Dynamic convex risk measures: time consistency, prudence, and sustainability

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Frau Dipl.-Math. Irina Penner
geboren am 27.03.1975 in Orsk

Gutachter:

1. Prof. Dr. H. Föllmer
2. Prof. Dr. P. Imkeller
3. Prof. Dr. S. Weber

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Introduction

Several high-profile failures in the last decades have raised the concern how to monitor and control risk exposure in the financial industry. The global range of action, intense competition, and the increasing involvement in derivative trading create new dangers and require new methods of risk measurement and management. This issue is addressed in the regulation of standards and guidelines for banking supervision, known under the keyword "Basel II". Basel II is an international initiative that requires financial services companies to have a more risk sensitive framework for the assessment of regulatory capital. The aim is to create a better link between minimum regulatory capital and risk, to establish and to maintain a minimum capital requirement sufficient to ensure financial stability, and to ground risk measurement and management in actual data and rigorous quantitative techniques.

These objectives create a new challenge on Mathematical Finance to provide appropriate risk quantification methods. Indeed, the problem of quantifying the risk associated to a financial position has emerged as a key topic in the recent mathematical finance research. It started with an axiomatic analysis of capital requirements and the introduction of coherent risk measures in Artzner et al. [ADEH97] and [ADEH99]. The theory of coherent risk measures was developed further in Delbaen [Del02] and [Del00]. Föllmer and Schied [FS04] and Frittelli and Rosazza Gianin [FRG02] replace positive homogeneity by convexity in the set of axioms and establish a more general concept of a convex risk measure.

The theory of coherent and convex risk measures was developed first in the static setting. In this setting the future net values of financial positions are described as random variables X on some probability space. A convex risk measure ρ is defined as a real-valued convex functional on a space of such positions, i.e., the risk measure assigns to each position X a real value $\rho(X)$ interpreted as the associated risk. The set $\mathcal{A}$ of all financial positions with non-positive risk is called the acceptance set of ρ. The axiom of cash invariance then implies the representation

$$\rho(X) = \inf \left\{ m \in \mathbb{R} \,\middle|\, m + X \in \mathcal{A} \right\}.$$

Thus the value $\rho(X)$ can be viewed as the minimal capital requirement sufficient to ensure the acceptability of a position X. Moreover, under some regularity conditions, the duality theory of Fenchel-Legendre yields a robust representation of the form

$$\rho(X) = \sup_{Q} \Big(E_Q[-X] - \alpha(Q) \Big).$$

In other words, the risk of a position is evaluated as the worst expected loss under a whole class of probabilistic models Q. These alternative models contribute to the evaluation at a different degree, and this is made precise by the non-negative penalty function $\alpha(Q)$.

In the static formulation, however, the role of information is not jet visible. Suppose that the information available at time t is described by a σ-field $\mathcal{F}_t$. Then it is natural to assume that the risk assessment depends on the events in $\mathcal{F}_t$. Thus updated risk assignment at time t is described by a conditional risk measure ρ_t which associates to each position X an $\mathcal{F}_t$-measurable random variable $\rho_t(X)$. Such risk measures were studied in Arztner et al. [ADE$^+$], Delbaen [Del06], Frittelli and Rosazza Gianin [FRG04]. A conditional risk measure provides a natural generalization of a static risk measure to the conditional setting. It satisfies the same axioms and it can be represented as a suitably modified worst conditional expected loss under a whole class of measures. Such robust representations for conditional coherent risk measures were obtained first on a finite probability space in Roorda and Schumacher [RSE05] for random variables and in Riedel [Rie04] for stochastic processes. On a general probability space, robust representations for conditional coherent and convex risk measures defined on random variables were proved in Detlefsen [Det03], Scandolo [Sca03], Detlefsen and Scandolo [DS05], Bion-Nadal [BN04], Burgert [Bur05], Klöppel and Schweizer [KS]. Cheridito et al. provide in [CDK06] a representation result in the more general setting of conditional risk measures for stochastic processes.

In Chapter 1 we review and refine the robust representation results of conditional convex risk measures for random variables. This chapter is mostly expository, but we include the proofs in order to give a self-contained presentation and to introduce some technical modifications that we will need later on. In particular the representations we obtain in Lemma 1.2.5 will be useful for the discussion of time consistency in Chapter 2, since they allow us to formulate supermartingale properties in terms of a suitable class of measures.

After this preparation we go on to the dynamic discrete-time setting and assume that the information flow is given by some filtration $(\mathcal{F}_t)_{t=0,1,\dots}$ on the underlying probability space. Since the risk assessments should be updated as new information is released, we consider a dynamic risk measure given by a sequence $(\rho_t)_{t=0,1,\dots}$ of conditional convex risk measures adapted to the filtration $(\mathcal{F}_t)$.

A key question in the dynamical setting is how the conditional risk assessments at different times are interrelated. This question has led to several notions of time consistency that have been discussed in the literature. One of todays most used notions is *strong time consistency* which amounts to the recursion

$$\rho_t(-\rho_{t+1}) = \rho_t.$$

This form of time consistency was studied in Riedel [Rie04], Arztner et al. [ADE$^+$], Delbaen [Del06], Detlefsen and Scandolo [DS05], Burgert [Bur05], Klöppel and Schweizer [KS], Cheridito et al. [CDK06], Föllmer and Penner [FP06], Cheridito and Kupper [CK06]. As explained in [ADE$^+$], strong time consistency may be viewed as a version of the Bellmann principle. Thus strongly time consistent dynamic risk measures provide a particularly convenient tool for risk quantification methods based on the recursive principle, as in the case of superhedging. The recursion formula allows one to construct strongly time consistent dynamic risk measures easily in finite discrete time as shown in [CK06], and to use backward stochastic differential equations as a tool in continuous time as indicated in Peng [Pen97] and Rosazza Gianin [RG03]. However, strong time consistency is a rather strict notion, and it fails in some natural examples of dynamic risk measures such as average value at risk. This was noted in [ADE$^+$]. Moreover, Tutsch [Tut06] argues that strong time consistency is not an appropriate criterion for "updating" risk measures.

The literature on alternative notions of time consistency is not so numerous. One weaker form of time consistency is based on the following idea: If some position is accepted (or rejected) for any scenario tomorrow, it should be already accepted (or rejected) today. This property has appeared under several names. We call it here weak acceptance (resp. rejection) consistency. As to our knowledge weak acceptance consistency appeared first in [ADE$^+$]. Both weak acceptance and weak rejection consistency were introduced and used in Weber [Web06] in the context of law-invariant risk measures. Both notions also appear in Roorda and Schumacher [RS07] under the name "sequential consistency". Some characterizations of weak acceptance consistency are given in Burgert [Bur05] and in Tutsch [Tut06].

It is shown in Tutsch [Tut06] that time consistency properties can be characterized via benchmark sets: If a financial position at some future time is always preferable to some element of the benchmark set, then it should also be preferable today. The bigger the benchmark set, the stronger is the resulting notion of time consistency. This idea leads to some other possible notions of time consistency. We recall the general argumentation from [Tut06] in Section 2.1. In particular we focus on middle acceptance and rejection consistency, properties that are weaker as time consistency but stronger than the corresponding weak notions.

The main subject of this thesis is to investigate various time consistency properties of a dynamic convex risk measure and the manifestation of these properties in the dynamics of the corresponding penalty functions and risk processes. We turn to this question in Chapter 2. First we focus on the strong notion of time consistency in Section 2.2. This section is based on joint work with Hans Föllmer [FP06]. Theorem 2.2.2 gives equivalent characterizations of strong time consistency in terms of acceptance sets, penalty functions and a joint supermartingale property of the risk measure and its penalty function under any reasonable model Q. This supermartingale property is our main contribution to the characterizations of strong time consistency and it will play a key role in analyzing asymptotic properties of a dynamic risk measure as time goes to infinity. The characterization of time consistency in terms of acceptance sets has already appeared in Delbaen [Del06], Cheridito et al. [CDK06], Klöppel and Schweizer [KS]. Some similar properties of penalty functions are given in Cheridito et al. [CDK06] for time consistent risk measures on stochastic processes, but not as an equivalent characterization. Our characterization of time consistency in terms of penalty functions was also shown independently in Bion-Nadal [BN06], but under the stronger assumption of continuity from below. We also provide the explicit form of the Doob and of the Riesz decomposition of the penalty function process in Section 2.3.

In Section 2.4 we introduce and study a weaker notion of time consistency that we call *prudence*. This property is described by the relation

$$\rho_t(-\rho_{t+1}) \leq \rho_t.$$

Economically this means that the future update $\rho_t(X) - \rho_{t+1}(X)$ of the risk process is acceptable at time t. Thus by using prudent risk measures one always stays on the safe side. We believe that it is a reasonable property for a dynamic risk measure, in particular from a point of view of a regulator. Similar to the time consistent case, we characterize prudent dynamic risk measures in terms of acceptance sets, of penalty functions, and by a combined supermartingale property of risk processes and one-step penalty functions. The supermartingale property we obtain holds more generally for any bounded process such that its negative increments are acceptable with respect to the risk measure (ρ_t), i.e., the process can be upheld without any additional risk. We call such processes *sustainable* with respect to the dynamic risk measure (ρ_t). The characterization of sustainability by the supermartingale property is given in Theorem 2.4.6 and in Theorem 2.5.4. It can be viewed as a generalized optimal decomposition under convex constraints in analogy to results of Föllmer and Kramkov [FK97]; see also Chapter 9 in Föllmer and Schied [FS04]. In Example 4.2 we show how the results from [FS04] can be recovered from our more general characterization of sustainability.

Given a prudent dynamic risk measure (ρ_t), any risk process $(\rho_t(X))$ is sustainable with respect to it, and covers the final loss $-X$ if the time horizon is finite. The question arises whether $(\rho_t(X))$ is the smallest process with these properties, in other words whether we do not pay too much by "hedging" X with the process $(\rho_t(X))$. The discussion in Section 2.5 shows that one could possibly do better by using the strongly time consistent risk measure $(\widetilde{\rho}_t)$ that can be constructed from any dynamic risk measure (ρ_t) with a finite time horizon via the recursion

$$\widetilde{\rho}_T(X) := -X, \qquad \widetilde{\rho}_t(X) := \rho_t(-\widetilde{\rho}_{t+1}(X)), \quad t = 0, \dots, T-1.$$

This construction was introduced in Section 4.2 of Cheridito et al. [CDK06] and studied in Section 3.2 of Drapeau [Dra06] and in Cheridito and Kupper [CK06]. Using the supermartingale characterization of sustainability we identify the strongly time consistent risk measure $(\widetilde{\rho}_t)$ as the smallest process that is sustainable with respect to (ρ_t) and covers the final loss. Thus our discussion provides a new reason for using strongly time consistent risk measures.

The supermartingale properties of time consistent and of prudent dynamic risk measures with an infinite time horizon ensure the existence of the limit

$$\rho_\infty(X) := \lim_{t\to\infty} \rho_t(X)$$

for all positions X. In Chapter 3 we study the asymptotic behavior of time consistent and of prudent dynamic risk measures. In particular we focus on the question whether the dynamic risk measure (ρ_t) is *asymptotically safe* in the sense that the limiting capital requirement $\rho_\infty(X)$ covers the actual final loss $-X$. As shown by Example 3.1.6, not every time consistent dynamic risk measure is asymptotically safe. Theorem 3.1.4 gives criteria for asymptotic safety of time consistent risk measures in terms of the asymptotic behavior of acceptance sets and penalty functions, and Proposition 3.2.2 provides a sufficient condition for asymptotic safety of prudent risk measures. We also discuss the case where $\rho_\infty(X)$ is exactly equal to $-X$. This property of *asymptotic precision* may be viewed as a non-linear analogue of martingale convergence. We provide a sufficient condition for asymptotic safety in the time consistent case and in the prudent case. The results of this chapter for the time consistent risk measures were obtained in joint work with Hans Föllmer [FP06].

In the final Chapter 4 we illustrate the general results of Chapter 2 and Chapter 3 by examples. First we study the entropic dynamic risk measure in Section 4.1. In contrast to the usual definition of the entropic risk measure we allow the risk aversion to depend both on time and on the available information. Proposition 4.1.4 shows that time consistency properties of the entropic dynamic risk measure are completely determined by the monotonicity of its risk aversion process: Strong time consistency is characterized by constant risk aversion, and prudence by the condition that the risk aversion is decreasing in time. We illustrate the joint supermartingale property of the risk process and the penalty function and their asymptotic behavior for the time consistent and for the prudent entropic dynamic risk measure. The time consistent case was discussed in [FP06].

In Section 4.2 we consider a model for a financial market with convex trading constraints as in Chapter 9 of [FS04]. We define the risk of a position X as the minimal investment needed to hedge X in this model, i.e. $\rho_t(X)$ is the superhedging price of X at time t under the given constraints. We show that this definition leads to a time consistent dynamic convex risk measure. We provide a robust representation of this risk measure and identify it as the upper Snell envelope under constraints; cf. Section 9.3 of [FS04]. Moreover, using the supermartingale characterization of sustainability we show that any bounded process is dominated by a value process of some admissible strategy in this model iff it has a certain supermartingale property. This result is known as the optional decomposition under constraints, cf. Theorem 3.1 in [FK97] for continuous time and Theorem 9.20 in [FS04] for discrete time. Thus the latter theorem is a special case of our more general discussion of sustainability in Section 2.5 and Theorem 2.5.4.

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Chapter 1

Conditional convex risk measures and their robust representations

In this chapter we summerize and extend some results on conditional convex risk measures and their robust representations. It is well known that an unconditional convex risk measure which is continuous from above is of the form

$$\rho(X) = \sup_{Q \in \mathcal{M}_1(P)} \left(E_Q[X] - \alpha(Q) \right)$$

with some penalty function $\alpha : \mathcal{M}_1(P) \to \mathbb{R} \cup \{+\infty\}$, see [FS04] for details.

Analogous representations for conditional coherent risk measures on a finite probability space were obtained in [RSE05] for random variables and in [Rie04] for stochastic processes. Robust representations of conditional convex risk measures for random variables in general setting were proved in [Det03], [Sca03], [DS05], [Bur05], [KS] and in [CDK06] in a more general setting of stochastic processes.

In Section 1.1 we summerize some results from these cited papers, in particular from [DS05], introducing some technical modifications which we will need later on. In Section 1.2. we focus on sensitivity of a conditional convex risk measure. This property allows to drop the dependence on time t for the representing set of measures and to work only with equivalent propability measures, which is more convenient for technical reasons. In this section we apply and modify some results from [CDK06] and [KS], adapting them to our needs. In particular the representations we obtain in Lemma 1.2.5 will be very useful for the discussion of time consistency in Chapter 2.

In the following we consider a probability space $(\Omega, \mathcal{F}, P)$ and a sigma-field $\mathcal{F}_t \subseteq \mathcal{F}$. A conditional risk measures will be defined on the space $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$, where $X \in L^\infty$ describes a discounted terminal value of a financial position. By L^∞_t we denote the set of all $\mathcal{F}_t$-measurable P-a.s. bounded random variables. All inequalities and equalities applied to random variables are meant to hold P-a.s. .

1.1 Robust representations

We define a conditional convex risk measure as in [DS05]:

Definition 1.1.1. *A map $\rho_t : L^\infty \to L^\infty_t$ is called a* conditional convex risk measure *if it satisfies the following properties for all $X, Y \in L^\infty$:*

- *Conditional cash invariance: For all $X_t \in L^\infty_t$*

$$\rho_t(X + X_t) = \rho_t(X) - X_t$$

- *Monotonicity: $X \leq Y \;\Rightarrow\; \rho_t(X) \geq \rho_t(Y)$*
- *Conditional convexity: For all $\lambda \in L^\infty_t$, $0 \leq \lambda \leq 1$:*

$$\rho_t(\lambda X + (1-\lambda)Y) \leq \lambda \rho_t(X) + (1-\lambda)\rho_t(Y)$$

- *Normalization: $\rho_t(0) = 0$.*

A conditional convex risk measure is called a conditional coherent risk measure *if it has in addition the following property:*

- *Conditional positive homogeneity: For all $\lambda \in L^\infty_t$, $\lambda \geq 0$:*

$$\rho_t(\lambda X) = \lambda \rho_t(X).$$

Remark 1.1.2.
1. *If $\mathcal{F}_t = \{\emptyset, \Omega\}$, we have $L^\infty_t = \mathbb{R}$, and so we recover the usual definition of a convex risk measure; cf. [FS04]*
2. *In [DS05] a conditional convex risk measure is called* regular*, if*

$$\rho_t(I_A X) = I_A \rho_t(X)$$

for all $A \in \mathcal{F}_t$ and $X \in L^\infty$. It was shown in Corollary 1 of [DS05] that any normalized conditional convex risk measure is regular. In

[CDK06] a local property *of a conditional convex risk measure is defined as* $\rho_t(I_A X + I_{A^c} Y) = I_A \rho_t(X) + I_{A^c} \rho_t(Y)$ *for all* $A \in \mathcal{F}_t$ *and all* $X, Y \in L^\infty$. *Proposition 3.3 of [CDK06] shows that monotonicity and cash invariance imply this local property. For a normalized conditional convex risk measure regularity and the local property are equivalent, as shown in Proposition 1 in [DS05].*

3. *A weaker definition of a conditional convex risk measure is given in [KS], where normalization is not required and conditional convexity is replaced by regularity and by convexity only for constant coefficients.*

4. *If* ρ_t *is a convex conditional risk measure, then* $-\rho_t$ *defines a monetary concave utility functional on* L^∞ *in the sense of [CDK06], [KS].*

With a conditional convex risk measure ρ_t we associate its *acceptance set*

$$\mathcal{A}_t := \left\{ X \in L^\infty \;\middle|\; \rho_t(X) \leq 0 \right\}.$$

In the next proposition we recall some properties of an acceptance set from Proposition 3 in [DS05].

Proposition 1.1.3. *If* ρ_t *is a conditional convex risk measure then its acceptance set* $\mathcal{A}_t$ *is conditionally convex, solid and such that* $0 \in \mathcal{A}_t$ *and* $\operatorname{ess\,inf} \left\{ X \in L_t^\infty \;\middle|\; X \in \mathcal{A}_t \right\} = 0$. *Moreover,* ρ_t *is uniquely determined through its acceptance set, since*

$$\rho_t(X) = \operatorname{ess\,inf} \left\{ Y \in L_t^\infty \;\middle|\; X + Y \in \mathcal{A}_t \right\}. \tag{1.1}$$

Conversely, if some set $\mathcal{A}_t \subseteq L^\infty(\mathcal{F})$ *satisfies the preceding conditions, then the functional* $\rho_t : L^\infty \to L_t^\infty$ *defined via (1.1) is a conditional convex risk measure.*

Proof. The properties of the acceptance sets can be easy verified. To prove (1.1) note that due to cash invariance $\rho_t(X) + X \in \mathcal{A}_t$ for all X, and this implies "$\geq$" in (1.1). On the over hand, for all $Z \in \left\{ Y \in L_t^\infty \;\middle|\; X + Y \in \mathcal{A}_t \right\}$ we have

$$0 \geq \rho_t(Z + X) = \rho_t(X) - Z,$$

thus $\rho_t(X \leq \operatorname{ess\,inf} \left\{ Y \in L_t^\infty \;\middle|\; X + Y \in \mathcal{A}_t \right\}$.
For the proof of the last part of the assertion we refer to Proposition 3 in [DS05]. □

A characterization of acceptance sets in more general setting of risk measures on stochastic processes can be found in Proposition 3.6 of [CDK06].

Property (1.1) shows that a conditional convex risk measure can be viewed as a conditional capital requirement needed to make a financial position acceptable at time t. We will use (1.1) to define risk measures in Examples 4.1 and 4.2.

By $\mathcal{M}_1(P)$ we denote the set of all probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to P, and by $\mathcal{M}^e(P)$ the set of all probability measures on $(\Omega, \mathcal{F})$, which are equivalent to P on $\mathcal{F}$. Moreover, we define the sets

$$\mathcal{P}_t := \left\{ Q \in \mathcal{M}_1(P) \,\middle|\, Q \approx P \text{ on } \mathcal{F}_t \right\}$$

and

$$\mathcal{Q}_t := \left\{ Q \in \mathcal{M}_1(P) \,\middle|\, Q = P \text{ on } \mathcal{F}_t \right\}.$$

The *penalty function* will be given by a map α_t from some set $\mathcal{P} \subseteq \mathcal{P}_t$ to the set of $\mathcal{F}_t$-measurable random variable with values in $\mathbb{R} \cup \{+\infty\}$ such that

$$\operatorname*{ess\,sup}_{Q \in \mathcal{P}} (-\alpha_t(Q)) = 0.$$

In our setting the typical form of a penalty function will be

$$\alpha_t^{\min}(Q) = \operatorname*{ess\,sup}_{X \in \mathcal{A}_t} E_Q[-X \,|\, \mathcal{F}_t\,]. \tag{1.2}$$

Note that this penalty function is well defined for $Q \in \mathcal{P}_t$. We will say that ρ_t has a robust representation if

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{P}} (E_Q[-X \,|\, \mathcal{F}_t\,] - \alpha_t(Q)) \qquad \text{for all } X \in L^\infty$$

with some set $\mathcal{P} \subseteq \mathcal{P}_t$ and some penalty function α_t on $\mathcal{P}$.

The next theorem relates robust representations to some continuity properties of conditional convex risk measures; it is a version of Theorem 1 in [DS05] and Theorem 2.27 in [Det03], cf. also Theorem 3.6 in [Bur05], Theorem 3.16 in [KS] and Theorem 3.16 in [CDK06].

Theorem 1.1.4. *For a conditional convex risk measure ρ_t the following are equivalent:*

1. *ρ_t has the robust representation*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} (E_Q[-X \,|\, \mathcal{F}_t\,] - \alpha_t^{\min}(Q)), \qquad X \in L^\infty, \tag{1.3}$$

where the penalty function $\alpha_t^{\min}$ is given by (1.2).

2. ρ_t *has the robust representation*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{P}_t} (E_Q[-X \,|\, \mathcal{F}_t\,] - \alpha_t^{\min}(Q)), \qquad X \in L^\infty, \tag{1.4}$$

where the penalty function $\alpha_t^{\min}$ *is given by (1.2).*

3. ρ_t *has a robust representation.*

4. ρ_t *has the "Fatou-property": For any bounded sequence* (X_n) *which converges* P*-a.s. to some* X*,*

$$\rho_t(X) \leq \liminf_{n\to\infty} \rho_t(X_n) \quad P\text{-a.s.}.$$

5. ρ_t *is continuous from above, i.e.*

$$X_n \searrow X \;\; P\text{-a.s.} \quad \Longrightarrow \quad \rho_t(X_n) \nearrow \rho_t(X) \;\; P\text{-a.s.}$$

for any sequence $(X_n) \subseteq L^\infty$ *and* $X \in L^\infty$*.*

Proof. 2) $\Rightarrow$ 3) is obvious.

3) $\Rightarrow$ 4) Dominated convergence implies that $E_Q[X_n|\mathcal{F}_t] \to E_Q[X|\mathcal{F}_t]$ for each $Q \in \mathcal{P}_t$, and $\liminf \rho_t(X_n) \geq \rho_t(X)$ follows by using a robust representation of ρ_t as in the unconditional setting, see, e.g., Lemma 4.20 in [FS04].

4) $\Rightarrow$ 5) Monotonicity implies $\limsup \rho_t(X_n) \leq \rho_t(X)$, and $\liminf \rho_t(X_n) \geq \rho_t(X)$ follows by 4).

5) $\Rightarrow$ 1) The inequality

$$\begin{aligned} \rho_t(X) &\geq \operatorname*{ess\,sup}_{Q \in \mathcal{P}_t} (E_Q[-X \,|\, \mathcal{F}_t\,] - \alpha_t^{\min}(Q)) \\ &\geq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} (E_Q[-X \,|\, \mathcal{F}_t\,] - \alpha_t^{\min}(Q)) \end{aligned} \tag{1.5}$$

follows immediately from the definition of α_t^{min} and $\mathcal{Q}_t \subseteq \mathcal{P}_t$.

In order to prove the equality we will show that

$$E_P[\rho_t(X)] \leq E_P\left[\operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} (E_Q[-X \,|\, \mathcal{F}_t\,] - \alpha_t^{\min}(Q))\right].$$

To this end, consider the map $\rho^P : L^\infty \to \mathbb{R}$ defined by $\rho^P(X) := E_P[\rho_t(X)]$. It is easy to check that ρ^P is a convex risk measure which is continuous from above. Hence Theorem 4.31 in [FS04] implies that ρ^P has the robust representation

$$\rho^P(X) = \sup_{Q\in\mathcal{M}_1(P)} (E_Q[-X] - \alpha(Q)) \qquad X \in L^\infty,$$

where the penalty function $\alpha(Q)$ is given by

$$\alpha(Q) = \sup_{X\in L^\infty : \rho^P(X)\le 0} E_Q[-X].$$

Next we will prove that $Q \in \mathcal{Q}_t$ if $\alpha(Q) < \infty$. Indeed, let $A \in \mathcal{F}_t$ and $\lambda > 0$. Then

$$-\lambda P[A] = E_P[\rho_t(\lambda I_A)] = \rho^P(\lambda I_A) \ge E_Q[-\lambda I_A] - \alpha(Q),$$

so

$$P[A] \le Q[A] + \frac{1}{\lambda}\alpha(Q) \quad \text{for all} \quad \lambda > 0,$$

and hence $P[A] \le Q[A]$ if $\alpha(Q) < \infty$. The same reasoning with $\lambda < 0$ implies $P[A] \ge Q[A]$, thus $P = Q$ on $\mathcal{F}_t$ if $\alpha(Q) < \infty$. Moreover,

$$E_P[\alpha_t^{\min}(Q)] \le \alpha(Q) \tag{1.6}$$

holds for every $Q \in \mathcal{P}_t$, which can be seen as follows. As we will prove in Lemma 1.1.8 below,

$$E_P[\alpha_t^{\min}(Q)] = \sup_{Y\in\mathcal{A}_t} E_P[-Y].$$

Since $\rho^P(Y) \le 0$ for all $Y \in \mathcal{A}_t$, inequality (1.6) follows from the definition of the penalty function $\alpha(Q)$.

Finally we obtain

$$\begin{aligned}
E_P[\rho_t(X)] = \rho^P(X) &= \sup_{Q\in\mathcal{M}_1(P),\alpha(Q)<\infty} (E_Q[-X] - \alpha(Q)) \\
&\le \sup_{Q\in\mathcal{Q}_t, E_P[\alpha_t^{\min}(Q)]<\infty} (E_Q[-X] - \alpha(Q)) \\
&\le \sup_{Q\in\mathcal{Q}_t, E_P[\alpha_t^{\min}(Q)]<\infty} E_P[E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)] \\
&\le E_P\left[\operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t, E_P[\alpha_t^{\min}(Q)]<\infty} \Big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\Big)\right] \\
&\le E_P\left[\operatorname*{ess\,sup}_{Q\in\mathcal{Q}_t} E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\right],
\end{aligned} \tag{1.7}$$

proving equality (1.3).

1) $\Rightarrow$ 2) Follows immediately from inequality (1.5). □

Remark 1.1.5. *Another characterization of a conditional convex risk measure ρ_t that is equivalent to the properties 1)-5) of Theorem 1.1.4 is the following:*

6) The acceptance set $\mathcal{A}_t$ is weak-closed, e.g., it is closed in L^∞ with respect to the topology $\sigma(L^\infty, L^1)$.*

This equivalence was shown in Theorem 3.16 of [CDK06] in a more general contex of risk measures for stochastic processes and in Theorem 3.16 of [KS] for risk measures on random variables as in our setting. Though in [KS] a slightly different definition of a conditional risk measure is used, the reasoning given there works just the same in our case, cf. proof of Theorem 3.16 "I $\Rightarrow$ IV" and "IV $\Rightarrow$ I" in [KS].

A closer look at the proof of Theorem 1.1.4 yields the following corollary, which will be useful later on.

Corollary 1.1.6. *A conditional convex risk measure ρ_t is continuous from above if and only if for any $P^* \in \mathcal{M}^e(P)$ it is representable in the form*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^f(P^*)} \left(E_Q[-X \,|\, \mathcal{F}_t] - \alpha_t^{\min}(Q)\right), \qquad X \in L^\infty, \tag{1.8}$$

where

$$\mathcal{Q}_t^f(P^*) := \left\{ Q \in \mathcal{M}_1(P) \,\middle|\, Q = P^* \text{ on } \mathcal{F}_t,\ E_{P^*}[\alpha_t^{\min}(Q)] < \infty \right\}.$$

Proof. The inequality

$$\rho_t(X) \geq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^f(P)} \left(E_Q[-X \,|\, \mathcal{F}_t] - \alpha_t^{\min}(Q)\right)$$

follows from (1.5) since $\mathcal{Q}_t^f(P) \subseteq \mathcal{P}_t$, and (1.7) proves the equality for $\mathcal{Q}_t^f(P)$. Moreover, since the definition of a conditional convex risk measure and the continuity property only depend on the zero sets of P, the same reasoning works for any $P^* \in \mathcal{M}^e(P)$. □

In the *coherent* case we obtain the following representation result:

Corollary 1.1.7. *A conditional coherent risk measure ρ_t is continuous from above if and only if for any $P^* \in \mathcal{M}^e(P)$ it is representable in the form*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^0(P^*)} E_Q[-X \,|\, \mathcal{F}_t], \qquad X \in L^\infty, \tag{1.9}$$

where

$$\mathcal{Q}_t^0(P^*) := \left\{ Q \in \mathcal{M}_1(P) \,\middle|\, Q = P^* \text{ on } \mathcal{F}_t,\ \alpha_t^{\min}(Q) = 0\ Q\text{-a.s.} \right\}.$$

Proof. Due to positive homogeneity of ρ_t the penalty function $\alpha_t^{\min}(Q)$ can only take values 0 or ∞ for all $Q \in \mathcal{P}_t$. Indeed, for $A := \{\alpha_t^{\min}(Q) > 0\}, X \in \mathcal{A}_t$ and all $\lambda > 0$ we have $\lambda I_A X \in \mathcal{A}_t$, and hence

$$\begin{aligned}\alpha_t^{\min}(Q) &= \operatorname*{ess\,sup}_{X\in\mathcal{A}_t} E_Q[-X|\mathcal{F}_t] \\ &\geq \operatorname*{ess\,sup}_{X\in\mathcal{A}_t} E_Q[-\lambda I_A X|\mathcal{F}_t] \\ &= \lambda I_A \alpha_t^{\min}(Q),\end{aligned}$$

where the lower bound converges to ∞ with $\lambda \to \infty$ on A. Thus $\alpha_t^{\min}(Q) = \infty$ on A and $\alpha_t^{\min}(Q) = 0$ on A^c. If $Q \in Q_t^f(P^*)$ for some $P^* \approx P$, the inequality $E_{P^*}[\alpha_t^{\min}(Q)] < \infty$ implies $P[A] = 0$, hence $Q \in \mathcal{Q}_t^0(P^*)$. Thus (1.8) is equivalent to (1.9). □

The following lemma was used in the proof of the Theorem 1.1.4. Similar arguments are used in the proofs of Theorem 2.27 in [Det03], Theorem 1 in [DS05], Theorem 3.5 in [Bur05], Theorem 3.16 in [KS], and Theorem 3.16 in [CDK06].

Lemma 1.1.8. *For $Q \in \mathcal{P}_t$ and $0 \leq s \leq t$,*

$$E_Q[\alpha_t^{\min}(Q)|\mathcal{F}_s] = \operatorname*{ess\,sup}_{Y\in\mathcal{A}_t} E_Q[-Y|\mathcal{F}_s],$$

and in particular

$$E_Q[\alpha_t^{\min}(Q)] = \sup_{Y\in\mathcal{A}_t} E_Q[-Y].$$

Proof. First we claim that the set

$$\left\{ E_Q[-X|\mathcal{F}_t] \;\middle|\; X \in \mathcal{A}_t \right\}$$

is directed upward for any $Q \in \mathcal{P}_t$. Indeed, for $X, Y \in \mathcal{A}_t$ we can define $Z := XI_A + YI_{A^c}$, where $A := \{E_Q[-X|\mathcal{F}_t] \geq E_Q[-Y|\mathcal{F}_t]\} \in \mathcal{F}_t$. Conditional convexity of ρ_t implies that $Z \in \mathcal{A}_t$, and by definition of Z

$$E_Q[-Z|\mathcal{F}_t] = \max\left(E_Q[-X|\mathcal{F}_t], E_Q[-Y|\mathcal{F}_t]\right).$$

Hence there exists a sequence (X_n^Q) in $\mathcal{A}_t$ such that

$$\alpha_t^{\min}(Q) = \lim_n E_Q[-X_n^Q|\mathcal{F}_t] \qquad P\text{-a.s..,} \tag{1.10}$$

and by monotone convergence we get

$$\begin{aligned}E_Q[\alpha_t^{\min}(Q)|\mathcal{F}_s] &= \lim_n E_Q\left[E_Q[-X_n^Q|\mathcal{F}_t] \;\middle|\; \mathcal{F}_s \right] \\ &\leq \operatorname*{ess\,sup}_{Y\in\mathcal{A}_t} E_Q[-Y|\mathcal{F}_s].\end{aligned}$$

The converse inequality follows directly from the definition of $\alpha_t^{\min}(Q)$. □

Remark 1.1.9. *The penalty function* $\alpha_t^{\min}(Q)$ *is minimal in the sense that any other penalty function* α_t *in a robust representation of* ρ_t *satisfies*

$$\alpha_t^{\min}(Q) \leq \alpha_t(Q) \;\; P\text{-}a.s.$$

for all $Q \in \mathcal{P}_t$*. An alternative formula for the minimal penalty function is given by*

$$\alpha_t^{\min}(Q) = \operatorname*{ess\,sup}_{X \in L^\infty} \left(E_Q[-X \mid \mathcal{F}_t\,] - \rho_t(X) \right) \quad \text{for all } Q \in \mathcal{P}_t. \tag{1.11}$$

This follows as in the unconditional case; see e.g. Theorem 4.15 and Remark 4.16 in [FS04].

1.2 Sensitivity

In this section we will show that under an assumption of *sensitivity* with respect to the reference measure P it is sufficient to use only equivalent probability measures in the robust representations of risk measures. This is more convenient for technical reasons, and it allows us to drop the dependence on time t for the representing set of measures.

