2b_0}\right\}. \tag{7.62}$$

has to be independent of θ. The lattice Λ-parameter is analogous to the Λ-parameter in (2.24), but with the renormalised coupling replaced with

the bare coupling and the beta-function replaced with the lattice beta-function. For the ratio of the lattice Λ-parameters in the discrete Gross-Neveu model at $\theta = 0, 0.5$ and $N = 2, 4$ we find

$$\Lambda_{\text{LAT, gw}} / \Lambda_{\text{LAT, w}} \Big|_{N=2,\, \theta=0\ [0.5]} = 0.987(7)\ [0.986(12)]\,, \tag{7.63}$$

$$\Lambda_{\text{LAT, gw}} / \Lambda_{\text{LAT, w}} \Big|_{N=4,\, \theta=0\ [0.5]} = 0.712(12)\ [0.713(21)]\,. \tag{7.64}$$

Chiral Gross-Neveu model: The chiral Gross-Neveu model has two coupling constants and the result for the renormalised couplings can be given in the following form for Wilson fermions

$$\tilde{g}^2 \overset{a\to 0}{=} g^2_{\text{w}} - g^4_{\text{w}} \frac{N}{\pi} \ln(a/L) + c^{\text{w}}_g + \text{O}(g^6_{\text{w}})\,, \tag{7.65}$$

$$\tilde{\delta}^2_V \overset{a\to 0}{=} \delta^2_{V,\text{w}} + c^{\text{w}}_V - c^{\text{w}}_g / N + \text{O}(g^6_{\text{w}})\,, \tag{7.66}$$

and Ginsparg-Wislon fermions

$$\tilde{g}^2 \overset{a\to 0}{=} g^2_{\text{gw}} - g^4_{\text{gw}} \frac{N}{\pi} \ln(a/L) + c^{\text{gw}}_g + \text{O}(g^6_{\text{gw}})\,, \tag{7.67}$$

$$\tilde{\delta}^2_V \overset{a\to 0}{=} \delta^2_{V,\text{gw}} + c^{\text{gw}}_V - c^{\text{gw}}_g / N + \text{O}(g^6_{\text{gw}})\,. \tag{7.68}$$

These are couplings of (5.24). The finite parts are composed of three terms

$$c^i_g = c^i_{g,gg}\, g^4_i + c^i_{g,VV}\, g^4_{V,i} + c^i_{g,Vg}\, g^2_{V,i} g^2_i\,, \tag{7.69}$$

$$c^i_V = c^i_{V,gg}\, g^4_i + c^i_{V,VV}\, g^4_{V,i} + c^i_{V,Vg}\, g^2_{V,i} g^2_i\,, \tag{7.70}$$

for $i = \text{w}, \text{gw}$ (no sum). The renormalised quantities must coincide. This yields relations between the bare couplings

$$g^2_{\text{gw}} = g^2_{\text{w}} + a_g + \text{O}(g^6_{\text{w}})\,, \quad a_g = c^{\text{w}}_g - c^{\text{gw}}_g\,, \tag{7.71}$$

$$\delta^2_{V,\text{gw}} = \delta^2_{V,\text{w}} + a_V + \text{O}(\delta^6_{V,\text{w}})\,, \quad a_V = c^{\text{w}}_V - c^{\text{gw}}_V - a_g/N\,. \tag{7.72}$$

Again the coefficients a_g and a_V have to be independent of θ. That this is the case can be seen in Fig. 7.12 and 7.13. There a_g and a_V are the differences between the curves and the corresponding symbols. It is finite, but constant.

The beta-function of the vector-vector coupling $\tilde{\delta}_V$ vanishes at one-loop (7.43). This is in agreement with two-loop calculations [23] and formal

continuum arguments [74, 75]. Furthermore the vector-vector coupling was not constrained by the restoration of chiral symmetry in the case of Wilson fermions (see Chapter 6). Therefore no value of the bare coupling δ_V is distinguished. For each fixed value the theory is an one coupling theory. The ratio of the lattice Λ-parameters can then be calculated as in the discrete Gross-Neveu model (Eqs. (7.61) and (7.62)). For convenience we take $\delta^2_{V,\mathrm{w}} = 0$. From (7.72) it follows that $\delta^2_{V,\mathrm{gw}} = \mathrm{O}(g^4_{\mathrm{w}})$ and the coefficient a_g simplifies to

$$a_g = a_1\, g^4_{\mathrm{w}} \quad a_1 = c^{\mathrm{w}}_{g,gg} - c^{\mathrm{gw}}_{g,gg}\,. \tag{7.73}$$

We list the ratio for $\theta = 0.1, 0.5$

$$\Lambda_{\mathrm{LAT,gw}}/\Lambda_{\mathrm{LAT,w}}\Big|_{N=2,\theta=0.1\ [0.5]} = 0.9893(7)\ [0.9892(10)]\,, \tag{7.74}$$

$$\Lambda_{\mathrm{LAT,gw}}/\Lambda_{\mathrm{LAT,w}}\Big|_{N=4,\theta=0.1\ [0.5]} = 0.776(1)\ [0.776(2)]\,. \tag{7.75}$$

The ratio of the Λ-parameters has been calculated in QCD on a lattice with periodic boundary conditions in [79]. The values there are similar to the ones obtained here.

Chapter 8

Conclusions

The main result of this work is the formulation of the Schrödinger functional (SF) for fermionic models of the Gross-Neveu type with a finite number N of fermion flavours. In 1-loop lattice perturbation theory we showed that the theory is renormalisable with Wilson and with Ginsparg-Wilson fermions. This is the first check of the recently proposed Dirac operator [32] beyond the free theory.

In two dimensions four fermion interactions have dimensionless coupling constants. We discussed the symmetry properties of the four fermion interaction terms and the relations among them. Due to its Abelian chiral symmetry the continuum chiral Gross-Neveu (CGN) model has two independent interaction terms. A possible choice is $g^2/2\,((\overline{\psi}\psi)^2 - (\overline{\psi}\gamma_5\psi)^2) + g_V^2/2\,(\overline{\psi}\gamma_\mu\psi)^2$. Because Wilson fermions explicitly break chiral symmetry, the most general lattice action for the chiral Gross-Neveu model has an independent four fermion interaction that breaks chiral invariance, for example $\delta_P^2/2\,(\overline{\psi}\gamma_5\psi)$, and a mass term. The model with three couplings becomes the discrete Gross-Neveu model (DGN), when two of the couplings are set appropriately.

Like in QCD chiral Ward identities can be used to define the critical mass. The Ward identity of the local axial current, strictly valid only in the continuum, is demanded to hold on the lattice up to corrections of $\mathrm{O}(a)$. The critical mass cancels the linear divergence that appears in the Ward identity because of operator mixing. However, for the restoration of chiral symmetry it is also necessary to tune the symmetry breaking coupling to its symmetry restoring value. We computed the critical mass and the symmetric coupling up to second order in lattice perturbation theory. We find no constraint on the vector-vector coupling g_V^2. Or phrased in another way, the vector-vector coupling has not to be tuned in order to restore chiral symmetry. This is the first determination of this parameters at finite-N.

The result is consistent with calculations in the large-N limit.

Renormalised couplings for g^2 and g_V^2 are defined at vanishing renormalised mass. The natural choice are boundary-to-boundary correlation functions with four external fermions. For Wilson fermions the non-trivial dependence of the first order diagrams on the phase θ, which parametrises the spatial boundary conditions, could be calculated analytically. This was crucial in order to define the renormalised couplings in such a way, that they are equal to the corresponding bare coupling at leading order perturbation theory. This definition has then been applied in the calculation with Ginsparg-Wilson fermions, where one has no analytic handle on the diagrams.

The coefficient of the logarithmic divergence and the finite part at 1-loop is computed numerically to a high precision using the method described in [45]. We correctly reproduced the first coefficient of the beta-function for the CGN and DGN. Both models have a asymptotically free coupling. Furthermore the couplings can be redefined such, that the vector-vector coupling does not renormalise, i.e. its beta-function vanishes. This is consistent with 2-loop calculations and formal continuum arguments. The finite part shows a mild dependence on θ. The cut-off effects are clearly $\mathrm{O}(a)$.

The definition of the renormalised couplings is suitable for a computation in Monte Carlo simulations. The boundary-to-boundary correlation functions are easily implemented and the θ dependence is strong enough to discriminate the couplings. Since the SF is a finite size regularisation scheme the spatial extension provides a natural scale in the system and it is possible to define step scaling functions [4] for the renormalised couplings. The renormalisation group invariant step scaling functions can then be used as benchmark observables for universality studies of different lattice actions.

We used the results of the computation with Wilson fermions to study a recently proposed modified Neuberger-Dirac operator [32] in 1-loop lattice perturbation theory. The operator is compatible with the SF boundary conditions in the sense of Section 4.1.2. The operator of the free theory is shown to be local (but not ultralocal) and to obey the Ginsparg-Wilson relation up to terms localised at the boundaries with exponentially decreasing tails. Thus the lattice chiral symmetry associated with Ginsparg-Wilson fermions is a symmetry in the interior of the lattice and correlations of local fields at physical distances from the boundaries obey the same Ward identities as they do on periodic lattices. After the substitution $\psi \to (1 - \frac{a}{2}D)\psi$ the new four fermion operators transform under the lattice chiral symmetry in the same way as the old ones under the continuum

symmetry. Therefore $\delta_P^2 = 0$ is protected by the symmetry and no tuning is necessary.

