Preservation of Quasiconvexity and Quasimonotonicity in Polynomial Approximation of Variational Problems

DISSERTATION

zur Erlangung des akademischen Grades
doctor rerum naturalium
(Dr. rer. nat.)
im Fach Mathematik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät II
Humboldt-Universität zu Berlin

von
Herrn Sebastian Heinz
geboren am 26.09.1977 in Berlin

Präsident der Humboldt-Universität zu Berlin:
Prof. Dr. Christoph Markschies

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät II:
Prof. Dr. Wolfgang Coy

Gutachter:

1. Prof. Dr. Andreas Griewank
2. Prof. Dr. Sören Bartels
3. Prof. Dr. Jan Kristensen

Tag der mündlichen Prüfung: 18. Juli 2008

Danksagung

Die vorliegende Dissertation ist das Ergebnis dreijähriger Arbeit an der Humboldt-Universität zu Berlin innerhalb des Graduiertenkollegs 1128 "Analysis, Numerics, and Optimization of Multiphase Problems" der Deutschen Forschungsgemeinschaft (DFG). Ich bin der DFG für ihre finanzielle Unterstützung sehr dankbar. Sie ermöglichte es mir, mich in diesen drei Jahren voll auf meine Promotion konzentrieren zu können.

Zum Gelingen meiner Arbeit beigetragen haben auch die vielen Gespräche über mathematische Probleme, die mir oft neue Impulse und Ideen für mein Projekt gaben. Ein großes Dankeschön ergeht deshalb an meine Betreuer, Andreas Griewank, Sören Bartels und Jan Kristensen, sowie an Bernd Bank, Bernd Kummer, Christof Melcher, René Henrion, Lutz Lehmann, Donat Wegner und Dirk Peschka.

Ein Vierteljahr vor meiner Verteidigung wurde ich Mitarbeiter am Weierstraß-Institut für Angewandte Analysis und Stochastik in der Forschergruppe von Alexander Mielke. Ihm möchte ich an dieser Stelle für seine Geduld und sein Verständnis während der Endphase meiner Promotion danken.

Schließlich gebührt ein herzlicher Dank meiner Frau und meiner ganzen Familie, die mich auf meinem wissenschaftlichen Weg stets begleitet und nach Kräften unterstützt haben.

Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit drei Klassen ausgewählter nichtlinearer Probleme, die Forschungsgegenstand der angewandten Mathematik sind. Diese Probleme behandeln die Minimierung von Integralen in der Variationsrechnung (Kapitel 3), das Lösen partieller Differentialgleichungen (Kapitel 4) und das Lösen nichtlinearer Optimierungsaufgaben (Kapitel 5). Mit deren Hilfe lassen sich unterschiedlichste Phänomene der Natur- und Ingenieurwissenschaften sowie der Ökonomie mathematisch modellieren. Als konkretes Beispiel werden mathematische Modelle der Theorie elastischer Festkörper betrachtet.

Das Ziel der vorliegenden Arbeit besteht darin, ein gegebenes nichtlineares Problem durch polynomiale Probleme zu approximieren. Anders ausgedrückt: Zu dem nichtlinearen Problem zugehörige nichtlineare Funktionen, die als Parameter fungieren und die wesentliche Information des Problems tragen, werden durch algebraische Polynome ersetzt. Beim Ersetzen sollen charakteristische Eigenschaften des Problems erhalten bleiben. Das Ziel, die polynomiale Approximation, ist interessant, da für das Studium polynomialer Probleme mehr mathematische Werkzeuge zur Verfügung stehen als für nichtpolynomiale (nichtlineare) Probleme. Um dieses Ziel zu erreichen, beschäftigt sich ein großer Teil der vorliegenden Arbeit mit der polynomialen Approximation von nichtlinearen Funktionen. Den Ausgangspunkt dafür bildet der Weierstraßsche Approximationssatz. Auf der Basis dieses bekannten Satzes und eigener Sätze wird als Hauptresultat der vorliegenden Arbeit gezeigt, dass im Übergang von einer gegebenen Funktion zum approximierenden Polynom wesentliche Eigenschaften der gegebenen Funktion erhalten werden können. Die wichtigsten Eigenschaften, für die dies bisher nicht bekannt war, sind: Quasikonvexität im Sinne der Variationsrechnung, Quasimonotonie im Zusammenhang mit partiellen Differentialgleichungen sowie Quasikonvexität im Sinne der nichtlinearen Optimierung (Theoreme 3.16, 4.10 und 5.5).

Zu den eigenen Sätzen in der vorliegenden Arbeit gehören insbesondere Sätze zur polynomialen Approximation von nichtlinearen Funktionen, die auf einem abstrakten Niveau bewiesen werden (Theoreme 3.7 und 4.5), das es ermöglicht, diese Sätze auf eine Vielzahl von Konvexitäts- und Monotoniebegriffen anzuwenden. Auf diese Weise wird auch die Grundlage gelegt für die polynomiale Approximation der ausgewählten nichtlinearen Probleme.

Schließlich wird gezeigt, dass die zu den untersuchten Klassen gehörenden nichtlinearen Probleme durch polynomiale Probleme approximiert werden können (Theoreme 3.26, 4.16 und 5.8). Die dieser Approximation zugrunde

liegende Konvergenz garantiert sowohl eine Approximation im Parameterraum als auch eine Approximation im Lösungsraum. Für letztere werden die Konzepte der Gamma-Konvergenz (Epi-Konvergenz) und der G-Konvergenz verwendet.

Abstract

In this thesis, we are concerned with three classes of non-linear problems that appear naturally in various fields of science, engineering and economics. In order to cover many different applications, we study problems in the calculus of variation (Chapter 3), partial differential equations (Chapter 4) as well as non-linear programming problems (Chapter 5). As an example of possible applications, we consider models of non-linear elasticity theory.

The aim of this thesis is to approximate a given non-linear problem by polynomial problems. In other words: A given non-linear problem is associated with a number of non-linear functions that serve as parameters and represent the non-linear problem. We show that these non-linear functions can be approximated by algebraic polynomials so that characteristic properties of the corresponding problems are preserved. Polynomial approximation is interesting, since tools that can be applied to polynomial problems are not available for non-polynomial (non-linear) problems in general.

In order to achieve the desired polynomial approximation of problems, a large part of this thesis is dedicated to the polynomial approximation of non-linear functions. The Weierstraß approximation theorem forms the starting point. Based on this well-known theorem, we prove theorems that eventually lead to our main result: A given non-linear function can be approximated by polynomials so that essential properties of the function are preserved. This result is new for three properties that are important in the context of the considered non-linear problems. These properties are: quasiconvexity in the sense of the calculus of variation, quasimonotonicity in the context of partial differential equations and quasiconvexity in the sense of non-linear programming (Theorems 3.16, 4.10 and 5.5).

Several theorems in this thesis deal with polynomial approximation of non-linear functions on an abstract level (Theorems 3.7 and 4.5). The abstract approach is useful, since its results can be applied to various notions of convexity and of monotonicity. Moreover, it forms the basis for the polynomial approximation of non-linear problems.

Finally, we show the following: Every non-linear problem that belongs to one of the three considered classes of problems can be approximated by polynomial problems (Theorems 3.26, 4.16 and 5.8). The underlying convergence guarantees both the approximation in the parameter space and the approximation in the solution space. In this context, we use the concepts of Gamma-convergence (epi-convergence) and of G-convergence.

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Chapter 1

Introduction

We will be concerned with non-linear problems of applied mathematics that appear naturally in various fields of science, engineering and economics. In order to cover many different applications, we will study problems in the calculus of variations, partial differential equations as well as non-linear programming problems. The aim of this thesis is to show that three problems that are stated below can be approximated by polynomial problems with the same characteristic properties.

The usefulness of such an approximation lies in the unique properties of polynomials in comparison with arbitrary non-linear functions. In fact, polynomials are subject to research in many different disciplines including commutative algebra, algebraic geometry and even complexity theory. This universal character of polynomials provides us with tools to analyze polynomial problems that are not available in the general non-linear case.

Before we specify the non-linear problems of interest and present our main results, we will consider the content of this thesis from an abstract point of view. A family of non-linear problems can be seen as a set-valued mapping from a parameter space[1] to a solution space. In this thesis, an element of the parameter space will always consist of one or more non-linear functions and real numbers that together represent a specific problem. We call it a *polynomial problem* if the non-linear functions are given by polynomials.

In this general framework, an appropriate approximation method should guarantee that the generated sequence of approximating problems converges to the limit problem simultaneously in the parameter and solution space. Our approximation procedure is based on the Stone-Weierstraß theorem, which will take care of the convergence in the parameter space. It remains as the main task of this thesis to construct sequences of polynomial problems that

[1]This is sometimes called *data space* in the literature.

converge to the limit problem in both the parameter and solution space.

We will study three non-linear problems and begin with one from the calculus of variations.

Problem 1

In the calculus of variations, one of the fundamental problems is of the form:

$$\text{Minimize} \int_\Omega f(x, u(x), \mathrm{D}u(x))\mathrm{d}x \text{ for } u \in W_0^{1,p}(\Omega, \mathbb{R}^m). \tag{1.1}$$

Here $m, n > 0$ denote positive integers, $\Omega \subseteq \mathbb{R}^n$ a non-empty bounded open set with Lipschitz boundary[2], $p > 1$ a real number, $W_0^{1,p}(\Omega, \mathbb{R}^m)$ a Sobolev space[3] and $f \in \mathcal{C}(\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n})$ a continuous real-valued function.

Non-linear elasticity provides various non-trivial examples of (1.1). The derivation of the following model of a hyperelastic material in three-dimensional space can be found in Ciarlet [1988]. We consider an elastic body that is made of hyperelastic material and has $\Omega \subseteq \mathbb{R}^3$ as its reference configuration. Let $W_0^{1,p}(\Omega, \mathbb{R}^3)$ be the set of admissible deformations. Moreover, let $W \in \mathcal{C}(\Omega \times \mathbb{R}^{3\times 3})$ represent the stored-energy function[4] and $F \in \mathcal{C}(\Omega \times \mathbb{R}^3)$ the body force potential. Finally, set

$$f(x, y, A) = W(x, A) + F(x, y), \; x \in \Omega, y \in \mathbb{R}^3, A \in \mathbb{R}^{3\times 3}.$$

Then the integral in (1.1) is called the *total energy.* The existence of deformations with minimal total energy is an important question in mathematical elasticity.[5]

The direct methods in the calculus of variations provide us with a strategy to prove the existence of minimizers. See Dacorogna [1989] for details. Apart from coercivity and growth conditions, we have to guarantee that the integral in (1.1) is sequentially weakly lower semicontinuous in order to make use of direct methods. The last condition on the integrand function f gives rise to the concept of quasiconvexity.

Here we mean the notion of quasiconvexity in sense of the calculus of variations. Morrey [1952] introduced this notion and established necessary and

[2]Such a set Ω is frequently called a Lipschitz domain in the literature.

[3]The space $W_0^{1,p}(\Omega, \mathbb{R}^m)$ automatically implies homogenous Dirichlet boundary conditions.

[4]In practical applications, the stored-energy function usually fulfills $W(x, A) = +\infty$ whenever $\det(A) \leq 0$. However, we will study continuous functions W only.

[5]Assume that the function f has the form $f(x, y, A) = W(A) + F(x, y)$. Under additional assumptions, Raymond [1991] showed Lipschitz regularity of solutions to (1.1).

sufficient conditions for the integral in (1.1) to be sequentially weakly lower semicontinuous. Acerbi and Fusco [1984] as well as Marcellini [1985] generalized these results to a wider class of integrand functions. Quasiconvexity is not only essential for the existence but also for the (partial) regularity of solutions as shown by Evans [1986], Acerbi and Fusco [1987] as well as Kristensen and Mingione [2007].

In Chapter 3, we will require quasiconvexity of the integrand f. Recall that we wish to approximate a non-linear problem of the form (1.1) by a sequence of polynomial problems. If the limit problem admits solutions, so should every approximating problem. Hence it makes sense to preserve quasiconvexity during the approximation process. As an important step to reach this goal, we will show that quasiconvex continuous functions can be approximated locally uniformly by quasiconvex polynomials. We will furthermore show analogous results for rank-one convexity and polyconvexity. These two notions were studied by Ball [1977] in the context of non-linear elasticity.

We will call a set $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^N)$ of continuous real-valued functions *admissible* if all of the following conditions hold:

(F1) $\mathcal{F}$ is a convex cone.

(F2) $\mathcal{F}$ is translation invariant.

(F3) $\mathcal{F}$ is closed in $\mathcal{C}(\mathbb{R}^N)$.

(F4) $\mathcal{F}$ contains all convex functions in $\mathcal{C}(\mathbb{R}^N)$.

This framework is sufficiently abstract to include most of the convexity notions that are used in the calculus of variations. In the first part of Chapter 3, we will show that every function that is of polynomial growth and lies in a given admissible set $\mathcal{F}$ can be approximated locally uniformly by polynomials in $\mathcal{F}$. In addition, we will invoke properties of the convexity notions that enable us to conclude the approximation results without assuming any growth conditions. One approximation result of such a kind was presented in Heinz [Published online: January 30, 2008] for quasiconvex functions in the calculus of variations.

The rest of Chapter 3 deals with the approximation of (1.1) via polynomial problems. As an important issue, we have to take care of the convergence in the solution space. Therefore we will use the concept of Γ-convergence introduced by De Giorgi [1977]. Our main result is: If we assume quasiconvexity of the integrand f as well as coercivity and growth conditions, a limit problem given by (1.1) can be approximated by polynomial problems in the sense of Γ-convergence (compare Theorem 3.27). We will see that our approximation result on the level of functions perfectly satisfies the requirements for the approximation on the level of problems.

Problem 2

We seek weak solutions $u \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ of the following non-linear partial differential equation:

$$\operatorname{div} \sigma(x, \mathrm{D}u(x)) = -f(x) \text{ almost everywhere in } \Omega. \tag{1.2}$$

Here $m, n > 0$ denote positive integers, $\Omega \subseteq \mathbb{R}^n$ a non-empty bounded open set with Lipschitz boundary, $p, q > 1$ real numbers so that $1/p + 1/q = 1$, $W_0^{1,p}(\Omega, \mathbb{R}^m)$ a Sobolev space, $f \in L^q(\Omega, \mathbb{R}^m)$ a q-integrable function and $\sigma\colon \Omega \times \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{m\times n}$ a continuous matrix-valued function. A Sobolev function $u \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ is a weak solution to (1.2) if and only if u is a solution to the weak formulation:

$$\int_\Omega \sigma(x, \mathrm{D}u(x)) : \mathrm{D}v(x)\mathrm{d}x = \int_\Omega \langle f, v\rangle \mathrm{d}x \text{ for every } v \in W_0^{1,p}(\Omega, \mathbb{R}^m).$$

Similar to Problem 1, we consider an example from three-dimensional elasticity. Again we refer to Ciarlet [1988] for details. In contrast to the above example, the elastic body in question does not have to be made of a hyperelastic material. Let $\Omega \subseteq \mathbb{R}^3$ be the reference configuration and $W_0^{1,p}(\Omega, \mathbb{R}^3)$ the set of admissible deformations, like before. Moreover, let $\sigma \in \mathcal{C}(\Omega \times \mathbb{R}^{3\times 3}, \mathbb{R}^{3\times 3})$ represent the stress-strain response and $f \in L^q(\Omega, \mathbb{R}^3)$ the applied body force. Then the requirement of *static equilibrium* coincides with (1.2).

When the material is hyperelastic, every deformation that minimizes the total energy is in static equilibrium and, hence, a weak solution to (1.2). In this case, we have results on the existence of weak solutions via the direct methods, which would lead us once more to quasiconvexity in the calculus of variations. Yet, if the material fails to be hyperelastic, we have to go another route in order to analyze the existence of weak solutions to (1.2).

Zhang [1988] introduced quasimonotonicity[6] and showed the existence of weak solutions to a family of partial differential equations including (1.2), provided that σ fulfills an ellipticity condition (strict quasimonotonicity) as well as coercivity and growth conditions. In fact, Landes [1996] pointed out that (strict) quasimonotonicity of σ is closely related to the pseudomonotonicity of the Nemytskij operator associated with the partial differential

[6] An equivalent concept was earlier introduced by Scheffer [1974].

equation (1.2). Pseudomonotonicity was introduced by Brézis [1968] and is a fundamental notion in the context of non-linear partial differential equations.

In Chapter 4, we will require strict quasimonotonicity of the integrand σ. The structure of Chapter 4 is very similar to that of Chapter 3. In order to approximate the non-linear partial differential equation (1.2), we first show that quasimonotonicity can be preserved during the approximation. We will prove that quasimonotone continuous functions of polynomial growth can be approximated locally uniformly by quasimonotone polynomial maps. A corresponding statement is true for monotone functions.

Here we will call a set $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$ of continuous vector-valued functions *admissible* if all of the following conditions hold:

(F1)' $\mathcal{F}$ is a convex cone.

(F2)' $\mathcal{F}$ is translation invariant.

(F3)' $\mathcal{F}$ is closed in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$.

(F4)' $\mathcal{F}$ contains all monotone functions in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$.

Within this abstract framework, we will show that every function that is of polynomial growth and lies in a given admissible set $\mathcal{F}$ can be approximated locally uniformly by polynomial maps in $\mathcal{F}$. After that, the approximation result for quasimonotone functions is immediate. Note that polynomial growth is not as restrictive as it may seem. The growth conditions that we will put on σ are polynomial anyway.

The rest of Chapter 4 deals with the approximation of (1.2) via polynomial problems. Like in Chapter 3, we have to take special care of the convergence in the solution space. A suitable concept has been introduced by De Giorgi [1977] and is called G-convergence. Our main result is: If we assume strict quasimonotonicity of the function σ as well as coercivity and growth conditions, a non-linear partial differential equation given by (1.2) can be approximated by polynomial problems in the sense of G-convergence (compare Theorem 4.16).

Problem 3

We study the non-linear programming problem over $\mathbb{R}^N$ of the form:

$$\text{Minimize } g(x) \text{ subject to } g_1(x) \leq 0, \ldots, g_l(x) \leq 0. \tag{1.3}$$

Here $N, l > 0$ denote positive integers and $g, g_1, \ldots, g_l \colon \mathbb{R}^N \longrightarrow \mathbb{R} \cup \{+\infty\}$ quasiconvex lower semicontinuous extended real-valued functions.

Whenever we speak about Problem 3, we have in mind the quasiconvexity in the sense of non-linear programming[7]. Frenk and Kassay [2005] pointed out that this property was studied the first time by von Neumann [1928][8]. An introduction to quasiconvexity and applications to economy can be found in Avriel et al. [1998].

Let us also mention that quasiconvexity has an impact on stability theory of non-linear parametric optimization problems. Results are given by Bank et al. [1982] (continuous optimization) as well as Bank and Mandel [1988] (integer optimization). Moreover, we have the following result in integer polynomial optimization: There exists an algorithm[9] that solves the corresponding integer programming problem given by (1.3) as long as the functions $g, g_1, \ldots, g_l$ are quasiconvex polynomials with integer coefficients. See Bank et al. [1990] for the first construction of such an algorithm and Heinz [2005] for the complexity analysis. Note that the general (non-quasiconvex) integer programming problem cannot be solved by an algorithm. This was shown by Jeroslow [1973].

Our aim is to approximate (1.3) with the help of polynomial problems. The functions $g, g_1, \ldots, g_l$ are assumed to be quasiconvex and we will require that the approximating polynomials be quasiconvex too. Since Problem 3 can be seen as a finite dimensional problem[10], the preservation of quasiconvexity is not motivated by results on the existence of solutions. Nevertheless, it is interesting in itself. In Chapter 5, we will prove that quasiconvex semicontinuous functions can be approximated by quasiconvex polynomials. This has not been done before. Here the underlying notion of convergence is related to Γ-convergence, which will also guarantee the convergence on the level of problems. We will see that our way of proving this result is related to what is sometimes called the Fenchel problem of level sets.

Preserving Quasiconvexity and Quasimonotonicity

Up to now, we have discussed the non-linear problems that we will approximate by polynomial problems. In all of the three different cases, the main issue is to preserve a particular property of functions during the approximation procedure:

- quasiconvexity in the sense of the calculus of variations,

[7]This notion is completely different to the quasiconvexity in the calculus of variations.

[8]Von Neumann did not call it quasiconvexity. Yet, he introduced and studied an equivalent property in the context of minimax problems.

[9]Here an algorithm is understood in the sense of Turing machines.

[10]Note that Problem 1 and Problem 2 are both infinite dimensional problems.

- quasimonotonicity in the context of partial differential equations and
- quasiconvexity in the sense of non-linear programming.

We would like to point out that our approach goes beyond shape-preserving approximation. Shape-preserving approximation via polynomials has been studied extensively over the years. A survey is given by Leviatan [2000]. In this context, the shape of an univariate real-valued function is understood in a geometric sense and refers to local properties of the function like monotonicity and convexity. The three properties of interest here turn out to be of non-local character. This globality represents the main difference between shape-preserving approximation and our polynomial approximations.

Our key idea is to exploit the basic geometric structure of the corresponding sets of functions. The set of quasiconvex functions in the calculus of variations as well as the set of quasimonotone functions in the context of partial differential equations are convex cones. Having this in mind, we will prove the desired approximation results on an abstract level. This includes other convexity notions in the calculus of variations as well as monotonicity. We will introduce the notion of admissible sets of functions in Chapter 3 and Chapter 4, exactly for this purpose. A first approximation result that makes use of this strategy is given in Heinz [Published online: January 30, 2008].

Only the quasiconvexity in non-linear programming requires a different approach, since the set of quasiconvex functions in non-linear programming is in general not convex. Yet, we can see it as a part of the general framework of this thesis. In fact, we will use results from Chapter 3 and Chapter 4 in order to show that quasiconvexity in non-linear programming can also be preserved during polynomial approximation.

Chapter 2

Preliminaries and Notation

This chapter introduces our basic notation and, in particular, the function spaces that we will use. We wish to work with spaces of continuous and differentiable functions, with spaces of lower semicontinuous functions and with Sobolev spaces. Here our special interest lies on topological issues. In addition, this chapter recalls some well-known approximation results. The Stone-Weierstraß Theorem is one of them. Another one deals with the approximation by smooth functions. We complete this chapter with a few remarks on the polynomial growth of functions.

2.1 Basic Notation

2.1.1 Euclidean Structure on $\mathbb{R}^N$

Let $N > 0$ be a positive integer. We will write a typical N-dimensional real vector x like $x = (x^{(1)}, \ldots, x^{(N)})$ where all entries $x^{(1)}, \ldots, x^{(N)} \in \mathbb{R}$ are real numbers. We denote by $\mathbb{R}^N$ the Euclidean space of all such vectors equipped with the scalar product $\langle x, y\rangle = x^{(1)}y^{(1)} + \ldots + x^{(N)}y^{(N)}$ for $x, y \in \mathbb{R}^N$. The corresponding norm $|x|$ of a vector $x \in \mathbb{R}^N$ is given by $|x| = \langle x, x\rangle^{1/2}$.
We have in mind the Euclidean topology whenever we speak about topological properties in $\mathbb{R}^N$:

- $\mathcal{B}_{x,r} = \{y \in \mathbb{R}^N \mid |y-x| < r\}$ is the open ball with the center in $x \in \mathbb{R}^N$ and radius $r > 0$ and $\overline{\mathcal{B}}_{x,r}$ the closed ball.

