Intersection Cohomology of Hypersurfaces

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von
Herr Dipl.-Math. Lorenz Wotzlaw
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Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. Christoph Markschies

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät I:
Prof. Dr. Wolfgang Coy

Gutachter:

1. Prof. Dr. Alexandru Dimca
2. Prof. Dr. Herbert Kurke
3. Prof. Dr. Duco van Straten

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Für Catrin, Anton und Fynn

Introduction

On a smooth hypersurface X in $\mathbb{P}^n =: Y$, intersection cohomology

$$\mathrm{IH}^k(X, \mathbb{C}) := \mathbb{H}^k(X, \mathrm{IC}_X(\mathbb{C})[-n+1])$$

is ordinary cohomology with values in the constant sheaf

$$\mathbb{C} = \mathrm{IC}_X(\mathbb{C})[-n+1].$$

It comes along with an interesting object: The primitive cohomology of middle dimension $\mathrm{H}_0^{n-1}(X)$. It is the proper cohomology of X, not coming from the ambient projective space, and owes its existence to the fact that vanishing theorems for cohomology, like Kodaira vanishing, explicitly exclude middle cohomology. At the same time, this failure makes the representation of middle cohomology so challenging. Any cohomology theory is as good as its vanishing theorems. On toric varieties and homogeneous spaces there is Bott vanishing for middle cohomology:

$$\mathrm{H}^k(\mathbb{P}^n, \Omega^p(l)) = 0; \quad \text{for } k, l \geq 0$$

and this provides an instrument to attack even advanced questions concerning the Hodge structure of the middle cohomology.

The idean of Clemens and Griffiths [CG80] was as follows: The usual Poincare residue induces an exact sequence

$$0 \to \Omega_Y^\bullet \to \Omega_Y^\bullet(\log X) \to \Omega_X^\bullet[-1] \to 0$$

and, as all Gysin-maps $\mathbb{H}^{k-2}(X, \Omega_X^\bullet)(-1) \to \mathbb{H}^k(Y, \Omega_Y^\bullet)$ are surjective, an isomorphism between the pure Hodge structures

$$\mathbb{H}^k(Y, \Omega_Y^\bullet(\log X)) \simeq \mathbb{H}_0^{k-1}(X, \Omega_X^\bullet)(-1)$$

for all k.

To give an explicit description of these Hodge structures, Griffiths essentially needs **4 ingredients**

(i) Rational forms with pole-order filtration As the inclusion of the log-complex into the complex $\Omega^\bullet(*X)$ of rational forms with poles only along X is a quasi-isomorphism, which is filtered when we put the (standard) stupid filtration on the log-complex and pole-filtration

$$\begin{aligned}\mathrm{P}^p\Omega^\bullet(*X) :=&(\Omega^p(X) \to \Omega^{p+1}(2X) \to \ldots \to \Omega^n((n-p+1)X))[-p],\\ =&(\mathrm{P}^p(\Omega^0(*X) \to \mathrm{P}^p(\Omega^1(*X)) \to \ldots \to \mathrm{P}^p(\Omega^n(*X)));\\ &\text{where } \mathrm{P}^p(\Omega^j(*X)) := \Omega^j(1+j-p) \text{ and zero for } j-p \le 0\end{aligned}$$

on $\Omega^\bullet(*X)$. That is, log-forms and the de Rham complex of X can be replaced as indicated in the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^\bullet_Y & \longrightarrow & \Omega^\bullet_Y(\log X) & \longrightarrow & \Omega^\bullet_X[-1] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Omega^\bullet_Y & \longrightarrow & \mathrm{P}^0\Omega^\bullet_Y(*X) & \longrightarrow & \Omega^\bullet_Y(*X)/\Omega^\bullet_Y & \longrightarrow & 0 \end{array}$$

and still induce the Deligne Hodge structure on the cohomology groups. (We will therefore hereafter still denote the induced filtration by F.)

(ii) Bott vanishing For all r, the spectral sequences

$$\mathrm{E}_1^{pq} = \mathrm{H}^q(Y,\ \mathrm{P}^r\Omega^p(*X)) \implies \mathbb{H}^{p+q}(Y,\ \mathrm{P}^r\Omega^\bullet(*X))$$

induce isomorphisms

$$\mathbb{H}^k(Y, \mathrm{P}^r\Omega^\bullet(*X)) = \mathrm{H}^k(\Gamma(Y,\ \mathrm{P}^r\Omega^\bullet(*X)), \Gamma(d))$$

for all r. Since there is no non-middle primitive cohomology, the $\Gamma(d)$-complex is exact, away from

$$\mathbb{H}^n(Y, \mathrm{P}^r\Omega^\bullet(*X)) = \Gamma(Y,\ \mathrm{P}^r\Omega^n(*X))/d\Gamma(Y, \mathrm{P}^r\Omega^{n-1}(*X)).$$

(iii) Rational top-forms Let $\mathrm{C}Y := \mathbb{C}^{n+1} - \{0\}$, $q : \mathrm{C}Y \to \mathbb{P}^n := Y$ be the cone over Y, $E := \sum x_i\partial_i$ the *Eulerfield* on $\mathrm{C}Y$. It characterizes those forms in $\Omega^p_{\mathrm{C}Y}(*\mathrm{C}X)$; $\mathrm{C}X := q^{-1}(X)$, that are induced from q via

$$q^{-1}\Omega^p_Y(*X) = (\ker \mathcal{L}_E \cap \ker i_E) \subset q^*\Omega^p_{\mathrm{C}Y}(*X) = \ker i_E \subset \Omega^p_{\mathrm{C}Y}(*\ \mathrm{C}X);$$

As the Koszul-complex $(\Omega^p_{\mathrm{C}Y}(*\mathrm{C}X), i_E)$ is exact for any cycle

$$\omega \in \Gamma(Y, \mathrm{P}^p\Omega^n(*X)),$$

there is a polynomial $A \in \mathbb{C}[x_0, \ldots, x_n]$ such that

$$\omega = i_E \frac{AdV}{F^{n+1-p}}\ ,$$

where $dV := dx_0 \wedge \ldots \wedge dx_n$ and F denotes the homogeneous equation of X.

Now $\mathcal{L}_E \omega = 0$ iff A is homogeneous with $\deg A = (n+1-p) \cdot \deg F - (n+1)$ so that if we put $\Omega := i_E dV$,

$$\omega = \frac{A\Omega}{F^{n+1-p}} \ .$$

Similarly, every $n-1$-form $\eta \in \Gamma(Y, \mathrm{P}^p \Omega^{n-1}(*X))$ is of the type $i_E \frac{\sum G_i dx^i}{F^{n-p}}$; $\deg G_i = (n-p)d - n$, hence every boundary-cycle is of the form

$$\begin{aligned} d\eta =& d i_E \frac{\sum G_i dx^i}{F^{n-p}} = -i_E d \frac{\sum G_i dx^i}{F^{n-p}} \\ \equiv& n-p \frac{\sum G_i \partial_i F \Omega}{F^{n+1-p}} \text{ modulo term of lower pole order .} \end{aligned}$$

But the pole filtration induces the Hodge filtration, therefore this is already the proof of

Theorem (Griffiths-calculus, [CG80]). *Let $X = V_+(F)$ be a smooth hypersurface in $Y := \mathbb{P}^n$. Then $\mathrm{H}^n(Y - X) \simeq \mathrm{H}_0^{n-1}(X)(-1) =: \mathrm{H}$ as Hodge structure and there is an isomorphism of vector spaces*

$$\begin{array}{rcl} \mathbb{C}[x_0, \ldots, x_n]/\langle F_0, \ldots, F_n \rangle_{d(n+1-p)-n-1} & \to & \mathrm{Gr}^p \mathrm{H}, \\ A & \mapsto & \frac{A\Omega}{F^{n+2-p}} \end{array}$$

$X = V_+(F)$ is smooth so that $\mathbb{P}^n$ is covered by the complements of the vanishing loci of the derivatives $F_i := \frac{\partial}{\partial x_i} F$ of F. In a long calculation in the Čech complex of this 'Jacobi cover'

$$J := \coprod_{i=0}^{n} \mathrm{D}_+(F_i),$$

they proved

Theorem (pairing, [CG80]). *Let*

$$\alpha = \frac{A\Omega}{F^{n-p+1}} \in \Gamma(Y, \ \mathrm{P}^p(\Omega^n(*X)[n]),$$

$$\beta = \frac{B\Omega}{F^p} \in \Gamma(Y, \ \mathrm{P}^{n-p+1}(\Omega^n(*X)[n])$$

represent classes in $\mathrm{F}^{p-1}\mathrm{H}_0^{n-1}(X)$, $\mathrm{F}^{n-p}\mathrm{H}_0^{n-1}(X)$.

Then

$$i_! \alpha \cup \beta = \frac{(-1)^{n-p} p!}{n-p!} \frac{AB\Omega}{F_0 \cdot \ldots \cdot F_n} \in \check{\mathrm{H}}^n(J, \Omega^n),$$

where $i_! : \mathrm{H}^{n-1}(\Omega_X^{n-1}) \to \mathrm{H}^n(\mathbb{P}^n, \Omega^n)$ is the Gysin map.

Regarding the relative situation of a family of hypersurfaces

$$\begin{array}{ccc} \mathcal{X} & \subset & S \times \mathbb{P}^n \\ \downarrow & & \downarrow \\ S & = & S \end{array}$$

they furthermore obtained the following

(iv) Description of the $\mathbf{H^1(X, \Theta_X)}$-action on $\mathbf{Gr_F\, H}$ If S is a double point $S \sim \mathrm{Spec}(k[\epsilon]/\langle\epsilon^2\rangle)$ and $\theta = \mathrm{ks}(\frac{\partial}{\partial\epsilon}) \in \mathrm{H}^1(X, \Theta_X)$ the Kodaira-Spencer class of $\mathcal{X} \to S$, $T \in \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}(\deg(X)))$ a lift of θ in the normal bundle sequence (which exists because the deformation is embedded) and $[\alpha] = [\frac{A\Omega}{F^{n-p+1}}] \in \mathrm{Gr}_\mathrm{F}^p(\mathrm{H})$ as above, then

$$\theta \cup [\alpha] \equiv \frac{TA\Omega}{F^{n-p+2}} \equiv \frac{T}{F} \cdot \alpha \in \mathrm{Gr}_\mathrm{P}^{p-1}\,\mathrm{H}$$

i.e. $\mathrm{Gr}_\mathrm{F}^{-1}(- \cup \theta)$ is given by multiplication with T/F on $\mathrm{Gr}_\mathrm{P}\,\mathrm{H}$.

Equipped with the tools $i) - iv)$, Carlson and Griffiths were able to prove a global Torelli theorem. For this they calculated the Yukawa coupling associated to a family of hypersurfaces

$$\begin{array}{ccc} \mathcal{X} & \subset & S \times \mathbb{P}^n \\ \downarrow & & \downarrow \\ S & = & S \end{array}$$

over a smooth base: Via the Kodaira-Spencer map $\mathrm{ks} : \Theta_S \to \mathrm{R}^1\pi_*\Theta_{X|S}$, the tangent sheaf of the base space acts on the Hodge bundle $\mathcal{H}^{n-1} := \mathrm{R}^{n-1}\pi_*\Omega^\bullet_{\mathcal{X}|S}$ in such a way that $\omega \to \mathrm{ks}\,X_1 \cup \ldots \cup \mathrm{ks}\,X_p \cup \omega$ defines a module structure over the symmetric algebra $S^{n-1}(\Theta_S)$ of Θ_S for $\mathrm{Gr_F}\,\mathcal{H}$, that is $\mathrm{Gr_F}\,\mathcal{H}$ is a Higgs bundle on S. Via the pairing on the middle cohomology, this structure comes along with a natural tensor $S^{n-1}\Theta_S \otimes \pi_*\Omega^n \to \mathcal{O}_S$

$$(X_1 \cdot \ldots \cdot X_{n-1}) \otimes \omega \to \langle \mathrm{ks}\,X_1 \cup \ldots \cup \mathrm{ks}\,X_p \cup \omega\rangle\ ,$$

the famous Yukawa coupling. The Torelli theorem says that in many cases the variety can be reconstructed from its Yukawa coupling.

This calculus was generalized to (quasi-) smooth complete intersections in Toric varieties. It is still the central tool in curve counting considerations of toric mirror symmetry associated to families of Calabi-Yau threefolds.

Here we come to the geometrical motivation for my work. It grew out of an attempt to calculate Yukawa couplings for new families of Calabi-Yau threefolds, which are not complete intersections: Small resolutions of nodal quintics X in $\mathbb{P}^4$. One soon misses the embedding as hypersurface to get a description of the middle cohomology. Blowing up the nodes will give a normal crossing divisor, but does not lead to satisfying representations of the cohomology via Leray Spectral sequence. The difference to the smooth case is that cohomology of coherent sheaves on smooth projective varieties is local cohomology at the *isolated* singularity defined by its affine cone. Apparently global questions have their local equivalent and are therefore well understood. The cone over a nodal variety is no longer an isolated singularity; it has a global invariant, the defect of the linear system of the nodes. The direct method via blow up and Leray spectral sequence leads to trouble with this phenomenon.

The method for unwinding the problem is to ignore the resolutions and stay with the singular space X but to calculate another cohomology: The intersection cohomology of X. A posteriori it will turn out that the cohomology of the small resolution and the intersection cohomology are both isomorphic to $\mathrm{Gr}_5^{\mathrm{W}}\,\mathrm{H}^4(\mathbb{P}^4 - X)(1)$. For families of singular varieties, it has the further advantage that there is no need for a simultaneous resolution. This is a problem for small resolutions and a general problem: Some notions of equisingular deformations contain additional data for a simultaneous simplicial resolution. But the apriory motivation for the use of intersection cohomology in our context is that it fulfills self duality:

If X is singular, there is no pairing on the cohomology groups

$$\mathrm{H}^k(X, \mathbb{C}_X) = \mathrm{H}^k(Y, i_*\mathbb{C}_X)$$

at all. The reason is that $i_*\mathbb{C}_X[n-1] = i_*i^*\mathbb{C}_Y[n-1]$ as constructible sheaf is not self dual. This means the following: In the derived category of constructible sheaves on Y, as we will see, there is the dualizing functor $\mathbb{D} : A \to \mathrm{R}\mathcal{H}\mathrm{om}(A, \mathbb{C}_Y[2n])$ such that for any complex of constructible sheaves A, the pairing induced from the natural evaluation map

$$\mathrm{H}^k(Y, A) \times \mathrm{H}^{-k}(Y, \mathbb{D}(A)) \to \ \mathrm{H}^{2n}(Y, \mathbb{C}) \sim \mathbb{C}$$

is nondegenerate. In this setting, we can expect a pairing of the cohomology groups of A only if A is self dual, i.e. naturally isomorphic to $\mathbb{D}(A)$.

For $A = i_*i^*(\mathbb{C}_Y)$, there is a comparison morphism to its dual: the fundamental class $c_{X|Y}$ of X in Y, going from $i_*i^*(\mathbb{C}_Y)$ to

$$\mathbb{D}(i_*i^*\mathbb{C}_Y[n-1]) = i_*i^!\mathbb{C}_Y[n+1]$$

(which is $\mathrm{R}\underline{\Gamma}_X(\mathbb{C}_Y[n+1])$. But it is an isomorphism only in the smooth case.

In this sense, Carlson and Griffiths calculated the second row in

$$\begin{array}{ccc} \mathrm{H}^{n-1}(X,\mathbb{C}) \times \ \mathrm{H}^{n-1}(X,\mathbb{C}) & \xrightarrow{i_! \circ \cup} & \mathrm{H}^{2n}(Y,\mathbb{C}) \\ \Big\downarrow{\scriptstyle id\otimes c_{X|Y}} & & \Big\| \\ \mathrm{H}^0(Y, i_* i^* \mathbb{C}_Y[n-1]) \times \ \mathrm{H}^0(Y, i_* i^! \mathbb{C}_Y[n+1]) & \longrightarrow & \mathrm{H}^0(Y, \mathbb{C}_Y[2n]) \end{array}$$

and could profit from the fact that the fundamental class was an isomorphism.

In general, the self-dual constructible sheaf complex on X, which is isomorphic to $\mathbb{C}_X[n-1]$ on the smooth locus, is the intersection complex IC_X, whose cohomology is the intersection cohomology. Our aim is to provide theorems, which allow an analog of the Griffiths calculus in the singular case for the intersection cohomology. The main issues are

1. Find an analog description of $i_* \mathrm{IC}_X$ by means of a mixed Hodge complex, which allows a presentation of middle primitive intersection cohomology by global n-forms.

2. Even if there is no Jacobi-cover of Y: Use Verdier duality to calculate the product on the middle cohomology.

For this goal, we follow the ideas of M. Saito [Sai88], [Sai89a], [Sai90b] and review the category of differential complexes $\mathrm{D}^{\mathrm{b}}(\mathcal{O}_Y, \mathrm{Diff})$ of complexes of $\mathcal{O}_Y$-modules with differential operators as morphisms. It is isomorphic to the derived category of right $\mathcal{D}_Y$-modules and equipped with a duality functor, preserving $\mathcal{O}_Y$-coherence. This is the category in which one can describe in a conceptional way the Verdier dual for any constructible sheaf complex given by the de Rham complex of any $\mathcal{D}_Y$-module, or by the log complex of a normal crossing divisor. We use it to construct a naive Verdier dual $\widetilde{A}$ of $A := \mathrm{P}^0\Omega^\bullet(*X)/\Omega^\bullet$ for later use in the calculation of the pairing.

In the case of isolated homogeneous singularities, we prove the following theorem

Proposition. *Let*

- *Y be a projective manifold,*
- *X a locally homogeneous Cartier divisor with only isolated singularities and affine complement,*
- $$\mathrm{W}_m(\Omega^\bullet_Y(*X)) := \begin{cases} 0 & ; \ m \leq -1 \\ \Omega^\bullet & ; \ m = 0 \\ \tau_{\leq n-1}(\Omega^\bullet_Y(*X)) & ; \ m = 1 \\ \Omega^\bullet_Y(*X) & ; \ m \geq 2 \ , \end{cases}$$

- $\mathrm{P}^p\Omega^q_Y(*X) := \begin{cases} \Omega^q((q-p+1)X) & ; \quad p \leq q \\ 0 & , \quad else \end{cases}$

Then

$$(\mathrm{W}, \mathrm{P}, j_* j^* \mathbb{Q}_Y, \Omega^\bullet(*X)); \quad j : (Y - X) \hookrightarrow Y \ ,$$

is a mixed Hodge complex calculating the Deligne MHS *on the cohomology of* $Y - X$.

Moreover, $\mathrm{Gr}_1 \Omega^\bullet_Y(*X) = \mathrm{IC}_X(\mathbb{C})$ *and* $(\mathrm{P}, \mathrm{IC}_X((Q)), \mathrm{Gr}_1 \Omega^\bullet_Y(*X)[n])$ *is a Hodge complex inducing a pure Hodge structure on the intersection cohomology.*

As an application we obtain a result on K. Saito's log complex:

Proposition (log comparison)**.** *Let* X *be an isolated homogeneous singularity. Then*

- $\Omega^\bullet(\log X) = \tau_{\leq -2}\Omega^\bullet(*X)$ *if* K_Q *has no global section.*
- $\Omega^\bullet(\log X) \subset \Omega^\bullet(*X)$ *is quis iff the exceptional locus has no primitive cohomology. In this case, it is filtered quis.*

We obtain an explicit formula for the cup product on the intersection cohomology:

Proposition (pairing)**.** *Let* X *be as above. Then*

- *any class in*

$$\mathrm{F}^p \, \mathrm{IH}^n(Y|X) = \mathrm{F}^{p-1} \, \mathrm{IH}^{n-1}_0(X),$$
$$\mathrm{F}^{n-p+1} \, \mathrm{IH}^n(Y|X) = \mathrm{F}^{n-p} \, \mathrm{IH}^{n-1}_0(X)$$

can be represented by global sections

$$\alpha = \frac{A\Omega}{F^{n-p+1}} \in \Gamma(Y, \mathrm{P}^p \tau_{\leq -1}(\Omega^n(*X)[n]),$$

$$\beta = \frac{B\Omega}{F^p} \in \Gamma(Y, \mathrm{P}^{n-p+1} \tau_{\leq -1}(\Omega^n(*X)[n]) \ .$$

-

$$\alpha \cup \beta = \frac{(-1)^{n-p} p!}{n-p!} \frac{AB\Omega}{F_0 \cdot \ldots F_n} \in \mathrm{H}^n(Y, \tau_{\leq -1} l_* l^*(\Omega^n)) = \ \mathrm{H}^n(Y, \Omega^n).$$

The last three chapters are devoted to the study of nodal hypersurfaces, in particular nodal quintics in $\mathbb{P}^4$. First, using M. Saito's theory of mixed Hodge modules [Sai90b], we derive explicit descriptions of the middle intersection cohomology groups.

In a case study, we give a result on the non-existence of a projective small resolution of the family of tangent hyperplane sections to the WE_6-invariant quintic in $\mathbb{P}^5$.

Finally, we give some known facts on deformation theory and describe the Kodaira-Spencer map for families of nodal quintics. Together with the description of the pairing above and an explicit trace morphism, we can give a simple expression for the Yukawa coupling for families of nodal quintics over an Artinian scheme.

Contents

Chapter 1

Riemann-Hilbert Correspondence

The central object of the calculations in the introduction was $\Omega^\bullet_Y(*X)$, which is (a shift of) the de Rham complex of the $\mathcal{D}_Y$-module $\mathcal{O}(*X)$. In order to generalize the results above to singular hypersurfaces, we first need to recall some basic theorems and notations about $\mathcal{D}_Y$-modules and constructible sheaves. For further details, we refer to [Bjo79], [Bjo93], [Bor87], [Meb89] and [Pha79].

1.1 Regular Holonomic Complexes of $\mathcal{D}_Y$-Modules

1.1.1 Analytic Case

Let Y be a complex manifold and denote by $\mathcal{D}_Y \subset \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_Y, \mathcal{O}_Y)$ the **sheaf of rings of differential operators** on Y. The natural filtration of $\mathcal{D}_Y$ by the order of a differential operator with $\operatorname{Gr}\mathcal{D}_Y = \operatorname{Sym}\Theta_Y$ has the property that if $\pi : T^*_Y \to Y$ is the (analytic) cotangent bundle, $\mathcal{O}_{T^*(Y)}$ is a flat $\pi^{-1}\operatorname{Gr}\mathcal{D}_Y$-algebra.

Definition 1.1.1. *A sheaf of left-modules over $\mathcal{D}_Y$ ("**$\boldsymbol{D_Y}$-module**") M is **holonomic** if, in a neighborhood of any point $y \in Y$, there is a filtration $\mathrm{F}_\bullet(M)$ of M making it a filtered sheaf of modules over the filtered sheaf of rings $\mathcal{D}_Y$ ("filtered $\mathcal{D}$-module"), such that all $\operatorname{Gr}^{\mathrm{F}}_q(M)$ are $\mathcal{O}_Y$-coherent ("good filtration") and there is an j depending on y such that*

$$\mathrm{F}_{i+j}M = \mathrm{F}_i\mathcal{D}_Y \cdot \mathrm{F}_jM$$

for all $i \in \mathbb{N}$.

$$\mathrm{Char}(M) := \mathrm{supp}(\mathrm{Gr}^{\mathrm{F}}(M) \otimes_{\pi^{-1}\mathrm{Gr}^{\mathrm{F}}\mathcal{D}_Y} \mathcal{O}_{T^*Y}) \quad \subset \quad \mathrm{T}^*_Y$$

is a Lagrangian subvariety of the cotangentbundel T^*_Y *of* Y*, which turns out to be independent of the particular choice of the good filtration* F.

Recall the definitions of cohomology with tempered support: $\mathcal{D}_Y$-modules form an abelian category $\mathrm{M}(\mathcal{D}_Y)$ so one can form its derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y)$ of bounded complexes. Let $M \in \mathrm{M}(\mathcal{D}_Y)$ and Z be a closed subset of Y with idealsheaf I.

Definition 1.1.2.

$$\underline{\Gamma}_{[Y|Z]}(M) := \varinjlim_k \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(I^k, M^\bullet);$$
$$\underline{\Gamma}_{[Z]}(M) := \varinjlim_k \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(\mathcal{O}_Y/I^k, M)$$

These carry a natural structure of $\mathcal{D}_Y$-module, and with these $\underline{\Gamma}_{[Y|Z]}$ and $\underline{\Gamma}_{[Z]}$ are left exact functors from $\mathrm{M}(\mathcal{D}_Y)$ to itself.

Definition 1.1.3.

$$\mathcal{H}^k_{[Y|Z]}(M^\bullet) := \mathcal{H}^k\ \mathrm{R}\underline{\Gamma}_{[Y|Z]}(M^\bullet)$$
$$\mathcal{H}^k_{[Z]}(M^\bullet) := \mathcal{H}^k\ \mathrm{R}\underline{\Gamma}_{[Z]}(M^\bullet)$$

Example 1.1.4. *Let* $X = V(z_1, \ldots, z_c)$ *be a complete intersection,* $\widetilde{M^\bullet}$ *an* $\mathcal{D}_Y$*-injective resolution of a complex of* $\mathcal{D}_Y$*-modules* $M^\bullet$*, then* $\langle z_1, \ldots, z_c\rangle^k$ *and* $\langle z_1^k, \ldots, z_c^k\rangle$ *define cofinal sequences of ideals, hence*

$$\varinjlim_k \mathcal{H}om_{\mathcal{O}_Y}(\mathrm{K}(z_1^k, \ldots, z_c^k)_\bullet, M^\bullet)$$

is a complex of $\mathcal{D}$*-modules, quasi-isomorphic to*

$$\varinjlim_k \mathcal{H}om_{\mathcal{O}_Y}(\mathrm{K}(z_1^k, \ldots, z_c^k)_\bullet, I^\bullet) = \mathrm{R}\underline{\Gamma}_{[Z]}(M^\bullet)$$

because $I^\bullet$ *is injective as* $\mathcal{O}_Y$*-module also and* $\mathrm{K}(z_1^k, \ldots, z_c^k)_\bullet$ *is a locally free resolution of* $\mathcal{O}_Y/\langle z_1^k, \ldots, z_c^k\rangle$ *(i.e. the standard spectral sequence converging to* $\mathrm{R}\underline{\Gamma}_{[Z]}(M^\bullet)$ *considered as the* $\mathcal{O}_Y$*-module*

$$\varinjlim_k \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathrm{K}(z_1^k, \ldots, z_c^k)_\bullet, M^\bullet)$$

degenerates).

In the same way, since $\mathrm{K}(z_1^k,\dots,z_c^k)_{\bullet\geq 1}$ *is a* $\mathcal{O}_Y$*-free resolution of* $\langle z_1^k,\dots,z_c^k\rangle[1]$,

$$\check{\mathrm{C}}^{\bullet}_{\mathrm{alg}}((\mathrm{D}(z_i))_{i=1,\dots,c}, M^\bullet) := \varinjlim_k \mathcal{H}om_{\mathcal{O}_Y}(\mathrm{K}(z_1^k,\dots,z_c^k)_{\bullet\geq 1}, M^\bullet)[-1]$$

realizes $M(*X)$ *and can be considered as the subcomplex of the ordinary Čech complex of forms with only poles at all* $V(z_i)$*. The exact sequences of complexes* $0 \to (\mathrm{K}(z_1^k,\dots,z_c^k)_0 \to (\mathrm{K}(z_1^k,\dots,z_c^k)_\bullet \to \mathrm{K}(z_1^k,\dots,z_c^k)_{\bullet\geq 1} \to 0$ *for all* $k \in \mathbb{N}$ *induce an exact sequence of their limits, which represents the exact triangle*

$$M(*X)[-1] \longrightarrow \mathrm{R}\underline{\Gamma}_{[X]}(M) \longrightarrow M \xrightarrow{[1]} .$$

Remark 1.1.5. *Frequently we will use the notation* $M^\bullet(*Z)$ *for* $\underline{\Gamma}_{[Z|Y]}(M^\bullet)$.

To introduce the concept of regularity, we need first the notion of the *de Rham complex* of a complex of $\mathcal{D}$-modules:

Definition 1.1.6. $\mathrm{DR}(M^\bullet) := \Omega_Y^\bullet(M^\bullet)[n] \in \mathrm{D^b}(\mathbb{C}_Y)$

Definition 1.1.7. *A coherent* $\mathcal{D}_Y$*-**module*** M *is **regular** iff* $\mathrm{DR}(\mathrm{R}\underline{\Gamma}_{[Y|Z]}(M)) = \mathrm{R}j_*j^*(\mathrm{DR}(M)) \in \mathrm{D^b}(\mathbb{C}_Y)$ *for all closed analytic subsets* Z*; where* $j : (Y - Z) \to Y$ *is the inclusion of the complement.* M *is said to be **regular holonomic**, iff it is regular and holonomic.*

Definition 1.1.8. *A **complex*** $M^\bullet \in \mathrm{D^b}(\mathcal{D}_Y)$ *is called **regular (regular holonomic)** if* $\mathrm{DR}(\mathrm{R}\underline{\Gamma}_{[Y|Z]}(M^\bullet)) = \mathrm{R}j_*j^*\,\mathrm{DR}(M^\bullet) \in \mathrm{D^b}(\mathbb{C}_Y)$ *for all* j *as above; that is if all cohomology sheaves are regular (and holonomic).*

The conditions to be regular, holonomic or regular holonomic define abelian subcategories of $\mathrm{M}(\mathcal{D}_Y)$ and $\mathrm{D^b}(\mathcal{D}_Y)$.

Definition 1.1.9. *Let* $\mathrm{RH}(\mathcal{D}_Y)$ *denote the abelian category of regular holonomic* $\mathcal{D}_Y$*-modules,* $\mathrm{D^b}\,\mathrm{RH}(\mathcal{D}_Y))$ *its derived category of bounded complexes and* $\mathrm{D_{rh}}(\mathcal{D}_Y)$ *the category of complexes of* $\mathcal{D}_Y$*-modules with regular holonomic cohomology sheaves.*

There are natural functors $\mathrm{RH}(\mathcal{D}_Y) \to \mathrm{D^b}\,\mathrm{RH}(\mathcal{D}_Y)) \to \mathrm{D^b_{rh}}(\mathcal{D}_Y)$*, the later is frequently called the 'realization functor' real.*

Example 1.1.10. *It is easy to see that for* M *to be regular, it suffices that* $\mathrm{DR}(\mathrm{R}\underline{\Gamma}_{[Y|Z]}(M)) = \mathrm{R}j_*j^*(\mathrm{DR}(M)) \in \mathrm{D^b}(\mathbb{C}_Y)$ *for all divisors* Z.

Hence the fact that $\Omega^\bullet(*X)[n] = \mathrm{DR}(\mathcal{O}(*X))$ *calculates the cohomology of the complement of* X *for all divisors* X *[Gro66] just expresses the regularity of* $\mathcal{O}_Y$.

1.1.2 Algebraic Case

A **complex algebraic variety** is a complex variety Y^{an} which is the associated complex variety of some reduced scheme Y of finite type over $\mathbb{C}$. The inclusion $\varphi : Y^{\mathrm{an}} \to Y$ is continuous and for any $\mathcal{O}_Y$-module M, there is the associated analytic object $M^{\mathrm{an}} := \varphi^*(M) = \mathcal{O}_{Y^{\mathrm{an}}} \otimes_{\mathcal{O}_Y} M^\bullet$ on Y^{an}. Comparison of M^{an} with $M^{\mathrm{alg}} := \varphi^{-1}(M)$ leads to the famous

Theorem (GAGA, [Ser56], [Gro71])**.** *Let Y denote a smooth proper scheme over* $\mathrm{Spec}(\mathbb{C})$*, M a coherent sheaf on Y, then* $\mathrm{H}^k(Y^{\mathrm{an}}, M^{\mathrm{an}}) = \mathrm{H}^k(Y, M)$*.*

$\mathcal{D}_Y$ is the sheaf of rings of differential operators with polynomial coefficients. $M^\bullet \in \mathrm{D^b}(\mathcal{D}_Y)$ is holonomic iff the associated analytic object $(M^{\mathrm{an}})^\bullet$ is. If Y is affine, then M is regular iff there is an embedding $j : Y \to \overline{Y}$ in some smooth projective scheme $\overline{Y}$ such that $\mathcal{O}_{\overline{Y}^{\mathrm{an}}} \otimes_{\mathcal{O}_{\overline{Y}}} \varphi_*(M^\bullet)$ is regular on $\overline{Y}^{\mathrm{an}}$.

If Y is not affine, then $M^\bullet$ is regular if it is locally regular, i.e. everywhere in affine charts in the sense above (cf. [Bor87], [Meb04]).