The modified Neuberger-Dirac operator (4.48) is rather complicated. The eigenfunctions of the operator under the square root were derived. We did not succeed in deriving an analytic expression for the Dirac operator. But the Dirac operator and its inverse can be computed numerically using standard techniques [72].

The computation of the previously defined renormalised couplings has been repeated with Ginsparg-Wilson fermions. For the coefficients of the logarithmic divergence we found perfect agreement with the Wilson result, thus proving the SF with Ginsparg-Wilson fermions to be renormalisable at the one loop level. As expected, the regularisation dependent finite part differs. The cut-off effects are $\mathrm{O}(a)$ and larger than in the Wilson case. To achieve $\mathrm{O}(a)$-improvement one would have to redefine the correlation functions [80] and to introduce four fermion interaction terms at the boundaries.

Finally we used the 1-loop calculation of the renormalised coupling with Wilson and Ginsparg-Wilson fermions to compute the ratio of corresponding Λ-parameters, which yields reasonable results.

With this work the Schrödinger functional for Gross-Neveu models is well established. It can be used as a benchmark system for fermion actions. It should be possible to simulate with Wilson as well as Ginsparg-Wilson fermions.

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Appendix A

Notation

A.1 Definitions

A.1.1 Dirac matrices

The γ-matrices are defined through the Clifford algebra in Euclidean space

$$\{\gamma_\mu^E, \gamma_\nu^E\} = 2\delta_{\mu\nu}\,.$$

Since the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

already have the right dimension and anticommutation properties, they can directly used to represent the Euclidean γ-matrices in $D = 2$ dimensions. One possible choice is

$$\gamma_0 = \sigma_2\,,$$

$$\gamma_1 = \sigma_1\,.$$

and for hermitian γ_5

$$\gamma_5 \equiv i\gamma_0\gamma_1 = \sigma_3\,.$$

In this representation the chiral projectors are diagonal. They are defined

$$P_{R,L} = \frac{1}{2}(1 \pm \gamma_5)\,, \tag{A.1}$$

with the properties

$$P_{R,L}^2 = P_{R,L}\,, \quad P_L P_R = P_R P_L = 0\,, \tag{A.2}$$

$$\gamma_5 P_{R,L} = \pm P_{R,L}\,, \tag{A.3}$$

$$P_R + P_L = 1\,, \quad P_R - P_L = \gamma_5\,, \tag{A.4}$$

We also need the projectors defined using γ_0

$$P_\pm = \frac{1}{2}(1 \pm \gamma_0)\,. \tag{A.5}$$

These projectors become especially simple in another representation

$$\gamma_0 = \sigma_3\,,$$

$$\gamma_1 = \sigma_1\,.$$

$$\gamma_5 \equiv i\gamma_0\gamma_1 = -\sigma_2\,.$$

Then

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\,, \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{A.6}$$

A.1.2 Generators of SU(N)

The Lie algebra su(N) of $SU(N)$ can be identified with the space of complex $N \times N$ matrices M satisfying

$$M^\dagger = M\,, \quad \mathrm{Tr}\,\{M\} = 0\,. \tag{A.7}$$

The generators of $SU(N)$ can be identified with a basis λ^a in this space, with $a = 1, 2, \ldots, N^2 - 1$ and normalised to

$$\mathrm{Tr}\left\{\lambda^a\,\lambda^b\right\} = C\delta_{ab}\,, \quad C = 2\,. \tag{A.8}$$

They obey commutation relation (repeated indices are summed over)

$$[\lambda^a, \lambda^b] = 2if^{abc}\lambda^c\,, \tag{A.9}$$

with the totally antisymmetric structure constants f^{abc}.

Now every complex $N \times N$ matrix X can be expanded in the complete basis

$$\lambda^A = \left\{\sqrt{\tfrac{2}{N}}\,1, \lambda^a\right\}\,, \quad A = 0, 1, \ldots, N^2 - 1\,. \tag{A.10}$$

The expansion reads

$$X = \tfrac{1}{2}\sum_A \lambda^A\ \mathrm{Tr}\left\{\lambda^A\,X\right\}\,. \tag{A.11}$$

Expanding in such a way the matrix $X_{ij}^{(lk)} = 2\delta_{ik}\delta_{jl}$ one finds the identity

$$(\lambda^a)_{ij}(\lambda^a)_{lk} = 2\delta_{ik}\delta_{jl} - \tfrac{2}{N}\delta_{ij}\delta_{lk}\,. \tag{A.12}$$

Thus the quadratic Casimir operator is

$$(\lambda^a\lambda^a)_{ij} = C_2\delta_{ij}\,, \quad C_2 = \frac{2(N^2-1)}{N}\,. \tag{A.13}$$

The following identities are usefull when evaluating flavour traces of Feynman diagrams

$$\mathrm{Tr}\left\{\lambda^a\lambda^a\right\} = N\,C_2\,, \tag{A.14}$$

$$\mathrm{Tr}\left\{\lambda^a\lambda^a\lambda^b\lambda^b\right\} = N\,(C_2)^2\,, \tag{A.15}$$

$$(\lambda^a\lambda^b\lambda^a)_{ij} = (C_2 - 2N)(\lambda^b)_{ij}\,, \tag{A.16}$$

$$\mathrm{Tr}\left\{\lambda^a\lambda^b\lambda^a\lambda^b\right\} = N\,C_2\,(C_2 - 2N)\,, \tag{A.17}$$

$$\mathrm{Tr}\left\{\lambda^a\lambda^b\right\}\ \mathrm{Tr}\left\{\lambda^a\lambda^b\right\} = C\,N\,C_2\,. \tag{A.18}$$

A.1.3 Lattice notation

Here we define the lattice difference operators. Via the factor λ_μ we are able to define the theory with general boundary conditions. (For details see [71].)

$$\partial_\mu\psi(x) = \tfrac{1}{a}[\lambda_\mu\psi(x+a\hat{\mu}) - \psi(x)] \tag{A.19}$$

$$\partial^*_\mu\psi(x) = \tfrac{1}{a}[\psi(x) - \lambda_\mu^{-1}\psi(x-a\hat{\mu})] \tag{A.20}$$

$$\lambda_\mu = \mathrm{e}^{ia\theta_\mu/L}\,, \quad \theta_0 = 0\,, \quad -\pi < \theta_1 \leq \pi\,, \tag{A.21}$$

$$p^\pm_\mu = p_\mu \pm \theta_\mu/L \tag{A.22}$$

$$\mathring{p}_\mu = \tfrac{1}{a}\sin(ap_\mu) \tag{A.23}$$

$$\hat{p}_\mu = \tfrac{2}{a}\sin(ap_\mu/2)\,, \tag{A.24}$$

The left action of difference operators is defined as

$$\overline{\psi}(x)\overleftarrow{\partial}_\mu = \tfrac{1}{a}[\overline{\psi}(x+a\hat{\mu})\lambda_\mu^{-1} - \overline{\psi}(x)] \tag{A.25}$$

$$\overline{\psi}(x)\overleftarrow{\partial}^*_\mu = \tfrac{1}{a}[\overline{\psi}(x) - \overline{\psi}(x-a\hat{\mu})\lambda_\mu]\,. \tag{A.26}$$

They are related to the right difference operators

$$\sum_x \overline{\psi}(x)\overleftarrow{\partial}_\mu \psi(x) = -\sum_x \overline{\psi}(x)\partial^*_\mu \psi(x) \tag{A.27}$$

$$\sum_x \overline{\psi}(x)\overleftarrow{\partial}^*_\mu \psi(x) = -\sum_x \overline{\psi}(x)\partial_\mu \psi(x)\,. \tag{A.28}$$

and can be used to define the left action of the Dirac operator

$$\overleftarrow{D} = \tfrac{1}{2}[(\overleftarrow{\partial}^*_\mu + \overleftarrow{\partial}_\mu)\gamma_\mu - a\overleftarrow{\partial}^*_\mu\overleftarrow{\partial}_\mu] \tag{A.29}$$

$$\overleftarrow{D}^\dagger = \tfrac{1}{2}[-(\overleftarrow{\partial}^*_\mu + \overleftarrow{\partial}_\mu)\gamma_\mu - a\overleftarrow{\partial}^*_\mu\overleftarrow{\partial}_\mu]\,. \tag{A.30}$$

In particular, the action in the interior of the lattice can be written

$$a^2 \sum_{x_0=a}^{T-a}\sum_{x_1=0}^{L-a} \overline{\psi}(x)\{D+m_0\}\psi(x) = a^2 \sum_{x_0=a}^{T-a}\sum_{x_1=0}^{L-a} \overline{\psi}(x)\{\overleftarrow{D}^\dagger + m_0\}\psi(x)\,. \tag{A.31}$$

A.2 Free theory

A.2.1 Formulae

The positive energy plane wave solutions of the Dirac equation are

$$\psi(x) = \mathrm{e}^{ipx}\,, \quad \mathrm{Im}\ p_0 > 0\,, \tag{A.32}$$

with spatial momentum p_1 integer multiple of $2\pi/L$ in the range

$$-\pi/a < p_1 \le \pi/a\,, \tag{A.33}$$

The energy

$$p_0 = p_0^+ = i\omega(p_1^+) \mod 2\pi/a\,, \tag{A.34}$$

is constrained by Dirac equation giving

$$(\mathring{p}^+)^2 + M(p^+)^2 = 0\,, \tag{A.35}$$

$$M(p) = m_0 + \tfrac{a}{2}\hat{p}^2\,, \tag{A.36}$$

which defines $\omega(q_1)$

$$\sinh\left[\tfrac{a}{2}\omega(q_1)\right] = \frac{a}{2}\left\{\frac{\mathring{q}_1^2 + (m_0 + \frac{a}{2}\hat{q}_1^2)^2}{1 + a(m_0 + \frac{a}{2}\hat{q}_1^2)}\right\}^{\frac{1}{2}}\,, \tag{A.37}$$

This implies $\omega(p_1^+) \geq 0$ for $m_0 \geq 0$.