Definition 1.2.1. *We call a conditional convex risk measure* sensitive *or* relevant*, if*

$$P[\rho_t(-\varepsilon I_A) > 0] > 0 \tag{1.12}$$

holds for all $\varepsilon > 0$ *and for any* $A \in \mathcal{F}$ *such that* $P[A] > 0$.

Remark 1.2.2.

1. *For coherent risk measures it is sufficient to require*

$$P[\rho_t(-I_A) > 0] > 0, \tag{1.13}$$

since (1.13) and (1.12) are equivalent under the assumption of positive homogeneity. This corresponds to the definition of relevance for coherent risk measures given in [Del02] for the unconditional case. For a convex risk measure, condition (1.12) is stronger than (1.13).

2. *Several slightly different definitions of relevance can be found in the literature. In [KS] relevance is defined as in (1.13). In [CDK06] the stronger property* $A \subseteq \{\rho_t(-\varepsilon I_A) > 0\}$ *for all* $\varepsilon > 0$ *is required in a more general setting. The arguments used in this section are similar to those in [KS] and [CDK06] up to some technical details.*

In the sequel we will assume that a conditional convex risk measure ρ_t has a robust representation. First we prove a "*σ-pasting property*" of the penalty functions which also appears in Lemma 3.12 of [KS].

Lemma 1.2.3. *Let (Q_n) be a sequence in $\mathcal{Q}_t$ and (A_n) a sequence of pairwise disjoint events in $\mathcal{F}_t$ such that $\cup_n A_n = \Omega$ P-a.s.. Then*

$$\widetilde{Z} := \sum_{n=1}^{\infty} I_{A_n} \frac{dQ_n}{dP}$$

defines a density of a probability measure $\widetilde{Q} \in \mathcal{Q}_t$ such that

$$\alpha_t^{\min}(\widetilde{Q}) = \sum_{n=1}^{\infty} I_{A_n} \alpha_t^{\min}(Q_n)$$

(here we define $I_{A_n}\alpha_t^{\min}(Q_n) := 0$, if $P[A_n] = 0$).

Proof. We will prove the first part of the lemma more generally for any sequence (λ_n) in L_t^∞ with $0 \leq \lambda_n \leq 1$ and $\sum_{n=1}^{\infty} \lambda_n = 1$ P-a.s.. Let $Z_n := dQ_n/dP$ and $\widetilde{Z} := \sum_{n=1}^{\infty} \lambda_n Z_n$. By monotone convergence,

$$E[\widetilde{Z}|\mathcal{F}_t] = \lim_n \sum_{k=1}^{n} \lambda_k E[Z_k|\mathcal{F}_t] = 1,$$

and so $\widetilde{Z}$ is indeed the density of a probability measure $\widetilde{Q} \in \mathcal{Q}_t$. Since

$$|\sum_{k=1}^{n} \lambda_k Z_k X| \leq \widetilde{Z}\,\|X\|_\infty \in L^1(P) \qquad \text{for all } n,$$

the dominated convergence theorem implies

$$E_{\widetilde{Q}}[X|\mathcal{F}_t] = \sum_{n=1}^{\infty} \lambda_n E_{Q_n}[X|\mathcal{F}_t] \tag{1.14}$$

for any $X \in L^\infty$. From the definition of the minimal penalty function we obtain immediately

$$\alpha_t^{\min}(\widetilde{Q}) \leq \sum_{n=1}^{\infty} \lambda_n \alpha_t^{\min}(Q_n).$$

In particular if $\lambda_n := I_{A_n}$ for a sequence (A_n) as above we obtain

$$\begin{aligned}\operatorname*{ess\,sup}_{X\in\mathcal{A}_t} E_{\widetilde{Q}}[-X|\mathcal{F}_t] &= \operatorname*{ess\,sup}_{X\in\mathcal{A}_t} \left(\sum_{n=1}^{\infty} I_{A_n} E_{Q_n}[-X|\mathcal{F}_t]\right)\\ &= \sum_{n=1}^{\infty} I_{A_n} \operatorname*{ess\,sup}_{X\in\mathcal{A}_t} E_{Q_n}[-X|\mathcal{F}_t]\end{aligned}$$

so

$$\alpha_t^{\min}(\widetilde{Q}) = \sum_{n=1}^{\infty} I_{A_n} \alpha_t^{\min}(Q_n).$$

□

In particular, for any $A \in \mathcal{F}_t$ and $Q_1, Q_2 \in \mathcal{Q}_t$,

$$\widetilde{Z} := I_A \frac{dQ_1}{dP} + I_{A^c} \frac{dQ_2}{dP}$$

defines a density of a probability measure $\widetilde{Q} \in \mathcal{Q}_t$ with

$$\alpha_t^{\min}(\widetilde{Q}) = I_A \alpha_t^{\min}(Q_1) + I_{A^c} \alpha_t^{\min}(Q_2). \tag{1.15}$$

This finite pasting property of the penalty functions, which corresponds to the local property of the risk measure, also appears in Remark 3.13 of [CDK06].

It follows from (1.15) that the set $\left\{ \alpha_t^{\min}(Q) \;\middle|\; Q \in \mathcal{Q}_t \right\}$ is downward directed, and hence there exists a sequence (Q_n) in $\mathcal{Q}_t$ such that

$$\alpha_t^{\min}(Q_n) \searrow \operatorname*{ess\,inf}_{Q \in \mathcal{Q}_t} \alpha_t^{\min}(Q) = 0 \qquad P\text{-a.s.}. \tag{1.16}$$

For $\varepsilon > 0$ we consider the set

$$\mathcal{Q}_t^{\varepsilon} := \left\{ Q \in \mathcal{Q}_t \;\middle|\; \alpha_t^{\min}(Q) < \varepsilon \;\; P\text{-a.s.} \right\},$$

and we use the same notation for the corresponding set of densities:

$$\mathcal{Q}_t^{\varepsilon} = \left\{ \frac{dQ}{dP} \;\middle|\; Q \in \mathcal{Q}_t, \; \alpha_t^{\min}(Q) < \varepsilon \; P\text{-a.s.} \right\}.$$

We now show that the set $\mathcal{Q}_t^{\varepsilon}$ is non-empty. Moreover, it contains an equivalent probability measure as soon as the risk measure is sensitive; this part is similar to Lemma 3.22 in [CDK06].

Lemma 1.2.4. *For any $\varepsilon > 0$ the set $\mathcal{Q}_t^{\varepsilon}$ is nonempty. For a sensitive conditional convex risk measure there exists a probability measure $P^* \approx P$ such that $P^* \in \mathcal{Q}_t^{\varepsilon}$.*

Proof. For $\varepsilon > 0$ and a sequence (Q_n) as in (1.16) with densities (Z_n) we define an $\mathcal{F}_t$-measurable $\mathbb{N}$-valued random variable

$$\tau^{\varepsilon} := \min \left\{ n \;\middle|\; \alpha_t^{\min}(Q_n) < \varepsilon \right\}.$$

It follows from (1.16) that $\tau^\varepsilon < \infty$ P-a.s.. Thus the sets $A_n := \{\tau^\varepsilon = n\}$ $(n = 1, 2, \ldots)$ form a disjoint partition of Ω with $A_n \in \mathcal{F}_t$ for all n. By Lemma 1.2.3

$$Z_{\tau^\varepsilon} := \sum_{n=1}^{\infty} Z_n I_{\{\tau^\varepsilon = n\}}$$

defines a density of a probability measure $Q^\varepsilon \in \mathcal{Q}_t$ with

$$\alpha_t^{\min}(\widetilde{Q}) = \sum_{n=1}^{\infty} I_{A_n} \alpha_t^{\min}(Q_n) < \varepsilon \quad P\text{-a.s.},$$

which proves $Q^\varepsilon \in \mathcal{Q}_t^\varepsilon$.
Next we use a standard exhaustion argument to conclude that $\mathcal{Q}_t^\varepsilon$ contains an equivalent measure P^* under the assumption of sensitivity. Let

$$c := \sup \left\{ P[Z > 0] \;\middle|\; Z \in \mathcal{Q}_t^\varepsilon \right\}$$

and take a sequence $(Z_n)_{n \in \mathbb{N}}$ in $\mathcal{Q}_t^\varepsilon$ such that $P[Z_n > 0] \to c$. Then

$$Z^* := \sum_{n=1}^{\infty} \frac{1}{2^n} Z_n$$

belongs to the set $\mathcal{Q}_t^\varepsilon$ by Lemma 1.2.3, and

$$\{Z^* > 0\} = \cup_n \{Z_n > 0\}.$$

Hence $P[Z^* > 0] = c$. Next we show that $c = 1$, and so the probability measure P^* defined via $dP^*/dP := Z^*$ has the desired properties. Suppose by way of contradiction that the set $A := \{Z^* = 0\}$ has positive probability. Sensitivity implies $P[\rho_t(-\varepsilon I_A) > 0] > 0$, where

$$\rho_t(-\varepsilon I_A) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} \left(E_Q[\varepsilon I_A | \mathcal{F}_t] - \alpha_t^{\min}(Q) \right).$$

Hence there exists $\widetilde{Q} \in \mathcal{Q}_t$ such that the set $B := \{\alpha_t^{\min}(\widetilde{Q}) < E_{\widetilde{Q}}[\varepsilon I_A | \mathcal{F}_t]\} \in \mathcal{F}_t$ satisfies $P[B] > 0$. In particular is $\alpha_t^{\min}(\widetilde{Q}) < \varepsilon$ on B. By $\widetilde{Z}$ we denote the density of $\widetilde{Q}$ with respect to P. Without loss of generality we assume that $\widetilde{Q} \in \mathcal{Q}_t^\varepsilon$; otherwise we can simply switch to a probability measure $\widehat{Q}$ defined via $d\widehat{Q}/dP := I_B \widetilde{Z} + I_{B^c} Z$, where Z is an arbitrary element of $\mathcal{Q}_t^\varepsilon$. Then $\widehat{Q}$ is in $\mathcal{Q}_t^\varepsilon$ by (1.15) and $\widehat{Q}$ and $\widetilde{Q}$ coincide on B.
Next we will show that the set $\{\widetilde{Z} > 0\} \cap A$ has positive probability. Indeed, it follows from the definition of B and $\alpha_t^{\min}(\widetilde{Q}) \geq 0$, that

$$E[\widetilde{Z} I_B I_A] = E_{\widetilde{Q}}[I_B I_A] = E_{\widetilde{Q}}[I_B E_{\widetilde{Q}}[I_A | \mathcal{F}_t]] > 0,$$

which implies $P[\{\widetilde{Z} > 0\} \cap A \cap B] > 0$ and in particular $P[\{\widetilde{Z} > 0\} \cap A] > 0$. Thus the probability measure $\widehat{Q}$ defined via

$$\frac{d\widehat{Q}}{dP} := \widehat{Z} := \frac{1}{2}Z^* + \frac{1}{2}\widetilde{Z},$$

belongs to $\mathcal{Q}_t^\varepsilon$, and we have

$$P[\widehat{Z} > 0] = P[Z^* > 0] + P[\{\widetilde{Z} > 0\} \cap A] > P[Z^* > 0],$$

in contradiction to the maximality of $P[Z^* > 0]$. □

Our goal is to obtain a robust representation for a conditional convex risk measure in terms of equivalent probability measures. The following lemma shows that this is possible if there exists some equivalent probability measure such that its penalty function is a.s. bounded. Similar arguments are used in Proposition 3.22 of [KS] and Theorem 3.22 of [CDK06]. In the second part of the lemma we reduce the class of the representing measures even further, and this reduced representation will be useful in our discussion of time consistency.

Lemma 1.2.5. *Let ρ_t be a conditional convex risk measure that is continuous from above, and let P^* be a probability measure such that $P^* \approx P$ and $\alpha_t^{\min}(P^*) < \infty$ P-a.s.. Then*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{M}^e(P)} \Big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \Big) \tag{1.17}$$

for all $X \in L^\infty$. Moreover, if $E_{P^*}[\alpha_t^{\min}(P^*)] < \infty$ then

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^{f,e}(P^*)} \Big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \Big) \tag{1.18}$$

for all $X \in L^\infty$, where

$$\mathcal{Q}_t^{f,e}(P^*) := \Big\{ Q \in \mathcal{M}^e(P) \,\Big|\, Q = P^* \text{ on } \mathcal{F}_t,\ E_{P^*}[\alpha_t^{\min}(Q)] < \infty \Big\}.$$

Proof. By Z^* we denote the density of P^* with respect to P, and for $\varepsilon \in (0,1)$ and $Q \in \mathcal{Q}_t$ we define a probability measure Q_ε via

$$\frac{dQ_\varepsilon}{dP} := (1-\varepsilon)\frac{dQ}{dP} + \varepsilon \frac{Z^*}{E[Z^*|\mathcal{F}_t]}.$$

Then $Q_\varepsilon \in Q_t$, $Q_\varepsilon \in \mathcal{M}^e(P)$ and

$$E_{Q_\varepsilon}[X|\mathcal{F}_t] = (1-\varepsilon)E_Q[X|\mathcal{F}_t] + \varepsilon E_{P^*}[X|\mathcal{F}_t]$$

for all $X \in L^\infty$. By definition of the minimal penalty function we obtain

$$\alpha_t^{\min}(Q_\varepsilon) \leq (1-\varepsilon)\alpha_t^{\min}(Q) + \varepsilon\alpha_t^{\min}(P^*).$$

Thus

$$\begin{aligned}
\rho_t(X) &= \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} \Big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \Big) \\
&\geq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t \cap \mathcal{M}^e(P)} \Big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \Big) \\
&\geq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} \Big(E_{Q_\varepsilon}[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q_\varepsilon) \Big) \\
&\geq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t} ((1-\varepsilon) E_Q[-X|\mathcal{F}_t] \\
&\quad + \varepsilon E_{P^*}[-X|\mathcal{F}_t] - (1-\varepsilon)\alpha_t^{\min}(Q) - \varepsilon\alpha_t^{\min}(P^*) \Big) \\
&= (1-\varepsilon)\rho_t(X) + \varepsilon \Big(E_{P^*}[-X|\mathcal{F}_t] - \alpha_t^{\min}(P^*) \Big) \\
&\geq \rho_t(X) - \varepsilon \Big(\rho_t(X) + \|X\|_\infty + \alpha_t^{\min}(P^*) \Big), \qquad (1.19)
\end{aligned}$$

where the lower bound converges a.s. to ρ_t with $\varepsilon \to 0$. Hence

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t \cap \mathcal{M}^e(P)} \Big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \Big).$$

On the other hand it follows from the representation (1.4) that

$$\begin{aligned}
\rho_t(X) &\geq \operatorname*{ess\,sup}_{Q \in \mathcal{M}^e(P)} \Big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \Big) \\
&\geq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t \cap \mathcal{M}^e(P)} \Big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \Big),
\end{aligned}$$

proving the representation (1.17).

If $E_{P^*}[\alpha_t^{\min}(P^*)] < \infty$ we define for $Q \in \mathcal{Q}_t^f(P^*)$ and $\varepsilon \in (0,1)$ a probability measure Q_ε via

$$\frac{dQ_\varepsilon}{dP^*} := (1-\varepsilon)\frac{dQ}{dP^*} + \varepsilon.$$

Then $Q_\varepsilon = P^*$ on $\mathcal{F}_t$, $Q_\varepsilon \in \mathcal{M}^e(P)$ and

$$E_{Q_\varepsilon}[X|\mathcal{F}_t] = (1-\varepsilon)E_Q[X|\mathcal{F}_t] + \varepsilon E_{P^*}[X|\mathcal{F}_t]$$

for all $X \in L^\infty$. This implies

$$\alpha_t^{\min}(Q_\varepsilon) \leq (1-\varepsilon)\alpha_t^{\min}(Q) + \varepsilon\alpha_t^{\min}(P^*)$$

and in particular $E_{P^*}\left[\alpha_t^{\min}(Q_\varepsilon)\right] < \infty$, so $Q_\varepsilon \in \mathcal{Q}_t^{f,e}(P^*)$. Thus we obtain using Corollary 1.1.6

$$\begin{aligned}
\rho_t(X) &= \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^f} \Big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \Big) \\
&\geq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^{f,e}} \Big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \Big) \\
&\geq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_t^f} \Big(E_{Q_\varepsilon}[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q_\varepsilon) \Big) \\
&\geq \rho_t(X) - \varepsilon \Big(\rho_t(X) + \|X\|_\infty + \alpha_t^{\min}(P^*) \Big)
\end{aligned}$$

and the representation (1.18) follows. □

In view of Lemma 1.2.4 and Lemma 1.2.5 we obtain the following corollary:

Corollary 1.2.6. *Any sensitive conditional convex risk measure that is continuous from above is representable as in (1.17) and (1.18).*

Chapter 2

Time consistency, prudence, and sustainability

2.1 Introduction and notation.

In this chapter we consider a discrete-time multiperiod information structure given by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, P)$. The time horizon T might be finite or infinite. We assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F} = \mathcal{F}_T$ if $T < \infty$ and $\mathcal{F} = \sigma(\cup_t \mathcal{F}_t)$ if $T = \infty$. We use the same notation $t = 0, 1, \dots$ in both cases $T = \infty$ and $T < \infty$, for $T < \infty$ this numeration is meant to stop by T.

We consider a sequence $(\rho_t)_{t=0,1,\dots}$, such that each ρ_t is a conditional convex risk measure defined on the set of all financial positions $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$. Such a sequence (with a finite time horizon) is called a dynamic convex risk measure in [DS05] or (with opposite sign) a dynamic monetary utility functional in [CDK06].

A dynamic risk measure (ρ_t) induces for each financial position X a risk process $(\rho_t(X))$ describing the conditional risk assessments associated to X over the time. A key question in the dynamic setting is how these risk assessments in different periods of time should be interrelated. This question has led to several notions of time consistency in the literature. We would like to give a short overview of various time consistency notions here and explain the terminology we are going to use in this chapter.

A unifying view of different time consistency notions was suggested in [Tut06] and extended in [Dra06]. In the sequel we will summarize the reasoning from [Tut06] and [Dra06] with some minor modifications.

Definition 2.1.1. *Assume that* $(\rho_t)_{t=0,1,\dots}$ *is a dynamic risk measure and let* $\mathcal{Y}_t$ *be a subset of* L^∞ *such that* $0 \in \mathcal{Y}_t$ *and* $\mathcal{Y}_t + \mathbb{R} = \mathcal{Y}_t$ *for each* $t \in \{0, 1, \dots\}$. *Then* $(\rho_t)_{t=0,1,\dots}$ *is called* acceptance (resp. rejection) consistent with respect to $(\mathcal{Y}_t)_{t=0,1,\dots}$, *if for all* $t, s \in \{0, 1, \dots\}$ *and for any* $X \in L^\infty$ *and* $Y \in \mathcal{Y}_{t+s}$ *the following condition holds:*

$$\rho_{t+s}(X) \le \rho_{t+s}(Y) \;\; (\textit{resp.} \ge) \quad \Longrightarrow \quad \rho_t(X) \le \rho_t(Y) \;\; (\textit{resp.} \ge). \tag{2.1}$$

The idea is that the degree of time consistency is determined by a sequence of benchmark sets $(\mathcal{Y}_t)_{t=0,1,\dots}$: If a financial position at some future time is always preferable to some element of the benchmark set, then it should also be preferable today. The bigger the benchmark set, the stronger is the resulting notion of time consistency. In the following we focus on three cases.

Definition 2.1.2. *We call a dynamic convex risk measure* $(\rho_t)_{t=0,1,\dots}$

1. (strongly) time consistent*, if it is either acceptance consistent or rejection consistent with respect to* $\mathcal{Y}_t = L^\infty$ *for all* $t = 0, 1, \dots$ *in the sense of Definition 2.1.1;*
2. middle acceptance (resp. middle rejection) consistent*, if for all* $t = 0, 1, \dots$ *we have* $\mathcal{Y}_t = L^\infty(\mathcal{F}_t)$ *in the Definition 2.1.1;*
3. weakly acceptance (resp. weakly rejection) consistent*, if for all* $t = 0, 1, \dots$ *we have* $\mathcal{Y}_t = \mathbb{R}$ *in the Definition 2.1.1.*

Note that there is no difference between rejection consistency and acceptance consistency with respect to L^∞, since the role of X and Y is symmetric in that case.
Obviously strong time consistency implies both middle rejection and middle acceptance consistency, and middle rejection (resp. middle acceptance) consistency implies weak rejection (resp. weak acceptance) consistency. This explains the names strong, middle and weak.

We focus first on the weak notion of time consistency. The definition of this property can be relaxed, since due to cash invariance of the risk measure it is sufficient to take 0 as a benchmark set $\mathcal{Y}_t$ for all t.

Proposition 2.1.3. *A sequence of conditional risk measures* $(\rho_t)_{t=0,1,\dots}$ *is weakly acceptance (resp. weakly rejection) consistent, if and only if for any* $X \in L^\infty$ *and for all* $t \ge 0$ *the following condition holds:*

$$\rho_{t+1}(X) \le 0 \quad (\textit{resp.} \ge) \quad \Longrightarrow \quad \rho_t(X) \le 0 \quad (\textit{resp.} \ge). \tag{2.2}$$

Condition (2.2) explains the intuition behind this weak notion of time consistency: If some position is accepted (or rejected) for any scenario (modulo nullsets) tomorrow, it should be already accepted (or rejected) today. As to our knowledge weak acceptance consistency appeared first in [ADE+]. Both weak acceptance and weak rejection consistency were introduced and used in [Web06] (without adding "weak" to the names) in the context of law-invariant risk measures. Both notions also appear in [RS07] under the name "sequential consistency". Some characterizations of weak acceptance consistency are given in [Bur05] and in [Tut06]. We will come across weak acceptance consistency in Section 2.3. and we will characterize it in terms of penalty functions.

Next we turn to the strong notion of time consistency. From now on we call it simply time consistency. This notion is rather strict, but it seems to be the easiest one to work with and the most applicable. Time consistency has been used extensively in the recent work on dynamic risk measures, see [ADE+], [Del06], [Rie04], [DS05], [CDK06], [KS], [Bur05] and references therein. In the next Proposition we will give some equivalent characterizations of time consistency that can be (and are) used as definitions of this property.

Proposition 2.1.4. *A dynamic convex risk measure $(\rho_t)_{t=0,1,\dots}$ is time consistent if and only if*

1. *for all $t \geq 0$ the following condition holds:*

$$\rho_{t+1}(X) \leq \rho_{t+1}(Y) \;\; P\text{-a.s.} \quad \Longrightarrow \quad \rho_t(X) \leq \rho_t(Y) \;\; P\text{-a.s.}; \tag{2.3}$$

2. *for any $X, Y \in L^\infty$ and for all $t \geq 0$ the following condition holds:*

$$\rho_{t+1}(X) = \rho_{t+1}(Y) \;\; P\text{-a.s.} \quad \Longrightarrow \quad \rho_t(X) = \rho_t(Y) \;\; P\text{-a.s.}; \tag{2.4}$$

3. *$(\rho_t)_{t=0,1,\dots}$ is one-step recursive, i.e.*

$$\rho_t = \rho_t(-\rho_{t+1}) \quad P\text{-a.s.} \tag{2.5}$$

   for all $t \in \{0, 1, \dots\}$;

4. *$(\rho_t)_{t=0,1,\dots}$ is recursive, i.e.*

$$\rho_t = \rho_t(-\rho_{t+s}) \quad P\text{-a.s.};$$

   for all $t, s \in \{0, 1, \dots\}$.

Proof. It is obvious that time consistency implies condition (2.3), and (2.3) implies (2.4). By cash invariance we have $\rho_{t+1}(-\rho_{t+1}(X)) = \rho_{t+1}(X)$ and hence one-step recursiveness follows from (2.4).
We will prove that one-step recursiveness implies recursiveness by induction on s. For $s = 1$ the claim is true for all t. Now we assume that the induction hypothesis holds for each t and all $k \leq s$ for some $s \geq 1$. Then we obtain

$$\begin{aligned}\rho_t(-\rho_{t+s+1}(X)) &= \rho_t(-\rho_{t+s}(-\rho_{t+s+1}(X))) \\ &= \rho_t(-\rho_{t+s}(X)) \\ &= \rho_t(X),\end{aligned}$$

where we have applied the induction hypothesis to the random variable $-\rho_{t+s+1}(X)$. Hence the claim follows.
Due to monotonicity recursiveness implies time consistency. ☐

Remark 2.1.5. *1. The equivalence of time consistency and one-step recursiveness was already proved in Proposition 5 of [DS05].*

2. As explained in [ADE⁺], recursiveness may be viewed as a version of the Bellman principle for dynamic risk measures.

3. In [Bur05] time consistency is defined as in (2.3) but in terms of stopping times. For coherent risk measures it is shown in [Bur05] that this is equivalent to recursiveness for stopping times.

4. A more general definition of time consistency in terms of recursiveness for stopping times is given in [CDK06] for risk measures on stochastic processes. Proposition 4.5 in [CDK06] shows that this definition is equivalent to recursiveness in the sense of (2.5) if the time horizon is finite or if all risk measures are continuous from above.

We will study time consistency in more detail in Section 2.2.

Now we consider middle acceptance and middle rejection consistency. As to our knowledge these properties have not appeared in the literature before. Similar to strong time consistency, middle time consistency can be characterized via recursive inequalities as we state in the next Proposition.

Proposition 2.1.6. *A sequence of conditional risk measures $(\rho_t)_{t=0,1,\dots}$ is middle rejection (resp. middle acceptance) consistent if and only if for all $t, s \geq 0$ the following condition holds:*

$$\rho_t(-\rho_{t+s}) \leq \rho_t \quad (\textit{resp.} \geq) \quad P\text{-a.s.}. \tag{2.6}$$

The proof of this proposition is given in Theorem 3.1.5 of [Tut06] and Proposition 3.5 in [Dra06]. In order to stay self-contained we repeat it here.

Proof. We argue for the case of middle rejection consistency; the case of middle acceptance consistency follows in the same manner. Assume first that $(\rho_t)_{t=0,1,\ldots}$ satisfies (2.6) and let $X \in L^\infty$ and $Y \in L^\infty(\mathcal{F}_{t+s})$ such that $\rho_{t+s}(X) \geq \rho_{t+s}(Y)$. Since $\rho_{t+s}(Y) = -Y$ for all $Y \in L^\infty(\mathcal{F}_{t+s})$ due to cash-invariance, we obtain using (2.6) and monotonicity

$$\rho_t(X) \geq \rho_t(-\rho_{t+s}(X)) \geq \rho_t(-\rho_{t+s}(Y)) = \rho_t(Y).$$

This proves middle rejection consistency.
To prove the opposite direction note that

$$\rho_{t+s}(X) = \rho_{t+s}(-\rho_{t+s}(X))$$

for all $X \in L^\infty$ due to cash-invariance of the risk measure ρ_{t+s}. Since $-\rho_{t+s}(X) \in L^\infty(\mathcal{F}_{t+s})$ we can apply (2.1) to $Y = -\rho_{t+s}(X)$ and obtain

$$\rho_t(X) \geq \rho_t(-\rho_{t+s}(X)),$$

which proves (2.6). □

Remark 2.1.7. *It was proved in Proposition 3.9 of [Dra06] that for a* coherent *dynamic risk measure* $(\rho_t)_{t=0,1,\ldots}$ *weak acceptance consistency and middle acceptance consistency are equivalent.*

We have seen that in the case of time consistency one-step recursiveness $\rho_t = \rho_t(-\rho_{t+1})$ already implies the general property $\rho_t = \rho_t(-\rho_{t+s})$ for all $s \geq 0$. In the case of middle rejection consistency we believe that the inequality $\rho_t(-\rho_{t+1}) \leq \rho_t$ is not sufficient to prove general rejection consistency, although we do not have a counter example. Let us therefore introduce special notation for one-step properties.

Definition 2.1.8. *We call a sequence of conditional risk measures* $(\rho_t)_{t=0,1,\ldots}$ one-step middle rejection consistent (resp. one-step middle acceptance consistent)*, if inequality (2.6) (or equivalently relation (2.1)) holds only with* $s = 1$.

We will say more about the economic interpretation of middle rejection consistency in Section 2.4, where we relate it with the notion of prudence of a dynamic risk measure. Moreover, one-step middle rejection and one-step middle acceptance consistency are interesting because their combination results in the strong time consistency property. Thus characterizations of middle time consistency allow us to understand better the different aspects of strong time consistency.

In the rest of this chapter we will give alternative characterizations of various time consistency properties. To this end we introduce some notation. If we restrict a conditional convex risk measure ρ_t to the space L^∞_{t+s} for some $s \geq 0$, the corresponding acceptance set is given by

$$\mathcal{A}_{t,t+s} := \left\{ X \in L^\infty_{t+s} \,\middle|\, \rho_t(X) \leq 0 \right\}.$$

If ρ_t is continuous from above, then this property holds on L^∞_{t+s}, and thus the restriction of ρ_t to L^∞_{t+s} is representable with the minimal penalty function

$$\alpha^{\min}_{t,t+s}(Q) := \operatorname*{ess\,sup}_{X \in \mathcal{A}_{t,t+s}} E_Q[-X \,|\, \mathcal{F}_t\,], \quad Q \in \mathcal{P}_t.$$

Note that $\mathcal{A}_{t,t} = L^\infty_+(\mathcal{F}_t)$ and $\alpha^{min}_{t,t}(Q) = 0$ Q-a.s. for all $Q \in \mathcal{P}_t$. In particular we will consider *one-step acceptance sets* $\mathcal{A}_{t,t+1}$ and *one-step penalty functions* $\alpha^{\min}_{t,t+1}(Q)$ for each $t = 0, 1, \ldots$.

2.2 Time consistency

In this section we will focus on the strong notion of time consistency as defined in the preceding section; cf. Definition 2.1.2 and Proposition 2.1.4. We omit "strong" and call it simply time consistency. We will give various characterizations of time consistency in terms of acceptance sets, of penalty functions and of a joint supermartingale property of the risk measure and its penalty function. This section is based on Section 4 of [FP06].

Lemma 2.2.1. *Let $(\rho_t)_{t=0,1,\ldots}$ be a time consistent sequence of conditional convex risk measures and let ρ_0 be sensitive. Then ρ_t is sensitive for all $t \geq 0$.*

Proof. Let $A \in \mathcal{F}$ with $P[A] > 0$ and $\varepsilon > 0$. Then by monotonicity $\rho_t(-\varepsilon I_A) \geq 0$ P-a.s.. Assume that $\rho_t(-\varepsilon I_A) = 0$ P-a.s.. Then recursiveness and normalization imply $\rho_0(-\varepsilon I_A) = \rho_0(-\rho_t(-\varepsilon I_A)) = 0$ in contradiction to the sensitivity of ρ_0. Hence $P[\rho_t(-\varepsilon I_A) > 0] > 0$. □

The next theorem, and in particular the equivalence of 1) and 4), is the main result of this section.

Theorem 2.2.2. *Let $(\rho_t)_{t=0,1,\dots}$ be a sequence of conditional convex risk measures such that each ρ_t is continuous from above, and assume that the set*

$$\mathcal{Q}^* := \left\{ Q \in \mathcal{M}^e(P) \;\middle|\; \alpha_0^{\min}(Q) < \infty \right\}$$

is nonempty. Then the following properties are equivalent:

1. *$(\rho_t)_{t=0,1,\dots}$ is time consistent.*
2. *The equality*
$$\mathcal{A}_t = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}$$
holds for all $t = 0, 1, \dots$.
3. *The equality*
$$\alpha_t^{\min}(Q) = \alpha_{t,t+1}^{\min}(Q) + E_Q[\,\alpha_{t+1}^{\min}(Q)\,|\,\mathcal{F}_t\,]$$
holds for all $t = 0, 1, \dots$ and all $Q \in \mathcal{M}^e(P)$.
4. *For all $Q \in \mathcal{Q}^*$ and all $X \in L^\infty$, the process*
$$V_t^Q(X) := \rho_t(X) + \alpha_t^{\min}(Q), \qquad t \geq 0$$
is a Q-supermartingale.

In each case the dynamic risk measure admits a robust representation in terms of the set $\mathcal{Q}^*$, i.e.,

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^*} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \right) \tag{2.7}$$

for all $X \in L^\infty$ and all $t \geq 0$.

Remark 2.2.3. *In view of Lemma 1.2.4, the assumption $\mathcal{Q}^* \neq \emptyset$ is satisfied if ρ_0 is sensitive.*

Before we begin the proof let us compare Theorem 2.2.2 to the existing literature. The equivalence of 1) and 2) is already known: It was proved in a more general setting in Theorem 4.5 in [CDK06] and also in Lemma 3.25 in [KS]. The decomposition property 2) in terms of stopping times appeared first in [Del06] in the context of Snell envelopes, and it was shown that this decomposition property is equivalent to m-stability of the set of measures and time consistency.

In terms of penalty functions some necessary and sufficient conditions for time consistency are given in Theorems 4.19 and 4.22 of [CDK06]. In the more general context of risk measures for stochastic processes, they involve concatenation of the representing dual functionals. In our setting of risk measures for random variables, it is natural to identify dual functionals with probability measures and to use 3) as a necessary and sufficient condition. With a slight modification of property 3) and under the assumption that the risk measures are continuous from below the equivalence of the first three properties also appears in [BN06].

The equivalence of recursiveness and the supermartingale property of the process $(\rho_t)_{t=0,1,\ldots}$ was shown in [ADE$^+$] for dynamic *coherent* risk measures which are given in terms of the same representing class $\mathcal{Q}$; see also [Bur05]. In the context of dynamic *convex* risk measures, the equivalence of time consistency and the supermartingale property 4) seems to be new.

The proof of Theorem 2.2.2 will be given in several steps. Note that we may assume that $P \in \mathcal{Q}^*$; otherwise we can simply replace P by some $P^* \in \mathcal{Q}^*$.

The equivalence of time consistency and property 2) follows from the next lemma, which holds for any sequence of conditional convex risk measures; here we do not need robust representations and the set $\mathcal{Q}^*$. The equivalences between set inclusions and inequalities provide equivalent characterizations for both middle acceptance and middle rejection consistency in terms of acceptance sets. Property 2.8 was already shown in [Del06].