1.2 Constructible Complexes of Sheaves

For $M^\bullet$ holonomic, the **de Rham complex of $M^\bullet$**

$$\mathrm{DR}(M^\bullet) := \Omega_Y^\bullet(M^\bullet)[n] \in \mathrm{D^b}(\mathbb{C}_Y)$$

will fulfill a constructibility condition, which we now explain.

Let Y be a complex (algebraic) variety. An **(algebraic) stratification** of Y is a finite partition $Y = \cup_i Y_i$ of Y into locally closed smooth (algebraic) sub-varieties, called the strata such that the closure of any stratum is an union of strata. Let k be any field of characteristic zero. **A complex of sheaves of k-vector-spaces K on Y is said to be (algebraic) k-constructible** if there exists a (algebraic) stratification such that all cohomology sheaves $\mathcal{H}^i(K)$ are **(algebraic) k-constructible sheaves**, i.e. locally constant on each (algebraic) stratum in the analytic topology with finite dimensional stalks.

In both (analytic/algebraic) settings, we consider the fully triangulated sub-category

$$\mathrm{D^b_c}(k_Y) \subset \mathrm{D^b}(k_Y),$$

whose objects are bounded (algebraic) $\mathbb{C}$-constructible sheaf complexes. If $\mathrm{Constr}(k_Y)$ is the abelian category of all k-constructible sheaves, we have natural functors

$$\mathrm{Constr}(k_Y) \to \mathrm{D^b}(\mathrm{Constr}(k_Y)) \to \mathrm{D^b_c}(k_Y)$$

$\mathrm{D}^{\mathrm{b}}(\mathrm{Constr}(k_Y))$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(k_Y)$ are equipped with the **dualizing complex** $\mathbb{D}_Y$ and the associated **Verdier duality** functor

$$\mathbb{D} := \mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{C}_Y}(-, \mathbb{D}_Y) \ .$$

All we need to know about $\mathbb{D}$ and the functor $f^!$ for a morphism of varieties defined over k, $f : X \to Y$, are the following properties:

- **Verdier duality** $f^!$ is by definition the right adjoint of the functor of sections with proper support $f_! \sim Lf_!$ on the derived category:

$$\mathrm{R}f_* \, \mathrm{R}\mathcal{H}\mathrm{om}(A, f^!B) = \mathrm{R}\mathcal{H}\mathrm{om}(f_!A, B).$$

- **Functoriality** $f^!\mathbb{D}_Y = \mathbb{D}_X$. (In particular $\mathbb{D}_Y = \alpha^!(k_{pt})$; $\alpha : Y \to \mathrm{pt.}$)
- $\mathbb{D}_Y = \mathbb{C}_Y[2n]$ if Y is a smooth complex variety of complex dimension n.

1.3 Riemann-Hilbert Correspondence

First of all, the DR-functor constitutes a correspondence between *complexes* of regular holonomic $\mathcal{D}_Y$-modules and *complexes* of sheaves of constructible sheaves:

$$\mathrm{DR} : \mathrm{D}^{\mathrm{b}}\,\mathrm{RH}(\mathcal{D}_Y) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Constr}(\mathbb{C}_Y))$$

As it is a correspondence, $\mathrm{DR}\,\mathrm{RH}(Y)) \subset \mathrm{D}^b_c(\mathbb{C}_Y)$ must be an abelian subcategory of the latter such that $\mathrm{D}^{\mathrm{b}}(\mathrm{Constr}(\mathbb{C}_Y)) = \mathrm{D}^{\mathrm{b}}(\mathrm{DR}\,\mathrm{RH}(\mathcal{D}_Y)))$. It turns out that it is NOT the category of constructible sheaves $\mathrm{Constr}(\mathbb{C}_Y)$ that one might expect at first glance, and hence gives rise to a definition: The image $\mathrm{DR}\,\mathrm{RH}(\mathcal{D}_Y))$ is the category of $\mathbb{C}$- perverse sheaves on Y of middle perversity, which we denote by $\mathrm{Perv}(\mathbb{C}_Y)$.

N.B. 1.3.1. *A perverse sheaf is not a sheaf, but a sheaf complex.*

There is a notion of k-perverse sheaf for any field k with $char(k) = 0$:

Definition 1.3.2. *A k-constructible complex K on Y is a k-**perverse** sheaf if for all $i \in \mathbb{Z}$ one has support conditions*

$$\dim(\mathrm{supp}(\mathcal{H}^i(K))), \quad \dim(\mathrm{supp}(\mathcal{H}^i(\mathbb{D}(K)))) \leq -i.$$

Theorem 1.3.3. *Let Y be an algebraic variety. For any field k of characteristic zero, the k-perverse sheaves on Y form an abelian, artinian and noetherian full subcategory* $\mathrm{Perv}(k_Y)$ *of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(k_Y)$ *[BBJ83].*

So far we have explained the correspondence between the columns of

$$\begin{array}{ccccc}
\mathrm{RH}(\mathcal{D}_Y) & \xrightarrow{\mathrm{DR}} & \mathrm{Perv}(\mathbb{C}_Y) & & \\
{\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow & & \\
\mathrm{D^b}\,\mathrm{RH}(\mathcal{D}_Y) & \xrightarrow{\mathrm{DR}} & \mathrm{D^b}(\mathrm{Constr}(\mathbb{C}_Y)) & = & \mathrm{D^b}\,\mathrm{Perv}(\mathbb{C}_Y)
\end{array}$$

given by DR. Applying the DR-functor more generally to complexes of (apriori non regular holonomic) analytic $\mathcal{D}$-modules with regular holonomic cohomology, one obtains complexes of $\mathbb{C}$-modules with constructible cohomology; in this way, the correspondence above extends to the **Riemann-Hilbert correspondence, weak form**

Theorem 1.3.4. *The* DR*-functor induces the following equivalences of subcategories and cohomology functors*

$$\begin{array}{ccc}
\mathrm{RH}(\mathcal{D}_Y) & \xrightarrow{\mathrm{DR}} & \mathrm{Perv}(\mathbb{C}_Y) \\
{\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow \\
\mathrm{D^b}\,\mathrm{RH}(\mathcal{D}_Y) & \xrightarrow{\mathrm{DR}} & \mathrm{D^b}\,\mathrm{Perv}(\mathbb{C}_Y) \\
{\scriptstyle \mathrm{real}}\downarrow & & {\scriptstyle \mathrm{real}}\downarrow \\
\mathrm{D^b_{rh}}(\mathcal{D}_Y) & \xrightarrow{\mathrm{DR}} & \mathrm{D^b_c}(\mathbb{C}_Y) \\
{\scriptstyle \mathcal{H}^k}\downarrow & & {\scriptstyle {}^p\mathcal{H}^k}\downarrow \\
\mathrm{RH}(\mathcal{D}_Y) & \xrightarrow{\mathrm{DR}} & \mathrm{Perv}(\mathbb{C}_Y)
\end{array}$$

Corollary 1.3.5. *For* $M^\bullet \in \mathrm{D^b_{rh}}(\mathcal{D}_Y)$*, there are natural isomorphisms*

$$\begin{aligned}
\mathrm{DR}(\mathcal{H}^m_{[Z]}(M^\bullet)) &= {}^p\mathcal{H}^m\ \mathrm{R}\underline{\Gamma}_Z(\mathrm{DR}(M^\bullet)) =: {}^p\mathcal{H}^m_Z(\mathrm{DR}(M^\bullet)), \\
\mathrm{DR}(\mathcal{H}^m_{[Y|Z]}(M^\bullet)) &= {}^p\mathcal{H}^m \mathrm{R}j_*j^*(\mathrm{DR}(M^\bullet)).
\end{aligned}$$

Here, i.e. over $\mathbb{C}$, ${}^p\mathcal{H}^k$ could be defined by this correspondence and extends the natural cohomology functor of $\mathrm{D^b}(\mathrm{Perv}(\mathbb{C}_Y))$. The adequate language to define perverse sheaves and the functors ${}^p\mathcal{H}^k$ for arbitrary fields of characteristic zero (and arbitrary perversities) by support-conditions as above, is the formalism of t-structures on triangulated categories, which can be found in [BBJ83], [Dim04].

The full and precise statement in these terms is [Meb89]

Theorem 1.3.6 (Riemann-Hilbert correspondence). *Consider the triangulated category $\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_Y)$, endowed with the natural t-structure, and the triangulated category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{C}_Y)$, endowed with the middle perversity t-structure. Then the de Rham functor*

$$\mathrm{DR} : \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_Y) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{C}_Y); \qquad M \mapsto \Omega^{\bullet}_Y(M)[n]$$

is t-exact and establishes an equivalence of categories, which commutes with duality and the six standard operators, which are $f_*, f^*, f_!, f^!$ for a morphism of smooth varieties $f : X \to Y$ and the nearby-cycle functor and vanishing-cycle functor φ, ψ.

Note that in this generality, the perverse cohomology functor is given by the truncation functors of the perverse t-structure by ${}^p\mathcal{H}^k := {}^p\tau_{\leq 0}{}^p\tau^{\geq 0} \circ (-)[k]$. Let us conclude with a remark on Riemann Hilbert correspondence and GAGA:

Remark 1.3.7 (GAGA). *Complexes of analytic $\mathcal{D}_Y$-modules, which underly algebraic $\mathcal{D}_Y$-modules, correspond to algebraic $\mathbb{C}$-constructible sheaf-complexes.*

In the algebraic setup, the realization functors real in remark 1.3.4 are isomorphisms, i.e. $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{C}_{Y^{\mathrm{alg}}})$ *can be considered as* $\mathrm{D}^{\mathrm{b}}(\mathrm{Perv}(\mathbb{C}_{Y^{\mathrm{alg}}}))$ *(and hence as* $\mathrm{D}^{\mathrm{b}}(\mathrm{Constr}(\mathbb{C}_{Y^{\mathrm{alg}}}))$*).*

1.4 Intersection Cohomology

Recall from the previous section that $\mathrm{Perv}(k_Y)$ is artinian and noetherian for any field k of characteristic zero. In particular, every perverse sheaf has a finite composition series and thus the question arises: what are the simple objects?

Let V be an irreducible smooth locally closed subvariety of Y, $d := \dim V$, $X := \overline{V}$ and L a local system of k-vectorspaces on V corresponding to an irreducible representation of $\pi_1(V)$. Then $L[d]$ is a simple perverse sheaf on V and we denote by

$$\begin{aligned} \mathrm{IC}_X(L, V) &:= {}^p j_{*!} L[d] \\ &= \mathrm{Im}({}^p j_! L[d] \to {}^p j_* L[d]) \in \mathrm{Perv}(k_X) \ ; \end{aligned}$$

(where ${}^p j_! = {}^p\mathcal{H}^0(j_!)$, ${}^p j_* = {}^p\mathcal{H}^0(j_*), j : V \to X$) its **intermediary extension** to X. The result will be a simple perverse sheaf on X. The same is true for $i_*(\mathrm{IC}_X(V,L)) \in \mathrm{Perv}(k_Y)$; $i : X \to Y$ by the t-exactness of i_*. In fact, any simple perverse sheaf on Y can be obtained in this way!

These **intersection cohomology sheaves** are characterized among the perverse sheaves by the additional support conditions

$$\dim(\mathrm{supp}(\mathcal{H}^i(K))),\ \dim(\mathrm{supp}(\mathcal{H}^i(\mathbb{D}(K))) < -i \quad \text{for } i > -d.$$

Given an explicit analytic or algebraic stratification of Y,

$$(Y^0 = Y) \supset Y^1 \supset \cdots \supset (Y^{n+1} = \emptyset)$$

such that the strata $S^c := Y^c \setminus Y^{c+1}$ are empty or smooth of complex codimension c and S^0 dense with $V = X \cap S^0$, one can represent $i_*(\mathrm{IC}_X(V,L)) \in \mathrm{Perv}(k_Y)$ in terms of the inclusions of the complements $U_c := Y^0 \setminus Y^c$

$$(U_0 = \emptyset) \overset{j_1}{\subset} U_1 \overset{j_2}{\subset} \cdots \overset{j_{n+1}}{\subset} (U_{n+1} = Y); \quad V \overset{j}{\subset} Y$$

as

$$\begin{aligned} i_*(\mathrm{IC}_X(V,L)) &= {}^p j_{*!}(i|_{V*}(L[d])) \\ &= (\tau_{\leq -1} j_{n+1\ *}) \circ (\tau_{\leq -2} j_{n\ *}) \circ \cdots \circ (\tau_{\leq -n-1} j_{1\ *})(i|_{V*}(L[d]))\ . \end{aligned} \tag{1.1}$$

Definition 1.4.1. *For any irreducible variety X, the intersection complex $\mathrm{IC}_X(X_{\mathrm{reg}}, k)$ is denoted simply by $\mathrm{IC}_X(k)$; X_{reg} the smooth locus of X.*

In particular if X is smooth, then $\mathrm{IC}_X(k) = k_X[\dim X]$ so that the following definition is a generalization of the ordinary sheaf cohomology with values in k_X:

Definition 1.4.2 (intersection cohomology)**.**

$$\mathrm{IH}^q(X,k) := \mathbb{H}^q(X, \mathrm{IC}_X(k)[\dim_{\mathbb{C}}(X)])$$

Chapter 2

Verdier-Duality

2.1 Self-Duality

By Verdier-duality (cf. p. 5),

$$\mathrm{id} \in \mathbb{H}^0(X, \mathrm{R}\mathcal{H}\mathrm{om}(\alpha^! k_{pt}, \alpha^! k_{pt})) = \mathrm{H}^0\, \mathrm{R}\mathcal{H}\mathrm{om}(\alpha_! \alpha^! k_{pt}, K) = \mathbb{H}^0_c(X, \mathbb{D}_X)^\vee$$

corresponds to a morphism $\mathrm{tr} : \mathbb{H}^0_c(X, \mathbb{D}_X) \to k$ such that for any complex A, the composition

$$\begin{array}{ccccc} \mathbb{H}^q(X,A) \times \mathbb{E}xt_k^{-q}(A, \mathbb{D}_X) & \xrightarrow{\mathrm{ev}} & \mathbb{H}^0(X, k_X[2d]) & \xrightarrow{\mathrm{tr}} & k \\ {\scriptstyle =}\uparrow & & {\scriptstyle =}\uparrow & & {\scriptstyle =}\uparrow \\ \mathbb{H}^q(X,A) \times \mathbb{H}^{-q}(X, \mathbb{D}(A)) & \xrightarrow{\smile} & \mathbb{H}^0(X, \mathbb{D}_X) & \xrightarrow{\mathrm{tr}} & k \end{array}$$

defines a nondegenerate pairing ($d = \dim X$).

If A is self-dual, i.e. naturally isomorphic to $\mathbb{D}(A)$, we get a cup-product pairing on the cohomology groups of A. The prototype of this situation is, of course, Poincare duality on a smooth subvariety $X \subset Y$, where $A = i_* \mathbb{C}_X[d]$, $\mathbb{D}_Y = \mathbb{C}_Y[2d]$ and $\mathbb{D}A = \mathcal{H}^{n-d}_X(\mathbb{C}[d])$. The identification $A \sim \mathbb{D}(A)$ in this case is the Thom class, which is cup product with the fundamental class of X in $\Gamma(Y, \mathcal{H}^{n-d}_X(\mathbb{C}))$ [Ive86].

A simple observation guaranties that this self-duality goes over to the intersection complex of an arbitrary variety X: If U is the regular locus of X, $\mathbb{D}(\mathrm{IC}_X)$ must be a simple constructible sheaf complex such that $\mathbb{D}(\mathrm{IC}_X)|U$ is naturally isomorphic to $\mathrm{IC}_X(U)$ by the above, hence it must coincide with ${}^p j_{!*}(\mathbb{D}(\mathrm{IC}_X)|U) = {}^p j_{!*}(\mathrm{IC}_X\,|U) = \mathrm{IC}_X$. In particular, there is a pairing on the intersection cohomology groups as above and our overall goal will be to calculate it explicitly. The first step is now to elaborate an appropriate explicit form of the isomorphism between IC_X and $\mathbb{D}(\mathrm{IC}_X)$ induced from the

Thom class, this requires some more conceptional work on duality operations on de Rham complexes.

2.2 Differential Operators as Morphisms

Let us start with an example for motivation:

Example 2.2.1. *If X is smooth or a normal crossing divisor on a smooth variety Y, $\Omega_Y^\bullet(\log X)[n]$ and $\mathrm{DR}(\mathcal{O}(*X)) = \Omega_Y^\bullet(*X)[n]$ are two representatives of $j_*j^*\mathbb{C}_Y[n] \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{C}_Y)$.*

The Verdier dual of $j_*j^*\mathbb{C}_Y[n] \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{C}_Y)$ is $j_!j^*\mathbb{C}_Y[n]$, and is represented by the de Rham complex of the dual $\mathcal{D}_Y$-module of $\mathcal{O}(*X)$,

$$\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Y}(\mathcal{O}(*X), \Omega^n[n] \otimes D_Y).$$

The sheaves in the Verdier dual of the log*-complex $\mathrm{R}\mathcal{H}\mathrm{om}_{\mathbb{C}_Y}(\Omega^\bullet(\log X), \Omega^\bullet[2n])$ are neither coherent as $\mathcal{O}_Y$-modules, nor have a structure as $\mathcal{D}_Y$ modules at all, and a complex like $\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(\Omega^\bullet(\log X), \Omega^n[n])$ is not defined because the* log*-complex is not an element of the derived category of $\mathrm{M}(\mathcal{O}_Y)$ as the differentials are not $\mathcal{O}_Y$-linear.*

Nevertheless, there is a representative of the Verdier-dual with $\mathcal{O}_Y$-coherent sheaves of the expected form: The dual complex

$$\mathbb{D}j_*j^*\mathbb{C}_Y[n] = j_!j^*\mathbb{C}_Y[n] = \mathrm{Cone}(\mathbb{C}_Y[n] \to i_*i^*\mathbb{C}_Y[n])$$

is given in terms of differential forms explicitly as $\ker(\Omega_Y^\bullet \to \Omega_X^\bullet) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{C}_Y)$, which is represented by (an injective resolution of) the sheaves of kernels $\Omega_Y^\bullet(\log X)(-X)$, with differential induced from the inclusion in $\Omega_Y^\bullet$. And this is indeed the complex with sheaves

$$\Omega_Y^p(\log X)(-X) = \mathrm{Hom}_{\mathcal{O}_Y}(\Omega^{n-p}(\log X), \Omega^n[n])$$

and differentials adjoint to those of the log*-complex. The pairing to $\Omega^n[n]$ is likewise given by wedge product or evaluation.*

The aim of this chapter is to present the ideas of M. Saito [Sai88], [Sai89a], [Sai90b] and to introduce the category of differential complexes $\mathrm{D}^{\mathrm{b}}(\mathcal{O}_Y, \mathrm{Diff})$ of complexes of $\mathcal{O}_Y$-modules with differential operators as morphisms. This category will be isomorphic to the derived category of $\mathcal{D}_Y$-modules and is equipped with a duality functor, preserving $\mathcal{O}_Y$-coherence. For convenience, he uses the category $\mathrm{M}(\mathcal{D}_Y^o)$ of right $\mathcal{D}_Y$-modules and its bounded derived category $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y^o)$, which are equivalent to the appropriate categories of left-modules.

$\mathrm{D^b}(\mathcal{O}_Y, \mathrm{Diff})$ is the category in which one can describe in a conceptional way the Verdier dual for any constructible sheaf complex given by the de Rham complex of any $\mathcal{D}_Y$-module, or by the log complex of a normal crossing divisor.

First of all, we need the notion of a differential operator between $\mathcal{O}_Y$-modules: Let Δ denote the ideal-sheaf of the diagonal in $Y \times Y$ and for each $k \in \mathbb{N}$, let π_1, π_2 denote the projection of $\Delta_k := \mathrm{Spec}(\mathcal{O}_{Y\times Y}/(\Delta^{k+1}))$ to the first and second factor, and let

$$\mathrm{P}^k(L) := \pi_{1*}\pi_2^*(L)$$

be the module of k-jets in L.

Reflection at the diagonal induces $\mathbb{C}$-linear isomorphisms

$$\begin{array}{rccc} & \pi_1^{-1}L & \to & \pi_2^{-1}L \ , \\ \sigma : & \pi_1^* L & \to & \pi_2^* L \end{array}$$

and thereby an $\mathcal{O}_Y$-linear isomorphism

$$\begin{aligned} \pi_{1*}\pi_2^*(L) &= \pi_{1*}(\mathcal{O}_{\Delta_k} \otimes_{\pi_2^{-1}\mathcal{O}_Y} \pi_2^{-1}(L) \\ &=_\sigma \ \pi_{1*}(\mathcal{O}_{\Delta_k} \otimes_{\pi_1^{-1}\mathcal{O}_Y} \pi_1^{-1}(L) \\ &= \pi_{1*}(\mathcal{O}_{\Delta_k}) \otimes_{\mathcal{O}_Y} L \ ; \end{aligned}$$

i.e.

$$\mathrm{P}^k(L) = \mathrm{P}^k(\mathcal{O}_Y) \otimes L \ ;$$

where $\mathrm{P}^k(\mathcal{O}_Y)$ is locally isomorphic to $\mathrm{Sym}^{\leq k}(\Omega^1)$. Moreover, σ induces the *universal differential operator* $\mathrm{D}_{L,k}$ of order $k \in \mathbb{N}$ on an $\mathcal{O}_Y$-module L: $\mathrm{D}_{L,k} = \sigma \circ adj$

$$\mathrm{D}_{L,k} : L \xrightarrow{adj} \pi_{1*}\pi_1^*(L) \xrightarrow{\sigma} \pi_{1*}\pi_2^*(L) = \mathrm{P}^k(L) \ .$$

Definition 2.2.2. *Let L, L' be $\mathcal{O}_Y$-modules. A* ***differential operator P of order k∈ℕ, from L to L'*** *is a map that factorizes over $D_{L,k}$, followed by an $\mathcal{O}_Y$-linear morphism $\widetilde{P}$ from $\mathrm{P}^k(L)$ to L'.*

For $k \in \mathbb{N}$ let

$$\mathrm{F}_k \, \mathrm{Diff}(L, L')$$

denote the group of differential operators of order k from L to L' and

$$\mathrm{Hom}_{\mathrm{Diff}}(L, L')^f := \bigcup_{k\geq 0} \mathrm{F}_k \, \mathrm{Diff}(L, L')$$

denote the group of differential operators from L to L'.

Example 2.2.3. *Let* $\mathbb{A}^n = \operatorname{Spec}\mathbb{C}[x_1, \ldots, x_n]$ *and* $P = \sum_{|I|=0}^{k} a_I \frac{\partial}{\partial x_I}$ *be a differential operator of oder* k *in the usual sense.*

We have

$$\mathrm{P}^k\mathcal{O}_{\mathbb{A}^n} = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]/\langle dx_1, \ldots, dx_n\rangle^{k+1};$$

where $dx_i := y_i - x_i$*, which as* $\mathcal{O}_{\mathbb{A}^n}$*-module is free with basis*

$$B := (dx_I)_{|I|\leq k} \ ;$$

where $dx_I := dx_{i_1} \cdot \ldots \cdot dx_{i_l}$ *for* $I = i_1, \ldots, i_l$*.*

Let $(\partial_I)_{|I|\leq k}$ *denote the dual basis to* B *and define* $\widetilde{P} := \sum_{|I|=0}^{k} a_I\partial_I$*. Then, for all* $f \in \mathbb{C}[x_1, \ldots, x_n]$*,*

$$D_{\mathcal{O}_{\mathbb{A}^n},k}(f(x)) = f(y) = \sum_I \frac{1}{I!}\frac{\partial}{\partial x_I} f(x)dx_I$$

by Taylor expansion, hence $P(f) = \widetilde{P}(D_{\mathcal{O}_{\mathbb{A}^n},k}(f(x))) \in \mathrm{F}_k \operatorname{Diff}(\mathcal{O}, \mathcal{O})$*.*

Definition 2.2.4 ([Sai89a])**.** *For* $\mathcal{O}$*-modules* L*,* L'*, Saito defines the* ***sheaf of differential morphisms*** $\mathcal{H}om_{\mathrm{Diff}}(L, L')$ *from* L *to* L' *to be the image of the (injective) map*

$$\mathcal{H}om_{\mathcal{D}_Y^\circ}(L \otimes \mathcal{D}_Y, L' \otimes \mathcal{D}_Y) \to \mathcal{H}om_{\mathbb{C}_X}(L, L')$$

Differential operators are differential morphisms of finite order in the following sense: By definition a differential operator of order k is a global section of

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}}(\mathrm{P}^k(L), L') &= \mathcal{H}om_{\mathcal{O}}(L \otimes \mathrm{P}^k(\mathcal{O}), L') \\ &= \mathcal{H}om_{\mathcal{O}}(L, \mathrm{Hom}_{\mathcal{O}}(\mathrm{P}^k(\mathcal{O}), L')) \\ &= \mathcal{H}om_{\mathcal{O}}(L, L' \otimes F_k\mathcal{D}_Y) \ ,\end{aligned}$$

which is contained in

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}}(L, L' \otimes \mathcal{D}_Y) &= \mathcal{H}om_{\mathcal{D}_Y^\circ}(L \otimes \mathcal{D}_Y, L' \otimes \mathcal{D}_Y) \\ &= \mathcal{H}om_{\mathrm{Diff}}(L, L') \ .\end{aligned}$$

Definition 2.2.5. *[Sai89a]*

- $\mathrm{M}(\mathcal{O}_Y, \mathrm{Diff})^f$ *is the category of* $\mathcal{O}_Y$*-modules with differential operators as morphisms and* $\mathrm{D}^{\mathrm{b}}(\mathcal{O}_Y, \mathrm{Diff})^f$ *the derived category of bounded complexes in* $\mathrm{M}(\mathcal{O}_Y, \mathrm{Diff})^f$*.*

- $\mathrm{M}(\mathcal{O}_Y, \mathrm{Diff})$ *is the category of* $\mathcal{O}_Y$*-modules with differential morphisms as morphisms and* $\mathrm{D}^{\mathrm{b}}(\mathcal{O}_Y, \mathrm{Diff})$ *the derived category of bounded complexes in* $\mathrm{M}(\mathcal{O}_Y, \mathrm{Diff})$*.*

The functor

$$\widetilde{\mathrm{DR}}^{-1} : \mathrm{M}(\mathcal{O}_Y, \mathrm{Diff}) \to \mathrm{M}(\mathcal{D}_Y^o);\ L \mapsto L \otimes_{\mathcal{O}_Y} \mathcal{D}_Y = \mathrm{Hom}_{\mathrm{Diff}}(\mathcal{O}_Y, L)$$

will induce an equivalence of categories between $\mathrm{M}(\mathcal{O}_Y, \mathrm{Diff})$ and its image

$$\mathrm{M}_i(\mathcal{D}_Y^o) := \{M | M = L \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \text{ for some } \mathcal{O}_Y - \text{ module } L\} \subset \mathrm{M}(\mathcal{D}_Y^o)$$

which is a full sub-category of the abelian category $\mathrm{M}(\mathcal{D}_Y^o)$ of right $\mathcal{D}_Y$-modules, called the category of induced $\mathcal{D}_Y$-modules.

Then $\widetilde{\mathrm{DR}}^{-1}$ extends to an equivalence

$$\widetilde{\mathrm{DR}}^{-1} : \mathrm{D}^{\mathrm{b}}(\mathcal{O}_Y, \mathrm{Diff}) \to \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y^o);$$

as every right $\mathcal{D}_Y$-module M has a standard resolution

$$M \otimes_{\mathcal{O}_Y} \Lambda^\bullet \Theta_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \ \in\ \mathrm{D}^{\mathrm{b}}(\mathcal{O}_Y, \mathrm{Diff})$$

and for any complex of $\mathcal{D}_Y^o$-modules $M^\bullet$ $\widetilde{\mathrm{DR}}(M^\bullet)$ is a complex of $\mathrm{M}(\mathcal{O}_Y, \mathrm{Diff})$ because the differential of $M^\bullet$ is $\mathcal{D}_Y$- hence $\mathcal{O}_Y$-linear, and can be considered a differential operator of order 0 and that of $\mathrm{DR}(M^q)$ is of order 1.

The inverse is given explicitly by

$$\widetilde{\mathrm{DR}} : \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y^o) \to\ \mathrm{D}^{\mathrm{b}}(\mathcal{O}_Y, \mathrm{Diff});\ M^\bullet \mapsto M^\bullet \otimes_{\mathcal{O}_Y} \Lambda^\bullet \Theta_Y\ .$$

He proves moreover that

Proposition 2.2.6. *[Sai89a]*

$$\widetilde{\mathrm{DR}} : \mathrm{D}^{\mathrm{b}}(\mathcal{D}_Y^o)^f \to\ \mathrm{D}^{\mathrm{b}}(\mathcal{O}_Y, \mathrm{Diff})$$

is an equivalence of categories.

Recall that $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X^o)$ (resp. $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_Y^o)$) are the categories of complexes of right $\mathcal{D}_Y$-modules with coherent (resp. holonomic) cohomology sheaves. By the equivalences above, one can define

Definition 2.2.7. *[Sai89a]*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_Y, \mathrm{Diff})^f := \widetilde{\mathrm{DR}}^{-1}(\mathrm{D}^b_{\mathrm{coh}}(\mathcal{D}_X^o)),$$

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{O}_Y, \mathrm{Diff})^f := \widetilde{\mathrm{DR}}^{-1}(\ \mathrm{D}^b_{\mathrm{hol}}(\mathcal{D}_X^o))$$

and similarly for $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_Y, \mathrm{Diff})$ *and* $\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{O}_Y, \mathrm{Diff})$*.*

For $L^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_Y, \mathrm{Diff})^f$, $M^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{D}_Y)$, the dual is defined by

$$\mathbb{D}L^\bullet = \mathcal{H}om^f_{\mathrm{Diff}}(L^\bullet, \widetilde{\mathrm{DR}}(K^\bullet_Y)) \in (\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{O}_Y, \mathrm{Diff})^f)^{\mathrm{op}}$$
$$\mathbb{D}M^\bullet = \mathcal{H}om^f_{\mathrm{Diff}}(\widetilde{\mathrm{DR}}(M^\bullet), K^\bullet_Y) \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(D_Y)^{\mathrm{op}} \ .$$

This definition of the dual of a complex of $\mathcal{D}_Y$-modules with coherent cohomology coincides with the classical one:

Proposition 2.2.8. *[Sai89a] For $M^\bullet \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}^o_Y)$, the natural morphism*

$$\mathbb{D}M^\bullet \to \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(\widetilde{\mathrm{DR}}^{-1}\widetilde{\mathrm{DR}}(M^\bullet), \widetilde{\mathrm{DR}}^{-1}K^\bullet_Y) = \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}(M^\bullet, \omega[d_Y]\otimes_{\mathcal{O}_Y}\mathcal{D}_Y)$$

is a quasi-isomorphism.