- $(a, b) \subseteq \mathbb{R}$ is the open interval with endpoints $a, b \in \mathbb{R}$, $a < b$, and $[a, b]$ the closed interval.

- ∂A is the boundary of a set $A \subseteq \mathbb{R}^N$.

In what follows, N, M, n and m will be fixed positive integers that specify the dimension of a Euclidean space.

2.1.2 Matrices and Bilinear Forms

We will write a typical real $(M \times N)$-matrix A like

$$A = \begin{pmatrix} A^{(1,1)} & \dots & A^{(1,N)} \\ \vdots & & \vdots \\ A^{(M,1)} & \dots & A^{(M,N)} \end{pmatrix}, \; A^{(1,1)}, \dots, A^{(M,N)} \in \mathbb{R}.$$

We denote the set of all such matrices by $\mathbb{R}^{M \times N}$. If the matrix structure is irrelevant, we will identify a matrix $A \in \mathbb{R}^{M \times N}$ with a vector $x \in \mathbb{R}^{M \cdot N}$. The equations $x^{(i+(j-1)M)} = A^{(i,j)}$, $i = 1, \dots, M$, $j = 1, \dots, N$, determine one possible identification. With regard to this identification, we understand all topological aspects in $\mathbb{R}^{M \times N}$ with respect to the Euclidean structure on $\mathbb{R}^{M \cdot N}$. For the convenience of the reader, we will sometimes write $A : B$ instead of $\langle A, B \rangle$ for the scalar product of matrices $A, B \in \mathbb{R}^{M \times N}$.

We know that every matrix $A \in \mathbb{R}^{N \times N}$ defines a bilinear form on $\mathbb{R}^N$ (and vice versa). We write

$$A[y_1, y_2] = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(A^{(i,j)} \cdot y_1^{(i)} \cdot y_2^{(j)} \right), \; y_1, y_2 \in \mathbb{R}^N. \tag{2.1}$$

The matrix A is called *positive semi-definite* if the inequality $A[y, y] \geq 0$ holds for every vector $y \in \mathbb{R}^N$. Moreover, A is called *positive definite* if the last inequality is strict for $y \neq 0$. Note that the bilinear form defined by A does not need to be symmetric.

2.2 Convergence in Function Spaces

Unless specified otherwise, every function space considered in this thesis carries the structure of a topological vector spaces and, in particular, the convergence in the function space is induced by a topology. For an introduction to topological vector spaces as well as fundamental properties, we refer to Bourbaki [1987] and Grothendieck [1973].

2.2.1 Diagonal Sequence Argument

We will use the diagonal sequence argument as a key tool to prove approximation results. However, note that the argument that we have in mind cannot be applied to every kind of convergence.

Let X be a space together with a notion of convergence[1]. We say that the *diagonal sequence argument* is applicable in X if the following holds: Let $x, x_i, x_{i,j} \in X$, $i, j = 1, 2, \ldots$, be given elements so that $x_{t,s} \to x_t$ holds in X for every $s = 1, 2, \ldots$ and $x_t \to x$ in X. Then there exist positive integers $0 < s(1) < s(2) < \ldots$ so that $x_{r,s(r)} \to x$ in X as r tends to $+\infty$. The sequence $x_{1,s(1)}, x_{2,s(2)}, \ldots \in X$ is called *diagonal sequence*.

It is well-known that the diagonal sequence argument can be applied to any convergence induced by a metrizable topology.

2.2.2 Continuous and Differentiable Functions

Let $k \geq 0$ be a non-negative integer. We denote by $\mathcal{C}^k(\mathbb{R}^N)$ the vector space of all k-times continuously differentiable functions $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ where we identify $\mathcal{C}^0(\mathbb{R}^N)$ with the space $\mathcal{C}(\mathbb{R}^N)$ of all continuous functions.

The topology in $\mathcal{C}^k(\mathbb{R}^N)$ is given by seminorms and so is the convergence in $\mathcal{C}^k(\mathbb{R}^N)$. Given a sequence $f_1, f_2, \ldots \in \mathcal{C}^k(\mathbb{R}^N)$ and a function $f \in \mathcal{C}^k(\mathbb{R}^N)$, we write $f_s \to f$ in $\mathcal{C}^k(\mathbb{R}^N)$ whenever we have locally uniform convergence of all partial derivatives up to order k. That means the following: Let $K \subseteq \mathbb{R}^N$ be an arbitrary non-empty compact set, $0 \leq l \leq k$ an integer and $i_1, \ldots, i_l \in \{1, \ldots, N\}$ indices. Then the seminorm

$$g \mapsto \sup \left\{ \left| \frac{\partial^l g}{\partial x^{(i_1)} \cdots \partial x^{(i_l)}}(x) \right| \mid x \in K \right\},\ g \in \mathcal{C}^k(\mathbb{R}^N), \tag{2.2}$$

evaluated at $g = f_s - f$ tends to 0 as $s \to +\infty$.

The space $\mathcal{C}^\infty(\mathbb{R}^N)$ of all smooth functions is defined as the intersection of the spaces $C^0(\mathbb{R}^N), \mathcal{C}^1(\mathbb{R}^N), \ldots$ and we write $f_s \to f$ in $\mathcal{C}^\infty(\mathbb{R}^N)$ if $f_s \to f$ in $\mathcal{C}^k(\mathbb{R}^N)$ holds for every $k = 0, 1, \ldots$.

In order to define the spaces $C(\mathbb{R}^N, \mathbb{R}^M)$ up to $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^M)$ of vector-valued functions $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$, we set $f = (f^{(1)}, \ldots, f^{(M)})$ and impose the corresponding conditions on the real-valued functions $f^{(1)}, \ldots, f^{(M)}$.

All of these spaces are metrizable. Take, for example, the space $\mathcal{C}^\infty(\mathbb{R}^N)$. In order to prove that $\mathcal{C}^\infty(\mathbb{R}^N)$ is metrizable, it is sufficient to consider the seminorms of the form (2.2) for the closed balls $K = \overline{\mathcal{B}}_{0,1}, \overline{\mathcal{B}}_{0,2}, \ldots$ and for the integers $l = 0, 1, \ldots$. They form a countable family of seminorms on $\mathcal{C}^\infty(\mathbb{R}^N)$, say $p_1, p_2, \ldots$. The topology in $\mathcal{C}^\infty(\mathbb{R}^N)$ is induced, for example, by the metric $d\colon \mathcal{C}^\infty(\mathbb{R}^N)^2 \longrightarrow \mathbb{R}$ where

$$d(f_1, f_2) = \sum_{i=1}^{\infty} \min \left\{ p_i(f_1 - f_2), 2^{-i} \right\},\ f_1, f_2 \in \mathcal{C}^\infty(\mathbb{R}^N).$$

[1] The convergence does not need to be induced by a topology.

In what follows, k will be a fixed non-negative integer that specifies the order of differentiability in $\mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^M)$.

Derivatives

In this thesis, our special interest lies in the first and second derivatives. Let $f \in \mathcal{C}^1(\mathbb{R}^N)$, $g \in \mathcal{C}^2(\mathbb{R}^N)$ and $h \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R}^N)$ be given functions. Then the values of the first derivative $\mathrm{D}f \colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ of f can be seen as linear forms whereas the values of the second derivative $\mathrm{D}^2 g \colon \mathbb{R}^N \longrightarrow \mathbb{R}^{N\times N}$ of g as well as the values of the first derivative $\mathrm{D}h(x) \colon \mathbb{R}^N \longrightarrow \mathbb{R}^{N\times N}$ of h can be seen as bilinear forms. We wish to make this precise via the following notation. Let $x, y_1, y_2 \in \mathbb{R}^N$ be real vectors. We will write

$$\mathrm{D}f(x)[y_1] = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x^{(i)}}(x) \cdot y_1^{(i)} \right),$$

$$\mathrm{D}^2 g(x)[y_1, y_2] = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{\partial^2 g}{\partial x^{(i)} \partial x^{(j)}}(x) \cdot y_1^{(i)} \cdot y_2^{(j)} \right) \text{ and}$$

$$\langle \mathrm{D}h(x)[y_1], y_2 \rangle = \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\frac{\partial h^{(j)}}{\partial x^{(i)}}(x) \cdot y_1^{(i)} \cdot y_2^{(j)} \right).$$

2.2.3 Mollifier Property

The following construction of a mollifier can be found for example in Königsberger [1997, pp. 316-319]. Let $\psi \in \mathcal{C}^\infty(\mathbb{R}^N)$ be the smooth function given by

$$\psi(x) = \begin{cases} \exp\left(\frac{-1}{1-|x|^2}\right) & |x| \le 1 \\ 0 & |x| > 1. \end{cases}$$

We define for every $s = 1, 2, \ldots$ the function $\psi_s \in \mathcal{C}^\infty(\mathbb{R}^N)$ by

$$\psi_s(x) = \frac{s^N}{c} \cdot \psi(sx) \text{ where } c = \int_{\mathbb{R}^N} \psi(x) \mathrm{d}x.$$

Then the function ψ_s is non-negative with compact support in the ball $\overline{\mathcal{B}}_{0,s^{-1}}$ and its integral equals 1 for every $s = 1, 2, \ldots$.

We can associate every continuous function $f \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$ with a sequence of smooth functions via the convolution

$$(f * \psi_s)(x) = \int_{\mathbb{R}^N} f(x-y)\psi_s(y)\mathrm{d}y,\ x \in \mathbb{R}^N,\ s = 1, 2, \ldots.$$

The space $\mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^M)$ shares the *mollifier property*. That means that we have $(f * \psi_s) \to f$ in $\mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^M)$ for every $f \in \mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^M)$.

2.2.4 Lower Semicontinuous Functions

Apart from continuous functions, we will also study functions that are only lower semicontinuous.

Definition 2.1 *An extended real-valued function* $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}\cup\{+\infty\}$ *is called lower semicontinuous if the lower level set* $\mathcal{N}_{f,\alpha} = \{x \in \mathbb{R}^N \mid f(x) \le \alpha\}$ *is a closed subset of* $\mathbb{R}^N$ *for every real number* $\alpha \in \mathbb{R}$.

We denote by $\mathcal{LSC}(\mathbb{R}^N)$ the set of all such functions and we write $f_s \to f$ in $\mathcal{LSC}(\mathbb{R}^N)$ whenever the following conditions are fulfilled:

(L1) For every compact subset $K \subseteq \mathbb{R}^N$ there exists a positive integer $s_K > 0$ so that $f_s \le f_{s+1}$ on K holds as long as $s \ge s_K$.

(L2) For every vector $x \in \mathbb{R}^N$ we have $f_s(x) \to f(x)$ in $\mathbb{R}\cup\{+\infty\}$.[2]

These conditions enforce a kind of convergence, which could be called *locally monotone convergence*. Note also that it is sufficient to consider the sets $K = \overline{\mathcal{B}}_{0,1}, \overline{\mathcal{B}}_{0,2}, \ldots$ in (L1). The pointwise limit in (L2) exists (possibly equal to $+\infty$) if (L1) holds true. Moreover, if a sequence $f_1, f_2, \ldots \in \mathcal{LSC}(\mathbb{R}^N)$ of lower semicontinuous functions fulfills (L1), there exists a lower semicontinuous function $f \in \mathcal{LSC}(\mathbb{R}^N)$ so that $f_s \to f$ in $\mathcal{LSC}(\mathbb{R}^N)$.

The convergence in $\mathcal{LSC}(\mathbb{R}^N)$ is relatively strong, compared, for example, with pointwise convergence. A connection to locally uniform convergence is shown in Lemma 2.2 below. In addition, the monotonicity property will guarantee variational convergence of minimization problems in Chapter 5. See Attouch [1984] for more about monotone schemes and their relation to Γ-convergence and epi-convergence.

However, if we work in $\mathcal{LSC}(\mathbb{R}^N)$, we have to face the difficulty that the convergence in $\mathcal{LSC}(\mathbb{R}^N)$ is not induced by a topology and the diagonal sequence argument is not available.

Embedding Lemma

We know that $\mathcal{C}(\mathbb{R}^N) \subseteq \mathcal{LSC}(\mathbb{R}^N)$. The corresponding embedding is not continuous. Nevertheless, we have the following lemma:

Lemma 2.2 *Let* $f_i, f_{i,j} \in \mathcal{C}(\mathbb{R}^N)$, $i, j = 1, 2, \ldots$, *be continuous functions and let* $g \in \mathcal{LSC}(\mathbb{R}^N)$ *be a lower semicontinuous function.*

[2] Recall that a sequence $x_1, x_2, \ldots \in \mathbb{R}\cup\{+\infty\}$ of extended real numbers converges to $+\infty$ if and only if none of its subsequences is bounded from above.

(i) If $f_{1,s} \to f_1$ holds in $\mathcal{C}(\mathbb{R}^N)$, there exist a subsequence $f_{1,t_1}, f_{1,t_2}, \ldots$ and constants $c_1, c_2, \ldots \in \mathbb{R}$ so that $(f_{1,t_s} + c_s) \to f_1$ in $\mathcal{LSC}(\mathbb{R}^N)$.

(ii) If $f_{t,s} \to f_t$ holds in $\mathcal{C}(\mathbb{R}^N)$ for every $t = 1, 2, \ldots$ and $f_t \to g$ in $\mathcal{LSC}(\mathbb{R}^N)$, there exist constants $c_1, c_2, \ldots \in \mathbb{R}$ as well as positive integers $0 < t_1 < t_2 < \ldots$ and $0 < s(t_1) < s(t_2) < \ldots$ so that $(f_{t_r,s(t_r)} + c_r) \to f$ in $\mathcal{LSC}(\mathbb{R}^N)$ as r tends to $+\infty$.

(iii) If $f_{1,s} \to f_1$ holds in $\mathcal{LSC}(\mathbb{R}^N)$, then also $f_{1,s} \to f_1$ in $\mathcal{C}(\mathbb{R}^N)$.

Proof. Point (iii) is a consequence of Dini's theorem and (i) is a special case of (ii). Let us prove (ii). Since $f_{t,s} \to f_t$ holds in $\mathcal{C}(\mathbb{R}^N)$ for every $t = 1, 2, \ldots$, we can choose positive integers $s(1) < s(2) < \ldots$ so that

$$\sup\left\{\left|f_{t,s(t)}(x) - f_t(x)\right| \mid x \in \overline{\mathcal{B}}_{0,t}\right\} \leq 3^{-t},\ t = 1, 2, \ldots.$$

We conclude that

$$f_{t,s(t)}(x) - 2 \cdot 3^{-t} \leq f_t(x) - 3^{-t} \leq f_{t+1,s(t+1)}(x) - 2 \cdot 3^{-(t+1)} \tag{2.3}$$

is true for every $x \in \overline{\mathcal{B}}_{0,t}$. Since $f_t \to g$ in $\mathcal{LSC}(\mathbb{R}^N)$, we can choose a subsequence $f_{t_1}, f_{t_2}, \ldots$ so that $f_{t_r} \leq f_{t_{r+1}}$ holds on $\overline{\mathcal{B}}_{0,r}$ for every $r = 1, 2, \ldots$. Together with (2.3), we have

$$f_{t_r,s(t_r)}(x) - 2 \cdot 3^{-t_r} \leq f_{t_{r+1},s(t_{r+1})}(x) - 2 \cdot 3^{-t_{r+1}},\ x \in \overline{\mathcal{B}}_{0,r}.$$

Set $c_r = -2 \cdot 3^{-t_r}$ and we get $(f_{t_r,s(t_r)} + c_r) \to f$ in $\mathcal{LSC}(\mathbb{R}^N)$ as desired. □

Note that the sequence in (ii) is almost diagonal (compare Section 2.2.1). Most of the properties of functions considered in this thesis are stable against linear perturbations. In particular, adding a constant to a function will not change its convexity properties. That is why this lemma can be applied on different occasions later on.

2.2.5 Sobolev Spaces

We only give a short introduction to Sobolev spaces. See Adams [1978] for a full discussion of definitions, properties and for the proofs that we skip here. Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty bounded open set with Lipschitz boundary and $1 < p \leq \infty$ an extended real number. The set $L^p(\Omega, \mathbb{R}^m)$ consists of all Lebesgue measurable functions[3] $u\colon \Omega \longrightarrow \mathbb{R}^m$ for that the L^p-norm $\|u\|_{L^p}$

[3] Every two functions are identified whenever they coincide outside a set of Lebesgue measure 0. As it is standard in the literature, we still use the term *function* to address elements of an L^p-space rather than *class of functions*.

exists and is finite. In the case $1 < p < \infty$, we have

$$\|u\|_{L^p} = \left(\int\limits_\Omega |u(x)|^p \mathrm{d}x \right)^{\frac{1}{p}}.$$

In the case $p = \infty$, we have

$$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|.$$

We are going to work in the Sobolev space

$$W^{1,p}(\Omega, \mathbb{R}^m) = \left\{ u \in L^p(\Omega, \mathbb{R}^m) \mid \frac{\partial u}{\partial x^{(1)}}, \ldots, \frac{\partial u}{\partial x^{(n)}} \in L^p(\Omega, \mathbb{R}^m) \right\}.$$

In the case $1 < p < \infty$, the norm is given by

$$\|u\|_{W^{1,p}} = \left(\|u\|_{L^p}^p + \left\| \frac{\partial u}{\partial x^{(1)}} \right\|_{L^p}^p + \ldots + \left\| \frac{\partial u}{\partial x^{(n)}} \right\|_{L^p}^p \right)^{\frac{1}{p}}.$$

In the case $p = \infty$, the norm is given by

$$\|u\|_{W^{1,\infty}} = \max \left\{ \|u\|_{L^\infty}, \left\| \frac{\partial u}{\partial x^{(1)}} \right\|_{L^\infty}, \ldots, \left\| \frac{\partial u}{\partial x^{(n)}} \right\|_{L^\infty} \right\}.$$

Here $\frac{\partial u}{\partial x^{(i)}}$, $i = 1, \ldots, n$ denote the weak derivatives (in the sense of distributions).

The space $W^{1,p}(\Omega, \mathbb{R}^m)$ together with the norm is a Banach space. We write $(W^{1,p}(\Omega, \mathbb{R}^m))^*$ for its dual space. Strong and weak convergence are understood in the usual way. Let $u, u_1, u_2, \ldots \in W^{1,p}(\Omega, \mathbb{R}^m)$ be Sobolev functions. We write $u_s \to u$ strongly in $W^{1,p}(\Omega, \mathbb{R}^m)$ whenever $\|u - u_s\| \to 0$ and $u_s \rightharpoonup u$ weakly in $W^{1,p}(\Omega, \mathbb{R}^m)$ if we have

$$\langle f, u_s \rangle_{W^{1,p}} \to \langle f, u \rangle_{W^{1,p}}$$

for every $f \in (W^{1,p}(\Omega, \mathbb{R}^m))^*$. Here $\langle ., . \rangle_{W^{1,p}}$ denotes the dual pairing defined on $(W^{1,p}(\Omega, \mathbb{R}^m))^* \times W^{1,p}(\Omega, \mathbb{R}^m)$.

We emphasize the following fact, because of its future usefulness in the context of Γ- and G-convergence.

Remark 2.3 *Let $1 < p$ be a real number and $u, u_1, u_2, \ldots \in W^{1,p}(\Omega, \mathbb{R}^m)$ Sobolev functions so that $u_s \to u$ strongly in $W^{1,p}(\Omega, \mathbb{R}^m)$. Then there exists a subsequence $u_{s_1}, u_{s_2}, \ldots$ and there exist functions $v, w \in L^p(\Omega)$ so that the following conditions are fulfilled:*

(i) $u_{s_t} \to u$ *and* $\mathrm{D}u_{s_t} \to \mathrm{D}u$ *pointwise almost everywhere*[4] *in* Ω*,*

(ii) $|u_{s_t}| \le v$ *almost everywhere in* Ω *for every* $t = 1, 2, \ldots$*,*

(iii) $|\mathrm{D}u_{s_t}| \le w$ *almost everywhere in* Ω *for every* $t = 1, 2, \ldots$*.*

Proof. We show (ii). We choose a subsequence $u_{\tilde{s}_1}, u_{\tilde{s}_2}, \ldots$ so that

$$\|u_{\tilde{s}_t} - u\|_{L^p}^p \le \frac{1}{2^t},\ t = 1, 2, \ldots. \tag{2.4}$$

Now the function $v \in L^p(\Omega)$ can be defined almost everywhere in Ω by setting

$$v = \left(|u|^p + \sum_{t=1}^{\infty} |u_{\tilde{s}_t} - u|^p \right)^{\frac{1}{p}}.$$

In fact, (2.4) implies that the L^p-norm of v is finite. Due to the convexity of the norm in $\mathbb{R}^n$, we have the pointwise estimate

$$|u_{\tilde{s}_t}|^p \le |u|^p + |u_{\tilde{s}_t} - u|^p \le v^p,\ t = 1, 2, \ldots,$$

almost everywhere in Ω. Hence we get (ii). It is well-known that the condition (i) can be realized too. See, for example, Königsberger [1997, p. 268, Satz 1(ii)]. As a matter of fact, we have already $u_{s_t} \to u$ pointwise almost everywhere in Ω by the choice of the subsequence $u_{\tilde{s}_1}, u_{\tilde{s}_2}, \ldots$. In order to show (iii), we choose a subsequence of $u_{\tilde{s}_1}, u_{\tilde{s}_2}, \ldots$ using an analogous argument. Hence we can pass to a subsequence $u_{s_1}, u_{s_2}, \ldots$ so that the three conditions are fulfilled simultaneously. □

A subsequence that fulfills the conditions (ii) and (iii) is called *equi-integrable* in $W^{1,p}(\Omega, \mathbb{R}^m)$. As a direct consequence of Remark 2.3, we get a corresponding result for the space $L^p(\Omega, \mathbb{R}^m)$ as long as $1 < p < \infty$.

In the case $1 < p < \infty$, the space $W^{1,p}(\Omega, \mathbb{R}^m)$ is reflexive as well as separable and so is its dual space. The space $W^{1,\infty}(\Omega, \mathbb{R}^m)$ can be identified with the space of all Lipschitz continuous functions $u \colon \Omega \longrightarrow \mathbb{R}^m$. We set

$$W_0^{1,p}(\Omega, \mathbb{R}^m) = \left\{ u \in W^{1,p}(\Omega, \mathbb{R}^m) \mid u = 0 \text{ on } \partial\Omega \right\}$$

in the sense of traces. Note also that the space $W^{1,p}(\Omega, \mathbb{R}^m)$ together with the strong topology is metrizable, whereas, together with the weak topology, it is not metrizable.