Lemma 2.2.4. *Let $(\rho_t)_{t=0,1,\ldots}$ be a sequence of conditional convex risk measures. Then the following equivalences hold for all $s, t \geq 0$ and all $X \in L^\infty$:*

$$X \in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \iff -\rho_{t+s}(X) \in \mathcal{A}_{t,t+s} \tag{2.8}$$
$$\mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \iff \rho_t(-\rho_{t+s}) \leq \rho_t \quad P\text{-a.s.} \tag{2.9}$$
$$\mathcal{A}_t \supseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \iff \rho_t(-\rho_{t+s}) \geq \rho_t \quad P\text{-a.s..} \tag{2.10}$$

Proof.

a) To prove "⇒" in (2.8) let $X = X_{t,t+s} + X_{t+s}$ with $X_{t,t+s} \in \mathcal{A}_{t,t+s}$ and $X_{t+s} \in \mathcal{A}_{t+s}$. Then

$$\rho_{t+s}(X) = \rho_{t+s}(X_{t+s}) - X_{t,t+s} \leq -X_{t,t+s}$$

by cash invariance, and monotonicity implies

$$\rho_t(-\rho_{t+s}(X)) \leq \rho_t(X_{t,t+s}) \leq 0.$$

The converse direction follows immediately from $X = X + \rho_{t+s}(X) - \rho_{t+s}(X)$ and $X + \rho_{t+s}(X) \in \mathcal{A}_{t+s}$ for all $X \in L^\infty$.

b) In order to show ⇒" in (2.9), take $X \in L^\infty$. Since $X + \rho_t(X) \in \mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$, we obtain

$$\rho_{t+s}(X) - \rho_t(X) = \rho_{t+s}(X + \rho_t(X)) \in -\mathcal{A}_{t,t+s},$$

by (2.8) and cash invariance. Hence

$$\rho_t(-\rho_{t+s}(X)) - \rho_t(X) = \rho_t(-(\rho_{t+s}(X) - \rho_t(X))) \leq 0 \quad P\text{-a.s.}.$$

To prove "⇐" let $X \in \mathcal{A}_t$. Then $-\rho_{t+s}(X) \in \mathcal{A}_{t,t+s}$ by the right hand side of (2.9), and hence $X \in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$ by (2.8).

c) Let $X \in L^\infty$ and assume $\mathcal{A}_t \supseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$. Then

$$\begin{aligned}\rho_t(-\rho_{t+s}(X)) + X &= \rho_t(-\rho_{t+s}(X)) - \rho_{t+s}(X) + \rho_{t+s}(X) + X \\ &\in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \subseteq \mathcal{A}_t.\end{aligned}$$

Hence

$$\rho_t(X) - \rho_t(-\rho_{t+s}(X)) = \rho_t(X + \rho_t(-\rho_{t+s}(X))) \leq 0$$

by cash invariance, and this proves "⇒" in (2.10). For the converse direction let $X \in \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$. Since $-\rho_{t+s}(X) \in \mathcal{A}_{t,t+s}$ by (2.8), we obtain

$$\rho_t(X) \leq \rho_t(-\rho_{t+s}(X)) \leq 0,$$

hence $X \in \mathcal{A}_t$. □

Proof of 2) ⇒ 3) of Theorem 2.2.2 follows from the next lemma. As in the preceding lemma we do not need robust representations and the assumption $\mathcal{Q}^* \neq \emptyset$ for the proof of this result.

Lemma 2.2.5. *Let $(\rho_t)_{t=0,1,\dots}$ be a sequence of conditional convex risk measures. Then the following implications hold for all $s, t \geq 0$ and all $Q \in \mathcal{M}^e(P)$:*

$$\mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \Rightarrow \alpha^{\min}_{t+1}(Q) \leq \alpha^{\min}_{t,t+s}(Q) + E_Q[\alpha^{\min}_{t+s}(Q)|\mathcal{F}_t] \qquad (2.11)$$

$$\mathcal{A}_t \supseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \Rightarrow \alpha^{\min}_{t+1}(Q) \geq \alpha^{\min}_{t,t+s}(Q) + E_Q[\alpha^{\min}_{t+s}(Q)|\mathcal{F}_t]. \qquad (2.12)$$

Proof. We fix $Q \in \mathcal{M}^e(P)$. To prove (2.11) we take $X \in \mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$ such that $X = X_{t,t+s} + X_{t+s}$ with $X_{t,t+s} \in \mathcal{A}_{t,t+s}$ and $X_{t+s} \in \mathcal{A}_{t+s}$. Using the definition of the minimal penalty functions $\alpha^{\min}_{t,t+s}(Q), \alpha^{\min}_{t+s}(Q)$ and Lemma 1.1.8 we obtain

$$\begin{aligned} E_Q[-X|\mathcal{F}_t] &= E_Q[-X_{t,t+s}|\mathcal{F}_t] + E_Q[-X_{t+s}|\mathcal{F}_t] \\ &\le \alpha^{\min}_{t,t+s}(Q) + E_Q[\,\alpha^{\min}_{t+s}(Q)\,|\,F_t\,]. \end{aligned}$$

Thus

$$\alpha^{\min}_t(Q) = \operatorname*{ess\,sup}_{X\in\mathcal{A}_t} E_Q[-X|\mathcal{F}_t] \le \alpha^{\min}_{t,t+s}(Q) + E_Q[\alpha^{\min}_{t+s}(Q)|\mathcal{F}_t]$$

for all $s, t \ge 0$.
For the proof of the implication (2.12) we take $X_{t,t+s} \in \mathcal{A}_{t,t+s}$ and $X_{t+s} \in \mathcal{A}_{t+s}$. Then $X_{t,t+s} + X_{t+s} \in \mathcal{A}_t$ and by definition of the penalty function $\alpha^{\min}_t(Q)$

$$\alpha^{\min}_t(Q) \ge E_Q[-X_{t,t+s}|\mathcal{F}_t] + E_Q[-X_{t+s}|\mathcal{F}_t].$$

Thus for all $s, t \ge 0$

$$\begin{aligned} \alpha^{\min}_t(Q) &\ge \operatorname*{ess\,sup}_{X_{t,t+s}\in\mathcal{A}_{t,t+s}} E_Q[-X_{t,t+s}|\mathcal{F}_t] + \operatorname*{ess\,sup}_{X_{t+s}\in\mathcal{A}_{t+s}} E_Q[-X_{t+s}|\mathcal{F}_t] \\ &= \alpha^{\min}_{t,t+s}(Q) + E_Q[\,\alpha^{\min}_{t+s}(Q)\,|\,F_t\,], \end{aligned}$$

where we have used Lemma 1.1.8 for the second equality. □

Remark 2.2.6. *Applying property 2) and property 3) of Theorem 2.2.2 step by step to the time consistent sequence $(\rho_n)_{t\le n\le t+s}$ on the space $L^\infty(\mathcal{F}_{t+s})$ for each $t \ge 0$ and $s \ge 1$ we obtain*

$$\mathcal{A}_{t,t+s} = \sum_{n=t}^{t+s-1} \mathcal{A}_{n,n+1}$$

and

$$\alpha^{\min}_{t,t+s}(Q) = E_Q\left[\sum_{n=t}^{t+s-1} \alpha^{\min}_{n,n+1}(Q)\,\Big|\,\mathcal{F}_t\right]$$

for all all $t, s \ge 0$ and all $Q \in \mathcal{M}^e(P)$ (note that we have not used that the initial σ-field is trivial in the preceding proofs). In particular it follows inductive that

$$\mathcal{A}_t = \sum_{k=t}^{t+s-1} \mathcal{A}_{k,k+1} + \mathcal{A}_{t+s} = \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \tag{2.13}$$

and

$$\begin{aligned} \alpha^{\min}_{t+1}(Q) &= E_Q\left[\sum_{k=t}^{t+s-1} \alpha^{\min}_{k,k+1}(Q)\,\Big|\,\mathcal{F}_t\right] + E_Q[\,\alpha^{\min}_{t+s}(Q)\,|\,\mathcal{F}_t\,] \\ &= \alpha^{\min}_{t,t+s}(Q) + E_Q[\,\alpha^{\min}_{t+s}(Q)\,|\,\mathcal{F}_t\,] \end{aligned} \tag{2.14}$$

for all $t, s = 0, 1, \ldots$ and all $Q \in \mathcal{M}^e(P)$. On the other hand equalities (2.13) and (2.14) imply the one-step properties 2) and 3) of Theorem 2.2.2. Thus time consistency is equivalent to the properties

2') $\mathcal{A}_t = \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$ for all $t, s = 0, 1, \ldots$

and to

3') $\alpha_t^{\min}(Q) = \alpha_{t,t+s}^{\min}(Q) + E_Q[\,\alpha_{t+s}^{\min}(Q)\,|\,\mathcal{F}_t\,]$
for all $t, s = 0, 1, \ldots$ and all $Q \in \mathcal{M}^e(P)$.

This is consistent with the fact that one-step recursiveness implies general recursiveness, as we have shown in Proposition 2.1.4.

Proof of 3) $\Rightarrow$ 4) of Theorem 2.2.2:

a) First we will show that the representations (1.17), (1.18) and (2.7) hold for any $t \geq 0$. Note that property 3') implies $E_{P^*}[\alpha_t^{\min}(P^*)] < \infty$ for $P^* \in \mathcal{Q}^*$, and thus the representations (1.17) and (1.18) of Lemma 1.2.5 hold for any $P^* \in \mathcal{Q}^*$. Now take $Q \in \mathcal{M}^e(P)$ such that $Q = P$ on $\mathcal{F}_t$ and $E_Q[\alpha_t^{\min}(Q)] < \infty$, that is $Q \in \mathcal{Q}_t^{f,e}(P)$. Using 3') we obtain

$$\begin{aligned} \alpha_0^{\min}(Q) &= E_Q\left[\alpha_{0,t}^{min}(Q)\right] + E_Q[\alpha_t^{\min}(Q)] \\ &= E_P\left[\alpha_{0,t}^{min}(P)\right] + E_Q[\alpha_t^{\min}(Q)] \\ &\leq \alpha_0^{min}(P) + E_Q[\alpha_t^{\min}(Q)] < \infty, \end{aligned}$$

hence $Q \in \mathcal{Q}^*$. Thus it follows from (1.18) that

$$\rho_t(X) \leq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^*} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\right) \quad \text{for all } X \in L^\infty.$$

The converse inequality "$\geq$" follows from (1.17) of Lemma 1.2.5.

b) In the next step we fix $\widetilde{Q} \in \mathcal{Q}^*$ and apply Lemma 1.2.3 to the set

$$\mathcal{Q}_{t+1}^{f,e}(\widetilde{Q}) = \left\{ Q \in \mathcal{M}^e(P) \;\middle|\; Q = \widetilde{Q} \text{ on } \mathcal{F}_{t+1},\ E_{\widetilde{Q}}[\alpha_{t+1}^{\min}(Q)] < \infty \right\}.$$

For $Q_1, Q_2 \in \mathcal{Q}_{t+1}^{f,e}(\widetilde{Q})$ and $B \in \mathcal{F}_{t+1}$ we define

$$\widehat{Z} := I_B \frac{dQ_2}{d\widetilde{Q}} + I_{B^c} \frac{dQ_1}{d\widetilde{Q}}.$$

Then by Lemma 1.2.3 the probability measure $\widehat{Q}$ defined via $d\widehat{Q}/d\widetilde{Q} := \widehat{Z}$ satisfies $\widehat{Q} = \widetilde{Q}$ on $\mathcal{F}_{t+1}$ and

$$\alpha_{t+1}^{min}(\widehat{Q}) = \alpha_{t+1}^{min}(Q_1)\, I_{B^c} + \alpha_{t+1}^{min}(Q_2)\, I_B,$$

hence $\widehat{Q} \in \mathcal{Q}_{t+1}^{f,e}(\widetilde{Q})$.

c) Using b) and the same reasoning as in the proof of Lemma 1.1.8 we can deduce that the set

$$\Big\{ E_Q[-X|\mathcal{F}_{t+1}] - \alpha_{t+1}^{\min}(Q) \;\Big|\; Q \in \mathcal{Q}_{t+1}^{f,e}(\widetilde{Q}) \Big\}$$

is directed upward for all $X \in L^\infty$. Since ρ_{t+1} can be represented as essential supremum over this set by a), there exists a sequence $(Q_n) \subseteq \mathcal{Q}_{t+1}^{f,e}(\widetilde{Q})$ depending on $\widetilde{Q}$ and X such that

$$E_{Q_n}[-X|\mathcal{F}_{t+1}] - \alpha_{t+1}^{min}(Q_n) \nearrow \rho_{t+1}(X) \quad P\text{-a.s.} \quad \text{with } n \to \infty.$$

The monotone convergence theorem implies

$$\begin{aligned} E_{\widetilde{Q}}[\rho_{t+1}(X)|\mathcal{F}_t] &= \lim_{n\to\infty} E_{\widetilde{Q}}[\, E_{Q_n}[-X|\mathcal{F}_{t+1}] - \alpha_{t+1}^{min}(Q_n) \,|\, \mathcal{F}_t] \\ &= \lim_{n\to\infty} \Big(E_{Q_n}[-X|\mathcal{F}_t] - E_{Q_n}[\alpha_{t+1}^{min}(Q_n)|\mathcal{F}_t] \Big), \end{aligned}$$

where we have used that Q_n and $\widetilde{Q}$ coincide on $\mathcal{F}_{t+1}$. Moreover, the same reasoning as in a) implies that $Q_{t+1}^{f,e}(\widetilde{Q}) \subseteq \mathcal{Q}^*$, and applying 3) to Q_n we obtain

$$\begin{aligned} E_{Q_n}[\alpha_{t+1}^{min}(Q_n)|\mathcal{F}_t] &= \alpha_t^{min}(Q_n) - \alpha_{t,t+1}^{min}(Q_n) \\ &= \alpha_t^{min}(Q_n) - \alpha_{t,t+1}^{min}(\widetilde{Q}) \quad \text{for all } n \in \mathbb{N}. \end{aligned}$$

d) In the final step we obtain for $\widetilde{Q} \in \mathcal{Q}^*$ and $X \in L^\infty$

$$\begin{aligned} E_{\widetilde{Q}}[V_{t+1}^{\widetilde{Q}}(X)|\mathcal{F}_t] &= E_{\widetilde{Q}}[\rho_{t+1}(X) + \alpha_{t+1}^{min}(\widetilde{Q})|\mathcal{F}_t] \\ &= E_{\widetilde{Q}}[\rho_{t+1}(X)|\mathcal{F}_t] - \alpha_{t,t+1}^{min}(\widetilde{Q}) + \alpha_t^{min}(\widetilde{Q}) \\ &= \lim_{n\to\infty} \Big(E_{Q_n}[-X|\mathcal{F}_t] - \alpha_t^{min}(Q_n) \Big) + \alpha_t^{min}(\widetilde{Q}) \\ &\leq \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^*} \Big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{min}(Q) \Big) + \alpha_t^{min}(\widetilde{Q}) \\ &= \rho_t(X) + \alpha_t^{min}(\widetilde{Q}) \\ &= V_t^{\widetilde{Q}}(X), \end{aligned}$$

where we have used 3), c), a) and $Q_n \in \mathcal{Q}^*$ for all n. Moreover, $(V_t^{\widetilde{Q}}(X))$ is adapted and integrable for all $\widetilde{Q} \in \mathcal{Q}^*$, and thus a $\widetilde{Q}$-supermartingale.

Proof of 4) $\Rightarrow$ 1) of Theorem 2.2.2:

In the first step we will show again that the representation (2.7) holds for all $t \geq 0$. Indeed, 4) implies $E_{P^*}[\alpha_t^{\min}(P^*)] < \infty$ for all $t \geq 0$ and $P^* \in \mathcal{Q}^*$, since $\rho_t(X) + \alpha_t^{\min}(P^*)$ is P^*-integrable and $\rho_t(X) \in L_t^\infty$ for all $X \in L^\infty$ and $t \geq 0$. Hence the representation (1.18) of Lemma 1.2.5 holds for all $t \geq 0$ and $P^* \in \mathcal{Q}^*$. Moreover, for $Q \in \mathcal{Q}_t^{f,e}(P)$ and $X \in \mathcal{A}_0$ we obtain

$$
\begin{aligned}
E_Q[-X] &\leq E_Q[-X - \rho_t(X)] + E_Q[\rho_t(X) + \alpha_t^{\min}(P)] \\
&\leq E_Q[\alpha_t^{\min}(Q)] + E_P[\rho_t(X) + \alpha_t^{\min}(P)] \\
&\leq E_Q[\alpha_t^{\min}(Q)] + \rho_0(X) + \alpha_0^{min}(P) \\
&\leq E_Q[\alpha_t^{\min}(Q)] + \alpha_0^{min}(P),
\end{aligned}
$$

where we have used representation (1.11) for $\alpha_t^{\min}(Q),\ Q \in \mathcal{Q}_t^{f,e}(P),\ P \in \mathcal{Q}^*$, 4), and $X \in \mathcal{A}_0$. Hence

$$
\alpha_0^{\min}(Q) \leq E_Q[\alpha_t^{\min}(Q)] + \alpha_0^{min}(P) < \infty
$$

which implies $Q \in \mathcal{Q}^*$. Now we can argue as in part a) of the proof 3) $\Rightarrow$ 4) to obtain representation (2.7).
In the next step we will prove time consistency. To this end let $X, Y \in L^\infty$ such that $\rho_{t+1}(X) \leq \rho_{t+1}(Y)$ P-a.s.. Using 4) we obtain for all $Q \in \mathcal{Q}^*$:

$$
\begin{aligned}
\rho_t(Y) + \alpha_t^{\min}(Q) &\geq E_Q[\rho_{t+1}(Y) + \alpha_{t+1}^{\min}(Q)|\mathcal{F}_t] \\
&\geq E_Q[\rho_{t+1}(X) + \alpha_{t+1}^{\min}(Q)|\mathcal{F}_t] \\
&\geq E_Q[E_Q[-X|\mathcal{F}_{t+1}] - \alpha_{t+1}^{\min}(Q) + \alpha_{t+1}^{\min}(Q)|\mathcal{F}_t] \\
&= E_Q[-X|\mathcal{F}_t].
\end{aligned}
$$

Thus

$$
\rho_t(Y) \geq E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)
$$

for all $Q \in \mathcal{Q}^*$, and hence

$$
\rho_t(Y) \geq \rho_t(X) \quad P\text{-a.s.},
$$

proving time consistency of the sequence (ρ_t) as characterized by (2.3). □

In the *coherent* case the characterization of time consistency is already well understood; see Theorem 5.1. in [ADE$^+$], Theorem 6.2 in [Del06], Lemma 3.29 in [KS], Corollary 3.18 in [Bur05], and Section 4.4 in [CDK06]. Let us show how the main results for coherent risk measures for final values can be obtained as special cases of our discussion of the convex case. This involves the following stability property for the representing set of measures, sometimes called *fork convexity* as in [Del06] and *multiplicative stability* or *m-stability* as in [ADE$^+$]. It is equivalent to Definition 6.44 in [FS04] and stronger than the *weak m-stability* in Definition 3.27 of [KS].

Definition 2.2.7. *We call a set $\mathcal{Q} \subseteq \mathcal{M}^e(P)$* stable *if it has the following property: For any $Q^1, Q^2, Q^3 \in \mathcal{Q}$, any $t \geq 0$ and any $A_t \in \mathcal{F}_t$ the probability measure Q given by*

$$Q[A] = E_{Q^1}\Big[\, I_{A_t}\, Q^2[\, A \,|\, \mathcal{F}_t\,] + I_{A_t^c}\, Q^3[\, A \,|\, \mathcal{F}_t\,]\,\Big], \tag{2.15}$$

called the pasting *of Q^1, Q^2 and Q^3 in t via A_t, belongs again to the set $\mathcal{Q}$.*

Note that the density of the pasting Q is given by

$$Z_T := I_{A_t} \frac{Z_t^1}{Z_t^2} Z_T^2 + I_{A_t^c} \frac{Z_t^1}{Z_t^3} Z_T^3, \tag{2.16}$$

where Z^i denotes the density process of Q^i with respect to P for $i = 1, 2, 3$.

It is also easy to see that a probability measure Q is a pasting of Q^1, Q^2 and Q^3 at time t via A_t iff it has the following property:

$$E_Q[X|\mathcal{F}_s] = \begin{cases} E_{Q^1}\Big[\, I_{A_t}\, E_{Q^2}[\, X \,|\, \mathcal{F}_t\,] + I_{A_t^c}\, E_{Q^3}[\, X \,|\, \mathcal{F}_t\,] \,|\, \mathcal{F}_s\Big] & ; \quad s < t \\ I_{A_t}\, E_{Q^2}[\, X \,|\, \mathcal{F}_s\,] + I_{A_t^c}\, E_{Q^3}[\, X \,|\, \mathcal{F}_s\,] & ; \quad s \geq t. \end{cases} \tag{2.17}$$

for all $s \geq 0$. In particular we have $Q = Q^1$ on $\mathcal{F}_t$.

If the initial risk measure ρ_0 is coherent then the penalty function $\alpha_0^{\min}(Q)$ can only take values 0 or ∞. Hence the set $\mathcal{Q}^*$ takes the form

$$\mathcal{Q}^* = \Big\{\, Q \in \mathcal{M}^e(P) \,\Big|\, \alpha_0^{\min}(Q) = 0 \Big\}.$$

Corollary 2.2.8. *Let $(\rho_t)_{t=0,1,\ldots}$ be a sequence of conditional convex risk measures such that each ρ_t is continuous from above. Assume that $\mathcal{Q}^* \neq \emptyset$ and that the initial risk measure ρ_0 is coherent. Then the following conditions are equivalent:*

1. *$(\rho_t)_{t=0,1,\ldots}$ is time consistent.*

2. *The representation*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^*} E_Q[-X|\mathcal{F}_t] \tag{2.18}$$

holds for all $X \in L^\infty$ and all $t \geq 0$, and the set $\mathcal{Q}^*$ is stable.

3. *The representation (2.18) holds for all $X \in L^\infty$ and all $t \geq 0$, and the process $(\rho_t(X))_{t=0,1,\ldots}$ is a Q-supermartingale for all $Q \in \mathcal{Q}^*$ and all $X \in L^\infty$.*

In each case $(\rho_t)_{t=0,1,\ldots}$ is a dynamic coherent *risk measure.*

Proof. As in the proof of Theorem 2.2.2 we may assume that $P \in \mathcal{Q}^*$.

1) $\Rightarrow$ 2) Time consistency implies property 3) of Theorem 2.2.2, and we will show that this implies property 2) of Corollary 2.2.8. Indeed, $\alpha_0^{\min}(Q) = 0$ implies $\alpha_t^{\min}(Q) = 0$ for all $t \geq 0$ due to property 3'). Hence the representation (2.7) reduces to (2.18). To prove stability of the set $\mathcal{Q}^*$, take $Q^1, Q^2, Q^3 \in \mathcal{Q}^*$, $t \geq 0$, $A_t \in \mathcal{F}_t$ and define Q via (2.15). Using (2.17) we obtain $\alpha_{0,t}^{min}(Q) = \alpha_{0,t}^{min}(Q^1) = 0$ and

$$\begin{aligned}
\alpha_t^{min}(Q) &= \operatorname*{ess\,sup}_{X \in \mathcal{A}_t} E_Q[-X|\mathcal{F}_t] \\
&= I_A \operatorname*{ess\,sup}_{X \in \mathcal{A}_t} E_{Q^2}[-X|\mathcal{F}_t] + I_{A^c} \operatorname*{ess\,sup}_{X \in \mathcal{A}_t} E_{Q^3}[-X|\mathcal{F}_t] \\
&= I_A \alpha_t^{\min}(Q^2) + I_{A^c} \alpha_t^{\min}(Q^3) = 0,
\end{aligned}$$

hence $\alpha_0^{\min}(Q) = \alpha_{0,t}^{min}(Q) + E_Q[\alpha_t^{\min}(Q)] = 0$, and thus $Q \in \mathcal{Q}^*$.

2) $\Rightarrow$ 3) We have to show that 2) implies

$$E_{\widetilde{Q}}[\operatorname*{ess\,sup}_{Q \in \mathcal{Q}^*} E_Q[-X|\mathcal{F}_{t+1}] \,|\, \mathcal{F}_t\,] \leq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^*} E_Q[-X|\mathcal{F}_t] \tag{2.19}$$

for all $t \geq 0$ and $\widetilde{Q} \in \mathcal{Q}^*$. To this end note first that the set

$$\Big\{ E_Q[-X|\mathcal{F}_{t+1}] \,\Big|\, Q \in \mathcal{Q}^* \Big\}$$

is directed upward due to the stability of the set $\mathcal{Q}^*$ and our assumption $P \in \mathcal{Q}^*$. Indeed, for any $Q^1, Q^2 \in \mathcal{Q}^*$ the pasting Q of P, Q^1 and Q^2 in $t+1$ via $A_{t+1} := \{E_{Q^1}[-X|\mathcal{F}_{t+1}] > E_{Q^2}[-X|\mathcal{F}_{t+1}]\}$ with the density

$$Z_T := I_{A_{t+1}} \frac{Z_T^1}{Z_{t+1}^1} + I_{A_{t+1}^c} \frac{Z_T^2}{Z_{t+1}^2}$$

belongs to $\mathcal{Q}^*$ and

$$E_Q[-X|\mathcal{F}_{t+1}] = \max(E_{Q^1}[-X|\mathcal{F}_{t+1}], E_{Q^2}[-X|\mathcal{F}_{t+1}]).$$

Hence the same argument as in the proof of Lemma 1.1.8 implies

$$E_{\widetilde{Q}}[\operatorname*{ess\,sup}_{Q\in\mathcal{Q}^*} E_Q[-X|\mathcal{F}_{t+1}] \,|\, \mathcal{F}_t] = \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^*} E_{\widetilde{Q}}[E_Q[-X|\mathcal{F}_{t+1}] \,|\, \mathcal{F}_t].$$

Moreover, the pasting of $\widetilde{Q}$ and Q in $t+1$ via $A_{t+1} = \Omega$ belongs to $\mathcal{Q}^*$, and hence we have

$$\operatorname*{ess\,sup}_{Q\in\mathcal{Q}^*} E_{\widetilde{Q}}[E_Q[-X|\mathcal{F}_{t+1}] \,|\, \mathcal{F}_t] \le \operatorname*{ess\,sup}_{Q\in\mathcal{Q}^*} E_Q[-X|\mathcal{F}_t],$$

and this proves (2.19).

3) $\Rightarrow$ 1) We show that property 3) of Corollary 2.2.8 implies property 4) of Theorem 2.2.2. Indeed, for $X \in \mathcal{A}_{t+1}$ representation (2.18) implies $E_Q[-X|\mathcal{F}_{t+1}] \le 0$ for all $Q \in \mathcal{Q}^*$. Hence $E_Q[-X|\mathcal{F}_t] \le 0$ for all $Q \in \mathcal{Q}^*$ and $X \in \mathcal{A}_t$ by (2.18). Thus the sequence $(\rho_t)_{t=0,1,\dots}$ is weakly acceptance consistent, and the process $(\alpha_t^{\min}(Q))_{t=0,1,\dots}$ is a non-negative Q-supermartingale for all $Q \in \mathcal{Q}^*$ as we will show in Proposition 2.3.4 in the next section. Moreover, since $\alpha_0^{\min}(Q) = 0$ we obtain $\alpha_t^{\min}(Q) = 0$ for all $t \ge 0$. Hence the process

$$\rho_t(X) = \rho_t(X) + \alpha_t^{\min}(Q), \qquad t \ge 0$$

is a Q-supermartingale for all $Q \in \mathcal{Q}^*$, and so we have verified property 4) of Theorem 2.2.2. □

2.3 Dynamics of penalty functions

In this section we discuss property 3) of Theorem 2.2.2 and its impact on the dynamics of penalty functions.

Remark 2.3.1. *It follows from property 3) of Theorem 2.2.2 that*

$$E_Q[\,\alpha_{t+1}^{\min}(Q) \,|\, F_t\,] \le \alpha_t^{\min}(Q) \quad P\text{-a.s.} \quad \text{for all } Q \in \mathcal{M}^e(P).$$

This implies in particular $E_Q[\alpha_t^{\min}(Q)] < \infty$ for all $t \ge 0$ if $\alpha_0^{\min}(Q) < \infty$, i.e. $(\alpha_t^{\min}(Q))$ is Q-integrable for all $Q \in \mathcal{Q}^*$. Thus the process $(\alpha_t^{\min}(Q))_{t=0,1,\dots}$ is a Q-supermartingale for all $Q \in \mathcal{Q}^*$. Moreover, property 3) yields an explicit description of its Doob decomposition in terms of the one-step penalty functions $\alpha_{t,t+1}^{\min}(Q)$, as explained in Remark 2.3.3.

Since $(\alpha_t^{\min}(Q))$ is a non-negative Q-supermartingale, it has a Riesz decomposition for each $Q \in \mathcal{Q}^*$. In the next proposition we give an explicit form of this decomposition.

Proposition 2.3.2. *Let $(\rho_t)_{t=0,1,\dots}$ be a time consistent dynamic risk measure such that each ρ_t is continuous from above. Then for each $Q \in \mathcal{Q}^*$ the process $(\alpha_t^{\min}(Q))_{t=0,1,\dots}$ is a non-negative Q-supermartingale with the Riesz decomposition*

$$\alpha_t^{\min}(Q) = Z_t^Q + M_t^Q, \qquad t = 0, 1, \dots,$$

where

$$Z_t^Q := E_Q\left[\sum_{k=t}^{\infty} \alpha_{k,k+1}^{\min}(Q) \,\Big|\, \mathcal{F}_t\right], \quad t = 0, 1, \dots$$

is a Q-potential and

$$M_t^Q := \lim_{s\to\infty} E_Q\left[\alpha_s^{\min}(Q) \,|\, \mathcal{F}_t\right], \quad t = 0, 1, \dots$$

is a non-negative Q-martingale.

Proof. We fix $Q \in \mathcal{Q}^*$ note that due to Remark 2.2.6 we have

$$\alpha_{t+s}^{\min}(Q) = E_Q\left[\sum_{k=t}^{t+s-1} \alpha_{k,k+1}^{\min}(Q) \,\Big|\, \mathcal{F}_t\right] + E_Q[\, \alpha_{t+s}^{\min}(Q) \,|\, \mathcal{F}_t\,] \tag{2.20}$$

for all $t, s = 0, 1, \dots$. By monotonicity there exists the limit

$$Z_t^Q = \lim_{s\to\infty} E_Q\left[\sum_{k=t}^{s} \alpha_{k,k+1}^{\min}(Q) \,\Big|\, \mathcal{F}_t\right] = E_Q\left[\sum_{k=t}^{\infty} \alpha_{k,k+1}^{\min}(Q) \,\Big|\, \mathcal{F}_t\right]$$

for all $t = 0, 1, \dots$, where we have used the monotone convergence theorem for the second equality. Equality (2.20) implies then that there exists

$$M_t^Q = \lim_{s\to\infty} E_Q\left[\alpha_{t+s}^{\min}(Q) \,|\, \mathcal{F}_t\right], \quad t = 0, 1, \dots$$

and

$$\alpha_t^{\min}(Q) = Z_t^Q + M_t^Q$$

for all $t = 0, 1, \dots$.
The process (Z_t^Q) is a non-negative Q-supermartingale. Indeed,

$$E_Q[\, Z_t^Q \,] \le E_Q\left[\sum_{k=0}^{\infty} \alpha_{k,k+1}^{\min}(Q)\right] \le \alpha_0^{\min}(Q) < \infty \tag{2.21}$$

since $Q \in \mathcal{Q}^*$ and

$$\begin{aligned} E_Q[\, Z^Q_{t+1} \,|\, \mathcal{F}_t \,] &= E_Q\left[\sum_{k=t+1}^{\infty} \alpha^{\min}_{k,k+1}(Q) \,\Big|\, \mathcal{F}_t \right] \\ &\le E_Q\left[\sum_{k=t}^{\infty} \alpha^{\min}_{k,k+1}(Q) \,\Big|\, \mathcal{F}_t \right] \\ &= Z^Q_t \end{aligned}$$

for all $t = 0, 1, \ldots$. Moreover, monotone convergence implies

$$\lim_{t\to\infty} E_Q[\, Z^Q_t \,] = E_Q\left[\lim_{t\to\infty} \sum_{k=t}^{\infty} \alpha^{\min}_{k,k+1}(Q) \right] = 0 \quad P\text{-a.s.},$$

since $\sum_{k=0}^{\infty} \alpha^{\min}_{k,k+1}(Q) < \infty$ Q-a.s. by (2.21). Hence the process (Z^Q_t) is a Q-potential.
The process (M^Q_t) is a non-negative Q-martingale, since

$$E_Q[\, M^Q_t \,] \le E_Q\left[\alpha^{\min}_t(Q) \right] \le \alpha^{\min}_0(Q) < \infty$$

and

$$\begin{aligned} E_Q[M^Q_{t+1} - M^Q_t | \mathcal{F}_t] &= E_Q[\alpha^{\min}_{t+1}(Q) | \mathcal{F}_t] - \alpha^{\min}_t(Q) - E_Q[Z^Q_{t+1} - Z^Q_t | \mathcal{F}_t] \\ &= \alpha^{\min}_{t,t+1}(Q) - \alpha^{\min}_{t,t+1}(Q) = 0 \qquad \mathcal{Q}\text{-a.s.} \end{aligned}$$

for all $t = 0, 1, \ldots$ by property 3) of Theorem 2.2.2 and the definition of (Z_t).

□

Remark 2.3.3. *1. Using Proposition 2.3.2 we can precisely identify the Q-martingale and the predictable process in the Doob decomposition of the Q-supermartingal $(\alpha^{\min}_t(Q))$. For $T < \infty$ we have*

$$\alpha^{\min}_t(Q) = E_Q\left[\sum_{k=0}^{T} \alpha^{\min}_{k,k+1}(Q) \,\Big|\, \mathcal{F}_t \right] - \sum_{k=0}^{t-1} \alpha^{\min}_{k,k+1}(Q) \tag{2.22}$$

for all $t = 0, \ldots, T$, since $\alpha^{\min}_T(Q) = 0$ Q-a.s for all Q.
For $T = \infty$ we obtain

$$\alpha^{\min}_t(Q) = E_Q\left[\sum_{k=0}^{\infty} \alpha^{\min}_{k,k+1}(Q) \,\Big|\, \mathcal{F}_t \right] + M^Q_t - \sum_{k=0}^{t-1} \alpha^{\min}_{k,k+1}(Q) \tag{2.23}$$

for all $t = 0, 1, \ldots$ with the Q-martingale

$$E_Q\left[\sum_{k=0}^{\infty} \alpha^{\min}_{k,k+1}(Q) \,\Big|\, \mathcal{F}_t \right] + M^Q_t, \quad t = 0, 1, \ldots$$

and the increasing predictable process $(\sum_{k=0}^{t-1} \alpha^{\min}_{k,k+1}(Q))_{t=0,1,\ldots}$.