Theorem 2.2.9 ([Sai89a]). *Let Y be a complex manifold. Let*

$$\mathrm{For} : \mathrm{D}^{\mathrm{b}}_{hol}(\mathcal{O}_Y, \mathrm{Diff})^f \to \ \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{C}_Y)$$

be the forgetful functor. Put

$$\mathrm{DR} := \mathrm{For} \circ \widetilde{\mathrm{DR}} : \mathrm{D}^{\mathrm{b}}_{hol}(\mathcal{D}_Y) \to \ \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{C}_Y).$$

Then for $L^\bullet \in \mathrm{D}^{\mathrm{b}}_{hol}(\mathcal{O}_Y, \mathrm{Diff})^f$, the natural morphism

$$\begin{aligned}\mathbb{D}L^\bullet &= \mathcal{H}om^f_{\mathrm{Diff}}(L^\bullet, \widetilde{\mathrm{DR}}(K^\bullet_Y))\\ &\to \mathcal{H}om_{\mathbb{C}_Y}(\mathrm{For}(L^\bullet),\ \mathrm{For} \circ \widetilde{\mathrm{DR}}(K^\bullet_Y)) = \mathbb{R}\mathcal{H}om_{\mathbb{C}_Y}(\mathrm{For}(L^\bullet), \mathbb{C}_Y[2d_Y])\\ &= \mathbb{D}\mathrm{For}(L^\bullet)\end{aligned}$$

is an isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{C}_Y)$, i.e. we get a natural equivalence

$$\mathrm{For} \circ \mathbb{D} = \mathbb{D} \circ \mathrm{For} :\ \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{O}_Y, \mathrm{Diff})^f \to \ \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbb{C}_Y)^{\mathrm{op}} \ .$$

2.3 Verdier- versus Serre-Duality

By the last theorem if $E^\bullet$ is a complex in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y, k)$ (i.e. with only k-linear differentials) such that each E^p is locally $\mathcal{O}_Y$-free of finite rank and the differentials are given by differential operators (for example $E^\bullet = \mathrm{P}^k\Omega^\bullet(*X)$), then the vertical maps below

$$\begin{array}{ccc} \mathbb{H}^k(E^\bullet) \times \mathbb{E}xt^{-k}_{\mathbb{C}}(E^\bullet, \Omega^\bullet[2n]) & \longrightarrow & \mathbb{H}^0(\Omega^\bullet[2n]) \\ \uparrow & & \uparrow \\ \mathbb{H}^k(E^\bullet) \times \mathrm{H}^{-k}\mathcal{H}om_{\mathrm{Diff}}(E^\bullet, \widetilde{\mathrm{DR}}(K^\bullet_Y)) & \longrightarrow & \mathbb{H}^0(\Omega^n[n]) \end{array} \qquad (2.1)$$

are isomorphisms. For Y smooth projective, we want to work out precisely how Verdier- and Serre-duality are related in that case:

$$\begin{aligned}\mathcal{H}om_{\mathrm{Diff}}(E^\bullet, \widetilde{\mathrm{DR}}(K_Y^\bullet)) &= \mathcal{H}om_{\mathcal{D}_Y}(E^\bullet \otimes_{\mathcal{O}_Y} \mathcal{D}_Y, \widetilde{\mathrm{DR}}(K_Y^\bullet) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y) \\ &= \mathcal{H}om_{\mathcal{D}_Y}(E^\bullet \otimes_{\mathcal{O}_Y} \mathcal{D}_Y, \Theta^\bullet \otimes K_Y^\bullet \otimes_{\mathcal{O}_Y} \mathcal{D}_Y) \\ &= \Theta^\bullet \otimes \mathcal{H}om_{\mathcal{D}_Y}(E^\bullet \otimes_{\mathcal{O}_Y} \mathcal{D}_Y, K_Y^\bullet \otimes_{\mathcal{O}_Y} \mathcal{D}_Y) \\ &= \mathrm{DR}(\mathcal{H}om_{\mathcal{D}_Y}(E^\bullet \otimes_{\mathcal{O}_Y} \mathcal{D}_Y, K_Y^\bullet \otimes_{\mathcal{O}_Y} \mathcal{D}_Y)) \\ &= \mathrm{DR}(\mathrm{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Y}(E^\bullet \otimes \mathcal{D}_Y, \Omega^n[n] \otimes_{\mathcal{O}_Y} \mathcal{D}_Y))\end{aligned}$$

represents the DR-complex of the dual $\mathcal{D}_Y$-module of the induced right $\mathcal{D}_Y$-module $E^\bullet \otimes \mathcal{D}_Y$. Moreover for all k,

$$\begin{aligned}\mathcal{H}om^p_{\mathcal{D}_Y}(E^k \otimes_{\mathcal{O}_Y} \mathcal{D}_Y, K_Y^\bullet \otimes_{\mathcal{O}_Y} \mathcal{D}_Y) &= \mathcal{H}om_{\mathcal{O}_Y}(E^k, K_Y^\bullet \otimes_{\mathcal{O}_Y} \mathcal{D}_Y) \\ &= \mathcal{H}om_{\mathcal{O}_Y}(E^k, \mathcal{O}_Y) \otimes (K_Y^\bullet \otimes_{\mathcal{O}_Y} \mathcal{D}_Y)\end{aligned}$$

which, as complex of $\mathcal{D}_Y$-modules, is an injective resolution of the single $\mathcal{D}_Y$-module

$$\mathcal{H}om_{\mathcal{O}_Y}(E^k, \mathcal{O}_Y) \otimes (\Omega^n[n] \otimes_{\mathcal{O}_Y} \mathcal{D}_Y),$$

which is nothing but

$$\mathcal{H}om_{\mathcal{O}_Y}(E^k, \Omega^n[n]) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$$

because E^k is locally free, whose $\widetilde{\mathrm{DR}}$-complex is quasi-isomorph to the single sheaf $\mathcal{H}om_{\mathcal{O}_Y}(E^k, \Omega^n[n])$.

All together, we get that there is a natural quasi-isomorphism given by inclusion between $\mathbb{D}(E^\bullet) = \mathcal{H}om_{\mathrm{Diff}}(E^\bullet, \widetilde{\mathrm{DR}}(K_Y^\bullet))$ and the complex ${}^tE^\bullet$ with sheaves

$${}^tE^p := \mathcal{H}om_{\mathcal{O}_Y}(E^{n-p}, \Omega^n[n]) \ ,$$

and differentials transposed to those of $E^\bullet$.

Of course because $\widetilde{\mathrm{DR}}(K_Y^\bullet)$ is a $\mathcal{O}_Y$-injective resolution of $\Omega^\bullet[2n]$, the map from theorem 2.2.9 is merely the map $\mathbb{D}(E^\bullet) \to Ext_{\mathbb{C}}(E^\bullet, \Omega^\bullet[2n])$ given by inclusion of sub-complexes $\Omega^n[n] \to \Omega^\bullet[2n]$ followed by the forgetful map. That this map must be an isomorphism is now a consequence from the existence of the natural Yoneda-pairings on both rows in (2.1). They are compatible with those stupid filtrations on the double complexes above with associated E_1-terms

$$\mathrm{E}_1^{pq} = \ \mathrm{H}^q\,\mathrm{R}\mathcal{H}\mathrm{om}^p_{\mathbb{C}}(E^\bullet, \Omega^\bullet[2n])$$

and

$$\mathrm{E}_1^{pq} = \mathrm{H}^q \mathcal{H}om_{\mathcal{O}}(E^{n-p}, \Omega^n[n])$$

respectively in the sense that if A_r, B_r, C_r are the E_r-terms occurring in one of these rows,

1. for each r, p, q, p', q' there is a bilinear pairing

$$\cdot : A_r^{p\ q} \times B_r^{p'q'} \to C_r^{p+p'\ q+q'} \ ,$$

which satisfies

$$d_r^{p+p'\ q+q'}(a \cdot b) = d_r^{p\ q}(a) \cdot b + (-1)^{p+q} a \cdot d_r^{p'\ q'} b \ ,$$

2. the pairing on level $r+1$ coincides with the pairing induced on the cohomology of level r ,

3. the Yoneda pairing on the abutment

$$A^n \times B^m \to C^{m+n} \ ;$$

where (A^n, B^m, C^{m+n}) is either $(\mathbb{H}^n(E^\bullet), \mathbb{E}xt^m_{\mathbb{C}}(E^\bullet, \Omega^\bullet[2n]))$ or $(\mathbb{H}^n(E^\bullet), \mathbb{H}^m(E^\bullet, \Omega^n t[n]))$ rsp., is compatible with the stupid filtrations in the sense that

$$\mathrm{F}^p(A^n) \times \mathrm{F}^q B^m \to \mathrm{F}^{p+q}(C^{m+n}) \ ,$$

4. the induced pairing on the associated graded objects

$$\mathrm{Gr}^p_{\mathrm{F}}(A^n) \times \mathrm{Gr}^q_{\mathrm{F}} B^m \to \mathrm{Gr}^{p+q}_{\mathrm{F}}(C^{m+n})$$

coincides with the pairings on the E_∞-term.

The pairing on the E_1-terms of the second row (i.e. the case $D = \Omega^n[n]$) in diagram (2.1) on page 14 is just Serre duality by definition, which is non-degenerate. That means, $(B_1, d_1) = \mathrm{Hom}((A_1, d_1), k)$ are dual complexes of finite dimensional vector-spaces (it follows from the definition of the Yoneda-pairing that the differentials are indeed adjoint) so that the E_2-term inherits a non-degenerate pairing which goes over by induction to the E_∞-terms by the regularity of the complexes.

But then between the E_∞-term in (2.1) the arrows in vertical direction must be isomorphisms, as it is true for the right hand side and the first factor on the left hand side. Again by induction and regularity, the same is true for the abutments.

Chapter 3

Local Considerations

3.1 A and $\widetilde{A}$

So far we know in the smooth case,

$$\begin{aligned} i_* i^! \mathbb{C}_Y[n+1] &= (\Omega^\bullet \to \Omega^\bullet(\log X))[n] \\ &= (\Omega^\bullet \to \Omega^\bullet(*X))[n] \end{aligned}$$

and for all p for the filtrations induced from stupid filtration and pole filtration

$$(\mathrm{F}^p(\Omega^\bullet) \to \mathrm{F}^p(\Omega^\bullet(\log X)))\,[n] \subset (\mathrm{P}^p(\Omega^\bullet) \to \mathrm{P}^p(\Omega^\bullet(*X)))\,[n]$$

are quis, in particular $\mathrm{P}^0(\Omega^\bullet \to \Omega^\bullet(*X))[n] = i_* i^! \mathbb{C}_Y[n+1]$ as a constructible sheaf-complex.

Definition 3.1.1. *For X an arbitrary hypersurface, let*

$$\begin{aligned} A &:= \mathrm{P}^0(\Omega^\bullet \to \Omega^\bullet(*X))[n] \\ &= \begin{pmatrix} \Omega^0(X) & \longrightarrow & \cdots & \longrightarrow & \Omega^n((n+1)X) \\ \uparrow & & & & \uparrow \\ \Omega^0 & \longrightarrow & \cdots & \longrightarrow & \Omega^n \end{pmatrix} [n+1] \end{aligned}$$

with induced filtration, i.e. $\mathrm{P}^p(A) = (\,\mathrm{P}^0(\Omega^\bullet) \to \mathrm{P}^0(\Omega^\bullet(*X)))[n]$ *for* $p < 0$ *and*

$$\mathrm{P}^p(A) = \begin{pmatrix} \Omega^0 & \longrightarrow & \cdots & \longrightarrow & \Omega^{p+1}(X) & \longrightarrow & \cdots & \longrightarrow & \Omega^n((n-p)X) \\ \uparrow & & & & \uparrow & & & & \uparrow \\ \Omega^0 & \longrightarrow & \cdots & \longrightarrow & \Omega^{p+1} & \longrightarrow & \cdots & \longrightarrow & \Omega^n \end{pmatrix} [n]$$

for $p \geq 0$.

A natural idea to define a 'dual filtration' P on the Verdier dual $\mathbb{D}(B) = \mathrm{Hom}_{\mathrm{Diff}}(B, \widetilde{\mathrm{DR}}(K_Y))$ of any filtered complex (B, P) would be to force

$$\begin{aligned}
\mathrm{P}^p\mathbb{D}(B) &= \{\varphi \in \mathbb{D}(B) | \varphi(\mathrm{P}^{k-p}(B)) \subset \mathrm{P}^k(\widetilde{\mathrm{DR}}(K_Y))) \quad \forall k\} \\
&= \{\varphi \in \mathbb{D}(B) | \varphi(\mathrm{P}^{n+1-p}(B)) \subset \mathrm{P}^{n+1}(\widetilde{\mathrm{DR}}(K_Y)) = 0\} \\
&= \mathbb{D}(B/\mathrm{P}^{n-p+1}B) \ .
\end{aligned}$$

One possibility to develop this idea consequently would be to study a filtered version of the derived category $\mathrm{D^b}(\mathcal{O}_Y, \mathrm{Diff})$ with filtered objects and all morphisms strict (to make filtered modules with filtered morphisms an abelian category and the filtered derived category the derived category of it). But pole-filtration apriori does not induce strict filtration on global cohomology groups: This indicates that either pole filtration would not exist in such a category, or there is no direct image functor f_*. Therefore, we leave it with pole-filtration on complexes, not classes of complexes.

Definition 3.1.2.

$$\widetilde{A} := \begin{pmatrix} \Omega^n(-X) & \longleftarrow & \cdots & \longleftarrow & \Omega^0((-n-1)X) \\ \downarrow & & & & \downarrow \\ \Omega^n & \longleftarrow & \cdots & \longleftarrow & \Omega^0 \end{pmatrix} [n] \ , \qquad (3.1)$$

For the individual complex $\widetilde{A}$ representing $\mathbb{D}(A) = \mathrm{Hom}_{\mathrm{Diff}}(A, \widetilde{\mathrm{DR}}(K_Y))$, the rule above leads to a filtration on $\widetilde{A}$ because the candidate for $\mathrm{P}^p(\widetilde{A})$,

$$\mathbb{D}(A/\mathrm{P}^{n-p+1}A) = \mathbb{D}(((\mathrm{P}^0(\Omega^\bullet) \to \mathrm{P}^0(\Omega^\bullet(*X)))/(\mathrm{P}^p(\Omega^\bullet) \to \mathrm{P}^p(\Omega^\bullet(*X))))\,[n])$$

$$= \mathbb{D}(\begin{pmatrix} \Omega^0(X) & \longrightarrow \cdots \longrightarrow & \Omega^{n-p+2}((n-p+1)X) & \longrightarrow \cdots \longrightarrow & \Omega^n((n+1)X) \\ \uparrow & & \uparrow & & \uparrow \\ \Omega^0 & \longrightarrow \cdots \longrightarrow & \Omega^{n-p+2}(X) & \longrightarrow \cdots \longrightarrow & \Omega^n((p-1)X) \end{pmatrix} [n])$$

$$= \begin{pmatrix} \Omega^n(-X) & \longleftarrow \cdots \longleftarrow & \Omega^{p-2}((-n+p-1)X) & \longleftarrow \cdots \longleftarrow & \Omega^0((-n-1)X) \\ \downarrow & & \downarrow & & \downarrow \\ \Omega^n & \longleftarrow \cdots \longleftarrow & \Omega^{p-2}(-X) & \longleftarrow \cdots \longleftarrow & \Omega^0((1-p)X) \end{pmatrix} [n]$$

is represented by a sub-complex of $\widetilde{A}$, which we take as **definition for** $\mathbf{P^p(\widetilde{A})}$. Vice-versa $\mathbb{D}(\widetilde{A}, P) = (A, P)$ gives back the original filtration on A by the analogous calculation.

Lemma 3.1.3. *Let $X \subset Y$ be an arbitrary hypersurface. The cup-product with the fundamental-class $c_{X|Y} \in \Gamma(Y, \mathcal{H}^1_X(\mathbb{C}_Y))$ defines a global morphism $\widetilde{A} \to A$ compatible with the filtrations such that locally*

$$\mathrm{Cone}(\mathrm{Gr}^p_{\mathrm{P}}) = \mathrm{Cone}\left(\mathrm{Gr}^p_{\mathrm{P}}(\widetilde{A})) \stackrel{2\pi i c_{X|Y}}{\longrightarrow} \mathrm{Gr}^p_{\mathrm{P}}(\mathbb{D}(A))\right) = K^\bullet(df/f) \otimes N^{1-p}[n]$$

as a complex of sheaves, where f is a local equation of X in Y and $K^\bullet(df/f)$ denotes the Koszul cochain-complex of $df/f \in \Omega^1_Y \otimes N = (\Theta_Y \otimes N^{-1})^*$ (cf. p. 73),

$$N^k = \begin{cases} (I_X/I_X^2)^{\otimes -k} & ; k \le -1 \\ \mathcal{O}_X & ; k = 0 \\ ((I_X/I_X^2)^\vee)^{\otimes k} & ; k \ge 1 \ ; \end{cases}$$

with I_X the ideal sheaf of X in Y.

Proof. For all $s \in \mathbb{Z}$ and differential forms ω,

$$d(\omega \cdot f^s) = f^s d\omega + (-s) df/f \wedge (\omega \cdot f^s) \equiv (-s) \cdot df/f \wedge (\omega \cdot f^s)$$

modulo terms of lower pole-order. Hence if we define isomorphisms

$$\varphi_{k,l} : \Omega^k \otimes N^l \to \Omega^k \otimes N^l$$

by

$$\varphi_{k,l} := \begin{cases} (-l-1)! & l < 0 \\ 1 & k = 0 \\ (-1)^{(l-1)}(l-1)! & l > 0 \end{cases}$$

there is a commutative diagram

$$\begin{array}{ccccccccccc}
(\Omega^0 \otimes N^{1-p} & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^p \otimes N^0) & \xrightarrow{df/f} & (\Omega^{p+1} \otimes N & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^n \otimes N^{n-p}) \\
\varphi_{0,1-p}\uparrow & & & & \varphi_{p,0}\uparrow & & \varphi_{p+1,1}\uparrow & & & & \varphi_{n,n-p}\uparrow \\
(\Omega^0 \otimes N^{1-p} & \xrightarrow{df/f} & \cdots & \xrightarrow{df/f} & \Omega^p \otimes N^0) & \xrightarrow{df/f} & (\Omega^{p+1} \otimes N & \xrightarrow{df/f} & \cdots & \xrightarrow{df/f} & \Omega^n \otimes N^{n-p})
\end{array} .$$

The first row is the cone of $df/f : \mathrm{Gr}^p_{\mathrm{P}}(\widetilde{A})[-n] \to \mathrm{Gr}^p_{\mathrm{P}}(A)[-n]$, the second is $K^\bullet(df/f) \otimes N^{1-p}$. This is the claim. $\square$

Corollary 3.1.4. *If X is a smooth hypersurface of a smooth n-dim variety Y,*

- $\widetilde{A} = i_* i^* \mathbb{C}_Y[n], \quad A = i_* i^! \mathbb{C}_Y[n],$

- $c_{X|Y} : \widetilde{A} \to A$ *is a filtered quasi-isomorphism.*

- *The induced filtration on the cohomology groups of $\widetilde{A}$ is the Deligne Hodge filtration on* $\mathrm{H}^{n-1+k}(X, \mathbb{C}_X) = \mathrm{H}^k(Y, i_* i^* \mathbb{C}_Y[n-1])$.

Proof. Locally, from the commutativity of

$$\begin{array}{ccc} \mathcal{O}_X^n(1-d) & \xrightarrow{(f\frac{\partial}{\partial x_1}\cdots f\frac{\partial}{\partial x_n})} & \Theta_Y \otimes \mathcal{O}_X(-X) \\ {\scriptstyle (f_1\cdots f_n)}\big\downarrow & & {\scriptstyle df/f}\big\downarrow \\ \mathcal{O}_X & = & \mathcal{O}_X \ , \end{array}$$

we see that $\mathrm{K}(f_1, \dots, f_n)$ is the Koszul-complex of a regular sequence on $\mathcal{O}_X$ generating the unit ideal, hence exact as a sequence of $\mathcal{O}_X$-modules. N^{1-p} is locally free on X, therefore globally

$$\mathrm{Cone}(\mathrm{Gr}_{\mathrm{P}}(c_{X|Y})) = 0 \in \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X),$$

which is equivalent to $\mathrm{Gr}_{\mathrm{P}}(c_{X|Y})$ being quis. Because P is regular on $\widetilde{A}$ and $\mathbb{D}(A)$, this says $c_{X|Y}$ is filtered quis.

Moreover, for all $p \in \mathbb{Z}$,

$$\begin{aligned} \mathcal{H}^k \mathrm{Gr}_{\mathrm{P}}^p(\widetilde{A}) &= \mathcal{H}^k(\mathrm{K}(f_1, \dots, f_n) \otimes N^{1-p}[n]) \\ &= \begin{cases} \ker(df/f : \Omega^p \otimes \mathcal{O}_X \to \Omega^{p+1} \otimes N) & \\ = \Omega_Y^p \otimes \mathcal{O}_X / df \wedge \Omega_Y^{p-1} = i_* \Omega_X^p & ; \ k = p \\ 0 & ; \ \text{else} \ , \end{cases} \end{aligned}$$

so that by induction $\mathrm{P}^p(\widetilde{A})$, $\mathrm{P}^p(A)$ as sheaf-complexes on Y both represent

$$i_* \mathrm{P}^p \Omega_X^\bullet[n-1] \subset i_* \Omega_X^\bullet[n-1] \ ,$$

whose global cohomology is $\mathrm{P}^p \ \mathrm{H}^{n-1+k}(X, \mathbb{C}_X)$ by definition. □

3.2 Calculus

Definition 3.2.1. *For any subset S of $Y = \mathbb{P}^n$, denote by* $\mathrm{C}S := \pi^{-1}(S)$ *the* cone over S, *where $\pi : (\mathbb{C}^{n+1} - \{0\}) \to \mathbb{P}^n$ is the canonical projection.*

If $E := \sum z_i \frac{\partial}{\partial z_i}$ denotes the *Eulerfield* on $\mathrm{C}Y$, we can characterize with it p-forms on $\mathrm{C}Y$, which are induced from q, as follows:

$$\pi^{-1}\Omega_Y^p(d) = (\ker(\mathcal{L}_E - d \cdot id) \cap \ker i_E) \subset \pi^* \Omega_Y^p = \ker i_E \subset \Omega_{\mathrm{C}Y}^p$$

Geometrically $\mathcal{L}_E(\omega) = 0$ means ω is constant along the fibers of π. In the algebraic category, the Eigenspace decomposition of the action of $\mathcal{L}_E$ gives the grading on Ω^p_{CY} if we let x_i, dx_i be of degree 1 (this can be extended to all tensor-powers by $\deg(\frac{\partial}{\partial x_i}) := (-1)$). The formulan above yields

$$\Omega^p_Y(d) = \pi_*(\ker i_E \cap (\Omega^p_{\mathrm{CY}})_{(d)}),$$

when $\Omega^p_{\mathrm{CY}})_{(d)}$ denotes the summand of degree d with respect to this grading.

In this sense if $F_0, \ldots, F_r$ are homogeneous polynomials of degrees $\deg F_i = d_i$, for all $k \in \mathbb{N}$, $(F_0^k, \ldots, F_r^k)$ defines a global section of $\pi_*\pi^{-1} \oplus_i \mathcal{O}_Y(kd_i) = \oplus_i \mathcal{O}_Y(kd_i)$. Let $(e_0, \ldots, e_n)$ be a free basis of $\mathcal{O}_Y^{n+1}$ and consider the Koszul cochain-complex of sheaves

$$\mathrm{K}(F_0^k, \ldots, F_r^k)^{\bullet \geq 1} = (\Lambda^{\bullet \geq 1} \mathcal{O}_Y^{n+1}(k(d-1)),\ D := \sum_i F_i e_i)$$

as defined on page 73. Mapping e_i to $\frac{1}{F_i^k} e_i$ yields an isomorphism of the latter to

$$(\Lambda^{\bullet \geq 1} \mathcal{O}_Y^{n+1}(kX_i),\ C := \sum_i e_i); \qquad X_i := V_+(X_i)\ .$$

If $X_i := V_+(F_i)$, $D(F_i) := Y - X_i$, $i = 0, \ldots, n$, $M^\bullet$ a complex of $\mathcal{O}_Y$-modules and

$$\vec{\mathrm{K}}(F_0, \ldots, F_r)^\bullet := \varinjlim_k K_0(F_0^k, \ldots, F_r^k)^\bullet$$

then

$$\vec{\mathrm{K}}(F_0, \ldots, F_r)^{\bullet \geq 1}[1] \otimes M^\bullet$$

is via the evaluation

$$ev : \sum_{i_0 \cdots i_q} m_{i_0 \cdots i_q} e_{i_0} \wedge \ldots \wedge e_{i_q} \mapsto (i_0 \cdots i_q \mapsto m_{i_0 \cdots i_q})$$

isomorphic to the sub-complex of the ordinary algebraic alternating Čech **sheaf**-complex for the cover $\coprod \mathrm{D}_+(F_i)$ of $Y - V_+(F_0, \ldots, F_r)$, consisting of section with only poles along the $X_i := V_+(F_i)$ (in particular ev exchanges Koszul- and Čech-differential).

Lemma 3.2.2. *Let $(F_0, \ldots, F_r) \in \Gamma(Y, \oplus_i \mathcal{O}_Y(kd_i))$. Then there is an exact sequence*

$$0 \to \vec{\mathrm{K}}(F_1, \ldots, F_n)^\bullet(*X_0) \to \vec{\mathrm{K}}(F_0, \ldots, F_n)^{\bullet \geq 1}[1] \to \vec{\mathrm{K}}(F_1, \ldots, F_n)^{\bullet \geq 1}[1] \to 0\ .$$

Proof. By definition,

$$\begin{aligned}\mathrm{K}(F_0^k, \ldots, F_n^k)_{\bullet \geq 1} &= (\mathrm{K}(F_1^k, \ldots, F_n^k)_\bullet(-k(d-1)) \xrightarrow{F_0^k} \mathrm{K}(F_1^k, \ldots, F_n^k)_{\bullet \geq 1}) \\ &= (\mathrm{K}(F_1^k, \ldots, F_n^k)_\bullet(-kX_0) \longrightarrow \mathrm{K}(F_1^k, \ldots, F_n^k)_{\bullet \geq 1})\end{aligned}$$

so that there is an exact sequence

$$0 \to \mathrm{K}(F_1^k, \ldots, F_n^k)_{\bullet \geq 1} \to \mathrm{K}(F_0^k, \ldots, F_n^k)_{\bullet \geq 1} \to \mathrm{K}(F_1^k, \ldots, F_n^k)_\bullet(-kX_0)[1] \to 0\ ,$$

or dually

$$0 \to \mathrm{K}(F_1^k, \ldots, F_n^k)^\bullet(kX_0)[-1] \to \mathrm{K}(F_0^k, \ldots, F_n^k)^{\bullet \geq 1} \to \mathrm{K}(F_1^k, \ldots, F_n^k)^{\bullet \geq 1} \to 0\ .$$

If we take the direct limit, which is an exact functor, and shift by one, we get the claim. □

In view of the associated analytic varieties, we are interested in the following special case: Let $X = V_+(F(z_0, \ldots, z_n))$ be a hypersurface with isolated singularities at $\Sigma = V_+(F_0, \ldots, F_n)$; $F_i := \frac{\partial}{\partial z_i} F$. Choose homogeneous coordinates $z_0, \ldots, z_n$ such that $V_+(z_i) \cap \Sigma = \emptyset$ on CY and let f_i be the homogenization of F_i wrt. z_0.

Over $\mathrm{D}_+(z_0)$, homogenization defines an isomorphism of the sheaf sequence from the previous lemma to

$$0 \to \vec{\mathrm{K}}(f_1, \ldots, f_n)(*X_0)[-1]^\bullet \to \vec{\mathrm{K}}(f_0, \ldots, f_n)^{\bullet \geq 1}[1] \to \vec{\mathrm{K}}(f_1, \ldots, f_n)^{\bullet \geq 1}[1] \to 0.$$

By the regularity of the sequence $(f_1, \ldots, f_n)$,

$$\vec{\mathrm{K}}(f_1, \ldots, f_n) = R_{[\Sigma]}(\mathcal{O}_Y)$$

and because the Koszul complex is locally free, for any complex $M^\bullet$ of quasi-coherent $\mathcal{O}_Y$-modules, $\vec{\mathrm{K}}(f_1, \ldots, f_n) \otimes M^\bullet(*X_0) = R_{[X_0|\mathrm{D}_+(z_0)]} R_{[\Sigma]}(M^\bullet)$. But this is homotopic to zero because $\Sigma \subset X_0 = V_+(F_0)$, hence

$$\begin{aligned}\mathcal{H}^k \vec{\mathrm{K}}(f_0, \ldots, f_n)^{\bullet \geq 1}[1] \otimes_{\mathcal{O}_Y} M^\bullet &= \mathcal{H}^k \vec{\mathrm{K}}(f_1, \ldots, f_n)^{\bullet \geq 1}[1] \otimes_{\mathcal{O}_Y} M^\bullet \\ &= \mathcal{H}^k_{[\Sigma|\mathrm{D}_+(z_0)]}(M^\bullet)\ .\end{aligned}$$

This expresses that (tempered) Čech cohomology on $\mathrm{D}_+(z_0)$ is independent of the cover chosen, as it should be. With the same argument for the other $\mathrm{D}_+(z_i)$, we have shown

Lemma 3.2.3. *Let $X = V_+(F)$ be a hypersurface with isolated singularities, $(z_0, \ldots, z_n)$ homogeneous coordinates such that $V_+(z_i) \cap X$ are smooth, $F_i := \frac{\partial}{\partial z_i} F$, Σ the singular locus of X and $M^\bullet$ a complex of quasi-coherent $\mathcal{O}_Y$-modules. Then*

$$\vec{\mathrm{K}}(F_0, \ldots, F_n)[1]^{\bullet \geq 0} \otimes M^\bullet = \mathrm{R}\underline{\Gamma}_{[\Sigma|Y]}(M^\bullet) \ .$$

3.3 Super-Calculus

If F is a homogeneous polynomial, $\frac{1}{2\pi i} dF/F \in {\Omega^\bullet}_{\mathrm{C}Y}(*\mathrm{C}X)/\Omega^\bullet$ represents the fundamental class of $\mathrm{C}X = V(F) - \{0\}$ in $\mathrm{C}Y = \mathrm{Spec}\ \mathbb{C}[z_0, \ldots, z_n] - \{0\}$. dF/F fulfils

$$\mathcal{L}_E(dF/F) = 0, \qquad i_E(dF/F) = \deg(X) \cdot 1 \equiv 0 \mod(\Omega^\bullet)$$

and thereby defines a global section in

$$\Gamma(\mathrm{C}Y, \pi^{-1}(\Omega^\bullet_Y(*X)/\Omega^\bullet)) = \Gamma(Y, \Omega^\bullet_Y(*X)/\Omega^\bullet);$$

$X = V_+(F) \subset Y = \mathbb{P}^n$, which we still denote by dF/F. Of course, in $\mathrm{D}_+(z_i) \subset Y := \mathbb{P}^n$, $dF/F = df/f$; f the homogenization wrt. z_i of F, so dF/F represents the fundamental class of X in the projective space Y.

We are going to study the associated morphism on Y

$$dF/F : \vec{\mathrm{K}}(F_0, \ldots, F_n)^{\bullet \geq 1}[1] \otimes \widetilde{A} \to \vec{\mathrm{K}}(F_0, \ldots, F_n)^{\bullet \geq 1}[1] \otimes A$$

using natural filtrations $\mathrm{P}^p(\vec{\mathrm{K}}(F_0, \ldots, F_n)^{\bullet \geq 1}[1] \otimes \widetilde{A}) := \vec{\mathrm{K}}(F_0, \ldots, F_n)^{\bullet \geq 1}[1] \otimes \mathrm{P}^p \widetilde{A}$ for $\widetilde{A}$ and similarly for A.

For all $p \in \mathbb{Z}$,

$$(\vec{\mathrm{K}}(F_0, \ldots, F_n)^{\bullet \geq 1}[1] \otimes \mathrm{Cone}(\mathrm{Gr}^p_{\mathrm{P}}(dF/F))) = \oplus_{i\ j} \vec{\mathrm{K}}(F_0, \ldots, F_n)^{q+1} \otimes \Omega^j_Y \otimes N^{j-p})$$

is as $\mathcal{O}_X$-module isomorphic to

$$K_p^{\bullet\ \bullet} := \Lambda^\bullet \left(\mathcal{O}_Y^{n+1}(d-1) \otimes_{\mathcal{O}_Y} \mathrm{Cone}(\mathrm{Gr}^{\mathrm{P}}_p((dF/F)) \right) \ ,$$

where e_i, dx_i anti-commute and are in degree 1.

$K_p^{\bullet\ \bullet}$ is a sheaf of bi-graded $\Omega^\bullet \otimes \mathcal{O}_X$- super-algebras with $\mathcal{O}_X$-linear differentials

$$\frac{dF}{F} : K_p^{i\ j} \to K_p^{i+1\ j}, \quad \alpha \mapsto \frac{dF}{F} \wedge \alpha$$

$$C : K_p^{i\ j} \to K_p^{i\ j+1}, \quad \alpha \mapsto C \wedge \alpha; \quad C = \sum e_i.$$

Hence $\frac{dF}{F}$ and C are homogeneous of bi-degree (1 0) and (0 1) and anti-commute so that

$$K_p^\bullet := \mathrm{tot}(K_p^{\bullet\ \bullet}) \tag{3.2}$$

is a complex with differential $(C + \frac{dF}{F})$ of degree one.

$\mathrm{End}_{\mathcal{O}_X}(K_p^\bullet)$ is a sheaf of graded super-Lie-algebras with bracket

$$[X, Y] := X \circ Y - (-1)^{XY} Y \circ X$$

for homogeneous endomorphisms X, Y wrt. the grading as single complex. We have the usual properties and identities

- $[X, Y] = -(-1)^{XY}[Y, X]$.
- $[X, [Y, Z]] = [[X, Y], Z] + (-1)^{XY}[Y, [X, Z]]$.
- $[X, Y \circ Z] = [X, Y] \circ Z + (-1)^{XY} Y \circ [X, Z]$.
- Graded $\mathcal{O}_X$-derivations form a sub-Lie-algebra.

With the variables e_i of our Koszul version of the Čech complex, there is the possibility to describe endomorphisms of $K_p^\bullet$ as follows:

Notation 3.3.1.