The following amplitudes appear in the free propagator

$$A(q_1) = 1 + a(m_0 + \tfrac{a}{2}\hat{q}_1^2)\,, \tag{A.38}$$

$$R(q) = M(q)\left\{1 - \mathrm{e}^{-2\omega(q_1)T}\right\} - i\mathring{q}_0\left\{1 + \mathrm{e}^{-2\omega(q_1)T}\right\}\,. \tag{A.39}$$

A.3 Four fermion operators

A.3.1 Fierz transformation

Fierz transformations connect products of Dirac bilinears by rearranging the order of the Dirac spinors. Thus some of these products are not independent.

In two dimensions the set

$$\Gamma^A = \{1, \gamma_0, \gamma_1, \gamma_5\}$$

is normalised to

$$\mathrm{tr}(\Gamma^A\Gamma^B) = 2\delta_{AB} \tag{A.40}$$

and forms a complete basis for complex 2×2 matrices $M_{\alpha\beta}$:

$$M_{\alpha\beta} = \frac{1}{2}\sum_A \Gamma^A_{\alpha\beta}\ \mathrm{tr}(\Gamma^A M)\,.$$

Considering four Dirac spinors $\overline{\psi}_1$, ψ_2, $\overline{\psi}_3$ and ψ_4 the general form of the Fierz identity is then

$$(\overline{\psi}_1\Gamma^A\psi_2)(\overline{\psi}_3\Gamma^B\psi_4) = \sum_{C,D} C^{AB}_{CD}(\overline{\psi}_1\Gamma^C\psi_4)(\overline{\psi}_3\Gamma^D\psi_2)\,. \tag{A.41}$$

To fix the unknown coefficients C^{AB}_{CD} consider the left-hand side with all Dirac indices explicit

$$\begin{aligned}
\overline{\psi}_{1\alpha}\Gamma^A_{\alpha\beta}\psi_{2\beta}\overline{\psi}_{3\lambda}\Gamma^B_{\lambda\sigma}\psi_{4\sigma} &= -\Gamma^A_{\alpha\beta}\Gamma^B_{\lambda\sigma}\overline{\psi}_{1\alpha}\psi_{4\sigma}\overline{\psi}_{3\lambda}\psi_{2\beta} \\
&= -\Gamma^A_{\alpha\beta}\Gamma^B_{\lambda\sigma}M_{\sigma\alpha}M'_{\beta\lambda} \\
&= -\frac{1}{4}\sum_{C,D}\Gamma^A_{\alpha\beta}\Gamma^B_{\lambda\sigma}\Gamma^C_{\sigma\alpha}\Gamma^D_{\beta\lambda}\,\mathrm{tr}(\Gamma^C M)\,\mathrm{tr}(\Gamma^D M') \\
&= -\frac{1}{4}\sum_{C,D}\mathrm{tr}(\Gamma^C\Gamma^A\Gamma^D\Gamma^B)(\overline{\psi}_1\Gamma^C\psi_4)(\overline{\psi}_3\Gamma^D\psi_2)
\end{aligned}$$

Γ^A	Γ^C	C_{CC}^{AA}
1	$1, \gamma_0, \gamma_1, \gamma_5$	$-\frac{1}{2}$
γ_μ	1	$-\frac{1}{2}$
	γ_5	$\frac{1}{2}$
	γ_ν	$(-1)^{\delta_{\mu\nu}}\frac{1}{2}$
γ_5	$1, \gamma_5$	$-\frac{1}{2}$
	γ_ν	$\frac{1}{2}$

TABLE A.1: Some coefficients of the Fierz transformation Eq. (A.41).

where $M_{\sigma\alpha} = \overline{\psi}_{1\alpha}\psi_{4\sigma}$ and $M'_{\beta\lambda} = \overline{\psi}_{3\lambda}\psi_{2\beta}$ are 2×2 matrices, that are expanded in the third line in the basis given above. In the last line we used

$$\mathrm{tr}(\Gamma^C M) = \Gamma^C_{\rho\nu}\overline{\psi}_{1\rho}\psi_{4\nu} = \overline{\psi}_1\Gamma^C\psi_4$$

and similar for $\mathrm{tr}(\Gamma^D M')$. The overall sign is due to the fact that the fields anticommute. Thus the coefficients are determined through

$$C_{CD}^{AB} \;=\; -\frac{1}{4}\,\mathrm{tr}(\Gamma^C\Gamma^A\Gamma^D\Gamma^A)\,. \tag{A.42}$$

We are mainly interested in the coefficients for $\Gamma^A = \Gamma^B$. One finds

$$C_{CD}^{AA} \propto \delta_{CD} \quad \text{for all} \quad \Gamma_C\,, \Gamma_D\,.$$

The 16 nonzero coefficients C_{CC}^{AA} are collected in Table A.1.

For the subtraction of $\Gamma^A = \Gamma^B = 1$ and $\Gamma^A = \Gamma^B = \gamma_5$ only the terms with γ_μ survive

$$(\overline{\psi}_1\psi_1)(\overline{\psi}_2\psi_2) - (\overline{\psi}_1\,\gamma_5\,\psi_1)(\overline{\psi}_2\,\gamma_5\,\psi_2) \;=\; -\sum_\mu(\overline{\psi}_1\,\gamma_\mu\,\psi_2)\,(\overline{\psi}_2\,\gamma_\mu\,\psi_1)\,.$$

In the case of $\Gamma^A = \gamma_\mu$ the terms with $\Gamma^C = \gamma_\nu$ cancel due to the implicit sum over μ and the fact that we have just two γ-matrices:

$$(\overline{\psi}_1\gamma_\mu\psi_1)(\overline{\psi}_2\gamma_\mu\psi_2) \;=\; -(\overline{\psi}_1\psi_2)(\overline{\psi}_2\psi_1) + (\overline{\psi}_1\,\gamma_5\,\psi_2)(\overline{\psi}_2\,\gamma_5\,\psi_1)\,.$$

A.3.2 Flavour mixing

Using the expansion introduced in Eq. (A.10) the matrix $M_{ij} = \overline{\psi}_i\,\Gamma\,\psi_j$, where Γ is a matrix contracting the Dirac indices of $\overline{\psi}$ and ψ, can be written as

$$M_{ij} = \frac{1}{2}\sum_A \lambda_{ij}^A\,(\overline{\psi}\,\Gamma\,\lambda^A\,\psi)\,,$$

and

$$\sum_{i,j}^{N} M_{ij}\, M_{ji} = \frac{1}{2^2} \sum_{A,B} \mathrm{Tr}\left\{\lambda^A\, \lambda^B\right\}\, (\overline{\psi}\, \Gamma\, \lambda^A\, \psi)\, (\overline{\psi}\, \Gamma\, \lambda^B\, \psi)\,.$$

Since the matrices λ^A are normalised to

$$\mathrm{Tr}\left\{\lambda^A\, \lambda^B\right\} = 2\delta_{AB}$$

one ends up with

$$\sum_{i,j}^{N} \overline{\psi}_i\, \Gamma\, \psi_j\, \overline{\psi}_j\, \Gamma\, \psi_i = \frac{1}{N}(\overline{\psi}\, \Gamma\, \psi)^2 + \frac{1}{2} \sum_{a} (\overline{\psi}\, \Gamma\, \lambda^a\, \psi)^2\,.$$

Appendix B

Correlation functions

B.1 Properties of the free Wilson propagator

B.1.1 Analytically known quantities

We work in a half Fourier transformed space, e.g. the free propagator in the interior of the SF is

$$a \sum_{x_1} \mathrm{e}^{ik_1(y_1-x_1)} \, [\psi(x)\overline{\psi}(y)] = \widetilde{S}(x_0, y_0, k_1) \, . \tag{B.1}$$