2. *It is an interesting question under which conditions the martingale M^Q in the Riesz decomposition of $(\alpha_t^{\min}(Q))$ vanishes and the penalty function process $(\alpha_t^{\min}(Q))$ is a Q-potential. This is always true if $T < \infty$, as we have seen in (2.22). For $T = \infty$ the martingale M^Q is not necessarily zero as we will show in Example 3.1.6 in Section 3.1. It turns out that M^Q vanishes if and only if the dynamic risk measure $(\rho_t)_{t=0,1,\dots}$ is* asymptotically safe, *meaning that*

$$\rho_\infty(X) := \lim_{t\to\infty} \rho_t(X) \geq -X \quad P\text{-a.s.}$$

for all $X \in L^\infty$.In this case it is sufficient to require $M^Q = 0$ for one $Q \in \mathcal{Q}^*$, and then it holds for all. We will study asymptotic safety and its equivalent characterizations in Section 3.1, cf. Theorem 3.1.4.

If we do not assume time consistency of a dynamic risk measure $(\rho_t)_{t=0,1,\dots}$, it is still possible that its minimal penalty function process is a supermartingale. The supermartingale property of the minimal penalty function corresponds to *weak acceptance consistency* property as defined in Section 2.1, Definition 2.1.2. This connection was noted first in Lemma 3.17 in [Bur05] under slightly different conditions. In the next proposition we will show the relation between the supermartingale property of the penalty function process and weak acceptance consistency in our setting. Moreover, we will state a corresponding characterization of weak acceptance consistency in terms of acceptance sets. This characterization was shown in [Tut06], Corollary 3.1.7.

Proposition 2.3.4. *Let $(\rho_t)_{t=0,1,\dots}$ be a sequence of conditional convex risk measures such that each ρ_t is continuous from above and consider the following properties:*

1. *$(\rho_t)_{t=0,1,\dots}$ is weakly acceptance consistent.*
2. *$\mathcal{A}_{t+1} \subseteq \mathcal{A}_t \quad$ for all $t = 0, 1, \dots$.*
3. *The inequality*
$$E_Q[\,\alpha_{t+1}^{\min}(Q)\,|\,F_t\,] \leq \alpha_t^{\min}(Q) \tag{2.24}$$
holds for all $Q \in \mathcal{M}^e(P)$ and all $t = 0, 1, \dots$.
4. *$(\alpha_t^{\min}(Q))_{t=0,1,\dots}$ is a Q-supermartingale for all $Q \in \mathcal{Q}^*$.*

Then 1) and 2) are equivalent and imply 3), and 3) implies 4).
Conversely, 3) implies 1) and 2) if the representation (1.17) holds, and 4) implies 1) and 2) if the representation (2.7) holds.

Proof. The equivalence of 1) and 2) follows directly from the definition of weak acceptance consistency. We will show that 2) implies 3). Indeed, 2) and Lemma 1.1.8 provide

$$\begin{aligned} E_Q[\,\alpha_{t+1}^{\min}(Q)\,|\,F_t\,] &= \operatorname*{ess\,sup}_{X_{t+1}\in\mathcal{A}_{t+1}} E_Q[-X_{t+1}|\mathcal{F}_t] \\ &\leq \operatorname*{ess\,sup}_{X\in\mathcal{A}_t} E_Q[-X|\mathcal{F}_t] = \alpha_t^{\min}(Q) \end{aligned}$$

for all $Q \in \mathcal{M}^e(P)$. This proves 3). If in addition $Q \in \mathcal{Q}^*$, we have $\alpha_0^{\min}(Q) < \infty$ it follows from 3) that $\alpha_t^{\min}(Q)$ is Q-integrable for all $t \geq 0$. Hence $(\alpha_t^{\min}(Q))_{t=0,1,\ldots}$ is a Q-supermartingale for all $Q \in \mathcal{Q}^*$ and 3) implies 4).
To prove that 3) implies 2) we take $X \in \mathcal{A}_{t+1}$ and note that

$$E_Q[-X|\mathcal{F}_{t+1}] \leq \alpha_{t+1}^{\min}(Q) \quad P\text{-a.s.} \quad \text{for all } Q \in \mathcal{M}^e(P)$$

by definition of the minimal penalty function. Using (2.24) we obtain

$$E_Q[-X|\mathcal{F}_t] \leq E_Q[\,\alpha_{t+1}^{\min}(Q)\,|\,F_t\,] \leq \alpha_t^{\min}(Q) \quad P\text{-a.s.}$$

for all $Q \in \mathcal{M}^e(P)$. If the representation (1.17) holds we obtain

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q\in\mathcal{M}^e(P)} \Big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\Big) \leq 0, \tag{2.25}$$

and hence $X \in \mathcal{A}_t$. This proves that 3) implies 2). If ρ_t has a representation (2.7), we use the supermartingale property for $Q \in \mathcal{Q}^*$ and argue as in (2.25) to show that 4) implies 2). □

2.4 Prudence and sustainability

In this section we consider dynamic risk measures that are not necessarily time consistent. We introduce and study a new property of a dynamic risk measure which we call prudence. We will show that this property corresponds to the weaker notion of time consistency that we have called middle rejection consistency in Section 2.1.

Definition 2.4.1. *A sequence of conditional risk measures $(\rho_t)_{t=0,1,\ldots}$ is called* prudent*, if for any $X \in L^\infty$ and for all $t, s \geq 0$ the following condition holds:*

$$\rho_t(\rho_t(X) - \rho_{t+s}(X)) \leq 0, \quad i.e. \quad \rho_t(X) - \rho_{t+s}(X) \in \mathcal{A}_{t,t+s}. \tag{2.26}$$

A sequence of conditional risk measures $(\rho_t)_{t=0,1,\ldots}$ is called one-step prudent*, if the relation (2.26) holds only for s=1.*

Let us explain the economic meaning of the preceding definition. Recall that due to Proposition 1.1.3 the random variable $\rho_t(X)$ can be viewed as a minimal capital requirement needed at time t to make a financial position X acceptable, that is

$$X + \rho_t(X) \in \mathcal{A}_t \quad \text{for all} \;\; X \in L^\infty.$$

In the context of *dynamic* risk measurement, however one wants to reevaluate the risk of the position $X + \rho_t(X)$ at some later period of time. At time $t+s$ the investment

$$C_{t,t+s}(X) := \rho_{t+s}(X + \rho_t(X)) = \rho_{t+s}(X) - \rho_t(X)$$

is required in order to keep the position $X + \rho_t(X)$ acceptable. Prudence makes sure that this future liability is also acceptable at time t:

$$\rho_t(-C_{t,t+s}(X)) \leq 0.$$

In other words, having secured the position X at time t by adding $\rho_t(X)$ to it, we stay on the safe side at any later period of time.

Note that any time consistent sequence of risk measures $(\rho_t)_{t=0,1,\ldots}$ is prudent, since $\rho_t(\rho_t(X) - \rho_{t+s}(X)) = 0$ for all $X \in L^\infty$ by cash invariance and recursiveness. In the next proposition we will show that prudence corresponds to the weaker notion of time consistency that we have called middle rejection consistency in Definition 2.1.2.

Proposition 2.4.2. *A dynamic risk measure $(\rho_t)_{t=0,1,\ldots}$ is middle rejection consistent if it and only if is prudent. Another equivalent characterization of prudence is the following:*

$$X \in \mathcal{A}_t \quad \Rightarrow \quad -\rho_{t+s}(X) \in \mathcal{A}_t \tag{2.27}$$

for any $X \in L^\infty$ and all $t, s \geq 0$.

Proof. Since

$$\rho_t(-\rho_{t+s}(X)) = \rho_t(\rho_t(X) - \rho_{t+s}(X)) + \rho_t(X) \tag{2.28}$$

due to cash invariance of the risk measure ρ_t, prudence implies middle rejection consistency. Obviously middle rejection consistency, applied to an acceptable position, implies condition (2.27). Now assume that (2.27) holds. Then for any $X \in L^\infty$ we use cash invariance of the risk measure ρ_{t+s}, the fact that $X + \rho_t(X) \in \mathcal{A}_t$ and (2.27) in order to conclude:

$$\rho_t(\rho_t(X) - \rho_{t+s}(X)) = \rho_t\left(-\rho_{t+s}(X + \rho_t(X))\right) \leq 0, \tag{2.29}$$

i.e. the sequence of risk measures $(\rho_t)_{t=0,1,\ldots}$ is prudent. □

Condition (2.27) formalizes another aspect of "stay on the safe side": If position X is acceptable at time t, then all its future "proxies" $-\rho_{t+s}(X)$ are also acceptable at time t. Since this seems to be a reasonable property for a dynamic risk measure, we will study prudence in more detail in this section.

We will focus first on one-step prudence, since it provides one half of the strong notion of time consistency, as characterized in Section 2.2, Theorem 2.2.2. It is interesting to see how the equivalent characterizations of Theorem 2.2.2 adjust for this weaker condition. This question will be clarified in the next theorem. Before we state and prove this result we need some preparation.

In order to characterize middle rejection consistency we will need some assumptions corresponding to the condition $\mathcal{Q}^* \neq \emptyset$ of Theorem 2.2.2. In our present setting we will use sensitivity of the risk measures ρ_t and we will use a probability measure $Q^* \approx P$ such that the sum of its one-step penalty functions is bounded or locally bounded. In its global form the boundedness condition takes the following form:

$$\exists \; Q^* \in \mathcal{M}^e(P) \text{ such that } \sum_{k=0}^{T} \alpha_{k,k+1}^{\min}(Q^*) \le C \quad P\text{-a.s.} \tag{2.30}$$

for some constant $C \geq 0$. For the local version we require

$$\exists \; Q^* \in \mathcal{M}^e(P) \text{ such that } \alpha_{t,t+1}^{\min}(Q^*) \in L^\infty(\mathcal{F}_t) \;\; \forall \;\; t = 0, 1, \ldots. \tag{2.31}$$

Both conditions are equivalent if the time horizon T is finite.
Moreover, it turns out that if $T < \infty$, conditions (2.30) and (2.31) are already implied by sensitivity, as we will show next.

Lemma 2.4.3. *Let $(\rho_t)_{t=0,1,\ldots}$ be a sequence of conditional convex risk measures such that each ρ_t is continuous from above and sensitive. Then for each $t \in \{1, 2, \ldots\}$ there exists $Q_t \in \mathcal{M}^e(P)$ such that $\sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q_t)$ is P-a.s. bounded.*

Proof. The proof relies on Lemma 1.2.4. We fix $t \in \{1, 2, \ldots\}$ and note that for all $s = 0, \ldots, t-1$ the risk measure ρ_s is sensitive on $\mathcal{F}_{s+1}$. Thus by Lemma 1.2.4 there exist a probability measure P_s on $\mathcal{F}_{s+1}$ with corresponding density $Y_s \in L^1(\mathcal{F}_{s+1})$ such that $P_s \approx P$ on $\mathcal{F}_{s+1}$, $P_s = P$ on $\mathcal{F}_s$ and $\alpha_{s,s+1}^{\min}(P_s) \le \frac{1}{2^s}$ P-a.s.. Consider the process

$$Z_k := \prod_{s=0}^{k-1} Y_s \qquad k = 0, \ldots, t-1.$$

It follows from the properties of the measures P_k that Z_k is $\mathcal{F}_k$-measurable, $Z_k > 0$ P-a.s. and

$$E[\, Z_{k+1} \,|\, \mathcal{F}_k \,] = \prod_{s=0}^{k-1} Y_s \cdot E[\, Y_k \,|\, \mathcal{F}_k \,] = Z_k, \quad k = 0, \dots, t-2,$$

i.e. the process $(Z_k)_{k=0,\dots,t-1}$ is a strictly positive martingale with $E[Z_{t-1}] = E[Z_0] = 1$. Thus we can define a probability measure Q_t on $\mathcal{F}$ via the density $dQ_t/dP := Z_{t-1}$. It follows that $Q_t \in \mathcal{M}^e(P)$ and

$$E_{Q_t}[\, X \,|\, \mathcal{F}_k \,] = E_P[\, Y_k X \,|\, \mathcal{F}_k \,] = E_{P_k}[\, X \,|\, \mathcal{F}_k \,]$$

for all $X \in \mathcal{F}_{k+1}$ and all $k = 0, \dots, t-1$.
In particular we obtain $\alpha^{\min}_{k,k+1}(Q_t) = \alpha^{\min}_{k,k+1}(P_k)$ and

$$\sum_{k=0}^{t-1} \alpha^{\min}_{k,k+1}(Q_t) \le \sum_{k=0}^{t-1} \frac{1}{2^k} < \infty \quad P\text{-a.s.}.$$

□

Thus for $T < \infty$ the probability measure $Q_T \in \mathcal{M}^e(P)$ from Lemma 2.4.3 satisfies condition (2.30). For $T = \infty$ we obtain by the same construction as above a probability measure Q^* defined locally on the filtration $(\mathcal{F}_t)$ via the density process (Z_t), i.e. $\frac{dQ^*}{dP}|_{\mathcal{F}_t} := Z_t$ for all t. Hence Q^* would fulfill the conditions (2.31) and (2.30), if it would be possible to extend the definition of Q^* on $\mathcal{F}$ in such a way that $Q^* \approx P$ on $\mathcal{F}$.
However, in order to do so we will need uniform integrability of the martingale (Z_t) and some additional assumption ensuring that the limit will be strictly positive. Since we cannot expect this properties to hold in general, we will impose that for $T = \infty$ one of the conditions (2.30) or (2.31) is satisfied.

Assumptions (2.30) or (2.31) are justified in particular if there exists a probability measure $Q^* \approx P$ that is not penalized, i.e. $\alpha^{\min}_{t,t+1}(Q^*) = 0$ P-a.s. for all t. This is clearly the case with $Q^* = P$, if penalty functions are given in terms of *conditional φ-divergences* with respect to P, since then we have $\alpha^{\min}_{t,t+1}(P) = 0$ P-a.s. for all t. For details on general representations of risk measures with φ-divergences we refer to Chapter 4 of [Dra06], cf. also our Example 4.1. in Chapter 4 for the special case of the entropic dynamic risk measure.

Now we are ready to state the main result of this section.

Theorem 2.4.4. *Let $(\rho_t)_{t=0,1,\dots}$ be a sequence of conditional convex risk measures such that each ρ_t is continuous from above and sensitive.*
Then the following properties are equivalent:

1. $(\rho_t)_{t=0,1,\dots}$ *is one-step prudent.*

2. $(\rho_t)_{t=0,1,\dots}$ *is one-step middle rejection consistent.*

3. *The inclusion*
$$\mathcal{A}_t \subseteq \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}$$
holds for all $t = 0, 1, \dots$.

4. *The inequality*
$$\alpha_t^{\min}(Q) \leq \alpha_{t,t+1}^{\min}(Q) + E_Q[\, \alpha_{t+1}^{\min}(Q) \,|\, \mathcal{F}_t \,]$$
holds for all $t = 0, 1, \dots$ *and all* $Q \in \mathcal{M}^e(P)$.

Moreover, properties 1)-4) imply the following:

5. *The process*
$$U_t^Q(X) := \rho_t(X) - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \qquad t = 0, 1, \dots$$
is a Q-supermartingale for all $X \in L^\infty$ *and all* $Q \in \mathcal{Q}_T$, *where*
$$\mathcal{Q}_T := \left\{ Q \in \mathcal{M}^e(P) \;\middle|\; E_Q \left[\sum_{k=0}^{T} \alpha_{k,k+1}^{\min}(Q) \right] < \infty \right\}.$$

Assume further that either $T < \infty$ *or*
$$\exists \;\; Q^* \in \mathcal{M}^e(P) \text{ such that } \sum_{k=0}^{\infty} \alpha_{k,k+1}^{\min}(Q^*) \leq C \quad P\text{-a.s.}$$
for some constant $C \geq 0$. *Then property 5) is equivalent to properties 1)-4).*

Proof. Equivalence of 1) and 2) was already proved in Proposition 2.1.6 and equivalence of 2) and 3) is part (2.9) of Lemma 2.2.4. The proof of 3) $\Rightarrow$ 4) is part (2.11) of Lemma 2.2.5.

Let us show that property 4) implies property 1). To this end we fix $t \in \{0, 1, \dots\}$ and consider a risk measure
$$\widetilde{\rho}_t(X) := \rho_t(-\rho_{t+1}(X)), \qquad X \in L^\infty.$$
It is easily seen that $\widetilde{\rho}_t$ inherits all the properties of ρ_t and ρ_{t+1}, i.e. it is a conditional convex risk measure that is continuous from above and sensitive.

Moreover, the sequence of risk measures $(\widetilde{\rho}_t, \rho_{t+1})$ is time consistent by definition and thus it fulfills properties 2) and 3) of Theorem 2.2.2. (Note that we did not use the assumption $\mathcal{Q}^* \neq \emptyset$ for the proofs of this properties.) We denote by $\widetilde{\mathcal{A}}_t$ and $\widetilde{\mathcal{A}}_{t,t+1}$ the acceptance sets of the risk measure $\widetilde{\rho}_t$ and by $\widetilde{\alpha}_t^{\min}$ its penalty function. Since

$$\widetilde{\rho}_t(X) = \rho_t(-\rho_{t+1}(X)) = \rho_t(X)$$

for all $X \in L^\infty(\mathcal{F}_{t+1})$, we obtain $\widetilde{\mathcal{A}}_{t,t+1} = \mathcal{A}_{t,t+1}$ and thus

$$\widetilde{\mathcal{A}}_t = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1}$$

by property 2) of Theorem 2.2.2. This implies

$$\widetilde{\alpha}_t^{\min}(Q) = \alpha_{t,t+1}^{\min}(Q) + \alpha_{t+1}^{\min}(Q) \geq \alpha_t^{\min}(Q) \tag{2.32}$$

for all $Q \in \mathcal{M}^e(P)$, where we have used properties 3) of Theorem 2.2.2 and 4) of Theorem 2.4.4. Due to continuity from above and sensitivity both risk measures ρ_t and $\widetilde{\rho}_t$ have robust representations in terms of $\mathcal{M}^e(P)$ by Corollaray 1.2.6. Thus we conclude using (2.32):

$$\begin{aligned}\rho_t(X) &= \operatorname*{ess\,sup}_{Q \in \mathcal{M}^e(P)} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)\right) \\ &\geq \operatorname*{ess\,sup}_{Q \in \mathcal{M}^e(P)} \left(E_Q[-X|\mathcal{F}_t] - \widetilde{\alpha}_t^{\min}(Q)\right) \\ &= \rho_t(-\rho_{t+1}(X))\end{aligned}$$

for all $X \in L^\infty$. This proves 1).

The proof of 1) $\Rightarrow$ 5) and the equivalence of properties 1) and 5) under imposed assumptions follow from the next Theorem 2.4.6 applied to the bounded adapted process $(\rho_t(X))$ for $X \in L^\infty$. Note that condition (2.30) required in Theorem 2.4.6 is satisfied for $T < \infty$ due to sensitivity by Lemma 2.4.3. This concludes the proof. $\square$

The equivalence of one-step prudence and the supermartingale property 5) is a special case of the next theorem, which we would like to formulate in a more general setting. To this end we will introduce the notion of sustainability.

Definition 2.4.5. *Let $(\rho_t)_{t=0,1,\dots}$ be a dynamic risk measure and let $X = (X_t)_{t=0,1,\dots}$ be a bounded adapted process. Then we call X* sustainable with respect to the risk measure *$(\rho_t)_{t=0,1,\dots}$, if*

$$\rho_t(X_t - X_{t+1}) \leq 0 \qquad \textit{for all } t = 0, 1, \dots. \tag{2.33}$$

Let us explain our motivation for the definition of sustainability. We consider X to be a cumulative investment process. If we have already invested X_t at time t, an adjustment $X_{t+1} - X_t$ has to be added at time $t+1$. If the process X is sustainable, then this future payment is acceptable with respect to the risk measure ρ_t, i.e. $\rho_t(-(X_{t+1}-X_t)) \leq 0$. Note that in this terminology one-step prudence of a dynamic risk measure (ρ_t) means that for each $X \in L^\infty$ the risk process $(\rho_t(X))$ is sustainable with respect to (ρ_t).

In the next Theorem we will give an equivalent characterization of sustainability with respect to any risk measure (ρ_t) in terms of a supermartingale property. Note that we assume neither prudence nor sensitivity of the dynamic risk measure here. This result can be seen as an analogon to the optional decomposition under constraints, cf. Theorem 3.1 in [FK97] for continuous time and Theorem 9.20 in [FS04] for discrete time. Indeed our reasoning here is inspired by the latter theorem.

Theorem 2.4.6. *Suppose that $(\rho_t)_{t=0,1,\dots}$ is a sequence of conditional convex risk measures such that each ρ_t is continuous from above and let $(X_t)_{t=0,1,\dots}$ be any bounded adapted process. Consider the following properties:*

a) The process (X_t) is sustainable with respect to the risk measure (ρ_t).

b) The process

$$X_t - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \qquad t = 0, 1, \dots$$

is a Q-supermartingale for all $Q \in \mathcal{Q}_T$.

Then property a) implies property b). Assume further that condition (2.30) is satisfied, i.e.

$$\exists \ Q^* \in \mathcal{M}^e(P) \text{ such that } \sum_{k=0}^{T} \alpha_{k,k+1}^{\min}(Q^*) \leq C \quad P\text{-a.s.}$$

for some constant $C \geq 0$. Then properties a) and b) are equivalent.

Proof. We prove first the easier direction $a) \Rightarrow b)$. We have to show that the process

$$M_t^Q := X_0 + \sum_{k=1}^{t}(X_k - X_{k-1}) - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \qquad t = 0, 1, \dots$$

is a Q-supermartingale for all $Q \in \mathcal{Q}_T$. Indeed, fix $Q \in \mathcal{Q}_T$ and note that $(M_t^Q)_{t=0,1,\dots}$ is adapted and $M_t^Q \in L^1(Q)$ for all t, since (X_t) is bounded and

$$E_Q\left[\sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q)\right] \leq E_Q\left[\sum_{k=0}^{T} \alpha_{k,k+1}^{\min}(Q)\right] < \infty \quad \text{for all } t = 0, 1, \dots .$$

Moreover, we have for all t

$$E_Q[M_{t+1}^Q - M_t^Q | \mathcal{F}_t] = E_Q[X_{t+1} - X_t | \mathcal{F}_t] - \alpha_{t,t+1}^{\min}(Q) \leq 0 \quad Q\text{-a.s.},$$

where we have used the definition of the minimal penalty function $\alpha_{t,t+1}^{\min}(Q)$ and the fact that $-(X_{t+1} - X_t) \in \mathcal{A}_{t,t+1}$ due to sustainability. Thus (M_t^Q) is a Q-supermartingale.

To prove $b) \Rightarrow a)$ assume that (2.30) holds and let (X_t) be a bounded adapted process such that $(X_t - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q))$ is a Q-supermartingale for all $Q \in \mathcal{Q}_T$. We have to show that

$$X_t - X_{t-1} =: A_t \in -\mathcal{A}_{t-1,t} \quad \text{for all} \quad t = 1, 2, \dots .$$

We can assume without loss of generality that P satisfies the condition (2.30), otherwise we can switch to any $Q^* \approx P$ that does fulfill (2.30). Note that (2.30) implies in particular $P \in \mathcal{Q}_T$.

Suppose by way of contradiction that $A_t \notin -\mathcal{A}_{t-1,t}$. Since the set $\mathcal{A}_{t-1,t}$ is convex and weak*-closed due to Remark 1.1.5, we can apply the Hahn-Banach separation theorem, e.g. Theorem A.56 in [FS04], and obtain a random variable $Z \in L^1(\mathcal{F}_t, P)$ such that

$$a := \sup_{X \in \mathcal{A}_{t-1,t}} E[\,Z(-X)\,] < E[\,Z\,A_t\,] =: b < \infty. \tag{2.34}$$

In the rest of the proof we will use Z to construct a density of a probability measure $Q \in \mathcal{Q}_T$ that violates the supermartingale property b). We will argue in several steps.

i) Since the non-negative random variable $\lambda I_{\{Z<0\}}$ belongs to the set $\mathcal{A}_{t-1,t}$ for every $\lambda \geq 0$, we have

$$\lambda E[\,-Z\,I_{\{Z<0\}}\,] < b < \infty$$

due to (2.34). This is only possible if $Z \geq 0$ P-a.s..

ii) We can even assume that Z is bounded away from zero, since it is possible to choose $\varepsilon > 0$ such that the random variable

$$Z^{\varepsilon} := (1-\varepsilon)Z + \varepsilon$$

still satisfies the inequality (2.34). Indeed, we have on the left-hand-side of (2.34)

$$\begin{aligned} E[Z^{\varepsilon}(-X)] &= \varepsilon E[-X] + (1-\varepsilon)E[Z(-X)] \\ &\le \varepsilon\, E[\,\alpha^{\min}_{t-1,t}(P)\,] + (1-\varepsilon)a \end{aligned}$$

for all $X \in \mathcal{A}_{t-1,t}$, where we have used the definition of the minimal penalty function $\alpha^{\min}_{t-1,t}(P)$ and the definition of a. For the right-hand-side of (2.34) we obtain

$$E[Z^{\varepsilon}A_t] = \varepsilon E[A_t] + (1-\varepsilon)b.$$

Thus Z^{ε} satisfies (2.34) if

$$\varepsilon\, E[\,\alpha^{\min}_{t-1,t}(P)\,] + (1-\varepsilon)a < \varepsilon E[A_t] + (1-\varepsilon)b$$

or equivalently

$$0 < \varepsilon < \frac{b-a}{b-a+E[\,\alpha^{\min}_{t-1,t}(P)\,] - E[A_t]}.$$

Clearly the choice of such an ε is possible if $E[\,A_t\,] \le E[\,\alpha^{\min}_{t-1,t}(P)\,] < \infty$. This inequality holds since

$$\begin{aligned} E[\,A_t\,] = E&\left[X_t - \sum_{k=0}^{t-1}\alpha^{\min}_{k,k+1}(P) - \left(X_{t-1} - \sum_{k=0}^{t-2}\alpha^{\min}_{k,k+1}(P)\right)\right] \\ &+ E[\,\alpha^{\min}_{t-1,t}(P)\,] \\ &\le E[\,\alpha^{\min}_{t-1,t}(P)\,] \end{aligned} \tag{2.35}$$

where we have used that $P \in \mathcal{Q}_T$ and the supermartingale property of the process $(X_t - \sum_{k=0}^{t-1}\alpha_{k,k+1})$ due to b). Thus we can assume that $Z \ge \varepsilon > 0$ P-a.s..

iii) We take Z as in part ii) and define a probability measure Q on $\mathcal{F}$ via the density

$$\frac{dQ}{dP} := \frac{Z}{E[\,Z\,|\,\mathcal{F}_{t-1}\,]} \le \frac{1}{\varepsilon}Z. \tag{2.36}$$

Then $Q \approx P$ by part ii) and $Q = P$ on the σ-field $\mathcal{F}_{t-1}$ by definition. Moreover, we have

$$\alpha^{\min}_{k,k+1}(Q) = \alpha^{\min}_{k,k+1}(P) \quad \text{for all} \quad k = 0, \dots, t-2 \tag{2.37}$$

and

$$\begin{aligned} E_Q[\, \alpha^{\min}_{t-1,t}(Q)\,] &= \sup_{X \in \mathcal{A}_{t-1,t}} E_Q[-X] \\ &\leq \frac{1}{\varepsilon} \sup_{X \in \mathcal{A}_{t-1,t}} E[Z(-X)] \\ &= \frac{1}{\varepsilon}\, a, \end{aligned} \tag{2.38}$$

where we have used Lemma 1.1.8, (2.36) and (2.34). So we already have shown that

$$E_Q\left[\sum_{k=0}^{t-1} \alpha^{\min}_{k,k+1}(Q)\right] < \infty \tag{2.39}$$

by (2.37) and (2.38).

iv) The crucial issue is now to show that $Q \in \mathcal{Q}_T$, and here we will need the assumption (2.30). Since Z is $\mathcal{F}_t$-measurable, we obtain

$$E_Q[\, X \,|\, \mathcal{F}_{t+s}\,] = E_P[\, X \,|\, \mathcal{F}_{t+s}\,]$$

for all $s \geq 0$ and all $X \in L^\infty$. Hence

$$\alpha^{\min}_{t+s,t+s+1}(Q) = \alpha^{\min}_{t+s,t+s+1}(P) \quad \text{for all } \; s = 0, 1, \dots \tag{2.40}$$

and

$$\begin{aligned} E_Q\left[\alpha^{\min}_{t+s,t+s+1}(Q)\right] &= E_P\left[\frac{Z}{E[Z|\mathcal{F}_{t-1}]}\, \alpha^{\min}_{t+s,t+s+1}(P)\right] \\ &\leq \frac{1}{\varepsilon} E_P\left[Z\, \alpha^{\min}_{t+s,t+s+1}(P)\right], \end{aligned} \tag{2.41}$$

where we have used (2.40) and (2.36). Thus we obtain using monotone convergence, (2.41) and (2.30):

$$\begin{aligned} E_Q\left[\sum_{k=t}^{T} \alpha^{\min}_{k,k+1}(Q)\right] &= \sum_{k=t}^{T} E_Q\left[\alpha^{\min}_{k,k+1}(Q)\right] \\ &\leq \frac{1}{\varepsilon} \sum_{k=t}^{T} E_P\left[Z \alpha^{\min}_{k,k+1}(P)\right] \\ &= \frac{1}{\varepsilon} E_P\left[Z \sum_{k=t}^{T} \alpha^{\min}_{k,k+1}(P)\right] \\ &\leq \frac{C}{\varepsilon} < \infty. \end{aligned} \tag{2.42}$$

Hence $Q \in \mathcal{Q}_T$.

v) We will show that the process $(X_t - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q))$ is not a Q-supermartingale, which is a contradiction to condition b). To this end note first that the set $\mathcal{A}_{t-1,t}$ is directed upward, which can be seen as in the proof of Lemma 1.1.8. Hence there exists a sequence $(Y_n) \subseteq \mathcal{A}_{t-1,t}$ such that

$$E_Q[-Y_n|\mathcal{F}_{t-1}] \nearrow \alpha_{t-1,t}^{\min}(Q) \quad \text{with} \quad n \to \infty.$$

Monotone convergence and (2.34) imply

$$\begin{aligned} E_Q\left[\, E[\, Z \,|\, \mathcal{F}_{t-1}\,]\, \alpha_{t-1,t}^{\min}(Q) \right] &= \lim_n E_Q\left[\, E[\, Z \,|\, \mathcal{F}_{t-1}\,]\, E_Q[\, -Y_n \,|\, \mathcal{F}_{t-1}\,]\,\right] \\ &= \lim_n E_Q\left[\, E[\, Z \,|\, \mathcal{F}_{t-1}\,]\,(-Y_n)\,\right] \\ &= \lim_n E[\, Z\,(-Y_n)\,] \\ &\le \sup_{X \in \mathcal{A}_{t-1,t}} E[\, Z\,(-X)\,] = a. \end{aligned} \tag{2.43}$$

Using (2.34) and (2.43) we obtain

$$\begin{aligned} &E_Q\left[E[\, Z \,|\, \mathcal{F}_{t-1}\,]\left(X_t - X_{t-1} - \alpha_{t-1,t}^{\min}(Q)\right)\right] = \\ &= E[ZA_t] - E_Q\left[E[\, Z \,|\, \mathcal{F}_{t-1}\,]\alpha_{t-1,t}^{\min}(Q)\right] \\ &\ge b - a > 0. \end{aligned} \tag{2.44}$$

Since $E[\, Z \,|\, \mathcal{F}_{t-1}\,] > 0$ Q-a.s., inequality (2.44) cannot hold if

$$E_Q\left[\, X_t - X_{t-1} - \alpha_{t-1,t}^{\min}(Q) \,|\, \mathcal{F}_{t-1} \right] \le 0 \qquad Q\text{-a.s.}, \tag{2.45}$$

which means $(X_t - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q))$ cannot be a Q-supermartingale. □

Note that due to Lemma 2.4.3 our assumption (2.30) required in Theorem 2.4.6 is in particular satisfied if $T < \infty$ and the risk measure $(\rho_t)_{t=0,\ldots,T}$ is sensitive.

In the sequel we will explain how the assumptions we have used for the proofs of Theorem 2.4.6 and Theorem 2.4.4 and can be relaxed. Note first that we do not need sensitivity to prove the equivalence of properties 1), 2) and 3) in Theorem 2.4.4, and also not for the proofs of 3) ⇒ 4) and 1) ⇒ 5). Sensitivity is needed only to prove that property 4) implies 1), and we use it in the proof of 5) $\Rightarrow$ 1) for $T < \infty$ in order to show that condition (2.30) is satisfied.

As we have said before, assumption (2.30) can be relaxed to the local boundedness condition (2.31), that is

$$\exists\, Q^* \in \mathcal{M}^e(P) \text{ such that } \alpha^{\min}_{t,t+1}(Q^*) \in L^\infty(\mathcal{F}_t) \;\; \forall \;\; t = 0, 1, \ldots .$$

Obviously (2.31) is equivalent to (2.30) if $T < \infty$ and it only makes a difference for $T = \infty$. In this case we obtain the supermartingale property for the set

$$\mathcal{Q}_{\infty,\mathrm{loc}} := \left\{ Q \in \mathcal{M}^e(P) \;\middle|\; E_Q\left[\sum_{k=0}^{t} \alpha^{\min}_{k,k+1}(Q)\right] < \infty \text{ for all } t = 0, 1, \ldots \right\}.$$

Since $\mathcal{Q}_T \subseteq \mathcal{Q}_{\infty,\mathrm{loc}}$, we obtain an equivalent characterization of one-step prudence in terms of a supermartingale property for a larger set of measures $\mathcal{Q}_{\infty,loc}$. Note, however, that the supermartingales appearing in the the next theorem are no longer bounded from below in L^1, in contrast to the situation of Theorem 2.4.4.

Theorem 2.4.7. *Let $(\rho_t)_{t=0,1,\ldots}$ be a sequence of conditional convex risk measures such that each ρ_t is continuous from above and sensitive. Assume further that $T = \infty$ and condition (2.31) is satisfied.*
Then properties 1)-4) of Theorem 2.4.4 are equivalent to the following property:

5". The process

$$U^Q_t(X) := \rho_t(X) - \sum_{k=0}^{t-1} \alpha^{\min}_{k,k+1}(Q), \qquad t = 0, 1, \ldots$$

is a Q-supermartingale for all $X \in L^\infty$ and all $Q \in \mathcal{Q}_{\infty,\mathrm{loc}}$.