- *Let* $(\frac{\partial}{\partial e_1} \dots \frac{\partial}{\partial e_n})$ *be the dual basis of* $(e_1, \dots, e_n)$.
- *Write* $\frac{\partial}{\partial e_i}$ *for the contraction operator* $i_{\frac{\partial}{\partial e_i}} \in End^{-1}_{\mathcal{O}_X}(K_p^\bullet)$ *and similarly*
- *write* v_k *for the contraction* $i_{\frac{F}{F_k}\frac{\partial}{\partial x_k}}$ *with the vectorfield* $\frac{F}{F_k}\frac{\partial}{\partial x_k} \in \Theta(X_k)$ *(cf. (3.3.4)).*

Definition 3.3.2.

$$H := \sum_k v_k \circ e_k \frac{\partial}{\partial e_k} \in End^{-1}_{\mathcal{O}_X}(K_p^\bullet)$$

$$\check{E} := \sum_k e_k \frac{\partial}{\partial e_k} \in Aut^0_{\mathcal{O}_X}(K_p^\bullet)),\ \textit{operating as } (i+1) \textit{ on all summands } K_p^{i\ j}.$$

Lemma 3.3.3.

$$[H, \frac{dF}{F}] = \check{E}$$

Proof.

$$\begin{aligned}
[H, \frac{dF}{F}] &= \sum_i [v_i \circ (e_i \frac{\partial}{\partial e_i}), \frac{dF}{F}] \\
&= \sum_i v_i \circ \underbrace{[(e_i \frac{\partial}{\partial e_i}), \frac{dF}{F}]}_{=0} + \sum_i [v_i, \frac{dF}{F}] \circ (e_i \frac{\partial}{\partial e_i}) \\
&= \sum_i [i_{\frac{F}{F_i}\frac{\partial}{\partial x_i}}, \frac{dF}{F}] \circ (e_i \frac{\partial}{\partial e_i}) \\
&= \sum_i e_i \frac{\partial}{\partial e_i}.
\end{aligned}$$

□

Definition 3.3.4.

$$\Theta := -[H, C] = \sum_k e_k v_k \in \Gamma(Y, End^{-1\ 1}(K^{\bullet\ \bullet}) \subset \Gamma(Y, End^0(K^\bullet)).$$

Proposition 3.3.5. *Let $i : X \to (Y = \mathbb{P}^n)$ be the inclusion of a hypersurface with isolated singularities at $\Sigma \subset Y$, $l : Y - \Sigma \to Y$ the inclusion of the complement of Σ and $\widetilde{A}$ as in (3.1.1).*

Then $dF/F : \mathrm{R}\underline{\Gamma}_{[\Sigma|Y]}(\widetilde{A}) \to \mathrm{R}\underline{\Gamma}_{[\Sigma|Y]}(\mathbb{D}\widetilde{A})$ is filtered quis, if we let

$$\mathrm{P}^p(\mathrm{R}\underline{\Gamma}_{[\Sigma|Y]}(\widetilde{A})) := \vec{\mathrm{K}}(F_0, \dots, F_n)^{\bullet \geq 1}[1] \otimes \mathrm{P}^p(\widetilde{A}))$$

and similarly for $A = \mathbb{D}\widetilde{A}$ be the be the induced filtrations.

Proof. For all p, at any stalk $(K_p^\bullet)_y$ at $y \in Y$ is a free $\mathcal{O}_{Y,y}$-module of finite rank, which as complex possesses a zero homotopic automorphism

$$\check{E} - \Theta = [(dF/F + C), H] :$$

Modulo maximal ideal of y, $\check{E} - \Theta$ is the Jordan decomposition of an automorphism so that by Nakayama lemman it is an automorphism of the stalk. □

3.4 (Quasi-)Homogeneous Case

[1] From Riemann-Hilbert correspondence we know that

$$\mathbb{H}^k(Y, \Omega^\bullet(*X)) = \mathbb{H}^k(Y, j_* j^* \mathbb{C}_Y) = \mathrm{H}^k(U, \mathbb{C}_U);$$

[1] Although intensively studied here, pole-filtration and Hodge-filtration do not conicide in general, cf. [DSW07].

where (as always) $j_* = Rj_*$, $k \in \mathbb{N}$, $U := Y - X$, $j : U \hookrightarrow X$. If we define for all $p \in \mathbb{Z}$, the pole-filtration $\mathrm{P}^p\Omega^q(*X) := \Omega^\bullet((p+1)X)$, it induces filtration on the cohomology groups by

$$\mathrm{P}^p(\mathrm{H}^k(U)) := \mathrm{Im}\left(\mathbb{H}^k(Y,\ \mathrm{P}^p\Omega^\bullet(*X)) \to \mathbb{H}^k(Y, \Omega^\bullet(*X))\right) .$$

Note that we have an exact sequence

$$0 \to \Omega^\bullet[n] \to \mathrm{P}^0\Omega^\bullet[n] \to A \to 0$$

compatible with the filtrations on $\Omega^\bullet$ (stupid filtration) and A, which we introduced. In what follows, we want to compare the induced filtration on $\mathrm{H}^k(U, \mathbb{C})$ with the usual one, occurring in the standard mixed Hodge-structure
$(\mathrm{F}_{\mathrm{Del}}, \mathrm{W}, \mathrm{H}^k(U, \mathbb{Q}_U))$ on $\mathrm{H}^k(U, \mathbb{C})$. The latter was defined choosing an embedded resolution of singularities $\pi : (\widetilde{Y}, D) \to (Y, X)$, obtained by blowing up successively smooth sub-varieties contained in the singular locus such that D is a normal crossing divisor [Hir64]. Then $U = Y - X = \widetilde{Y} - D$ and $\mathrm{H}^k(U, \mathbb{C}_Y) = \mathbb{H}^k(\widetilde{Y}, \Omega^\bullet(\log D)))$.

Definition 3.4.1 ([Del71]). *Let U be smooth quasi-projective scheme and $\widetilde{Y}$ a smooth projective compactification such that $D := \widetilde{Y} - U$ is a divisor with normal crossings on $\widetilde{Y}$.*

- $\mathrm{F}_{\mathrm{Del}}\Omega^\bullet(\log D) := \Omega^{\bullet \geq p}(\log D)$
- $\mathrm{W}_m\Omega^p(\log D) := \Omega^m(\log D) \wedge \Omega^{p-m}$
- $\mathrm{F}_{\mathrm{Del}}\mathrm{H}^k(U, \mathbb{C}_Y) := \mathrm{Im}\left(\mathbb{H}^k(\widetilde{Y}, \mathrm{F}^p_{\mathrm{Del}}\Omega^\bullet(\log D)) \to \mathbb{H}^k(\widetilde{Y}, \Omega^\bullet(\log D))\right)$
- $\mathrm{W}(k)_m\mathrm{H}^k(U, \mathbb{C}_Y) := \mathrm{H}^k(\widetilde{Y},\ \mathrm{W}_{m-k}\Omega^\bullet(\log D))$

It is a theorem that the induced filtrations are independent of the choices made and give $(\mathrm{F}_{\mathrm{Del}}, \mathrm{W}(k),\ \mathrm{H}^k(U, \mathbb{Q}))$ the structure of a mixed Hodge structure (cf. appendix). In general, the following relation between the two filtrations is known:

Theorem 3.4.2 ([DD90]). *Let Y be proper and smooth, $X \subset Y$ a divisor. Then for all p and $q \in \mathbb{Z}$,*

$$\mathrm{F}^p_{\mathrm{Del}}(\mathrm{H}^q(U, \mathbb{C})) \subset\ \mathrm{P}^p(\mathrm{H}^q(U, \mathbb{C})) .$$

Definition 3.4.3. *An analytic hypersurface X of a smooth manifold Y of dimension n is* locally quasi-homogeneous *if for each point $x \in X$ there exist* good charts*, i.e. local analytic coordinates $(V; (x_1, \dots x_n))$ centered at x and* weights $w_1, \dots, w_n \in \mathbb{N}$*, with respect to which $X \cap V$ is given by a polynomial $f(x_1, \dots, x_n)$ which is quasi-homogeneous, i.e. homogeneous when the variable x_i is considered to have degree w_i for $i = 1, \dots, n$.*

If all weights are positive, we call X locally strict quasi-homogeneous *and* locally homogeneous *if they are all equal to* 1*.*

Around smooth points the quasi-homogeneity condition is trivially fulfilled.

Extending ideas of [HM98], [CJNMM96a], we will show the following theorem:

Theorem 3.4.4. *Let Y be proper and smooth, $X \subset Y$ a locally homogeneous divisor with isolated singularities. Then for $n := \dim Y$, the canonical morphism*

$$\tau_{\leq n-1}\Omega^\bullet(*X) \to \tau_{\leq n-1}\mathrm{R}\sigma_*(\Omega^\bullet(*D))$$

respects pole filtration and Deligne filtration respectively such that

$$\mathrm{P}^p(\mathrm{IH}^n(Y|X, \mathbb{C})) \simeq \mathrm{F}^p_{\mathrm{Del}}(\mathrm{IH}^n(Y|X, \mathbb{C}))$$

for all $p \in \mathbb{Y}$.

Equipped with this filtration and together with the canonical lattice, $\mathrm{IH}^n(Y|X)$ forms a pure Hodge structure of weight $n+1$, isomorphic to the Deligne Hodge structure $\mathrm{Gr}^{\mathrm{W}}_{n+1}\ \mathrm{H}^n(U)$, which in turn is isomorphic to $\mathrm{H}^{n-1}(\widetilde{X})(-1)$ if $\widetilde{X}$ is the blow-up of X in the singular points.

Remark 3.4.5. *If X is a nodal threefold, $\hat{X}$ a small resolution of X, then moreover*

$$\mathrm{IH}^n(Y|X) \simeq \mathrm{H}^{n-1}(\hat{X})(-1)$$

as a Hodge structure.

Proposition 3.4.6. *We have E_2 degeneracy of the spectral sequence associated to* F*, abutting to the global hypercohomologies. For the spectral sequence abutting to the local hypercohomologies, we will even have E_1-degeneration.*

On Y, away from the singularities of X, $\tau_{\leq n-1}\Omega^\bullet(*X) \to \tau_{\leq n-1}\mathrm{R}\sigma_*(\Omega^\bullet(*D))$ is clearly a filtered quism. Now let x be an isolated singularity of X. Geometrically we can interpret our assumptions as follows: There is a quasi-homogeneous polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ defining a cone $V(f)$ in all of $\mathbb{C}^n$, smooth away from its vertex at the origin and a biholomorphic map φ from V onto some open set $U \subset \mathbb{C}^n$ such that $V \cap X = \varphi^{-1}(V(F))$ and

$\varphi(x) = 0$. We can assume U is an open polycylinder centered at zero. The pair $(\mathbb{C}^n, V(f))$ is then a deformation retract of the pair $(U, U \cap V(f)$. Hence, $\mathrm{H}^k(S, \mathbb{C})$, $\mathrm{H}^k_{V(f)}(S, \mathbb{C})$ and $\mathrm{H}^k_{\{0\}}(S, \mathbb{C})$ are the same for $S = U$, $\mathbb{C}^n$ and all k. For the purpose of cohomological calculations, we are able to enter the algebraic category again: **We are now reduced to the case**

$$X \subset Y = \mathbb{A}^n$$

is given by a quasi-homogeneous polynomial with positive weights.

Let us fix some notations for the rest of this section:

Notation 3.4.7. *Fix weights* $(w_1, \ldots, w_n) \in \mathbb{N}$.

- *Let* $E := \sum_i w_i x_i \frac{\partial}{\partial x_i}$ *denote the* Eulerfield *with weights* w_i *on* Y.
- *The Lie-derivative* $\mathcal{L}_E$ *defines an endomorphism of the sheaves* Ω^p_Y *and* $\Omega^p_Y(*X)$ *for all* p.

  Let $\Omega^p_{Y(d)}$, $(\Omega^p_Y(*X))_{(d)}$ *denote the Eigenspace* $\ker(\mathcal{L}_E - d \cdot id)$, *of* $\mathcal{L}_E$ *for* $d \in \mathbb{Z}$.
- *For a* $\mathcal{O}_Z$*-module* M *on an algebraic variety* Z *we introduced the sheaves* M^{alg}, M^{an} *on* Z^{an} *(cf. p.4). We will omit the superscript alg/an in every statement that addresses both sheaves on* Z^{an}.

Definition 3.4.8 (Saito, K. [Sai80])**.**

$$\Omega^p(\log X) = \{\omega \in \Omega^p(*X) | \omega,\ d\omega \in \Omega^\bullet(X)\}$$

Proposition 3.4.9. *Let* $X = V(f) \subset \mathbb{C}^n$ *be defined by a quasi-homogeneous polynomial* f,

$$B \in \{\Omega^\bullet, \Omega^\bullet(\log X), \Omega^\bullet(*X)\},$$
$$B' \in \{\Omega^\bullet, \Omega^\bullet(\log X), \Omega^\bullet(*X), \Omega^\bullet(\log X)/\Omega^\bullet, \Omega^\bullet(*X)/\Omega^\bullet\},$$

then

(i) For $\omega \in B^{\mathrm{an}}$ *there is an unique local expansion in weighted homogeneous terms*

$$\omega = \sum_{i=-N}^{\infty} \omega_i \in B^{\mathrm{an}}_x \quad N \gg 0;$$

with $\omega_i \in (B^{\mathrm{an}})_{(i)}$ *and* $\omega_i \in (B^{\ \mathrm{alg}})_{(i)}$ *if* f *is strictly quasi-homogeneous (i.e. positive weights). In particular* $\omega \mapsto \omega_0$ *defines a projector* $B \to B_{(0)}$ *and if we set*

$$\mathrm{F}^q_E(B) := \{\omega | \omega_i = 0 \ for\ i \leq q\},$$

we get an exhausting filtration on B.

(ii) The inclusion $B_{(0)} \subset B$ are filtered quis wrt. the pole-filtration (stupid filtration rsp.)

(iii) $B^{\text{alg}} \subset B^{\text{an}}$, is filtered quis (wrt. pole-filtration (stupid filtration rsp.)) if f is strictly quasi-homogeneous.

Proof. Ad (i): In both algebraic/analytic settings, $B^{\text{'an}}$ is the direct limit (union) of the $\mathrm{F}^q(B^{\text{'an}})$ and for finite pole-order the first claim is clear: If $\omega \in \mathrm{F}^q(B^{\text{'an}})$, then write $\omega_d = \frac{1}{f^{q+1}} \cdot \eta$, with η holomorphic. By absolute convergence, we can reorder the power series expansion of η into to an unique expansion $\eta = \sum_{i=0}^{\infty} \eta_i$, with η_i quasi-homogeneous of weight i. This yields the existence of an expansion $\omega = \sum_{i=-d(q+1)}^{\infty} \omega_i$ as claimed, which must be unique.

Ad (ii): For all k and p, the map $\mathcal{H}^k\ \mathrm{F}^p B'_{(0)} \to \mathcal{H}^k \mathrm{F}^p B^{\text{'an}}$ is

- injective because given a class $[\omega_0]$ in the kernel, i.e. $\omega_0 = d\sigma$, by the uniqueness of the expansion in quasi-homogeneous terms, then $\omega = (d\sigma)_0$. And because, in addition, the differential of the complex d is homogeneous of degree zero ($[\mathcal{L}_E, d] = 0$), then $(d\sigma)_0 = d(\sigma_0)$.

- surjective: Given a cycle $\omega = \sum_{k=-N}^{\infty} \omega_k$, then $d\omega_k = 0$ for all k and the homogeneity condition $\mathcal{L}_E \omega_k = d \circ i_E \omega_k = k \cdot \omega_i$ means for the terms with $k \neq 0$, $\omega_k = \frac{1}{k} d i_E \omega_k$ is exact so that

$$\omega = \omega_0 + d(\sum_{i \neq 0}^{\infty} \frac{1}{i} \omega_i) ,$$

  where the latter converges because for $i >= 0$, $|\frac{1}{i}\omega_i| \leq |\omega_i|$ and there are only finitely many $i < 0$. By a cone construction, or 5-lemma, the result follows for a quotient complex B too.

Ad (iii): Clear, because all inclusions in $B^{\text{alg}} \supset (B^{\text{alg}})_{(0)} = (B^{\text{an}})_{(0)} \subset B^{\text{an}}$ are equivalences up to homotopy. □

Example 3.4.10. *Even in the smooth case, i.e. $X = V(x_1)$, homogeneous wrt. the Eulerfield $E = \sum x_i \frac{\partial}{\partial x_i}$, the previous proposition is instructive: Then*

$$(\Omega^\bullet(\log X)^{\text{an}})_{(0)} = \langle 1, \frac{1}{2\pi i} dx_1/x_1 \rangle_{\mathbb{C}},$$

from which one can read off that the weight filtration W_m coincides with $\tau_{\leq m}$ and in particular is defined over $\mathbb{Q}$.

Definition 3.4.11 (cf. (3.1.10.1) in [Del71])**.** *For a normal crossing divisor*

$$D = D_1 \cup \ldots \cup D_l$$

the pole-filtration on the meromorphic p-forms with poles along the components of D, $\Omega^p(*D)$, is defined as

$$\mathrm{P}^q(\Omega^p(*D)) = \sum_{i=1}^{l} \sum_{(q_i)\in A_q} \Omega^p((p-q_i+1)D_i); \quad \textit{where}$$

$$A_q = \left\{ (q_i)_{i=1,\ldots,l} \,\middle|\, \sum_i q_i = q, \quad q_i \le p \quad \forall i \right\}$$

Proposition ((3.1.11), [Del70])**.**

$$(\Omega^\bullet(\log D), \mathrm{F}_{\mathrm{Del}}) \subset (\Omega^\bullet(*D), \mathrm{P})$$

is a filtered quasi-isomorphism.

For later use, we give a variant:

Proposition 3.4.12 (and definition)**.** *Let D be a normal crossing divisor, locally given as $= V(z_1, \ldots z_l) \subset U \subset \mathbb{C}^n$. and $E = V(z_1, \ldots z_k)$, $\widetilde{X} = V(z_{k+1}, \ldots, z_l)$ a decomposition $D = \widetilde{X} + E$. Define*

- $\mathrm{W}_m\Omega^p(*D)$ $:= \left\{ \omega = \sum_{NL} a_{NL} z^N dz_L \,\middle|\, \begin{array}{l} N = n_1, \ldots n_l \in \mathbb{Z}^l, \quad a_N \in \mathcal{O}_U \ \textit{s.th.} \\ a_{n_1,\ldots n_l\ L} = 0 \quad \textit{when} \quad (\sharp\{i|n_i < 0\}) > m \end{array} \right\}$
  i.e. maximal m components of D in polar locus of each summand.
- $\Omega^\bullet(\log E)(*\widetilde{X}) := \Omega^\bullet(\log E) \otimes \Omega^\bullet(*\widetilde{X})$*, with* W*- and* P*-filtration induced from the inclusion* $\Omega^\bullet(\log E)(*\widetilde{X}) \hookrightarrow \Omega^\bullet(*D)$.

Then $\Omega^p(\log E)(*X) \subset \Omega^p(*D)$ is a bi-filtered quasi-isomorphism.

Proof. For $1 \le i \le k$, $E_i := x_i \frac{\partial}{\partial x i} \in \Theta(\log V(x_i) = (\Omega^1(\log V(x_1))^\vee$ is a logarithmic vectorfield acting on

$$\Omega^p(\log V(x_1, \ldots, x_i))(*V(z_{i+1}, \ldots z_l))$$

and

$$\Omega^p(\log V(x_1, \ldots, x_{i+1}))(*V(z_i, \ldots z_l))$$

respecting the filtration W. As above, expansion in $\mathcal{L}_{E_i}$-homogeneous terms shows that

$$\mathrm{F}^0_E(\Omega^p(\log V(x_1,\dots,x_{i+1}))(*V(z_i,\dots z_l)) \subset \Omega^p(\log V(x_1,\dots,x_{i+1}))(*V(z_i,\dots z_l))$$

is a filtered quasi-isomorphic subcomplex, but

$$\mathrm{F}^0_{E_i}(\Omega^p(\log V(x_1,\dots,x_{i+1}))(*V(z_i,\dots z_l))$$

is already $\Omega^p(\log V(x_1,\dots,x_i))(*V(z_{i+1},\dots z_l))$.
The result follows by induction on $i \in \{1,\dots,k\}$. □

Corollary 3.4.13. *With the notations above,*

$$\Omega^p(\log E)(*X) = \bigcap_{i=1,\dots,k} \mathrm{F}^0_{E_i}(\Omega^p(*D)) \subset \Omega^p(*D)\ ; \quad E_i := x_i \frac{\partial}{\partial x_i}\ .$$

Definition 3.4.14. *For $p \in \mathbb{N}$, let*

$$\Omega^p(\log f) := \ker\ i_E \subset \Omega^p(\log X),$$
$$\Omega^p(*f) := \ker\ i_E \subset \Omega^p(*X).$$

A key observation is that for any rational form ω,

$$\deg(f)\cdot\omega = i_E(\frac{\mathrm{d}f}{f}\wedge\omega) + \frac{\mathrm{d}f}{f}\wedge i_E\omega\ .$$

It follows that for $B \in \{\Omega^\bullet(\log X), \Omega^\bullet(*X)\}$, the complexes (B, i_E) and $(B, \frac{\mathrm{d}f}{f})$ are split-exact. In particular, one has decompositions

$$\begin{array}{lcllcl} \Omega^p(\log X) & = & \Omega^p(\log f) \oplus \frac{\mathrm{d}f}{f}\wedge\Omega^{p-1}(\log f) & \simeq & \Omega^p(\log f)\oplus\Omega^{p-1}(\log f) \\ \Omega^p(*X) & = & \Omega^p(*f)\oplus\frac{\mathrm{d}f}{f}\wedge\Omega^{p-1}(*f) & \simeq & \Omega^p(*f)\oplus\Omega^{p-1}(*f) \end{array} \tag{3.3}$$

compatible with F_E because $\frac{\mathrm{d}f}{f}$ and i_E are F_E-homogeneous endomorphisms of degree 0.

From now on, we assume X is locally homogeneous, i.e. $(w_1,\dots,w_n) = (1,\dots,1)$. The Hironaka resolution of singularities for an isolated homogeneous singularity x is simply the blow up $\sigma : (\widetilde{Y}, \widetilde{X}\cup E) \to (Y, X)$; with center x, exceptional locus E and $\widetilde{X}$ the (smooth) direct transform of X:

Definition 3.4.15.
- *Let $X = V(f) \subset Y :=$ Spec $\mathbb{C}[x_1,\dots,x_n]$ be the zero locus of a homogeneous polynomial f such that X has an isolated singularity at zero.*

- E *be the projective space* $E := \operatorname{Proj}\mathbb{C}[y_1, \ldots, y_n]$.
- $Q := \operatorname{Proj}(\mathbb{C}[y_1, \ldots, y_n]/\langle f(y)\rangle)$. *$Q$ can be identified with the smooth subvariety*
$$V_+(f) \subset E \subset \widetilde{Y},$$
where
- $$\begin{array}{rcl} \widetilde{Y} := V((x_i y_j - x_j y_i)_{i\ j}) & \subset & \mathbb{P}(0, \ldots, 0, 1, \ldots, 1) \\ & = & \operatorname{Proj}\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \\ & = & \mathbb{A}^n \times E. \end{array}$$
As for $i \neq j$, we have $x_i = x_j \frac{y_i}{y_j} \in \mathcal{O}_{\widetilde{Y} \cap D(y_j)}$,
$$\left((U_i := \widetilde{Y} \cap D(y_i)), (x_i; \frac{y_1}{y_i}, \ldots, \widehat{(\frac{y_i}{y_i})}, \ldots, \frac{y_n}{y_i})\right)_{i=1,\ldots,n}$$
is an algebraic atlas (of bundle charts for π as in the next item).
- $\widetilde{Y}$ *can be identified with the bundle* $\mathbb{V}(\mathcal{O}_E(1)) := \operatorname{Spec}(\operatorname{Sym}(\mathcal{O}_E(1)))$. *The embedding in $E \times \mathbb{A}^n$ is the one associated to $\mathcal{O}_E^n \overset{(y_1 \ldots y_n)}{\twoheadrightarrow} \mathcal{O}_E(1)$ and applying $\mathbb{V}$. The modification map σ and the bundle projection*
$$\pi = \pi_2 \circ i : \widetilde{Y} \to E$$
onto E are given by
$$\begin{array}{ccccc} (\widetilde{Y} = \mathbb{V}(\mathcal{O}_E(1))) & \xrightarrow{i} & (\mathbb{V}(\mathcal{O}_E^n) = \mathbb{A}^n \times E) & \xrightarrow{\pi_2} & E \\ \sigma \downarrow & & \downarrow & & \\ Y & = & \mathbb{A}^n . & & \end{array}$$
- *For any subset $S \subset E$, let $\overline{C}S := \pi^{-1}(S)$ denote the closed cone of S. $\overline{C}S$ is indeed the closure of $CS \simeq \sigma^{-1}CS \subset \widetilde{Y}$, with C taken as in definition 3.2.1, p.20.*
- $\widetilde{X} := \mathbb{V}(\mathcal{O}_Q(1)) = \overline{C}Q \subset \widetilde{Y}$, $\sigma|_{\widetilde{X}} : \widetilde{X} \to X$ *is the blow up of X with exceptional locus Q.*
- *The $\mathbb{C}^*$-action on $CE \subset \overline{C}E = \widetilde{Y}$ extends to $\widetilde{Y}$ (with fix-locus the exceptional locus E of σ). We denote its generating vectorfield by* Eulerfield $\widetilde{E}$ *(and hope there is no confusion with the exceptional locus E).*

- *Let $D := \widetilde{X} \cup E$ and define morphisms $i, j, \widetilde{i}, \widetilde{j}$ as in the diagram*

$$\begin{array}{ccccc} D & \xrightarrow{\widetilde{i}} & \widetilde{Y} & \xleftarrow{\widetilde{j}} & \widetilde{Y} - D \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{i} & Y & \xleftarrow{j} & Y - X \ . \end{array}$$

Remark 3.4.16.
- *Together with the Eulerfield, the filtration F_E extends to a filtration $\mathrm{F}_{\widetilde{E}}$ on forms on $\widetilde{Y}$.*
- *In a bundle chart $(U, (x; y_1, \ldots, y_{n-1}))$ for π (like the U_i above) the action is $t \cdot (p, p_1, \ldots, p_n) = (tp, p_1, \ldots, p_n)$, hence $\widetilde{E} = x\frac{\partial}{\partial x}$.*

Lemma 3.4.17. *1. $\Omega^1_{\widetilde{Y}}(\log E)/\pi^*\Omega^1_E$ is globally generated by a section $\frac{dx_\bullet}{x_\bullet}$, which maps to*

$$2\pi i \cdot c_{E|\widetilde{Y}} \in \Gamma(\widetilde{Y}, \Omega^1_{\widetilde{Y}}(\log E)/\Omega^1_{\widetilde{Y}}).$$

2. $\omega \mapsto \frac{dx_\bullet}{x_\bullet}\omega$ defines an isomorphism

$$\pi^*\Omega^\bullet_E(-E)[-1](-1) \simeq \frac{dx_\bullet}{x_\bullet} \wedge \pi^*\Omega^\bullet_E(-E)[-1](-1) \subset \Omega^\bullet.$$

3. $\omega \mapsto \frac{dx_\bullet}{x_\bullet}\omega$ defines an isomorphism

$$\pi^*\Omega^\bullet_E[-1](-1) \simeq \frac{dx_\bullet}{x_\bullet} \wedge \pi^*\Omega^\bullet_E[-1](-1) \simeq \Omega^\bullet_{\widetilde{Y}}(\log E)/\pi^*(\Omega^\bullet_E).$$

There is commutative diagram $(*)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*\Omega^\bullet_E & \longrightarrow & \Omega^\bullet_{\widetilde{Y}} & \xrightarrow{i_{\widetilde{E}}} & \pi^*(\Omega^\bullet_E)(-E)[-1] & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi^*\Omega^\bullet_E & \longrightarrow & \Omega^\bullet_{\widetilde{Y}}(\log E) & \xrightarrow{i_{\widetilde{E}}} & \pi^*\Omega^\bullet_E[-1] & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \Omega^\bullet_E[-1] & = & \Omega^\bullet_E[-1] & & \end{array}$$

4. $\pi^*(\Omega^\bullet_E)(-E) \simeq \pi^*(\Omega^\bullet_E(Q))$

Proof. 1. If $d_|$ is the π-relative (i.e. $\pi^{-1}\mathcal{O}_E$-linear) differential, $\frac{d_| x_i}{x_i} - \frac{d_| x_j}{x_j} = \frac{d_|(x_i/x_j)}{x_i/x_j} = \frac{0}{x_i/x_j}$.

2. On every U_i, $f \cdot d_| x_i \to f \cdot x_i$ defines a natural isomorphism $\varphi_i : \Omega^1_{\widetilde{Y}|E} = \mathcal{O}_{\widetilde{Y}}(-E)$. $d_| x_j = d_| \frac{x_j}{x_i} \cdot x_j = \frac{x_j}{x_i} d_| x_j$ so that the φ_i glue to a global isomorphism. Therefore the local lifts of the given morphism defined by composition of

$$\frac{dx_i}{x_i} \wedge \pi^* \Omega^\bullet_E(-E)[-1](U_i) \hookrightarrow (\Omega^\bullet_{\widetilde{Y}})(U_i) \twoheadrightarrow (\Omega^\bullet_{\widetilde{Y}|E})(U_i)$$

glue to a global isomorphism $\frac{dx_\bullet}{x_\bullet} \wedge \pi^* \Omega^\bullet_E(-E)[-1] \to \Omega^\bullet_{\widetilde{Y}}$.

3. is analog.

4. is clear.

5. follows from $\mathcal{O}_E(-E) \simeq \mathcal{O}_E(Q)$. □

Lemma 3.4.18. *Let*

- $\gamma := \sigma^* \frac{\mathrm{d}f}{f} \in \Gamma(\Omega^\bullet(\log D))$, $m := \deg(f)$,
- $q : \Omega^\bullet_{\widetilde{Y}}(\log E)(*\widetilde{X}) \to \Omega^\bullet_{\widetilde{Y}}(\log E)(*\widetilde{X})/\Omega^\bullet$ *be the canonical projection,*
- $$\begin{array}{rccc} h: & \pi^*\Omega^\bullet_E(*Q) & \longrightarrow & \Omega^\bullet_{\widetilde{Y}}(\log E)(*\widetilde{X}) \\ & \omega & \mapsto & \frac{1}{m}\gamma \wedge \omega \end{array}.$$

Then

(i)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi^*\Omega^\bullet_E(*Q) & \longrightarrow & \Omega^\bullet_{\widetilde{Y}}(\log E)(*\widetilde{X}) & \xrightarrow{q} & \Omega^\bullet_{\widetilde{Y}}(\log E)(*\widetilde{X})/\pi^*\Omega^\bullet_E(*Q) & \longrightarrow & 0 \\ & & & & h\uparrow & & \frac{dx_\bullet}{x_\bullet}\uparrow\wr & & \\ & & & & \pi^*\Omega^\bullet_E(*Q)[-1] & = & \pi^*\Omega^\bullet_E(*Q)[-1] & & \end{array}$$

commutes, that is $\frac{dx_\bullet}{x_\bullet} = \frac{1}{\deg(f)}\sigma^*(\frac{df}{f}) \in \Omega^\bullet_{\widetilde{Y}}(\log E)(*\widetilde{X})/\pi^*\Omega^\bullet_E(*Q)$;

(ii) i_E *is an inverse of* $\frac{dx_\bullet}{x_\bullet}$,

(iii) $s := h \circ i_E$

(a) is a semi-split of q,

(b) is a split q *on the subcomplex of homogeneous form of degree zero wrt.* $\mathcal{L}_E$,

(c) is compatible with the Hodge and weight filtration on $\Omega^\bullet(\log D) \subset \Omega^\bullet_{\widetilde{Y}}(\log E)(*\widetilde{X})$ *and hence*

(d) defines a split of MHS *on the level of cohomology groups.*

Proof. (i) We have to show that $\gamma \wedge \omega \equiv m\frac{dx_\bullet}{x_\bullet} \wedge \omega$ modulo $\pi^*\Omega^\bullet_E(*Q)$ for $\omega \in \pi^*\Omega^\bullet_E(*Q)$. This follows from $\gamma|_{U_i} = df(x/x_i)/f(x/x_i) + mdx_i/x_i$.

(ii) For $\omega \in \pi^*(\Omega^\bullet(*Q)) = \ker i_E$, we have $i_E\frac{dx_\bullet}{x_\bullet} \wedge \omega = \omega$.

(iii) (a) Follows from the commutativity in (i).

(b) We need to show that $s = h \circ i_E$ super commutes with the differentials. h is multiplication with a closed form and $i_E d = -di_E$, when the Lie-derivative vanishes.

(c) is clear.