To evaluate the zeroth and first order diagrams we need the free propagators from boundary to boundary

$$a \sum_{x_1} \mathrm{e}^{ip_1(y_1-x_1)} \, [\zeta'(x_1)\overline{\zeta}(y_1)] = k(p_1) \, P_+ \equiv K(p_1) \, , \tag{B.2}$$

$$a \sum_{x_1} \mathrm{e}^{ip_1(y_1-x_1)} \, [\zeta(x_1)\overline{\zeta}'(y_1)] = \gamma_5 K^\dagger(p_1)\gamma_5 \, , \tag{B.3}$$

and from the boundaries to the bulk (and vice versa)

$$a\sum_{y_1} \mathrm{e}^{ip_1(y_1-x_1)}\,[\psi(x)\overline{\zeta}(y_1)] = \frac{2\mathrm{e}^{-2\omega(p_1^+)T}}{R(p^+)}\,h(x_0,p_1)\,P_+ \tag{B.4}$$

$$\equiv H(x_0,p_1)\,, \tag{B.5}$$

$$a\sum_{y_1} \mathrm{e}^{ip_1(y_1-x_1)}\,[\zeta(y_1)\overline{\psi}(x)] = \gamma_5 H^\dagger(x_0,p_1)\gamma_5\,, \tag{B.6}$$

$$a\sum_{y_1} \mathrm{e}^{ip_1(y_1-x_1)}\,[\psi(x)\overline{\zeta}'(y_1)] = \frac{2\mathrm{e}^{-2\omega(p_1^+)T}}{R(p^+)}\,h(T-x_0,p_1)\,P_- \tag{B.7}$$

$$\equiv H'(x_0,p_1)\,, \tag{B.8}$$

$$a\sum_{y_1} \mathrm{e}^{ip_1(y_1-x_1)}\,[\zeta'(y_1)\overline{\psi}(x)] = \gamma_5 H'^\dagger(x_0,p_1)\gamma_5\,. \tag{B.9}$$

From the definition of the Wilson propagator (4.41) one infers

$$k(k_1) = -2i\mathring{k}_0^+\,\frac{A(k_1^+)}{R(k^+)}\,\mathrm{e}^{-\omega(k_1^+)T}\,, \tag{B.10}$$

$$h(y_0,k_1) = h_1(y_0,k_1) + i\gamma_1 h_2(y_0,k_1)\,, \tag{B.11}$$

$$h_1(y_0,k_1) = M(k^+)\sinh(\omega(k_1^+)(T-y_0)) - i\mathring{k}_0^+\cosh(\omega(k_1^+)(T-y_0))\,, \tag{B.12}$$

$$h_2(y_0,k_1) = -\mathring{k}_1^+\sinh(\omega(k_1^+)(T-y_0))\,. \tag{B.13}$$

For $k_1 = 0$ and at zero bare mass we find

$$k(0)\big|_{m_0=0} = \frac{1}{\cosh(\theta T/L)} + \mathrm{O}(a^2)\,, \tag{B.14}$$

$$h_1(y_0,0)\big|_{m_0=0} = \theta/L\cosh(\theta(T-y_0)/L) + \mathrm{O}(a^2)\,, \tag{B.15}$$

$$h_2(y_0,0)\big|_{m_0=0} = \theta/L\sinh(\theta(T-y_0)/L) + \mathrm{O}(a^2)\,, \tag{B.16}$$

$$\left.\frac{2\mathrm{e}^{-2\omega(0^+)T}}{R(0^+)}\right|_{m_0=0} = \frac{1}{\theta/L\cosh(\theta T/L)} + \mathrm{O}(a^2)\,. \tag{B.17}$$

B.1.2 The propagator at zero distance

Consider the free propagator

$$S(x,y) = [\psi(x)\overline{\psi}(y)] = \left\langle\psi(x)\overline{\psi}(y)\right\rangle_0\,, \tag{B.18}$$

which is one of the basic contractions in our computation. For coinciding arguments $y \to x$ and in the continuum theory, just by Euclidean invariance, one infers that $S(x,x)$ must be diagonal in Dirac space. This can be made more explicit by looking at the free propagator on the lattice in a periodic box. Since $S(x,x)$ it is a local quantity, it is not sensitive to the special kind of boundary conditions, once the continuum limit has been taken. For an infinite periodic box the free propagator for Wilson fermions reads

$$S_{\mathrm{PB}}(x,y) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^2 p}{2\pi}\, \mathrm{e}^{ip(x-y)}\, \frac{-i\gamma_\mu \mathring{p}_\mu + M(p)}{\mathring{p}^2 + M(p)^2}, \tag{B.19}$$

where $M(p)$ is defined as usual (cf. my notes). Except for $-i\gamma_\mu \mathring{p}_\mu$, which is odd, and $\mathrm{e}^{ip(x-y)}$, which is one for $y \to x$, all terms in this expression are even. Thus $S_{\mathrm{PB}}(x,x)$ has no Dirac structure, i.e. is a multiple of the unit matrix in Dirac space, even at finite lattice spacing. This will change if we allow for general boundary conditions including a phase $\mathrm{e}^{i\theta}$. But it will still converge to the identity in the continuum limit.

For the Schrödinger functional free propagator the same behaviour can be shown to hold. In particular this means for the dimensionless product

$$a\, S(x,x) = \frac{a}{L} \sum_{k_1} \widetilde{S}(x_0, x_0, k_1) = B(x_0) + \mathrm{O}(a^2/L^2\, \theta_1)\,, \tag{B.20}$$

where $B(x_0) = B_1 + B_2(x_0, \theta_1)\, a/L + \mathrm{O}(a^2/L^2)$ is a c-number valued function and $\widetilde{S}(x_0, y_0, k_1)$ is the propagator in the half Fourier transformed space we do the actual calculation in. Its deviation from a diagonal form in Dirac space is of order $a^2/L^2\, \theta_1$. For the diagonal part we find

$$B(x_0) = 0.3849001 + [0.5000 + \mathrm{O}((T - 2x_0)/L) + \mathrm{O}(\theta_1)]\ a/L + \mathrm{O}(a^2/L^2)\,, \tag{B.21}$$

B.1.3 Bubble reduction

In the perturbative expansion the free propagator at zero distance appears sandwiched between two Dirac structures when two spinors of one and the same four fermion interaction are contracted and produce a bubble in the diagram (see also Fig. B.1)

$$\Gamma_I a\ \left[\psi(x)\overline{\psi}(x)\right] \Gamma_I = a\, S(x,x) + \Gamma_I\ [a\, S(x,x), \Gamma_I] \tag{B.22}$$

$$= B(x_0) + F_I(x)\ a^2/L^2\, \theta_1 + \dots\,, \tag{B.23}$$

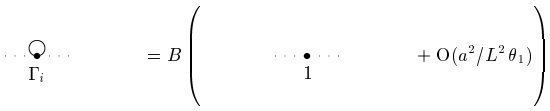

FIGURE B.1: A four fermion interaction contracted to a bubble can be reduced to a insertion of the scalar density.

where on the left hand side the brackets denote a contraction while on the right hand side they denote a commutator. This means that in these cases the insertion of the four fermion interactions can be reduced to the insertion of the scalar density

$$a^2 \sum_x \left\langle \dots\ \overline{\psi}(x)\Gamma_I \left[\psi(x)\ \overline{\psi}(x)\right] \Gamma_I \psi(x)\ \dots \right\rangle_0 = a \sum_x \left\langle \dots\ \overline{\psi}(x) \left(B(x_0) + F_I(x)\ a^2/L^2\,\theta_1 + \dots \right) \psi(x)\ \dots \right\rangle_0 . \quad \text{(B.24)}$$

For $\theta_1 = 0$ (periodic boundary conditions in space) this reduction is exact.

The insertion of the scalar density in a correlation function in turn can be written as the derivation of this correlation function with respect to the mass parameter m_0

$$\frac{\partial}{\partial a m_0} \langle O \rangle_0 = -a \sum_x \left\langle O\ \overline{\psi}(x)\psi(x) \right\rangle_0 . \quad \text{(B.25)}$$

B.2 Boundary-boundary correlation functions

B.2.1 Free theory

In a diagrammatic expansion we find at zeroth order one diagram for the two- and the four-point function (7.2) and (7.3). In terms of the propagators introduced in the last section they read

$$f_2^{(0)} = \mathrm{tr}_d \left\{ K(0) \right\} = k(0) , \quad \text{(B.26)}$$

$$f_4^{(0)} = \mathrm{tr}_d \left\{ K(0) K^\dagger(0) \right\} = k^2(0) . \quad \text{(B.27)}$$

Note that $f_4^{(0)} = \left(f_2^{(0)} \right)^2$. At zero bare mass we find

$$f_2^{(0)} \Big|_{m_0=0} = \frac{1}{\cosh(\theta T/L)} + \mathrm{O}(a^2) . \quad \text{(B.28)}$$

B.2.2 First order

At first order there are two diagrams for f_4 and one diagram for f_2 (cf. Fig. 7.1). Only diagram a is needed in the computation of the ratio (7.5). That is because with the help of Lemma 1 diagram b can be written as $f_2^{(1)} \cdot f_2^{(0)}$ and is hence canceled in the expansion. In terms of the above introduced propagators diagram a reads ($H(y_0) = H(y_0, 0)$)

$$f_{4,\mathrm{a}}^{(1)} = \sum_I c_I \, f_{4,I,\mathrm{a}}^{(1)} \,, \quad I = S, P, V \,, \tag{B.29}$$

$$f_{4,I,\mathrm{a}}^{(1)} = \frac{a}{L} \sum_{x_0} \; \mathrm{tr}_d \left\{ \Gamma_B \gamma_5 H^\dagger(x_0) \gamma_5 \Gamma_I H'(x_0) \Gamma'_B \gamma_5 H'^\dagger(x_0) \gamma_5 \Gamma_I H(x_0) \right\} . \tag{B.30}$$