Proof. The proof follows from the more general version of Theorem 2.4.6 below. ☐

Corollary 2.4.8. *Suppose that $(\rho_t)_{t=0,1,\ldots}$ is a sequence of conditional convex risk measures such that each ρ_t is continuous from above. Assume further that condition (2.31) is satisfied.*

Then for any bounded adapted process $(X_t)_{t=0,1,\ldots}$ the following conditions are equivalent:

a) The process (X_t) *is sustainable with respect to the risk measure* (ρ_t)*.*

b") The process

$$X_t - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \qquad t = 0, 1, \ldots$$

is a Q-supermartingale for all $Q \in \mathcal{Q}_{\infty,\mathrm{loc}}$*.*

Proof. We can argue exactly like in the proof of Theorem 2.4.6 to show *a)* ⇒ *b")*. For the proof of *b")* ⇒ *a)* only the following minor corrections are needed in case $T = \infty$: We assume that P satisfies the condition (2.31), in particular $P \in \mathcal{Q}_{\infty,loc}$. Then we continue as in the proof of Theorem 2.4.6 with the only difference that in the inequality (2.42) in part iv) we consider $E_Q\left[\sum_{t=0}^{t+s} \alpha_{k,k+1}^{\min}(Q)\right]$ and show that it is finite for all $s \geq 0$. Thus $Q \in \mathcal{Q}_{\infty,loc}$ and part v) provides a contradiction to 5"). □

Without the restrictions $T < \infty$, (2.30) or (2.31) property 5) of Theorem 2.4.4 takes the following form:

Corollary 2.4.9. *Let* $(\rho_t)_{t=0,1,\ldots}$ *be a sequence of conditional convex risk measures such that each* ρ_t *is continuous from above and sensitive. Then properties 1)-4) of Theorem 2.4.4 are equivalent to the following property:*

5'. The inequality

$$E_Q\left[\,\rho_{t+1}(X)\,|\,\mathcal{F}_t\,\right] \leq \rho_t(X) + \alpha_{t,t+1}^{\min}(Q) \tag{2.46}$$

holds for all $t = 0, 1, \ldots$ *and all* $Q \in \widetilde{\mathcal{Q}}_{t+1}$*, where*

$$\widetilde{\mathcal{Q}}_{t+1} := \left\{ Q \in \mathcal{M}_1(P) \;\middle|\; Q \approx P \;\text{ on }\; \mathcal{F}_{t+1},\; E_Q\left[\alpha_{t,t+1}^{\min}(Q)\right] < \infty \right\}.$$

Proof. The proof of this equivalence follows from Corollary 2.4.10 below. Condition $\widetilde{\mathcal{Q}}_{t+1} \neq \emptyset$ is satisfied here for all t due to sensitivity of the risk measure (ρ_t). □

Corollary 2.4.10. *Suppose that* $(\rho_t)_{t=0,1,\ldots}$ *is a sequence of conditional convex risk measures such that each* ρ_t *is continuous from above. Assume further that the sets*

$$\widetilde{\mathcal{Q}}_{t+1} := \left\{ Q \in \mathcal{M}_1(P) \;\middle|\; Q \approx P \;\text{ on }\; \mathcal{F}_{t+1},\; E_Q\left[\alpha_{t,t+1}^{\min}(Q)\right] < \infty \right\}$$

are not empty for all $t = 0, 1, \ldots$*. Then for any bounded adapted process* $(X_t)_{t=0,1,\ldots}$ *the following conditions are equivalent:*

a) The process (X_t) *is sustainable with respect to the risk measure* (ρ_t).

b') The inequality

$$E_Q\left[\, X_{t+1} \,|\, \mathcal{F}_t \,\right] \leq X_t + \alpha^{\min}_{t,t+1}(Q) \tag{2.47}$$

holds for all $t = 0, 1, \ldots$ *and all* $Q \in \widetilde{\mathcal{Q}}_{t+1}$.

Proof. We will adjust the proof of Theorem 2.4.6 to our present setting.

1) To prove $a) \Rightarrow b')$ we fix $t \in \{0, 1, \ldots\}$ and $Q \in \widetilde{\mathcal{Q}}_{t+1}$. Then sustainability of X and definition of the minimal penalty function imply the inequality

$$E_Q[X_{t+1} - X_t|\mathcal{F}_t] \leq \alpha^{\min}_{t,t+1}(Q) \quad Q\text{-a.s..}$$

which is equivalent to (2.47).

2) To prove $b') \Rightarrow a)$ we can argue almost exactly as in the proof of $b) \Rightarrow a)$ of Theorem 2.4.6, we just skip part iv) of that proof. We fix $t \in \{1, 2, \ldots\}$ and show that $X_t - X_{t-1} \in A_{t-1,t}$ by way of contradiction. We assume that $P \in \widetilde{\mathcal{Q}}_t$. We apply the separation theorem and argue further as in parts i)-iii) of the proof of Theorem 2.4.6. In part ii) we use condition (2.47) for X under P instead of the supermartingale property b) to prove inequality (2.35). In part iii) we omit (2.37) and use (2.38). This implies $Q \in \widetilde{\mathcal{Q}}_t$. And (2.45) in part v) of the proof of Theorem 2.4.6 shows that the inequality (2.47) (with t instead of $t+1$) does not hold for Q. □

Obviously condition b') of Corollary 2.4.10 implies the supermartingale property b) of Theorem 2.4.6 for all $Q \in \mathcal{Q}_T$. But the relaxed version b') does no longer involve a fixed class of measures independent of time, and so we cannot formulate the result in terms of supermartingales.

So far we have studied one-step prudence and its equivalent characterizations. In the sequel we will give a version of Theorem 2.4.4 for the general case of prudence, that involves looking ahead at any finite time $t + s$ and not just at the next step.

Corollary 2.4.11. *Let* $(\rho_t)_{t=0,1,\ldots}$ *be a sequence of conditional convex risk measures such that each* ρ_t *is continuous from above and sensitive.*

Then the following properties are equivalent:

1. $(\rho_t)_{t=0,1,\ldots}$ *is prudent.*

2. $(\rho_t)_{t=0,1,\dots}$ *is middle rejection consistent.*

3. *The inclusion*
$$\mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}$$
holds for all $t, s = 0, 1, \dots$.

4. *The inequality*
$$\alpha_t^{\min}(Q) \le \alpha_{t,t+s}^{\min}(Q) + E_Q[\,\alpha_{t+s}^{\min}(Q)\,|\,\mathcal{F}_t\,]$$
holds for all $t, s = 0, 1, \dots$ *and all* $Q \in \mathcal{M}^e(P)$.

5. *The inequality*
$$E_Q\left[\,\rho_{t+s}(X)\,|\,\mathcal{F}_t\,\right] \le \rho_t(X) + \alpha_{t,t+s}^{\min}(Q) \tag{2.48}$$
holds for all $t, s = 0, 1, \dots$ *and all* $Q \in \widetilde{\mathcal{Q}}_{t,t+s}$, *where*
$$\widetilde{\mathcal{Q}}_{t,t+s} := \left\{ Q \in \mathcal{M}^e(P) \;\middle|\; E_Q\left[\alpha_{t,t+s}^{\min}(Q)\right] < \infty \right\}.$$

Proof. The proofs of 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3) $\Leftrightarrow$ 4) are exactly like in Theorem 2.4.4 with $t+1$ replaced by $t+s$. The proof of 1) $\Rightarrow$ 5) follows straight from the definition of the minimal penalty function $\alpha_{t,t+s}^{\min}(Q)$:
$$E_Q\left[\rho_{t+s}(X) - \rho_t(X)|\mathcal{F}_t\right] \le \alpha_{t,t+s}^{\min}(Q) \tag{2.49}$$
for all $Q \in \mathcal{M}^e(P)$. And since $\alpha_{t,t+s}^{\min}(Q) < \infty$ P-a.s. for all $Q \in \widetilde{\mathcal{Q}}_{t,t+s}$, we can rearrange (2.49) to (2.48).
To prove 5) $\Rightarrow$ 1) we argue like in the proof of *b'*) $\Rightarrow$ *a*) of Corollary 2.4.10. We assume that there exists $X \in L^\infty$ and $t, s \in \{0, 1, \dots\}$ such that $A_{t,s} := \rho_{t+s}(X) - \rho_t(X) \notin \mathcal{A}_{t,t+s}$ in contradiction to prudence. Sensitivity implies that the sets $\widetilde{\mathcal{Q}}_{t,t+s}$ are not empty for all t, s and thus there is no loss of generality in assuming $P \in \widetilde{\mathcal{Q}}_{t,t+s}$. Applying the separation theorem and arguing further like in parts i)-iii) of the proof of Theorem 2.4.6 we obtain a probability measure $Q \approx P$ such that $Q \in \widetilde{\mathcal{Q}}_{t,t+s}$. And the same reasoning as in part v) of Theorem 2.4.6 shows that (2.48) does not hold under Q, providing a contradiction to 5). □

Of course property 5) of Corollary 2.4.11 implies property 5') of Corollary 2.4.10 and the supermartingale properties 5) of Theorem 2.4.4 and 5") of Theorem 2.4.7. In itself, however, it cannot be formulated as a supermartingale property for some given process and some class of measures.

2.5 Sustainability and time consistency

In the following we will assume that the time horizon T is finite and $(\rho_t)_{t=0,\dots,T}$ is a dynamic convex risk measure such that each ρ_t is continuous from above. If (ρ_t) is prudent, then for each $X \in L^\infty$ the risk process $(\rho_t(X))$ is sustainable with respect to (ρ_t) and covers the final loss: $\rho_T(X) = -X$ P-a.s.. We can ask whether $(\rho_t(X))$ is the smallest process with these properties, in other words whether we do not pay too much by "hedging" X with the process $(\rho_t(X))$. It turns out that we could possibly do better if we hedge "step by step" as explained below.

Consider a new risk measure $(\widetilde{\rho}_t)_{t=0,\dots,T}$ defined recursively by

$$\begin{aligned} \widetilde{\rho}_T(X) &:= \rho_T(X) = -X \\ \widetilde{\rho}_t(X) &:= \rho_t(-\widetilde{\rho}_{t+1}(X)), \quad t = 0, \dots, T-1, \quad X \in L^\infty. \end{aligned}$$

Thus we have $\widetilde{\rho}_t = \rho_t(-\rho_{t+1}(-\rho_{t+2}(\dots)))$, what we call "step by step" hedging. It is easy to see that $(\widetilde{\rho}_t)_{t=0,\dots,T}$ is again a dynamic convex risk measure, such that each $\widetilde{\rho}_t$ is continuous from above. Moreover, the sequence $(\widetilde{\rho}_t)$ is time consistent by definition. Such a recursive construction of time consistent risk measures was introduced in Section 4.2 of [CDK06] and studied in Section 3.2 of [Dra06] and in [CK06].

Lemma 2.5.1. *If the original risk measure $(\rho_t)_{t=0,\dots,T}$ is prudent, then the time consistent risk measure $(\widetilde{\rho}_t)$ defined via (2.50) lies below (ρ_t).*

Proof. This is easily proved by backward induction on t: We have $\widetilde{\rho}_T(X) = \rho_T(X)$ by definition and if the inequality $\widetilde{\rho}_{t+1}(X) \le \rho_{t+1}(X)$ holds, we obtain

$$\rho_t(X) \ge \rho_t(-\rho_{t+1}(X)) \ge \rho_t(-\widetilde{\rho}_{t+1}(X)) = \widetilde{\rho}_t(X)$$

using prudence and monotonicity. Thus the claim follows for all t. □

Moreover, it was shown in Theorem 3.10 of [Dra06], that if the original risk measure (ρ_t) is prudent, then $(\widetilde{\rho}_t)$ is the biggest time consistent dynamic convex risk measure that lies below (ρ_t).

The previous lemma shows that the process $(\widetilde{\rho}_t(X))$ is "cheaper" than $(\rho_t(X))$ for all X. It is interesting, that though $(\widetilde{\rho}_t(X))$ is cheaper, it still has the desired properties:

$$\widetilde{\rho}_T(X) \ge -X \tag{2.50}$$

by definition and by cash invariance

$$\rho_t(\widetilde{\rho}_t(X) - \widetilde{\rho}_{t+1}(X)) = -\widetilde{\rho}_t(X) + \rho_t(-\widetilde{\rho}_{t+1}(X)) = 0 \tag{2.51}$$

for all t, i.e. the process $(\widetilde{\rho}_t(X))$ is sustainable with respect to (ρ_t).

This is true not only for a prudent risk measure. Using Corollary 2.4.10 we will prove in the next proposition that for any dynamic risk measure (ρ_t) the process $(\widetilde{\rho}_t(X))$ defined via (2.50) is the smallest process that is sustainable with respect to (ρ_t) and covers the final loss $-X$. This shows in particular that time consistent dynamic risk measures are "Snell envelope - type" constructions. In the coherent case, a related result was proved in Theorem 6.4. of [Del06] for Snell envelopes in terms of m-stable sets of measures, but without using the notion of sustainability.

Proposition 2.5.2. *Suppose that $T < \infty$ and let $(\rho_t)_{t=0,\dots,T}$ be a sequence of conditional convex risk measures such that each ρ_t is continuous from above. Assume further that the sets $\widetilde{\mathcal{Q}}_t$ are not empty for all $t = 1, \dots, T-1$.*
Let $(\widetilde{\rho}_t)_{t=0,\dots,T}$ denote the time consistent sequence of conditional convex risk measures that arises from (ρ_t) via definition (2.50). Then for each $X \in L^\infty$ the risk process $(\widetilde{\rho}_t(X))_{t=0,\dots,T}$ is the smallest bounded adapted process $(U_t)_{t=0,\dots,T}$ such that (U_t) is sustainable with respect to (ρ_t) and $U_T \geq -X$ P-a.s..

Proof. We have already seen that $(\widetilde{\rho}_t)_{t=0,\dots,T}$ satisfies (2.50) and is sustainable with respect to (ρ_t) due to (2.51).
Let $(U_t)_{t=0,\dots,T}$ be another bounded adapted process such that (U_t) is sustainable with respect to (ρ_t) and $U_T \geq -X$ P-a.s.. We have to show that

$$U_t \geq \widetilde{\rho}_t(X) \quad P\text{-a.s.} \tag{2.52}$$

for all $t = 0, \dots, T$. The proof of this inequality will follow by backward induction on t.

We have

$$U_T \geq -X = \widetilde{\rho}_T(X) \qquad P\text{-a.s.}$$

to begin with. Assume that we have proved (2.52) for $t+1$ already. To proceed we use equivalent characterization of sustainability from Corollary 2.4.10. Since (U_t) is sustainable w.r.t. (ρ_t) we obtain for all $Q \in \widetilde{\mathcal{Q}}_{t+1}$:

$$\begin{aligned} U_t &\geq E_Q\left[\, U_{t+1} - \alpha^{\min}_{t,t+1}(Q) \,|\, \mathcal{F}_t \,\right] \\ &\geq E_Q\left[\, \widetilde{\rho}_{t+1}(X) - \alpha^{\min}_{t,t+1}(Q) \,|\, \mathcal{F}_t \,\right] \quad P\text{-a.s.}, \end{aligned} \tag{2.53}$$

where we have used (2.47) and the induction hypothesis. To proceed with the proof we will have to show that risk measure ρ_t restricted to the space L^∞_{t+1} has a robust representation in terms of the set $\widetilde{\mathcal{Q}}_{t+1}$. We will postpone the proof of this result to the next Lemma 2.5.3. Provided the representation holds, we obtain from (2.53)

$$\begin{aligned} U_t &\geq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_{t+1}} \left(E_Q \left[\widetilde{\rho}_{t+1}(X) | \mathcal{F}_t \right] - \alpha^{\min}_{t,t+1}(Q) \right) \\ &= \rho_t(-\widetilde{\rho}_{t+1}(X)) = \widetilde{\rho}_t(X) \quad P\text{-a.s.}, \end{aligned}$$

and this proves the inequality (2.52) for all t. □

The following Lemma completes the proof of Proposition 2.5.2.

Lemma 2.5.3. *Let $(\rho_t)_{t=0,1,\ldots}$ be a sequence of conditional convex risk measures such that each ρ_t is continuous from above and let $\widetilde{\mathcal{Q}}_t \neq \emptyset$ for all $t = 1, 2, \ldots$. Then for each $t \in \{0, 1, \ldots\}$ the risk measure ρ_t restricted to the space L^∞_{t+1} has a robust representation*

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \widetilde{\mathcal{Q}}_{t+1}} \left(E_Q \left[-X | \mathcal{F}_t \right] - \alpha^{\min}_{t,t+1}(Q) \right), \qquad X \in L^\infty(\mathcal{F}_{t+1}), \tag{2.54}$$

where

$$\widetilde{\mathcal{Q}}_{t+1} := \left\{ Q \in \mathcal{M}_1(P) \;\middle|\; Q \approx P \;\; on \;\; \mathcal{F}_{t+1}, \; E_Q \left[\alpha^{\min}_{t,t+1}(Q) \right] < \infty \right\}.$$

Proof. We fix $t \in \{0, 1, \ldots\}$ and take $P^* \in \widetilde{\mathcal{Q}}_{t+1}$. Since $\alpha^{\min}_{t,t+1}(P^*) < \infty$ P-a.s., (1.17) of Lemma 1.2.5 provides the representation

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \approx P|_{\mathcal{F}_{t+1}}} \left(E_Q \left[-X | \mathcal{F}_t \right] - \alpha^{\min}_{t,t+1}(Q) \right), \quad X \in L^\infty(\mathcal{F}_{t+1}),$$

and hence we obtain "$\geq$" in (2.54).
To prove the opposite inequality note that $E_{P^*}[\alpha^{\min}_{t,t+1}(P^*)] < \infty$. Thus we can apply (1.18) of Lemma 1.2.5 and obtain a representation

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \widetilde{\mathcal{Q}}^{f,e}_t} \left(E_Q \left[-X | \mathcal{F}_t \right] - \alpha^{\min}_{t,t+1}(Q) \right), \quad X \in L^\infty(\mathcal{F}_{t+1}), \tag{2.55}$$

where

$$\widetilde{\mathcal{Q}}^{f,e}_t := \left\{ Q \in \mathcal{M}_1(P) \;\middle|\; Q \approx P^*|_{F_{t+1}}, \; Q = P^*|_{\mathcal{F}_t}, \; E_Q \left[\alpha^{\min}_{t,t+1}(Q) \right] < \infty \right\}.$$

Since $\widetilde{\mathcal{Q}}^{f,e}_t \subseteq \widetilde{\mathcal{Q}}_{t+1}$ we obtain "$\leq$" in (2.54). □

We believe that Proposition 2.5.2 provides a good reason for using *time consistent* dynamic risk measures which can be constructed from any dynamic risk measure via (2.50) for a finite time horizon. If the original risk measure

(ρ_t) is middle rejection consistent or prudent, the arising risk measure $(\widetilde{\rho}_t)$ is cheaper, as shown in Lemma 2.5.1. If the original risk measure is middle acceptance consistent, then the risk process $(\widetilde{\rho}_t(X))$ may be more expensive, as shown in Theorem 3.10 of [Dra06]. But on the other hand $(\widetilde{\rho}_t(X))$ is sustainable with respect to (ρ_t) for all X, whereas the process $(\rho_t(X))$ itself might not be sustainable.

The next theorem is a version of Theorem 2.4.6 and provides an equivalent characterization of sustainability with respect to *time consistent* risk measures. Using time consistency we do not need the additional assumption (2.30) of Theorem 2.4.6, and we obtain the supermartingale property for the set $\mathcal{Q}^*$.

Theorem 2.5.4. *Suppose that $(\rho_t)_{t=0,1,\dots}$ is a time consistent sequence of conditional convex risk measures such that each ρ_t is continuous from above. Assume further that $\mathcal{Q}^* \neq \emptyset$.*
Then for any bounded adapted process $(X_t)_{t=0,1,\dots}$ the following conditions are equivalent:

a) The process (X_t) is sustainable with respect to the risk measure (ρ_t).

b) The process

$$X_t - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \qquad t = 0, 1, \dots$$

is a Q-supermartingale for all $Q \in \mathcal{Q}^*$.

Proof. We will modify the proof of Theorem 2.4.6 once more in order to use time consistency.
To prove that sustainability implies the supermartingale property b) for all $Q \in \mathcal{Q}^*$ note that

$$E_Q\left[\sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q)\right] = E_Q\left[\alpha_{0,t}^{\min}(Q)\right] \leq E_Q\left[\alpha_0^{\min}(Q)\right] < \infty$$

for all $Q \in \mathcal{Q}^*$ due to property 3) of Theorem 2.2.2. This implies integrability of the process $(X_t - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q))$. The rest of the reasoning is the same as in the proof of $a) \Rightarrow b)$ of Theorem 2.4.6.
To prove that b) implies a) we argue similar to the proof of $b) \Rightarrow a)$ of Theorem 2.4.6 with the following modifications:
We assume without loss of generality that $P \in \mathcal{Q}^*$ and suppose by the way of contradiction that $X_t - X_{t-1} = A_t \notin -\mathcal{A}_{t-1,t}$. Then by cash invariance

$-\rho_t(-A_t) \notin \mathcal{A}_{t-1,t}$ and hence (2.8) of Lemma 2.2.4 and property 2) of Theorem 2.2.2 imply

$$-A_t \notin \mathcal{A}_{t-1,t} + \mathcal{A}_t = \mathcal{A}_{t-1}.$$

Thus we apply the Hahn-Banach separation theorem to the set $\mathcal{A}_t$ instead of $\mathcal{A}_{t,t-1}$ and obtain a random variable $Z \in L^1(P)$ such that

$$a := \sup_{X \in \mathcal{A}_{t-1}} E[\, Z(-X)\,] < E[\, Z\, A_t\,] =: b < \infty. \tag{2.56}$$

Then we use Z to define a density of a probability measure $Q \in \mathcal{Q}^*$ that violates property b). We argue in several steps.

i) Condition (2.56) implies that $Z \geq 0$ P-a.s. exactly as in part i) of the proof of Theorem 2.4.6.

ii) To prove that Z can be chosen bounded away from zero we consider again the random variable

$$Z^\varepsilon := (1-\varepsilon)Z + \varepsilon$$

and show that it still satisfies the inequality (2.56) for an appropriate $\varepsilon > 0$. Indeed, we have on the left-hand-side of (2.56)

$$\begin{aligned} E[Z^\varepsilon(-X)] &= \varepsilon E[-X] + (1-\varepsilon)E[Z(-X)] \\ &\leq \varepsilon\, \alpha_0^{\min}(P) + (1-\varepsilon)a \end{aligned}$$

for all $X \in \mathcal{A}_{t-1}$, where we have used that $\mathcal{A}_{t-1} \subseteq \mathcal{A}_0$ by property 2) of Theorem 2.2.2, the definition of the minimal penalty function $\alpha_0^{\min}(P)$ and the definition of a. For the right-hand-side of (2.56) we obtain

$$E[Z^\varepsilon A_t] = \varepsilon E[A_t] + (1-\varepsilon)b.$$

Thus Z^ε satisfies (2.56) if

$$0 < \varepsilon < \frac{b-a}{b-a+\alpha_0^{\min}(P) - E[A_t]}.$$

The choice of such an ε is possible if $E[\, A_t\,] \leq \alpha_0^{\min}(P)$. This inequality holds since

$$E[\, A_t\,] \leq E[\, \alpha_{t-1,t}^{\min}(P)\,] \leq \alpha_0^{\min}(P),$$

where we have used the supermartingale property under $P \in \mathcal{Q}^*$ as in (2.35) in the first inequality and the property 3) of Theorem 2.2.2 in the second.

iii) We define a probability measure Q on $\mathcal{F}$ via density

$$\frac{dQ}{dP} := \frac{Z}{E[\, Z \,|\, \mathcal{F}_{t-1}\,]}.$$

Then it follows as in part iii) of the proof of Theorem 2.4.6

$$\alpha^{\min}_{k,k+1}(Q) = \alpha^{\min}_{k,k+1}(P) \quad \text{for all} \quad k = 0, \dots, t-2 \tag{2.57}$$

and

$$E_Q[\, \alpha^{\min}_{t-1}(Q)\,] = \sup_{X \in \mathcal{A}_{t-1}} E_Q[-X] \leq \frac{1}{\varepsilon}\, a. \tag{2.58}$$

Hence we obtain

$$\begin{aligned} \alpha^{\min}_0(Q) &= E_Q\left[\sum_{k=0}^{t-2} \alpha^{\min}_{k,k+1}(Q)\right] + E_Q[\, \alpha^{\min}_{t-1}(Q)\,] \\ &\leq \alpha^{\min}_0(P) + \frac{1}{\varepsilon}\, a \\ &< \infty, \end{aligned}$$

due to property 3) of Theorem 2.2.2, (2.57) and (2.58). Thus $Q \in \mathcal{Q}^*$.

iv) We will show that the process $(X_t - \sum_{k=0}^{t-1} \alpha^{\min}_{k,k+1}(Q))$ is not a Q-supermartingale. The proof of this claim goes almost exactly as in part v) of the proof of Theorem 2.4.6. The only difference is that in the inequality (2.43) we use that $\mathcal{A}_{t-1,t} \subseteq \mathcal{A}_t$ by property 2) of Theorem 2.2.2 and conclude that

$$\begin{aligned} E_Q\Big[\, E[\, Z \,|\, \mathcal{F}_{t-1}\,]\, \alpha^{\min}_{t-1,t}(Q) \Big] &\leq \sup_{X \in \mathcal{A}_{t-1,t}} E[\, Z\,(-X)\,] \\ &\leq \sup_{X \in \mathcal{A}_t} E[\, Z\,(-X)\,] = a. \end{aligned}$$

The rest of the reasoning is the same. □

If a dynamic risk measure (ρ_t) is time consistent, it is also prudent and thus the risk process $(\rho_t(X))$ is sustainable with respect to (ρ_t) for all $X \in L^\infty$. Hence the process

$$U^Q_t(X) = \rho_t(X) - \sum_{k=0}^{t-1} \alpha^{\min}_{k,k+1}(Q), \quad t = 0, 1, \dots$$

is a Q-supermartingale for all $Q \in \mathcal{Q}^*$ by Theorem 2.5.4. Moreover, since time consistency implies prudence, the process U^Q is a Q-supermartingale for all $Q \in \mathcal{Q}_T$ due to Theorem 2.4.4. In the sequel we will discuss how these supermartingale properties are related to the supermartingale property of the process

$$V_t^Q(X) = \rho_t(X) + \alpha_t^{\min}(Q) \quad t = 0, 1, \ldots$$

for all $Q \in \mathcal{Q}_T$ that we have proved in Theorem 2.2.2.

Let $(\rho_t)_{t=0,1,\ldots}$ be a time consistent dynamic risk measure such that each ρ_t is continuous from above. Assume further that the set $\mathcal{Q}^*$ is not empty.

We first consider the case $T < \infty$. Then we obtain using the Doob decomposition (2.22) of $\alpha_t^{\min}(Q)$ from Remark 2.3.3:

$$\begin{aligned} V_t^Q(X) - U_t^Q(X) &= \alpha_t^{\min}(Q) + \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q) \\ &= E_Q\left[\sum_{k=0}^{T-1} \alpha_{k,k+1}^{\min}(Q) \,\Big|\, \mathcal{F}_t \right] \end{aligned}$$

for all $t = 0, \ldots, T-1$ and all $Q \in \mathcal{Q}^*$. Thus the difference between the two processes $V^Q(X)$ and $U^Q(X)$ is identified as the Q-martingale

$$E_Q\left[\sum_{k=0}^{T-1} \alpha_{k,k+1}^{\min}(Q) \,\Big|\, \mathcal{F}_t \right], \quad t = 0, \ldots, T.$$

This shows that $V^Q(X)$ is a supermartingale iff $U^Q(X)$ is a supermartingale for all $Q \in \mathcal{Q}^*$. Moreover, since

$$\alpha_0^{\min}(Q) = E_Q\left[\sum_{k=0}^{T-1} \alpha_{k,k+1}^{\min}(Q) \right]$$

for all $Q \in \mathcal{M}^e(P)$ by property 3) of Theorem 2.2.2, we have $\mathcal{Q}^* = \mathcal{Q}_T$. In particular time consistency implies the supermartingale property of $U^Q(X)$ for all $Q \in \mathcal{Q}_T$ if $T < \infty$, which is consistent with the fact that time consistency implies prudence.

For $T = \infty$ we use the Doob decomposition (2.23) of $\alpha_t^{\min}(Q)$ from Remark 2.3.3 and obtain

$$\begin{aligned} V_t^Q(X) - U_t^Q(X) &= \alpha_t^{\min}(Q) + \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q) \\ &= E_Q\left[\sum_{k=0}^{\infty} \alpha_{k,k+1}^{\min}(Q) \,\Big|\, \mathcal{F}_t \right] + M_t^Q \end{aligned}$$

for all $t = 0, 1, \ldots$ and all $Q \in \mathcal{Q}^*$, where M^Q denotes the martingale appearing in the Riesz decomposition of the penalty function process $(\alpha_t^{\min}(Q))$ as stated in Proposition 2.3.2. Thus the difference between the two processes $V^Q(X)$ and $U^Q(X)$ is again a Q-martingale, and their supermartingale properties are equivalent for all $Q \in \mathcal{Q}^*$.

To explain the relation between the sets $\mathcal{Q}^*$ and $\mathcal{Q}_\infty$ for $T = \infty$ note that property 3) of Theorem 2.2.2 implies

$$\alpha_t^{\min}(Q) = E_Q\left[\sum_{k=t}^{t+s} \alpha_{k,k+1}^{\min}(Q) \,\Big|\, \mathcal{F}_t\right] + E_Q\left[\alpha_{t+s}^{\min}(Q) \,\Big|\, \mathcal{F}_t\right]$$

for all $s \geq 0$ and all $Q \in \mathcal{M}_1(P)$. Since all penalty functions are non-negative, monotone convergence implies

$$\alpha_t^{\min}(Q) = E_Q\left[\sum_{k=t}^{\infty} \alpha_{k,k+1}^{\min}(Q) \,\Big|\, \mathcal{F}_t\right] + \lim_{s\to\infty} E_Q\left[\alpha_{t+s}^{\min}(Q) \,\Big|\, \mathcal{F}_t\right]$$

for all $t = 0, 1, \ldots$ and all $Q \in \mathcal{Q}^*$. In particular

$$\alpha_0^{\min}(Q) \geq E_Q\left[\sum_{k=0}^{\infty} \alpha_{k,k+1}^{\min}(Q)\right],$$

and thus $\mathcal{Q}^* \subseteq \mathcal{Q}_\infty$. The converse inclusion is not clear: In order to obtain the equality of these sets we would need some additional assumption making sure that

$$\limsup_{s\to\infty} E_Q\left[\alpha_{t+s}^{\min}(Q)\right] < \infty \quad P\text{-a.s.}$$

for all $Q \in \mathcal{Q}_\infty$.

We summarize the results of the preceding discussion in the next proposition.

Proposition 2.5.5. *Let $(\rho_t)_{t=0,1,\ldots}$ be a time consistent dynamic risk measure such that each ρ_t is continuous from above. Assume further that the set $\mathcal{Q}^*$ is not empty. Then the process*

$$U_t^Q(X) = \rho_t(X) - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \quad t = 0, 1, \ldots$$

is a Q-supermartingale for all $Q \in \mathcal{Q}^*$ if and only if the process

$$V_t^Q(X) = \rho_t(X) + \alpha_t^{\min}(Q) \quad t = 0, 1, \ldots$$

is a Q-supermartingale for all $Q \in \mathcal{Q}^*$.
If $T < \infty$, the sets

$$\mathcal{Q}^* = \left\{ Q \in \mathcal{M}^e(P) \,\Big|\, \alpha_0^{\min}(Q) < \infty \right\}$$

and

$$\mathcal{Q}_T = \left\{ Q \in \mathcal{M}^e(P) \;\middle|\; E_Q\left[\sum_{k=0}^{T-1} \alpha_{k,k+1}^{\min}(Q)\right] < \infty \right\}$$

are equal. If $T = \infty$ *we have* $\mathcal{Q}^* \subseteq \mathcal{Q}_\infty$.

Remark 2.5.6. *Proposition 2.5.5 provides in particular an alternative way to prove* 1) $\Rightarrow$ 4) *of Theorem 2.2.2: Time consistency implies one-step prudence, and one-step prudence implies the supermartingale property of the process* U^Q *for all* $Q \in \mathcal{Q}_\infty$ *by Theorem 2.4.4. Since* $\mathcal{Q}^* \subseteq \mathcal{Q}_\infty$*, the process* U^Q *is in particular a* Q*-supermartingale for all* $Q \in \mathcal{Q}^*$*. Thus the process* V^Q *is a* Q*-supermartingale for all* $Q \in \mathcal{Q}^*$ *by Proposition 2.5.5.*

Chapter 3

Asymptotic safety and asymptotic precision

In this chapter we consider a dynamic convex risk measure $(\rho_t)_{t=0,1,\ldots}$ with infinite time horizon $T = \infty$. In the preceding chapter we have seen that various time consistency properties imply various supermartingale properties of a dynamic risk measure and its penalty function process. In this chapter we will study the asymptotic behavior of the arising supermartingales. In particular we will show that for time consistent and for prudent dynamic risk measures there exists a limit

$$\rho_\infty(X) := \lim_{t\to\infty} \rho_t(X) \quad P\text{-a.s.}$$

for all $X \in L^\infty$.

We characterize the functional ρ_∞ on L^∞. In particular it is natural to ask when is the functional ρ_∞ a conditional convex risk measure itself, i.e. when is $\rho_\infty(X) = -X$ for all $X \in L^\infty$. We call this property *asymptotic precision* of a dynamic risk measure $(\rho_t)_{t=0,1,\ldots}$. In finite time horizon every dynamic risk measure is naturally asymptotically precise. In infinite time horizon this condition does not always hold, as we will show in Example 3.1.11. We will give some sufficient conditions for asymptotic precision for time consistent dynamic risk measures in Proposition 3.1.12 and for prudent risk measures in Proposition 3.2.3.

Another interesting question is when is the limit $\rho_\infty(X)$ enough to cover the final loss $-X$. We call this property *asymptotic safety*. We will show in Example 3.1.6 that not every dynamic risk measure is asymptotically safe. We will give equivalent characterizations of asymptotic safety for time consistent risk measures in Theorem 3.1.4 and sufficient conditions for asymptotic safety for prudent risk measures in Proposition 3.2.2.