(d) s is a split of

$$0 \to \pi^*\Omega^\bullet_E(\log Q))_{(0)} \to \Omega^\bullet(\log D)_{(0)} \to (\Omega^\bullet(\log D)/\pi^*\Omega^\bullet_E(\log Q))_{(0)} \to 0$$

compatible with the filtrations.

Moreover, the k-th hyper-cohomology group of

$$\Omega^\bullet(\log D)/\pi^*\Omega^\bullet_E(\log Q))_{(0)} = \frac{dx_\bullet}{x_\bullet} \wedge \pi^*\Omega^\bullet_E(\log Q))_{(0)}(-1)[-1]$$

is $\mathbb{H}^k(E, \Omega^\bullet_E(\log Q)(-1)[-1])$.

For $k \neq 1$, this coincides with $\mathrm{H}^k_0(E, \mathbb{C}(-2)[-2])$, a shift of the pure Hodge structure on the primitive cohomology of Q.

Hence for all k, the MHS

$$\Big(\mathrm{W}(k), \mathrm{F}, \mathbb{H}^k(\widetilde{Y}, \Omega^\bullet(\log D))\Big) = \mathrm{H}^k(E{-}Q, \mathbb{C}) \oplus \mathrm{H}^k(E{-}Q, \mathbb{C}(-1)[-1])$$

is a direct sum of pure Hodge structures.

□

Proposition 3.4.19.

$$\begin{aligned}
\underline{\underline{\mathbb{H}}}^k(Y^{\mathrm{an}}, \mathrm{F}^p\Omega^\bullet(*X)^{\mathrm{an}})_x &= \mathrm{F}^p\underline{\underline{\mathbb{H}}}^k(Y^{\mathrm{an}}, \Omega^\bullet(*X)^{\mathrm{an}})_x \\
&= \mathrm{F}^p\mathbb{H}^k(\widetilde{Y}, \Omega^\bullet(*(\widetilde{D}))) \\
&= \mathrm{F}^p\mathrm{H}^k(E-Q, \mathbb{C}) \oplus \mathrm{F}^p\mathrm{H}^k(E-Q, \mathbb{C}(-1)[-1]) \\
&\quad (\textit{weights } k+1, k+2) \\
&= \begin{cases} \mathrm{F}^{p-1}\mathrm{H}^{n-2}_0(Q, \mathbb{C}) & ;\ k = n-1 \\ \mathrm{F}^{p-2}\mathrm{H}^{n-2}_0(Q, \mathbb{C}) & ;\ k = n \\ \mathbb{C} & ;\ k = 0, 1 \textit{ and } p = 0 \\ 0 & ;\ \textit{else.} \end{cases}
\end{aligned}$$

Hereby $\mathrm{H}_0^{n-2}(Q,\mathbb{C})$ *denotes the* $n-2$*-th primitive cohomology of* $Q \subset E$ *wrt. to the polarization* $\mathcal{O}_E(1)$.

In particular, we have E_1 *degeneracy* $\mathrm{E}_1^{p\ q} = \underline{\mathbb{H}}^{p+q}(Y^{\mathrm{an}}, \mathrm{Gr}_{\mathrm{F}}^p\, \Omega^\bullet(*X)^{\mathrm{an}}) = \mathrm{E}_\infty$ *of the spectral sequence associated to the filtration* F *abutting to the hypercohomology sheaves.*

Proof. We already know that $\mathbb{H}^k(\mathrm{F}^p\Omega^\bullet(*X)^{\ \mathrm{an}})_x = \mathbb{H}^k(Y, \mathrm{F}^p\Omega^\bullet(*X))$ and this is the k-th cohomology of the complex of groups $(\Gamma(Y,\ \mathrm{F}^p\Omega^\bullet(*X)), \Gamma(d))$ by the $\Gamma(Y,-)$-acyclicity of $\Omega^l(kX)$ for all l, $k \geq 0$ (Bott vanishing).

Similarly for the normal crossing divisor $D := \widetilde{X} \cup E$, $U := Y - X = \widetilde{Y} - D$, the morphism $\widetilde{j} : U \hookrightarrow \widetilde{Y}$ is affine, hence

$$0 = \mathrm{H}^k(U, \Omega_U^p) = \mathbb{H}^k(\widetilde{Y}, R\widetilde{j}_*\Omega^p) = \mathbb{H}^k(\widetilde{Y}, \mathrm{R}^0\widetilde{j}_*\Omega^p); k > 0.$$

But in the algebraic category $\mathrm{R}^0\widetilde{j}_*\Omega^p = \Omega^p(*D)$ that means that $\Omega^p(*D)$ is $\Gamma(Y,-)$-acyclic for all p and $\mathrm{H}^k(U,\mathbb{C}) = (\Gamma(Y, \Omega^\bullet(*D)), \Gamma(d))$.

Nevertheless, the canonical map

$$\Gamma(Y, \Omega_Y^\bullet(*X)) \to \Gamma(\widetilde{Y}, \Omega_{\widetilde{Y}}^\bullet(*D)); \qquad \omega \mapsto \sigma^*\omega$$

seems apriori not to respect the pole filtration.

If we tensor the whole diagram $(*)$ with the flat sheaf $\mathcal{O}_{\widetilde{Y}}(*\widetilde{X})$, we get $(**)$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi^*(\Omega_E^\bullet(*Q)) & \longrightarrow & \Omega_{\widetilde{Y}}^\bullet(*\widetilde{X}) & \longrightarrow & \pi^*(\Omega_E^\bullet(*Q))(-E)[-1] & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*(\Omega_E^\bullet(*Q)) & \longrightarrow & \Omega_{\widetilde{Y}}^\bullet(\log E)(*\widetilde{X}) & \longrightarrow & \pi^*(\Omega_E^\bullet(*Q))[-1] & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow 1 & & \\
 & & & & \Omega_E^\bullet(*Q)[-1] & = & \Omega_E^\bullet(*Q)[-1]\ . & &
\end{array}$$

Now we make use of the fact that projecting $\omega \in \Gamma(Y,\ \mathrm{F}^p\Omega^l(*X))$ to its homogeneous component ω_0 on Y does not change the cohomology class on both sides of the pullback morphism. Indeed, in degree zero, the pullback defines a morphism

$$\Gamma(Y, \mathrm{F}^p\Omega_Y^\bullet(*X)_{(0)}) \to \Gamma(\widetilde{Y},\ \mathrm{F}^p\Omega_Y^\bullet(\log E)(*\widetilde{X})_{(0)})\ ,$$

which is an isomorphism and respects the filtrations: For this, note that

$$0 \to \Omega_Y^p(*f) \to \Omega_Y^p(*X) \to \Omega_Y^{p-1}(*f) \to 0$$

is exact for all p so that by induction on p for the cohomology with support in the maximal ideal $\mathfrak{m}$ corresponding to $\{0\}$, $\mathrm{H}_m^0(\Omega_Y^p(*f)) = \ \mathrm{H}_m^1(\Omega_Y^p(*f)) = 0$

so that $\Omega^p_Y(*f) \to l_* l^* \Omega^p_Y(*f)$ is an isomorphism. In particular $\Omega^p_Y(*f)_{(0)} = \pi^{-1}(\Omega^p_E(*Q))$ (Griffiths calculus).

Now check that

$$\begin{array}{ccccc}
\Gamma(\widetilde{Y}, \pi^{-1}\mathrm{F}^p\Omega^\bullet_E(*Q)) & \longrightarrow & \Gamma(\widetilde{Y}, \mathrm{F}^p\Omega^\bullet_{\widetilde{Y}}(\log E)(*\widetilde{X})_{(0)}) & \xrightarrow{i_{\widetilde{E}}} & \Gamma(\widetilde{Y}, \pi^{-1}\mathrm{F}^p\Omega^\bullet_E(*Q)(-1)[-1]) \\
\| & & \uparrow \sigma^* & & | \\
\Gamma(Y, \mathrm{F}^p\Omega^\bullet_Y(*f)_{(0)}) & \longrightarrow & \Gamma(Y, \ \mathrm{F}^p\Omega^\bullet_Y(*X)_{(0)}) & \xrightarrow{i_E} & \Gamma(Y, \ \mathrm{F}^p\Omega^\bullet_Y(*f)(-1)[-1]_{(0)})
\end{array}$$

(zeroes on both sides omitted) commutes, so σ^* is as claimed on the degree zero part.

By lemma 3.4.12, $\Omega^\bullet_{\widetilde{Y}}(\log E)(*\widetilde{X}) \subset \Omega^\bullet(*(E+\widetilde{X}))$ is (bi-)filtered quis, so we will have shown that the pullback map is a quasi-isomorphism, once we know that all $\mathrm{P}^p\Omega^q_{\widetilde{Y}}(\log E)(*\widetilde{X})_{(0)}$ are $\Gamma(\widetilde{Y}, -)$-acyclic (ie. $\Gamma = \mathrm{R}\Gamma$). But this we can read off from the exact sequence above also, since

$$\mathrm{H}^k(\widetilde{Y}, \pi^*(\Omega^\bullet_E(lQ))_{(0)}) = \mathrm{H}^k(\widetilde{Y}, \pi^{-1}(\Omega^\bullet_E(lQ))) = \mathrm{H}^k(E, \Omega^\bullet_E(lQ))) = 0$$

vanish for $k, l > 0$. □

Corollary 3.4.20. *With $D := \widetilde{X} + E$ in the notation as above,*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^\bullet(*\widetilde{X}) & \longrightarrow & \Omega^\bullet_{\widetilde{Y}}(\log E)(*\widetilde{X}) & \longrightarrow & \frac{dx_\bullet}{x_\bullet} \wedge \Omega^\bullet(*\widetilde{X})[-1] & \longrightarrow & 0 \\
& & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathrm{W}_1\Omega^\bullet(*D) & \longrightarrow & \mathrm{W}_2\Omega^\bullet_{\widetilde{Y}}(*D) & \longrightarrow & \mathrm{Gr}^{\mathrm{W}}_2 \Omega^\bullet(*D) & \longrightarrow & 0
\end{array}$$

induces isomorphisms between the global cohomology groups in the columns. In particular $\mathrm{Gr}^{\mathrm{W}(k)}_k \mathrm{H}^k(Y - X) = 0$ for all $k \in \mathbb{N}$.

Proof. The morphism at 1 in diagram (∗∗) is a quasi-isomorphism, as one sees going over to the degree zero subcomplex of sections constant in the fibers of π. Therefore, $\Omega^\bullet_E(*Q)(-E)$ has all global hyper-cohomology sheaves zero so that the first row above and the protagonist of the previous proposition are term to term bi-filtered quasi-isomorph. □

Remark 3.4.21. *Wrt. π, we have globally trivialized the direct image sheaves of the Hopf-fibration restricted to $\widetilde{Y} - \widetilde{X}$: The long exact sequence associated to $(*) \otimes \mathcal{O}(*\widetilde{X})$ can be considered as an explicit split of the long exact sequence of the (Wang-type) Leray spectral sequence for $R\pi_*$*

$$\begin{array}{ccccc}
\cdots \mathrm{H}^k(\tau_{\leq 0}\mathbb{R}\pi_*\Omega^\bullet_{\widetilde{Y}}(\log E)(\widetilde{X})) & \longrightarrow & \mathrm{H}^k(\tau_{\leq 1}R\pi_*\Omega^\bullet_{\widetilde{Y}}(\log E)(\widetilde{X})) & \longrightarrow & \mathrm{H}^k(\mathrm{Gr}^{\tau^1} R\pi_*\Omega^\bullet_{\widetilde{Y}}(\log E)) \cdots \\
\| & & \| & & \| \\
\cdots \mathrm{H}^k\pi^*\Omega^\bullet_E(*Q)) & \longrightarrow & \mathrm{H}^k(\mathbb{R}\pi_*\Omega^\bullet_{\widetilde{Y}}(\log E)) & \longrightarrow & \mathrm{H}^k(\frac{dx_\bullet}{x_\bullet} \wedge \pi^*\Omega^\bullet_E(*Q)(-1)[-1]) \cdots \\
\| & & \| & & \| \\
\cdots \mathrm{H}^k(E - Q, \mathbb{C}) & \longrightarrow & \mathrm{H}^k(\widetilde{Y} - \widetilde{X}, \mathbb{C}) & \longrightarrow & \mathrm{H}^{k-1}(E - Q, \mathbb{C})(-1) \cdots
\end{array}$$

Definition 3.4.22.

$$\mathrm{W}_m(\Omega_Y^\bullet(*X)) := \begin{cases} 0 & ;\ m \leq -1 \\ \Omega^\bullet & ;\ m = 0 \\ \tau_{\leq -n-1}(\Omega_Y^\bullet(*X)) & ;\ m = 1 \\ \Omega_Y^\bullet(*X) & ;\ m \geq 2\ , \end{cases}$$

where $\tau_{\leq -n-1}(\Omega_Y^\bullet(*X))$ is the subcomplex of forms which are locally exact in degree n (cf. Appendix).

Note that the notions 'locally exact' and 'locally closed' differ if there is no Poincare lemma.

Proposition 3.4.23. *Let M be a projective manifold, N a strict quasi-homogeneous Cartier divisor with only isolated singularities and affine complement.*

Then with P *and* W *locally defined as above,*

$$(\mathrm{W}, \mathrm{P}, j_* j^* \mathbb{Q}_M, \Omega^\bullet(*N)); \quad j : (M - N) \hookrightarrow M$$

is a mixed Hodge complex calculating the Deligne MHS *on the cohomology of $M - N$.*

Proof. Of course $\tau_{\leq n-1}(\Omega_M^\bullet(*N))$ represents $\tau_{\leq n-1} j_* j^* \mathbb{C}_M$ in the derived category because all $\Omega_M^p(*N)$ are direct limits of coherent sheaves (in particular W_1 is already defined over $\mathbb{Q}$). Using good charts centered in a singular point s, we can locally reduce to the homogeneous case $V(f) \subset \mathbb{C}^n$ as above.

With the last corollary, we proved therefore that

$$\begin{aligned} \mathbb{C}_s &= \mathrm{Im}(\mathbb{R}\sigma_* \mathrm{W}_0 \Omega_M^\bullet(*D)) \to \mathbb{R}\sigma_* \Omega_M^\bullet(*D))_s \\ &= \mathrm{Im}(\mathrm{W}_0 \Omega_M^\bullet(*N)) \to \mathbb{R}\sigma_* \Omega_{\widetilde{M}}^\bullet(*D))_s \end{aligned}$$

and

$$\begin{aligned} \mathrm{W}_1(\Omega_M^\bullet(*N)) &= \tau_{\leq n-1}(\Omega_M^\bullet(*N)) \\ &= \mathbb{R}\sigma_*(\Omega_{\widetilde{M}}^\bullet(*\widetilde{N})) \\ &= \mathbb{R}\sigma_*(\mathrm{W}_1 \Omega_{\widetilde{M}}^\bullet(*D))\ . \end{aligned}$$

By definition

$$\begin{aligned} \mathrm{W}_2(\Omega_M^\bullet(*N)) &= \Omega_M^\bullet(*N) \\ &= \mathbb{R}\sigma_*(\mathrm{W}_2 \Omega_{\widetilde{M}}^\bullet(*\widetilde{N})) \\ &= \mathbb{R}\sigma_*(\Omega_{\widetilde{M}}^\bullet(*D))[n])\ . \end{aligned}$$

Proposition 3.4.19 assures the compatibility of the filtration F. Hence

$$(\mathrm{W}, \mathrm{F}, j_* j^* \mathbb{Q}_M, \Omega^\bullet(*N))$$

induces the same filtrations on the cohomology of $M - N$ as

$$(\mathrm{W}, \mathrm{F}, \widetilde{j}_* \widetilde{j}^* \mathbb{Q}_M, \Omega^\bullet_{\widetilde{M}}(*D)) \ ;$$

thus defining a mixed Hodge complex. □

Let us give an application of lemma 3.4.18 to the theory of logarithmic forms:

Corollary 3.4.24 (log comparison)**.** *Let X be an isolated homogeneous singularity. Then*

- $\Omega^\bullet(\log X) = \tau_{\leq -2}\Omega^\bullet(*X)$ *if K_Q has no global section.*
- $\Omega^\bullet(\log X) \subset \Omega^\bullet(*X)$ *is quis iff the exceptional locus has no primitive cohomology. In this case, it is filtered quis.*

Proof. As $l_* l^* \Omega^\bullet(\log X) = l_* l^* \Omega^\bullet(*X) = \Omega^\bullet(*X)$, we can compare $\Omega^\bullet(\log X)$ and $l_* l^* \Omega^\bullet(*X)$ via the spectral sequence

$$\mathrm{E}_1^{p\ q} = \mathrm{R}^q l_* l^* (\Omega^p(\log X))_x = \underline{\mathbb{H}}^{q+p} \, \mathrm{Gr}^p_{\mathrm{F}}(\Omega^\bullet(*X))_x.$$

(Recall that $\mathrm{Gr}^p_{\mathrm{F}}\, \Omega^\bullet(\log X)$ is $\Omega^p(\log X)[-p]$ and not Ω^p.)

By proposition 3.4.19, the ${}_{\mathrm{F}}\mathrm{E}_1 = {}_{\mathrm{F}}\, \mathrm{E}_\infty$ table is

$$\left[\begin{array}{l|lllllll}
n & 0 & \mathrm{H}_0^{0\ n-2} & \mathrm{H}_0^{0\ n-2} & 0 & & \cdots & 0 \\
n-1 & & 0 & \mathrm{H}_0^{1\ n-3} & \mathrm{H}_0^{1\ n-3} & 0 & & \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 & \mathrm{H}_0^{n-3\ 1} & \mathrm{H}_0^{n-3\ 1} & 0 \\
0 & \mathbb{C} & \mathbb{C} & 0 & \cdots & 0 & \mathrm{H}_0^{n-2\ 0} & \mathrm{H}_0^{n-2\ 0} \\
\hline
q/p & 0 & 1 & 2 & \cdots & & n-1 & n
\end{array}\right]$$

with $\mathrm{H}_0^{p\ q} := \mathrm{H}_0^{p\ q}(Q, \mathbb{C})$. Because the log complex is reflexive [Sai80], it has depth of at least two, thus the the lower row is $\Gamma(Y, \Omega^\bullet(\log X))$ and this already proves the claim. □

Chapter 4

Global Considerations

4.1 Mixed Hodge Structures

Notation 4.1.1. *Throughout this chapter, let*

- $X \subset Y = \mathbb{P}^n$ *be a Cartier divisor with locally homogeneous isolated singularities at* $\Sigma \subset X$.
- *Let* $\sigma : (\widetilde{Y}, D) \to (Y, X)$ *be the resolution of singularities obtained by blowing up* Σ,
- $\widetilde{X}$ *the (smooth) direct transform of* X,
- $E = \cup_{s\in\Sigma} \mathrm{E}_s$ *the exceptional divisor of* σ,
- $Q = \cup_{s\in\Sigma} Q_s$ *the exceptional divisor of* $\sigma|_{\widetilde{X}}$,
- D *the divisor with normal crossings* $\widetilde{X} \cup E$.
- *Define morphisms* $i, j, \widetilde{i}, \widetilde{j}$ *as in the diagram*

$$\begin{array}{ccccccccc} Q & \subset & \widetilde{X} & \subset & D & \overset{\widetilde{i}}{\subset} & \widetilde{Y} & \overset{\widetilde{j}}{\supset} & (\widetilde{Y} - D) \\ \downarrow & & \downarrow & & \sigma\downarrow & & \downarrow & & \downarrow \\ \Sigma & \subset & X & = & X & \overset{i}{\subset} & \widetilde{Y} & \overset{j}{\supset} & (Y - X) \end{array}$$

- *and let* $k : \Sigma \hookrightarrow Y$, $l : (Y - \Sigma) \hookrightarrow Y$.

On the intersection cohomology of X, primitive cohomology wrt. a hyperplane section H of X is defined as

$$\mathrm{IH}^k_0(X) := \ker\big((H \cup -) : \mathrm{IH}^k_0(X) \to \mathrm{IH}^{k+2}_0(X)\big)$$

and fulfills Lefschetz decomposition and is part of an $\mathrm{sl}_2(\mathbb{C})$ action as in the smooth case ("Kaehler package for intersection cohomology"). The question of Griffiths calculus is whether we can relate primitive intersection cohomology with rational forms on Y with poles along X and calculate the intersection pairing on the middle cohomology.

Proposition 4.1.2. *Let* $(\mathrm{W}, \mathrm{F}, L, H) := (\mathrm{W}, \mathrm{F}, j_*j^*\mathbb{Q}_Y, \Omega^\bullet(*X))[n]$ *be the mixed Hodge complex defined as in the last chapter but shifted* n *places to the left.*

Then

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{W}_0 L & \longrightarrow & \mathrm{W}_1 L & \longrightarrow & \mathrm{Gr}^{\mathrm{W}}_1 L & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{W}_0 L & \longrightarrow & \mathrm{W}_2 L & \longrightarrow & \mathrm{W}_2 L/\mathrm{W}_0 L & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Gr}^{\mathrm{W}}_2 L & = & \mathrm{Gr}^{\mathrm{W}}_2 L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

is term-wise quasi-isomorphic to

$$
\begin{array}{ccccccc}
\tau_{\leq -n} L & \longrightarrow & \tau_{\leq -1} L & \longrightarrow & \tau^{\geq 1-n}\tau_{\leq -1} L & \xrightarrow{[1]} & \\
\| & & \downarrow & & \downarrow & & \\
\tau_{\leq -n} L & \longrightarrow & L & \longrightarrow & \tau^{\geq 1-n} L & \xrightarrow{[1]} & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathrm{Gr}^0_\tau L & \xrightarrow{\sim} & \mathrm{Gr}^0_\tau L\ , & & \\
 & & {\scriptstyle [1]}\downarrow & & {\scriptstyle [1]}\downarrow & &
\end{array}
$$

which is

$$\begin{array}{ccccccc}
\mathrm{IC}_Y(\mathbb{Q}) & \longrightarrow & \mathrm{IC}_{Y|X}(\mathbb{Q}) & \longrightarrow & i_*\,\mathrm{IC}_X(\mathbb{Q}) & \xrightarrow{[1]} & \\
\Vert & & \downarrow & & \downarrow & & \\
\mathbb{Q}_Y[n] & \longrightarrow & j_*j^*\mathbb{Q}_Y[n] & \longrightarrow & i_*i^!(\mathbb{Q}_Y[n+1]) & \xrightarrow{[1]} & \\
 & & \downarrow & & \downarrow & & \\
 & & S & \xrightarrow{\sim} & S_0\ , & & \\
 & & {\scriptstyle [1]}\downarrow & & {\scriptstyle [1]}\downarrow & &
\end{array} \tag{4.1}$$

where $\mathrm{IC}_{Y|Z} := \tau_{\leq -1} j_* j^* \mathbb{Q}_Y[n]$.

S is a perverse sheaf with zero dimensional support, therefore it is (quasi-isomorph to) a complex $S^\bullet$ concentrated in degree n. S^n is a skyscraper sheaf supported at Σ and stalk at $s \in \Sigma$ isomorphic to $\mathrm{H}^{n-1}(\mathrm{E}_s - Q_s, \mathbb{C})(-1)$.

S_0 equals S in the derived category; the isomorphism from S^n to $S_0^n = \mathrm{H}_0^{n-2}(Q, \mathbb{Q})(-2)$ is the residue on E along Q.

Proof. We have

$$\begin{aligned}
&\mathrm{W}_0 j_*j^*\mathbb{Q}_Y[n] = \tau_{\leq -n} j_*j^*\mathbb{Q}_Y[n] = \mathrm{Gr}_\tau^{-n}\, j_*j^*\mathbb{Q}_Y[n] = \left(\mathrm{R}^0 j_*j^*\mathbb{Q}_Y\right)[n] = \mathbb{Q}_Y[n]\\
&\mathrm{W}_1 j_*j^*\mathbb{Q}_Y[n] = \tau^{-1} j_*j^*\mathbb{Q}_Y[n]\\
&\mathrm{W}_2 j_*j^*\mathbb{Q}_Y[n] = j_*j^*\mathbb{Q}_Y[n]
\end{aligned}$$

by definition and S, S_0 are as claimed by proposition 3.4.19, so that

$$\mathrm{W}_2 L / \mathrm{W}_0 L = i_* i^! \mathbb{Q}[n+1]$$

by the existence of an exact triangle

$$i_* i^! \mathbb{Q}[n] \to \mathbb{Q}[n] \to j_* j^* \mathbb{Q}[n] \xrightarrow{[1]} \ .$$

For the assertion on the intersection cohomology, consider the algebraic stratification

$$(Y^0 = Y) \supset (Y^1 = \Sigma) = \cdots = (Y^n = \Sigma) \supset (Y^{n+1} = \emptyset)$$

of Y. The strata $S^k := Y^k \setminus Y^{k+1}$ are empty or smooth of complex codimension k and S^0 is dense in Y with $X - \Sigma = X \cap S^0$. By formula (1.1) on page 8 if $U_k := Y^0 \setminus Y^k$, $j_{k+1} : U_k \to U_{k+1}$, the inclusion

$$i_*(\mathrm{IC}_X(\mathbb{Q})) = (\tau_{\leq -1} j_{n+1\ *}) \circ (\tau_{\leq -2} j_{n\ *}) \circ \cdots \circ (\tau_{\leq -n-1} j_{1\ *})(i|_{X-\Sigma *}(\mathbb{Q}_{X-\Sigma}[n-1]))\ ,$$

and this is $\tau_{\leq -1}(i|_{X-\Sigma_*}\mathbb{Q}_{X-\Sigma}[n-1]) = \tau_{\leq -1} l_* l^* i_* i^! \mathbb{Q}_X[n+1]$.

Because Y is smooth, only $\mathrm{R}^0(l_*\mathbb{Q}_{Y-\Sigma}) = \mathbb{Q}_Y$ and $\mathrm{R}^{2n-1}(l_*\mathbb{Q}_{Y-\Sigma}) = \mathbb{Q}_\Sigma$ differ from zero among the sheaves $\mathrm{R}^k(l_*\mathbb{Q}_{Y-\Sigma})$ and therefore

$$\mathrm{Gr}_0^\tau(l_* l^* \mathbb{Q}_Y[n]) = \mathrm{R}^n(l_*\mathbb{Q}_{Y-\Sigma}) = 0 \ ,$$

so that there is an exact triangle

$$\tau_{\leq -1} l_* l^* \mathbb{Q}[n] \to \tau_{\leq -1} l_* l^* j_* j^* \mathbb{Q}[n] \to \tau_{\leq -1} l_* l^* i_* i^! \mathbb{Q}[n+1] \xrightarrow{[1]} \ ,$$

which can be identified with

$$\mathrm{IC}_Y(\mathbb{Q}) \to \mathrm{IC}_{Y|X}(\mathbb{Q}) \to i_* \,\mathrm{IC}_X(\mathbb{Q}) \xrightarrow{[1]} \ . \tag{4.2}$$

Here we used the relation $j_* j^* = l_* l^* j_* j^*$, which is a standard fact because for any constructible sheaf complex C,

$$j_* j^* C \to l_* l^* j_* j^* C \to k_* k^! j_* j^* C \xrightarrow{[1]}$$

is an exact triangle and $k^! j_* C = 0$ because $\mathrm{supp}(j_* C) \cap \Sigma = \emptyset$. □

Proposition 4.1.3. $\mathrm{IC}_{Y|X}(\mathbb{Q}) \simeq {}^p l_{!*} j_* j^* \,\mathrm{IC}_Y(\mathbb{Q})$.

Proof. The intermediary extension $\mathcal{G} := l_{!*}\mathcal{F}$ of a perverse sheaf $\mathcal{F}$ on $Y - \Sigma$ is characterized by [BBJ83]

$$\begin{array}{lcll} l^*\mathcal{G} & = & l^*\mathcal{F} & \\ {}^p\mathcal{H}^m k^*\mathcal{G} & = & 0 & m \geq 0 \\ {}^p\mathcal{H}^m k^!\mathcal{G} & = & 0 & m \leq 0 \end{array}$$

so that the claim follows from five-lemman and (4.2). □

Remark 4.1.4. $S \simeq k_* k^! \,\mathrm{IC}_{Y|X}[1]$.

Proof. Applying $k_* k^!$ to (4.2) gives an exact triangle

$$k_* k^! \,\mathrm{IC}_{Y|X}(\mathbb{Q}[n]) \to k_* k^! (j_* j^* \mathbb{Q}_Y) \to H \xrightarrow{[1]}$$

which proves the claim by the fact that $k^! j_* = 0$. □

Definition 4.1.5. *Let $\mathbb{F}$ be a field of characteristic zero. In analogy to the definition of intersection cohomology groups, let us introduce for $q \in \mathbb{Z}$*

$$\mathrm{IH}^q(Y|X, \mathbb{F}) := \mathbb{H}^q(Y, i_* \,\mathrm{IC}_{Y|X}(\mathbb{F})[-n]) \ ,$$

In case $\mathbb{F} = \mathbb{Q}$, we will also write $\mathrm{IH}^i(Y|X)$.

With this notion, we have long exact sequences of groups

$$\cdots \to \mathrm{H}^k(Y) \to \mathrm{IH}^k(Y|X) \to \mathrm{IH}^k(X) \to \cdots$$

and

$$\cdots \to \mathrm{H}^k(Y) \to \mathrm{H}^k(Y-X) \to \ \mathrm{H}^{k+1}_X(Y) \to \cdots$$

associated to diagram (4.1).

Lemma 4.1.6. *The morphisms*

$$\begin{aligned} \mathrm{IH}^k(X,\mathbb{Q}) &\to \mathrm{H}^{k+2}(Y,\mathbb{Q}); \quad (\textit{Gysin map } i_!) \\ \mathrm{H}^{k+2}_X(Y,\mathbb{Q}) &\to \mathrm{H}^{k+2}(Y,\mathbb{Q}); \quad k \in \mathbb{N}\ , \end{aligned}$$

associated to the distinguished triangles in (4.2) are surjective.

Proof. By (4.1) it is enough to show the surjectivety of $i_!$. The absolute fundamental-class $c_X \in \mathrm{H}^2(Y,\mathbb{Q})$ of X in Y is the image in absolute cohomology of $c_{X|Y} \in \mathrm{H}^2_X(Y,\mathbb{Q})$.

Of course $c_{Y|X} = \deg(X)\cdot H \neq 0$ (H, the class of a hyperplane in Y) and as $i_!$ fulfills module relations

$$\alpha \cup i_!(\beta) = i_!(i^*(\alpha)\cup\beta) \qquad \forall \alpha \in \mathrm{H}^*(Y), \beta \in \mathrm{IH}^*(X)\ .$$

We see that $i_!$ is given by "cup-product with the fundamental class", i.e. for all l:

$$\deg(X)\cdot H^l = H^{l-1}\cup c_X = H^{l-1}\cup i_!(c_{X|Y}) = i_!\left(c_{X|Y}\cup i^*(H^{l-1})\right)$$

is in the image of $i_!$, which proves the surjectivety over $\mathbb{Q}$.

As this is equivalent (dual) to the injectivity of $i^*:\ \mathrm{H}^{2n-k}(Y) \to \mathbb{H}^{2n-k}(X)$ and the Lefschetz operator $c_X\cup$ factors over $\mathrm{IH}^k(X) \to\ \mathrm{H}^{k+2}(Y) \to \mathrm{IH}^{k+2}(X)$, we indeed get that $\mathrm{IH}^k_0(X) = \ker i_!$. □

Proposition 4.1.7. *The mixed Hodge complex* $(\mathrm{W},\mathrm{P},L,H)$ *from proposition 4.1.2 induces, for all k, mixed Hodge structures on* $\mathrm{H}^k(U)$, $\mathrm{H}^k_X(Y)$ *and pure Hodge structures on* $\mathrm{IH}^k(Y|X) = \mathrm{IH}^{k-1}_0(X)(-1)$ *and* $\mathrm{IH}^k(X)$.

Proof. The mixed Hodge structure on $\mathrm{H}^k(U)$ is of course the one from (W, P, L, H).

Let $({}^{\backprime}\mathrm{W}, {}^{\backprime}\mathrm{P}, {}^{\backprime}L, {}^{\backprime}H) := (\mathrm{W}|_{\mathrm{W}_0(H)}, \mathrm{P}|_{\ \mathrm{W}_0(H)}, \mathrm{W}_0(L), \mathrm{W}_0(H))$. For the inclusion

$$i : ({}^{\backprime}\mathrm{W}, {}^{\backprime}\mathrm{P}, {}^{\backprime}L, {}^{\backprime}H) \hookrightarrow (\mathrm{W},\mathrm{P},L,H),$$

there is a mixed Hodge complex Cone(i) given by the filtrations

$$\begin{aligned}\mathrm{W}_m \operatorname{Cone}(i_{\mathbb{Q}})^p &:= {}^{\lq}\mathrm{W}_{m-1}({}^{\lq}L)^{p+1} \oplus \mathrm{W}_m(L)^p \\ \mathrm{P}^q \operatorname{Cone}(i_{\mathbb{C}})^p &:= {}^{\lq}\mathrm{P}^q({}^{\lq}H)^{p+1} \oplus \mathrm{P}^q(H)^p\end{aligned}$$

on the ordinary cones of L and H. This mixed Hodge complex induces a mixed Hodge structure on $\mathbb{H}^k(\operatorname{Cone}((\mathrm{W}_0 L \to WL)[1]) = \mathrm{H}^k_X(Y)$ for all k.