[4] A property is said to hold *almost everywhere* in Ω if it is satisfied outside a set with Lebesgue measure 0.

2.3 Stone-Weierstraß Theorem

We can easily embed algebraic polynomials in the space of smooth functions. Let $\mathbb{R}[X_1, \ldots, X_N]$ be the ring of polynomials in N variables over the real numbers and $P = (P^{(1)}, \ldots, P^{(M)})$ a given polynomial map with components $P^{(1)}, \ldots, P^{(M)} \in \mathbb{R}[X_1, \ldots, X_N]$. Then P can be identified with a smooth function $P\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$. We denote by $\mathcal{P}(\mathbb{R}^N, \mathbb{R}^M)$ the set of all such smooth functions[5].

The Stone-Weierstraß theorem is fundamental in approximation theory and an important tool in this thesis. The next theorem can be seen as a direct application of the Stone-Weierstraß theorem to simultaneous approximation of derivatives.

Theorem 2.4 *Every smooth function $f \in \mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^M)$ can be approximated by polynomial maps $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^N, \mathbb{R}^M)$ so that $P_s \to f$ in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^M)$.*

Proof. The topology in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^M)$ is metrizable. Hence it suffices to show that there exists a sequence of polynomial maps $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^N, \mathbb{R}^M)$ so that $P_s \to f$ in $\mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^M)$ holds for every non-negative integer $k \geq 0$. See Sauvigny [2006, pp. 6-7] for the case $M = 1$. The higher dimensional case is an immediate consequence. $\square$

2.4 Properties of Sets of Functions

We will be concerned with two different regimes: vector-valued functions $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ and extended real-valued functions $f\colon \mathbb{R}^N \longrightarrow \mathbb{R} \cup \{+\infty\}$. In order to give a list of some basic properties, let $\mathcal{F}$ be a given set of functions of the form $f\colon \mathbb{R}^N \longrightarrow \mathcal{R}$ where $\mathcal{R} = \mathbb{R}^M$ or $\mathcal{R} = \mathbb{R} \cup \{+\infty\}$.

- The set $\mathcal{F}$ is called a *cone* if for every $f \in \mathcal{F}$ and every non-negative real number $\lambda \geq 0$, the function λf is contained in $\mathcal{F}$.[6]

- The set $\mathcal{F}$ is called *convex* if for every $f, g \in \mathcal{F}$ and every real number $\lambda \in [0, 1]$, the function $\lambda f + (1 - \lambda) g$ is contained in $\mathcal{F}$.

- The set $\mathcal{F}$ is called *translation invariant* if for every $f \in \mathcal{F}$ and every real vector $x_0 \in \mathbb{R}^N$, the function $g\colon \mathbb{R}^N \longrightarrow \mathcal{R}$ defined by $g(x) = f(x + x_0)$, $x \in \mathbb{R}^N$, is contained in $\mathcal{F}$.

[5] We will sometimes write $\mathcal{P}(\mathbb{R}^N)$ if $M = 1$.

[6] As usual, we set $+\infty \cdot 0 = 0 \cdot (+\infty) = 0$.

With the help of $\mathcal{F}$, we define a set $\mathcal{F}_{\mathrm{loc}}$ that can be seen as a local version of $\mathcal{F}$. A function $f\colon \mathbb{R}^N \longrightarrow \mathcal{R}$ is an element of $\mathcal{F}_{\mathrm{loc}}$ if there exists an index set I (possibly uncountable), a family $U_\alpha \subseteq \mathbb{R}^N$, $\alpha \in I$, of open subsets of $\mathbb{R}^N$ and a family $f_\alpha \in \mathcal{F}$, $\alpha \in I$, of functions so that the following holds:

(i) The sets U_α, $\alpha \in I$, cover the whole of $\mathbb{R}^N$, meaning $\bigcup\limits_{\alpha\in I} U_\alpha = \mathbb{R}^N$.

(ii) We have $f = f_\alpha$ on U_α for every $\alpha \in I$.

Clearly $\mathcal{F}$ is a subset of $\mathcal{F}_{\mathrm{loc}}$.
If we have $\mathcal{F} = \mathcal{F}_{\mathrm{loc}}$, then the set $\mathcal{F}$ is called *locally definable* and the property of functions associated with $\mathcal{F}$ is called *local*[7].

2.5 Basic Approximation Results

The results that we collect in this section (except maybe Remark 2.9) are well-known in approximation theory. We give the details, since we will point out special properties of the approximation procedures later on.

2.5.1 Approximation by Continuous Functions

We start with the approximation of lower semicontinuous function by continuous functions. Let $f \in \mathcal{LSC}(\mathbb{R}^N)$ be a lower semicontinuous function. Then we write $\mathcal{N}_{f,\alpha} = \{x \in \mathbb{R}^N \mid f(x) \le \alpha\}$, $\alpha \in \mathbb{R}$, for the lower level sets of f.

Lemma 2.5 *Every lower semicontinuous function $f \in \mathcal{LSC}(\mathbb{R}^N)$ can be approximated by continuous functions $f_1, f_2, \ldots \in \mathcal{C}(\mathbb{R}^N)$ so that $f_s \to f$ in $\mathcal{LSC}(\mathbb{R}^N)$.*

Proof.[8] In this proof, we use the conditions (L1) and (L2) from Section 2.2.4. If $f = +\infty$ on the whole of $\mathbb{R}^N$, we set $f_s = s$ for $s = 1, 2, \ldots$. Now let $f \ne +\infty$. Without loss of generality, we assume that $f(0) < +\infty$ holds. If necessary, we translate the coordinate system.
Fix a positive integer $s > 0$. We define a function by specifying its lower

[7]In an abstract sense, this corresponds to the question whether $\mathcal{F}$ defines a sheaf or just a presheaf.

[8]There is a short proof of Lemma 2.5, which we do not give here. Instead, we will use a strategy that fits also to the requirements of Chapter 5.

level sets. Therefore we consider the family $\mathcal{M}_\alpha \subseteq \mathbb{R}^N$, $\alpha \in \mathbb{R}$, of sets given by

$$\mathcal{M}_\alpha = \begin{cases} (\mathcal{N}_{f,\alpha} + \overline{\mathcal{B}}_{0,5^{-s}(\alpha+s)}) \cap \overline{\mathcal{B}}_{0,s+5^{-s}(\alpha+s)} & \alpha \geq -s \\ \emptyset & \alpha < -s. \end{cases} \tag{2.5}$$

Here we used the Minkowski sum: $A + B = \{x + y \in \mathbb{R}^N \mid x \in A,\ y \in B\}$, defined for sets $A, B \subseteq \mathbb{R}^N$.

All the sets $\mathcal{M}_\alpha$, $\alpha \in \mathbb{R}$, are compact. Moreover, we have $\mathcal{M}_\alpha \subseteq \mathcal{M}_\beta$ whenever $\alpha \leq \beta$. Recall that $f(0) < +\infty$. In particular, $0 \in \mathcal{N}_{f,\alpha}$ holds for every $\alpha \geq \max\{f(0), -s\}$. Hence, for every $x \in \mathbb{R}^n$, there exists a real number $\alpha \in \mathbb{R}$ so that $x \in \mathcal{M}_\alpha$ and

$$\alpha \leq \max\{f(0), -s\} + 5^s|x|.$$

In fact, the following inequality holds:

$$-s \leq \inf\{\alpha \in \mathbb{R} \mid x \in \mathcal{M}_\alpha\} \leq \max\{f(0), -s\} + 5^s|x|,\ x \in \mathbb{R}^N.$$

The infimum is attained (and, therefore, is a minimum), since the sets $\mathcal{N}_{f,\alpha}$, $\alpha \in \mathbb{R}$, are the lower level sets of a function. We conclude that

$$f_s(x) = \min\{\alpha \in \mathbb{R} \mid x \in \mathcal{M}_\alpha\}$$

is well-defined for every $x \in \mathbb{R}^N$. The real-valued function f_s is specified by its lower level sets $\mathcal{N}_{f_s,\alpha} = \mathcal{M}_\alpha$, $\alpha \in \mathbb{R}$. Equation (2.5) also permits the estimate

$$|f_s(x) - f_s(y)| \leq 5^s|x - y|,\ x, y \in \mathbb{R}^N. \tag{2.6}$$

Hence the function f_s is Lipschitz continuous and, in particular, lies in $\mathcal{C}(\mathbb{R}^N)$. In the remaining part, we prove that $f_s \to f$ in $\mathcal{LSC}(\mathbb{R}^N)$ as $s \to +\infty$.

Fix a positive integer $r > 0$. The function f is lower semicontinuous and does not take the value $-\infty$. Hence there exists a real number $c_r \in \mathbb{R}$ that realizes the minimum of f over the ball $\overline{\mathcal{B}}_{0,r}$. Set $s_r = \max\{r, 1 + |c_r|\}$. We show that we have

$$f(x) \geq f_{s+1}(x) \geq f_s(x),\ x \in \overline{\mathcal{B}}_{0,r} \tag{2.7}$$

for every $s \geq s_r$. The choice of s_r implies that $f \geq f_{s+1}$ on $\overline{\mathcal{B}}_{0,r}$ for every $s \geq s_r$. In order to prove (2.7), we still have to show that $f_{s+1} \geq f_s$ holds on $\overline{\mathcal{B}}_{0,r}$. It suffices to show

$$\overline{\mathcal{B}}_{0,5^{-(s+1)}(\alpha+s+1)} \subseteq \overline{\mathcal{B}}_{0,5^{-s}(\alpha+s)},\ \alpha \geq c_r. \tag{2.8}$$

We have $\alpha + s + 1 \leq 5(\alpha + s)$ as long as $\alpha + s \geq 1$. Since $s \geq 1 + |c_r|$, we get $5^{-(s+1)}(\alpha + s + 1) \leq 5^{-s}(\alpha + s)$ for every $\alpha \geq c_r$. This implies (2.8) and finishes the proof of (2.7).

Fix a vector $x_0 \in \overline{\mathcal{B}}_{0,r}$. We show that $f_s(x_0) \to f(x_0)$ in $\mathbb{R} \cup \{+\infty\}$, which is equivalent to (L2). We know that $f_{s_r}(x_0), f_{s_r+1}(x_0), \ldots$ yields a non-decreasing sequence of real numbers that converges to some limit $y_0 \in \mathbb{R} \cup \{+\infty\}$ due to (2.7). Assume that $y_0 \neq f(x_0)$. Set $\alpha_0 = \min\{\frac{y_0 + f(x_0)}{2}, y_0 + 1\}$. We then have $y_0 < \alpha_0 < f(x_0)$. In particular, α_0 is a real number (even for $f(x_0) = +\infty$). Set $d_0 = \min\{|x_0 - y| \mid y \in \mathcal{N}_{f,\alpha_0}\}$. The vector x_0 cannot lie in the lower level set $\mathcal{N}_{f,\alpha_0}$. Hence we must have $d_0 > 0$, since the function f is lower semicontinuous.
Choose the integer $s \geq s_r$ so that $s \geq \alpha_0$ and $5^{-s}(\alpha_0 + s) < \frac{d_0}{2}$ hold. Then (2.5) implies that $\mathcal{N}_{f_s,\alpha_0} \subseteq \mathcal{N}_{f,\alpha_0} + \overline{\mathcal{B}}_{0,d_0/2}$. Hence $f_s(x_0) \geq \alpha_0$ holds and the sequence $f_{s_r}(x_0), f_{s_r+1}(x_0), \ldots$ cannot be non-decreasing, since $y_0 < \alpha_0$ and $f_s(x_0) \to y_0$ as s tends to $+\infty$. This is a contradiction to (2.7).

Fix a compact set $K \subseteq \mathbb{R}^N$. If r is chosen so that $K \subseteq \overline{\mathcal{B}}_{0,r}$ holds, we have proven two things: $f_s \leq f_{s+1}$ on K as long as $s \geq s_r$ and $f_s(x) \to f(x)$ in $\mathbb{R} \cup \{+\infty\}$ for every $x \in K$. As a consequence, we get that $f_s \to f$ holds in $\mathcal{LSC}(\mathbb{R}^N)$. □

2.5.2 Approximation by Smooth Functions

The main concern of this thesis is polynomial approximation. As a first step, let us concentrate on the approximation by smooth functions. We will see that a large class of properties of continuous functions can be preserved during the approximation by smooth functions. In particular, the framework of admissible sets in Chapter 3 and Chapter 4 will meet the requirements of the results in this section.

In the remaining part of this chapter, we make use of the functions $\psi_1, \psi_2, \ldots$ given in Section 2.2.3.

Lemma 2.6 *Let $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$ be a convex cone, translation invariant and closed in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$. Let $f \in \mathcal{F}$ be a given function. Then the convolution $f * \psi_s$ lies in $\mathcal{F}$ for every $s = 1, 2, \ldots$.*

Proof. Fix positive integers $s, t > 0$, set $s' = 2^{N(s+1)}$ and let $\{y_1, y_2, \ldots, y_{s'}\}$ be the set of all vectors in the cube $[-1, 1]^N$ so that

$$y_j^{(i)} \in \{-1, -1 + 2^{-s}, -1 + 2 \cdot 2^{-s}, \ldots, 1 - 2^{-s}\}$$

holds for every $i = 1, \ldots, N$, $j = 1, \ldots, s'$. With the help of these vectors, we divide $[-1, 1]^N$ into subcubes $Q_1, Q_2, \ldots, Q_{s'} \subseteq \mathbb{R}^N$ given by

$$Q_j = [y_j^{(1)}, y_j^{(1)} + 2^{-s}] \times \ldots \times [y_j^{(N)}, y_j^{(N)} + 2^{-s}],\ j = 1, \ldots, s'.$$

We consider the function $g_s\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ defined by

$$g_s(x) = \frac{1}{s'}\sum_{j=1}^{s'} f(x - y_j)\psi_t(y_j),\ x \in \mathbb{R}^N.$$

The function ψ_t is non-negative. Hence all real numbers $\psi_t(y_1), \ldots, \psi_t(y_{s'})$ are non-negative and we have $g_s \in \mathcal{F}$, since the set $\mathcal{F}$ is a convex cone and translation invariant.

Let s tend to infinity, while $t > 0$ remains fixed. We are going to show that $g_s \to (f * \psi_t)$ in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$. Let $K \subseteq \mathbb{R}^N$ be an arbitrary non-empty compact set and $\epsilon > 0$ a positive real number. Note that the support of the function ψ_t is bounded in $\overline{\mathcal{B}}_{0,1} \subseteq [-1, 1]$. Hence for every $x \in K$ we have the estimate

$$|(f * \psi_t)(x) - g_s(x)| \leq \sum_{j=1}^{s'} \int_{Q_j} \underbrace{|f(x - y)\psi_t(y) - f(x - y_j)\psi_t(y_j)|}_{\alpha_j(x,y)}\,\mathrm{d}y.$$

The triangle inequality implies that

$$\alpha_j(x, y) \leq |f(x - y) - f(x - y_j)| \cdot |\psi_t(y)| + |f(x - y_j)| \cdot |\psi_t(y) - \psi_t(y_j)|.$$

Since $K, [-1, 1] \subseteq \mathbb{R}^N$ are compact subsets and f as well as ψ_t continuous functions, there exists a positive integer $s_{K,\epsilon}$ so that

$$\sup\{|(f * \psi_t)(x) - g_s(x)| \mid x \in K\} < \epsilon,\ s \geq s_{K,\epsilon}.$$

Thus $g_s \to (f * \psi_t)$ holds in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$ and the convolution $f * \psi_t$ lies in the set $\mathcal{F}$, since $\mathcal{F}$ is closed in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$. □

Note that the assertion remains valid if $\mathcal{F}$ is not a convex cone but only convex.

With the help of Lemma 2.6, the proof of the next theorem is immediate.

Theorem 2.7 *Let $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$ be a convex cone, translation invariant and closed in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$. Then every function $f \in \mathcal{F} \cap \mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^M)$ can be approximated by smooth functions $f_1, f_2, \ldots \in \mathcal{F} \cap \mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^M)$ so that $f_s \to f$ in $\mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^M)$.*

Proof. We set $f_s = f * \psi_s$ for every $s = 1, 2, \ldots$. Then Lemma 2.6 implies that the functions $f_1, f_2, \ldots$ lie in $\mathcal{F} \cap \mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^M)$. Since the function space $\mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^M)$ has the mollifier property we conclude that $f_s \to f$ in $\mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^M)$. □

2.5.3 About the Polynomial Growth of Functions

In this section and further on, we will use the following notation.
A real-valued function $f\colon \mathbb{R}^N \longrightarrow \mathbb{R}^M$ is said to be *of polynomial growth* if there exists a polynomial $P \in \mathcal{P}(\mathbb{R}^N)$ so that $|f(x)| \leq P(x)$ holds for every $x \in \mathbb{R}^N$.

We begin with a lemma.

Lemma 2.8 *Let $f \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$ be a continuous function of polynomial growth. Then the convolution $f * \psi_s$ and its partial derivatives are also of polynomial growth for every $s = 1, 2, \ldots$.*

Proof. Fix a positive integer $s > 0$. We will show that $f * \psi_s$ is of polynomial growth. Fix an integer $j \in \{1, \ldots, M\}$.
The function $\psi_s \in \mathcal{C}^\infty(\mathbb{R}^N)$ is smooth with compact support in $\overline{\mathcal{B}}_{0,1}$. In particular, ψ_s is bounded. Thus there exists a positive constant $c > 0$ so that

$$|(f * \psi_s)^{(j)}(x)| \leq c \int_{\overline{\mathcal{B}}_{0,1}} |f^{(j)}(x - y)| \mathrm{d}y, \; x \in \mathbb{R}^N. \tag{2.9}$$

By assumption, the function f is of polynomial growth. Hence there exists a polynomial $P \in \mathcal{P}(\mathbb{R}^N)$ so that $|f^{(j)}(x - y)| \leq P(x - y)$ for every $x, y \in \mathbb{R}^N$. Together with (2.9), this implies that there exists a polynomial $Q \in \mathcal{P}(\mathbb{R}^N)$, having the same absolute degree like P, so that

$$|(f * \psi_s)^{(j)}(x)| \leq Q(x), \; x \in \mathbb{R}^N.$$

Consequently, $f * \psi_s$ is of polynomial growth.

Since ψ_s is a smooth function, we have

$$\frac{\partial (f * \psi_s)^{(j)}}{\partial x^{(i)}} = f^{(j)} * \frac{\partial \psi_s}{\partial x^{(i)}}, \; i = 1, \ldots, N. \tag{2.10}$$

By iteration of (2.10), we get corresponding equations for higher-order partial derivatives of $(f * \psi_s)^{(j)}$. All partial derivatives of the function ψ_s are smooth with compact support in $\overline{\mathcal{B}}_{0,1}$. Thus we can replace ψ_s by any of its partial derivatives in the above argument. □

In the next remark, we discuss an interesting case where we can strengthen the convergence.

Remark 2.9 *Let $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$ be a convex cone, translation invariant and closed in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$. Let $f \in \mathcal{F} \cap \mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^M)$ be a smooth function that can be approximated by functions $f_1, f_2, \ldots \in \mathcal{F}$ of polynomial growth so that $f_s \to f$ in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$. Then there exist smooth functions $g_1, g_2, \ldots \in \mathcal{F} \cap \mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^M)$ of polynomial growth so that $g_s \to f$ in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^M)$.*

Proof. Set $h_s = f_s - f$ for every $s = 1, 2, \ldots$. We then have $h_s \to 0$ in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$. Fix integers $k > 0$ and $j \in \{1, \ldots, M\}$.

Since the function ψ_k is smooth with compact support, there exists a real number $c_k \in \mathbb{R}$ that bounds the partial derivatives

$$c_k \geq \sup \left\{ \left| \frac{\partial^k \psi_k}{\partial x^{(i_1)} \cdots \partial x^{(i_k)}}(x) \right| \mid x \in \mathbb{R}^N,\ i_1, \ldots, i_k \in \{1, \ldots, N\} \right\}.$$

We set $\tilde{c}_k = \max \{c_l \mid l \leq k\}$.

Together with (2.10), we can refine (2.9) so that we have

$$\left| \frac{\partial^k (h_s * \psi_k)^{(j)}}{\partial x^{(i_1)} \cdots \partial x^{(i_k)}}(x) \right| \leq \tilde{c}_k \int\limits_{\overline{\mathcal{B}}_{0,1}} |h_s^{(j)}(x - y)| \mathrm{d}y,\ x \in \mathbb{R}^N \tag{2.11}$$

for every $s = 1, 2, \ldots$ and indices $i_1, \ldots, i_k \in \{1, \ldots, N\}$.

Since $h_s \to 0$ in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$, we find a positive integer $t_k > 0$ so that

$$\sup \left\{ \left| h_t^{(j)}(x) \right| \mid x \in \overline{\mathcal{B}}_{0,k+1} \right\} \leq \frac{1}{k \cdot 2^N \cdot \tilde{c}_k}$$

holds for every $t \geq t_k$. Here 2^N can be replaced by any number larger than the volume of $\overline{\mathcal{B}}_{0,1}$. By (2.11), we have

$$\sup \left\{ \left| \frac{\partial^k (h_t * \psi_k)^{(j)}}{\partial x^{(i_1)} \cdots \partial x^{(i_k)}}(x) \right| \mid x \in \overline{\mathcal{B}}_{0,k},\ i_1, \ldots, i_k \in \{1, \ldots, N\} \right\} \leq \frac{1}{k}$$

for every $t \geq t_k$. Hence $(h_{t_k} * \psi_k) \to 0$ in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^M)$ as $k \to +\infty$.

Recall that $h_{t_s} = f_{t_s} - f$ for every $s = 1, 2, \ldots$. The sequence of convolutions $f * \psi_1, f * \psi_2, \ldots$ converges not only in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$ but also in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^M)$. This implies that $(f_{t_s} * \psi_s) \to f$ holds in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^M)$.

In addition to that, by Lemma 2.6 and Lemma 2.8, all convolutions $f_{t_1} * \psi_1, f_{t_2} * \psi_2, \ldots$ lie in $\mathcal{F}$ and are of polynomial growth. Hence we can set $g_s = f_{t_s} * \psi_s$ for every $s = 1, 2, \ldots$. □

Chapter 3

Quasiconvexity and Variational Minimization

We first recall Problem 1 of the introduction, which will be at the heart of this chapter:

$$\text{Minimize} \int_\Omega f(x, u(x), \mathrm{D}u(x))\mathrm{d}x \text{ among all } u \in W_0^{1,p}(\Omega, \mathbb{R}^m).$$

Here $\Omega \subseteq \mathbb{R}^n$ denotes a non-empty bounded open set with Lipschitz boundary, $p > 1$ a real number, $W_0^{1,p}(\Omega, \mathbb{R}^m)$ a Sobolev space and $f \in \mathcal{C}(\Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n})$ a continuous real-valued function.