The convention for coupling constants used in this appendix translates into the one of (5.27) via

$$c_S = g^2 \,, \quad c_P = g^2 - \delta_P^2 \,, \quad \text{and} \quad c_V = g_V^2 \,. \tag{B.31}$$

Since $\{\gamma_1, h(y_0, p_1)\} = 0$ in case of diagram a, it is easier to do the computation with $\Gamma'_B = \Gamma_B = \gamma_1$ (of course the result must be the same with $\Gamma'_B = \Gamma_B = \gamma_5$, this can be used as a check). Then

$$f_{4,I,\mathrm{a}}^{(1)} = \left(\frac{2\mathrm{e}^{-2\omega(0^+)T}}{R(0^+)} \right)^4 \frac{a}{L} \sum_{x_0} \mathrm{tr}_d \left\{ h(x_0) P_+ \gamma_1 P_- h(x_0) \Gamma_I h(T - x_0) P_- \gamma_1 P_+ h(T - x_0) \Gamma_I \right\} , \tag{B.32}$$

and

$$\mathrm{tr}_d \{ \dots \} = \; \mathrm{tr}_d \left\{ \gamma_1 \Gamma_I \gamma_1 \; h(x_0) P_- h(x_0) \; \Gamma_I \; h(T - x_0) P_- h(T - x_0) \right\} . \tag{B.33}$$

Using

$$h P_- h = h_1^2 P_- - h_2^2 P_+ + i \gamma_1 h_1 h_2 \,, \tag{B.34}$$

this results in

$$\begin{aligned}
\Gamma_S = 1 \;\rightarrow\; \mathrm{tr}_d \{ \dots \} &= \left(h_1(x_0) h_1(T - x_0) - h_2(x_0) h_2(T - x_0) \right)^2 \\
\Gamma_P = i\gamma_5 \;\rightarrow\; \mathrm{tr}_d \{ \dots \} &= - \left(h_1(x_0) h_2(T - x_0) - h_2(x_0) h_1(T - x_0) \right)^2 \\
\Gamma_V = \gamma_0 + \gamma_1 \;\rightarrow\; \mathrm{tr}_d \{ \dots \} &= - \Big\{ \left(h_1(x_0) h_1(T - x_0) + h_2(x_0) h_2(T - x_0) \right)^2 \\
&\quad + \left(h_1(x_0) h_2(T - x_0) + h_2(x_0) h_1(T - x_0) \right)^2 \Big\}
\end{aligned}$$

At zero bare mass we find

$$\begin{aligned}
\Gamma_S = 1 \;\to\; \mathrm{tr}_d\,\{\dots\} &= (\theta/L)^4 \cosh^2(\theta(T-2x_0)/L) + \mathrm{O}(a^2) \\
\Gamma_P = i\gamma_5 \;\to\; \mathrm{tr}_d\,\{\dots\} &= -(\theta/L)^4 \sinh^2(\theta(T-2x_0)/L) + \mathrm{O}(a^2) \\
\Gamma_V = \gamma_0 + \gamma_1 \;\to\; \mathrm{tr}_d\,\{\dots\} &= -(\theta/L)^4 \cosh(2\theta T/L) + \mathrm{O}(a^2)\,.
\end{aligned}$$

Taking the naive continuum limit of the sum in (B.30) for $I = S, P$ we find

$$\begin{aligned}
\frac{a}{L}\sum_{y_0=a}^{T-a} \cosh^2(\theta(T-2y_0)L) &= \frac{1}{L}\int_0^T \mathrm{d}y_0 \cosh^2(\theta(T-2y_0)/L) + \mathrm{O}(a) \\
&= \frac{1}{4\theta}\sinh(2\theta T/L) + \frac{T}{2L} + \mathrm{O}(a)\,, \\
\frac{a}{L}\sum_{y_0=a}^{T-a} \sinh^2(\theta(T-2y_0)L) &= \frac{1}{L}\int_0^T \mathrm{d}y_0 \sinh^2(\theta(T-2y_0)/L) + \mathrm{O}(a) \\
&= \frac{1}{4\theta}\sinh(2\theta T/L) - \frac{T}{2L} + \mathrm{O}(a)\,.
\end{aligned}$$

Finally using (B.17)

$$f^{(1)}_{4,\mathrm{a}}\Big|_{m_0=0} = \frac{T}{2L\,(C(\theta))^2}\,\{(c_S - c_P)\,A(\theta) + c_S + c_P - 2c_V B(\theta)\} + \mathrm{O}(a)\,, \tag{B.35}$$

$$A(\theta) = \frac{L}{2\theta T}\sinh(2\theta T/L) \overset{\theta\to 0}{\to} 1\,, \tag{B.36}$$

$$B(\theta) = \cosh(2\theta T/L) \overset{\theta\to 0}{\to} 1\,, \tag{B.37}$$

$$C(\theta) = \cosh^2(\theta T/L) \overset{\theta\to 0}{\to} 1\,. \tag{B.38}$$

B.2.3 Second order

At second order there are nine diagrams for f_4 and three diagram for f_2 (cf. Fig. 7.2). But because of Lemma 1 we need to evaluate only the five non-reducible diagrams a-e.

The second order diagrams involve one momentum loop and are therefore treated numerically. For the logarithmic divergent part we find

$$\begin{aligned}
f^{(2)}_{4,\mathrm{a}}\Big|_{m_0=0} = \frac{T}{2L\,(C(\theta))^2}\,\frac{\ln(a/L)}{2\pi}\,\Big\{&(-c_S^2 - c_P^2 + 2\,c_S\,c_P)\,B(\theta) \\
&- 4\,c_V^2\,B(\theta) + 4\,c_V(c_S - c_P)\,A(\theta)\Big\} + \mathrm{O}(1)\,,
\end{aligned} \tag{B.39}$$

$$f^{(2)}_{4,\mathrm{b}}\Big|_{m_0=0} = \frac{T}{2L\ (C(\theta))^2}\,\frac{\ln(a/L)}{2\pi}\,\Big\{2\,c_S^2\ (A(\theta)+1) - 2\,c_P^2\ (A(\theta)-1) - 4\,c_S\,c_P + 4\,c_V\,(c_S\,(A(\theta)+1) - c_P\,(A(\theta)-1))\Big\} + \mathrm{O}(1)\,, \quad \text{(B.40)}$$

$$f^{(2)}_{4,\mathrm{c}}\Big|_{m_0=0} = \frac{T}{2L\ (C(\theta))^2}\,\frac{\ln(a/L)}{2\pi}\,\Big\{(c_S^2 + c_P^2 + 2\,c_S\,c_P)\,B(\theta) + 4\,c_V^2\,B(\theta) - 4\,c_V(c_S + c_P)\Big\} + \mathrm{O}(1)\,, \quad \text{(B.41)}$$

$$f^{(2)}_{4,\mathrm{d}}\Big|_{m_0=0} = \frac{-NT}{2L\ (C(\theta))^2}\,\frac{\ln(a/L)}{2\pi}\,\Big\{2\,c_S^2\ (A(\theta)+1) - 2\,c_P^2\ (A(\theta)-1)\Big\} + \mathrm{O}(1)\,, \quad \text{(B.42)}$$

$$\left[4f^{(2)}_{4,\mathrm{e}} + am_c^{(1)}\partial_m f^{(1)}_{4,\mathrm{a}} - 2\frac{f_2^{(1)} + am_c^{(1)}\partial_m f_2^{(0)}}{f_2^{(0)}}\,f^{(1)}_{4,\mathrm{a}}\right]_{m_0=0} = \mathrm{O}(1) \quad \text{(B.43)}$$

B.2.4 Proof of Lemma 1

A diagram $f^n_{2,i}$ at order $n \geq 0$ in the expansion of f_2 (cf. Eq. (7.2)) takes the general form (the subscript i is only to label the diagram)

$$f^{(n)}_{2,i} = \left(\frac{a}{L}\right)^n \sum_{x_0^t\,;\,k_1^t\,;\,I_t} \mathrm{tr}_d\left\{V_n\,P_+\right\}\,, \quad t = 1,\ldots,n\,, \quad \text{(B.44)}$$

where for $n = 0$

$$V_0 = K(0)\,, \quad \text{(B.45)}$$

and for $n \geq 1$

$$V_n = F^2\,h(T - y_0)\,V(x_0^t\,;\,k_1^t\,;\,I_t)\,h(z_0)\,, \quad t = 1,\ldots,n\,, \quad \text{(B.46)}$$

$$y_0\,, z_0 \in \{x_0^1, \ldots x_0^n\}\,, \quad F = \frac{2\mathrm{e}^{-2\omega(0^+)T}}{R(0^+)}\,, \quad \text{(B.47)}$$

With $h(y_0) = h(y_0, 0)$ as defined in Section B.1.1. For example, if $n = 1$ there is only one possibility

$$V_1 = F^2\,h(T - x_0^1)\,\Gamma_{I_1} \sum_{k_1} \widetilde{S}(x_0^1, x_0^1, k_1)\,\Gamma_{I_1}\,h(x_0^1)\,. \quad \text{(B.48)}$$