Similarly $(\mathrm{W}|_{\mathrm{W}_0(H)}, \mathrm{P}|_{\mathrm{W}_0(H)}, \ \mathrm{W}_0(L), \mathrm{W}_0(H))$ is by definition a MHC inducing a MHS on $\mathrm{H}_X(Y)$ such that the cone of the inclusion ι of $({}^{\lq}\mathrm{W}, {}^{\lq}\mathrm{P}, {}^{\lq}L, {}^{\lq}H)$ into it gives a MHC for the intersection cohomology.

Indeed, it is pure (HC) because $\operatorname{Cone} \iota_{\mathbb{Q}} = \operatorname{Gr}^{\mathrm{W}}_1 L$ and induces a pure Hodge structure on all intersection cohomology groups.

By the last lemma, $\mathrm{IH}^k(Y|X) = \ker i_! = \mathrm{IH}^{k-1}_0(X)(-1)$ is pure of weight $k+1$. □

4.2 The Pairing

Thus, we have for $m \leq n-2$, $n+1 \leq m$ commuting isomorphisms

$$\begin{array}{ccc} \mathrm{IH}^{m+1}(Y|X) & = & \mathrm{IH}^m_0(X) \\ \| & & \| \\ \mathrm{H}^{m+1}(U) & = & \mathrm{H}^{m+2}_{X\,0}(Y) \end{array},$$

and the diagram with exact rows and columns

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathrm{IH}^n(Y|X) & = & \mathrm{IH}^{n-1}_0(X) \\ \downarrow & & \downarrow \\ \mathrm{H}^n(U) & = & \mathrm{H}^{n+1}_{X\,0}(Y) \\ \downarrow & & \downarrow \\ \mathrm{H}^n & = & \mathrm{H}^n_0 \\ \downarrow & & \downarrow \\ \mathrm{IH}^{n+1}(Y|X) & = & \mathrm{IH}^n_0(X) \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array},$$

where we denote by $\mathrm{H}^k_{X\ 0}(Y) := \ker(\mathrm{H}^k_X(Y) \to \mathrm{H}^k(Y))$ the *putative primitive cohomology with support.* Of course it does not belong to an $\mathrm{sl}_2\mathbb{C}$ representation, nor it is self dual, etc.

We can read off from this the well known following theorem

Proposition 4.2.1 (Lefschetz theorem for intersection cohomology for a divisor with isolated singularities in $\mathbb{P}^n$)**.** $\mathrm{IH}^k_0(X) = 0$ *for* $|k - \dim(X)| > 1$.

Proof. Indeed we do not need the local quasi-homogeneity for this, the analog sheaf H is still perverse with zero dimensional support and hence concentrated in degree n.

As U is affine, $\mathrm{IH}^m_0(X) = \mathrm{H}^{m+1}(U) = 0$ for $m > n$. Selfduality of intersection cohomology implies vanishing for $m < n-2$. □

Let us take a closer look at the middle primitive cohomology:

Lemma 4.2.2.

$$\mathrm{Gr}^{p-1}\,\mathrm{IH}^{n-1}_0(X) = \mathrm{Gr}^p_{\mathrm{P}}\,\mathbb{H}^n\mathrm{W}_1(H) = \mathbb{H}^n\,\mathrm{Gr}^p_{\mathrm{P}}\ \mathrm{W}_1(H)$$

Proof. For notation we refer to the appendix, which is based on [Del71], [Del74], [PS99] and [BDIC96]. On $\mathbb{H}^n(H)$, for each m the spectral sequence starting with

$$ {}^{\mathrm{W}}\mathrm{E}_1^{m\ q} = \mathbb{H}^{m+q}\,\mathrm{Gr}^{\mathrm{W}}_{-m} H$$

abuts to $\mathrm{Gr}^{\mathrm{W}}_{-m}\,\mathbb{H}^{m+q}(H)$, which is a pure Hodge structure of weight q. It degenerates at $\mathrm{E}_2 = \mathrm{E}_\infty$.

This means $\mathrm{Gr}^{\mathrm{W}(n)}_{n+1}\,\mathbb{H}^n(H)$ is a pure Hodge structure of weight $n+1$, which we can calculate by the cohomology of

$$\begin{array}{ccccc} {}^{\mathrm{W}}\mathrm{E}_1^{-2\ \ n+1} & \to & {}^{\mathrm{W}}\mathrm{E}_1^{-1\ \ n+1} & \to & {}^{\mathrm{W}(n)}\mathrm{E}_1^{0\ \ n+1} \\ \| & & \| & & \| \\ \mathrm{H}^{n-1}\,\mathrm{Gr}^{\mathrm{W}}_2 H & \to & \mathrm{H}^n\,\mathrm{Gr}^{\mathrm{W}}_1 H & \xrightarrow{d_1} & \mathrm{H}^{n+1}\,\mathrm{Gr}^{\mathrm{W}}_0 H \end{array}$$

in the middle. $\mathrm{Gr}^{\mathrm{W}}_0 H = \mathrm{W}_0 H$ and by the long exact sequence associated to

$$\mathrm{W}_0 H \to \mathrm{W}_1 H \to \mathrm{Gr}^{\mathrm{W}}_1 H \xrightarrow{[1]},$$

which we already studied, we see that $\ker d_1 = \mathrm{H}^n\ \mathrm{W}_1 H = \mathrm{IH}^{n-1}_0(X)(-1)$.

Moreover, $\mathrm{H}^{n-1}\,\mathrm{Gr}_2 H = \mathrm{H}^{n-1} S = 0$ because S is perverse with zero dimensional support.

By the E_1 degeneracy of ${}_{\mathrm{P}}\mathrm{E}^{p\ q}$ and ${}^1_{\mathrm{P}}\mathrm{E}^{p\ q}$, this is the claim. □

With proposition 3.4.19, we proved E_1 degeneracy for spectral sequence ${}_{\mathrm{P}}\mathrm{E}_1$ abutting to the local cohomology sheaves. We have

$$\mathrm{H}^k \operatorname{Gr}^p_{\mathrm{P}} H = \operatorname{Gr}^p_{\mathrm{P}} \mathrm{H}^k H;$$

$$\mathrm{H}^k \operatorname{Gr}^p_{\mathrm{P}} \operatorname{Gr}_1 H = \operatorname{Gr}^p_{\mathrm{P}} \mathrm{H}^k \operatorname{Gr}_1 H;$$

this means in particular that the truncation functor τ and $\operatorname{Gr}_{\mathrm{P}}$ commute.

Lemma 4.2.3. *There are the following identities:*

$$\begin{array}{ccccccc}
\operatorname{Gr}^p_{\mathrm{P}} \mathrm{W}_0 H & \to & \operatorname{Gr}^p_{\mathrm{P}} \mathrm{W}_1 H & \to & \operatorname{Gr}^p_{\mathrm{P}} \operatorname{Gr}^{\mathrm{W}}_1 H & \xrightarrow{[1]} & \\
\| & & \| & & \| & & \\
\tau_{\leq -n} \operatorname{Gr}^p_{\mathrm{P}} H & \to & \tau_{\leq -1} \operatorname{Gr}^p_{\mathrm{P}} H & \to & \tau^{\geq 1-n}\tau_{\leq -1} \operatorname{Gr}^p_{\mathrm{P}} H & \xrightarrow{[1]} & ,
\end{array}$$

where $\tau^{\geq 1-n}\tau_{\leq -1} \operatorname{Gr}^p_{\mathrm{P}} H = \tau_{\leq -1} l_* l^* \operatorname{Gr}^p_{\mathrm{P}} i_* i^! H$ *equals*

$$\tau_{\leq -1} l_* l^* \operatorname{Gr}^p_{\mathrm{P}} \Omega^\bullet(*X)/\Omega^\bullet = \tau_{\leq -1} \vec{\mathrm{K}}(F_0, \dots, F_n)[1]^{\bullet \geq 0} \otimes \operatorname{Gr}^p_{\mathrm{P}} \Omega^\bullet(*X)/\Omega^\bullet))$$

Thus, we can apply our calculus for global calculations. The first step is

Proposition 4.2.4. *Any class in* $\mathrm{P}^p \operatorname{IH}^n(Y|X(\mathbb{C}))$ *or* $\mathrm{P}^{p-1} \operatorname{IH}^{n-1}_0(X(\mathbb{C}))$ *can be represented by a global n-form*

$$\alpha := \frac{A\Omega}{F^{n-p+1}} \in \Gamma(Y, \Omega^n(n-p+1)); \quad A \in \mathbb{C}[x_0, \dots, x_n]_{(n-p+1)d-n-1},$$

$d = \deg(F)$, $\Omega = i_E dV$, $dV = dx_0 \wedge \dots \wedge dx_n$.

Proof. We have

$$\begin{aligned}
&\tau_{\leq n} \mathrm{P}^p \Omega^\bullet(*X) \\
&\qquad = \left(\Omega^p(X) \to \Omega^{p+1}(2X) \cdots \Omega^{n-1}((n-p)X) \to d\Omega^{n-1}((n-p)X)\right),
\end{aligned}$$

which are acyclic sheaves except for $d\Omega^{n-1}((n-p)X)$ by Bott vanishing.

This means, the E_1 spectral sequence for the stupid filtration on this complex has $\mathrm{E}_1^{p\ q} = 0$ for all p, q, such that $p + q \leq n$ and $q > 0$. Hence $\mathrm{P}^p \operatorname{IH}^n(Y|X(\mathbb{C})) = \Gamma(Y, d\Omega^{n-1}((n-p)X)/d\Gamma(Y, \Omega^{n-1}((n-p)X)$.

As $\mathrm{P}^p \operatorname{IH}^n(Y|X(\mathbb{C})) = \mathrm{P}^{p-1} \operatorname{IH}^{n-1}_0(X, \mathbb{C})$ we get precisely the intersection classes in this way. □

The prototype in the smooth case of the following calculation is in Carlson and Griffiths article [CG80] from 1979. Given $\alpha \in \Gamma(Y, \Omega^n((n-p+1)X)$, $\beta \in$

$\Gamma(Y, \Omega^n(pX)$ representing primitive forms in $\mathrm{Gr}_{\mathrm{P}}^{p-1} \mathrm{H}_0^{n-1}(X)$, $\mathrm{Gr}_{\mathrm{P}}^{n-p} \ \mathrm{H}_0^{n-1}(X)$ respectively, they used the Jacobi cover

$$J = (\mathrm{D}_+(F_i))_{i=0,\ldots,n} \,;$$

of $Y = \mathbb{P}^n$, given by the derivatives $F_i := \frac{\partial}{\partial x_i} F$ of a defining equation F of X to reduce the pole-order of the representatives step by step and received their representatives in the log complex and their residues in $\check{\mathrm{H}}^{n-p}(J, \Omega_X^{p-1})$ and $\check{\mathrm{H}}^{p-1}(J, \Omega_X^{n-p})$.

After that, they calculated the cup product $\operatorname{Res} \alpha \cup \operatorname{Res} \beta \in \check{\mathrm{H}}^{n-1}(\Omega_X^{n-1})$ and received that, up to a constant factor

$$i_! \left(\operatorname{Res} \alpha \cup \operatorname{Res} \beta\right) = \frac{AB\Omega}{F_0 \cdots F_n} \in \mathrm{v}\ \mathrm{H}^n(J, \Omega_Y^n)$$

represents the product.

We want to perform the analog calculation, but without Jacobi cover, log forms and Kaehler differential on X. Instead we work with $\widetilde{A}$ and A. Please recall the setting in proposition 3.3.5. There we have seen that $\frac{\mathrm{d}f}{f} : \widetilde{A}' \to A'$, where $\widetilde{A}' := \vec{\mathrm{K}}(F_0, \ldots, F_n) \otimes \widetilde{A}$, $A' := \vec{\mathrm{K}}(F_0, \ldots, F_n) \otimes A$ is a filtered quasi-isomorphism. Assume we had a fixed inverse Res. By the natural pairing on $A \times \widetilde{A} \to \mathbb{D}_Y = \Omega^\bullet[2n]$ induced from wedge product $\Omega^p(kX) \times \Omega^{n-p}(-k) \to \Omega^n$, as in the smooth case, we get a morphism

$$\tau_{\leq -1}\widetilde{A}' \times \tau_{\leq -1}A' \to \mathbb{D}_Y \ .$$

Under the corresponding map

$$\tau_{\leq -1}\widetilde{A}' \to \mathbb{D}(\tau_{\leq -1}A'); \quad a \mapsto \langle \ , a\rangle$$

$\tau_{\leq -1}\widetilde{A}'$ gets identified with $\mathbb{D}\tau_{\leq -1}A' = \mathbb{D}\tau_{\leq -1}l_* l^* A = \widetilde{\tau}^{\geq 1} l_! l^* \widetilde{A}$.

We can hence calculate the pairing via

$$\begin{array}{ccc}
\mathbb{H}^0(Y, \tau_{\leq -1} \mathrm{Gr}_{\mathrm{P}} A') \times \mathbb{E}xt^0_{\mathbb{C}}(\tau_{\leq -1} \mathrm{Gr}_{\mathrm{P}} \widetilde{A}', \mathbb{D}_Y) & \xrightarrow{ev} & \mathbb{H}^0(Y, \tau_{\leq -1}\vec{\mathrm{K}}(F_0, \ldots, F_n) \otimes \mathrm{Gr}_{\mathrm{P}} \mathbb{D}_Y)) \\
{\scriptstyle id \times \mathrm{Res}} \uparrow & & \| \\
\mathbb{H}^0(Y, \tau_{\leq -1} \mathrm{Gr}_{\mathrm{P}} A') \times \mathbb{H}^0(X, \tau_{\leq -1} \mathrm{Gr}_{\mathrm{P}} A') & \xrightarrow{\smile} & \mathbb{H}^0(\Omega^n[n])
\end{array}$$

using this naive pairing. Recall the notations from lemma 3.1.3, definition 3.3.4 and equation (3.2) on page 24.

Definition 4.2.5. *Let* ${}'K_p^\bullet := \mathrm{Cone}\left(\mathrm{Gr}_{\mathrm{P}}^p(dF/F) \to K_p^\bullet\right)$ *denote the augmentation of* $K_p^\bullet$.

Proposition 4.2.6. [1]

$$Dexp(\Theta) = exp(\Theta)\frac{dF}{F} \quad \in End_{\mathbb{C}}({}^{(\prime)}K^{\bullet})$$

Proof. First of all, from the formula $\Theta = \sum e_k v_k$ we get that Θ operates on the augmentation. From the basic identities for the bracket, it follows that in $End_{\mathbb{C}}({}^{(\prime)}K_p^{\bullet})$ we have

$$[\Theta, C] = 0,$$

$$[\Theta, \frac{dF}{F}] = [\sum e_k v_k, \frac{dF}{F}] = C,$$

hence by induction also $[\Theta^k, \frac{dF}{F}] = kC\Theta^{k-1} \; \forall k \in \mathbb{N}$ so that

$$[exp(\Theta), \frac{dF}{F}] = Cexp(\Theta).$$

□

Lemma 4.2.7. *Let*

$$C\alpha := C\frac{A\Omega}{F^{n-p+1}} \in \Gamma(Y, \tau_{\leq -1}(\Omega^n(*X)[n]); \quad A \in R_{(n-p+1)d-n-1}.$$

represent a class in $\mathbb{H}^n \tau_{\leq -1} \operatorname{R}\underline{\Gamma}_{[\Sigma|Y]} \operatorname{Gr_P} A$.

Then

$$\operatorname{Res} C\alpha := \frac{\Theta^n}{n!}C\alpha \in \mathbb{H}^n(\tau_{\leq -1}\operatorname{R}\underline{\Gamma}_{[\Sigma|Y]} \operatorname{Gr_P} \widetilde{A}),$$

fulfills $dF/F(\operatorname{Res} C\alpha) = C\alpha$, *where*

$$dF/F : \tau_{\leq -1}\operatorname{R}\underline{\Gamma}_{[\Sigma|Y]} \operatorname{Gr_P} \widetilde{A} \to \tau_{\leq -1}\operatorname{R}\underline{\Gamma}_{[\Sigma|Y]} \operatorname{Gr_P} A \ .$$

Proof. It is enough to show that $C\alpha$ and $C\frac{\Theta^n}{n!}\alpha = \frac{\Theta^n}{n!}C\alpha$ define the same classes on the cone $K_p^{\bullet}$. For this we consider the relation

$$dF/F \wedge \alpha = 0,$$

on the augmented cone ${}'K_p^{\bullet}$: From the last proposition it follows that

$$\begin{aligned} 0 = & \exp(\Theta)dF/F\alpha \\ & = D\exp(\Theta)\alpha \\ & = C\frac{\Theta^n}{n!}\alpha + D(\sum_{k=1}^{n-1}\frac{\Theta^k}{k!})\alpha + C\alpha \ , \end{aligned}$$

[1] ${}^{(\prime)}K_p^{\bullet}$ shall denote $K_p^{\bullet}$ and ${}'K_p^{\bullet}$ at the same time.

which is the desired relation in the non augmented complex K_p.

The same argument applied locally implies that if α is locally exact, then $\frac{\Theta^n}{n!}C\alpha$ is. Therefore $\frac{\Theta^n}{n!}C\alpha$ can be considered as element of $\tau_{\leq -1}\mathrm{R}\underline{\Gamma}_{[\Sigma|Y]}\,\mathrm{Gr}^p_P\,\widetilde{A}$ which maps to $C\alpha$ under dF/F. □

Proposition 4.2.8 (pairing)**.** *Let*

$$\alpha = \frac{A\Omega}{F^{n-p+1}} \in \Gamma(Y, \mathrm{P}^p\tau_{\leq -1}(\Omega^n(*X)[n]),$$

$$\beta = \frac{B\Omega}{\mathrm{P}^p} \in \Gamma(Y, \mathrm{P}^{n-p+1}\tau_{\leq -1}(\Omega^n(*X)[n])$$

represent classes in

$$\mathrm{P}^p\,\mathrm{IH}^n(Y|X) = \mathrm{P}^{p-1}\,\mathrm{IH}^{n-1}_0(X), \quad \mathrm{P}^{p+1}\,\mathrm{IH}^n(Y|X) = \mathrm{P}^{n-p}\,\mathrm{IH}^{n-1}_0(X).$$

Then

$$\alpha \cup \beta = \frac{(-1)^{n-p}p!}{n-p!}\frac{AB\Omega}{F_0\cdot\ldots F_n} \in \mathrm{H}^n(Y, \tau_{\leq -1}l_*l^*(\Omega^n)) = \mathrm{H}^n(Y, \Omega^n)$$

Proof. The product must factorize over $\mathrm{Gr}^{p-1}_{\mathrm{P}}\,\mathrm{IH}^{n-1}_0(X) \times \mathrm{Gr}^{n-p}_{\mathrm{P}}\,\mathrm{IH}^{n-1}_0(X)$, so we send α to $C\alpha \in \mathrm{Gr}^p_{\mathrm{P}}\,A'$ and β to $C\frac{\Theta^n}{n!}\beta$ as discussed above and have to interpret the expression

$$C\frac{A\Omega}{F^{n-p+1}} \cup C\frac{\Theta^n}{n!}\frac{B\Omega}{F^p}.$$

Here the cup product is given by ordering the monomials in the e_i and using the usual first p-face right q-face formula defined via Whitney diagonal approximation, i.e. the standard cup product for simplicial modules.

Calculating in the augmented complex, we see

$$\begin{aligned} C\frac{\Theta^n}{n!}\frac{B\Omega}{F^p} &= \frac{B}{F^p}C\frac{\Theta^n}{n!}i_E dV \\ &= \frac{B}{F^p}i_E\frac{\Theta^n}{n!}CdV \\ &= \frac{B}{F^p}i_E[\frac{dF}{F}, \frac{\Theta^{n+1}}{n+1!}]dV \\ &= \frac{B}{F^p}i_E\frac{dF}{F}\frac{\Theta^{n+1}}{n+1!}dV \\ &= \frac{\deg(F)}{F^p}\cdot B\cdot F^{n-p+1}\frac{e_0}{F_0}\wedge\cdots\wedge\frac{e_n}{F_n} \end{aligned}$$

and hence

$$C\frac{A\Omega}{F^{n-p+1}}\cup C\frac{\Theta^n}{n!}\frac{B\Omega}{F^p} = C\frac{A\Omega}{F^{n-p+1}}\cup\frac{\deg(F)\cdot Be_0\wedge\cdots\wedge e_n}{F^pF_0\dots F_n}$$
$$= \deg(F)AB\Omega\frac{e_0}{F_0}\wedge\cdots\wedge\frac{e_n}{F_n}\ .$$

If we lift this expression back to $\mathrm{H}^n(Y,\tau_{\leq -1}\vec{\mathrm{K}}(F_0,\dots,F_n)[1]^{\bullet\geq 0}\Omega^n)$, we get an element, representing β because of the isomorphy

$$\mathrm{P}^p\,\mathrm{IH}^n(Y|X) = \mathrm{P}^{p-1}\,\mathrm{IH}_0^{n-1}(X).$$

Besides: Another way to interpret this lift is that it performs the natural isomorphism $\mathrm{H}^{n-1}(X,\omega_X) = \mathrm{Ext}^{n-1}(\mathcal{O}_X,\mathcal{E}xt^1(\mathcal{O}_X,\Omega^n)) = \mathrm{Ext}^n(\mathcal{O}_Y,\Omega^n) = \mathrm{H}^n(Y,\Omega^n)$.
Finally compare this class with the de Rham complex by formula (3.1), to get the factor $\frac{(-1)^{n-p}p!}{n-p!}$ □

To describe a trace morphism, we need a generator of

$$\mathrm{H}^n(Y,\tau_{\leq -1}l_*l^*(\Omega^n)) \xrightarrow{tr} \mathbb{C}.$$

There is a natural candidate

$$\frac{1}{(2\pi i)^n}\cdot i_E\left(\frac{dF_0}{F_0}\wedge\cdots\wedge\frac{dF_n}{F_n}\right) = \frac{1}{(2\pi i)^n}\cdot\frac{H\Omega}{F_0\cdot\ldots\cdot F_n};$$

$$H = det(F_{i\ j});\quad F_{i\ j} = \frac{\partial^2}{\partial x_i\partial x_j}F.$$

It depends on the local multiplicity of H if $\omega := \frac{H\Omega}{F_0\cdot\ldots\cdot F_n}$ is locally exact, i.e. defines an intersection cohomology class of $\tau_{\leq -1}l_*l^*(\Omega^n)$).

$$H = \frac{1}{x_0^2}\begin{vmatrix} d(d-1)F & (d-1)F_1 & \cdots & F_n \\ (d-1)F_1 & F_{1\ 1} & \cdots & F_{1\ n} \\ \vdots & \vdots & \vdots & \vdots \\ (d-1)F_n & F_{n\ 1} & \cdots & F_{n\ n} \end{vmatrix}$$

has in $D(x_0)$ the homogenization

$$h := \begin{vmatrix} d(d-1)f & (d-1)f_1 & \cdots & f_n \\ (d-1)f_1 & f_{1\ 1} & \cdots & f_{1\ n} \\ \vdots & \vdots & \vdots & \vdots \\ (d-1)f_n & f_{n\ 1} & \cdots & f_{n\ n} \end{vmatrix}$$

so that for example at a node s at zero in $D(x_0)$, the restriction of ω is

$$\frac{h dx_1/x_0 \wedge \ldots \wedge dx_n/x_0}{f_0 \cdot \ldots \cdot f_n} ,$$

which is homogeneous wrt. the local Eulerfield $E = \sum_{i=1}^{n} x_i/x_0 \frac{\partial}{\partial x_i/x_0}$ with degree $(2+n) - (n+1) = 1 \neq 0!$ So

$$1 \cdot \omega|_{D(x_0)} = d(i_E \omega|_{D(x_0)})$$

is locally exact and gives an intersection class, indeed (see [HM98][1.6.] that the decomposition by $\mathcal{L}_E$ holds for cycles).

If it is everywhere defined, it is nonzero because

$$\alpha_{i\ j} := \frac{1}{2\pi i} \sum_{i<j} d\log(F_i/F_j) e_i \wedge e_j = \frac{1}{2\pi i} \sum_{i<j} (\frac{dF_i}{F_i} - \frac{dF_j}{F_j}) e_i \wedge e_j$$

is the first Chern class of $\mathcal{O}_Y(d-1)$, hence

$$\frac{1}{(2\pi i)^n} H\Omega \frac{e_0}{F_0} \wedge \cdots \wedge \frac{e_n}{F_n} = (\alpha_{i\ j})^{\cup n} = (d-1)^n \neq 0$$

and defines a trace morphism.

Chapter 5

Nodal Varieties

5.1 Hodge Filtration

In the case of isolated quasi-homogeneous singularities, everything can be made very explicitly. For example if locally $X = V(f)$, f homogeneous and the degree does not equal the dimension of the ambient space, $L_{Y|X}$ is just the sub $\mathcal{D}_Y$-module of the sheaf of meromorphic functions, generated by $1/f$; taken this $\mathcal{D}_Y$-module modulo holomorphic functions, one gets L_X. That is

$$\begin{array}{lcll} L_{Y|X} & = & D_Y \cdot \mathcal{O}_Y(X) & \subset \mathcal{O}(*X) \\ L_X & = & D_Y \cdot \mathcal{O}_Y(X)/\mathcal{O} & \subset \mathcal{O}(*X)/\mathcal{O} \; . \end{array}$$

Remember that L_X is simple, so for the latter, one only has to check that $1/f$ is a section of $L_X \subset \mathcal{O}(*X)/\mathcal{O}$ at all. For this, we can check if $l := 1/f$ passes the "Vilonen test":

Proposition (Vilonen test [Vil85]). *$l \in (\mathcal{O}_Y(*X)/\mathcal{O}_Y)_s$, $s \in \Sigma$ belongs to $(L_X)_s$ if and only*

$$\int_\gamma l \cdot h \mathrm{d}V = 0$$

for $\mathrm{d}V = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, h holomorphic and all classes $\gamma \in \mathrm{H}_n(B - s)$.

This condition is of course equivalent to the condition that

$$hl\mathrm{d}V \in \tau_{\leq n-1}\Omega^\bullet(*X)/\Omega^\bullet$$

for all holomorphic h.

In the case of a node, we can assume $f = \sum_i^n x_i^2$, for $n \geq 3$, we see that $hl\mathrm{d}V$ is homogeneous wrt. $\mathcal{L}_E$ of degree ≥ 1 for all h, hence has no local residue and passes the Vilonen test.

Example 5.1.1. *Let* $f = \sum_i^4 x_i^2$ *describe a node in* $\mathbb{C}^4$, $E = \sum_1^4 x_i \frac{\partial}{\partial x_i}$.

Let $E = \mathbb{P}^3$ *be the exceptional locus of the blow up of* $\mathbb{C}^4$ *at* s. *For any* $\gamma \in \mathrm{H}_n(B - s)$, *we can assume* $\gamma = \partial T(\eta)$ *is the boundary of a tubular neighborhood of a cycle* η *in* $\mathrm{H}_{n-1}(E)$. *Then*

$$\begin{aligned}\int_\gamma \frac{\mathrm{d}V}{f^2} &= \frac{1}{2}\int_\gamma \frac{\mathrm{d}f}{f} \wedge \frac{i_E \mathrm{d}V}{f^2} \\ &= \pi i \int_\eta \frac{i_E \mathrm{d}V}{f^2}\end{aligned}$$

And this is not zero for all γ *because by* $\frac{i_E \mathrm{d}V}{f^2}$ *generates* $\mathrm{H}^{n-1}(E - Q)$; $Q = V_+(f) \subset E$. *Taking* $h = f^{k-2}$, *we see* $1/f^k \notin L_X$ *for* $k \geq 2$. *Nevertheless,*

$$\int_\gamma \frac{\mathrm{d}V}{f^k} = 0$$

for all $k \neq 2$, *so we see that*

$$\Omega^\bullet(L_X) \subsetneq \tau_{\leq -1}(\Omega^\bullet(*X)/\Omega^\bullet[n])$$

The construction of the good filtration on $\mathcal{O}(*X)$ and L_X making them mixed Hodge modules was done in [Sai96] [1]. We extract from this paper the explicit filtration in case of our main interest, namely the case of a nodal hypersurface:

Proposition 5.1.2 (special cases of thm 0.7, 0.8 in [Sai96])**.** *Let* X *be a nodal hypersurface in a smooth* n*-dim manifold* Y, $m := \lfloor n \rfloor$ *(the biggest integer not bigger than* $n/2$ *('off-round')). Then*

$$\begin{aligned}
\mathrm{F}_p(\mathcal{O}(*X)) &= \begin{cases} \mathcal{O}_Y((p+1)X) & p+1 \leq m \\ \mathrm{F}_{p-m+1}\mathcal{D}_Y \cdot \mathcal{O}_Y(mX) & p+1 \geq m \end{cases} \\
\mathrm{F}_p(\mathcal{O}(*X)/\mathcal{O}) &= \begin{cases} \mathcal{O}_Y((p+1)X)/\mathcal{O} & p+1 \leq m \\ \mathrm{F}_{p-m+1}\mathcal{D}_Y \cdot (\mathcal{O}_Y(mX)/\mathcal{O}) & p+1 \geq m \end{cases} \\
\mathrm{F}[-1]_p(L_X) &= \begin{cases} \mathcal{O}_Y((p+1)X)/\mathcal{O} & p+1 \leq m-1 \\ \mathrm{F}_{p-m+2}\mathcal{D}_Y \cdot (\mathcal{O}_Y((m-1)X)/\mathcal{O}) & p+1 \geq m-1 \end{cases}
\end{aligned}$$

For $p < 0$, $\mathrm{F}_p(L_X) = \mathrm{F}_p(\mathcal{O}_Y(X)/\mathcal{O}) = \mathrm{F}_p(\mathcal{O}(*X)) = 0.$

Corollary 5.1.3. $L_{Y|X} = \mathcal{D}_Y \cdot \mathcal{O}_Y(X)$ *and*

$$\mathrm{F}_p L_{Y|X} = \begin{cases} \mathcal{O}_Y((p+1)X) & p \leq m-2 \\ \mathrm{F}_{p-m+2}\mathcal{D}_Y \cdot \mathcal{O}_Y((m-1)X) & p \geq m-2 \end{cases}$$

[1] In the notation of [Sai96] $M = L_X, \mathrm{F}M = \mathrm{F}[1]L_X; M' = \mathcal{O}(*X); M'' = \mathcal{O}(*X)/\mathcal{O}$

Proof. Morphisms of mixed Hodge modules are strict, therefore $\mathrm{F}_p L_{Y|X} = \pi^{-1}(\mathrm{F}[-1]_p L_X)$ is as claimed; $\pi : L^H_{Y|X} \to L^H_X$ the canonical projection. □

Corollary 5.1.4. *If $n = \dim Y$, $\mathcal{I}$ is the ideal-sheaf of the nodes, $m = \lfloor n \rfloor$, then*

$$\begin{aligned}
\mathrm{F}_p(\mathcal{O}(*X)) &= \begin{cases} \mathcal{O}_Y((p+1)X) & p+1 \le m \\ \mathcal{I}^{p+1-m}((p+1)X) & p+1 \ge m \end{cases} \\
\mathrm{F}_p(\mathcal{O}(*X)/\mathcal{O}) &= \begin{cases} \mathcal{O}_Y((p+1)X)/\mathcal{O} & p+1 \le m \\ \mathcal{I}^{p+1-m}((p+1)X)/\mathcal{O} & p+1 \ge m \end{cases} \\
\mathrm{F}[-1]_p(L_X) &= \begin{cases} \mathcal{O}_Y((p+1)X)/\mathcal{O} & p+1 \le m-1 \\ \mathcal{I}((p+1)X)/\mathcal{O} & p+1 = m-1 \\ \mathcal{I}^{p+2-m}((p+1)X)/\mathcal{O} & p+1 \ge m-1 \textit{ and } n \textit{ odd} \end{cases}
\end{aligned}$$

Proof. [2] In analytic charts, we may assume, $X = V(f)$, $f = \sum_i^n x_i^2$. We show on induction on $q \ge 0$ that

$$\mathrm{F}_q \mathcal{D}_Y \cdot f^{-m} = \mathcal{I}^q f^{-m-q};\ m = \lfloor n \rfloor.$$

The inclusion "$\subset$" being clear, for this we have to show that $x^a f^{-m-q} \in \mathcal{I}^q f^{-m-q}$ belongs to the left-hand side if $|a| = q$, where $a = (a_1, \dots, a_n)$. Here we may assume $a_i \ne 1$ for all i and $q > 1$ because the assertion is easy otherwise. Then we have $x^a = x_i^2 x^b$ for some i.