Before we study the approximation of this problem, we will concentrate on the preservation of convexity notions in the context of polynomial approximation of continuous functions. We present an abstract framework that includes many convexity notions and, in particular, quasiconvexity in the calculus of variations. We begin with a few properties of convex functions.

3.1 Some Properties of Convex Functions

The proofs of the abstract results of this chapter make use of two properties of convex functions. They are both well-known. Nevertheless, we would like to give the arguments here. See Rockafellar [1970], Hörmander [1994] as well as Stoer and Witzgall [1970] for more about convex functions over finite dimensional vector spaces.

Definition 3.1 *A lower semicontinuous function $f \in \mathcal{LSC}(\mathbb{R}^N)$ is called convex if the inequality*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

holds true for every $x, y \in \mathbb{R}^N$ *and every real number* $\lambda \in [0,1]$.

We can characterize a convex twice continuously differentiable function with the help of its second derivative, compare for example Rockafellar [1970, Theorem 4.5].

Remark 3.2 *Let* $f \in \mathcal{C}^2(\mathbb{R}^N)$ *be a given function. Then* f *is convex if and only if its second derivative* $\mathrm{D}^2 f$ *is positive semi-definite on* $\mathbb{R}^N$.

Proof. Let $f \in \mathcal{C}^2(\mathbb{R}^N)$ be convex. Fix vectors $x_0, y_0 \in \mathbb{R}^N$. Set $x = x_0 + ty_0$ and $y = x_0 - ty_0$ for some real number $t > 0$. Then the convexity of the function f implies

$$\frac{f(x_0 + ty_0) - 2f(x) + f(x_0 - ty_0)}{t^2} \geq 0.$$

Let t tend to 0. Then the last inequality just reads $\mathrm{D}^2 f(x_0)[y_0, y_0] \geq 0$.

Conversely, let $f \in \mathcal{C}^2(\mathbb{R}^N)$ be a function so that its second derivative $\mathrm{D}^2 f$ is positive semi-definite on $\mathbb{R}^N$. Fix arbitrary vectors $x_0, x_1, y_0 \in \mathbb{R}^N$ and a real number $\lambda_0 \in [0,1]$ so that $y_0 = \lambda_0 x_0 + (1-\lambda_0)x_1$. We are going to show that $f(y_0) \leq \lambda_0 f(x_0) + (1-\lambda_0) f(x_1)$.

We define the function $g \in \mathcal{C}^2(\mathbb{R}^N)$ by a linear perturbation of f

$$g(x) = f(x) - \mathrm{D}f(y_0)[x], \; x \in \mathbb{R}^N.$$

Then $\mathrm{D}g(y_0) = 0$ and the second-order Taylor expansions around y_0 for the vectors x_0 and x_1 read

$$g(x_i) = g(y_0) + \frac{1}{2}\mathrm{D}^2 f(\xi_i)[x_i - y_0, x_i - y_0], \; i \in \{0,1\},$$

for some vectors $\xi_0, \xi_1 \in \mathbb{R}^N$.

By assumption, the second derivative $\mathrm{D}^2 f$ is positive semi-definite on $\mathbb{R}^N$. The above Taylor expansions imply that $g(y_0) \leq \lambda_0 g(x_0) + (1-\lambda_0) g(x_1)$ holds. Finally, we have $f(y_0) \leq \lambda_0 f(x_0) + (1-\lambda_0) f(x_1)$, since linear perturbations have no effect on this inequality. □

Fix positive real numbers $r, t > 0$. We consider the function $\theta_{r,t} \in \mathcal{C}(\mathbb{R}^N)$ defined by

$$\theta_{r,t}(x) = \frac{t}{6}\left(d_r(x)\right)^3, \; x \in \mathbb{R}^N,$$

where $d_r(x) = \max\{0, |x|^2 - (\frac{3}{2}r)^2\}$. The function $\theta_{r,t}$ is twice continuously differentiable on $\mathbb{R}^N$. Its derivatives can be written in the following form:

$$\mathrm{D}\theta_{r,t}(x)[y] = t\langle x, y\rangle \left(d_r(x)\right)^2, \; x, y \in \mathbb{R}^N,$$

$$\mathrm{D}^2\theta_{r,t}(x)[y,y] = t|y|^2(d_r(x))^2 + 4t\langle x,y\rangle^2 d_r(x),\ x,y \in \mathbb{R}^N.$$

Since d_r is a non-negative function, the derivative $\mathrm{D}^2\theta_{r,t}$ is positive semi-definite on $\mathbb{R}^N$ and, by Remark 3.2, the function $\theta_{r,t}$ is convex. We have the estimate

$$\mathrm{D}^2\theta_{r,t}(x)[y,y] \geq t|y|^2\left(|x|^2 - \left(\frac{3}{2}r\right)^2\right)^2 \geq 3t|y|^2r^4 \tag{3.1}$$

for every $x \in \mathbb{R}^N \setminus \overline{\mathcal{B}}_{0,2r}$ and every $y \in \mathbb{R}^N$. With the help of the functions $\theta_{r,t}$, $r,t > 0$, we extend convex smooth functions to the whole space.

Remark 3.3 *Let $K \subseteq \mathbb{R}^N$ be an open subset and $f \in \mathcal{C}^2(\mathbb{R}^N)$ a function so that $\mathrm{D}^2 f$ is positive semi-definite on K. Then for every vector $x_0 \in K$, there exists a positive real number $r > 0$ and a convex function $g \in \mathcal{C}^2(\mathbb{R}^N)$ so that $f = g$ on $\mathcal{B}_{x_0,r}$.*

Proof. Without loss of generality, we assume that $x_0 = 0$. Fix a radius $r > 0$ so that $\overline{\mathcal{B}}_{0,3r} \subseteq K$. Let $\phi \in \mathcal{C}^\infty(\mathbb{R}^N)$ be a cut-off function so that $\phi = 1$ on $\overline{\mathcal{B}}_{0,2r}$ and $\phi = 0$ on $\mathbb{R}^N \setminus \mathcal{B}_{0,3r}$.

The second-order partial derivatives of the function $f \cdot \phi$ are bounded. Thus there exists a positive real number $t_0 > 0$ so that $\mathrm{D}^2(f \cdot \phi)(x)[y,y] \geq -t_0|y|^2$ for every $x,y \in \mathbb{R}^N$. Set $t_1 = \frac{1}{3}r^{-4}t_0$ and $g = f \cdot \phi + \theta_{r,t_1}$. Then $g = f$ on $\mathcal{B}_{0,r}$ and, in view of (3.1), $\mathrm{D}^2 g$ is positive semi-definite on $\mathbb{R}^N$. Remark 3.2 implies that the function $g \in \mathcal{C}^2(\mathbb{R}^N)$ is convex. □

3.2 Abstract Results

In this chapter, we call a set $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^N)$ of continuous functions *admissible* if all of the following conditions hold:

(F1) $\mathcal{F}$ is a convex cone.

(F2) $\mathcal{F}$ is translation invariant.

(F3) $\mathcal{F}$ is closed in $\mathcal{C}(\mathbb{R}^N)$.

(F4) $\mathcal{F}$ contains all convex functions in $\mathcal{C}(\mathbb{R}^N)$.

We will see later on that the convexity notions in the calculus of variations are included in this framework.

3.2.1 Functions of Polynomial Growth

The growth of a function plays an important role, as we will see in Section 3.2.2. The aim of this section is to approximate functions of arbitrary growth with the help of functions of polynomial growth. We begin with a useful observation.

Lemma 3.4 *Let $A \in \mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^{N\times N})$ be a smooth matrix-valued function of polynomial growth and $r > 0$ a positive real number. Then there exists a convex non-negative polynomial $P_r \in \mathcal{P}(\mathbb{R}^N)$ so that $A + \mathrm{D}^2(P_r)$ is positive semi-definite on $\mathbb{R}^N \setminus \mathcal{B}_{0,2r}$ and we have*

$$\sup\left\{\left|\frac{\partial^k P_r}{\partial x^{(i_1)}\cdots\partial x^{(i_k)}}(x)\right| \mid x \in \overline{\mathcal{B}}_{0,r}\right\} < \frac{1}{r}$$

for every integers $0 \le k \le r$ and indices $i_1, \ldots, i_k \in \{1, \ldots, N\}$.

Proof. We consider convex non-negative polynomials $R_2, R_3, \ldots \in \mathcal{P}(\mathbb{R}^N)$ defined by

$$R_t(x) = \left(\frac{2}{3r}\right)^{2t} |x|^{2t},\ x \in \mathbb{R}^N, t \ge 2. \tag{3.2}$$

We show that there exists an integer $t_1 \ge 2$ so that we have

$$\sup\left\{\left|\frac{\partial^k R_t}{\partial x^{(i_1)}\cdots\partial x^{(i_k)}}(x)\right| \mid x \in \overline{\mathcal{B}}_{0,r}\right\} < \frac{1}{r},\ t \ge t_1 \tag{3.3}$$

for every integer $0 \le k \le r$ and indices $i_1, \ldots, i_k \in \{1, \ldots, N\}$. The polynomials $R_2, R_3, \ldots$ and their partial derivatives are homogeneous. Hence the supremum in (3.3) is attained on the boundary of $\overline{\mathcal{B}}_{0,r}$. In fact, as long as $t \ge r$, there exists a non-negative constant $c > 0$ independent of t so that

$$\sup\left\{\left|\frac{\partial^k R_t}{\partial x^{(i_1)}\cdots\partial x^{(i_k)}}(x)\right| \mid x \in \overline{\mathcal{B}}_{0,r}\right\} < c\, \underbrace{\left[\left(\frac{2}{3r}\right)^{2t} r^{2t-k}\right]}_{\alpha(t)} \prod_{j=0}^{k-1}(2t-j).$$

holds for every integer $0 \le k \le r$ and indices $i_1, \ldots, i_k \in \{1, \ldots, N\}$. Clearly we have $\alpha(t) \to 0$ as $t \to +\infty$. Since the growth of the term $\alpha(t)$ dominates the right-hand side for large t, there must be an integer $t_1 \ge 2$ so that (3.3) holds true.

The calculation of the second derivative shows that we have

$$\mathrm{D}^2 R_t(x)[y,y] = \left(\frac{2}{3r}\right)^{2t} \left(4t(t-1)|x|^{2t-4}\langle x, y\rangle^2 + 2t|x|^{2t-2}|y|^2\right)$$

and, by neglecting a non-negative term on the right-hand side,

$$\mathrm{D}^2 R_t(x)[y,y] \geq 2t\left(\frac{2}{3r}\right)^{2t} |x|^{2t-2}|y|^2 \tag{3.4}$$

for every $x, y \in \mathbb{R}^N$ and $t \geq 2$.

Since A is of polynomial growth, there exists a polynomial $P \in \mathcal{P}(\mathbb{R}^N)$ so that $|A[y,y]| \leq P(x) \cdot |y|^2$ holds for every $x, y \in \mathbb{R}^N$.
Hence, by (3.4), we can find an integer $t_2 \geq t_1$ so that $A + \mathrm{D}^2(R_{t_2})$ is positive semi-definite on $\mathbb{R}^N \setminus \mathcal{B}_{0,2r}$. Consequently we can set $P_r = R_{t_2}$. □

As a direct consequence we get the following corollary:

Corollary 3.5 *Let $f \in \mathcal{C}^\infty(\mathbb{R}^N)$ be a smooth real-valued function so that the second derivative $\mathrm{D}^2 f$ is of polynomial growth. Let $r > 0$ be a positive real number. Then there exists a convex non-negative polynomial $P_r \in \mathcal{P}(\mathbb{R}^N)$ so that $\mathrm{D}^2 f + \mathrm{D}^2(P_r)$ is positive semi-definite on $\mathbb{R}^N \setminus \mathcal{B}_{0,2r}$, $f + P_r$ is non-negative on $\mathbb{R}^N \setminus \mathcal{B}_{0,2r}$ and we have*

$$\sup\left\{\left|\frac{\partial^k P_r}{\partial x^{(i_1)} \cdots \partial x^{(i_k)}}(x)\right| \mid x \in \overline{\mathcal{B}}_{0,r}\right\} < \frac{1}{r}$$

for every integer $0 \leq k \leq r$ and indices $i_1, \ldots, i_k \in \{1, \ldots, N\}$.

Proof. If we set $A = \mathrm{D}^2 f$ and apply Lemma 3.4, we get that $\mathrm{D}^2 f + \mathrm{D}^2(R_{t_2})$ is positive semi-definite on $\mathbb{R}^N \setminus \mathcal{B}_{0,2r}$ for some integer t_2 large enough. Since $\mathrm{D}^2 f$ is of polynomial growth, so is f itself. We conclude that f is dominated by R_t outside $\mathcal{B}_{0,2r}$ for large integers $t \geq 2$. Hence there exists a positive integer $t_3 > t_2$ so that $P_r = R_{t_3}$ is as desired. □

With the help of Lemma 3.4, we can prove the density of functions that are of polynomial growth. Note that the proof heavily relies on the fact that the set of functions under consideration is locally definable.

Theorem 3.6 *Let $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^N)$ be an admissible and locally definable subset of continuous functions. Then every function $f \in \mathcal{F} \cap \mathcal{C}^k(\mathbb{R}^N)$ can be approximated by functions $f_1, f_2, \ldots \in \mathcal{F} \cap \mathcal{C}^k(\mathbb{R}^N)$ of polynomial growth so that $f_s \to f$ in $\mathcal{C}^k(\mathbb{R}^N)$.*

Proof. Theorem 2.7 together with a diagonal sequence argument implies that it suffices to consider the case $f \in \mathcal{C}^\infty(\mathbb{R}^N)$.

Fix a positive integer $r > 0$. Let $\phi \in \mathcal{C}^\infty(\mathbb{R}^N)$ be a cut-off function so that $\phi = 1$ on $\overline{\mathcal{B}}_{0,3r}$ and $\phi = 0$ on $\mathbb{R}^N \setminus \mathcal{B}_{0,4r}$. Then the function $(f \cdot \phi)\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ is smooth and its second derivative $\mathrm{D}^2(f\cdot\phi)\in\mathcal{C}^\infty(\mathbb{R}^N,\mathbb{R}^{N\times N})$ is of polynomial growth (it is even bounded).

By Lemma 3.4, there exists a convex polynomial $P_r\in\mathcal{P}(\mathbb{R}^N)$ so that $\mathrm{D}^2(f\cdot\phi)+\mathrm{D}^2(P_r)$ is positive semi-definite on $\mathbb{R}^N\setminus\mathcal{B}_{0,2r}$ and we have

$$\sup\left\{\left|\frac{\partial^k P_r}{\partial x^{(i_1)}\cdots\partial x^{(i_k)}}(x)\right| \mid x\in\overline{\mathcal{B}}_{0,r}\right\}<\frac{1}{r} \tag{3.5}$$

for every integer $0\le k\le r$ and indices $i_1,\dots,i_k\in\{1,\dots,N\}$. Set $f_r=f\cdot\phi+P_r$. Then the function f_r is smooth and of polynomial growth.

Next we show that $f_r\in\mathcal{F}$. In order to do that, we use that the open sets $\mathcal{B}_{0,3r}$ and $\mathbb{R}^N\setminus\overline{\mathcal{B}}_{0,2r}$ cover the whole of $\mathbb{R}^N$ and that the set $\mathcal{F}$ is locally definable. On the one hand, we have $f+P_r=f_r$ on $\mathcal{B}_{0,3r}$ and we know that the function $f+P_r$ lies in $\mathcal{F}$ because of (F1) and (F4). On the other hand, the second derivative of f_r is positive semi-definite on $\mathbb{R}^N\setminus\overline{\mathcal{B}}_{0,2r}$. Hence Remark 3.3 together with (F4) implies that $f_r\in\mathcal{F}$.

We know that $f_s\to f$ in $\mathcal{C}^\infty(\mathbb{R}^N)$ due to (3.5). This finishes the proof. □

3.2.2 Approximation by Polynomials

We show that every property of continuous functions that corresponds to an admissible and locally definable subset can be preserved under the approximation by polynomials (compare Corollary 3.8). The next theorem states that this remains true even without assuming locality of the property, as long as the limit function in question is of polynomial growth.

The key idea of the proof is the following: In order to guarantee that an approximating polynomial lies in the set $\mathcal{F}$, we will use only such polynomials whose difference to the limit function is convex.

Theorem 3.7 *Let $\mathcal{F}\subseteq\mathcal{C}(\mathbb{R}^N)$ be an admissible subset of continuous functions. Then every function $f\in\mathcal{F}\cap\mathcal{C}^k(\mathbb{R}^N)$ of polynomial growth can be approximated by polynomials $P_1,P_2,\ldots\in\mathcal{F}\cap\mathcal{P}(\mathbb{R}^N)$ so that $P_s\to f$ in $\mathcal{C}^k(\mathbb{R}^N)$ and the difference P_s-f is convex and non-negative*[1] *for every $s=1,2,\ldots$.*

Proof. The diagonal sequence argument can be applied in $\mathcal{C}^k(\mathbb{R}^N)$. By Lemma 2.6 and Lemma 2.8, it is sufficient to consider functions $f\in\mathcal{F}\cap\mathcal{C}^\infty(\mathbb{R}^N)$ with second order partial derivatives of polynomial growth.

Let $f_t\in\mathcal{C}^\infty(\mathbb{R}^N)$, $t=1,2,\ldots$, be the functions defined by a convex perturbation of f

$$f_t(x)=f(x)+\frac{1}{t}|x|^2+\frac{1}{t},\ x\in\mathbb{R}^N.$$

[1]The fact that the difference can be required to be non-negative will become very important for the approximation on the level of non-linear problems.

We have $f_t \to f$ in $\mathcal{C}^\infty(\mathbb{R}^N)$. According to (F1) and (F4), the function f_t lies in $\mathcal{F}$ for every $t = 1, 2, \ldots$. Hence it suffices to fix an arbitrary positive integer $t > 0$ and construct a sequence $P''_1, P''_2, \ldots \in \mathcal{F} \cap \mathcal{P}(\mathbb{R}^N)$ of polynomials so that $P''_s \to f_t$ in $\mathcal{C}^\infty(\mathbb{R}^N)$.

Theorem 2.4 implies that there exist polynomials $Q_1, Q_2, \ldots \in \mathcal{P}(\mathbb{R}^N)$ so that $Q_s \to f_t$ in $\mathcal{C}^\infty(\mathbb{R}^N)$. Fix a positive integer $r > 0$. The derivative $\mathrm{D}^2(f_t - f)$ is positive definite and the function $f_t - f$ positive on $\overline{\mathcal{B}}_{0,2r}$. Hence we can find a positive integer $s_r > 0$ so that $\mathrm{D}^2(Q_s - f)$ is also positive definite and $Q_s - f$ also positive on $\overline{\mathcal{B}}_{0,2r}$ for every $s \geq s_r$. In fact, there exists such a number s_r, since the polynomials $Q_1, Q_2, \ldots$ and their second derivatives $\mathrm{D}^2 Q_1, \mathrm{D}^2 Q_2, \ldots$ converge to f_t and $\mathrm{D}^2 f_t$, respectively, uniformly on the compact set $\overline{\mathcal{B}}_{0,2r}$.

The polynomial Q_{s_r} does not have to lie in the set $\mathcal{F}$. In order to construct polynomials in $\mathcal{F}$, we will use (F1) and (F4) again. All second order partial derivatives of the difference $Q_{s_r} - f$ are of polynomial growth. By Corollary 3.5, there exists a convex non-negative polynomial $P'_r \in \mathcal{P}(\mathbb{R}^N)$ so that $\mathrm{D}^2(Q_{s_r} - f + P'_r)$ is positive semi-definite on $\mathbb{R}^N \setminus \mathcal{B}_{0,2r}$, $Q_{s_r} - f + P'_r$ is non-negative on $\mathbb{R}^N \setminus \mathcal{B}_{0,2r}$ and we have

$$\sup \left\{ \left| \frac{\partial^k P'_r}{\partial x^{(i_1)} \cdots \partial x^{(i_k)}}(x) \right| \mid x \in \overline{\mathcal{B}}_{0,r} \right\} < \frac{1}{r} \tag{3.6}$$

for every integer $0 \leq k \leq r$ and indices $i_1, \ldots, i_k \in \{1, \ldots, N\}$.

As a consequence, we see that $\mathrm{D}^2(Q_{s_r} - f + P'_r)$ is positive semi-definite and $Q_{s_r} - f + P'_r$ non-negative on the whole of $\mathbb{R}^N$. Hence by Remark 3.2, the function $Q_{s_r} - f + P'_r$ is convex. Then (F1) and (F4) imply that $Q_{s_r} + P'_r \in \mathcal{F}$.

We know that $Q_{s_r} \to f_t$ and $P'_r \to 0$ in $\mathcal{C}^\infty(\mathbb{R}^N)$ as $r \to +\infty$. If we set $P''_r = Q_{s_r} + P'_r$ for every $r = 1, 2, \ldots$, then the polynomials $P''_1, P''_2, \ldots$ lie in $\mathcal{F}$ and, by (3.6), we have $P''_s \to f_t$ in $\mathcal{C}^\infty(\mathbb{R}^N)$. We can chose polynomials $P_1, P_2, \ldots \in \mathcal{F} \cap \mathcal{P}(\mathbb{R}^N)$ as desired with the help of the diagonal sequence argument. □

The following corollary is a direct consequence of Theorem 3.6 and Theorem 3.7.

Corollary 3.8 *Let $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^N)$ be an admissible and locally definable subset of continuous functions. Then every function $f \in \mathcal{F} \cap \mathcal{C}^k(\mathbb{R}^N)$ can be approximated by polynomials $P_1, P_2, \ldots \in \mathcal{F} \cap \mathcal{P}(\mathbb{R}^N)$ so that $P_s \to f$ in $\mathcal{C}^k(\mathbb{R}^N)$.*

3.3 Convexity Notions Characterized by Polynomials

We collect approximation results that can be seen as corollaries of the abstract results in Section 3.2. In the quasiconvex case, we generalize a result by Heinz [Published online: January 30, 2008]. Although some of the other results might be known already, they cannot be found easily in the literature. We emphasize that the approximation results stated below give characterizations of the different convexity notions, since the considered subsets of functions are closed under the convergence in question.

We will concentrate on polyconvexity, quasiconvexity and rank-one convexity in the calculus of variations. These concepts as well as relations between them are studied, for example, in Dacorogna [1989]. The following hierarchy of properties of real-valued functions is well known:

$$\text{convexity} \Rightarrow \text{polyconvexity} \Rightarrow \text{quasiconvexity} \Rightarrow \text{rank-one convexity}. \quad (3.7)$$

The close connection between these convexity notions and the analysis of crystalline microstructure involving (gradient) Young measures is studied, for example, in Dolzmann [2003] as well as Müller [1999c]. See also Luskin [1996] as well as Bartels et al. [2004] for a numerical viewpoint.