For arbitrary $n \geq 1$ the kernel $V(x_0^t\,;\,k_1^t\,;\,I_t)$ is a product of free propagators and $2n$ Dirac structures $\Gamma_{I_t} = \{1, \gamma_5, \gamma_0 + \gamma_1\}$ coming from the vertices. The free propagator is γ_5 hermitian $\gamma_5 \widetilde{S}(x_0, y_0, k_1)^\dagger \gamma_5 = \widetilde{S}(y_0, x_0, k_1)$. And since besides the propagators there is an even number of γ-matices in V, it also is γ_5 hermitian. Then V_n has the property

$$\gamma_5 V_n^\dagger \gamma_5 = F^2\, h(z_0)\, V(x_0^t\,;\,k_1^t\,;\,I_t)\, h(T - y_0)\,, \quad t = n, \ldots, 1\,. \tag{B.49}$$

Using properties of the trace and that f_2 is real we have the identity

$$\mathrm{tr}_d\,\{V_n\, P_+\} = \mathrm{tr}_d\left\{\gamma_5 V_n^\dagger \gamma_5\, P_-\right\}\,. \tag{B.50}$$

A reducible diagram $f_{4,i}^{(n)}$ at order $n \geq 0$ takes the general form

$$f_{4,i}^{(n)} = \left(\frac{a}{L}\right)^n \sum_{x_0^t\,;\,k_1^t\,;\,I_t} \mathrm{tr}_d\left\{\Gamma_B P_- D_s P_- \Gamma_B' P_+ U_r P_+\right\}\,,\; t = 1, \ldots, n\,,\; r + s = n\,. \tag{B.51}$$

Because of the projectors $P_\pm = \frac{1}{2}(1 \pm \gamma_0)$ only for boundary Dirac structures $\Gamma_B\,,\, \Gamma_B' \in \{\gamma_1\,,\, \gamma_5\}$ Eq. (B.51) does not vanish and reduces to

$$f_{4,i}^{(n)} = \left(\frac{a}{L}\right)^n \sum_{x_0^t\,;\,k_1^t\,;\,I_t} \mathrm{tr}_d\left\{\Gamma_B D_s \Gamma_B' P_+ U_r P_+\right\}\,. \tag{B.52}$$

First we concentrate on proving the case $n = 0$. For $r, s = 0$ we have

$$U_0 = K(0)\,, \quad D_0 = \gamma_5 K^\dagger(0) \gamma_5\,. \tag{B.53}$$

and there is only one diagram

$$f_4^{(0)} = \mathrm{tr}_d\left\{\Gamma_B \gamma_5 K^\dagger(0) \gamma_5 \Gamma_B' P_+ K(0) P_+\right\}\,. \tag{B.54}$$

For $\Gamma_B = \Gamma_B'$ we find

$$f_4^{(0)} = \mathrm{tr}_d\left\{K^\dagger(0) P_+ K(0) P_+\right\} \tag{B.55}$$

$$= \mathrm{tr}_d\left\{K^\dagger(0) P_+\right\} \cdot\; \mathrm{tr}_d\left\{K(0) P_+\right\} \tag{B.56}$$

$$= f_2^{(0)} \cdot f_2^{(0)}\,, \tag{B.57}$$

where we used $K^\dagger(0) = K(0)$ and which proves the Lemma in this case. For mixed choices for $\Gamma_B\,,\, \Gamma_B'$ there are factors $\pm i$ left which can be absorbed into Γ_B or Γ_B'.

Now we turn to arbitrary $n \geq 1$. For $r, s \geq 1$ we find

$$U_r = F^2\, h(T - y_0)\, U(x_0^v\,;\, k_1^v\,;\, I_v)\, h(z_0)\,, \quad v = 1, \ldots, r \tag{B.58}$$

$$y_0\,, z_0 \in \{x_0^1, \ldots x_0^r\} \tag{B.59}$$

and

$$D_s = F^2\, h(u_0)\, D(x_0^w\,;\, k_1^w\,;\, I_w)\, h(T - v_0)\,, \quad w = r+1, \ldots, r+s \tag{B.60}$$

$$u_0\,, v_0 \in \{x_0^{r+1}, \ldots x_0^{r+s}\} \tag{B.61}$$

Because of the projectors $P_\pm$ in (B.52) and the fact that the x_0^t and k_1^t-summations do not mix U_r and $D_s^\dagger$ the overall trace factorises into two independent traces

$$f_{4,i}^{(n)} = \left(\frac{a}{L}\right)^s \sum_{x_0^w\,;\, k_1^w\,;\, I_w} \mathrm{tr}_d\left\{\Gamma_B D_s \Gamma_B' P_+\right\} \cdot \left(\frac{a}{L}\right)^r \sum_{x_0^v\,;\, k_1^v\,;\, I_v} \mathrm{tr}_d\left\{U_r P_+\right\}\,. \tag{B.62}$$

The second factor is evidently an f_2 diagram. Because of $\Gamma_B'\,, \Gamma_B \in \{\gamma_1\,,\, \gamma_5\}$ and the cyclic properties of the trace the trace of the first factor can always be written as (one may have to absorb some factor $\pm i$ into Γ_B' or Γ_B)

$$\mathrm{tr}_d\left\{\Gamma_B D_s \Gamma_B' P_+\right\} = \mathrm{tr}_d\left\{D_s\, P_-\right\}\,. \tag{B.63}$$

Comparing Eqs. (B.63), (B.60) with (B.50), (B.49) also the first factor in Eq. (B.62) can be identified with an f_2 diagram and we arrive at

$$f_{4,i}^{(n)} = f_{2,j}^{(r)} \cdot f_{2,k}^{(s)}\,. \tag{B.64}$$

It should be obvious that this holds also for $s \geq 1\,, r = 0$, or vice versa, which completes the proof of Lemma 1.

B.3 Boundary-to-interior correlation functions

B.3.1 Free theory

Here we consider tree level correlation functions of the type

$$C_{2\mathrm{f,i}}(\Gamma_\mathrm{I}, \Gamma_\mathrm{R}; p_1, x_0) = -a^2 \sum_{y_1 z_1} \mathrm{e}^{i p_1 (y_1 - z_1)} \left\langle \overline{\psi}(x) \Gamma_\mathrm{I} \psi(x)\, \overline{\zeta}(y_1) \Gamma_\mathrm{R} \zeta(z_1) \right\rangle_0\,, \tag{B.65}$$

Γ_R	Γ_I
γ_1, γ_5	γ_1, γ_5
$\gamma_1\lambda^a, \gamma_5\lambda^a$	$\gamma_1\lambda^a, \gamma_5\lambda^a$

TABLE B.1: Nonvanishing Dirac and flavour structure for correlation functions with current/density insertions.

where Γ_I is a matrix with Dirac and flavour indices specifying the quantum numbers of the current or density that is inserted at x. The disconnected part of Eq. (B.65) is proportional to

$$\mathrm{tr}_{d,f}\left\{S(x,x)\Gamma_I\right\}\ \ \mathrm{tr}_{d,f}\left\{P_- S P_+ \Gamma_R - a P_- \gamma_1 i \mathring{p}_1^+ \Gamma_R\right\} \tag{B.66}$$

and the connected part

$$\mathrm{tr}_{d,f}\left\{P_- S\,\Gamma_I\, S P_+ \Gamma_R\right\}\,. \tag{B.67}$$

Both expression vanish for $\Gamma_R \propto 1, \gamma_0$. The disconnected part vanishes for nontrivial flavour structure of either Γ_R or Γ_I. The connected part can be written in terms of propagators between boundary and interior using (B.4–B.9)

$$\mathrm{tr}_{d,f}\left\{H^\dagger(x_0,p_1)\,\gamma_5\Gamma_I\,H(x_0,p_1)\,\Gamma_R\gamma_5\right\}\,. \tag{B.68}$$

The nonvanishing combinations of Γ_R, Γ_I are listed in the table Table B.1. $\Gamma_I \propto 1, \gamma_0$ are excluded by parity.

If we introduce the shorthand

$$L(p_1) = a\sum_{y_1} \mathrm{e}^{ip_1(y_1-z_1)}\, P_- S(a,z_1;a,y_1) P_+\,, \tag{B.69}$$

in the disconnected part the the correlation function Eq. (B.65) reads

$$C_{2\mathrm{f,i}}(\Gamma_I,\Gamma_R;p_1,x_0) = C^{(1)}_{2\mathrm{f,i}} - C^{(2)}_{2\mathrm{f,i}}\,, \tag{B.70}$$

$$C^{(1)}_{2\mathrm{f,i}} = \ \mathrm{tr}_{d,f}\left\{H^\dagger(x_0,p_1)\,\gamma_5\Gamma_I\,H(x_0,p_1)\,\Gamma_R\gamma_5\right\}\,, \tag{B.71}$$

$$C^{(2)}_{2\mathrm{f,i}} = \sum_{k_1}\ \mathrm{tr}_{d,f}\left\{\widetilde{S}(x_0,x_0,k_1)\Gamma_I\right\}\ \ \mathrm{tr}_{d,f}\left\{L(p_1)\Gamma_R\right\}\,. \tag{B.72}$$

The half Fourier transformed of the propagator $\widetilde{S}$ is defined in Eq. (5.56). The minus sign infront of $C^{(2)}_{2\mathrm{f,i}}$ is due to the additional trace of the disconnected diagram (cf. Section 5.5.2).