Now consider

$$\partial_j(x_j x^b f^{-(m+q-1)}) = ((b_j+1)f - (m+q-1)x_j f_j)x^b f^{-m-q}$$

for all $j = 1, \dots, n$.

Adding this over j, we see that $x^b f^{-m-q+1}$ belongs to the left-hand side because $|b| + n - 2(m+q-1) = n - 2m - q < 0$. So the assertion follows, as $\partial_i(x_i x^b f^{-(m+q-1)}) = (b_i+1)x^b f^{-m-q+1} - (m+q-1)x^a f^{-m-q}$. □

Putting all things together, we get the following result:

Proposition 5.1.5. *Let X be a nodal hypersurface of degree k in $Y := \mathbb{P}^n$, $U := Y - X$. Then*

$$\mathrm{Gr}^p_{\mathrm{F}} \mathrm{H}^n(U) = \begin{cases} (R/J)_{(n-p+1)k-n-1} & p > n-m \\ (I/J)_{(n-p+1)k-n-1} & p = n-m \end{cases}$$

[2] This trick originates from a mail from M. Saito.

where
$$J = \langle \partial_0 F, \ldots, \partial_n F \rangle \subset R := \mathbb{C}[x_0, \ldots, x_n]$$
denotes the Jacobi ideal and $I = (I : \mathfrak{m}_0^*) \subset R$ *is the saturation of* J*, i.e. the ideal of the lines in* $\mathbb{A}^{n+1}(\mathbb{C})$ *corresponding to the nodes (in particular* $I_{(d)} = \mathrm{H}^0(Y, \mathcal{I}(d))$ *for all* $d \in \mathbb{Z}$*).*

Proof. By proposition 3.4 and corollary 5.1.4

$$\mathrm{F}^p \mathrm{H}^n(U) = \Gamma(Y, \Omega^n(\mathrm{F}_{n-p} \mathcal{O}(*X)))/d\Gamma(Y, \Omega^{n-1}(\mathrm{F}_{n-p-1} \mathcal{O}(*X))).$$

To unify the notation, put $I^k := R$, $\mathcal{I}^k := \mathcal{O}_Y$ for $k \leq 0$ so that

$$\mathrm{F}_q \mathcal{O}(*X) = \mathcal{I}^{q-m+1}((q+1)X) \text{ for all } q.$$

For $n \geq n-m$, every

$$\omega \in \Gamma(Y, \Omega^n(\mathrm{F}_{n-p}\mathcal{O}(*X))) = \Gamma(Y, \mathcal{I}^{n-p+1-m}\Omega^n((n-p+1)X)))$$

can be written as

$$\omega = i_E \frac{AdV}{F^{n-p+1}}; \quad A \in (I^{n-p+1-m})_{(n-p+1)k-n-1} \subset R_{(n-p+1)k-n-1} \ ,$$

thus there is a surjective map

$$(I^{n-p+1-m})_{(n-p+1)k-n-1} \to \mathrm{Gr}_{\mathrm{F}}^p \mathrm{H}^n(Y - X) \ .$$

Every $\eta \in \Gamma(Y, \Omega^{n-1}(\mathrm{F}_{n-p-1}\mathcal{O}(*X))) = \Gamma(Y, \mathcal{I}^{n-p-m}\Omega^{n-1}((n-p)X)))$ is of the type
$$i_E \frac{\sum G_i dx^i}{F^{n-p}}; G_i \in (I^{n-p-m})_{(n-p)k-n} \ .$$
Hence every boundary-cycle in this group is of the form

$$\begin{aligned} d\eta = di_E \frac{\sum G_i dx^i}{F^{n-p}} &= i_E d \frac{\sum G_i dx^i}{F^{n-p}} \\ &= \frac{\sum \partial_i G_i \Omega}{F^{n-p}} - (n-p) \frac{\sum G_i \partial_i F \Omega}{F^{n-p+1}} \end{aligned}$$

For $p \geq n-m$ $\mathrm{F}_p \mathcal{O}(*X) = \mathcal{O}(n-m+1)X$, hence

$$\frac{\sum \partial_i G_i \Omega}{F^{n-p}}$$

is a section in $\Gamma(Y, \mathrm{F}^{p+1}\Omega^n(*X))$ so that

$$d\eta \equiv -(n-p)\frac{\sum G_i \partial_i F \Omega}{F^{n-p+1}} \in \mathrm{Gr}_{\mathrm{F}}^p \mathrm{H}^n(Y-X) \ ;$$

i.e.

$$\omega = i_E \frac{AdV}{F^{n-p+1}}$$

is zero in $\mathrm{Gr}_{\mathrm{F}}^p \mathrm{H}^n(Y-X)$ iff $A \in J_{(n-p+1)k-n-1}$.

□

Proposition 5.1.6.

$$\mathrm{Gr}_{\mathrm{F}}^p \mathrm{H}^n(Q) = \begin{cases} \mathbb{C}^\Sigma & p = (n-m+1) \\ 0 & else \end{cases}$$

where Σ denotes the singular locus.

Proof. Clear, as $\mathrm{F}_p L_{Y|X} = \mathrm{F}_p \mathcal{O}(*X)$ for $p < m-1$, for $p = m-1$ we have the inclusion in the direction "$\subsetneq$" , and $\mathcal{H}^n(Q) \simeq \mathbb{C}_\Sigma$ has only one-dimensional stalks. □

Proposition 5.1.7. *Let $X = V(F)$ be a nodal hypersurface of degree k in $Y := \mathbb{P}^n$, $U := Y - X$. Then*

$$\mathrm{Gr}_{\mathrm{F}}^{p-1} \mathrm{IH}_0^{n-1}(X) = \begin{cases} (R/J)_{(n-p+1)k-n-1} & p > n-m+1 \\ (I/J)_{mk-n-1} & p = n-m+1 \\ (I/J)_{(m+1)k-n-1} & p = n-m \end{cases}$$

Proof. By the exact sequence

$$0 \to \mathrm{Gr}_{\mathrm{F}}^{p-1} \mathrm{IH}_0^{n-1}(X) \to \mathrm{Gr}_{\mathrm{F}}^p \ \mathrm{H}^n(U, \mathbb{C}) \to \mathrm{Gr}_{\mathrm{F}}^p \mathbb{H}^n(Y, Q)$$

and the proposition above, we see that we are ready for $p \neq n-m+1$.

In the remaining case, we have to study

$$\mathrm{F}^{n-m} \mathrm{IH}_0^{n-1}(X) = \mathrm{H}^1\mathrm{tot} \begin{pmatrix} \Gamma\Omega^n(mX) & \xrightarrow{q} & \Gamma\Omega^n(mX)/\mathcal{I}\Omega^n(mX) \\ \uparrow & & \uparrow{\scriptstyle \Gamma(d_{n-1})} \\ \Gamma\Omega^{n-1}((m-1)X) & \xrightarrow{q} & 0 \end{pmatrix} .$$

Hence $\omega \in \Gamma\Omega^n \mathrm{F}_{m-1}\mathcal{O}(*X)$ defines a class in this double complex iff $q(\omega) = 0$, i.e. precisely if $\omega \in \Gamma\Omega^n \ \mathrm{F}_{m-1}(L_{Y|X}) = \Gamma\mathcal{I}\Omega^n(mX)[n] = \Gamma\tau_{\leq -1}\Omega^n(mX)[n]$, $\mathcal{I}$ the idealsheaf of the singular locus. In the same way as above, one deduces that $\mathrm{Gr}_{\mathrm{F}}^{n-m+1} \mathrm{IH}_0^{n-1}(X) = (I/J)_{mk-n-1}$. □

5.2 Nodal Hypersurfaces in $\mathbb{P}^4$

Let X be a nodal hypersurface in $\mathbb{P}^4$ and $\pi\hat{X} \to X$ a small resolution of X, which replaces each node by a copy of $\mathbb{P}^1$. Although $\hat{X}$ may be not Kaehler, it is Moisheson with a dominant and generically finite map from a projective variety (e.g. the blow up $\widetilde{X} = \hat{X} \times_X \hat{X}$ of X at the nodes). It is known that than there is a functorial Hodge structure on the cohomology of $\hat{X}$ such that $\mathrm{H}^k(\hat{X})$ is a sub-Hodge structure of $\mathrm{H}^k(\widetilde{X})$, and $\mathrm{H}^{pq}(\hat{X}) \simeq \mathrm{H}^q(\hat{X}, \Omega^p_{\hat{X}})$. In particular

$$\mathrm{H}^3(\hat{X}) = \mathrm{H}^3(\widetilde{X})$$

as a Hodge structure because they have the same dimension [Wer87]

Let $h^{pq} := \dim \mathrm{H}^{pq}$, $d := h^{11} - 1$. Because $h^{01} = h^{02} = 0$, we have

$$\left(h(\hat{X})^{p\ q}\right)_{p\ q} = \begin{array}{c|c|c|c} h^{03}(\hat{X}) & 0 & 0 & 1 \\ \hline 0 & h^{1\ 2}(\hat{X}) & h^{2\ 2}(\hat{X}) & 0 \\ \hline 0 & h^{1\ 1}(\hat{X}) & h^{2\ 1}(\hat{X}) & 0 \\ \hline 1 & 0 & 0 & h^{30}(\hat{X}) \end{array}$$

If X_t denotes a smooth hypersurface of the same degree than X and s the number of nodes, than for the topological Euler characteristics of X_t and $\hat{X}$ we have the relation

$$\chi(X_t) - \chi(\hat{X}) = -2s \ .$$

This reflects that each vanishing cycle $\sim S^3$ gets replaced by an S^2.

As a consequence the Hodgenumbers of $\hat{X}$ can be expressed by those of X_t: Recall that

$$(h(X_t)^{p\ q})_{p\ q} = \begin{array}{c|c|c|c} h^{03}(X_t) & 0 & 0 & 1 \\ \hline 0 & h^{1\ 2}(X_t) & 1 & 0 \\ \hline 0 & 1 & h^{2\ 1}(X_t) & 0 \\ \hline 1 & 0 & 0 & h^{30}(X_t) \end{array}$$

by the Lefschetz theorem for smooth hypersurfaces. Hence

$$-2s = \chi(X_t) - \chi(\hat{X}) = 2 - 2h^{2\ 1}(X_t) + 2h^{2\ 1}(\hat{X}) - 2h^{1\ 1}(\hat{X}).$$

Altogether we get that the Hodgenumbers of $\hat{X}$ are expressible in terms of the Hodgenumbers of X_t, s and $d := h^{22}(\hat{X}) - 1$:

$$h^{2\ 1}(\hat{X}) = h^{2\ 1}(X_t) - s + d$$
$$h^{3\ 0}(\hat{X}) = h^{3\ 0}(X_t) \ .$$

The relation to intersection cohomology is as follows:

Lemma 5.2.1. *Let $\pi : \hat{X} \to X$ be a small resolution of a nodal 3-fold. Then*

$$\mathrm{R}\pi_*\mathbb{C}_{\hat{X}} = \mathrm{IC}_X \quad .$$

Proof. Let $k : L \to \hat{X}$ denote the inclusion of the π-exceptional locus, $j : \hat{X} - L \to \hat{X}$ the inclusion of the complement.

Pushing down the distinguished triangle

$$k_*\mathrm{R}k^!\mathbb{C}_{\hat{X}} \to \mathbb{C}_{\hat{X}} \to \mathrm{R}j_*j^*\mathbb{C}_{\hat{X}}$$

we get that

$$\mathrm{R}\pi_*\mathbb{C}_{\hat{X}} = \tau_{\leq 3}\mathrm{R}\pi_*\mathrm{R}j_*j^*\mathbb{C}_{\hat{X}}$$

because $\mathrm{R}^l\pi_*\mathbb{C}_{\hat{X}} = 0$ for $l > \dim_{\mathbb{R}} L = 2$ and $\mathrm{R}^l\pi_*k_*\mathrm{R}k^!\mathbb{C}_{\hat{X}} = \mathrm{R}^l\pi_* \ \mathrm{R}\Gamma_L\mathbb{C}_{\hat{X}}0$ for $k \leq \operatorname{codim}_{\mathbb{R}} L = 4$. Substituting $\mathrm{R}\pi_*j_*j^*\mathbb{C}_{\hat{X}} = \mathrm{R}j_{\Sigma *}j_{\Sigma}^*\mathbb{C}_X$, where $j_\Sigma : X - \Sigma \to X$ is the inclusion of the complement of the nodes, we get the claim. □

So intersection cohomology is the cohomology of the small resolution.

Lemma 5.2.2. *Let $\hat{X}$ denote a small resolution of X, $\widetilde{X}$ the blowup of X in the nodes ("big resolution"). Then $\mathrm{H}^3(\hat{X})$ has an Hodge structure such that $\mathrm{H}^3(\hat{X}) = \mathrm{H}^3(\widetilde{X}) = \mathrm{Gr}_5^{\mathrm{W}}\,\mathrm{H}^4(Y - X)(1)$.*

Proof. $\mathrm{H}^k\hat{X}$ is a sub-Hodge structure of $\mathrm{H}^k\widetilde{X}$. Big- and small resolution have the same 3rd Betti number [Werner] so indeed $\mathrm{H}^3(\hat{X}) = \mathrm{H}^3(\widetilde{X})$.

Let $\widetilde{Y}$ be the blowup of Y, then we have a sequence of MHS

$$\mathrm{H}^4(\widetilde{Y}) \to \mathrm{H}^4(\widetilde{Y} - \widetilde{X}) \to \mathrm{H}^3(\widetilde{X})(-1) \to 0,$$

hence $\mathrm{Gr}_5^{\mathrm{W}}\,\mathrm{H}^4(Y - X) = \mathrm{H}^3(\widetilde{X})(-1)$ as claimed. □

In particular $\mathrm{H}^3(\hat{X}) = \mathrm{IH}^3(X)$ are indeed isomorphic Hodge structures, where the latter Hodge structure is the one defined by mixed Hodge modules. In this way, we know that $\mathrm{Gr}_{\mathrm{F}}^2(\mathrm{IH}^3(X)) \simeq I/J_{2\deg X-5} = \mathrm{H}^1(\Omega^2_{\hat{X}})$ by the last chapters. This particular fact can still be obtained in elementary way:

Proposition 5.2.3. $\mathrm{H}^1(\Omega^2_{\hat{X}}) \simeq I/J_{2\deg X-5}$

Proof. Let $k := \deg X$. From the exact sequence

$$\begin{aligned} \mathrm{H}^0\Theta_X(k-5) \to \mathrm{H}^0\Theta_Y \otimes \mathcal{O}(k-5)_X \to \mathrm{H}^0\widetilde{I}(2k-5) \\ \to \underbrace{\mathrm{H}^1\Theta_X \otimes \omega_X}_{=I/J_{(2k-5)}} \to \underbrace{\mathrm{H}^1\Theta_Y \otimes \mathcal{O}_X(2k-5)}_{=0} \end{aligned}$$

we get $\mathrm{H}^1\Theta_X \otimes \omega_X \simeq I_{(2k-5)}/J_{(2k-5)} \otimes R/\langle F\rangle = I_{(2k-5)}/J_{(2k-5)}$.

All there is to show is that $\mathrm{H}^1\Theta_X \otimes \omega_X = \mathrm{H}^1\Omega^2_{\hat{X}}$. Let $\pi : \hat{X} \to X$ be the canonical projection. Of course $\Omega^2_{\hat{X}} = \Theta_{\hat{X}} \otimes \pi^*\omega_X$. Once we know that $\pi_*\Theta_{\hat{X}} = \Theta_X$ and $\mathrm{R}^1\pi_*\Theta_{\hat{X}} = 0$, we are done, applying Leray spectral sequence.

For the vanishing of $\mathrm{R}^1\pi_*\Theta_{\hat{X}}$, recall that $\hat{X} \to X$ locally looks like $\hat{A} := \operatorname{Spec} S\mathcal{O}^2_{\mathbb{P}^1}(1) \to A$, A some affine quadric cone with vertex x. The bundle projection $p : \hat{A} \to \mathbb{P}^1$ yields a presentation

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Theta_{\hat{A}|\mathbb{P}^1} & \longrightarrow & \Theta_{\hat{A}} & \longrightarrow & p^*\Theta_{\mathbb{P}^1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & p^*\mathcal{O}^2_{\mathbb{P}^1}(-1) & \longrightarrow & \Theta_{\hat{A}} & \longrightarrow & p^*\mathcal{O}_{\mathbb{P}^1}(2) & \longrightarrow & 0,
\end{array}$$

from which the vanishing follows.

$\pi_*\Theta_{\hat{X}} = \Theta_X$ gets clear, considering the blow down $\hat{A} \to A$:

$$\begin{array}{ccccccc}
\mathrm{H}^0_{\mathbb{P}^1}(\hat{A}, \Theta_{\hat{A}}) & \longrightarrow & \mathrm{H}^0(\hat{A}, \Theta_{\hat{A}}) & \xrightarrow{\rho^{\hat{A}}_U} & \mathrm{H}^0(U, \Theta_{\hat{X}}) & \longrightarrow & \mathrm{H}^1_{\mathbb{P}^1}(\Theta_{\hat{A}}) \\
\downarrow & & \downarrow \varphi_A & & \downarrow \varphi_U & & \downarrow \\
\mathrm{H}^0_x(\Theta_A) & \longrightarrow & \mathrm{H}^0(A, \Theta_{\hat{A}}) & \xrightarrow{\rho^{\hat{A}}_U} & \mathrm{H}^0(U, \Theta_{\hat{U}}) & \longrightarrow & \mathrm{H}^1_x(\Theta_{\hat{A}});
\end{array}$$

where $U := \hat{A} - \mathbb{P}^1$. All local cohomologies involved in this diagram vanish because $\operatorname{depth}_{\mathbb{P}^1}\Theta_{\hat{A}} = \operatorname{codim}(\mathbb{P}^1) = 2$ and $\operatorname{depth}_x \Theta_A = \operatorname{depth}_x \mathcal{H}om(\Omega^1_X, \mathcal{O}_X) \geq 2$. This is the claim. □

Now as we know that $\Theta_X = \mathrm{R}\pi_*(\Theta_{\hat{X}})$, it is easy to derive that d, defined as $h^{22}(\hat{X}) - 1$ is the defect of the linear system of forms of degree $2\deg X - 5$ going through the nodes, $d = h^1(Y, I) = h^1(X, I)$:

The short exact sequence

$$0 \to I_X/I_X^2 \to \Omega^1 \otimes \mathcal{O}_X \to \Omega^1_X \to 0$$

induces morphisms of spectral sequences $\mathrm{E}_2^{p\ q} = \mathrm{H}^p\mathcal{E}xt^q(-, \omega_X)$; $\omega_X = \mathcal{O}_X(k-5)$ which glue the long exact sequence associated to

$$0 \to \Theta_X \otimes \omega_X \to \Theta_Y \otimes \omega_X \to \mathcal{O}_X(2k-5) \to 0$$

and the long exact sequence of the $\mathrm{Ext}(-,\omega_X)$-groups:

$$
\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \uparrow & & \uparrow \\
\mathrm{Ext}^1(\Omega^1_X \otimes \mathcal{O}_X, \omega_X) & \longrightarrow & \underbrace{\mathrm{Ext}^1(N^\vee, \omega_X)}_{=0} & \longrightarrow & \mathrm{Ext}^2(\Omega^1_X, \omega_X) & \xrightarrow{\sim} & \underbrace{\mathrm{Ext}^2(\Omega^1_X \otimes \mathcal{O}_X, \omega_X)}_{1-\dim} \\
=\uparrow & & \uparrow & & \uparrow & & =\uparrow \\
\underbrace{\mathrm{H}^1\Theta_Y \otimes \omega_X}_{=0} & \longrightarrow & \underbrace{\mathrm{H}^1\widetilde{I}(2k-5)}_{d} & \longrightarrow & \underbrace{\mathrm{H}^2\Theta_X \otimes \omega_X}_{\mathrm{H}^2\Omega^2_{\tilde{X}}} & \longrightarrow & \underbrace{\mathrm{H}^2\Theta_Y \otimes \omega_X}_{\simeq \mathrm{H}^3\Omega^3_Y} \\
 & & \uparrow & & \uparrow & & \\
 & & \mathrm{H}^0\mathbb{C}_\Sigma & = & \mathrm{H}^0\mathcal{E}xt^1(\Omega^1_X, \omega_X) & & \\
 & & \uparrow & & \uparrow & & \\
\mathrm{Ext}^0(\Omega^1_X \otimes \mathcal{O}_X, \omega_X) & \longrightarrow & \mathrm{Ext}^0(N^\vee, \omega_X) & \longrightarrow & \underbrace{\mathrm{Ext}^1(\Omega^1_X, \omega_X)}_{=R/J} & \longrightarrow & 0 \\
=\uparrow & & =\uparrow & & \uparrow & & \uparrow \\
\mathrm{H}^0\Theta_Y \otimes \omega_X & \longrightarrow & \mathrm{H}^0\widetilde{I}(2k-5) & \longrightarrow & \underbrace{\mathrm{H}^1\Theta_X \otimes \omega_X}_{=I/J_{(2k-d)}} & \longrightarrow & \underbrace{\mathrm{H}^1\Theta_Y \otimes \omega_X}_{=0} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & & 0 & & 0 & & 0
\end{array}
$$

This covers well with the result of the preceding chapters, namely the isomorphy

$$
\begin{aligned}
\mathrm{Gr_F}\,\mathrm{IH}^3(X) = {} & \mathrm{Gr_F}(\mathrm{IH}^4(Y|X)(1))) \\
\simeq {} & \big((I^{(2)}/K)_{(4d-5)}, (I/J)_{(3d-5)}, (I/J)_{(2d-5)}, R_{(d-5)}\big) \\
\subset {} & \mathrm{Gr_F}\,\mathrm{H}^4(Y-X)(1)\ ;
\end{aligned}
$$

where $I^{(1)}$ is the saturation of I^2 and K the kernel of the map $I^{(2)}_{(15)} \to \mathrm{Gr}^1_\mathrm{F}\,\mathrm{H}^4(Y-X)$. Of course the first two summands are not at all explicit. This gets better if we restrict to quintics in $\mathbb{P}^4$. One reason is that the deformation functor of a quintic is unobstructed. Therefore we make a short excursion on deformation theory.

5.3 Functors of Artin Rings

Definition 5.3.1. *For a scheme X with isolated singularities Σ, let $\mathbf{D}_{X|\Sigma}$ denote the* ***functor of equisingular deformations of*** *X, which associates to an Artinian Ring A (or likewise its spectrum) the set of all flat deformations* *of X over* $\mathrm{Spec}(A)$*, which are formally locally trivial. I.e. the completion of each local ring is isomorphic to the base change by $\hat{A} \to \mathbb{C}$.*

One might express this as an exact sequence of functors of pointed sets

$$* \to D_{X|\Sigma} \to D_X \to \prod_{s\in\Sigma} D_{X,s} \ , \tag{5.1}$$

where $D_{X,s}$ denotes the deformations of the germ X, s.

Moreover if D_X^Y is the usual functor of embedded deformations of X in Y, i.e. $D_X^Y(S)$ is the set of flat

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & Y \times S \\ \big\downarrow & & \big\downarrow \\ S & = & S \end{array} \quad ,$$

let $D^Y_{X|\Sigma}(S)$ be the subset of those deformations, which map to $D_{X|\Sigma}(S)$ under the forgetful map $D_X^Y \to D_X$ i.e. the associated functor $\mathbf{D^Y_{X|\Sigma}}$ is defined by fibered product

$$\begin{array}{ccc} D^Y_{X|\Sigma} & \longrightarrow & D^Y_X \\ \big\downarrow & & \big\downarrow \\ D_{X|\Sigma} & \longrightarrow & D_X \end{array} \ ,$$

and shall be denoted ***the functor of embedded equisingular deformations***.

Let $X = V(F) \subset Y = \mathbb{P}^n$, $I_X := \langle F \rangle$. Recall there is a correspondence

$$\begin{array}{rcl} \Gamma(X, N) = \mathrm{Hom}_Y(I_X, \mathcal{O}_X) & \to & D_X^Y(\mathbb{C}[\epsilon]) \\ \varphi & \mapsto & V(F + \epsilon\varphi(F)). \end{array}$$

The inverse is given by the homomorphism from the right upper corner to the left bottom corner induced from the splitting of the middle row in

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \epsilon I_X & \longrightarrow & \mathcal{I}_X & \longrightarrow & I_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \epsilon \mathcal{O}_Y & \longrightarrow & \mathcal{O}_Y[\epsilon] & \longrightarrow & \mathcal{O}_Y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \epsilon \mathcal{O}_X & \underset{j}{\longrightarrow} & \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

or, which is the same, the map $\varphi : I_X \to \mathcal{O}_X; h(x) \mapsto -\frac{\partial}{\partial \epsilon} H(x, \epsilon)$ for some lift $H(x, \epsilon)$ of $h(x)$ to $\mathcal{I}_X$, which is well defined by flatness.

Proposition 5.3.2. *Let $\mathcal{J}$ be the Jacobian ideal sheaf of $X \subset Y$ on Y. Then $\Gamma(X, \mathcal{J} \otimes N) \subset \Gamma(X, N)$ corresponds to infinitesimal equisingular deformations of X, in the sense that $D^Y_{X|\Sigma}(\mathbb{C}[\epsilon]) = \Gamma(X, \mathcal{J} \otimes N)$.*

Proof. We have to discuss the tangent map of the forgetful functor $D^Y_X \to D_X$. For this let us first review the correspondence between $\mathrm{Ext}^1_X(\Omega^1_X, \mathcal{O}_X)$ and $D_X(\mathbb{D})$; $\mathbb{D} := \mathrm{Spec}(\mathbb{C}[\epsilon])$:

The data $(\mathcal{X} \to \mathbb{D})$ define an extension $0 \to \pi^*\Omega_{\mathbb{D}} \to \Omega_{\mathcal{X}} \to \Omega_{\mathcal{X}|\mathbb{D}} \to 0$ in $\mathrm{Ext}^1_{\mathcal{X}}(\Omega_{\mathcal{X}|\mathbb{D}}, \mathcal{O}_X)$ because $\pi^*\Omega_{\mathbb{D}} = \mathcal{O}_X d\epsilon$. Moreover $\Omega_{\mathcal{X}|\mathbb{D}} = \Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{X}}$ and hence the class above corresponds $1:1$ to $0 \to \mathcal{O}_X \to \Omega_{\mathcal{X}} \otimes \mathcal{O}_X \to \Omega_X \to 0$ in $\mathrm{Ext}^1_X(\Omega_X, \mathcal{O}_X) = \mathrm{Ext}^1_{\mathcal{X}}(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{X}}, \mathcal{O}_X)$.

On the other hand, an arbitrary extension $0 \to \mathcal{O}_X \to * \to \Omega^1_X \to 0$ in $\mathrm{Ext}^1_X(\Omega_X, \mathcal{O}_X)$ defines by pullback with $d : \mathcal{O}_X \to \Omega^1_X$ a sheaf $\mathcal{O}_{\mathcal{X}}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle d} & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & * & \longrightarrow & \Omega^1_X & \longrightarrow & 0
\end{array}$$

This sheaves carries a canonical structure as a sheaf of $\mathbb{C}[\epsilon]$-algebras such that with this $\mathbb{D}$-scheme structure $* = \Omega_{\mathcal{X}} \otimes \mathcal{O}_X$.

Now consider for $\sigma \in \Gamma N$ the section

$$\delta(\sigma) := [\sigma : L_1 \to \mathcal{O}_X] \in \mathrm{H}^1(\mathcal{H}om(L., \mathcal{O}_X), d^*) = \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X),$$

where $(L_1 \xrightarrow{d} L_0)$ is the projective resolution of Ω_X, given by $(I/I^2 \to \Omega^1_Y \otimes \mathcal{O}_X)$ itself. The Yoneda picture for this map is

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega^1_Y \otimes \mathcal{O}_X & \longrightarrow & \Omega^1_X & \longrightarrow & 0 \\ & & {\scriptstyle\sigma}\downarrow & & {\scriptstyle\varphi}\downarrow & & \| & & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & * & \longrightarrow & \Omega^1_X & \longrightarrow & 0. \end{array}$$

Everything commutes if we take $\Omega_{\mathcal{X}} \otimes \mathcal{O}_X$ for $*$ so that the image class is reconstructed from the forgetful data $\mathcal{X} \to \mathbb{D}$. Hence δ realizes the forgetful map from $D^Y_X(\mathbb{D})$ to $D_X(\mathbb{D})$, which is the tangent map we want.

σ induces locally everywhere the trivial class of $\mathcal{H}^1(\mathcal{H}om(L., \mathcal{O}_X), d^*)$ iff locally $\sigma = d^*(\theta)$ for some local section of θ of $(\Theta_Y \otimes \mathcal{O}_X)$ iff $\sigma \in \Gamma(\mathcal{J} \otimes N)$! This can be seen in the Yoneda Picture also: On the locus where θ is defined, $\varphi - \theta$ splits the extension:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_1 & \longrightarrow & L_0 & \longrightarrow & \Omega^1_X & \longrightarrow & 0 \\ & & \downarrow \sigma & \swarrow \theta & \downarrow & \swarrow \varphi - \theta & \| & & \\ 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & * & \longrightarrow & \Omega^1_X & \longrightarrow & 0\,. \end{array}$$

Vice versa, the image of σ under the forgetful map is in $D_{X|\Sigma}(\mathbb{D})$ iff the second row above locally has a retract $r : * \to \mathcal{O}_X$. But then $\sigma = d^*(\varphi^*(r)) = 0 \in \mathcal{H}^1(\mathcal{H}om(L., \mathcal{O}_X))$ and $\sigma \in \Gamma(\mathcal{J} \otimes N)$.

Together this demonstrates that the tangent maps δ of the forgetful functors $D^Y_X \to D_X$, or $D^Y_{X|\Sigma} \to D_{X|\Sigma}$ respectively, coincide with the usual connecting morphisms induced from R_{Hom} applied to $0 \to I/I^2 \to \Omega^1_Y \otimes \mathcal{O}_X \to \Omega^1_X \to 0$, or R_Γ applied to $0 \to \Theta_X \to \Theta_Y \otimes \mathcal{O}_X \to \mathcal{J} \otimes N \to 0$ rsp. and fit into the following diagram

$$\begin{array}{ccccc} & & \Gamma\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X) & = & \Gamma\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X) \\ & & \uparrow & & \uparrow \\ \Gamma\Theta_Y \otimes \mathcal{O}_X & \longrightarrow & \Gamma N & \xrightarrow{\delta} & \mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) \\ \| & & \uparrow & & \uparrow \\ \Gamma\Theta_Y \otimes \mathcal{O}_X & \longrightarrow & \Gamma(\mathcal{J} \otimes N) & \xrightarrow{\delta} & \mathrm{H}^1(\Theta_X) \\ & & \uparrow & & \uparrow \\ & & 0 & & 0 \end{array} .$$

□

Proposition 5.3.3. *Let X be a nodal 3-fold and choose a small resolution $\pi : \hat{X} \to X$ of X.*

- *There is a natural injection of $D_{\hat{X}}$ in $D_{X|\Sigma}$.*
- *If X is a nodal quintic in $\mathbb{P}^4$, then $D_{\hat{X}}$, D_X and $D_{X,s}$ are unobstructed.*

Proof. Let $\mu : \mathcal{X} \to S$ in $D_{\hat{X}}(S)$ and choose an open affine cover $(U_i)_{i\in I}$ of X. The vanishing of $R^1_{\pi_*}(\mathcal{O}_{\hat{X}})$ induces that for each U_i, the induced deformation $\hat{\mathcal{U}_i}$ of $\hat{U}_i := \pi^{-1}U_i$ (which makes sense, since the underlying space is still U_i), has the property that $\Gamma\mathcal{O}_{\hat{\mathcal{U}}_i}$ is flat over S, hence $\mathcal{U}_i := \mathrm{Spec}(\Gamma\mathcal{O}_{\mathcal{U}_i})$ is a deformation of U_i over S such that $\hat{\mathcal{U}_i}$ and $\mathcal{U}_i$ are isomorphic away from the singularities (we refer to [Wah76] for details). In particular, the affine sets $\mathcal{U}_i$ glue to a scheme $\mathcal{X} := (\cup_{i\in I}U_i)/\sim$, flat over S, with special fiber X. Hence every deformation of $\hat{X}$ blows down to a deformation of X in $D_{X|\Sigma}$.