In order to motivate the relevance of approximation results in $\mathcal{C}^k$-spaces, we remark that higher differentiability of convex (polyconvex, quasiconvex) functions is necessary in regularity theory as well as in numerical schemes to solve variational minimization problems using Newton-type methods. Moreover, $\mathcal{C}^1$-functions occur naturally in the relaxation of non-convex minimization problems. We recall two results in this context. Fix an arbitrary function[2] in $\mathcal{C}^{1,1}(\mathbb{R}^N)$. Then Griewank and Rabier [1990] have shown that its convex envelope (if it exists) lies also in $\mathcal{C}^{1,1}(\mathbb{R}^N)$. A corresponding result has been shown by Ball et al. [2000] about the $\mathcal{C}^1$-smoothness of quasiconvex envelopes in the presence of polynomial growth.

Note also that every real-valued rank-one convex function is locally Lipschitz continuous[3] and, in particular, continuous.

Remark on Subharmonic Functions

Our main interests lie in the convexity notions that occur in (3.7). They are the most relevant convexity notions in the calculus of variations. Nevertheless, there are other properties of functions that are compatible with

[2] A function lies in a $\mathcal{C}^{1,1}$-space if and only if it has a locally Lipschitz continuous first derivative.

[3] See Dacorogna [1989, Theorem 1.1] for a proof.

the abstract framework of this chapter. We wish to make a short remark on subharmonicity.

The definition and fundamental properties of subharmonic functions can be found, for example, in Hörmander [1994]. Similar to the convex case below, we can prove that every subharmonic continuous function in $\mathcal{C}(\mathbb{R}^N)$ can be approximated locally uniformly by subharmonic polynomials. Švedov [1985] even showed, via the Riesz representation theorem, that any subharmonic function on a simply connected domain $D \subseteq \mathbb{R}^N$ can be approximated in this way. We mention that this stronger approximation result cannot be proven by the tools that were provided in Section 3.2.[4]

3.3.1 Convex Functions

We will consider convex continuous and convex extended real-valued functions[5]. The fact that convex functions can be approximated by convex polynomials has been known for years. The rate of convergence, for example, is studied by Hu et al. [1994] for the polynomial approximation of convex continuous functions on the interval $[-1, 1]$. The convex case is included in this thesis for the sake of completeness. Moreover, the results will be used later on.

The next remark is almost a direct consequence of Definition 3.1.

Remark 3.9 *The subset of all convex functions in $\mathcal{C}(\mathbb{R}^N)$ is admissible and locally definable.*

Proof. See Hörmander [1994, Corollary 1.1.12] for the proof of locality in the case $N = 1$. The higher dimensional case can be dealt with in the same way.

We show that the set of all convex functions in $\mathcal{C}(\mathbb{R}^N)$ forms a closed subset and argue by contradiction. Assume that there exist convex functions $f_1, f_2, \ldots \in \mathcal{C}(\mathbb{R}^N)$ so that $f_s \to f$ holds in $\mathcal{C}(\mathbb{R}^N)$ for some non-convex function $f \in \mathcal{C}(\mathbb{R}^N)$. Then there exist vectors $x, y \in \mathbb{R}^N$ and a real number $\lambda \in [0, 1]$ so that we have

$$f(\lambda x + (1 - \lambda)y) > \lambda f(x) + (1 - \lambda)f(y). \tag{3.8}$$

Recall that all functions $f_1, f_2, \ldots$ are convex and, hence, we get

$$f_s(\lambda x + (1 - \lambda)y) \leq \lambda f_s(x) + (1 - \lambda)f_s(y), \; s = 1, 2, \ldots.$$

[4]Subharmonicity is not further discussed in this thesis.

[5]See Definition 3.1.

This contradicts (3.8), since $f_s \to f$ in $\mathcal{C}(\mathbb{R}^N)$. In fact, the same argument even implies that the set of all convex functions in $\mathcal{LSC}(\mathbb{R}^N)$ is closed under the convergence in $\mathcal{LSC}(\mathbb{R}^N)$. The rest of the proof is immediate. □

As a first step to the approximation result, we prove the following lemma:

Lemma 3.10 *Every convex lower semicontinuous function $f \in \mathcal{LSC}(\mathbb{R}^N)$ can be approximated by convex continuous functions $f_1, f_2, \ldots \in \mathcal{C}(\mathbb{R}^N)$ so that $f_s \to f$ in $\mathcal{LSC}(\mathbb{R}^N)$.*

Proof. Fix a positive integer $s > 0$. We follow the proof of Lemma 2.5 and define the function $f_s \in \mathcal{C}(\mathbb{R}^N)$ by specifying its lower level sets

$$\mathcal{N}_{f_s,\alpha} = \begin{cases} (\mathcal{N}_{f,\alpha} + \overline{\mathcal{B}}_{0,5^{-s}(\alpha+s)}) \cap \overline{\mathcal{B}}_{0,s+5^{-s}(\alpha+s)} & \alpha \geq -s \\ \emptyset & \alpha < -s. \end{cases} \tag{3.9}$$

It remains to show that f_s is a convex function. Fix vectors $x_1, x_2 \in \mathbb{R}^N$ and a real number $\lambda \in [0,1]$. Set $\beta_1 = f_s(x_1)$, $\beta_2 = f_s(x_2)$, $\beta = \lambda\beta_1 + (1-\lambda)\beta_2$ and $x = \lambda x_1 + (1-\lambda)x_2$. We have to show that $f(x) \leq \beta$. We have $x_i \in \mathcal{N}_{f_s,\beta_i}$ and, in view of (3.9), $x_i \in \overline{\mathcal{B}}_{0,s+5^{-s}(\beta_i+s)}$ for $i = 1,2$. This implies that $x \in \overline{\mathcal{B}}_{0,s+5^{-s}(\beta+s)}$. Moreover, for $i = 1,2$ we can write $x_i = v_i + w_i$ so that $v_i \in \mathcal{N}_{f,\beta_i}$ and $w_i \in \overline{\mathcal{B}}_{0,5^{-s}(\beta_i+s)}$. Hence $\lambda w_1 + (1-\lambda)w_2 \in \overline{\mathcal{B}}_{0,5^{-s}(\beta+s)}$ and $\lambda v_1 + (1-\lambda)v_2 \in \mathcal{N}_{f,\beta}$, since f is convex. Altogether we get $x \in \mathcal{N}_{f,\beta}$ and, thus, $f(x) \leq \beta$. □

We are now in the position to prove the approximation result on convex functions.

Theorem 3.11 *Convex functions can be approximated in the following way:*

(i) Every convex function $f \in \mathcal{LSC}(\mathbb{R}^N)$ can be approximated by convex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^N)$ so that $P_s \to f$ in $\mathcal{LSC}(\mathbb{R}^N)$.

(ii) Every convex function $f \in \mathcal{C}^k(\mathbb{R}^N)$ can be approximated by convex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^N)$ so that $P_s \to f$ in $\mathcal{C}^k(\mathbb{R}^N)$.

Proof. In order to show (i), by Lemma 3.10 and Lemma 2.2, it is sufficient to show that every convex continuous function $f \in \mathcal{C}(\mathbb{R}^N)$ can be approximated by convex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^N)$ so that $P_s \to f$ in $\mathcal{C}(\mathbb{R}^N)$. Let $\mathcal{F}$ be the set of all convex continuous functions in $\mathcal{C}(\mathbb{R}^N)$. Then $\mathcal{F}$ is admissible and locally definable by Remark 3.9. Hence we are in the position to apply Corollary 3.8. This proves (i) and, at the same time, (ii). □

3.3.2 Polyconvex Functions

We denote by $T\colon \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{\tau(m,n)}$ the function that associates every matrix $A \in \mathbb{R}^{m\times n}$ with the vector of all minors of A in a fixed order[6].

Definition 3.12 *A lower semicontinuous function $f \in \mathcal{LSC}(\mathbb{R}^{m\times n})$ is called polyconvex if there exists a convex function $g \in \mathcal{LSC}(\mathbb{R}^{\tau(m,n)})$ so that $f = g \circ T$.*

Ball [1977] has introduced polyconvexity in non-linear elasticity theory.

In order to prove the approximation result for polyconvex functions, we will apply what we already know about the convex case.

Theorem 3.13 *Polyconvex functions can be approximated in the following way:*

(i) Every polyconvex lower semicontinuous function $f \in \mathcal{LSC}(\mathbb{R}^{m\times n})$ can be approximated by polyconvex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^{m\times n})$ so that $P_s \to f$ in $\mathcal{LSC}(\mathbb{R}^{m\times n})$.

(ii) Every polyconvex function $f \in \mathcal{C}^k(\mathbb{R}^{m\times n})$ can be approximated by polyconvex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^{m\times n})$ so that $P_s \to f$ in $\mathcal{C}^k(\mathbb{R}^{m\times n})$.

Proof. Choose a convex function $g \in \mathcal{LSC}(\mathbb{R}^{\tau(m,n)})$ so that $f = g \circ T$. Lemma 3.10 implies that we can approximate g with the help of convex continuous functions $g_1, g_2, \ldots \in \mathcal{C}(\mathbb{R}^{\tau(m,n)})$ so that $g_s \to g$ in $\mathcal{LSC}(\mathbb{R}^{\tau(m,n)})$. The functions constructed in the proof of Lemma 3.10 are of polynomial growth. Fix a positive integer $s > 0$. We consider the polyconvex function $f_s \in \mathcal{C}(\mathbb{R}^{m\times n})$ defined by $f_s = g_s \circ T$. The function T is of polynomial growth and continuous (hence compact sets are mapped on compact sets). We conclude that f_s is also of polynomial growth and that $f_s \to f$ in $\mathcal{LSC}(\mathbb{R}^{m\times n})$ as s tends to $+\infty$. Let $\mathcal{F}$ be the set of all polyconvex continuous functions in $\mathcal{C}(\mathbb{R}^{m\times n})$. Set $N = m \cdot n$. By definition, $\mathcal{F}$ considered as a subset of $\mathcal{C}(\mathbb{R}^N)$ is admissible. This can be shown similarly to the convex case. Together with Theorem 3.7 and Lemma 2.2, we conclude (i).

The set $\mathcal{F}$ is not locally definable. This was shown by Kristensen [1999b]. In order to apply Theorem 3.7 again, we have to show that every polyconvex function $f \in \mathcal{C}^k(\mathbb{R}^{m\times n})$ can be approximated in $\mathcal{C}^k(\mathbb{R}^{m\times n})$ by polyconvex functions of polynomial growth. The diagonal sequence argument can be applied in $\mathcal{C}^k(\mathbb{R}^{m\times n})$. It is sufficient to consider the case $f \in \mathcal{C}^\infty(\mathbb{R}^{m\times n})$ due to Lemma 2.6 and Lemma 2.8. Following the above argument and Lemma 2.2, we can approximate f by polyconvex functions $f_1, f_2, \ldots \in \mathcal{C}(\mathbb{R}^{m\times n})$ of

[6] See, for example, Dacorogna [1989, p. 99] for details.

polynomial growth so that $f_s \to f$ in $\mathcal{C}(\mathbb{R}^{m\times n})$. Now Remark 2.9 can be applied, which finishes the proof of (ii).[7] $\square$

3.3.3 Quasiconvex Functions

We study quasiconvexity in the sense of Morrey [1952], which is quasiconvexity in the sense of the calculus of variations. We focus on quasiconvex continuous functions.

Definition 3.14 *A continuous function $f \in \mathcal{C}(\mathbb{R}^{m\times n})$ is called quasiconvex if we have*

$$\int_{\mathbb{R}^n} [f(A + \mathrm{D}\phi(x)) - f(A)]\,\mathrm{d}x \geq 0 \tag{3.10}$$

for every matrix $A \in \mathbb{R}^{m\times n}$ and every smooth function $\phi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$ of compact support[8]*.*

Like in the convex case, the remark states a fact that is a direct consequence of the definition.

Remark 3.15 *The subset of all quasiconvex functions in $\mathcal{C}(\mathbb{R}^{m\times n})$ can be seen as an admissible subset of $\mathcal{C}(\mathbb{R}^N)$ for $N = m \cdot n$.*

Proof. We show that the set of all quasiconvex functions in $\mathcal{C}(\mathbb{R}^{m\times n})$ is a closed subset. We argue by contradiction. Assume that there exist quasiconvex functions $f_1, f_2, \ldots \in \mathcal{C}(\mathbb{R}^{m\times n})$ so that $f_s \to f$ holds in $\mathcal{C}(\mathbb{R}^{m\times n})$ for some non-quasiconvex function $f \in \mathcal{C}(\mathbb{R}^{m\times n})$. Then there exist a matrix $A \in \mathbb{R}^{m\times n}$ and a smooth function $\phi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$ of compact support so that (3.10) is violated. There exists a positive real number $\epsilon > 0$ so that

$$\int_S [f(A + \mathrm{D}\phi(x)) - f(A)]\,\mathrm{d}x < -\epsilon$$

where $S \subseteq \mathbb{R}^{m\times n}$ denotes the support of the function ϕ. Since $f_s \to f$ holds in $\mathcal{C}(\mathbb{R}^{m\times n})$ and the set S is compact, we must have

$$\int_S [f_s(A + \mathrm{D}\phi(x)) - f_s(A)]\,\mathrm{d}x < 0,\ s \geq s_0,$$

for some positive integer $s_0 > 0$ that is large enough. This contradicts the quasiconvexity of the functions $f_{s_0}, f_{s_0+1}, \ldots$. The rest of the assertion is immediate. $\square$

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[7] The result of Theorem 3.13(ii) does not follow directly from the convex case. See the example of a polyconvex function given by Bevan [2003].

[8] The integral is well-defined, since the integrand vanishes outside the support of ϕ.

The theorem is a generalization of the result in Heinz [Published online: January 30, 2008, Theorem 6.1].

Theorem 3.16 *Every quasiconvex function $f \in \mathcal{C}^k(\mathbb{R}^{m\times n})$ can be approximated by quasiconvex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^{m\times n})$ so that $P_s \to f$ in $\mathcal{C}^k(\mathbb{R}^{m\times n})$.*

Proof. Let $\mathcal{F}$ be the set of all quasiconvex continuous functions in $\mathcal{C}(\mathbb{R}^{m\times n})$. Set $N = m \cdot n$. Then $\mathcal{F}$ considered as a subset of $\mathcal{C}(\mathbb{R}^N)$ is admissible due to Remark 3.15. The diagonal sequence argument can be applied in $\mathcal{C}^k(\mathbb{R}^{m\times n})$. By Lemma 2.6 and Lemma 2.8, it is sufficient to consider the case $f \in \mathcal{C}^\infty(\mathbb{R}^{m\times n})$.

Like in the polyconvex case, the set $\mathcal{F}$ is not locally definable, which was proven by Kristensen [1999a]. Hence Corollary 3.8 cannot be applied here. In view of Heinz [Published online: January 30, 2008, Lemma 5.1], we know that every quasiconvex function $f \in \mathcal{C}^\infty(\mathbb{R}^{m\times n})$ can be approximated in $\mathcal{C}(\mathbb{R}^{m\times n})$ by quasiconvex continuous functions $f_1, f_2, \ldots \in \mathcal{C}(\mathbb{R}^{m\times n})$ of polynomial growth. This is a consequence of a result in Müller [1999a, Corollary 9] about quasiconvex functions that take the value $+\infty$ outside a convex body[9]. Now Remark 2.9 implies that there exist quasiconvex smooth functions $g_1, g_2, \ldots \in \mathcal{C}^\infty(\mathbb{R}^{m\times n})$ of polynomial growth so that $g_s \to f$ in $\mathcal{C}^\infty(\mathbb{R}^{m\times n})$. Theorem 3.7 gives the desired sequence of quasiconvex polynomials. $\square$

A definition of quasiconvex functions can also be given for lower semicontinuous functions on $\mathbb{R}^{m\times n}$. With regard to (3.7), every quasiconvex real-valued function is rank-one convex and, hence, locally Lipschitz continuous, which has been proven in Dacorogna [1989, Theorem 1.1]. Therefore a quasiconvex function $f \in \mathcal{LSC}(\mathbb{R}^{m\times n})$ that is not covered by Definition 3.14 must take the value $+\infty$ somewhere.[10] A closer look at the result in Müller [1999a, Corollary 9] shows that the principal arguments in the proof of Theorem 3.16 can still be applied as long as the domain of the function f is convex[11]. However, the case where the domain is non-convex is not understood until now. In this context, see also Wagner [forthcoming].

[9]Müller's result is essential for our approach.

[10]Note that quasiconvexity does not imply rank-one convexity in the context of lower semicontinuous functions.

[11]The domain of f is the subset of $\mathbb{R}^{m\times n}$ where f is real-valued.

3.3.4 Rank-One Convex Functions and Morrey's Conjecture

One reason to study rank-one convex functions is that the rank-one convex envelope of a given function forms an upper bound to the quasiconvex envelope. See Dacorogna [1989] for the characterization and for properties of the relevant envelopes in the calculus of variations.

Definition 3.17 *A continuous function $f \in \mathcal{C}(\mathbb{R}^{m\times n})$ is called rank-one convex if the inequality*

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda)f(B)$$

holds for every matrices $A, B \in \mathbb{R}^{m\times n}$ so that the rank of $A - B$ equals 1.

We can approximate rank-one convex functions in the following way:

Theorem 3.18 *Every rank-one convex function $f \in \mathcal{C}^k(\mathbb{R}^{m\times n})$ can be approximated by rank-one convex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^{m\times n})$ so that $P_s \to f$ in $\mathcal{C}^k(\mathbb{R}^{m\times n})$.*

Proof. A comparison between Definition 3.17 and Definition 3.1 shows that rank-one convexity and convexity are closely related. As a consequence, we can argue like we did in the proof of Remark 3.9. This directly implies that the subset of all rank-one convex functions in $\mathcal{C}(\mathbb{R}^{m\times n})$ can be seen as an admissible subset of $\mathcal{C}(\mathbb{R}^N)$ for $N = m \cdot n$. Moreover, this subset is locally definable. Hence we can apply Corollary 3.8. This finishes the proof. $\square$

Morrey's Conjecture

Morrey's conjecture says that "there is no condition [...], which involves f and only a finite number of its derivatives, and that is both necessary and sufficient for quasi-convexity in the general case" Morrey [1952, p. 26]. With the help of a counterexample, Šverák [1992] showed that rank-one convexity does not imply quasiconvexity for functions in $\mathcal{C}(\mathbb{R}^{m\times n})$ whenever $m \geq 3$ and $n \geq 2$. As a consequence, Kristensen [1999a] proved non-locality of quasiconvexity. In particular, he showed that Morrey's conjecture holds true in the case $m \geq 3$, $n \geq 2$. Yet, the case $m = 2$, $n \geq 2$ is still open (the case $m = n = 1$ being trivial). In the context of Morrey's conjecture, see also Müller [1999b], Pedregal and Šverák [1998] as well as Faraco and Székelyhidi [2006].

Let us remark that, in view of Theorem 3.16 and Theorem 3.18, it is sufficient to analyze polynomials in order to decide whether rank-one convexity and quasiconvexity coincide in $\mathcal{C}(\mathbb{R}^{2\times n})$. It is well-known that the properties quasiconvexity and rank-one convexity coincide for polynomials of (absolute) degree at most 3. However, this is still an open question for polynomial functions over $\mathbb{R}^{2\times n}$ of higher degree. See Alibert and Dacorogna [1992] for an example of a polynomial of degree 4.

3.3.5 Remarks on Objectivity and Isotropy

We are going to study two properties of functions that are motivated by elasticity theory. In what follows, let SO(3) denote the special orthogonal group in three dimensions. We will identify its elements with matrices in $\mathbb{R}^{3\times 3}$ so that

$$\mathrm{SO}(3) = \left\{ Q \in \mathbb{R}^{3\times 3} \mid Q \cdot Q^T = I \text{ and } \det(Q) = 1 \right\}.$$

Here Q^T is the transposed of the matrix $Q \in \mathbb{R}^{3\times 3}$, $\det(Q)$ its determinant and $I \in \mathbb{R}^{3\times 3}$ the identity matrix.

In the next definition, we think of $W \in \mathcal{C}(\mathbb{R}^{3\times 3})$ as a stored-energy function of a hyperelastic homogeneous[12] material.

Definition 3.19 *A function $W \in \mathcal{C}(\mathbb{R}^{3\times 3})$ is called objective*[13] *if the following holds:*

$$W(F) = W(Q \cdot F), \ F \in \mathbb{R}^{3\times 3}, Q \in \mathrm{SO}(3).$$

A function $W \in \mathcal{C}(\mathbb{R}^{3\times 3})$ is called isotropic if the following holds:

$$W(F) = W(Q \cdot F \cdot Q^T), \ F \in \mathbb{R}^{3\times 3}, Q \in \mathrm{SO}(3).$$

See Ciarlet [1988] and Ball [1977] for details about objectivity and isotropy in elasticity.

The next two theorems deal with quasiconvex functions in the calculus of variations. Corresponding statements about convex, polyconvex and rank-one convex functions are also true and can be shown in a similar way.

Theorem 3.20 *Every objective quasiconvex function $W \in \mathcal{C}^k(\mathbb{R}^{3\times 3})$ can be approximated by objective quasiconvex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^{3\times 3})$ so that $P_s \to W$ in $\mathcal{C}^k(\mathbb{R}^{3\times 3})$.*

---

[12] In this context, homogeneous means that W is independent of the spatial variable. This restriction simplifies the arguments. It is not hard to see that the theorems remain valid also in the inhomogeneous case.

[13] Objectivity is sometimes called *frame-indifference* in the literature.

Proof. Due to Theorem 3.16, there exists a sequence $P'_1, P'_2, \ldots \in \mathcal{P}(\mathbb{R}^{3\times3})$ of quasiconvex polynomials so that $P'_s \to W$ in $\mathcal{C}^k(\mathbb{R}^{3\times3})$. Without loss of generality, we can assume that the estimate

$$\sup\left\{\left|\frac{\partial^l(P'_s - W)}{\partial F^{(i,j)}}(F)\right| \mid F \in \overline{\mathcal{B}}_{0,s}\right\} \le \frac{1}{s} \tag{3.11}$$

holds for every indices $i, j \in \{1,2,3\}$, every integer $0 \le l \le k$ and every $s = 1, 2, \ldots$. Fix a positive integer $t > 0$. We define the function $P_t \colon \mathbb{R}^{3\times3} \longrightarrow \mathbb{R}$ via

$$P_t(F) = \frac{1}{c}\int\limits_{\mathrm{SO}(3)} P'_t(Q\cdot F)\mathrm{dSO}(3),\ F \in \mathbb{R}^{3\times3},\ \text{where } c = \int\limits_{\mathrm{SO}(3)} \mathrm{dSO}(3).$$

The set SO(3) forms a smooth manifold in $\mathbb{R}^9$ and the integrals are understood with respect to the integration on manifolds[14].