3.1 Asymptotic properties of time consistent risk measures

This section is based on Section 5 of [FP06]. We consider a time consistent dynamic convex risk measure $(\rho_t)_{t=0,1,\dots}$ with infinite time horizon $T=\infty$. We assume that $\mathcal{F}=\mathcal{F}_\infty := \sigma\left(\cup_{t\geq 0}\mathcal{F}_t\right)$, that each ρ_t is continuous from above and that $\mathcal{Q}^*\neq\emptyset$.

For $Q\in\mathcal{Q}^*$ and $X\in L^\infty$, the process

$$V_t^Q(X)=\rho_t(X)+\alpha_t^{\min}(Q), \qquad t=0,1,\dots$$

is a Q-supermartingale due to Theorem 2.2.2, and the process $(\alpha_t^{\min}(Q))$ is a non-negative Q-supermartingale by Remark 2.3.1. Moreover, the $(V_t^Q(X))$ is bounded from below since

$$V_t^Q(X)\geq E_Q[-X|\mathcal{F}_t] \quad Q\text{-a.s.}$$

due to the robust representation (2.7) of the risk measure ρ_t. Hence the processes $(V_t^Q(X))$ and $(\alpha_t^{\min}(Q))$ are both Q-a.s. convergent to some finite limits $\alpha_\infty^{\min}(Q)$ and $V_\infty^Q(X)$.
In particular, the limit

$$\rho_\infty(X):=\lim_{t\to\infty}\rho_t(X)=V_\infty^Q(X)-\alpha_\infty^{\min}(Q) \tag{3.1}$$

exists P-a.s..

Lemma 3.1.1. *The functional $\rho_\infty:\ L^\infty\to L^\infty$ defined by (3.1) is normalized, monotone, conditionally convex and conditionally cash invariant with respect to $\mathcal{F}_t$ for any $t\geq 0$, and it satisfies*

$$\rho_\infty(X)\geq -X-\operatorname*{ess\,inf}_{Q\in\mathcal{Q}^*}\alpha_\infty^{\min}(Q) \quad P\text{-a.s..}$$

Proof. Normalization, monotonicity, conditional convexity and conditional cash invariance w.r.t. any $\mathcal{F}_{t_0}$ follow from the corresponding properties of ρ_t for $t\geq t_0$. Since

$$\rho_t(X)\geq E_Q[-X|\mathcal{F}_t]-\alpha_t^{\min}(Q)$$

for all t, we obtain

$$\rho_\infty(X) \geq -X - \alpha_\infty^{\min}(Q) \quad Q\text{-a.s.} \tag{3.2}$$

by martingale convergence for any $Q \in \mathcal{Q}^*$. □

Clearly, ρ_∞ is a conditional convex risk measure if and only if it reduces to the trivial monetary risk measure

$$\rho_\infty(X) = -X, \tag{3.3}$$

since this is equivalent to cash invariance w.r.t. $\mathcal{F}_\infty = \mathcal{F}$. But this property does not always hold as shown by examples 3.1.6 and 3.1.11 below.

Let us first focus on the weaker property

$$\rho_\infty(X) \geq -X,$$

i.e., the asymptotic capital requirement ρ_∞ is enough to cover the actual final loss $-X$:

Definition 3.1.2. *We say that the sequence $(\rho_t)_{t=0,1,\dots}$ is* asymptotically safe *if the limit ρ_∞ defined by (3.1) satisfies*

$$\rho_\infty(X) \geq -X$$

for any $X \in L^\infty$.

In order to characterize asymptotic safety we recall that the classes

$$\mathcal{A}_{0,t} = \mathcal{A}_0 \cap L_t^\infty \qquad t = 0, 1, \dots$$

and the corresponding penalty functions

$$\alpha_{0,t}^{min}(Q) = \sup_{X \in \mathcal{A}_{0,t}} E_Q[-X] \qquad t = 0, 1, \dots$$

satisfy the relations

$$\mathcal{A}_0 = \mathcal{A}_{0,t} + \mathcal{A}_t$$

and

$$\alpha_0^{min}(Q) = \alpha_{0,t}^{min}(Q) + E_Q[\alpha_t^{\min}(Q)],$$

for all $Q \in \mathcal{Q}^*$ by Remark 2.2.6. In particular,

$$\alpha_{0,t}^{min}(Q) = E_Q\left[\sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q)\right]$$

is increasing in t and bounded from above by $\alpha_0^{min}(Q)$ for $Q \in \mathcal{Q}^*$. Thus the limit

$$\alpha_{0,\infty}^{min}(Q) := \lim_{t \to \infty} \alpha_{0,t}^{min}(Q) \leq \alpha_0^{min}(Q) \tag{3.4}$$

exists for all $Q \in \mathcal{Q}^*$.

Definition 3.1.3. *Let us say that* $X \in L^\infty$ *is* predictably acceptable *if there exists a uniformly bounded and* P*-a.s. convergent sequence* $(X_t) \subseteq L^\infty$ *such that* $X_t \in \mathcal{A}_{0,t}$ *for all* $t \geq 0$ *and*

$$X \geq \lim_{t\to\infty} X_t.$$

We denote by $\mathcal{A}_{0,\infty}$ *the class of all predictably acceptable positions* X*.*

Note that

$$\mathcal{A}_{0,\infty} \subseteq \mathcal{A}_0, \tag{3.5}$$

since $X \geq \lim_t X_t$ implies

$$\rho_0(X) \leq \rho_0(\lim_t X_t) \leq \liminf_t \rho_0(X_t) \leq 0$$

by monotonicity and by the Fatou property of the unconditional risk measure ρ_0.

Theorem 3.1.4. *The following properties are equivalent:*

1. $\bigcap_{t\geq 0} \mathcal{A}_t = L^\infty_+$.
2. $\mathcal{A}_{0,\infty} = \mathcal{A}_0$.
3. $\lim_{t\to\infty} \alpha^{\min}_{0,t}(Q) = \alpha^{\min}_0(Q)$ *for all* $Q \in \mathcal{Q}^*$.
4. $\lim_{t\to\infty} \alpha^{\min}_t(Q) = 0$ Q*-a.s. and in* $L^1(Q)$ *for all* $Q \in \mathcal{Q}^*$.
5. $\lim_{t\to\infty} \alpha^{\min}_t(Q) = 0$ Q*-a.s. and in* $L^1(Q)$ *for at least one* $Q \in \mathcal{Q}^*$.
6. $(\rho_t)_{t=0,1,\dots}$ *is asymptotically safe.*

Proof. 1) $\Rightarrow$ 2) In view of (3.5) we have to show that property 1 implies $\mathcal{A}_0 \subseteq \mathcal{A}_{0,\infty}$. For $X \in \mathcal{A}_0$ define $X_t := -\rho_t(X)$. Then $X_t \in \mathcal{A}_{0,t}$ by property (2.8) of Lemma 2.2.4. Moreover, for $0 \leq n \leq t$ we have

$$X + \rho_t(X) \in \mathcal{A}_n,$$

since $\rho_t(X + \rho_t(X)) = 0$ and thus $\rho_n(X + \rho_t(X)) = 0$ for all $n \leq t$ by time consistency.
Using the Fatou property of ρ_n we obtain

$$\rho_n(X + \rho_\infty(X)) \leq \liminf_{t\to\infty} \rho_n(X + \rho_t(X)) = 0$$

for any $n \geq 0$, hence

$$X + \rho_\infty(X) \in \bigcap_{n \geq 0} \mathcal{A}_n = L^\infty_+.$$

Thus $\lim_t X_t = -\rho_\infty(X) \leq X$ P-a.s., and this shows $X \in \mathcal{A}_{0,\infty}$.

2) ⇒ 3) If $X \in \mathcal{A}_0 = \mathcal{A}_{0,\infty}$, then there exists a bounded convergent sequence $X_t \in \mathcal{A}_{0,t},\ t \geq 0$, such that $\lim_t X_t \leq X$ P-a.s.. For any $Q \in \mathcal{Q}^*$ we have

$$\alpha_0^{min}(Q) \geq \alpha_{0,\infty}^{min}(Q) = \lim_{t\to\infty} \alpha_{0,t}^{min}(Q) \geq \liminf_{t\to\infty} E_Q[-X_t] \geq E_Q[-X],$$

where we have used (3.4), the definition of $\alpha_{0,t}^{min}(Q)$ and Lebesgue's convergence theorem for Q. But

$$\alpha_0^{min}(Q) = \sup_{X \in \mathcal{A}_0} E_Q[-X],$$

and this implies the equality $\alpha_0^{min}(Q) = \alpha_{0,\infty}^{min}(Q)$.

3) ⇒ 4) Note that property (2.14) in Remark 2.2.6 implies

$$\alpha_0^{min}(Q) = \alpha_{0,t}^{min}(Q) + E_Q[\alpha_t^{\min}(Q)]$$

for $Q \in \mathcal{Q}^*$. Thus the convergence of $\alpha_{0,t}^{min}(Q)$ to $\alpha_0^{min}(Q)$ implies that the Q-expectation of $\alpha_t^{\min}(Q)$ converges to 0 as $t \to \infty$. This yields our claim since $(\alpha_t^{\min}(Q))_{t=0,1,\ldots}$ is a non-negative Q-supermartingale by Remark 2.3.1.

4) ⇒ 5) This is obvious.

5) ⇒ 6) Property 5) and Lemma 3.1.1 imply $\rho_\infty(X) \geq -X$ P-a.s..

6) ⇒ 1) We have to show that the inequality $\rho_\infty(X) \geq -X$ implies $\bigcap_{t\geq 0} \mathcal{A}_t \subseteq L^\infty_+$. Indeed,

$$\begin{aligned} X \in \bigcap_{t \geq 0} \mathcal{A}_t &\Rightarrow \rho_t(X) \leq 0 \quad \text{for all } t \geq 0 \\ &\Rightarrow -X \leq \rho_\infty(X) \leq 0 \\ &\Rightarrow X \in L^\infty_+. \end{aligned}$$

□

Recall that due to Proposition 2.3.2 the minimal penalty function process $(\alpha_t^{\min}(Q))$ has the Riesz decomposition

$$\alpha_t^{\min}(Q) = Z_t^Q + M_t^Q, \qquad t = 0, 1, \ldots$$

with the Q-potential

$$Z_t^Q = E_Q\left[\sum_{k=t}^{\infty} \alpha_{k,k+1}^{\min}(Q)\,\Big|\,\mathcal{F}_t\right] = \lim_{s\to\infty} \alpha_{t,t+s}^{\min}(Q), \qquad t = 0, 1, \ldots \tag{3.6}$$

and the non-negative Q-martingale

$$M_t^Q = \lim_{s\to\infty} E_Q[\,\alpha_{t+s}^{\min}(Q)\,|\mathcal{F}_t\,], \qquad t = 0, 1, \ldots$$

for all $Q \in \mathcal{Q}^*$. Using this decomposition we obtain the following equivalent characterizations of asymptotic safety.

Corollary 3.1.5. *Conditions 1)-6) of Theorem 3.1.4 are equivalent to the following:*

7. *The martingale M^Q in the Riesz decomposition of the process $(\alpha_t^{\min}(Q))$ vanishes for all $Q \in \mathcal{Q}^*$.*
8. $\lim_{s\to\infty} \alpha_{t,t+s}^{\min}(Q) = \alpha_t^{\min}(Q)$ *for all $Q \in \mathcal{Q}^*$ and all $t = 0, 1, \ldots$.*
9. $\lim_{s\to\infty} \alpha_{t,t+s}^{\min}(Q) = \alpha_t^{\min}(Q)$ *for at least one $Q \in \mathcal{Q}^*$ and at least one $t \in \{0, 1, \ldots\}$.*

Proof. Condition 4) of Theorem 3.1.4 implies $M_0^Q = 0$ for all $Q \in \mathcal{Q}^*$. Since M^Q is a non-negative supermartingale, we obtain $M_t^Q = 0$ for all $t = 0, 1, \ldots$ and all $Q \in \mathcal{Q}^*$, which proves 7).
Property 7) implies $Z_t^Q = \alpha_t^{\min}(Q)$ for all $t = 0, 1, \ldots$ and all $Q \in \mathcal{Q}^*$. In view of (3.6) this is equivalent to property 8).
Obviously 8) implies 9).
Conversely, if 9) holds we have $Z_t^Q = \alpha_t^{\min}(Q)$ for one $t \in \{0, 1, \ldots\}$ and one $Q \in \mathcal{Q}^*$ due to (3.6). Thus $M_t^Q = 0$ and the martingale property implies $M_0^Q = 0$ for this $Q \in \mathcal{Q}^*$. This implies condition 5) of Theorem 3.1.4. □

Not every time consistent dynamic risk measure is asymptotically safe, as illustrated by the following example.

Example 3.1.6. *Let P denote Lebesgue measure on the unit interval $\Omega := (0, 1]$, and let $\mathcal{F}_t$ denote the finite σ-field generated by the t-th dyadic partition into the intervals $J_{t,k} := (k2^{-t}, (k+1)2^{-t}]$ $(k = 0, \ldots, 2^t - 1)$. Take a set $A \in \mathcal{F} := \sigma(\cup_{t\geq 0}\mathcal{F}_t)$ such that $P[A] > 0$ and $P[A^c \cap J_{t,k}] \neq 0$ for any dyadic interval, for example*

$$A^c = \bigcup_{t=1}^{\infty} \bigcup_{k=1}^{2^t-1} U_{\varepsilon_t}(k2^{-t})$$

with $\varepsilon_t \in (0, 2^{-2t}]$*. For any* $t \geq 0$ *we fix the same acceptance set*

$$\mathcal{A}_t := \left\{ X \in L^\infty \,\middle|\, X \geq -I_A \right\}.$$

The corresponding conditional convex risk measure ρ_t *is given by*

$$\rho_t(X) = -\operatorname{ess\,sup}\left\{ m \in L^\infty_t \,\middle|\, m \leq X + I_A \right\}.$$

Note that ρ_t *is indeed normalized since* $m \leq 0$ *for any* $m \in L^\infty_t$ *such that* $m \leq I_A$*, due to our assumption that* $P[A^c \cap J_{t,k}] > 0$ *for any atom of the* σ*-field* $\mathcal{F}_t$*. The corresponding penalty function is given by*

$$\alpha_t^{\min}(Q) = E_Q[\, I_A \,|\, \mathcal{F}_t \,].$$

Since $\alpha_0^{\min}(Q) = Q[A]$*, we have* $\mathcal{Q}^* = \mathcal{M}^e(P)$*, and in particular* $\mathcal{Q}^* \neq \emptyset$ *as required in Theorem 2.2.2.*

The sequence $(\rho_t)_{t=0,1,\dots}$ *is time consistent. Indeed,* $\mathcal{A}_{t,t+1} = L^\infty_+(\mathcal{F}_{t+1})$ *for* $t \geq 0$*, and so we have*

$$\mathcal{A}_t = \mathcal{A}_{t+1} = \mathcal{A}_{t+1} + L^\infty_+(\mathcal{F}_{t+1}) = \mathcal{A}_{t+1} + \mathcal{A}_{t,t+1}$$

in accordance with property 2 of Theorem 2.2.2. On the other hand, we have $\rho_t(-I_A) = 0$ *for all* $t = 0, 1, \dots$ *and thus*

$$\rho_\infty(-I_A) = 0 \not\geq I_A,$$

i.e., the sequence $(\rho_t)_{t=0,1,\dots}$ *is not asymptotically safe. In order to illustrate the criteria of Theorem 3.1.4, note that*

$$\bigcap_{t \geq 0} \mathcal{A}_t = \mathcal{A}_0 \neq L^\infty_+,$$

that

$$\alpha_{0,t}^{\min}(Q) \equiv 0 \neq \alpha_0^{\min}(Q),$$

and that

$$\alpha_\infty^{\min}(Q) = \lim_{t \to \infty} \alpha_t^{\min}(Q) = I_A \neq 0.$$

Remark 3.1.7. *Every dynamic conditional* coherent *risk measure that satisfies the conditions of Theorem 2.2.2 is asymptotically safe. Indeed, property 4) of Theorem 3.1.4 is clearly satisfied, since* $\alpha_t^{\min}(Q) = 0$ *for all* $Q \in \mathcal{Q}^*$ *as shown in the proof of Corollary 2.2.8.*

Lemma 3.1.8. *Asymptotic safety holds if the initial risk measure ρ_0 satisfies the condition*

$$\rho_0\left(E_{P^X}[X|\mathcal{F}_t]\right) \leq \rho_0(X) \tag{3.7}$$

for any $X \in L^\infty$, all $t \geq 0$ and for some measure $P^X \approx P$.

Proof. Let us verify that condition (3.7) implies property 2) of Theorem 3.1.4. Indeed, for any $X \in \mathcal{A}_0$ the sequence $X_t := E_{P^X}[X|\mathcal{F}_t] \in L_t^\infty\,(t \geq 0)$ is uniformly bounded and P-a.s. convergent to X. Moreover, $X_t \in \mathcal{A}_{0,t}$ for all $t \geq 0$ since $\rho_0(X_t) \leq \rho_0(X) \leq 0$ due to (3.7). □

Remark 3.1.9. *Condition (3.7) is satisfied for $P^X = P$ if ρ_0 is* law-invariant *w.r.t. P; see Corollary 4.59 in [FS04].*

Let us now return to the question whether the asymptotic capital requirement ρ_∞ is exactly equal to the actual final loss.

Definition 3.1.10. *We say that the sequence $(\rho_t)_{t=0,1,\ldots}$ is* asymptotically precise *if the limit ρ_∞ defined by (3.1) satisfies*

$$\rho_\infty(X) = -X$$

for any $X \in L^\infty$.

The following example shows that the sequence $(\rho_t)_{t=0,1,\ldots}$ may be asymptotically safe without being asymptotically precise.

Example 3.1.11. *In the situation of example 3.1.6 we now define the acceptance sets*

$$\mathcal{A}_t := \left\{ X \in L^\infty \;\middle|\; X \geq 0 \right\}$$

and the corresponding conditional coherent risk measures

$$\rho_t(X) = -\operatorname{ess\,sup}\left\{ m \in L_t^\infty \;\middle|\; m \leq X \right\}.$$

The sequence $(\rho_t)_{t=0,1,\ldots}$ is time consistent and satisfies the conditions of Theorem 2.2.2. Moreover, it is asymptotically safe due to Remark 3.1.7. But it is not asymptotically precise, since the set A defined in example 3.1.6 satisfies $\rho_t(I_A) = 0$ for all $t \geq 0$, hence $\rho_\infty(I_A) = 0 \neq -I_A$.

Let us now formulate a simple sufficient condition for asymptotic precision.

Proposition 3.1.12. *Suppose that the time consistent dynamic risk measure $(\rho_t)_{t=0,1,\dots}$ is asymptotically safe, and that the supremum in the robust representation of the initial risk measure ρ_0 is in fact a maximum, i.e.,*

$$\rho_0(X) = E_{Q^X}[-X] - \alpha_0^{\min}(Q^X) \tag{3.8}$$

for any $X \in L^\infty$ and for some $Q^X \approx P$. Then the sequence $(\rho_t)_{t=0,1,\dots}$ is asymptotically precise.

Proof. Let us fix $X \in L^\infty$. Since we are assuming asymptotic safety, it remains to show $\rho_\infty(X) \leq -X$. Due to time consistency as characterized by property 4) of Theorem 2.2.2, the process

$$U_t^Q := V_t^Q(X) + E_Q[X|\mathcal{F}_t], \qquad t \geq 0,$$

is a non-negative Q-supermartingale for any $Q \in \mathcal{Q}^*$. For $Q = Q^X$, we have $Q \in \mathcal{Q}^*$ and

$$U_0^Q = \rho_0(X) + \alpha_0(Q) + E_Q[X] = 0$$

due to (3.8). This implies $U_t^Q = 0$ for any $t > 0$ and

$$U_\infty^Q = \rho_\infty(X) + \alpha_\infty(Q) + X = 0,$$

hence $\rho_\infty(X) \leq -X$, P-a. s.. □

Remark 3.1.13. *In Proposition 2.9 of [Nau07] it was shown that it is not necessary to require asymptotic safety of the risk measure $(\rho_t)_{t=0,1,\dots}$ in Proposition 3.1.12 to prove asymptotic precision.*

Another sufficient condition for asymptotic safety is given in [Nau07]. It is a stronger form of sensitivity called *hypersensitivity* in [Nau07]. This property appeared in [Pen97] in the context of g-expectations under the name "strong monotonicity".

Definition 3.1.14. *A conditional convex risk measure ρ_t is called* hypersensitive *if for all $Y, X \in L^\infty$*

$$X \geq Y \ \text{ and } \ \rho_t(X) = \rho_t(Y) \implies X = Y \quad P\text{-a.s.}.$$

Obviously hypersensitivity implies sensitivity. For more details on hypersensitivity we refer to Chapter 3 of [Nau07]. In particular it is proved in Proposition 3.12 of [Nau07] that if the dynamic risk measure $(\rho_t)_{t=0,1,\dots}$ is asymptotically safe and ρ_t is hypersensitive for one $t \in \{0, 1, \dots\}$, then $(\rho_t)_{t=0,1,\dots}$ is asymptotically precise. We will give the proof of this assertion in the more general setting of the next section, where time consistency is replaced by prudence.

3.2 Asymptotic properties of prudent risk measures

We consider a one-step prudent dynamic convex risk measure $(\rho_t)_{t=0,1,\dots}$ with infinite time horizon $T=\infty$. We assume that each ρ_t is continuous from above and sensitive.

For $Q \in \mathcal{Q}_\infty$ and $X \in L^\infty$, the process

$$U_t^Q(X) = \rho_t(X) - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \qquad t = 0, 1, \dots$$

is a Q-supermartingale by Theorem 2.4.4. Moreover, the process $(U_t^Q(X))$ is bounded in $L^1(Q)$ for all $Q \in \mathcal{Q}_\infty$ since

$$\sup_t E_Q[\,|U_t^Q(X)|\,] \le ||X||_\infty + E_Q\left[\sum_{k=0}^{\infty} \alpha_{k,k+1}^{\min}(Q)\right] < \infty$$

due to the fact that $|\rho_t(X)| \le ||X||_\infty$ and by definition of the set $\mathcal{Q}_\infty$. Hence the process $(U_t^Q(X))$ is Q-a.s. convergent to some finite limit $U_\infty^Q(X)$.
In addition we have

$$\lim_{t\to\infty} \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q) = \sum_{k=0}^{\infty} \alpha_{k,k+1}^{\min}(Q) < \infty \quad Q\text{-a.s. and in } L^1(Q)$$

due to monotonicity and monotone convergence.
In particular, the limit

$$\rho_\infty(X) := \lim_{t\to\infty} \rho_t(X) = U_\infty^Q(X) + \sum_{k=0}^{\infty} \alpha_{k,k+1}^{\min}(Q) \tag{3.9}$$

exists P-a.s..

As in the case of time consistency we have

Lemma 3.2.1. *The functional $\rho_\infty : L^\infty \to L^\infty$ defined by (3.9) is normalized, monotone, conditionally convex and conditionally cash invariant with respect to $\mathcal{F}_t$ for any $t \ge 0$, and it satisfies*

$$\rho_\infty(X) \ge -X - \operatorname*{ess\,inf}_{Q\in\mathcal{Q}_\infty} (\limsup_t \alpha_t^{\min}(Q)) \quad P\text{-a.s.}. \tag{3.10}$$

Proof. To prove monotonicity, conditional convexity and conditional cash-invariance we can argue as in the proof of Lemma 3.1.1. To prove (3.10) note that due to sensitivity each ρ_t has the robust representation

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{M}^e(P)} \left(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \right), \qquad X \in L^\infty$$

by Corollary 1.2.6. Thus

$$\rho_t(X) \geq E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q)$$

for all $t = 0, 1, \ldots$ and all $Q \in \mathcal{Q}_\infty$. The inequality (3.10) follows by martingale convergence as in Lemma 3.1.1. The only difference is that we have to replace $\alpha_\infty^{\min}(Q)$ by $\limsup_t \alpha_t^{\min}(Q)$ in (3.10), since the limit $\alpha_\infty^{\min}(Q) = \lim_{t\to\infty} \alpha_t^{\min}(Q)$ might not exist in case of one-step prudence. □

In the following propostion we give a simple sufficient condition for asymptotic safety.

Proposition 3.2.2. *Let $(\rho_t)_{t=0,1,\ldots}$ be a one-step prudent dynamic risk measure such that each ρ_t is continuous from above and sensitive. Assume further that there exists $Q \in \mathcal{Q}_\infty$ such that $\alpha_t^{\min}(Q) \to 0$ Q-a.s. with $t \to \infty$. Then $(\rho_t)_{t=0,1,\ldots}$ is asymptotically safe.*

Proof. If there exists $Q \in \mathcal{Q}_\infty$ with $\lim_{t\to\infty} \alpha_t^{\min}(Q) = 0$ Q-a.s., inequality (3.10) of the preceding lemma implies

$$\rho_\infty(X) \geq -X \quad P\text{-a.s.}.$$

□

In the next proposition we give a sufficient condition for asymptotic safety of a prudent dynamic risk measure $(\rho_t)_{t=0,1,\ldots}$. This result is similar to Proposition 3.12 in [Nau07], where it was proved for time consistent dynamic risk measures. It involves hypersensitivity of a conditional convex risk measure as defined in part 2) of Remark 3.1.14. Note that we assume prudence and not just one-step prudence here.

Proposition 3.2.3. *Let $(\rho_t)_{t=0,1,\ldots}$ be an asymptotically safe prudent dynamic risk measure such that each ρ_t is continuous from above and sensitive. Assume further that ρ_s is hypersensitive for one $s \in \{0, 1, \ldots\}$. Then $(\rho_t)_{t=0,1,\ldots}$ is asymptotically precise.*

Proof. Since the sequence $(\rho_t(X))$ is bounded and converges P-a.s. to $\rho_\infty(X)$, the Fatou-property of the conditional risk measure ρ_s implies

$$\rho_s(-\rho_\infty(X)) \leq \liminf_t \rho_s(-\rho_{s+t}(X)) \leq \rho_s(X),$$

where we have used prudence for the last inequality. On the other hand, $\rho_\infty(X) \geq -X$ due to asymptotic safety and hence

$$\rho_s(-\rho_\infty(X)) \geq \rho_s(X)$$

by monotonicity. Thus $\rho_s(-\rho_\infty(X)) = \rho_s(X)$ and hypersensitivity implies

$$\rho_\infty(X) = -X \quad P\text{-a.s.}.$$

□

Chapter 4

Examples

4.1 The entropic dynamic risk measure

Suppose that preferences of some economic agent at time $t \in \{0, 1, \ldots\}$ are characterized by an exponential utility function $u_t(x) = 1 - \exp(-\gamma_t x)$. In contrast to the usual definition of the entropic risk measure we allow the risk aversion to depend both on time and on the available information. More precisely, we assume that γ_t is a bounded $\mathcal{F}_t$-measurable random variable such that

$$\gamma_t > 0 \quad P\text{-a.s. and} \quad \frac{1}{\gamma_t} \in L^\infty(\mathcal{F}_t).$$

Thus the conditional expected utility of a financial position $X \in L^\infty$ at time t is given by the $\mathcal{F}_t$-measurable random variable

$$U_t(X) = E[1 - e^{-\gamma_t X} | \mathcal{F}_t].$$

The set

$$\mathcal{A}_t := \left\{ X \in L^\infty \;\middle|\; U_t(X) \geq U_t(0) \right\} = \left\{ X \in L^\infty \;\middle|\; E[e^{-\gamma_t X} | \mathcal{F}_t] \leq 1 \right\}$$

satisfies the necessary conditions for a convex acceptance set, and hence due to Proposition 1.1.3 we can define a sequence of conditional convex risk measures $(\rho_t)_{t=0,1,\ldots}$ via

$$\begin{aligned} \rho_t(X) &:= \operatorname{ess\,inf} \left\{ Y \in L_t^\infty \;\middle|\; Y + X \in \mathcal{A}_t \right\} \\ &= \operatorname{ess\,inf} \left\{ Y \in L_t^\infty \;\middle|\; E[e^{-\gamma_t X} | \mathcal{F}_t] \leq e^{\gamma_t Y} \right\} \\ &= \frac{1}{\gamma_t} \log E[e^{-\gamma_t X} | \mathcal{F}_t]. \end{aligned} \tag{4.1}$$

We call a risk measure defined via (4.1) a *conditional entropic risk measure* with risk aversion γ_t and the sequence $(\rho_t)_{t=0,1,\dots}$ a *dynamic entropic risk measure* with risk aversion $(\gamma_t)_{t=0,1,\dots}$. In the case of constant risk aversion, these risk measures are also discussed in Section 4 of [DS05] and in Section 5.6 of [CDK06].

It is easy to see that a conditional entropic risk measure is continuous from above and hence representable for all $t \geq 0$. To identify the minimal penalty function in the robust representation (1.4) we will need the notion of *conditional relative entropy.*

Recall that the *relative entropy* of $Q \in \mathcal{M}_1(P)$ with respect to P on the σ-field $\mathcal{F}_t$ is defined as

$$H_t(Q|P) := E_Q[\log Z_t] = E_P[Z_t \log Z_t] \in [0, \infty],$$

where Z_t denotes a density of Q with respect to P on $\mathcal{F}_t$. By Jensen's inequality we have $H_t(Q|P) \geq 0$, with equality iff $Q = P$ on $\mathcal{F}_t$.

Definition 4.1.1. *For $Q \in \mathcal{M}_1(P)$ we define the* conditional relative entropy *of Q with respect to P at time $t \geq 0$ as the $\mathcal{F}_t$-measurable random variable*

$$\begin{aligned}\widehat{H}_t(Q|P) &:= E_Q\left[\log \frac{Z_T}{Z_t} \,\Big|\, \mathcal{F}_t\right] \\ &= E_P\left[\frac{Z_T}{Z_t} \log \frac{Z_T}{Z_t} \,\Big|\, \mathcal{F}_t\right] I_{\{Z_t>0\}}\end{aligned}$$

(note that $Z_t > 0$ Q-a.s.).

If Ω is a polish space, then for all $Q \in \mathcal{P}_t$ there exists a regular conditional probability of Q given $\mathcal{F}_t$, that is a probability kernel $Q_t : \Omega \times \mathcal{F} \to [0, 1]$ such that $Q_t(\cdot, B) = Q[B|\mathcal{F}_t]$ Q-a.s. for all $B \in \mathcal{F}$. In this case the conditional relative entropy can be calculated pointwise as the relative entropy of $Q_t(\omega, \cdot)$ with respect to $P_t(\omega, \cdot)$.

The next lemma is a version of Proposition 4 in [DS05]:

Lemma 4.1.2. *For all $t \geq 0$ the conditional entropic risk measure ρ_t has the robust representations (1.4) with the minimal penalty function*

$$\alpha_t^{\min}(Q) = \frac{1}{\gamma_t} \widehat{H}_t(Q|P), \qquad Q \in \mathcal{P}_t.$$

Proof. To calculate the minimal penalty function we use formula (1.11):

$$\begin{aligned}\alpha_t^{\min}(Q) &= \operatorname*{ess\,sup}_{X\in L^\infty}\left(E_Q[-X|\mathcal{F}_t] - \rho_t(X)\right)\\ &= \operatorname*{ess\,sup}_{X\in L^\infty}\left(E_Q[-X|\mathcal{F}_t] - \frac{1}{\gamma_t}\log E_P[e^{-\gamma_t X}|\mathcal{F}_t]\right)\\ &= \frac{1}{\gamma_t}\operatorname*{ess\,sup}_{Y\in L^\infty}\Big(E_Q[-Y|\mathcal{F}_t] - \log E_P[e^Y|\mathcal{F}_t]\Big), \qquad Q\in\mathcal{P}_t.\end{aligned}$$

Now we use the conditional version of a well-known variational formula for relative entropy:

$$\operatorname*{ess\,sup}_{Y\in L^\infty}\Big(E_Q[-Y|\mathcal{F}_t] - \log E_P[e^Y|\mathcal{F}_t]\Big) = \widehat{H}_t(Q|P)$$

for $Q\in\mathcal{P}_t$. This follows as in the unconditional case; see, e.g., Lemma 3.29 in [FS04] and Lemma 2 in [DS05]. □

In order to identify the one-step minimal penalty functions we introduce the "one-step" conditional entropy

$$\widehat{H}_{t,t+1}(Q|P) := E_Q\left[\log\frac{Z_{t+1}}{Z_t}\,\Big|\,\mathcal{F}_t\right],$$

i.e., the conditional entropy at time t if Q and P are regarded as measures on $\mathcal{F}_{t+1}$. Then Lemma 4.1.2 applied to ρ_t restricted to $\mathcal{F}_{t+1}$ implies

$$\alpha_{t,t+1}^{\min}(Q) = \frac{1}{\gamma_t}\widehat{H}_{t,t+1}(Q|P), \qquad Q\in\mathcal{P}_t.$$

Remark 4.1.3. *If the risk aversion γ_t is a constant, Lemma 4.1.2 can be seen as a special case of a more general representation result for an* optimized certainty equivalent *in the sense of Ben-Tal and Teboulle; cf. [BTT05]. A conditional version of the optimized certainty equivalent was studied in [Dra06]. It is given for a concave utility function u_t by*

$$S_t^{u_t}(X) := \operatorname*{ess\,sup}_{\eta_t\in L_t^\infty}\left(\eta_t + E\left[u_t\left(X-\eta_t\right)\Big|\mathcal{F}_t\right]\right), \quad X\in L^\infty.$$

It was shown in Theorem 4.8 of [Dra06], that an optimized certainty equivalent has a robust representation in terms of penalty function given by the conditional φ-divergence *with respect to $\varphi(x) = -u_t^*(-x)$, where u_t^* denotes the conjugate of the utility function. If φ is a proper closed convex function, the $\mathcal{F}_t$-conditional φ-divergence of a probability measure Q with respect to P is defined as*

$$I_\varphi(Q|P) = \begin{cases} E\left[\varphi\left(\frac{Z_T}{Z_t}\right)I_{\{Z_t>0\}}\Big|\mathcal{F}_t\right] & \text{if } Q\ll P\\ +\infty & \text{else,}\end{cases}$$

where Z_t denotes the density of Q with respect to P on $\mathcal{F}_t$.
In fact the entropic risk measure with minus sign can be identified as the optimized certainty equivalent for the exponential utility function $u_t(x) = \frac{1-e^{-\gamma_t x}}{\gamma_t}$. Then we have the conjugate $\varphi(x) = -u_t^*(-x) = \frac{1}{\gamma_t} x \log x$, and the conditional φ-divergence of the conjugate is just the conditional relative entropy; cf. Example 4.3.1 in [Dra06].