Moreover, the induced map of functors $D_{\hat{X}} \to D_{X|\Sigma}$ is injective because $R^1_{\pi}(\Theta_{\hat{X}}) = 0$ [Wah76]. I believe, the surjectivity can be shown as well, but this is still open. An equivalence $D_{\hat{X}} \sim D_{X|\Sigma}$ would imply the unobstructedness of $D_{X|\Sigma}$:

The unobstructedness for $D_{\hat{X}}$ follows (without Kaehler assumption) from T^1 lifting [Ran92], for $D_{X,s}$ this is classical and for D_X it follows from the latter, together with the abstract T^1 lifting property [Kaw92, theorem 4]. □

Let $\mathbb{D} := \mathrm{Spec}\,\mathbb{C}[\epsilon]$ and $G \in \Gamma(Y, \mathcal{J} \otimes N)$ correspond to the

$$\mathcal{X} := (V(F + \epsilon G) \subset Y \times \mathbb{D}) \in D^Y_{X|\Sigma}$$

and $v \in \mathrm{H}^1(X, \Theta_X)$ be the image of G under δ. Because any third cohomology is primitive and isomorphic to a subspace of $\mathrm{H}^4(U)$, we can define the Kodaira-Spencer map ks on the middle primitive cohomology as $\mathrm{ks} = \mathrm{Gr}^{-1}_{\mathrm{F}}(\nabla)$, where ∇ is the connecting morphism

$$\nabla : \mathbb{H}^4\Omega^\bullet(*X) \to \mathbb{H}^5(\Omega^\bullet(*X)[-1]d\epsilon) = \mathbb{H}^4\Omega^\bullet(*X)$$

associated to

$$0 \to \Omega^\bullet(*X)[-1]d\epsilon \to \Omega^\bullet(*\mathcal{X}) \to \Omega^\bullet(*X) \to 0$$

restricted to forms in $\mathbb{H}^4\tau_{\leq -1}\Omega^\bullet(*X)$.

Proposition 5.3.4 (Kodaira-Spencer map). *Let X be a nodal 3-fold in $Y = \mathbb{P}^4$, $\alpha = [\frac{A\Omega}{F^{5-p}}] \in \mathrm{Gr}^p_{\mathrm{F}}\,\mathrm{IH}^4(Y|X) = \mathrm{Gr}^{p-1}_{\mathrm{F}}\,\mathrm{IH}^3_0(X)$ a primitive intersection cohomology class, then*

$$\mathrm{ks}(v)(\alpha) = [(p-5)\frac{GA\Omega}{F^{6-p}}] \in \mathrm{Gr}^p_{\mathrm{F}}\,\mathrm{IH}^3_0(X).$$

Proof. $\nabla\alpha$ equals d, applied to a lift of α and taken modulo terms of higher order wrt. F. Hence we have

$$\begin{aligned}\nabla\frac{A\Omega}{F^{5-p}} &= d(\frac{A\Omega}{(F+\epsilon G)^{6-p}}) \\ &= (p-5)\frac{GA\Omega d\epsilon}{F^{6-p}}\end{aligned}$$

because A does not depend on ϵ. The result $\mathrm{ks}(v)(\alpha) = (p-5)\frac{GA\Omega}{F^{6-p}}$ lies indeed in $\tau_{\leq -1}\Omega^\bullet(*X)$ because G is a section of the ideal sheaf of the nodes $\mathcal{I} = \mathcal{J}$. □

5.4 Nodal Quintics in $\mathbb{P}^4$

As mentioned in the introduction if M is a Calabi Yau manifold (i.e. 3-dimensional projective manifold with $\Omega^3_M = \mathcal{O}_M$) and ω a generator of $\mathrm{H}^0(M,\Omega^3_M)$, one considers the normalized Yukawa coupling, defined as the map

$$\begin{aligned}YC : S^3\mathrm{H}^1(\Theta_M) &\to \mathrm{H}^{3\ 3}(M) \\ X_1\cdot X_2\cdot X_3 &\mapsto \mathrm{ks}(X_1)\circ\mathrm{ks}(X_2)\circ\mathrm{ks}(X_3)(\omega)\cup\omega\ .\end{aligned}$$

If $\hat{X}$ is a small resolution of a nodal quintic in $\mathbb{P}^4$, $\hat{X}$ may not be projective i.e. Calabi Yau, nevertheless it has trivial canonical sheaf so that the tensor above is defined.

Moreover $\mathrm{H}^1\Theta_{\hat{X}} \sim \mathrm{H}^1(\Theta_X)$ and $\mathrm{H}^3(\hat{X}) = \mathrm{IH}^3(X)$ as Hodge structures so that it can be expressed entirely in terms of X. As a consequence of the formula for the Kodaira Spencer map and the pairing, we can state

Corollary 5.4.1. *Let $X_1, X_2, X_3 \in \mathrm{H}^1(\Theta_X)$, which is naturally isomorphic to $\mathrm{H}^0(\mathcal{J}\otimes N)/J_{(5)} = I/J_{(5)}$. Considering the X_i in the latter way, the Yukawa coupling by the section $\omega = \mathrm{Res}\,\Omega/F$, followed by the Gysin map $\mathrm{IH}^{33}(X)\to\mathbb{H}^4\Omega^4_Y$ is given by*

$$\begin{aligned}S^3 I/J_{(5)} &\longrightarrow \mathrm{IH}^{33}(X) \\ X_1\cdot X_2\cdot X_3 &\mapsto [\frac{X_1X_2X_3\Omega}{F_0\cdots F_4}]\ .\end{aligned}$$

Proof. Indeed we have

$$\begin{aligned}i_!\left(\mathrm{ks}(X_1)\circ\mathrm{ks}(X_2)\circ\mathrm{ks}(X_3)(\omega)\cup\omega\right) &= i_!\left(\mathrm{Res}(\frac{X_1X_2X_3\Omega}{6F^4}))\cup\mathrm{Res}(\frac{\Omega}{F})\right) \\ &= \frac{X_1X_2X_3\Omega}{F_0\cdots F_4}\end{aligned}$$

in $\mathbb{H}^4(Y, \tau_{\leq -1} l_* l^* \Omega^\bullet) = \mathbb{H}^4(Y, \Omega^4)$. □

Corollary 5.4.2. *If X is a nodal quintic in $\mathbb{P}^4$ such that the Yukawa coupling is nonzero. Then there is an isomorphism*

$$\begin{aligned}\mathrm{Gr_F}\,\mathrm{IH}^3(X) &= \mathrm{Gr_F}(\mathrm{IH}^4(Y|X)(1)\\ &\simeq \big((I^3/K)_{(15)}, (I^2/(I^2 \cap J))_{(10)}, (I/J)_{(5)}, R_{(0)}\big)\ .\end{aligned}$$

Proof. By a theorem of Griffiths [BG83], iterated application of $\mathrm{H}^1(\Theta_X)$ to a generator of $\mathrm{F}^3(\mathrm{IH}^3(X))$ generates $\mathrm{F}^1(\mathrm{IH}^3(X))$. If the Yukawa coupling is nonzero, it generates all of $\mathrm{IH}^3(X)$).

As we have seen, the action of an element of $\mathrm{H}^1(\Theta_X)$ on the generator $\Omega/F \equiv 1 \in R$ is given by multiplication with A/F for some $A \in I_{(5)}$. Hence we get that $\mathrm{Gr}_\mathrm{F}^{3-k}\,\mathrm{IH}^3(X))$ is a quotient of $(I^k/K)_{(5k)}$, $k = 0, \ldots, 3$.

It remains to show that $(I^2/(I^2 \cap J))_{(10)} \simeq \mathrm{Gr}^1\,\mathrm{IH}^3(X))$. We know that $I/J_{(10)} \simeq \mathrm{Gr}^1\,\mathrm{IH}^3(X)$ by the canonical map so that the kernel of the surjective map $I^2_{(10)} \to \mathrm{Gr}^1\,\mathrm{IH}^3(X)$ must be $I^2_{(10)} \cap J_{(10)}$. □

As a conjecture, the middle intersection cohomology should be isomorphic to a subquotient of the Rees algebra $\mathbb{C}[tI]$, namely $\mathbb{C}[tI_{X(5)}]/tJ_{X(5)}$; where $I_X = I/\langle F\rangle$.

One indication for this conjecture is that it would be in analogy to the smooth case, as the middle intersection cohomology equals the middle cohomology of the big resolution $\widetilde{X}$ which is given by $\mathrm{Biproj}\mathbb{C}[tI_X]$.

The other indication is that in the case of the Hunt-Straten quintic (c.f. next chapter) the predictions of the formulan above are confirmed by a computer calculation performed with the computer program Macaulay2 [GS].

Maybe this relation between middle primitive intersection cohomology of a nodal variety and the Rees algebra of the ideal of the nodes is a fruitful object for further investigation [LV04].

5.5 The Hunt-Straten Quintic

It is a delicate question if a given nodal 3-fold has a projective small resolution or not. A bit aside from the main theme of this text, we will study this question for a special family of nodal quintics.

In [Hun96] Bruce Hunt threats a 4-dimensional quintic I_5 in $\mathbb{P}^5$ which is singular along a configuration of 120 lines. It is the unique quintic, invariant under the standard action of the $\mathrm{W(E_6)}$ as reflection group on $\mathbb{P}^5$. The singular locus of the I_5 is a configuration of 120 lines, and the singularities are “transversal type A_1” away from the intersection of these lines, i.e. a general hyperplane section of I_5 will have 120 nodes.

He discusses this 5-dim family of generic hyperplanesections parametrized by (some open set of) the dual $\mathbb{P}^5$ and the 4-dim family of tangent hyperplanesections parametrized by (some open set of) the dual variety of I_5, which are families of 120 rsp. 121-nodal (120 + 1 tangent node) quintics in $\mathbb{P}^4$. He states implicitly that both families are versal.

These families wouldn't be of interest if they were not versal. Nevertheless is only a computer result that they are indeed, which I checked with Macaulay2 [GS], calculating the dimension of

$$\mathrm{H}^1(X, \mathcal{I}(10)) = \left(\mathrm{H}^2_m(R, (\langle F_1, \ldots, F_4\rangle : \mathfrak{m}_0^*))\right)_{(10)},$$

if $X = I^5 \cap V_+(X_0)$, $V_+(x_0)$ a tangent hyperplane and $R = \mathbb{C}[x_0, \ldots, x_n/\langle x_0\rangle$ the homogeneous coordinate ring of the hyperplane. This is possible because I_5 is rational and Bruce Hunt gave an explicit birational map from $\mathbb{P}^4$ onto I^5, so on can randomly choose a smooth point on I^5, calculate its tangent hyperplane $H = V_+(x_0)$, restrict the equation to it and calculate the defect $d = \dim \mathrm{H}^2_{\mathfrak{m}}(R, I)_{10}$, where R is the homogeneous coordinate ring of H and I the ideal of the lines corresponding to the nodes.

Namely one needs that the Hodge numbers of the small resolutions are $h^1\Omega^2_{\hat{X}} = h^1(\theta_X) = 4 = \dim I_5$ in the tangent and $h^{12} = 5$ in the generic case. The versality then follows from a diagram chase. In particular, the defects are the same, this is, what we need here: Let T be a tangent hyperplanesection of I_5 with 121 nodes, G a generic one (with 120 nodes) then

$$d(T) = d(G) = 24.$$

Blow down in $\hat{T}$ the exceptional line L over the tangent 121^{st} node to receive a variety T' with only one node and small resolution $\hat{T} \to T'$. A small resolution $\hat{G}$ of G can be considered a smoothing, i.e. smooth deformation -or Milnor fiber-, of T'.

Therefore it exists by an 'Mayer-Vietoris argument' as in [Wer87]

- an exact sequence

$$0 \to \mathrm{H}_3\hat{T} \to \mathrm{H}_3T' \to \mathrm{H}_2(L) \to \mathrm{H}_2\hat{T} \to \mathrm{H}_2(T') \to 0$$

- and an isomorphism $\mathrm{H}_2(T') \simeq \mathrm{H}_2(\hat{G})$.

It follows $h_2(T') = h_2(\hat{G}) = 1 + d(\hat{G}) = 1 + d(\hat{T}) = h_2(\hat{T})$. Hence $i_* : \mathrm{H}_2(L) \to \mathrm{H}_2\hat{T}$ is zero, i.e. *L is a zerohomotopic curve*, which could not be if $\hat{T}$ were a Kähler manifold. In particular $\hat{T}$ is not Calabi Yau.

Appendix

Basic Facts and Notations

Complexes Let $\mathcal{A}$ be an abelian category. A chain complex of objects in $\mathcal{A}$ is a sequence of objects indexed by $\mathbb{Z}$ and homomorphisms

$$C_\bullet := (\cdots \xrightarrow{d_{k+1}} C_k \xrightarrow{d_k} \cdots)$$

with the property $d_k \circ d_{k+1} = 0$ for all k.
Similar a cochain complex is a sequence

$$C^\bullet := (\cdots \xrightarrow{d^{k-1}} C_k \xrightarrow{d^k} \cdots)$$

such that $d^k \circ d^{k-1} = 0$.
There is a conversion from chain complexes to cochain complexes and vice versa given by

$$\begin{aligned}(C^\bullet)^k &:= C_{-k};\ d^k := d_{-k-1}\\ (C_\bullet)_k &:= C^{-k};\ d_k := d^{k+1}\end{aligned}$$

and shift operators

$$\begin{aligned}(C_\bullet[n])_k &:= C_{k-n}\\ (C^\bullet[n])^k &:= C^{k+n}\end{aligned}$$

commuting with conversion.
A homomorphism $\varphi : A^\bullet \to B^\bullet$ of chain complexes is a sequence of morphism $(\varphi : A^k \to B^k)_{k\in\mathbb{Z}}$ commuting with the differentials. A homomorphism φ is called quasi-isomorphism if is induces isomorphisms on the cohomology

$$\mathrm{H}^k(C^\bullet) := \ker d^k / \operatorname{Im} d^{k-1}$$

for all $k \in \mathbb{Z}$. There is the usual notion of exact sequences of complexes and a long exact sequence of cohomology groups associated to a short exact sequence of complexes.

A sequence of morphisms $\sigma = \left(\sigma_k : B^k \to A^k\right)$ is called a semisplit of φ if $\varphi \circ \sigma = id$ and split, if σ is furthermore a morphism of complexes.
Given a homomorphism $\varphi : A^\bullet \to B^\bullet$, one can form $\mathrm{Cone}(\varphi)$ the cone of φ, the unique complex such that the canonical inclusion and projection morphisms give an short exact sequence of complexes

$$0 \to B[-1]^\bullet \to \mathrm{Cone}(\varphi) \to A^\bullet \to 0$$

with connecting morphism equal to φ. $\mathrm{Cone}(\varphi)$ is often denoted by $\mathrm{Cone}(A^\bullet \to B^\bullet)$.
If $C^\bullet$ is a cochain complex, we define new cochain complexes

$$(\tau_{\leq q} C^\bullet)^k := \begin{cases} C^k & \text{if } k < q+1 \\ \mathrm{Im}\, d^{q+1} & \text{if } k = q+1 \\ 0 & \text{if } k > q+1 \end{cases}$$

$$\left(\tau^{\geq q} C^\bullet\right)^k := \left(C^\bullet / \tau_{\leq q-1} C^\bullet\right)^k = \begin{cases} 0 & \text{if } k < q \\ \mathrm{coker}\, d^q & \text{if } k = q \\ C^k & \text{if } k > q \end{cases}$$

There are also dual notions

$$(\widetilde{\tau}_{\leq q} C^\bullet) := \begin{cases} C^k & \text{if } k < q \\ \ker d^q & \text{if } k = q \\ 0 & \text{if } k > q \end{cases}$$

$$\left(\widetilde{\tau}^{\geq q} C^\bullet\right) := \begin{cases} 0 & \text{if } k < q-1 \\ \mathrm{coim}\, d^{q-1} & \text{if } k = q-1 \\ C^k & \text{if } k > q-1 \end{cases}$$

with the property that $\widetilde{\tau}_{\leq q} C^\bullet \hookrightarrow \tau_{\leq q} C^\bullet$ and $\tau^{\geq q} C^\bullet \twoheadrightarrow \tau^{\geq q} C^\bullet$ are quasi-isomorphisms.
These functors determine *truncation functors* on the derived category. For example if Y is a smooth complex variety of complex dimension n, and $A^\bullet \in \mathrm{D}^{\mathrm{b}}_c(\mathbb{C}_Y)$ is a constructible sheaf complex,

$$\begin{aligned} \mathbb{D}(\tau_{\leq q} A^\bullet[n]) &:= \mathrm{R}\underline{\mathrm{Hom}}(\tau_{\leq q} A^\bullet[n], \mathbb{C}[2n]) \\ &= \widetilde{\tau}^{\geq -q} \mathrm{R}\underline{\mathrm{Hom}}(\tau_{\leq q} A^\bullet[n], \mathbb{C}[2n]) \\ &= \widetilde{\tau}^{\geq -q} \mathrm{R}\underline{\mathrm{Hom}}(A^\bullet[n], \mathbb{C}[2n]) \\ &= \widetilde{\tau}^{\geq -q} \mathbb{D} A^\bullet[n] \ . \end{aligned}$$

Filtrations A decreasing filtration $\mathrm{F} = \mathrm{F}^\bullet C^\bullet$ indexed by $\mathbb{Z}$ of a complex $C^\bullet$ is a family of sub-objects (= object + monomorphism)

$$\cdots \supset \mathrm{F}^p C^k \supset \mathrm{F}^{p+1} C^k \cdots$$

for all k such that $d^k \mathrm{F}^p C^k \subset \mathrm{F}^p C^{k+1}$. F is called regular if $\bigcap_p \mathrm{F}^p C^\bullet = 0$ and $\bigcup_p \mathrm{F}^p C^\bullet = C^\bullet$. F is called bi-regular if F induces a finite filtration on each component C^k. A morphism f between filtered objects A, B is strict if $f\mathrm{F}^p A = \mathrm{Im}(f) \cap \mathrm{F}^p B$. f is a filtered quasi-isomorphism if $\mathrm{Gr}_\mathrm{F}(f)$ is a quasi-isomorphism. A morphism f between complexes with two bi-regular filtrations W and F is called bi-filtered quasi-isomorphism if $\mathrm{Gr}_\mathrm{W} \mathrm{Gr}_\mathrm{F}(f)$ is a quasi-isomorphism.
If F is a decreasing filtration of C, then $\mathrm{F}(k)$ is the filtration with $\mathrm{F}(k)^p C := \mathrm{F}^{k+p} C$ of C. $\mathrm{Gr}^p_\mathrm{F}\, C := \mathrm{F}^p C / \mathrm{F}^{p+1} C$.
If W is an increasing filtration of C, then $\mathrm{W}(k)$ is the filtration with $\mathrm{W}(k)_m C := \mathrm{W}_{n-m} C$ of C. $\mathrm{Gr}^\mathrm{W}_m\, C := \mathrm{W}_m C / \mathrm{W}_{m-1} C$.

Hodge theory A rational Hodge structure **(HS)** of weight m is a triple (F, L, H) consisting of

- an n-dimensional $\mathbb{C}$-vectorspace H,
- an n dimensional $\mathbb{Q}$ sub-vectorspace ("lattice") L, such that $\mathbb{R} \cdot L$ is totally real and hence defines an involution $\overline{(\quad)}$,
- a filtration F on H such that $\mathrm{F}^\bullet$ and $\overline{\mathrm{F}}^\bullet$ are m-opposed, meaning $H = \oplus_{p+q=m} H^{p\ q}$ (the Hodge decomposition) and $H^{p\ q} = 0$ for $p+q = m+1$; where $H^{p\ q} := \mathrm{F}^p H \cap \overline{\mathrm{F}^q H}$ are complex subspaces of H.

If (F, L, H) is a Hodge structure of weight m, $(\mathrm{F}, L, H)(k) := (\mathrm{F}[-k], (2\pi i)^k \cdot L, H)$ is a Hodge structure of weight $m - 2k$ ("Tate twist").
A morphism of HS is a morphism compatible with F. It turns out that such morphisms are automatically strict and that Hodge structures form an abelian category.
A rational mixed Hodge structure **(MHS)** is a quadruple $(\mathrm{W}, \mathrm{F}, L, H)$ of

- n-dimensional $\mathbb{C}$-vectorspace H,
- an decreasing filtration $\mathrm{F}^\bullet H$ on H,
- an n dimensional $\mathbb{Q}$ sub-vectorspace ("lattice") L,
- an increasing filtration $\mathrm{W}_\bullet L$ on L

such that for each m, F induces a pure Hodge structure of $(\mathrm{F}, \mathrm{Gr}_m^{\mathrm{W}} L, \mathrm{Gr}_m^{\mathrm{W}} H)$ weight m.
A morphism of MHS is a morphism compatible with W and F. With such, MHS is an abelian category.

A $\mathbb{Q}$-Hodge complex of sheaves of weight m **(HC)** on a complex manifold Y is a triple (F, L, H) consisting of

- A bounded below complex H of sheaves of $\mathbb{C}$-vectorspaces with finite dimensional hyper-cohomology groups.
- A bounded below complex of sheaves L of $\mathbb{Q}$-vectorspaces with a comparison quasi-isomorphism $L \otimes \mathbb{C} = H$.
- A decreasing filtration F on H

such that

$$_{\mathrm{F}}\mathrm{E}_1^{p\ q} := \mathbb{H}^{q+p}(\mathrm{Gr}_{\mathrm{F}}^p H) \Rightarrow \mathbb{H}^{p+q} H$$

degenerates at E_1, i.e. $\mathrm{E}_1 = \mathrm{E}_\infty$ and $(\mathbb{H}^k\mathrm{F}, \mathbb{H}^k L, \mathbb{H}^k H)$ form a pure Hodge structure of weight $k+m$ for all k.
A mixed $\mathbb{Q}$-Hodge complex **(MHC)** of sheaves on a complex manifold Y is a triple $(\mathrm{W}, \mathrm{F}, L, H)$ consisting of

- A bounded below complex H of sheaves of $\mathbb{C}$-vectorspaces with finite dimensional hyper-cohomology groups.
- A bounded below complex of sheaves L of $\mathbb{Q}$-vectorspaces with a comparison quasi-isomorphism $L \otimes \mathbb{C} = H$.
- A decreasing filtration F on H,
- An increasing filtration W on L

such that $(\mathrm{Gr}_m^{\mathrm{W}} \mathrm{F}, \mathrm{Gr}_m^{\mathrm{W}} L, \mathrm{Gr}_m^{\mathrm{W}} H)$ is a Hodge complex of weight m for all m.
It follows that the shifted weight filtration $\mathrm{W}(k)$ on $\mathbb{H}^k(L)$, defined by

$$\mathrm{W}(k)_m \mathbb{H}^k(L) := \mathrm{Im}(\mathbb{H}^k(\mathrm{W}_{m-k} L) \to \mathbb{H}^k(L))$$

together with F induce a MHS $(\mathrm{W}(k), \mathrm{F}, \mathbb{H}^k(L), \mathbb{H}^k(H))$.
By definition, the spectral sequence $^m_{\mathrm{F}}\mathrm{E}$, starting at

$$^m_{\mathrm{F}}\mathrm{E}_1^{p\ q} := \mathbb{H}^{p+q}(\mathrm{Gr}_{\mathrm{F}}^p \mathrm{Gr}_m H) \Rightarrow \mathbb{H}^{p+q} \mathrm{Gr}_m H$$

degenerates at E_1, and one can show by this that the sequence $_{\mathrm{F}}\mathrm{E}$, starting at

$$_{\mathrm{F}}\mathrm{E}_1^{p\ q} := \mathbb{H}^{p+q}(\mathrm{Gr}_{\mathrm{F}}^p H) \Rightarrow \mathbb{H}^{p+q} H$$

still degenerates at E_1. Moreover, the canonical spectral sequence for the increasing filtration W, denoted by ${}^{\mathrm{W}}\mathrm{E}$, with

$$ {}^{\mathrm{W}}\mathrm{E}_1^{m\ q} := \mathbb{H}^{q+m}(\mathrm{Gr}^{\mathrm{W}}_{-m} L) \Rightarrow \mathbb{H}^{q+m} L $$

degenerates at E_2, as well as the spectral sequences ${}^{\mathrm{W}}_{p}\mathrm{E}$ with

$$ {}^{\mathrm{W}}_{p}\mathrm{E}_1^{m\ q} := \mathbb{H}^{q+m}(\mathrm{Gr}^{\mathrm{W}}_{-m} \mathrm{Gr}^{p}_{\mathrm{F}} H) \Rightarrow \mathbb{H}^{q+m} \mathrm{Gr}^{p}_{\mathrm{F}} H $$

for all p.
That means, the subquotient of $\mathbb{H}^k(L)$, which is pure of weight m, is

$$ \begin{aligned} \mathrm{Gr}^{\mathrm{W(k)}}_{m} \mathbb{H}^k(L) &= \mathrm{Gr}^{\mathrm{W}}_{m-k} \mathbb{H}^k(L) \\ &= \mathrm{E}_{\infty}^{k-m\ m} \\ &= \ker d_{m,k} / \operatorname{Im} d_{m+1,k-1} \ , \end{aligned} $$

where $d_{m,k}$ is the differential of the E_1-term

$$ \begin{array}{ccc} {}^{\mathrm{W}}\mathrm{E}_1^{k-m\ m} & \xrightarrow{d_{m,k}} & {}^{\mathrm{W}}\mathrm{E}_1^{k-m+1\ m} \\ \| & & \| \\ \mathbb{H}^k \mathrm{Gr}^{\mathrm{W}}_{m-k}(L) & \longrightarrow & \mathbb{H}^{k+1} \mathrm{Gr}^{\mathrm{W}}_{m-k-1}(L) \ . \end{array} $$

The morphism $d_{m,k}$ is the connecting morphism from the long exact sequence of global hyper-cohomology groups associated to

$$ 0 \to \mathrm{Gr}^{\mathrm{W}}_{m-k-1} L \to \mathrm{W}_{m-k}L/\mathrm{W}_{m-k-2}L \to \mathrm{Gr}^{\mathrm{W}}_{m-k} L \to 0. $$

$$ \mathrm{Gr}^{p}_{\mathrm{F}} \mathrm{Gr}^{\mathrm{W}(k)}_{m} \mathbb{H}^k(Y,L) = {}^{\mathrm{W}}_{p}\ \mathrm{E}_{\infty}^{k-mm} = \ker {}^{\text{‘}}d_{m,k} / \operatorname{Im} {}^{\text{‘}}d_{m+1,k-1} $$

with ${}^{\text{‘}}d_{m,k}$ associated to

$$ 0 \to \mathrm{Gr}^{\mathrm{W}}_{m-k-1} \mathrm{Gr}^{p}_{\mathrm{F}}(L) \to \mathrm{W}_{m-k}L/\ \mathrm{W}_{m-k-2} \mathrm{Gr}^{p}_{\mathrm{F}}(L) \to \mathrm{Gr}^{\mathrm{W}}_{m-k} \mathrm{Gr}^{p}_{\mathrm{F}}(L) \to 0. $$

In all of the cases above, the filtration F is referred to as the *Hodge filtration*, W as the *weight filtration*.

Koszul complex Let Y be a variety, M a locally free $\mathcal{O}_Y$ module of rank r and $\alpha \in M^* := \mathrm{Hom}_{\mathcal{O}_Y}(M, \mathcal{O}_Y)$.

$$ \begin{aligned} \mathrm{K}(\alpha)_{\bullet} &:= (\Lambda^r(M) \xrightarrow{i_\alpha} \Lambda^{r-1}(M) \to \cdots \to \Lambda^0(M)) \\ \mathrm{K}(\alpha)^{\bullet} &:= (\Lambda^0(M^*) \xrightarrow{\wedge\alpha} \Lambda^1(M^*) \to \cdots \to \Lambda^r(M^*)) \\ &= \mathrm{Hom}(K_{\bullet}(\alpha), \mathcal{O}_Y[r]) \end{aligned} $$

are the *Koszul (co-) chain-complexes of* α. The Koszul complex is self-dual in the sense that $\mathrm{K}(\alpha)_\bullet = \mathrm{Hom}(\mathrm{K}(\alpha)_\bullet, \Lambda^r M[r])$.
If $Y = \mathbb{P}^n$ and given $s_i \in \mathcal{O}_Y(d_i)$; $i = 1, \dots, r$, we consider $\alpha := (s_1, \dots, s_r) \in M^*$ for $M := \oplus_{i=1}^r \mathcal{O}_Y(-s_i)$ and write $\mathrm{K}(s_1, \dots, s_r)_\bullet$ for $\mathrm{K}(\alpha)_\bullet$, $\mathrm{K}(s_1, \dots, s_r)^\bullet$ for $\mathrm{K}(\alpha)^\bullet$.

Fundamental class If Y is a topological manifold, let or_X denote the **orientation sheaf** associated to the pre-sheaf $(U \mapsto \mathrm{H}^n_c(U, k)$. By the vanishing $\mathcal{H}^k \mathbb{D}_Y = 0, k \neq 0$ there are quasi-isomorphisms $\mathbb{D}_Y = \mathcal{H}^{-n}\mathbb{D}_Y = \mathrm{or}_X$.
A $\mathbb{Q}$-orientation -if it exists- is a quism $\mathbb{Q} = \mathrm{or}_Y$. On smooth complex varieties Y there is a natural $\mathbb{Q}$-orientation, hence $\mathbb{D}_Y = \mathbb{Q}[\dim_{\mathbb{R}} Y]$ and for any closed subvariety X, $\mathbb{D}_X = i^! \mathbb{Q}[\dim_{\mathbb{R}} Y]$, by the functoriality of the dualizing complex. Hence there is the canonical isomorphism ("Thom class") $(\tau_{Z_{\mathrm{reg}}} : \mathbb{Q}_{Z_{\mathrm{reg}}} \to i^! \mathbb{Q}_X[2c]) \in \Gamma(Z_{\mathrm{reg}}, \mathrm{R}^{2c} i^! \mathbb{Q}_Y)$. Cup-product by $\tau_{X_{\mathrm{reg}}}$ induces the Thom isomorphism $\mathrm{H}^k X_{\mathrm{reg}} \simeq \mathrm{H}^{k+d}_{X_{\mathrm{reg}}} Y_{\mathrm{Sing} X}$.

Definition (Borel Moore homology [Ive86])**.** $\mathrm{H}^{BM}_p(Z, \mathbb{Q}) := \mathrm{H}^{-p}(Z, \mathbb{D}_Z)$

If Z is embedded in an n-dim smooth variety X, by Verdier duality, for H_c the cohomology with compact support,

$$\mathrm{H}^{BM}_p(X, \mathbb{Q}) = \mathrm{H}^p_c(X)^\vee = \mathrm{H}^{2n-p}_X(Y)$$

like usual homology in the smooth case.
Now let X be a local complete intersection in Y of pure $\mathrm{codim}_Y(X) =: c$. By a depth argument, it is known that $\mathrm{H}^p_X(Y) = 0, \quad p < 2c$, or, which is the same, $\mathrm{H}^{BM}_p(X, \mathbb{Q}) = 0, \quad p > 2c$. It follows first that the canonical isomorphism ("Thom class")

$$\left(\tau_X : \mathbb{Q}_{X_{\mathrm{reg}}} \xrightarrow{\sim} i^! \mathbb{Q}_X[2c]\right) \in \Gamma(X_{\mathrm{reg}}, \mathrm{R}^{2c} i^! \mathbb{Q}_Y)$$

extends uniquely to an element of $\Gamma(X, i_* \mathrm{R}^{2c} i^! \mathbb{Q}_Y) = \Gamma(\mathcal{H}^{2c}_X(X, \mathbb{Q})$, (like integration of 2 $\dim X$-forms over 2 $\dim X$-cycles does not depend on higher codimensional sets) and second that the latter group equals $\mathrm{H}^{2c}_X(X, \mathbb{Q})$ as the local to global spectral sequence degenerates.
In other words, the isomorphism $\mathbb{Q}_{X_{\mathrm{reg}}}[2 \dim X] \to \mathbb{D}_{X_{\mathrm{reg}}}$ extends uniquely to a morphism on X, hence to a global section of $\mathcal{H}^{-2\dim X}(\mathbb{D}_X)$ which is an element $c_{X|Y} \in \mathrm{H}^{-2\dim X}(\mathbb{D}_X)$.

Definition (fundamental class of a subvariety $X \subset Y$)**.**

$$c_{X|Y} \in \mathrm{H}^{2c}_X(X, \mathbb{Q}) = \mathrm{H}^{BM}_{2n-2c}(X).$$

Its image in absolute cohomology will be denoted $c_X \in \mathrm{H}^{2c}(X, \mathbb{Q})$.

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List of Symbols

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Eidesstattliche Erklärung

Hiermit versichere ich, die vorliegende Dissertation eigenständig verfasst und keine anderen als die angegebenen Quellen und Hilfmittel verwendet zu haben. Diese Arbeit wurde keiner anderen Prüfungsbehörde vorgelegt.

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