By definition, P_t is an objective polynomial. We will show that P_t is also quasiconvex. Fix an arbitrary rotation $Q \in \mathrm{SO}(3)$. In view of Definition 3.14, the quasiconvexity of P'_t implies that

$$\int\limits_{\mathbb{R}^n} \left[P'_t(Q\cdot F + Q\cdot \mathrm{D}\phi(x)) - P'_t(Q\cdot F)\right]\mathrm{d}x \ge 0$$

for every matrix $Q \cdot F \in \mathbb{R}^{3\times3}$ and every smooth function $Q\cdot\phi \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$ of compact support. Hence the polynomial that is given by $F \mapsto P'_t(Q\cdot F)$ must be quasiconvex too[15]. Now we proceed like we did in the proof of Lemma 2.6. We know that the set of quasiconvex functions in $\mathcal{C}(\mathbb{R}^3)$ forms a closed convex cone. As a consequence, P_t has to be quasiconvex.

It remains to show that $P_s \to W$ holds in $\mathcal{C}^k(\mathbb{R}^3)$. We have $Q\cdot F \in \overline{\mathcal{B}}_{0,s}$ as long as $F \in \overline{\mathcal{B}}_{0,s}$ and, hence, (3.11) holds if P'_s is replaced by $P'_s(Q\cdot .)$. This is true for every $s = 1, 2, \ldots$ and every rotation $Q \in \mathrm{SO}(3)$. Since the seminorm in (3.11) is convex, we get still the same estimate if P'_s is replaced by P_s. This finishes the proof. □

With the help of the same arguments, we get the corresponding result for isotropic functions.

Theorem 3.21 *Every isotropic quasiconvex function $W \in \mathcal{C}^k(\mathbb{R}^{3\times3})$ can be approximated by isotropic quasiconvex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^{3\times3})$ so that $P_s \to W$ in $\mathcal{C}^k(\mathbb{R}^{3\times3})$.*

---

[14] Details about the integration on manifolds can be found, for example, in Agricola and Friedrich [2001].

[15] A similar result holds for convex, polyconvex and rank-one convex functions.

Without giving the proof here, we note that the strategy to show Theorem 3.20 can also be applied to the following two cases:

- Definition 3.19 can easily be generalized to extended real-valued functions. It can be shown that every objective (isotropic) polyconvex function $W \in \mathcal{LSC}(\mathbb{R}^{3\times 3})$ can be approximated by objective (isotropic) polyconvex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^{3\times 3})$ so that $P_s \to W$ in $\mathcal{LSC}(\mathbb{R}^{3\times 3})$.

- Objectivity and isotropy can also be expressed for a stress-strain response function $\sigma \in \mathcal{C}(\mathbb{R}^{3\times 3}, \mathbb{R}^{3\times 3})$. This becomes important in the context of non-hyperelastic materials, where a stored-energy function is absent. See Ciarlet [1988] for details. It can be shown that every objective (isotropic) quasimonotone[16] function $\sigma \in \mathcal{C}^k(\mathbb{R}^{3\times 3}, \mathbb{R}^{3\times 3})$ of polynomial growth can be approximated by objective (isotropic) quasimonotone polynomial maps $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^{3\times 3}, \mathbb{R}^{3\times 3})$ so that $P_s \to \sigma$ in $\mathcal{C}^k(\mathbb{R}^{3\times 3}, \mathbb{R}^{3\times 3})$.

3.4 Polynomial Approximation of Variational Minimization Problems

We have seen that approximation results can be obtained for a variety of different convexity notions. Now we will concentrate on the approximation of variational minimization problems, such as Problem 1.

As the main result of this section, we will construct polynomial problems that converge to Problem 1 in a variational sense. Before we do that, we recall the abstract notions of direct methods and of Γ-convergence.

3.4.1 Direct Methods in the Calculus of Variation

The direct methods in the calculus of variations provide us with a powerful tool to prove the existence of solutions of variational problems. See Dacorogna [1989] and the references therein.

We recall two properties of functions.

Definition 3.22 *Let X be a reflexive Banach space and $\mathcal{I}\colon X \longrightarrow \mathbb{R}$ a real-valued function. We call $\mathcal{I}$ sequentially weakly lower semicontinuous if we have*

$$\liminf_{s\to+\infty} \mathcal{I}(u_s) \geq \mathcal{I}(u)$$

[16] This property is studied in Chapter 4.

for every $u, u_1, u_2, \ldots \in X$ *with* $u_s \rightharpoonup u$ *weakly in* X.
We call $\mathcal{I}$ *coercive if there exists a positive real number* $c > 0$ *so that*

$$\liminf_{\|u\|_X \to +\infty} \frac{\mathcal{I}(u)}{\|u\|_X} \geq c.$$

The direct methods are based on the following well-known result[17]:

Theorem 3.23 *Let* X *be a reflexive Banach space and* $\mathcal{I}\colon X \longrightarrow \mathbb{R}$ *a real-valued function that is sequentially weakly lower semicontinuous and coercive. Then the minimization problem given by*

$$\inf \{\mathcal{I}(u) \mid u \in X\}$$

admits at least one solution in X.

3.4.2 A Variational Minimization Problem

Now we wish to apply the abstract approximation results to a problem in the calculus of variations. We fix a non-empty bounded open set $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary. Let $f \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times n})$ be a continuous function[18] and $p > 1$ a real number. We study the following variational minimization problem with homogeneous Dirichlet boundary conditions, denoted by $MP(f,p)$:

$$\inf \left\{\mathcal{I}_f(u) \mid u \in W_0^{1,p}(\Omega, \mathbb{R}^m)\right\}$$

where

$$\mathcal{I}_f(u) = \int_\Omega f(x, u(x), \mathrm{D}u(x))\mathrm{d}x,\ u \in W^{1,p}(\Omega, \mathbb{R}^m). \tag{3.12}$$

This non-linear problem coincides with Problem 1.

In order to guarantee that there exists at least one solution to $MP(f,p)$, we make the following assumptions:

(Cp) Coercivity: There exists a positive real number $\alpha > 0$ so that

$$\alpha |A|^p \leq f(x, y, A)$$

for every $x \in \Omega$, $y \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$.

[17] Compare, for example, Dacorogna [1989, Theorem 1]).

[18] For technical reasons, we assume that f is defined and continuous on the whole of $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m \times n}$ rather than only on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times n}$.

(Gp) Growth condition: There exists a positive real number $\beta > 0$ so that

$$|f(x, y, A)| \leq \beta(1 + |y|^p + |A|^p)$$

for every $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$.

(QC) Quasiconvexity: The function $f(x_0, y_0, .) \in \mathcal{C}(\mathbb{R}^{m \times n})$ is quasiconvex for every $x_0 \in \mathbb{R}^n$ and $y_0 \in \mathbb{R}^m$.

We get the following result on the existence of solutions to $MP(f, p)$:

Theorem 3.24 *Let $MP(f, p)$ be a minimization problem so that (Cp), (Gp) and (QC) hold. Then the function $\mathcal{I}_f \colon W_0^{1,p}(\Omega, \mathbb{R}^m) \longrightarrow \mathbb{R}$ is sequentially weakly lower semicontinuous and coercive. In particular, $MP(f, p)$ admits at least one solution.*

Proof. The condition (Gp) implies that $\mathcal{I}_f$ is a real-valued function and (Cp) together with the Poincaré inequality implies the coercivity of $\mathcal{I}_f$ on $W_0^{1,p}(\Omega, \mathbb{R}^m)$. Marcellini [1985] shows that $\mathcal{I}_f$ is sequentially weakly lower semicontinuous. See also Acerbi and Fusco [1984]. By Theorem 3.23, we conclude that $MP(f, p)$ admits at least one solution. $\Box$

The assumptions for the result by Marcellini [1985] are less restrictive than what we assumed here. In particular, the continuity of the function f is not necessary. Nevertheless, we will use the stronger assumptions to make f fit in the abstract framework of Section 3.2.

3.4.3 Approximation via Γ-Convergence

Γ-convergence in the context of abstract topological spaces was introduced by De Giorgi [1977]. In contrast to this abstract form, we are concerned with minimization problems over a topological space that, for reasons of simplicity, is assumed to be metrizable. Here Γ-convergence can be written in the following sequential form:

Definition 3.25 *Let $\mathcal{I}, \mathcal{I}_1, \mathcal{I}_2, \ldots : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be extended real-valued functions over a metrizable topological space X. We say that the sequence $\mathcal{I}_1, \mathcal{I}_2, \ldots$ Γ-converges to $\mathcal{I}$ with respect to the topology in X if both of the following conditions are fulfilled for every $u \in X$:*

(Γ1) $\mathcal{I}(u) \leq \liminf_{s \to +\infty} \mathcal{I}_s(u_s)$ for every sequence $u_1, u_2, \ldots \in X$ so that $u_s \to u$ in X.

(Γ2) $\mathcal{I}(u) \geq \limsup_{s\to+\infty} \mathcal{I}_s(u_s)$ *for at least one sequence* $u_1, u_2, \ldots \in X$ *so that* $u_s \to u$ *in* X.

Note that Γ-convergence is, in general, not induced by a topology. We refer the reader to De Giorgi [1979], Attouch [1984] and Braides [2002] for applications of Γ-convergence[19] to all branches of optimization theory.

The next theorem is taken from Attouch et al. [2006, Theorem 12.1.1(i)]. It shows the fundamental property of Γ-convergence for minimization problems.

Theorem 3.26 *Let* $\mathcal{I}, \mathcal{I}_1, \mathcal{I}_2, \ldots : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ *be extended real-valued functions over a metrizable topological space* X *so that* $\mathcal{I}_1, \mathcal{I}_2, \ldots$ Γ*-converges to* $\mathcal{I}$*. Let* $u_1, u_2, \ldots X$ *be a given sequence and* $\epsilon_1, \epsilon_2, \ldots \in \mathbb{R}$ *positive real numbers so that* $\epsilon_s \to 0$ *as well as*

$$\mathcal{I}_s(u_s) \leq \inf \{\mathcal{I}_s(u) \mid u \in X\} + \epsilon_s,\ s = 1, 2, \ldots$$

Assume that the sequence $u_1, u_2, \ldots$ *forms a relatively compact subset of* X*. Then every cluster point* $\tilde{u} \in X$ *of the sequence* $u_1, u_2, \ldots$ *is a minimizer of* $\mathcal{I}$ *and*

$$\lim_{s\to+\infty} \inf \{\mathcal{I}_s(u) \mid u \in X\} = \mathcal{I}(\tilde{u}).$$

Now we are in the position to prove the main result of this section.

Theorem 3.27 *Let* $MP(f, p)$ *be a minimization problem so that (Cp), (Gp) and (QC) hold. Then there exist a sequence* $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n})$ *of polynomials and a sequence* $p_1, p_2, \ldots \geq p$ *of real numbers so that all following conditions hold:*

(i) *We have* $P_s \to f$ *in* $\mathcal{C}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n})$.

(ii) *The polynomial* P_s *fulfills (*Cp_s*), (*Gp_s*) and (QC) for every* $s = 1, 2, \ldots$*. In particular, the problem* $MP(P_s, p_s)$ *admits at least one solution.*

(iii) *Let* $u_1, u_2, \ldots \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ *be a sequence so that* u_s *is a solution to* $MP(P_s, p_s)$ *for every* $s = 1, 2, \ldots$*. Then any weak cluster point* $u \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ *of this sequence is a solution to* $MP(f, p)$.

(iv) *The minima converge in* $\mathbb{R}$*, meaning, as* $s \to +\infty$*, we get that*

$$\inf \left\{\mathcal{I}_{P_s}(u) \mid u \in W_0^{1,p_s}(\Omega, \mathbb{R}^M)\right\} \to \inf \left\{\mathcal{I}_f(u) \mid u \in W_0^{1,p}(\Omega, \mathbb{R}^M)\right\}.$$

Proof. We prove (i) and (ii). After that, we show a Γ-convergence result that will imply that (iii) and (iv) hold.

[19] Γ-convergence coincides with epi-convergence in the context of minimization problems. Both notions are commonly used in the literature.

Step 1: Proof of (i) and (ii)

Let $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times n})$ be the subset of all functions that fulfill (QC). Set $N = n + m + m \cdot n$. As a consequence of Remark 3.15, the set $\mathcal{F}$ considered as a subset of $\mathcal{C}(\mathbb{R}^N)$ is admissible. Since, by (Gp), the function f is of polynomial growth, we can apply Theorem 3.7. We conclude that there exists a sequence $P_1', P_2', \ldots \in \mathcal{F} \cap \mathcal{P}(\mathbb{R}^N)$ of polynomials so that $P_s' \to f$ in $\mathcal{C}(\mathbb{R}^N)$.

Fix a positive integer $s > 0$. With the help of Theorem 3.7, we can assume that the difference $P_s' - f$ is both convex and non-negative. Let $\alpha_s > 0$ be a positive real number and $p_s \geq \max\{p, 2\}$ an even integer. We consider the polynomial $P_s \in \mathcal{P}(\mathbb{R}^N)$ given by

$$P_s(x, y, A) = P_s'(x, y, A) + \alpha_s |A|^{p_s},\ x \in \mathbb{R}^n, y \in \mathbb{R}^m, A \in \mathbb{R}^{m\times n}.$$

Then P_s satisfies (QC) and $P_s - f$ is convex and non-negative. We can choose positive real numbers $\alpha_1, \alpha_2, \ldots > 0$ as well as even integers $p_1, p_2, \ldots \geq \max\{p, 2\}$ so that $P_s \to f$ holds in $\mathcal{C}(\mathbb{R}^N)$ and P_s fulfills (Cp_s) and (Gp_s) for every $s = 1, 2, \ldots$. Together with Theorem 3.24, this implies (i) and (ii). In order to simplify the following arguments, we pass to a subsequence (still denoted by $P_1, P_2, \ldots$) so that we have

$$\sup\Big\{|P_s(z) - f(z)| \mid z \in \mathbb{R}^N, \max\{|z^{(1)}|, \ldots, |z^{(N)}|\} \leq s\Big\} \to 0. \tag{3.13}$$

Step 2: Construction of a metrizable topological space

Note that the whole space $W^{1,p}(\Omega, \mathbb{R}^m)$ is not metrizable with respect to the weak topology. However, we will construct a metrizable topological subspace X that is suitable for a Γ-convergence argument. Since the function f fulfills (Cp) and the difference $P_s - f$ is non-negative, all functions $f, P_1, P_2, \ldots \in \mathcal{C}(\mathbb{R}^N)$ share (Cp) with the same constant $\alpha > 0$. Moreover, the infima can be bounded uniformly from above by some constant $c > 0$:

$$\inf\Big\{\mathcal{I}_{P_s}(u) \mid u \in W_0^{1,p_s}(\Omega, \mathbb{R}^M)\Big\} \leq \mathcal{I}_{P_s}(0) \leq c,\ s = 1, 2, \ldots.$$

Now a standard estimate via Poincaré inequality implies that there exists a constant $\gamma > 0$ so that the following holds: All solutions of $MP(f, p)$ and all solutions of $MP(P_s, p_s)$, $s > 0$, have a norm in $W^{1,p}(\Omega, \mathbb{R}^m)$ that is bounded above by γ.

We consider the set of Sobolev functions defined by

$$X = \Big\{u \in W_0^{1,p}(\Omega, \mathbb{R}^m) \mid \|u\|_{W^{1,p}} \leq \gamma + 1\Big\}$$

together with the topology that is induced by the weak topology of the ambient space $W^{1,p}(\Omega, \mathbb{R}^m)$. The set X is convex, bounded and closed with respect to the strong topology in $W^{1,p}(\Omega, \mathbb{R}^m)$. Hence, by Mazur lemma, X is a weakly compact subset. Since the dual space $(W^{1,p}(\Omega, \mathbb{R}^m))^*$ is separable, we conclude that X is metrizable. A proof can be found, for example, in Conway [1990, Theorem 5.1].

In order to make the functionals $\mathcal{I}_{P_1}, \mathcal{I}_{P_2}, \ldots$ fit into the requirements of Definition 3.25, we extend $\mathcal{I}_{P_s}$ formally in the following way for every $s = 1, 2, \ldots$:

$$\mathcal{I}_{P_s}(u) = \begin{cases} \mathcal{I}_{P_s}(u) & u \in W^{1,p_s}(\Omega, \mathbb{R}^m) \\ +\infty & u \in W^{1,p}(\Omega, \mathbb{R}^m) \setminus W^{1,p_s}(\Omega, \mathbb{R}^m). \end{cases}$$

By the above arguments, X is compact and, in particular, every sequence in X is relatively compact. In order to prove (iii) and (iv), by Theorem 3.26, it suffices to show that the functionals $\mathcal{I}_{P_1}, \mathcal{I}_{P_2}, \ldots$ Γ-converge to $\mathcal{I}_f$ with respect to the topology in X.

Step 3: Proof of (Γ1)

The difference $P_s - f$ is non-negative for every $s = 1, 2, \ldots$. This implies that $\mathcal{I}_f(u) \leq \mathcal{I}_{P_s}(u)$ for every $u \in X$ and every $s = 1, 2, \ldots$. By Theorem 3.24, the function $\mathcal{I}_f$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^m)$ and, hence, we have (Γ1).

Step 4: Proof of (Γ2)

Fix $u \in X$. Since the set $\Omega \subseteq \mathbb{R}^n$ is bounded and has a Lipschitz boundary, we can approximate u in X with the help of Lipschitz continuous functions. See, for example, Adams [1978, 3.18 Theorem]. With the help of Remark 2.3, we can choose a sequence $u_1, u_2, \ldots \in X \cap W^{1,\infty}(\Omega, \mathbb{R}^m)$ that is equi-integrable in $W^{1,p}(\Omega, \mathbb{R}^m)$ and for that we have $u_s \to u$ strongly in $W^{1,p}(\Omega, \mathbb{R}^m)$ as well as $u_s \to u$ and $\mathrm{D}u_s \to \mathrm{D}u$ pointwise almost everywhere in Ω. Without loss of generality, we can assume that

$$\|u_s\|_{W^{1,\infty}} \leq s, \; s \geq s_0 \tag{3.14}$$

for some positive integer $s_0 > 0$. If necessary, we change the sequence slightly by using certain of its elements more than once[20]. Now (3.13) and (3.14) imply that

$$\limsup_{s \to +\infty} \mathcal{I}_f(u_s) = \limsup_{s \to +\infty} \mathcal{I}_{P_s}(u_s). \tag{3.15}$$

[20]We construct sequences like $u_1, u_1, u_2, u_2, u_2, u_3, \ldots$.

The special choice of the functions $u_1, u_2, \ldots$ implies that the sequence $f(., u_1, \mathrm{D}u_1),\ f(., u_2, \mathrm{D}u_2), \ldots$ is equi-integrable in $L^1(\Omega)$, since $f \in \mathcal{C}(\mathbb{R}^N)$ fulfills condition (Gp). In addition, by continuity of f, we conclude that

$$f(., u_s, \mathrm{D}u_s) \to f(., u, \mathrm{D}u) \text{ pointwise}$$

almost everywhere in Ω. By Lebesgue's dominated convergence theorem, we get that

$$\limsup_{s\to+\infty} \mathcal{I}_f(u_s) = \lim_{s\to+\infty} \mathcal{I}_f(u_s) = \mathcal{I}_f(u).$$

Together with (3.15), this implies (Γ2) and finishes the proof. $\square$

We have concentrated on homogeneous Dirichlet boundary conditions. Inhomogeneous Dirichlet boundary conditions (like $u = u_0$ on $\partial\Omega$ for some given $u_0 \in W^{1,p}(\Omega, \mathbb{R}^m)$) can be considered as well. The statement of the theorem and the proof would be basically the same. However, the function $u_0 \in W^{1,p}(\Omega, \mathbb{R}^m)$, which encodes the Dirichlet data, is not necessarily contained in the spaces $W^{1,p_s}(\Omega, \mathbb{R}^m)$, $s = 1, 2, \ldots$, and, hence, has to be approximated as well.

Theorem 3.27 looks similar to an approximation result established in Marcellini [1985, Theorem 1.2][21] in order to obtain sequentially weakly lower semicontinuity for multiple integrals of the form (3.12). Unlike in Marcellini's result, the approximation in Theorem 3.27 is not and cannot be made monotone, since we are working with polynomials. This generates a major difference. In particular, we cannot show that the function $\mathcal{I}_f$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega, \mathbb{R}^m)$ via Theorem 3.27 .

[21] We made use of this result in the context of Theorem 3.24.

Chapter 4

Quasimonotonicity and Elliptic Differential Operators

This chapter concentrates on Problem 2, which is: Find weak solutions $u \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ to the non-linear partial differential equation

$$\operatorname{div} \sigma(x, \mathrm{D}u(x)) = -f(x) \text{ almost everywhere in } \Omega.$$

Here $\Omega \subseteq \mathbb{R}^n$ denotes a non-empty bounded open set with Lipschitz boundary, $p, q > 1$ real numbers so that $1/p + 1/q = 1$, $W_0^{1,p}(\Omega, \mathbb{R}^m)$ a Sobolev space, $f \in L^q(\Omega, \mathbb{R}^m)$ a q-integrable function and $\sigma \colon \Omega \times \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}^{m \times n}$ a continuous matrix-valued function.

There are many parallels between this chapter and Chapter 3. At first, we will study the preservation of monotonicity and quasimonotonicity in the context of polynomial approximation of continuous functions. In order to do so, we present an abstract framework that is very similar to that in Chapter 3. After that, we will focus on non-linear partial differential equations. The main goal of this chapter is to approximate the above problem with the help of polynomial problems.

4.1 Some Properties of Monotone Functions

Let us collect two well-known properties of monotone functions.

Definition 4.1 *A function $f \colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is called monotone if the inequality*

$$\langle f(x) - f(y), x - y \rangle \geq 0$$

holds true for every $x, y \in \mathbb{R}^N$.

In particular, a real-valued function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is monotone if and only if f is non-decreasing.

We can characterize a monotone continuously differentiable function with the help of its first derivative.

Remark 4.2 *Let $f \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R}^N)$ be a given function. Then f is monotone if and only if the derivative $\mathrm{D}f$ is positive semi-definite on $\mathbb{R}^N$.*

Proof. Note that the derivative $\mathrm{D}f(x)$ at a vector $x \in \mathbb{R}^N$ defines a bilinear form A via the equality $A[y_1, y_2] = \langle \mathrm{D}f(x)[y_1], y_2\rangle$, $y_1, y_2 \in \mathbb{R}^N$.

Let $f \in \mathcal{C}^1(\mathbb{R}^N)$ be monotone. Fix vectors $x_0, y_0 \in \mathbb{R}^N$. Set $x = x_0 + ty_0$ and $y = x_0$ for some positive real number $t > 0$. Consequently, the monotonicity of the function f implies

$$\left\langle \frac{f(x_0 + ty_0) - f(x_0)}{t}, y_0 \right\rangle \geq 0.$$

Let t tend to 0. Then the last inequality just reads $\langle \mathrm{D}f(x_0)[y_0], y_0\rangle \geq 0$.