Since

$$H(x_0, p_1)\gamma_5 = H(x_0, p_1)i\gamma_0\gamma_1 = iH(x_0, p_1)\gamma_1\,, \tag{B.73}$$

there are only two independent zero momentum correlation functions with current/density insertion and vanishing disconnected diagrams. Namely the insertion of the space component of the vector current γ_1 and pseudoscalar density γ_5. Note that in two dimensions $\gamma_\mu\gamma_5 = i\epsilon_{\mu\nu}\gamma_\nu$, therefore $A_\mu = i\epsilon_{\mu\nu}V_\nu$ and $f_A = if_V$, where f_A is the correlation function of the insertion of the time component of the axial current. Comparing (B.65) with (6.7) we have

$$f_A^{(0)}(x_0) = \frac{1}{2N}\,C_{\mathrm{2f,i}}(\gamma_0\gamma_5, \gamma_5; p_1, x_0)\,, \tag{B.74}$$

and

$$f_P^{(0)}(x_0) = \frac{1}{2N}\,C_{\mathrm{2f,i}}(\gamma_5, \gamma_5; p_1, x_0)\,. \tag{B.75}$$

Using eqs. (4.42), (B.4), (B.69) and performing some algebra these correlation functions explicitly read

$$f_X^{(0)}(x_0) = f_X^{(0,1)} + f_X^{(0,2)}\,, \tag{B.76}$$

with

$$f_A^{(0,1)} = \frac{1}{R(p^+)^2}\Bigg\{2M_+(p^+)M_-(p^+)\mathrm{e}^{-2\omega(p_1^+)T} - M(p^+)\left[M_-(p^+)\mathrm{e}^{-2\omega(p_1^+)x_0} + M_+(p^+)\mathrm{e}^{-2\omega(p_1^+)(2T-x_0)}\right]\Bigg\}\,, \tag{B.77}$$

$$f_A^{(0,2)} = N\,\mathring{p}_1^+\,\frac{A(p_1^+)}{R(p^+)}\left(1-\mathrm{e}^{-2\omega(p_1^+)T}\right)\sum_{k_1}\frac{-\mathring{k}_1^+}{2i\mathring{k}_0^+A(k_1^+)R(k^+)} \Bigg\{M_-(k^+) + M_+(k^+)\mathrm{e}^{-2\omega(k_1^+)T} - M(k^+)\left(\mathrm{e}^{-2\omega(k_1^+)x_0} + \mathrm{e}^{-2\omega(k_1^+)(T-x_0)}\right)\Bigg\}\,, \tag{B.78}$$

and

$$f_P^{(0,1)} = -\frac{i\,\mathring{p}_0^+}{R(p^+)^2}\left\{M_-(p^+)\mathrm{e}^{-2\omega(p_1^+)x_0} - M_+(p^+)\mathrm{e}^{-2\omega(p_1^+)(2T-x_0)}\right\}, \tag{B.79}$$

$$f_P^{(0,2)} = N\,\mathring{p}_1^+\,\frac{A(p_1^+)}{R(p^+)}\left(1-\mathrm{e}^{-2\omega(p_1^+)T}\right)\sum_{k_1}\frac{\mathring{k}_1^+}{2A(k_1^+)R(k^+)} \tag{B.80}$$

$$\left\{\mathrm{e}^{-2\omega(k_1^+)x_0} - \mathrm{e}^{-2\omega(k_1^+)(T-x_0)}\right\}. \tag{B.81}$$

Here we used the abbreviation

$$M_\pm(p^+) = M(p^+) \pm i\,\mathring{p}_0^+\,. \tag{B.82}$$

Note that

$$\tfrac{1}{2}(\partial_0^* + \partial_0)\,f_A^{(0,1)}(x_0) = 2\,M(p^+)\cosh\left[a\omega(p_1^+)\right]\;f_P^{(0,1)}(x_0)\,, \tag{B.83}$$

and

$$\tfrac{1}{2}(\partial_0^* + \partial_0)\,f_A^{(0,2)}(x_0) = N\,\mathring{p}_1^+\,\frac{A(p_1^+)}{R(p^+)}\left(1-\mathrm{e}^{-2\omega(p_1^+)T}\right)$$

$$\sum_{k_1}\frac{\mathring{k}_1^+\,M(k^+)\cosh\left[a\omega(k_1^+)\right]}{A(k_1^+)R(k^+)}\;\left\{\mathrm{e}^{-2\omega(k_1^+)x_0} - \mathrm{e}^{-2\omega(k_1^+)(T-x_0)}\right\}. \tag{B.84}$$

Appendix C

Perturbation theory vs. Monte Carlo simulation

C.1 Full theory with bosonic auxiliary fields

Consider the chiral Gross-Neveu model in two dimensions with Wilson fermions and the interaction terms O_{SS}, O_{PP}, O_{VV}. In the Schrödinger functional set up correlation functions are calculated from the generating functional Z

$$\langle O \rangle = \left\{ \frac{1}{Z} O Z \right\}_{\bar{\rho}\ldots\eta=0}, \tag{C.1}$$

where

$$Z = \int \mathrm{D}[\psi]\mathrm{D}[\overline{\psi}] \exp\left\{ -S_0 - S_I + (\overline{\psi}, \eta) + (\overline{\eta}, \psi) \right\}, \tag{C.2}$$

and

$$S_I = -a^2 \sum_{x_0=a}^{T-a} \sum_{x_1=0}^{L-a} \left\{ \frac{g_S^2}{2} (\overline{\psi}\psi)^2 + \frac{g_P^2}{2} (\overline{\psi} i\gamma_5 \psi)^2 + \frac{g_V^2}{2} (\overline{\psi}\gamma_\mu \psi)^2 \right\}. \tag{C.3}$$

In Eq. (C.3) the interaction terms are defined only for $0 < x_0 < T$. We could extent the definition to the boundary, but the fields there are no dynamical variables and do not contribute to the following considerations. In fact the fermionic integration in Eq. (C.2) is only over fields $\overline{\psi}(x)$, $\psi(x)$ with $0 < x_0 < T$. The free action S_0 is defined in Eq. (4.34) with the Wilson-Dirac operator in presence of the boundaries (4.35). The choice of coupling constants translates into the one of (5.27) via

$$g_S^2 = g^2 \quad \text{and} \quad g_P^2 = g^2 - \delta_P^2 . \tag{C.4}$$

On the right hand side of Eq. (C.1) the operators in O on the left hand side have been replaced by the corresponding functional derivatives. In perturbation theory one expands the interaction part of the generating functional in the coupling constant g_I^2. In this way all sorts of vertices are added to the external operators in O. Here we want to introduce bosonic auxiliary fields absorbing the interaction terms and leaving a free fermionic theory with a modified Dirac operator.

Introducing auxiliary fields defined by

$$\sigma \equiv -g_S^2\,\overline{\psi}\psi \tag{C.5}$$

$$\pi \equiv -g_P^2\,\overline{\psi} i\gamma_5\psi \tag{C.6}$$

$$B_\mu \equiv -g_V^2\,\overline{\psi}\gamma_\mu\psi \tag{C.7}$$

into the generating functional via

$$1 \propto \int \mathrm{D}[X] \mathrm{e}^{-\frac{1}{2g_X^2}\left(X+g_X^2\overline{\psi}\Gamma_X\psi\right)^2} \tag{C.8}$$

the fermionic integration becomes Gaussian.

But the generating functional now contains also the integration over the auxiliary fields and the free action now depends on them

$$\begin{aligned} S_0[\sigma,\pi,B_\mu] \;=\; & a\sum_{x_1=0}^{L-a}\overline{\psi}(0,x_1)P_-\{a\gamma_1\widetilde{\partial}_1\psi(0,x_1)-\psi(a,x_1)\} \\ & + a^2\sum_{x_0=a}^{T-a}\sum_{x_1=0}^{L-a}\overline{\psi}(x)\{D+m_0+\sigma+i\gamma_5\pi+\gamma_\mu B_\mu\}\psi(x) \\ & + a\sum_{x_1=0}^{L-a}\overline{\psi}(T,x_1)P_+\{a\gamma_1\widetilde{\partial}_1\psi(T,x_1)-\psi(T-a,x_1)\} \end{aligned} \tag{C.9}$$

The the generating functional is up to constant factor

$$Z \propto \int \mathrm{D}[\sigma]\mathrm{D}[\pi]\mathrm{D}[B_\mu]\, Z_0[\sigma,\pi,B_\mu]\,\mathrm{e}^{-S_a}\,, \tag{C.10}$$

with the fermionic part of the generating functional

$$Z_0[\sigma,\pi,B_\mu] = \int \mathrm{D}[\psi]\mathrm{D}[\overline{\psi}]\;\exp\left\{-S_0[\sigma,\pi,B_\mu]+(\overline{\psi},\eta)+(\overline{\eta},\psi)\right\}\,, \tag{C.11}$$

and the kinetic terms for the auxiliary fields

$$S_a = a^2\sum_{x_0=a}^{T-a}\sum_{x_1=0}^{L-a}\left\{\frac{1}{2g_S^2}\sigma^2+\frac{1}{2g_P^2}\pi^2+\frac{1}{2g_V^2}B_\mu^2\right\}\,. \tag{C.12}$$