Time consistency properties of the dynamic entropic risk measure are completely determined by the adapted process of risk aversion $(\gamma_t)_{t=0,1,\ldots}$, as we will show in the next proposition. It is a more general version of Proposition 3.13 in [Dra06], where a similar result is shown for constant γ_t, $t = 0, 1, \ldots$. The proof of the "only if" part of the Proposition 4.1.4 is due to Samuel Drapeau.

Proposition 4.1.4. *Let $(\rho_t)_{t=0,1,\ldots}$ be the dynamic entropic risk measure with risk aversion given by an adapted process $(\gamma_t)_{t=0,1,\ldots}$ such that $\gamma_t \in L^\infty(\mathcal{F}_t)$, $1/\gamma_t \in L^\infty(\mathcal{F}_t)$, $\gamma_t > 0$ P-a.s.. Then the following assertions hold:*

1. *$(\rho_t)_{t=0,1,\ldots}$ is middle rejection consistent if $\gamma_t \geq \gamma_{t+1}$ P-a.s. for all $t = 0, 1, \ldots$;*
2. *$(\rho_t)_{t=0,1,\ldots}$ is middle acceptance consistent if $\gamma_t \leq \gamma_{t+1}$ P-a.s. for all $t = 0, 1, \ldots$;*
3. *$(\rho_t)_{t=0,1,\ldots}$ is time consistent if $\gamma_t = \gamma_0 \in \mathbb{R}$ P-a.s. for all $t = 0, 1, \ldots$.*

Moreover, the assertions 1), 2) and 3) hold with "if and only if", if $\gamma_t \in \mathbb{R}$ for all t, or if the filtration $(\mathcal{F}_t)_{t=0,1,\ldots}$ is rich enough in the sense that for all t and for all $B \in \mathcal{F}_t$ such that $P[B] > 0$ there exists $A \subset B$ such that $A \notin \mathcal{F}_t$ and $P[A] > 0$.

Proof. Fix $t \in \{0, 1, \ldots\}$ and $X \in L^\infty$. Then

$$\begin{aligned}\rho_t(-\rho_{t+1}(X)) &= \frac{1}{\gamma_t} \log \left(E \left[\exp \left\{ \frac{\gamma_t}{\gamma_{t+1}} \log \left(E \left[e^{-\gamma_{t+1} X} | \mathcal{F}_{t+1} \right] \right) \right\} \Big| \mathcal{F}_t \right] \right) \\ &= \frac{1}{\gamma_t} \log \left(E \left[E \left[e^{-\gamma_{t+1} X} | \mathcal{F}_{t+1} \right]^{\frac{\gamma_t}{\gamma_{t+1}}} \Big| \mathcal{F}_t \right] \right). \qquad (4.2)\end{aligned}$$

Thus $\rho_t(-\rho_{t+1}) = \rho_t$ if $\gamma_t = \gamma_{t+1}$ and this proves time consistency. One-step middle rejection (resp. acceptance) consistency follow by the generalized Jensen inequality that we will prove in the next Lemma 4.1.5. We apply this inequality at time $t+1$ to the bounded random variable $Y := e^{-\gamma_{t+1}X}$ and the $\mathcal{B}((0,\infty)) \otimes \mathcal{F}_{t+1}$-measurable function

$$u : (0,\infty) \times \Omega \to \mathbb{R}, \qquad u(x,\omega) := x^{\frac{\gamma_t(\omega)}{\gamma_{t+1}(\omega)}}.$$

Note that $u(\cdot,\omega)$ is convex if $\gamma_t(\omega) \geq \gamma_{t+1}(\omega)$ and concave if $\gamma_t(\omega) \leq \gamma_{t+1}(\omega)$. Moreover, $u(X,\cdot) \in L^\infty$ for all $X \in L^\infty$ and $u(\cdot,\omega)$ is differentiable on $(0,\infty)$ with

$$|u'(x,\cdot)| = \frac{\gamma_t}{\gamma_{t+1}} x^{\frac{\gamma_t}{\gamma_{t+1}}-1} \leq a x^b \quad P\text{-a.s.}$$

for some $a,b \in \mathbb{R}$ if $\gamma_t \geq \gamma_{t+1}$ P-a.s. due to our assumption $\frac{\gamma_t}{\gamma_{t+1}} \in L^\infty$. For $\gamma_t \leq \gamma_{t+1}$ P-a.s. we obtain

$$|u'(x,\cdot)| = \frac{\gamma_t}{\gamma_{t+1}} x^{\frac{\gamma_t}{\gamma_{t+1}}-1} \leq a \frac{1}{x^c} \quad P\text{-a.s.}$$

for some $a,c \in \mathbb{R}$. Thus the assumptions of Lemma 4.1.5 are satisfied and we obtain

$$\rho_t(-\rho_{t+1}) \leq \rho_t \quad \text{if} \quad \gamma_t \geq \gamma_{t+1} \quad P\text{-a.s. for all } t = 0,1,\ldots$$

and

$$\rho_t(-\rho_{t+1}) \geq \rho_t \quad \text{if} \quad \gamma_t \leq \gamma_{t+1} \quad P\text{-a.s. for all } t = 0,1,\ldots.$$

Moreover, $\gamma_t \geq \gamma_{t+1}$ (resp. $\leq$) for all t implies $\gamma_t \geq \gamma_{t+s}$ (resp $\leq$) for all $t,s \in \{0,1,\ldots\}$ and hence by the same reasoning as above we obtain $\rho_t(-\rho_{t+s}) \leq \rho_t$ (resp $\geq$) in (4.2). Thus for the entropic dynamic risk measures one-step rejection (resp acceptance) consistency imply general rejection (resp. acceptance) consistency.
The "only if" direction for constant γ_t follows by the classical Jensen inequality.
Now we assume that the sequence $(\rho_t)_{t=0,1,\ldots}$ is middle rejection consistent and our assumption on the filtration $(\mathcal{F}_t)$ holds. We will show that the sequence (γ_t) is decreasing in this case. Indeed, for $t \in \{0,1,\ldots\}$ consider $B := \{\gamma_t < \gamma_{t+1}\}$ and suppose that $P[B] > 0$. Our assumption on the filtration allows us to choose $A \subset B$ with $P[B] > P[A] > 0$ and $A \notin \mathcal{F}_{t+1}$. We define a random variable $X := -xI_A$ for some $x > 0$. Then

$$\begin{aligned}
\rho_t(-\rho_{t+1}(X)) &= \frac{1}{\gamma_t} \log\left(E\left[\exp\left(\frac{\gamma_t}{\gamma_{t+1}} \log\left(E\left[e^{\gamma_{t+1}xI_A}\middle|\mathcal{F}_{t+1}\right]\right)\right)\middle|\mathcal{F}_t\right]\right) \\
&= \frac{1}{\gamma_t} \log\left(E\left[\exp\left(\frac{\gamma_t}{\gamma_{t+1}} I_B \log\left(E\left[e^{\gamma_{t+1}xI_A}\middle|\mathcal{F}_{t+1}\right]\right)\right)\middle|\mathcal{F}_t\right]\right),
\end{aligned}$$

where we have used that $A \subset B$. Setting

$$Y := E\left[e^{\gamma_{t+1}xI_A}\middle|\mathcal{F}_{t+1}\right] = e^{\gamma_{t+1}x}P\left[A|\mathcal{F}_{t+1}\right] + P\left[A^c|\mathcal{F}_{t+1}\right]$$

and bringing $\frac{\gamma_t}{\gamma_{t+1}}$ inside of the logarithm we obtain

$$\rho_t\left(-\rho_{t+1}\left(X\right)\right) = \frac{1}{\gamma_t}\log\left(E\left[\exp\left(I_B\log\left(Y^{\frac{\gamma_t}{\gamma_{t+1}}I_B}\right)\right)\middle|\mathcal{F}_t\right]\right). \tag{4.3}$$

The function $x \mapsto x^{\gamma_t(\omega)/\gamma_{t+1}(\omega)}$ is strictly concave for almost each $\omega \in B$, and thus

$$\begin{aligned} Y^{\frac{\gamma_t}{\gamma_{t+1}}} &= \left(e^{\gamma_{t+1}x}P\left[A|\mathcal{F}_{t+1}\right] + \left(1 - P\left[A|\mathcal{F}_{t+1}\right]\right)\right)^{\frac{\gamma_t}{\gamma_{t+1}}} \\ &\geq e^{\gamma_t x}P\left[A|\mathcal{F}_{t+1}\right] + \left(1 - P\left[A|\mathcal{F}_{t+1}\right]\right) \qquad P\text{-a.s on } B \end{aligned} \tag{4.4}$$

with strict inequality on the set

$$C := \left\{P\left[A|\mathcal{F}_{t+1}\right] > 0\right\} \cap \left\{P\left[A|\mathcal{F}_{t+1}\right] < 1\right\} \cap B.$$

Our assumptions $P[A] > 0$, $A \subset B$ and $A \notin \mathcal{F}_{t+1}$ imply $P[C] > 0$ and using

$$e^{\gamma_t x}P\left[A|\mathcal{F}_{t+1}\right] + \left(1 - P\left[A|\mathcal{F}_{t+1}\right]\right) = E\left[e^{\gamma_t x I_A}|\mathcal{F}_{t+1}\right] \tag{4.5}$$

we obtain from (4.3), (4.4) and (4.5)

$$\rho_t\left(-\rho_{t+1}\left(X\right)\right) \geq \frac{1}{\gamma_t}\log\left(E\left[\exp\left(I_B\log\left(E\left[e^{\gamma_t x I_A}|\mathcal{F}_{t+1}\right]\right)\right)\middle|\mathcal{F}_t\right]\right) \tag{4.6}$$

with the strict inequality on some set of positive probability due to strict monotonicity of the exponential and the logarithmic functions. For the right hand side of (4.6) we have

$$\begin{aligned} &\frac{1}{\gamma_t}\log\left(E\left[\exp\left(I_B\log\left(E\left[e^{\gamma_t x I_A}|\mathcal{F}_{t+1}\right]\right)\right)\middle|\mathcal{F}_t\right]\right) = \\ &= \frac{1}{\gamma_t}\log\left(E\left[I_B E\left[e^{\gamma_t x I_A}|\mathcal{F}_{t+1}\right] + I_{B^c}\middle|\mathcal{F}_t\right]\right) \\ &= \frac{1}{\gamma_t}\log\left(E\left[\exp\left(\gamma_t x I_A\right)\middle|\mathcal{F}_t\right]\right) \\ &= \rho_t\left(X\right), \end{aligned}$$

where we have used $A \subset B$ and $B \in \mathcal{F}_{t+1}$. This is a contradiction to middle rejection consistency of (ρ_t), and we conclude that $\gamma_{t+1} \leq \gamma_t$ P-a.s. for all t. The proof in the case of middle acceptance consistency follows in the same manner. And since time consistent dynamic risk measure is both middle acceptance and middle rejection consistent, we obtain $\gamma_{t+1} = \gamma_t$ P-a.s. for all t in this case. □

The following lemma concludes the proof of Proposition 4.1.4.

Lemma 4.1.5. *Assume that* $(\Omega, \mathcal{F}, P)$ *is a probability space and* $\mathcal{F}_t \subseteq \mathcal{F}$ *a* σ*-field. Let* $I \subseteq \mathbb{R}$ *be an open interval and*

$$u \;:\; I \times \Omega \;\rightarrow\; \mathbb{R}$$

be a $\mathcal{B}(I) \otimes \mathcal{F}_t$*-measurable function such that* $u(\cdot, \omega)$ *is convex (resp. concave) and finite on* I *for* P*-a.e.* ω*. Assume further that*

$$|u'_+(x, \cdot)| \leq c(x) \quad P\text{-a.s. with some } c(x) \in \mathbb{R} \text{ for all } x \in I,$$

where $u'_+(\cdot, \omega)$ *denotes the right-hand derivative of* $u(\cdot, \omega)$*.*
Let $X \;:\; \Omega \;\rightarrow\; [a, b] \subseteq I$ *be an* $\mathcal{F}$*-measurable bounded random variable such that* $E\left[\,|u(X, \;)|\,\right] < \infty$*. Then*

$$E\left[\, u(X, \;) \,|\, \mathcal{F}_t \,\right] \geq u\left(E[X|\mathcal{F}_t], \;\right) \quad (\textit{resp} \leq) \quad P\text{-a.s.}.$$

Proof. We will prove the assertion for the convex case; the concave follows in the same manner. Fix $\omega \in \Omega$ such that $u(\cdot, \omega)$ is convex. Due to convexity we obtain for all $x_0 \in I$

$$u(x, \omega) \geq u(x_0, \omega) + u'_+(x_0, \omega)(x - x_0) \quad \text{for all} \quad x \in I.$$

Take $x_0 = E[X|\mathcal{F}_t](\omega)$ and $x = X(\omega)$. Then

$$u(X(\omega), \omega) \geq u(E[X|\mathcal{F}_t](\omega), \omega) + u'_+(E[X|\mathcal{F}_t](\omega), \omega)(X(\omega) - E[X|\mathcal{F}_t](\omega)) \tag{4.7}$$

for P-almost all $\omega \in \Omega$. Note further that $\mathcal{B}(I) \otimes \mathcal{F}_t$-measurability of u implies $\mathcal{B}(I) \otimes \mathcal{F}_t$-measurability of u_+. Thus

$$\omega \;\rightarrow\; u(E[X|\mathcal{F}_t](\omega), \omega) \quad \text{and} \quad \omega \;\rightarrow\; u'_+(E[X|\mathcal{F}_t](\omega), \omega)$$

are $\mathcal{F}_t$-measurable random variables, and $\omega \rightarrow u(X(\omega), \omega)$ is $\mathcal{F}$-measurable. Moreover, due to our assumption on X there are constants $a, b \in I$ such that $a \leq E[X|\mathcal{F}_t] \leq b$ P-a.s.. Since $u'_+(\cdot, \omega)$ is increasing by convexity, we obtain using our assumption on the boundedness of u'_+

$$-c(a) \leq u'_+(a, \omega) \leq u'_+(E[X|\mathcal{F}_t], \omega) \leq u'_+(b, \omega) \leq c(b),$$

i.e. $u'_+(E[X|\mathcal{F}_t], \;)$ is bounded. Since $E\left[\,|u(X, \;)|\,\right] < \infty$ we can build conditional expectation on the both sides of (4.7) and we obtain

$$\begin{aligned} E[\, u(X, \;) \,|\, \mathcal{F}_t \,] &\geq E\left[\, u(E[X|\mathcal{F}_t], \;) + u'_+(E[X|\mathcal{F}_t], \;)(X - E[X|\mathcal{F}_t]) \,|\, \mathcal{F}_t \,\right] \\ &= E\left[\, u(E[X|\mathcal{F}_t], \;) \,|\, \mathcal{F}_t \,\right] \quad P\text{-a.s.}, \end{aligned}$$

where we have used $\mathcal{F}_t$-measurability of $u(E[X|\mathcal{F}_t],)$ and of $u'_+(E[X|\mathcal{F}_t],)$ and the boundedness of $u'_+(E[X|\mathcal{F}_t],)$. This proves our claim. □

In the sequel we will illustrate the results of Chapter 2 and Chapter 3 by applying them to the time consistent and to the prudent versions of the dynamic entropic risk measure.

We first focus on *time consistency.* Consider the dynamic entropic risk measure $(\rho_t)_{t=0,1,\dots}$ with constant risk aversion γ for all t. Then $(\rho_t)_{t=0,1,\dots}$ is time consistent by Proposition 4.1.4 and the set

$$\mathcal{Q}^* = \left\{ Q \in \mathcal{M}^e(P) \;\middle|\; H_T(Q|P) < \infty \right\}$$

is obviously not empty. Thus we could apply Theorem 2.2.2 and Theorem 3.1.4. Asymptotic precision follows directly by martingale convergence w.r.t. P. But let us rather illustrate the main criteria for time consistency and asymptotic precision by verifying them directly in our present case.

Recall that due to Lemma 4.1.2 we have

$$\alpha_t^{\min}(Q) = \frac{1}{\gamma}\widehat{H}_t(Q|P) \quad \text{and} \quad \alpha_{t,t+1}^{\min}(Q) = \frac{1}{\gamma}\widehat{H}_{t,t+1}(Q|P)$$

for all $t = 0, 1, \dots$ and $Q \in \mathcal{P}_t$. Clearly,

$$\widehat{H}_t(Q|P) = \widehat{H}_{t,t+1}(Q|P) + E_Q[\widehat{H}_{t+1}(Q|P)|\mathcal{F}_t],$$

and this illustrates property 3) in Theorem 2.2.2.
In the next theorem we prove directly the supermartingale property 4) of the process

$$V_t^Q(X) = \rho_t(X) + \alpha_t^{\min}(Q), \qquad t = 0, 1, \dots$$

in the entropic case. Moreover, we clarify the structure of the corresponding Doob decomposition, i.e., we identify the increasing predictable process $(A_t^Q(X))$ such that

$$V_t^Q(X) - A_t^Q(X), \qquad t = 0, 1, \dots$$

is a martingale under Q.

Theorem 4.1.6. *Consider the case of constant risk aversion where $\gamma_t = \gamma$ P-a.s. for all t. Then for any $Q \in \mathcal{M}_1(P)$ such that $H_T(Q|P) < \infty$ and for any $X \in L^\infty$ the process*

$$V_t^Q(X) = \frac{1}{\gamma}\log E_P[e^{-\gamma X}|\mathcal{F}_t] + \frac{1}{\gamma}\widehat{H}_t(Q|P), \qquad t = 0, 1, \dots$$

is a supermartingale under Q. Its Doob decomposition is given by the predictable increasing process

$$A_t^Q(X) := \frac{1}{\gamma}\sum_{s=0}^{t-1}\widehat{H}_{s,s+1}(Q|P^X), \qquad t = 0, 1, \ldots, \tag{4.8}$$

where $P^X \in \mathcal{M}^e(P)$ is defined by

$$\frac{dP^X}{dP} := \frac{e^{-\gamma X}}{E_P[e^{-\gamma X}]}.$$

The process $(V_t^Q(X))_{t=0,1,\ldots}$ is in fact a martingale iff $Q = P^X$. Moreover, $V_T^Q(X) = -X$ for $T < \infty$, and

$$\lim_{t\to\infty}\rho_t(X) = \lim_{t\to\infty}V_t^Q(X) = -X \quad Q\text{-a.s. and in } L^1(Q)$$

for $T = \infty$. In particular,

$$\lim_{t\to\infty}\alpha_t^{\min}(Q) = \lim_{t\to\infty}\frac{1}{\gamma}\widehat{H}_t(Q|P) = 0 \quad Q\text{-a.s. and in } L^1(Q).$$

Proof. Since $P^X \approx P$, we can write

$$\begin{aligned}\widehat{H}_t(Q|P) &= \widehat{H}_t(Q|P^X) + E_Q\left[\log\frac{e^{-\gamma X}}{E_P[e^{-\gamma X}|\mathcal{F}_t]}\,\Big|\,\mathcal{F}_t\right]\\ &= \widehat{H}_t(Q|P^X) - \gamma E_Q[X|\mathcal{F}_t] - \gamma\rho_t(X),\end{aligned}$$

and this implies

$$V_t^Q(X) = E_Q[-X|\mathcal{F}_t] + \frac{1}{\gamma}\widehat{H}_t(Q|P^X).$$

Lemma 4.1.7, applied to P^X instead of P, shows that $(V_t^Q(X))_{t=0,1,\ldots}$ is a supermartingale under Q which converges to $-X$ Q-a.s and in $L^1(Q)$. It also shows that the increasing predictable process $(A_t^Q(X))$ defined by (4.8) is such that

$$V_t^Q(X) - A_t^Q(X), \qquad t = 0, 1, \ldots$$

is a Q-martingale. In particular, $(V_t^Q(X))_{t=0,1,\ldots}$ is a Q-martingale if and only if $\widehat{H}_{t,t+1}(Q|P^X) = 0$ Q-a.s. for all $t \geq 0$, and this is the case iff $Q = P^X$ on $\mathcal{F} = \mathcal{F}_T$. □

The following lemma was used in the preceding proof.

Lemma 4.1.7. *For any $Q \in \mathcal{M}_1(P)$ such that $H_T(Q|P) < \infty$, the process of conditional relative entropies*

$$\widehat{H}_t(Q|P), \quad t = 0, 1, \ldots$$

is a supermartingale under Q. It is in fact a potential in the sense that $\widehat{H}_T(Q|P) = 0$ for $T < \infty$ and

$$\lim_{t\to\infty} \widehat{H}_t(Q|P) = 0 \quad Q\text{-a.s. and in } L^1(Q) \tag{4.9}$$

for $T = \infty$. Its Doob decomposition is given by the predictable increasing process

$$A_t := \sum_{s=0}^{t-1} \widehat{H}_{s,s+1}(Q|P), \qquad t = 0, 1, \ldots, \tag{4.10}$$

i.e., the process $\widehat{H}_t(Q|P) + A_t$, $t \geq 0$ is a martingale under Q.

Proof. We have

$$\begin{aligned} \widehat{H}_{t+1}(Q|P) &= E_Q\left[\log \frac{Z_T}{Z_{t+1}} \,\Big|\, \mathcal{F}_{t+1}\right] \\ &= E_Q\left[\log \frac{Z_T}{Z_t} \,\Big|\, \mathcal{F}_{t+1}\right] - \log \frac{Z_{t+1}}{Z_t}, \end{aligned}$$

hence

$$E_Q[\, \widehat{H}_{t+1}(Q|P) \,|\, \mathcal{F}_t\,] = \widehat{H}_t(Q|P) - \widehat{H}_{t,t+1}(Q|P).$$

Since $\widehat{H}_{t,t+1}(Q|P) \geq 0$ Q-a.s. by Jensen's inequality, it follows that $(\widehat{H}_t(Q|P)$ is a supermartingale under Q, and that the predictable increasing process in its Doob decomposition is given by (4.10). Moreover, (4.9) follows from

$$H_T(Q|P) = H_t(Q|P) + E_Q[\widehat{H}_t(Q|P)],$$

since $\lim_{t\to\infty} H_t(Q|P) = H_T(Q|P)$. Indeed, we have $H_t(Q|P) \leq H_T(Q|P)$ by Jensen's inequality, and the convergence follows by Fatou's lemma applied to the P-a.s. convergent sequence $(u(Z_t))_{t=0,1,\ldots}$ with $u(x) = x \log x$. □

Now we assume that the sequence $(\gamma_t)_{t=0,1,\ldots}$ is decreasing in time. The corresponding dynamic entropic risk measure $(\rho_t)_{t=0,1,\ldots}$ defined by

$$\rho_t(X) = \frac{1}{\gamma_t} \log E[e^{-\gamma_t X}|\mathcal{F}_t], \quad t = 0, 1, \ldots$$

is prudent due to Proposition 4.1.4. Moreover, each ρ_t is sensitive and even hypersensitive due to strict monotonicity of the exponential and logarithmic functions. And since $\alpha^{\min}_{t,t+1}(P) = 0$ P-a.s. for all t, condition (2.30) is satisfied. Thus all the characterizations 1)-5) of Theorem 2.4.4 are equivalent.

To illustrate property 4) note that

$$\begin{aligned}\alpha^{\min}_t(Q) &= \frac{1}{\gamma_t} E_Q\left[\log \frac{Z_T}{Z_t} | \mathcal{F}_t\right] \\ &= \frac{1}{\gamma_t}\widehat{H}_{t,t+1}(Q|P) + \frac{1}{\gamma_t} E_Q\left[\widehat{H}_{t+1}(Q|P)|\mathcal{F}_t\right] \\ &= \alpha^{\min}_{t,t+1}(Q) + E_Q\left[\frac{\gamma_{t+1}}{\gamma_t}\alpha^{\min}_{t+1}(Q)|\mathcal{F}_t\right] \end{aligned} \tag{4.11}$$

for all $t = 0, 1, \ldots$ and all $Q \in \mathcal{M}^e(P)$.

Remark 4.1.8. *Equality (4.11) provides an alternative proof of assertion 1) of Lemma 4.1.4. Indeed, if $\gamma_{t+1} \le \gamma_t$ P-a.s., (4.11) implies that*

$$\alpha^{\min}_t(Q) \le \alpha^{\min}_{t,t+1}(Q) + E_Q\left[\alpha^{\min}_{t+1}(Q)|\mathcal{F}_t\right], \tag{4.12}$$

and this is equivalent to middle rejection consistency by Theorem 2.4.4. Moreover, if the risk aversion process (γ_t) is predictable, (4.11) shows that the inequality (4.12) holds iff $\gamma_{t+1} \le \gamma_t$ P-a.s.. This proves that if (γ_t) is predictable, the entropic dynamic risk measure $(\rho_t)_{t=0,1,\ldots}$ is prudent iff the sequence of risk aversions (γ_t) is decreasing in time. In contrast to Proposition 4.1.4 we do not need any additional assumption on the filtration in this case.

Property 5) of Theorem 2.4.4 provides the following corollary.

Corollary 4.1.9. *If the sequence (γ_t) is decreasing in time, the process*

$$\begin{aligned} U^Q_t(X) &= \rho_t(X) - \sum_{k=0}^{t-1} \alpha^{\min}_{k,k+1}(Q) \\ &= \frac{1}{\gamma_t}\log E\left[e^{-\gamma_t X}|\mathcal{F}_t\right] - \sum_{k=0}^{t-1}\frac{1}{\gamma_k}\widehat{H}_{k,k+1}(Q|P), \quad t = 0, 1, \ldots \end{aligned}$$

is a Q-supermartingale for all $X \in L^\infty$ and all $Q \in \mathcal{Q}_T$, where

$$\mathcal{Q}_T = \left\{ Q \in \mathcal{M}^e(P) \;\Big|\; E_Q\left[\sum_{k=0}^{\infty}\frac{1}{\gamma_k}\widehat{H}_{k,k+1}(Q|P)\right] < \infty \right\}.$$

The converse implication is true if the filtration is rich enough in the sense of Proposition 4.1.4.

Proof. The proof is a combination of Theorem 2.4.4 and Proposition 4.1.4. □

For $T = \infty$ we can discuss the asymptotic properties of the prudent entropic dynamic risk measure $(\rho_t)_{t=0,1,...}$. Since $\alpha_t^{\min}(P) = 0$ P-a.s. for all t, Proposition 3.2.2 implies asymptotic safety of (ρ_t). More directly, asymptotic safety follows from the Jensen inequality

$$\rho_t(X) = \frac{1}{\gamma_t} \log E\left[e^{-\gamma_t X} | \mathcal{F}_t\right] \geq E[-X|\mathcal{F}_t]$$

combined with the martingale convergence theorem. Moreover, by applying the Blackwell-Dubins modification of the martingale convergence theorem we obtain

$$\lim_{t\to\infty} \frac{1}{\gamma_t} \log E\left[e^{-\gamma_t X} | \mathcal{F}_t\right] = \frac{1}{\gamma_\infty} \log E\left[e^{-\gamma_\infty X} | \mathcal{F}\right] = -X \quad P\text{-a.s.}$$

at least for $\gamma_\infty := \lim_{t\to\infty} \gamma_t > 0\, P$-a.s.. From the general point of view asymptotic precision follows from Proposition 3.2.3, since ρ_0 is hypersensitive. Thus

$$\lim_{t\to\infty} \frac{1}{\gamma_t} \log E\left[e^{-\gamma_t X} | \mathcal{F}_t\right] = -X \quad P\text{-a.s.}$$

for all $X \in L^\infty$ if the sequence (γ_t) is decreasing.

Assume now that $T < \infty$. If the risk aversion (γ_t) is constant over time, then by Proposition 2.5.2 the time consistent entropic dynamic risk measure $(\rho_t)_{t=0,...,T}$ provides an "optimal hedge" in the sense that the risk process $(\rho_t(X)_{t=0,...,T}$ is the smallest process that is sustainable with respect to (ρ_t) and covers the final loss $-X$. If the risk aversion (γ_t) is not constant, Proposition 2.5.2 implies that we might do better by using the time consistent dynamic risk measure $(\widetilde{\rho}_t)_{t=0,...,T}$ defined via (2.50)

$$\widetilde{\rho}_T(X) := -X, \quad \widetilde{\rho}_t(X) = \frac{1}{\gamma_t} \log E\left[e^{-\gamma_t \widetilde{\rho}_{t,t+1}(X)} | \mathcal{F}_t\right], \quad t = 0, \ldots, T-1,$$

thus iterating the computation in (4.2); see also Example 3.3.2 in [CK06]. However, $(\widetilde{\rho}_t)$ is in general not entropic.

4.2 Hedging under constraints

In this section we consider a model for a financial market with convex trading constraints. In this model we call a position X acceptable if it can be hedged by means of some self-financing admissible strategy in the market. We define the risk of a position X at time t as the minimal investment needed to make X acceptable, i.e. $\rho_t(X)$ is a superhedging price of a European claim X at time t under the given constraints. We will show that this definition leads to a time consistent dynamic convex risk measure. Theorem 2.2.2 provides a robust representation of this risk measure and identifies it as the upper Snell envelope under constraints; cf. Section 9.3 of [FS04]. Moreover, applying Theorem 2.5.4 we characterize the sustainability in this model: Proposition 4.2.10 shows that any bounded process is dominated by a value process of some admissible strategy iff it has a certain supermartingale property. This result is known as the optional decomposition under constraints, cf. Theorem 3.1 in [FK97] for continuous time and Theorem 9.20 in [FS04] for discrete time. Thus the latter theorem is a special case of our more general discussion of sustainability in Section 2.5 and Theorem 2.5.4. Moreover, using Proposition 2.5.2 we identify the risk process arising from the dynamic convex risk measure as the "optimal hedge" in the sense that it is the smallest process, that can be financed by means of some admissible strategy and covers the final loss.

The results we obtain by identifying the superhedging price process under constraints as a time consistent dynamic risk measure are already well known, since the market model has been studied before, cf. e.g. [FK97] for continuous time and Chapter 9 of [FS04] for discrete time. But the characterizations of the superhedge price from this particular model provided the intuition for the general case of a dynamic risk measure. Thus we believe that it is interesting to see how the results from [FS04] can be recovered from the general results we have obtained in Chapter 2.

We use here the setting and notation from Chapter 9 of [FS04]. We consider a discrete time market model in which $d+1$ assets are priced at times $t = 1, \dots, T$. We assume that the time horizon T is finite. The price process of the assets is modelled by an adapted stochastic process $(S_t^0, \dots, S_t^d)$, $t = 0, \dots, T$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, P)$ with $\mathcal{F}_0 = \{\Omega, \emptyset\}$, $\mathcal{F} = \mathcal{F}_T$. We assume further that $S_t^0 > 0$ P-a.s. for all t and using it as a numeraire we switch to the discounted price process $(X_t^1, \dots, X_t^d)$ with $X_t^i = S_t^i / S_t^0$ for $t = 0, \dots, T$ and $i = 1, \dots, d$. The portfolio constraints are modelled by a set $\mathcal{S}$ of d-dimensional predictable processes, viewed as admissible investment strategies into risky assets. We will assume that $\mathcal{S}$ satisfies the following conditions:

1. $0 \in \mathcal{S}$.

2. $\mathcal{S}$ is predictably convex: If $\xi, \eta \in \mathcal{S}$ and h is a predictable process with $0 \le h \le 1$, then the process $h_t\xi_t + (1-h_t)\eta_t$ $(t = 1, \dots, T)$ belongs to $\mathcal{S}$.

3. For each $t \in \{1, \dots, T\}$ the set $\mathcal{S}_t := \left\{ \xi_t \,\middle|\, \xi \in \mathcal{S} \right\}$ is closed in $L^0(\Omega, \mathcal{F}_{t-1}, P, \mathbb{R}^d)$ with respect to almost sure convergence.

4. Non-redundancy: For all $t \in \{1, \dots, T\}$ and $\xi_t \in \mathcal{S}_t$, $\xi_t(X_t - X_{t-1}) = 0$ P-a.s. implies $\xi_t = 0$ P-a.s. .

The first two conditions have a clear economic interpretation, whereas the last two are more of a technical nature, they are needed for the proof of "fundamental theorem of asset pricing" in this setting. We refer to Chapter 9 of [FS04] for the details and examples for the set $\mathcal{S}$.

Let $\bar{\mathcal{S}}$ denote the set of all self-financing trading strategies $\bar{\xi} = (\xi^0, \xi)$ which arise from an investment strategy $\xi \in \mathcal{S}$, i.e.,

$$\bar{\mathcal{S}} := \left\{ \bar{\xi} = (\xi^0, \xi) \,\middle|\, \bar{\xi} \text{ is self-financing and } \xi \in \mathcal{S} \right\}.$$

We call a position $X \in L^\infty(\mathcal{F}_T)$ acceptable at time $t \in \{0, \dots, T\}$ if it can be hedged with some admissible strategy $\xi \in \bar{\mathcal{S}}$ at no additional cost. Thus, we obtain the class of the sets of acceptable positions

$$\mathcal{A}_t^{\mathcal{S}} := \left\{ X \in L^\infty(\mathcal{F}_T) \,\middle|\, \exists \xi \in \mathcal{S} \,:\, \sum_{k=t+1}^{T} \xi_k(X_k - X_{k-1}) \ge -X \; P\text{-a.s.} \right\}$$

for $t = 0, \dots, T$ with the convention $\mathcal{A}_T^{\mathcal{S}} = L_+^\infty(\mathcal{F}_T)$. Moreover, we introduce the "one-step" sets

$$\mathcal{A}_{t,t+1}^{\mathcal{S}} := \left\{ X \in L^\infty(\mathcal{F}_{t+1}) \,\middle|\, \exists \, \xi \in \mathcal{S} \;:\; \xi_{t+1}(X_{t+1} - X_t) \ge -X \; P\text{-a.s.} \right\}$$

of all positions, which are acceptable for the next period of time in $t = 0, \dots, T-1$.