Conversely, let $f \in \mathcal{C}^1(\mathbb{R}^N)$ be a function so that its derivative $\mathrm{D}f$ is positive semi-definite on $\mathbb{R}^N$. Fix arbitrary vectors $x_0, y_0 \in \mathbb{R}^N$. We are going to show that $\langle f(x_0) - f(y_0), x_0 - y_0\rangle \geq 0$.

The first-order Taylor expansion around y_0 for the vector x_0 reads

$$f(x_0) = f(y_0) + \mathrm{D}f(\xi_0)[x_0 - y_0]$$

for some vector $\xi_0 \in \mathbb{R}^N$. Now we can write

$$\langle f(x_0) - f(y_0), x_0 - y_0\rangle = \langle \mathrm{D}f(\xi_0)[x_0 - y_0], x_0 - y_0\rangle. \tag{4.1}$$

By assumption, the right-hand side of (4.1) is non-negative. □

Fix positive real numbers $r, t > 0$. We consider the function $\Theta_{r,t} \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$ defined by

$$\Theta_{r,t}(x) = tx\,(d_r(x))^2\,,\ x \in \mathbb{R}^N,$$

where $d_r(x) = \max\{0, |x|^2 - (\frac{3}{2}r)^2\}$. The function $\Theta_{r,t}$ is continuously differentiable on $\mathbb{R}^N$. In fact, it is the first derivative of the function $\theta_{r,t}$ defined in Section 3.1. Hence we have

$$\langle \mathrm{D}\Theta_{r,t}(x)[y], y\rangle = t|y|^2(d_r(x))^2 + 4t\langle x, y\rangle^2 d_r(x),\ y \in \mathbb{R}^N.$$

The derivative $\mathrm{D}\Theta_{r,t}$ is positive semi-definite on $\mathbb{R}^N$ and, by Remark 4.2, the function $\Theta_{r,t}$ is monotone. We have the estimate

$$\langle \mathrm{D}\Theta_{r,t}(x)[y], y\rangle \geq t|y|^2 \left(|x|^2 - \left(\frac{3}{2}r\right)^2\right)^2 \geq 3t|y|^2 r^4 \tag{4.2}$$

for every $x \in \mathbb{R}^N \setminus \overline{\mathcal{B}}_{0,2r}$ and every $y \in \mathbb{R}^N$. With the help of the functions $\Theta_{r,t}$, $r, t > 0$, we extend monotone smooth functions to the whole space.

Remark 4.3 *Let $K \subseteq \mathbb{R}^N$ be an open subset and $f \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R}^N)$ a function so that $\mathrm{D}f$ is positive semi-definite on K. Then for every vector $x_0 \in K$ there exists a positive real number $r > 0$ and a monotone function $g \in \mathcal{C}^1(\mathbb{R}^N, \mathbb{R}^N)$ so that $f = g$ on $\mathcal{B}_{x_0,r}$.*

Proof. Without loss of generality, we assume that $x_0 = 0$. Fix a radius $r > 0$ so that $\overline{\mathcal{B}}_{0,3r} \subseteq K$. Let $\phi \in \mathcal{C}^\infty(\mathbb{R}^N)$ be a cut-off function so that $\phi = 1$ on $\overline{\mathcal{B}}_{0,2r}$ and $\phi = 0$ on $\mathbb{R}^N \setminus \mathcal{B}_{0,3r}$.

The partial derivatives of the function $f \cdot \phi$ are bounded. Thus there exists a positive real number $t_0 > 0$ so that $\langle \mathrm{D}(f \cdot \phi)(x)[y], y\rangle \geq -t_0|y|^2$ for every $x, y \in \mathbb{R}^N$. Set $t_1 = \frac{1}{3}r^{-4}t_0$ and $g = f \cdot \phi + \Theta_{r,t_1}$. We then have $g = f$ on $\mathcal{B}_{0,r}$ and, in view of (4.2), $\mathrm{D}g$ is positive semi-definite on $\mathbb{R}^N$. Remark 4.2 implies that the function $g \in \mathcal{C}^1(\mathbb{R}^N)$ is monotone. □

4.2 Abstract Results

In this chapter, we call a set $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$ of continuous functions *admissible* if all of the following conditions hold:

(F1)' $\mathcal{F}$ is a convex cone.

(F2)' $\mathcal{F}$ is translation invariant.

(F3)' $\mathcal{F}$ is closed in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$.

(F4)' $\mathcal{F}$ contains all monotone functions in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N)$.

4.2.1 Functions of Polynomial Growth

The aim of this section is to approximate functions of arbitrary growth with the help of functions with polynomial growth.

Theorem 4.4 *Let $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$ be an admissible and locally definable subset of continuous functions. Then every function $f \in \mathcal{F} \cap \mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^N)$ can be approximated by functions $f_1, f_2, \ldots \in \mathcal{F} \cap \mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^N)$ of polynomial growth so that $f_s \to f$ in $\mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^N)$.*

Proof. Theorem 2.7 together with a diagonal sequence argument implies that it suffices to consider the case $f \in \mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N)$.

Fix a positive integer $r > 0$. Let $\phi \in \mathcal{C}^\infty(\mathbb{R}^N)$ be a cut-off function so that $\phi = 1$ on $\overline{\mathcal{B}}_{0,3r}$ and $\phi = 0$ on $\mathbb{R}^N \setminus \mathcal{B}_{0,4r}$. Then the function $(f \cdot \phi)\colon \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is smooth and its first derivative $\mathrm{D}(f \cdot \phi) \in \mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^{N\times N})$ is of polynomial growth (it is even bounded).

By Lemma 3.4, there exists a convex polynomial $P_r \in \mathcal{P}(\mathbb{R}^N)$ so that $\mathrm{D}(f \cdot \phi) + \mathrm{D}^2 P_r$ is positive semi-definite on $\mathbb{R}^N \setminus \mathcal{B}_{0,2r}$ and we have

$$\sup\left\{\left|\frac{\partial^k P_r}{\partial x^{(i_1)} \cdots \partial x^{(i_k)}}(x)\right| \mid x \in \overline{\mathcal{B}}_{0,r}\right\} < \frac{1}{r} \tag{4.3}$$

for every integer $0 \leq k \leq r$ and indices $i_1, \ldots, i_k \in \{1, \ldots, N\}$. Set $f_r = f \cdot \phi + \mathrm{D}P_r$. Then the function f_r is smooth and of polynomial growth.

Next we show that $f_r \in \mathcal{F}$. In order to do that, we use the fact that the open sets $\mathcal{B}_{0,3r}$ and $\mathbb{R}^N \setminus \overline{\mathcal{B}}_{0,2r}$ cover the whole of $\mathbb{R}^N$ and that the set $\mathcal{F}$ is locally definable. On the one hand, we have $f + \mathrm{D}P_r = f_r$ on $\mathcal{B}_{0,3r}$. Since the polynomial P_r is convex, Remark 3.2 and Remark 4.2 imply that the vector-valued function $\mathrm{D}P_r$ is monotone. Hence $f + \mathrm{D}P_r$ lies in $\mathcal{F}$ because of (F1)' and (F4)'. On the other hand, the first derivative of f_r is positive semi-definite on $\mathbb{R}^N \setminus \overline{\mathcal{B}}_{0,2r}$. Remark 4.3 together with (F4)' implies that $f_r \in \mathcal{F}$.

By (4.3), we know that $\mathrm{D}P_s \to 0$ in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N)$ and, consequently, $f_s \to f$ in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N)$. □

4.2.2 Approximation by Polynomial Maps

The main idea to prove the approximation results is the same as in Chapter 3. The arguments remain almost unaltered to those of Section 3.2.2. However, let us point out that the key idea is slightly different. In order to guarantee that an approximating polynomial map lies in the set $\mathcal{F}$, we will use only such polynomial maps whose difference to the limit function is monotone.

Theorem 4.5 *Let $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$ be an admissible subset of continuous functions. Then every function $f \in \mathcal{F} \cap \mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^N)$ of polynomial growth can be approximated by polynomial maps $P_1, P_2, \ldots \in \mathcal{F} \cap \mathcal{P}(\mathbb{R}^N, \mathbb{R}^N)$ so that $P_s \to f$ in $\mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^N)$ and the difference $P_s - f$ is monotone for every $s = 1, 2, \ldots$.*

Proof. The diagonal sequence argument can be applied in $\mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^N)$. By Lemma 2.6 and Lemma 2.8, it is sufficient to show the assertion for functions $f \in \mathcal{F} \cap \mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N)$ with first-order partial derivatives of polynomial growth.

Consider the functions $f_t \in \mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N)$, $t = 1, 2, \ldots$, defined by a monotone perturbation of f

$$f_t(x) = f(x) + \frac{1}{t}x,\ x \in \mathbb{R}^N.$$

We have $f_t \to f$ in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N)$. By (F1)' and (F4)', the function f_t lies in $\mathcal{F}$ for every $t = 1, 2, \ldots$. Hence it suffices to fix an arbitrary positive integer $t > 0$ and construct a sequence $P_1'', P_2'', \ldots \in \mathcal{F} \cap \mathcal{P}(\mathbb{R}^N, \mathbb{R}^N)$ of polynomial maps so that $P_s'' \to f_t$ in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N)$.

In view of Theorem 2.4, we can find a sequence $Q_1, Q_2, \ldots \in \mathcal{P}(\mathbb{R}^N, \mathbb{R}^N)$ of polynomial maps so that $Q_s \to f_t$ in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N)$. Fix a positive integer $r > 0$. The derivative $\mathrm{D}(f_t - f)$ is positive definite on $\overline{\mathcal{B}}_{0,2r}$. Hence we can find a positive integer $s_r > 0$ so that $\mathrm{D}(Q_s - f)$ is also positive definite on $\overline{\mathcal{B}}_{0,2r}$ for every $s \geq s_r$. In fact, there exists such a number s_r, since the derivatives $\mathrm{D}Q_1, \mathrm{D}Q_2, \ldots$ converge to $\mathrm{D}f_t$ uniformly on the compact set $\overline{\mathcal{B}}_{0,2r}$.

The polynomial Q_{s_r} does not have to lie in the set $\mathcal{F}$. In order to construct polynomial maps in $\mathcal{F}$, we will use (F1)' and (F4)' again. All first-order partial derivatives of the difference $Q_{s_r} - f$ are of polynomial growth. Due to Lemma 3.4, there exists a convex polynomial $P_r' \in \mathcal{P}(\mathbb{R}^N)$ so that $\mathrm{D}(Q_{s_r} - f + \mathrm{D}P_r')$ is positive semi-definite on $\mathbb{R}^N \setminus \mathcal{B}_{0,2r}$ and we have

$$\sup\left\{\left|\frac{\partial^k P_r'}{\partial x^{(i_1)} \cdots \partial x^{(i_k)}}(x)\right| \mid x \in \overline{\mathcal{B}}_{0,r}\right\} < \frac{1}{r} \tag{4.4}$$

for every integer $0 \leq k \leq r$ and indices $i_1, \ldots, i_k \in \{1, \ldots, N\}$.

As a consequence, the bilinear form $\mathrm{D}(Q_{s_r} - f + \mathrm{D}P_r')$ is positive semi-definite on the whole of $\mathbb{R}^N$. Hence by Remark 4.2, the function $Q_{s_r} - f + \mathrm{D}P_r'$ is monotone. Then (F1)' and (F4)' imply that $Q_{s_r} + \mathrm{D}P_r' \in \mathcal{F}$.

In view of (4.4), we get that $Q_{s_r} \to f_t$ and $\mathrm{D}P_r' \to 0$ in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N)$ as $r \to +\infty$. If we set $P_r'' = Q_{s_r} + \mathrm{D}P_r'$ for every $r = 1, 2, \ldots$, then the polynomial maps $P_1'', P_2'', \ldots$ lie in $\mathcal{F}$ and we have $P_s'' \to f_t$ in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N)$. With the help of the diagonal sequence argument in $\mathcal{C}^\infty(\mathbb{R}^N, \mathbb{R}^N)$, we can choose polynomial maps $P_1, P_2, \ldots \in \mathcal{F} \cap \mathcal{P}(\mathbb{R}^N, \mathbb{R}^N)$ as desired. $\square$

The following corollary is a direct consequence of Theorem 4.4 and Theorem 4.5:

Corollary 4.6 *Let $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$ be an admissible and locally definable subset of continuous functions. Then every function $f \in \mathcal{F} \cap \mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^N)$ can be approximated by polynomial maps $P_1, P_2, \ldots \in \mathcal{F} \cap \mathcal{P}(\mathbb{R}^N, \mathbb{R}^N)$ so that $P_s \to f$ in $\mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^N)$.*

4.3 Monotonicity Notions Characterized by Polynomials Maps

We study two monotonicity concepts: monotonicity and quasimonotonicity in the sense of Zhang [1988]. In view of Definitions 4.1 and 4.9, we have the implication

$$\text{monotonicity} \Rightarrow \text{quasimonotonicity.}$$

Both of them are closely related to the ellipticity of partial differential equations. The most general framework, at least for monotonicity, would certainly consider set-valued functions. Yet, we concentrate on continuous functions in this part of this thesis[1].

4.3.1 Monotone Functions

We consider monotone functions like in Definition 4.1. The next remark collects some facts about monotone functions.

Remark 4.7 *The subset of all monotone functions in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$ is admissible and locally definable.*

Proof. The locality of monotonicity is closely related to the locality of convexity. In fact, a function $f \in \mathcal{C}(\mathbb{R})$ is monotone if and only if f is the first derivative of a convex function in $\mathcal{C}^1(\mathbb{R})$. Hence monotonicity is a local property in the case $N = 1$. The higher dimensional case is immediate.

We show that the set of all monotone functions in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$ forms a closed subset and argue by contradiction. Assume that there exist monotone functions $f_1, f_2, \ldots \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$ so that $f_s \to f$ holds in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$ for some non-monotone function $f \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$. Then there exist vectors $x, y \in \mathbb{R}^N$ so that we have

$$\langle f(x) - f(y), x - y \rangle < 0. \tag{4.5}$$

Since all functions $f_1, f_2, \ldots$ are monotone we get

$$\langle f_s(x) - f_s(y), x - y \rangle \geq 0, \; s = 1, 2, \ldots .$$

This contradicts (4.5), since $f_s \to f$ in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$. The rest of the proof is immediate. $\square$

Now the approximation result is a direct consequence of Corollary 4.6.

[1] Only for technical reasons, we will work with set-valued functions later on.

Theorem 4.8 *Every monotone function $f \in \mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^N)$ can be approximated by monotone polynomial maps $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^N, \mathbb{R}^N)$ so that $P_s \to f$ in $\mathcal{C}^k(\mathbb{R}^N, \mathbb{R}^N)$.*

Proof. Let $\mathcal{F}$ be the set of all monotone continuous functions in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$. Then $\mathcal{F}$ is admissible and locally definable by Remark 4.7. Hence the assertion is a consequence of Corollary 4.6. $\square$

This is not surprising. See, for example, the result by DeVore and Yu [1985] concerning the rate of convergence for the approximation of monotone continuous functions via monotone polynomials in the one-dimensional case ($N = 1$).

4.3.2 Quasimonotone Functions

Zhang [1988] showed the existence of solutions for a family of elliptic partial differential equations. In order to do so, he introduced the following property of matrix-valued functions:

Definition 4.9 *A function $f \in \mathcal{C}(\mathbb{R}^{m\times n}, \mathbb{R}^{m\times n})$ is called quasimonotone (in the sense of Zhang [1988]) if we have*

$$\int_{\mathbb{R}^n} f(A + \mathrm{D}\phi(x)) : \mathrm{D}\phi(x)\mathrm{d}x \geq 0$$

for every matrix $A \in \mathbb{R}^{m\times n}$ and every smooth function $\phi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$ of compact support[2].

If we can write $f = \mathrm{D}g$ for some function $g \in \mathcal{C}^1(\mathbb{R}^{m\times n})$, the quasimonotonicity of f implies that g is quasiconvex in the sense of the calculus of variations. However, the converse fails to be true in general. The quasiconvexity of g does not imply that f is quasimonotone.[3]

Let $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^{m\times n}, \mathbb{R}^{m\times n})$ be the subset of all quasimonotone continuous functions. Then $\mathcal{F}$ is admissible. This can be shown in the same way as we did in Remark 3.15. Hence Theorem 4.5 directly implies the following approximation result:

Theorem 4.10 *Every function $f \in \mathcal{C}^k(\mathbb{R}^{m\times n}, \mathbb{R}^{m\times n})$ that is quasimonotone and of polynomial growth can be approximated by quasimonotone polynomial maps $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^{m\times n}, \mathbb{R}^{m\times n})$ so that $P_s \to f$ in $\mathcal{C}^k(\mathbb{R}^{m\times n}, \mathbb{R}^{m\times n})$.*

[2] The integral is well-defined, since the integrand vanishes outside the support of ϕ.

[3] Christoph Hamburger gave the following counter-example in $\mathbb{R}^{2\times 2}$: Set $g(A) = (\det(A))^2$, $A \in \mathbb{R}^{2\times 2}$. Then g is quasiconvex (even polyconvex), but $f = \mathrm{D}g$ is not quasimonotone.

Note that the set $\mathcal{F}$ is not locally definable. A proof is implicitly given by Kristensen [1999a][4]. In addition, the result for quasiconvex functions given in Müller [1999a, Corollary 9] does not seem to work in the context of quasimonotone functions, which is why we have to assume polynomial growth of the function in Theorem 4.10.

4.4 Polynomial Approximation of Elliptic Differential Operators

Now we will approximate non-linear partial differential equations. Our focus lies on the partial differential equation that is related to Problem 2. We will show that Problem 2 can be approximated with the help of polynomial problems. To begin with, we recall the notions of pseudomonotonicity and G-convergence.

4.4.1 Pseudomonotone Operators

Before we give an abstract result on the existence of solutions in the context of non-linear partial differential equations, we recall two properties of operators.

Definition 4.11 *Let V be a reflexive Banach space, V^* its topological dual and $\mathcal{A}\colon V \longrightarrow V^*$ an operator that is defined on the whole of V. Then $\mathcal{A}$ is called pseudomonotone (in the sense of Brézis [1968]) if the following two conditions are fulfilled:*

(i) Let $r > 0$ be a positive real number. Then there exists a real number $L(r) \in \mathbb{R}$ so that

$$\|\mathcal{A}u\|_{V^*} \leq L(r)$$

for every $u \in V$, $\|u\|_V \leq r$. In other words, $\mathcal{A}$ is a bounded operator.

(ii) Let $u, u_1, u_2, \ldots \in V$ be given so that $u_s \rightharpoonup u$ weakly in V and

$$\limsup_{s \to +\infty} \langle \mathcal{A}u_s, u_s - u \rangle_V \leq 0.$$

[4] Sketch of the proof: Let $f \in \mathcal{C}^2(\mathbb{R}^{m\times n})$ be a given function that satisfies the strict Legendre-Hadamard condition. Then the first derivative $\mathrm{D}f$ considered as a function in $\mathcal{C}^1(\mathbb{R}^{m\times n}, \mathbb{R}^{m\times n})$ is locally quasimonotone in the sense of Section 2.4. Yet, Kristensen shows that f does not have to be quasiconvex and, hence, $\mathrm{D}f$ is not necessarily quasimonotone.

Then we have

$$\liminf_{s \to +\infty} \langle \mathcal{A} u_s, u_s - v \rangle_V \geq \langle \mathcal{A} u, u - v \rangle_V, \ v \in V.$$

Definition 4.12 *Let V, V^* and $\mathcal{A}$ be given like above. Then $\mathcal{A}$ is called coercive if*

$$\lim_{\|u\|_V \to +\infty} \frac{\langle \mathcal{A} u, u \rangle_V}{\|u\|_V} = +\infty.$$

References for pseudomonotonicity and many applications to non-linear partial differential equations can be found in Roubíček [2005].

Recall that the direct methods in the calculus of variations heavily depend on the sequentially weakly lower semicontinuity of the integral function (compare Theorem 3.23). Pseudomonotonicity plays an equally important role in the context of non-linear operator equations.

Theorem 4.13 *Let V be a reflexive Banach space, $f \in V^*$ an element of the dual and $\mathcal{A} \colon V \longrightarrow V^*$ a coercive and pseudomonotone operator that is defined on the whole of V. Then the operator equation*

$$\mathcal{A} u = f$$

admits at least one solution in V.

Proof. See Brézis [1968]. □

4.4.2 An Elliptic Partial Differential Equation

We fix a non-empty bounded open set $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary. Let $\sigma \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^{m \times n}, \mathbb{R}^{m \times n})$ be a continuous function, $p > 1$ a real number and $f \in L^q(\Omega, \mathbb{R}^m)$ for $1/p + 1/q = 1$. We study the weak formulation of the following non-linear partial differential equation with homogeneous Dirichlet boundary conditions:

$$\operatorname{div} \sigma(., \mathrm{D}u) = -f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega.$$

This coincides with Problem 2. In the corresponding weak formulation, denoted by $PDE(\sigma, p)$, we seek an element $u \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ that fulfills the condition

$$\int_\Omega \sigma(x, \mathrm{D}u(x)) : \mathrm{D}v(x) \mathrm{d}x = \int_\Omega \langle f, v \rangle \mathrm{d}x, \ v \in W_0^{1,p}(\Omega, \mathbb{R}^m).$$

In order to guarantee that there exists at least one solution to $PDE(\sigma, p)$, we make the following assumptions:

(Cp)' Coercivity: There exists a positive real number $\alpha > 0$ so that

$$\alpha|A|^p \leq \sigma(x, A) : A$$

for every $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$.

(Gp)' Growth condition: There exists a positive real number $\beta > 0$ so that

$$|\sigma(x, A)| \leq \beta(1 + |A|^{p-1})$$

for every $x \in \Omega$ and $A \in \mathbb{R}^{m \times n}$.

(QM) Ellipticity: The function σ is uniformly strictly quasimonotone in the last variable, meaning that there exists a positive constant $\gamma > 0$ so that

$$\int\limits_{\mathbb{R}^n} \sigma(x_0, A + \mathrm{D}\phi(x)) : \mathrm{D}\phi(x)\mathrm{d}x \geq \gamma \int\limits_{\mathbb{R}^n} |\mathrm{D}\phi(x)|^p \mathrm{d}x$$

holds for every $x_0 \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and every compactly supported smooth function $\phi \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$.

The Nemytskij operator $\mathcal{A}_\sigma \colon W_0^{1,p}(\Omega, \mathbb{R}^m) \longrightarrow (W_0^{1,p}(\Omega, \mathbb{R}^m))^*$ associated with the elliptic partial differential equation $PDE(\sigma, p)$ is given by

$$\langle \mathcal{A}_\sigma u, v \rangle_{W^{1,p}} = \int\limits_{\Omega} \sigma(x, \mathrm{D}u(x)) : \mathrm{D}v(x)\mathrm{d}x, \ u, v \in W_0^{1,p}(\Omega, \mathbb{R}^m). \tag{4.6}$$

To solve $PDE(\sigma, p)$ means to seek a solution $u \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ of the equation $\mathcal{A}_\sigma u = f$.