The operator O contains functional derivatives only with respect to fermionic fields. We can therefore write the expectation value Eq. (C.1)

$$\langle O \rangle = \frac{1}{Z} \int \mathrm{D}[\sigma]\mathrm{D}[\pi]\mathrm{D}[B_\mu] \; \langle O \rangle_a \, Z_0[\sigma, \pi, B_\mu]_{\bar{\rho}\ldots\eta=0} \, \mathrm{e}^{-S_a} \,, \tag{C.13}$$

where $\langle \cdot \rangle_a$ is taken with $Z_0[\sigma, \pi, B_\mu]$ and the subscript a indicates, that it is still dependent on the auxiliary fields

$$\langle O \rangle_a = \left\{ O \, Z_0[\sigma, \pi, B_\mu] \right\}_{\bar{\rho}\ldots\eta=0} \,. \tag{C.14}$$

Introducing the Dirac operator D_a depending on the auxiliary fields

$$D_a = D + \sigma + i\gamma_5\pi + \gamma_\mu B_\mu \,, \tag{C.15}$$

the fermionic functional integral $Z_0[\sigma, \pi, B_\mu]$ at vanishing source fields and vanishing boundary fields is the determinant

$$Z_0[\sigma, \pi, B_\mu]_{\bar{\rho}\ldots\eta=0} = \left(\det(D_a + m_0)\right)^N \,. \tag{C.16}$$

The expectation value Eq. (C.14) is just a sum of tree diagrams with the propagator $S_a(x, y)$ defined through

$$(D_a + m_0) \, S_a(x, y) = a^{-2}\delta_{xy} \,, \quad 0 < x_0 < T \,. \tag{C.17}$$

Using the contractions derived in Section 4.4 every combination of field operators in O can be expressed in terms of the propagator $S_a(x, y)$. The remaining functional integral over the c-number valued auxiliary fields is accessible by numerical methods

$$\langle O \rangle = \frac{1}{Z} \int \mathrm{D}[\sigma]\mathrm{D}[\pi]\mathrm{D}[B_\mu] \; \langle O \rangle_a \; \left(\det(D_a + m_0)\right)^N \, \mathrm{e}^{-S_a} \,, \tag{C.18}$$

given the determinant can be well defined. Since we can choose a representation of the γ-matrices where γ_5 is imaginary and $\gamma_0 \,, \gamma_1$ real, $Q_{xy} = (D_a + m_0)(x, y)$ is a real matrix. Therefore the eigenvalues have to come in complex conjugate pairs and the determinant is real

$$\det Q = (\det Q)^* = \det Q^\dagger \,. \tag{C.19}$$

But because the determinant in Eq. (C.18) can be written as $\mathrm{e}^{\log(\det Q)^N}$, it has to be strictly positive. This, however, is the case for even N.

C.2 Results

We calculate at finite lattice spacing

$$am(x_0) = \frac{a\tilde{\partial}_0 f_A(x_0)}{2f_P(x_0)} \tag{C.20}$$

in Monte Carlo simulations and in perturbation theory (PT). In the simulations we use the representation (C.18) and a standard fermion algorithm [42]. Choosing

$$g_P^2 = g_S^2\,, \quad a\Delta m = am_0 - am_c\,, \tag{C.21}$$

with

$$am_c = -0.7698004(1)\,\left(N\,g_S^2 - g_V^2\right) + \mathrm{O}(g^4) \tag{C.22}$$

we find in leading order perturbation theory (cf. Chapter 6)

$$am(x_0) = \frac{h_0}{2f_P^{(0)}} + \sum_I g_I^2 \left(\frac{h_{1,I}}{2f_P^{(0)}} - \frac{h_0\, f_{P,I}^{(1)}}{2(f_P^{(0)})^2}\right) + \mathrm{O}(g^4) \tag{C.23}$$

$$= \left.\frac{h_0}{2f_P^{(0)}}\right|_{am_0=a\Delta m} \tag{C.24}$$

$$+ \sum_I g_I^2 \left(am_c^{(1)} \left(\frac{h_2}{2f_P^{(0)}} - \frac{h_0\,\partial_{am_0} f_P^{(0)}}{2(f_P^{(0)})^2}\right) + \frac{h_{1,I}}{2f_P^{(0)}} - \frac{h_0\, f_{P,I}^{(1)}}{2(f_P^{(0)})^2}\right)\Bigg|_{am_0=a\Delta m} + \mathrm{O}(g^4) \tag{C.25}$$

where

$$h_0 = a\tilde{\partial}_0 f_A^{(0)}(x_0)\,, \quad h_1 = a\tilde{\partial}_0 f_{A,I}^{(1)}(x_0)\,, \quad h_2 = \partial_{am_0} h_0\,, \tag{C.26}$$

and in the second line we expanded am_0 around $a\Delta m$.

For $p_1 = 0$ (external momentum) and $\theta_1 = 0$ (for these values the disconnected diagram vanishes) the tree level amplitudes can be computed analytically, giving

$$am(x_0) = aM(p^+)\cosh(a\omega(p_1^+)) + \mathrm{O}(g^2),. \tag{C.27}$$

.

Monte Carlo data and PT are plotted in Fig. C.1. First order PT seems to be valid at $g^2 \lessapprox 0.15$. See [34] for the same plot for a 12×13 lattice.

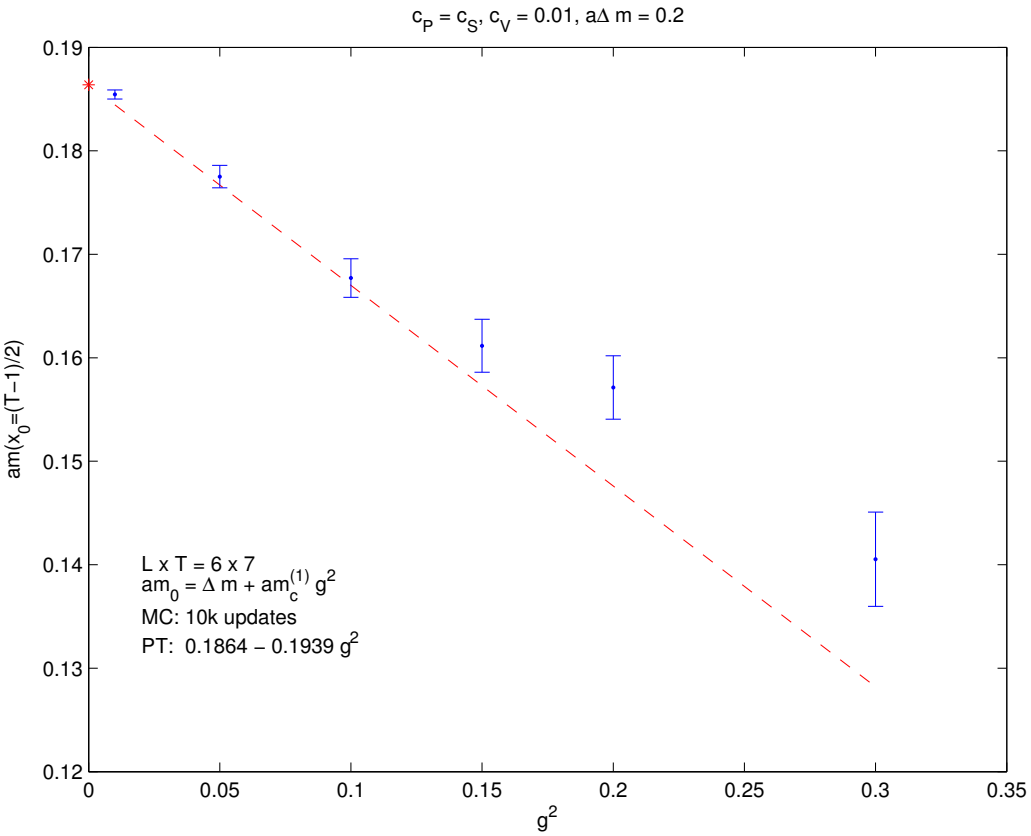

FIGURE C.1: MC vs. PT.

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Lebenslauf

Name:	Björn Leder
1985 - 91	Otto-Engert-Oberschule Posa
1991 - 97	Friedrichgymnasium Altenburg
1997	Abitur
1997 - 2003	Diplomstudiengang Physik an der Humboldt Universität Berlin
WS 00/01	Physikstudium in Großbritannien an der University of Wales Swansea
Okt. 2003	Diplom Physik
Okt. 2002 - Dez. 2003	Studentische Hilfskraft bei Prof. U. Wolff, HU Berlin
Jan. 2004 - Dez. 2006	Wissenschaftliche Hilfskraft, Theory Group, DESY Zeuthen

Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Arbeit selbständig ohne fremde Hilfe verfaßt und nur die angegebene Literatur und Hilfsmittel verwendet zu haben.

Björn Leder
30. Januar 2007