It follows straight from the definition of the sets that

$$\mathcal{A}_t^{\mathcal{S}} = \mathcal{A}_{t,t+1}^{\mathcal{S}} + \dots + \mathcal{A}_{T-1,T}^{\mathcal{S}} + L_+^\infty(\mathcal{F}_T) \tag{4.13}$$

for all $t = 0, \dots, T$. Moreover, it is easy to see that due to our assumptions on the set of admissible strategies $\mathcal{S}$ the sets $\mathcal{A}_t^{\mathcal{S}}$ and $\mathcal{A}_{t,t+1}^{\mathcal{S}}$ are conditionally convex, solid and contain 0 for all t. If we assume in addition that

$$\operatorname{ess\,inf} \left\{ X_t \in L_t^\infty \,\middle|\, X_t \in \mathcal{A}_t^{\mathcal{S}} \right\} = 0 \tag{4.14}$$

for all t, then by Proposition 1.1.3 the sequence of the sets $\mathcal{A}_t^{\mathcal{S}}$ $(t = 0, \ldots, T)$ induces a dynamic convex risk measure defined via (1.1):

$$\rho_t^{\mathcal{S}}(X) = \operatorname{ess\,inf}\left\{ Y \in L_t^\infty \;\middle|\; Y + X \in \mathcal{A}_t^{\mathcal{S}} \right\}, \quad X \in L^\infty(\mathcal{F}_T), \quad t = 0, \ldots, T.$$

Thus, $\rho_t^{\mathcal{S}}(X)$ is a minimal cost for the hedge of the position X at time t by means of some admissible strategy in $\mathcal{S}$, and so it coincides with the superhedge-price for X at time t.

It turns out that many important properties of the dynamic risk measure $(\rho_t^{\mathcal{S}})_{t=0,\ldots,T}$ are related to the *no-arbitrage condition* on the market model.

Definition 4.2.1. *We say that the set $\mathcal{S}$ satisfies the* no-arbitrage condition*, if there is no $\xi \in \mathcal{S}$ which generates a free lunch in the sense that*

$$\sum_{t=1}^{T} \xi_t (X_t - X_{t-1}) \geq 0 \quad P\text{-a.s.}$$

and

$$P\left[\sum_{t=1}^{T} \xi_t (X_t - X_{t-1}) > 0\right] > 0.$$

We call a market model arbitrage-free, if $\mathcal{S}$ satisfies no-arbitrage condition.

In the rest of this section we will assume that the no-arbitrage condition holds and we will study properties of the dynamic risk measure $(\rho_t^{\mathcal{S}})_{t=0,\ldots,T}$ under this assumption.

Proposition 4.2.2. *Condition (4.14) holds if the market model is arbitrage-free.*

Proof. Since $0 \in \mathcal{A}_t^{\mathcal{S}}$ we have $\operatorname{ess\,inf}\left\{ X_t \in L_t^\infty \;\middle|\; X_t \in \mathcal{A}_t^{\mathcal{S}} \right\} \leq 0$. To prove the converse inequality let $X \in \mathcal{A}_t^{\mathcal{S}} \cap L_t^\infty$ and $\xi \in \mathcal{S}$ such that $\sum_{k=t+1}^{T} \xi_k (X_k - X_{k-1}) \geq -X$. Consider the set $A := \{X < 0\} \in \mathcal{F}_t$ and the strategy

$$\widetilde{\xi} := \begin{cases} 0 & ; \quad k = 0, \ldots, t \\ I_A \xi_k & ; \quad k = t+1, \ldots, T. \end{cases}$$

Then $\widetilde{\xi} \in \mathcal{S}$ due to conditional convexity and we have

$$\sum_{k=1}^{T} \widetilde{\xi}_k (X_k - X_{k-1}) \geq -I_A X \geq 0$$

where the last inequality is strict on the set A. Hence, no-arbitrage condition implies $P[A] = 0$ and (4.14) follows. $\square$

In order to characterize the dynamic risk measure $(\rho_t^{\mathcal{S}})_{t=0,\dots,T}$ it is important to identify its acceptance sets

$$\mathcal{A}_t := \left\{ X \in L^\infty(\mathcal{F}_T) \,\middle|\, \rho_t^{\mathcal{S}}(X) \le 0 \; P\text{-a.s.} \right\}, \quad t = 0, \dots, T.$$

Clearly, $\mathcal{A}_t^{\mathcal{S}} \subseteq \mathcal{A}_t$ for all t. We will show that also the converse inclusion holds if the market is arbitrage-free, i.e., $\mathcal{A}_t^{\mathcal{S}} = \mathcal{A}_t$ for all t. Moreover, we will prove that the sets $\mathcal{A}_t^{\mathcal{S}}$ are weak*-closed. As noted in Remark 1.1.5, weak*-closedness of the acceptance set is equivalent to the existence of a robust representation for a conditional convex risk measure. Thus we obtain a robust representation for the risk measure $(\rho_t^{\mathcal{S}})_{t=0,\dots,T}$ under the no-arbitrage condition. We will argue in several steps.

First we identify the "one-step" sets $\mathcal{A}_{t,t+1}^{\mathcal{S}}$ as acceptance sets of the risk measure $\rho_t^{\mathcal{S}}$ restricted to the space $L^\infty(\mathcal{F}_{t+1})$.

Lemma 4.2.3. *The risk measure $\rho_t^{\mathcal{S}}$ restricted to the space $L^\infty(\mathcal{F}_{t+1})$ is determined through the set $\mathcal{A}_{t,t+1}^{\mathcal{S}}$ under the no-arbitrage condition:*

$$\rho_t(Y_{t+1}) = \operatorname{ess\,inf} \left\{ X_t \in L_t^\infty \,\middle|\, X_t + Y_{t+1} \in \mathcal{A}_{t,t+1}^{\mathcal{S}} \right\}$$

holds for all $Y_{t+1} \in L^\infty(\mathcal{F}_{t+1})$ and all $t = 0, \dots, T-1$.

Proof. We will prove the equality

$$\left\{ X_t \in L_t^\infty \,\middle|\, X_t + Y_{t+1} \in \mathcal{A}_{t,t+1}^{\mathcal{S}} \right\} = \left\{ X_t \in L_t^\infty \,\middle|\, X_t + Y_{t+1} \in \mathcal{A}_t^{\mathcal{S}} \right\}. \quad (4.15)$$

The inclusion "$\subseteq$" holds since $\mathcal{A}_{t,t+1}^{\mathcal{S}} \subseteq \mathcal{A}_t^{\mathcal{S}}$ due to (4.13). To prove the converse inclusion let $X_t \in \left\{ Z_t \in L_t^\infty \,\middle|\, Z_t + Y_{t+1} \in \mathcal{A}_t^{\mathcal{S}} \right\}$ and $\xi \in \mathcal{S}$ such that

$$X_t + \xi_{t+1}(X_{t+1} - X_t) + \sum_{k=t+2}^{T} \xi_k (X_{k+1} - X_k) \ge -Y_{t+1}.$$

Since the set $A := \{X_t + \xi_{t+1}(X_{t+1} - X_t) < -Y_{t+1}\}$ belongs to $\mathcal{F}_{t+1}$, we can construct arbitrage opportunity on A as in the proof of Proposition 4.2.2. No-arbitrage condition implies $P[A] = 0$ and thus $Y_{t+1} + X_t \in \mathcal{A}_{t,t+1}^{\mathcal{S}}$, which proves the equality (4.15) and the lemma. $\square$

In the next step we introduce the sets

$$\mathcal{K}_t^{\mathcal{S}} := \left\{ \xi_t (X_t - X_{t-1}) \,\middle|\, \xi_t \in \mathcal{S}_t \right\} \quad \text{and} \quad \mathcal{H}_t^{\mathcal{S}} = \mathcal{K}_t^{\mathcal{S}} - L_+^0(\mathcal{F}_t)$$

for $t = 1, \dots, T$. It is shown in Lemma 9.12 in [FS04] that the sets $\mathcal{H}_t^{\mathcal{S}}$ are closed convex subsets of $L^0(\mathcal{F}_t)$ for all t under the no-arbitrage condition. Moreover, we have $\mathcal{A}_{t,t+1}^{\mathcal{S}} = -\mathcal{H}_{t+1}^{\mathcal{S}} \cap L^\infty(\mathcal{F}_{t+1})$. This implies that the sets $\mathcal{A}_{t,t+1}^{\mathcal{S}}$ are weak*-closed, as we will prove in the the following propostion.

Proposition 4.2.4. *Let $\mathcal{A}^0$ be a closed convex subset of $L^0(\Omega, \mathcal{F}, P)$ and let $\mathcal{A} := \mathcal{A}^0 \cap L^\infty(\Omega, \mathcal{F}, P)$. Then $\mathcal{A}$ is weak*-closed.*

Proof. By Lemma A.64 in [FS04] it suffices to show that for every $r > 0$ the set

$$\mathcal{A}_r =: \left\{ X \in \mathcal{A} \,\middle|\, ||X||_\infty \le r \right\}$$

is closed in $L^1(\Omega, \mathcal{F}, P)$. In order to prove this let (X_n) be a sequence in $\mathcal{A}_r$ converging in L^1 to some random variable X. Then there is a subsequence that converges P-a.s. to X. This implies $||X||_\infty \le r$ and $X \in \mathcal{A}^0$ since $\mathcal{A}^0$ is closed with respect to almost sure convergence. Hence $\mathcal{A}_r$ is closed in L^1. □

Lemma 4.2.5. *Assume that the market model is arbitrage-free. Then the risk measure $\rho_t^{\mathcal{S}}$ restricted to the space $L^\infty(\mathcal{F}_{t+1})$ is continuous from above with the acceptance set $\mathcal{A}_{t,t+1}^{\mathcal{S}}$ for all $t = 0, \dots, T-1$, i.e.,*

$$\mathcal{A}_{t,t+1}^{\mathcal{S}} = \left\{ X \in L^\infty(\mathcal{F}_{t+1}) \,\middle|\, \rho_t(X) \le 0 \ P\text{-a.s.} \right\}. \tag{4.16}$$

Proof. We fix some $t \in \{0, \dots, T-1\}$. The set $\mathcal{A}_{t,t+1}^{\mathcal{S}}$ is weak*-closed by Lemma 9.12 in [FS04] and Proposition 4.2.4. Thus, Lemma 4.2.3 and Lemma 3.8. b) in [KS] imply the equality (4.16). Moreover, $\rho_t^{\mathcal{S}}$ is continuous from above on the space $L^\infty(\mathcal{F}_{t+1})$ by Remark 1.1.5. □

In the next theorem we will extend the results of the previous lemma to the sets $\mathcal{A}_t$.

Theorem 4.2.6. *Assume that the market model is arbitrage-free. Then the risk measure $\rho_t^{\mathcal{S}}$ is continuous from above with the acceptance set $\mathcal{A}_t^{\mathcal{S}}$, i.e.,*

$$\mathcal{A}_t^{\mathcal{S}} = \left\{ X \in L^\infty(\mathcal{F}_T) \,\middle|\, \rho_t(X) \le 0 \ P\text{-a.s.} \right\} \tag{4.17}$$

for each $t \in \{0, \dots, T\}$. Moreover, the sequence of risk measures $(\rho_t^{\mathcal{S}})_{t=0,\dots,T}$ is time consistent.

Proof. The proof will follow by backward induction on t.

First note that $\mathcal{A}_{T-1}^{\mathcal{S}} = \mathcal{A}_{T-1,T}^{\mathcal{S}} + L_+^\infty(\mathcal{F}_T) = \mathcal{A}_{T-1,T}^{\mathcal{S}}$ by (4.13) and solidness of the set $\mathcal{A}_{T-1,T}^{\mathcal{S}}$. Hence $\rho_{T-1}^{\mathcal{S}}$ is continuous from above with the acceptance set $\mathcal{A}^{\mathcal{S}}_{T-1}$ by Lemma 4.2.5 and we have the beginning of the induction. In the induction hypothesis we assume that $\rho^{\mathcal{S}}_{t+1}$ is continuous from above with the acceptance set $\mathcal{A}^{\mathcal{S}}_{t+1}$.

In order to apply this hypothesis we will show first that the sequence of risk measures $(\rho^{\mathcal{S}}_t, \rho^{\mathcal{S}}_{t+1})$ is time consistent, i.e., $\rho^{\mathcal{S}}_t = \rho^{\mathcal{S}}_t(-\rho^{\mathcal{S}}_{t+1})$. Indeed, we have

$$\mathcal{A}_t \supseteq \mathcal{A}^{\mathcal{S}}_t = \mathcal{A}^{\mathcal{S}}_{t,t+1} + \mathcal{A}^{\mathcal{S}}_{t+1} = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1},$$

where we have used (4.13), Lemma 4.2.5 and induction hypothesis. Hence $\rho^{\mathcal{S}}_t \le \rho^{\mathcal{S}}_t(-\rho^{\mathcal{S}}_{t+1})$ by (2.10) of Lemma 2.2.4. To prove the converse inequality fix $X \in L^\infty(\mathcal{F}_T)$, let $Y_t \in \left\{ Y \in L^\infty_t \,\middle|\, Y + X \in \mathcal{A}^{\mathcal{S}}_t \right\}$ and let $\xi \in \mathcal{S}$ such that

$$Y_t + \xi_{t+1}(X_{t+1} - X_t) + \sum_{k=t+2}^{T} \xi_k (X_{k+1} - X_k) \ge -X. \tag{4.18}$$

We argue that there is no loss of generality in assuming that $Y_t + \xi_{t+1}(X_{t+1} - X_t) =: V_{t+1} \in L^\infty(\mathcal{F}_{t+1})$. Indeed, since $0 \in \mathcal{S}$ it is sufficient to consider $V_{t+1} \wedge ||X||_\infty \ge X$ to hedge X. On the other hand, on the set $A := \{V_{t+1} < -||X||_\infty\} \in \mathcal{F}_{t+1}$ we have

$$I_A V_{t+1} + \sum_{k=t+2}^{T} I_A \xi_k (X_{k+1} - X_k) \ge -I_A X \ge -I_A ||X||_\infty$$

and we obtain an arbitrage opportunity on A:

$$\sum_{k=t+2}^{T} I_A \xi_k (X_{k+1} - X_k) \ge -I_A(||X||_\infty + V_{t+1}).$$

No-arbitrage condition implies $P[A] = 0$ and thus $||V_{t+1}||_\infty \le ||X||_\infty$ as we have claimed. Hence it follows from (4.18) that

$$Y_t + \xi_{t+1}(X_{t+1} - X_t) \ge \rho^{\mathcal{S}}_{t+1}(X) \tag{4.19}$$

by definition of $\rho^{\mathcal{S}}_{t+1}$. Inequality (4.19) implies further that $Y_t - \rho^{\mathcal{S}}_{t+1}(X) \in \mathcal{A}^{\mathcal{S}}_{t,t+1}$ and thus

$$Y_t \ge \rho^{\mathcal{S}}_t(-\rho^{\mathcal{S}}_{t+1}(X)) \tag{4.20}$$

due to Lemma 4.2.3. Since (4.20) holds for all $Y_t \in \left\{ Y \in L^\infty_t | Y + X \in \mathcal{A}^{\mathcal{S}}_t \right\}$, we conclude that it also holds for the essential infinum, i.e.,

$$\rho^{\mathcal{S}}_t(X) \ge \rho^{\mathcal{S}}_t(-\rho^{\mathcal{S}}_{t+1}(X)).$$

Thus we have proved the time consistency of $\rho^{\mathcal{S}}_t$ and $\rho^{\mathcal{S}}_{t+1}$.

Now we can identify the acceptance set of $\rho_t^{\mathcal{S}}$. We have

$$\mathcal{A}_t = \mathcal{A}_{t,t+1} + \mathcal{A}_{t+1} = \mathcal{A}^{\mathcal{S}}_{t,t+1} + \mathcal{A}^{\mathcal{S}}_{t+1} = \mathcal{A}^{\mathcal{S}}_t,$$

where we have used Lemma 2.2.4 for the first equality, Lemma 4.2.5 and induction hypothesis for the second, and (4.13) for the third. Moreover, $\rho_t^{\mathcal{S}}$ is continuous from above: If (X_n) is a decreasing sequence in $L^\infty(\mathcal{F}_T)$ converging P-a.s. to some $X \in L^\infty(\mathcal{F}_T)$, then $\rho^{\mathcal{S}}_{t+1}(X_n) \nearrow \rho^{\mathcal{S}}_{t+1}(X_n)$ P-a.s. by induction hypothesis, and thus

$$\rho_t^{\mathcal{S}}(X_n) = \rho_t^{\mathcal{S}}(-\rho^{\mathcal{S}}_{t+1}(X_n)) \nearrow \rho_t^{\mathcal{S}}(-\rho^{\mathcal{S}}_{t+1}(X)) = \rho_t^{\mathcal{S}}(X) \quad P\text{-a.s.}$$

due to time consistency and Lemma 4.2.5. □

In the next step we will identify minimal penalty functions in the robust representation of $\rho_t^{\mathcal{S}}$. By definition of the penalty functions and due to Theorem 4.2.6 we have

$$\begin{aligned}\alpha_t^{\min}(Q) &= \operatorname{ess\,sup}\Big\{\, E_Q[\,-X \,|\, \mathcal{F}_t\,] \;\Big|\; X \in \mathcal{A}_t\Big\} \\ &= \operatorname{ess\,sup}\left\{ E_Q[-X|\mathcal{F}_t] \;\Big|\; \exists\, \xi \in \mathcal{S} \,:\, \sum_{k=t+1}^{T} \xi_k(X_k - X_{k-1}) \geq -X \; P\text{-a.s}\right\}\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\alpha_{t,t+1}^{\min}(Q) &= \operatorname{ess\,sup}\Big\{\, E_Q[\,-X \,|\, \mathcal{F}_t\,] \;\Big|\; X \in \mathcal{A}_{t,t+1}\Big\} \\ &= \operatorname{ess\,sup}\Big\{E_Q[-X|\mathcal{F}_t] \;\Big|\; \exists\, \xi \in \mathcal{S} \,:\, \xi_{t+1}(X_{t+1} - X_t) \geq -X \; P\text{-a.s}\Big\}\end{aligned}$$

for $Q \in \mathcal{P}_t$ and all $t \in \{0, \dots, T\}$. For $Q \in \mathcal{M}^e(P)$ and $\xi \in \mathcal{S}$ such that $\sum_{k=t+1}^{T} \xi_k(X_k - X_{k-1}) \geq C$ P-a.s. for some constant $C \in \mathbb{R}$ the conditional expectation $E_Q[\sum_{k=t+1}^{T} \xi_k(X_k - X_{k-1}) \,|\, \mathcal{F}_t\,]$ is well defined, and thus we obtain alternative characterization of the minimal penalty function in the following propostion.

Proposition 4.2.7. *The minimal penalty functions $\alpha_t^{\min}$ and $\alpha_{t,t+1}^{\min}$ in the robust representation of the risk measure $\rho_t^{\mathcal{S}}$ are of the form*

$$\alpha_t^{\min}(Q) = \operatorname*{ess\,sup}_{\xi \in \mathcal{S}^b} E_Q\left[\sum_{k=t+1}^{T} \xi_k(X_k - X_{k-1}) \,\Big|\, \mathcal{F}_t\right] \tag{4.22}$$

and

$$\alpha_{t,t+1}^{\min}(Q) = \operatorname*{ess\,sup}_{\xi \in \mathcal{S}^b_{t+1}} E_Q[\,\xi_{t+1}(X_{t+1} - X_t) \,|\, \mathcal{F}_t\,], \tag{4.23}$$

where

$$\mathcal{S}^b := \left\{ \xi \in \mathcal{S} \;\middle|\; \exists\, C \in \mathbb{R} \;:\; \sum_{k=t+1}^{T} \xi_k (X_k - X_{k-1}) \geq C \;P\text{-}a.s. \right\}$$

and

$$\mathcal{S}^b_{t+1} := \left\{ \xi \in \mathcal{S} \;\middle|\; \exists\, C \in \mathbb{R} \;:\; \xi_{t+1}(X_{t+1} - X_t) \geq C \;P\text{-}a.s. \right\}$$

for $Q \in \mathcal{M}^e(P)$ *and all* $t \in \{0, \dots, T-1\}$.

Proof. Fix $Q \in \mathcal{M}^e(P)$ and $t \in \{0, \dots, T-1\}$. We will prove the equality (4.22), the proof of (4.23) is analog.
"$\leq$" follows straight from (4.21).
To prove the converse inequality let $\xi \in \mathcal{S}^b$ and

$$X_n := \left[\sum_{k=t+1}^{T} \xi_k (X_k - X_{k-1}) \right] \wedge n, \qquad n \in \mathbb{N}.$$

Then $-X_n \in \mathcal{A}_t$ for all n by definition, $X_n \nearrow \sum_{k=t+1}^{T} \xi_k (X_k - X_{k-1})$ P-a.s. with $n \to \infty$ and $E_Q[X_n | \mathcal{F}_t] \nearrow E_Q[\sum_{k=t+1}^{T} \xi_k (X_k - X_{k-1}) \,|\, \mathcal{F}_t]$ by monoton convergence. Thus we obtain "$\geq$" in (4.22). □

Remark 4.2.8. *If* $E_Q[\, X_{t+1} - X_t \,|\, \mathcal{F}_t \,]$ *is well defined for some probability measure* $Q \in \mathcal{M}^e(P)$, *(4.23) takes the form*

$$\alpha^{\min}_{t,t+1}(Q) = \operatorname*{ess\,sup}_{\xi \in \mathcal{S}} \left[\xi_{t+1} (E_Q[X_{t+1} | \mathcal{F}_t] - X_t) \right].$$

Thus the one-step penalty function $\alpha^{\min}_{t,t+1}(Q)$ *corresponds to the increment of the so called* "upper variation process" A^Q *of a probability measure* Q*, which was introduced in [FK97]. We recall here Definition 9.15 from [FS04]: For a measure* $Q \in \mathcal{M}_1(P)$*, the upper variation process is the increasing process* A^Q *defined by* $A^Q_0 := 0$ *and*

$$A^Q_{t+1} - A^Q_t := \operatorname*{ess\,sup}_{\xi \in \mathcal{S}} \left[\xi_{t+1} (E_Q[X_{t+1} | \mathcal{F}_t] - X_t) \right] \quad \text{for} \quad t = 0, \dots, T-1.$$

The set of all $Q \in \mathcal{M}^e(P)$ *such that*

$$E_Q[A^Q_T] < \infty \quad \text{and} \quad E_Q[\, |X_{t+1} - X_t| \,|\, \mathcal{F}_t \,] < \infty \;P\text{-}a.s. \text{ for all } t$$

is denoted by $Q_{\mathcal{S}}$ *in [FS04]. In our notation we obtain*

$$\alpha^{\min}_{t,t+1}(Q) = A^Q_{t+1} - A^Q_t \tag{4.24}$$

for all $t = 0, \dots, T-1$ *and all* $Q \in \mathcal{Q}_{\mathcal{S}}$.

We have already seen that the dynamic risk measure $(\rho_t^{\mathcal{S}})_{t=0,\dots,T}$ is time consistent and continuous from above if the market modell is arbitrage-free. Thus we can apply Theorem 2.2.2 in order to obtain equivalent characterizations of time consistency for $(\rho_t^{\mathcal{S}})_{t=0,\dots,T}$, if the set

$$\mathcal{Q}^* = \left\{ Q \in \mathcal{M}^e(P) \,\middle|\, \alpha_0^{\min}(Q) < \infty \right\}$$

is not empty. The next Lemma shows that this conditions is satisfied if the market is arbitrage-free. Moreover, we will show that here exists $P^* \in \mathcal{Q}^*$ such that $\alpha_0^{min}(P^*) = 0$ and this is equivalent to the no-arbitrage condition and to the sensitivity of the initial risk measure $\rho_0^{\mathcal{S}}$.

Lemma 4.2.9. *The following conditions are equivalent:*

1. *The risk measure $\rho_0^{\mathcal{S}}$ is sensitive.*
2. *There are no arbitrage opportunities in $\mathcal{S}$.*
3. *The risk measure $\rho_0^{\mathcal{S}}$ is continuous from above and there exists a probability measure $P^* \approx P$ such that $\alpha_0^{\min}(P^*) = 0$.*

Proof. 1) $\Rightarrow$ 2): Assume that $\rho_0^{\mathcal{S}}$ is sensitive, let $\xi \in \mathcal{S}$ such that

$$\sum_{k=1}^{T} \xi_k (X_k - X_{k-1}) \geq 0 \quad P\text{-a.s.}$$

and consider for $n \in \mathbb{N}$

$$X_n := \sum_{k=0}^{T} \xi_k (X_k - X_{k-1}) \wedge n.$$

Then $-X_n \in \mathcal{A}_0^{\mathcal{S}}$ and hence $\rho_0^{\mathcal{S}}(-X_n) \leq 0$. Since $X_n \geq 0$ P-a.s., we have $\rho_0^{\mathcal{S}}(-X_n) = 0$ by monotonicity of $\rho_0^{\mathcal{S}}$. We claim that sensitivity implies $X_n = 0$ P-a.s.. Indeed, if $P[X_n > 0] > 0$, we can find $\varepsilon > 0$ and $A \in \mathcal{F}_T$ with $P[A] > 0$ such that $X_n \geq \varepsilon I_A$ and thus $0 = \rho_0^{\mathcal{S}}(-X_n) \geq \rho_0^{\mathcal{S}}(-\varepsilon I_A)$ in contradiction to sensitivity of $\rho_0^{\mathcal{S}}$. Therefore $X_n = 0$ P-a.s. for all $n \in \mathbb{N}$ and

$$\sum_{k=1}^{T} \xi_k (X_k - X_{k-1}) = \lim_n X_n = 0 \quad P\text{-a.s.}.$$

Hence there are no arbitrage opportunities in $\mathcal{S}$.
2) $\Rightarrow$ 3): The risk measure $\rho_0^{\mathcal{S}}$ is continuous from above by Theorem 4.2.6. Moreover, by Theorem 9.9. in [FS04] there exists a probability measure

$P^* \approx P$ such that the value process of any trading strategy in $\bar{\mathcal{S}}$ is a local P^*-supermartingale. We will show that $\alpha_0^{min}(P^*) = 0$. To this end it is sufficient to prove that

$$E_{P^*}[-X] \leq 0 \quad \text{for all } X \in \mathcal{A}_0. \tag{4.25}$$

Let $X \in \mathcal{A}_0$. Since $\mathcal{A}_0 = \mathcal{A}_0^{\mathcal{S}}$ by Theorem 4.2.6, there exists $\xi \in \mathcal{S}$ such that

$$\sum_{k=1}^{T} \xi_k (X_k - X_{k-1}) \geq -X.$$

We consider the the associated value process

$$V_0 := 0, \quad V_t := \sum_{k=1}^{t} \xi_k (X_k - X_{k-1}), \quad t = 1, \dots, T.$$

Then (V_t) is a local supermartingale under P^*. Moreover, $V_T \geq -||X||_\infty$ and the same reasoning as in the proof of Theorem 4.2.6 applied to V_t for $t = 1, \dots, T-1$ implies $V_t \geq -||X||_\infty$ for all t. Thus (V_t) is a P^*-supermartingale by Proposition 9.6 in [FS04] and we obtain in particular

$$E_{P^*}[-X] \leq E_{P^*}[V_T] \leq E_{P^*}[V_0] = 0.$$

This proves (4.25).
3) $\Rightarrow$ 1): To prove sensitivity of the risk measure $\rho_0^{\mathcal{S}}$, let $\varepsilon > 0$ and $A \in \mathcal{F}_T$ with $P[A] > 0$. Since $\rho_0^{\mathcal{S}}$ has a robust representation in terms of $\mathcal{M}^e(P)$ due to Theorem 1.1.4 and Lemma 1.2.5, we obtain

$$\rho_0^{\mathcal{S}}(-\varepsilon I_A) = \sup_{Q \in \mathcal{M}^e(P)} \Big(E_Q[\varepsilon I_A] - \alpha_0^{\min}(Q) \Big) \geq E_{P^*}[\varepsilon I_A] > 0.$$

Thus $\rho_0^{\mathcal{S}}$ is sensitive. □

Now we can apply Theorem 2.2.2 to the time consistent dynamic risk measure $(\rho_t^{\mathcal{S}})_{t=0,\dots,T}$. Condition 3) amounts to

$$\alpha_t^{\min}(Q) = E_Q \left[\sum_{k=t}^{T-1} \alpha_{k,k+1}^{\min}(Q) \,|\, \mathcal{F}_t \right]$$

for all $t = 0, \dots, T-1$ and all $Q \in \mathcal{M}^e(P)$, and due to (4.24) we obtain the Doob decomposition of the minimal penalty function in terms of the upper variational process A^Q:

$$\alpha_t^{\min}(Q) = E_Q[\, A_T^Q \,|\, \mathcal{F}_t \,] - A_t^Q \tag{4.26}$$

for $Q \in \mathcal{Q}_{\mathcal{S}}$ and for all $t = 0, \dots, T$. In particular

$$\alpha_0^{\min}(Q) = E_Q[A_T^Q].$$

Thus

$$\mathcal{Q}^* = \mathcal{Q}_T = \left\{ Q \in \mathcal{M}^e(P) \;\Big|\; E_Q\left[\sum_{k=0}^{T-1} \alpha_{k,k+1}^{\min}(Q)\right] < \infty \right\}$$

and $\mathcal{Q}_{\mathcal{S}} \subseteq \mathcal{Q}^*$ with the difference that we do not require

$$E_Q[\,|X_{t+1} - X_t|\,|\,\mathcal{F}_t\,] < \infty \quad P\text{-a.s.}$$

for all t in the definition of the set $\mathcal{Q}^*$.

The robust representation

$$\rho_t(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}^*} \Big(E_Q[-X|\mathcal{F}_t] - \alpha_t^{\min}(Q) \Big), \quad t = 0, \dots, T \tag{4.27}$$

identifies for each $X \in L^\infty(\mathcal{F}_T)$ the process $\rho_t^{\mathcal{S}}(X)$ as the *upper $\mathcal{Q}^*$-Snell envelope* of a discounted European claim $-X$. For the details on upper Snell envelopes we refer to Definition 9.21 and Proposition 9.23 of [FS04]. It is shown there that the upper $\mathcal{Q}_{\mathcal{S}}$-Snell envelope of a discounted European claim $H \in L_+^\infty(\mathcal{F}_T)$ takes the form

$$\widetilde{U}_t^{\uparrow} = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}_{\mathcal{S}}} \Big(E_Q[\,H - A_T^Q\,|\,\mathcal{F}_t\,] + A_t^Q \Big), \quad t = 0, \dots, T,$$

which is the representation (4.27) if we replace $\mathcal{Q}_{\mathcal{S}}$ with $\mathcal{Q}^*$ and apply (4.26). Moreover, Theorem 9.29 in [FS04] identifies $\widetilde{U}_t^{\uparrow}$ as the minimal amount for which a superhedging strategy for H is available, i.e.,

$$\widetilde{U}_t^{\uparrow} = \operatorname*{ess\,inf} \Big\{ U_t \in L_+^0(\mathcal{F}_t) \Big| \exists\, \xi \in \mathcal{S} \text{ such that } U_t + \sum_{k=t+1}^{T} \xi_k (X_k - X_{k-1}) \geq H \ P\text{-a.s.} \Big\}. \tag{4.28}$$

This corresponds to our definition of the risk measure $\rho_t^{\mathcal{S}}$, since for $H \in L^\infty(\mathcal{F}_T)$ it is sufficient to consider $U_t \in L_t^\infty$ in (4.28). In contrast to the reasoning in [FS04] we have *defined* the risk measure $\rho_t^{\mathcal{S}}$ via (4.28) and have shown that it has the robust representation (4.27) under the no-arbitrage condition. Another difference in our reasoning is that we use the set $\mathcal{Q}^*$ instead of $\mathcal{Q}_{\mathcal{S}}$ in the robust representation of $\rho_t^{\mathcal{S}}$ since it appears more natural in our setting.

Condition 4) of Theorem 2.2.2 provides the supermartingale property of the process

$$V_t^Q(X) = \rho_t^{\mathcal{S}}(X) + \alpha_t^{\min}(Q), \quad t = 0, \dots, T,$$

and Proposition 2.5.5 implies the supermartingale property of the process

$$U_t^Q(X) = \rho_t^{\mathcal{S}}(X) - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \quad t = 0, \dots, T$$

for all $Q \in \mathcal{Q}^*$ and $X \in L^\infty$. This corresponds to sustainability of the risk process $(\rho_t^{\mathcal{S}}(X))$ and means that it can be financed by means of some admissible strategy in the market, as we show in the next proposition.

Proposition 4.2.10. *Suppose that the no-arbitrage condition holds. Then for any bounded adapted process $Y = (Y_t)$ the following conditions are equivalent:*

1. *The process*
$$Y_t - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \qquad t = 0, \dots, T$$
is a Q-supermartingale for all $Q \in \mathcal{Q}^*$.

2. *There exists $\xi \in \mathcal{S}$ and a non-negative adapted increasing process B such that $B_0 = 0$ and*
$$Y_t = Y_0 + \sum_{k=0}^{t} \xi_k (X_k - X_{k-1}) - B_t, \qquad t = 0, \dots, T. \tag{4.29}$$

Moreover, $(\rho_t^{\mathcal{S}}(X))$ is the smallest bounded adapted process that has the representation (4.29) and covers the final loss $-X$ for all $X \in L^\infty$.

Proof. By definition of the one-step acceptance sets $\mathcal{A}_{t,t+1}^{\mathcal{S}}$ sustainability of a bounded process $Y = (Y_t)$ with respect to $(\rho_t^{\mathcal{S}})$ in this model is equivalent to the representation (4.29). Since $\mathcal{Q}^* \neq \emptyset$ under the no-arbitrage condition by Lemma 4.2.9, the equivalence of 1) and 2) follows from Theorem 2.5.4. Moreover, since $E_Q[\alpha_{t,t+1}^{\min}(Q)] < \infty$ for all $t = 0, \dots, T$ if $Q \in \mathcal{Q}^*$, we have $\widetilde{\mathcal{Q}}_t \neq \emptyset$ for all t and thus Proposition 2.5.2 identifies $(\rho_t^{\mathcal{S}}(X))$ as the smallest bounded adapted process that it is sustainable with respect to $(\rho_t^{\mathcal{S}})$ and covers the final loss $-X$ for all $X \in L^\infty$. □

The equivalence of 1) and 2) was proved in Theorem 9.20 in [FS04], where it is called the *uniform Doob decomposition under constraints*. Since sustainability with respect to $(\rho_t^{\mathcal{S}})$ is equivalent to the representation (4.29), Theorem

9.20 in [FS04] is a special case of our general Theorem 2.5.4. The second part of Proposition 4.2.10, i.e. the identification of $(\rho_t^{\mathcal{S}}(X))$ as the smallest bounded adapted process that has the representation (4.29) and covers the final loss, is a special case of Theorem 9.22 in [FS04]. There the same result was proved more generally for upper $\mathcal{Q}_S$-Snell envelopes of American options.

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Selbständigkeitserklärung

Hiermit erkläre ich, dass ich die Arbeit selbständig und nur unter Verwendung der angegebenen Hilfsmittel und Hilfen angefertigt habe.

Berlin, den 1. August 2007