We get the following result on the existence of solutions.

Theorem 4.14 *Let $PDE(\sigma, p)$ be an elliptic partial differential equation so that (Cp)', (Gp)' and (QM) hold. Then $PDE(\sigma, p)$ admits at least one solution.*

Proof. It was Zhang [1988] who gave a proof in the first place (without using the pseudomonotonicity of $\mathcal{A}_\sigma$ directly). In view of the abstract result in Theorem 4.13, the proof can be written in the following form:

The condition (Gp)' implies that $\mathcal{A}_\sigma$ is well-defined. Moreover, (Cp)' together with the Poincaré inequality implies that the operator $\mathcal{A}_\sigma$ is coercive on $W_0^{1,p}(\Omega, \mathbb{R}^m)$. A result by Landes [1996] shows that $\mathcal{A}_\sigma$ is pseudomonotone. Hence, by Theorem 4.13, we conclude that $PDE(\sigma, p)$ admits at least one solution. □

We admit that the result by Zhang [1988] is much stronger than Theorem 4.14. In order to make σ fit in the abstract framework of Section 4.2 and in order to simplify the approximation procedure, we neither use the most general form nor the weakest possible assumptions here.

4.4.3 Approximation via G-Convergence

G-convergence was introduced by De Giorgi [1977]. Since we wish to work with a condition in sequential form, we assume that the topological spaces involved are metrizable (like we have done for Γ-convergence in Section 3.4.3). In this context, G-convergence can be characterized with the help of the next definition. We use the framework of set-valued operators, which incorporates the case where an operator is not defined everywhere.

Definition 4.15 *Let X, Y be metrizable topological spaces and 2^Y the set of all subsets of Y. Let $\mathcal{A}^{\mathrm{set}}, \mathcal{A}_1^{\mathrm{set}}, \mathcal{A}_2^{\mathrm{set}}, \ldots : X \longrightarrow 2^Y$ be set-valued operators. We say that the sequence $\mathcal{A}_1^{\mathrm{set}}, \mathcal{A}_2^{\mathrm{set}}, \ldots$ G-converges to $\mathcal{A}^{\mathrm{set}}$ with respect to the topologies in X and Y if for every $u \in X$ and every $f \in Y$ both the following conditions are fulfilled:*

(Γ*1)'* *Let $u_1, u_2, \ldots \in X$ and $b_1, b_2, \ldots \in Y$ be sequences so that $u_s \to u$ in X, $b_s \to b$ in Y and $b_s \in \mathcal{A}_s^{\mathrm{set}} u_s$ for infinitely many $s \in \{1, 2, \ldots\}$. Then $b \in \mathcal{A}^{\mathrm{set}} u$.*

(Γ*2)'* *If $b \in \mathcal{A}^{\mathrm{set}} u$ then there exist sequences $u_1, u_2, \ldots \in X$ and $b_1, b_2, \ldots \in Y$ so that $u_s \to u$ in X, $b_s \to b$ in Y and $b_s \in \mathcal{A}_s^{\mathrm{set}} u_s$ for every $s = 1, 2, \ldots$.*

G-convergence and its relation to homogenization of non-linear partial differential operators is studied, for example, by Pankov [1997].

Now we are in the position to prove our main result.

Theorem 4.16 *Let $PDE(\sigma, p)$ be a non-linear partial differential equation so that (Cp)', (Gp)' and (QM) hold. Then there exist sequences $p_1, p_2, \ldots \geq p$ and $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^n \times \mathbb{R}^{m \times n}, \mathbb{R}^{m \times n})$ of integers and of polynomial maps, respectively, so that all following conditions hold:*

(i) *We have $P_s \to \sigma$ in $\mathcal{C}(\mathbb{R}^n \times \mathbb{R}^{m \times n}, \mathbb{R}^{m \times n})$.*

(ii) *The polynomial map P_s fulfills (Cp$_s$)', (Gp$_s$)' and (QM) for every $s = 1, 2, \ldots$. In particular, the equation $PDE(P_s, p_s)$ is elliptic and admits at least one solution.*

(iii) *Let $u_1, u_2, \ldots \in W^{1,p}(\Omega, \mathbb{R}^m)$ be a sequence so that u_s is a solution to $PDE(P_s, p_s)$ for every $s = 1, 2, \ldots$. Then any weak cluster point $u \in W^{1,p}(\Omega, \mathbb{R}^m)$ of this sequence is a solution to $PDE(\sigma, p)$.*

Proof. We prove (i) and (ii). After that, we show a G-convergence result that will imply, in particular, that (iii) holds.

Step 1: Proof of (i) and (ii)

We set $N = n + m \cdot n$, $M = m \cdot n$ and embed $\mathcal{C}(\mathbb{R}^n \times \mathbb{R}^{m\times n}, \mathbb{R}^{m\times n})$ in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^n \times \mathbb{R}^M)$ via the injection given by

$$\tilde{\sigma} \mapsto (0, \tilde{\sigma}),\ \tilde{\sigma} \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^{m\times n}, \mathbb{R}^{m\times n}).$$

Let $\mathcal{F} \subseteq \mathcal{C}(\mathbb{R}^N, \mathbb{R}^n \times \mathbb{R}^M)$ be the subset of all functions of the form $(\tilde{\tau}, \tilde{\sigma})$ where $\tilde{\tau}\colon \mathbb{R}^N \longrightarrow \mathbb{R}^n$ is a continuous function and $\tilde{\sigma}\colon \mathbb{R}^n \times \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}^{m\times n}$ continuous and quasimonotone in the last variable. Note that $\tilde{\sigma}$ does not have to fulfill (QM). Since the set of all quasimonotone functions in $\mathcal{C}(\mathbb{R}^{m\times n}, \mathbb{R}^{m\times n})$ is admissible in the sense of Section 4.2, the set $\mathcal{F}$ considered as a subset of $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^N)$ is admissible too. With the help of Theorem 4.5, we conclude that there exists a sequence $(Q_1', P_1'), (Q_2', P_2'), \ldots \in \mathcal{F} \cap \mathcal{P}(\mathbb{R}^N, \mathbb{R}^n \times \mathbb{R}^M)$ of polynomial maps so that, in particular, $P_s' \to \sigma$ holds in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$.

Fix a positive integer $s > 0$. Theorem 4.5 also shows that the difference $(Q_s, P_s') - (0, \sigma)$ can be assumed to be monotone. This implies that $P_s' - \sigma$ becomes monotone in the last variable. Hence P_s' is not only quasimonotone but also fulfills (QM). Let $\alpha_s > 0$ be a positive real number and $p_s \geq \max\{p, 4\}$ an even integer. We consider the polynomial map $P_s \in \mathcal{P}(\mathbb{R}^N, \mathbb{R}^M)$ given by

$$P_s(x, A) = P_s'(x, A) - P_s'(x, 0) + \alpha_s |A|^{p_s - 2} \cdot A,\ x \in \mathbb{R}^n, A \in \mathbb{R}^{m\times n}.$$

Then the difference $P_s - \sigma$ is monotone in the last variable, P_s satisfies (QM) and we even get the estimate

$$((P_s - \sigma)(x, A) - (P_s - \sigma)(x, 0)) : A \geq \alpha_s |A|^{p_s},\ x \in \Omega, A \in \mathbb{R}^{m\times n}. \quad (4.7)$$

Since σ fulfills (Cp)' and σ is continuous, we must have $\sigma(x, 0) = 0$ for every $x \in \Omega$. Hence $(P_s - \sigma)(x, 0) = 0$ holds for every $x \in \Omega$ by construction. This implies that

$$(P_s - \sigma)(x, A) : A \geq \alpha_s |A|^{p_s},\ x \in \Omega, A \in \mathbb{R}^{m\times n}. \quad (4.8)$$

In addition, $P_s'(., 0) \to 0$ uniformly on Ω. Now we can choose positive real numbers $\alpha_1, \alpha_2, \ldots > 0$ as well as even integers $p_1, p_2, \ldots \geq \max\{p, 4\}$ so that $P_s \to \sigma$ holds in $\mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$ and P_s fulfills (Cp_s)' and (Gp_s)'. Moreover, $p_s \geq p$ implies that $f \in L^{q_s}(\Omega, \mathbb{R}^m)$ for $1/p_s + 1/q_s = 1$. Now we can apply Theorem 4.14. If we collect all above results, we obtain (i) and (ii). In order to simplify the following arguments, we pass to a subsequence (still denoted by $P_1, P_2, \ldots$) so that we have

$$\sup\left\{|P_s(z)) - \sigma(z))| \mid z \in \mathbb{R}^N, \max\{|z^{(1)}|, \ldots, |z^{(N)}|\} \leq s\right\} \to 0. \quad (4.9)$$

Step 2: Construction of a metrizable topological space

Since the function σ satisfies (Cp)' and the difference $P_s - \sigma$ fulfills (4.8) for every $s = 1, 2, \ldots$, all functions $\sigma, P_1, P_2, \ldots \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}^M)$ share (Cp)' with the same constant $\alpha > 0$.

A standard estimate via Poincaré inequality implies that there exists a constant $\gamma > 0$ so that the following holds: All solutions of $PDE(\sigma, p)$ and all solutions of $PDE(P_s, p_s)$, $s > 0$, have a norm in $W^{1,p}(\Omega, \mathbb{R}^m)$ that is bounded above by γ.
We consider the set of Sobolev functions defined by

$$X = \left\{ u \in W_0^{1,p}(\Omega, \mathbb{R}^m) \mid \|u\|_{W^{1,p}} \leq \gamma + 1 \right\}$$

together with the topology that is induced by the weak topology of the ambient space $W^{1,p}(\Omega, \mathbb{R}^m)$. Now the arguments that prove that X is metrizable are the same as in the proof of Theorem 3.27.

In order to meet the requirements of Definition 4.15, we consider the space $Y = (W_0^{1,p}(\Omega, \mathbb{R}^m))^*$ equipped with the strong topology. Then both X and Y are metrizable topological spaces. For every $s = 1, 2, \ldots$ we associate the operator $\mathcal{A}_{P_s}$ with a set-valued version $\mathcal{A}_{P_s}^{\text{set}} \colon X \longrightarrow 2^Y$ given by

$$\mathcal{A}_{P_s}^{\text{set}} u = \begin{cases} \{\mathcal{A}_{P_s} u\} & u \in W_0^{1,p_s}(\Omega, \mathbb{R}^m) \text{ and } \mathcal{A}_{P_s} u \in (W_0^{1,p}(\Omega, \mathbb{R}^m))^* \\ \emptyset & \text{else.} \end{cases}$$

Likewise we set $\mathcal{A}_\sigma^{\text{set}} u = \{\mathcal{A}_\sigma u\}$ for every $u \in X$. We are going to show that the set-valued operators $\mathcal{A}_{P_1}^{\text{set}}, \mathcal{A}_{P_2}^{\text{set}}, \ldots$ G-converge to $\mathcal{A}_\sigma^{\text{set}}$. We remark that the condition (Γ1)' alone would be sufficient to prove (iii).

Step 3: Proof of (Γ2)'

Fix $u \in W_0^{1,p}(\Omega, \mathbb{R}^m)$. Since the set $\Omega \subseteq \mathbb{R}^n$ is bounded and has a Lipschitz boundary, the space $W_0^{1,\infty}(\Omega, \mathbb{R}^m)$ must be dense in $W_0^{1,p}(\Omega, \mathbb{R}^m)$ with respect to the strong topology. See, for example, Adams [1978, 3.18 Theorem]. Together with Remark 2.3, this implies that there exists a sequence $u_1, u_2, \ldots \in W_0^{1,\infty}(\Omega, \mathbb{R}^m)$ that is equi-integrable in $W_0^{1,p}(\Omega, \mathbb{R}^m)$ and that can be chosen so that $u_s \to u$ strongly in $W_0^{1,p}(\Omega, \mathbb{R}^m)$ as well as $u_s \to u$ and $\mathrm{D}u_s \to \mathrm{D}u$ pointwise almost everywhere in Ω. Without loss of generality, we can assume that

$$\|u_s\|_{W^{1,\infty}} \leq s, \; s \geq s_0 \tag{4.10}$$

holds for some positive integer $s_0 > 0$. If necessary, we change the sequence slightly by using certain of its elements more than once (like, for example, $u_1, u_1, u_2, u_2, u_2, u_3, \ldots$). Having (4.6) in mind, (4.9) and (4.10) imply that

$$(\mathcal{A}_{P_s} - \mathcal{A}_\sigma)\, u_s \to 0 \text{ strongly in } (W_0^{1,p}(\Omega, \mathbb{R}^m))^*. \tag{4.11}$$

Note that, for every $s = 1, 2, \ldots$, the function $\mathcal{A}_{P_s} u_s$ lies in $(W_0^{1,p}(\Omega, \mathbb{R}^m))^*$, since u_s is a Lipschitz continuous functions.

Recall that the sequence $u_1, u_2, \ldots$ is equi-integrable in $W^{1,p}(\Omega, \mathbb{R}^m)$ and that the function σ fulfills condition (Gp)'. Hence the sequence $\sigma(., \mathrm{D}u_1)$, $\sigma(., \mathrm{D}u_2), \ldots$ is equi-integrable in $L^q(\Omega, \mathbb{R}^{m\times n})$ as long as $1/p + 1/q = 1$. In addition, we have chosen $u_1, u_2, \ldots$ so that $u_s \to u$ and $\mathrm{D}u_s \to \mathrm{D}u$ pointwise almost everywhere in Ω. By continuity of σ, we conclude that $\sigma(., \mathrm{D}u_s) \to \sigma(., \mathrm{D}u)$ pointwise almost everywhere in Ω. Lebesgue's dominated convergence theorem implies that

$$\lim_{s\to+\infty} \sigma(., \mathrm{D}u_s) = \sigma(., \mathrm{D}u) \text{ strongly in } L^q(\Omega, \mathbb{R}^{m\times n}).$$

The space $L^q(\Omega, \mathbb{R}^{m\times n})$ is continuously embedded in $(W_0^{1,p}(\Omega, \mathbb{R}^m))^*$. Hence we conclude that

$$\mathcal{A}_\sigma u_s \to \mathcal{A}_\sigma u \text{ strongly in } (W_0^{1,p}(\Omega, \mathbb{R}^m))^*.$$

Together with (4.11), this implies (Γ2)'.

Step 4: Proof of (Γ1)'

Let $u, \tilde{u}_1, \tilde{u}_2, \ldots \in W_0^{1,p}(\Omega, \mathbb{R}^m)$ and $b, b_1, b_2, \ldots \in (W_0^{1,p}(\Omega, \mathbb{R}^m))^*$ be Sobolev functions and elements of the dual space, respectively, so that $\tilde{u}_s \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, \mathbb{R}^m)$, $b_s \to b$ strongly in $(W_0^{1,p}(\Omega, \mathbb{R}^m))^*$ and $\mathcal{A}_{P_s}\tilde{u}_s = b_s$ for infinitely many $s \in \{1, 2, \ldots\}$. To begin with, we assume that $\mathcal{A}_{P_s}\tilde{u}_s = b_s$ holds for every $s = 1, 2, \ldots$.

Fix a Lipschitz continuous function $v \in W_0^{1,\infty}(\Omega, \mathbb{R}^m)$ and a positive integer $s > 0$. We can write

$$\langle b_s, \tilde{u}_s - v\rangle_{W^{1,p_s}} = \langle \mathcal{A}_\sigma \tilde{u}_s, \tilde{u}_s - v\rangle_{W^{1,p_s}} + I_s + II_s$$

where the two terms I_s and II_s are given by

$$I_s = \langle (\mathcal{A}_{P_s} - \mathcal{A}_\sigma)\tilde{u}_s - (\mathcal{A}_{P_s} - \mathcal{A}_\sigma)v, \tilde{u}_s - v\rangle_{W^{1,p_s}},$$

$$II_s = \langle (\mathcal{A}_{P_s} - \mathcal{A}_\sigma)v, \tilde{u}_s - v\rangle_{W^{1,p_s}}.$$

Note that the dual parings are well-defined, since $\mathcal{A}_\sigma\tilde{u}_s$ lies in $(W_0^{1,p}(\Omega, \mathbb{R}^m))^*$ and, hence, in $(W_0^{1,p_s}(\Omega, \mathbb{R}^m))^*$. The function $P_s - \sigma$ is monotone in the last variable. In view of (4.6), this results in $I_s \geq 0$. In addition, (4.9) implies $II_s \to 0$ as s tends to $+\infty$. Since $\tilde{u}_s \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, \mathbb{R}^m)$ and $b_s \to b$ strongly in $(W_0^{1,p}(\Omega, \mathbb{R}^m))^*$, we conclude that

$$\limsup_{s\to+\infty} \langle \mathcal{A}_\sigma \tilde{u}_s, \tilde{u}_s - v\rangle_{W^{1,p}} \leq \limsup_{s\to+\infty} \langle b_s, \tilde{u}_s - v\rangle_{W^{1,p_s}} = \langle b, u - v\rangle_{W^{1,p}}. \quad (4.12)$$

Similar to the proof in Step 3, we choose Lipschitz continuous functions $u_1, u_2, \ldots \in W_0^{1,\infty}(\Omega, \mathbb{R}^m)$ so that (4.11) holds and $u_s \to u$ strongly in $W_0^{1,p}(\Omega, \mathbb{R}^m)$. We repeat the above argument setting $v = u_s$. By (4.11) and, since $(\tilde{u}_s - u_s) \rightharpoonup 0$ weakly in $W_0^{1,p}(\Omega, \mathbb{R}^m)$, we still have $II_s \to 0$. Hence, similar to (4.12), we get

$$\limsup_{s\to+\infty} \langle \mathcal{A}_\sigma \tilde{u}_s, \tilde{u}_s - u_s\rangle_{W^{1,p}} \leq \limsup_{s\to+\infty} \langle b_s, \tilde{u}_s - u_s\rangle_{W^{1,p_s}} = 0.$$

This directly implies that

$$\limsup_{s\to+\infty} \langle \mathcal{A}_\sigma \tilde{u}_s, \tilde{u}_s - u\rangle_{W^{1,p}} \leq 0. \tag{4.13}$$

The operator $\mathcal{A}_\sigma$ is pseudomonotone. See Landes [1996] for a proof. The definition of pseudomonotone operators (Definition 4.11) together with (4.13) implies that we obtain the estimate

$$\liminf_{s\to+\infty} \langle \mathcal{A}_\sigma \tilde{u}_s, \tilde{u}_s - v\rangle_{W^{1,p}} \geq \langle \mathcal{A}_\sigma u, u - v\rangle_{W^{1,p}} \tag{4.14}$$

for every Lipschitz continuous function $v \in W_0^{1,\infty}(\Omega, \mathbb{R}^m)$. Putting (4.12) and (4.14) together, we conclude that

$$\langle \mathcal{A}_\sigma u, u - v\rangle_{W^{1,p}} \leq \langle b, u - v\rangle_{W^{1,p}},\ v \in W_0^{1,\infty}(\Omega, \mathbb{R}^m).$$

This holds true for every $v \in W_0^{1,p}(\Omega, \mathbb{R}^m)$, since the Lipschitz continuous functions form a dense subset with respect to the strong topology in $W_0^{1,p}(\Omega, \mathbb{R}^m)$. If we replace v by $2u - v$, we get the other direction of the last inequality and, hence, $\mathcal{A}_\sigma u = b$ (meaning $b \in \mathcal{A}_\sigma^{\mathrm{set}} u$).

Since the arguments can be applied for any subsequence of $P_1, P_2, \ldots$, we conclude that $\mathcal{A}_\sigma u = b$ holds as long as $\mathcal{A}_{P_s} \tilde{u}_s = b_s$ for infinitely many $s \in \{1, 2, \ldots\}$. This implies $(\Gamma 1)$' and finishes the proof. $\square$

Chapter 5

Quasiconvexity and Non-linear Programming

We will now study quasiconvexity in the sense of non-linear programming. Our aim is to preserve quasiconvexity during the process of polynomial approximation. The idea is to apply results of both Chapter 3 and Chapter 4. In order to do so, we will construct approximating functions that can be written as the composition of a convex and a monotone function. Finally, we will show that every quasiconvex continuous function can be approximated locally uniformly by quasiconvex polynomials.

After that, we will focus on Problem 3, which is to solve the non-linear programming problem of the form:

$$\inf \left\{ g(x) \mid x \in \mathbb{R}^N, g_1(x) \leq 0, \ldots, g_l \leq 0 \right\}.$$

Here $l > 0$ denotes a positive integer and $g, g_1, \ldots, g_l \colon \mathbb{R}^N \longrightarrow \mathbb{R} \cup \{+\infty\}$ denote quasiconvex lower semicontinuous extended real-valued functions. We will show that this problem can be approximated via polynomial problems. Here we will work with the concept of Γ-convergence (like we did in Chapter 3). At the end of this chapter, we will shortly discuss a connection to the so-called Fenchel problem of level sets.

To begin with, we emphasize a major difference between quasiconvexity and convexity.

5.1 The Lack of Convexity

The abstract results in Chapter 3 cannot be applied directly, since the set of quasiconvex functions is not convex. We will give an example of a quasiconvex function that even shows that the mollifier argument of Theorem 2.7 does not work in the context of quasiconvex functions. We begin with the definition of quasiconvex functions.

Recall that a set $K \subseteq \mathbb{R}^N$ is *convex* if we have $\lambda x + (1-\lambda)y \in K$ for every $x, y \in K$ and every $\lambda \in [0,1]$.

Definition 5.1 *A lower semicontinuous function $f \in \mathcal{LSC}(\mathbb{R}^N)$ is called quasiconvex if all lower level sets $\mathcal{N}_{f,\alpha} = \{x \in \mathbb{R}^N \mid f(x) \leq \alpha\}$, $\alpha \in \mathbb{R}$, are convex subsets of $\mathbb{R}^N$.*

Quasiconvexity can be defined over arbitrary (real) vector spaces but we are concerned with the finite dimensional case only. Note that, by definition, every composition $h \circ g$ of a convex function $g \in \mathcal{LSC}(\mathbb{R}^N)$ and of a monotone function $h \in \mathcal{C}(\mathbb{R})$ is quasiconvex.

In order to simplify the arguments, we will concentrate on radially symmetric functions. We denote by $\Psi(\mathbb{R}^N)$ the set of all non-negative integrable functions $\psi\colon \mathbb{R}^N \longrightarrow \mathbb{R}$ that fulfill the following properties: ψ vanishes outside the ball $\overline{\mathcal{B}}_{0,1}$, the equation $\psi(x) = \psi(y)$ holds for every $x, y \in \mathbb{R}^N$ with $|x| = |y|$ and

$$\int\limits_{\mathbb{R}^N} \psi(x)\mathrm{d}x = 1.$$

The smooth functions $\psi_1, \psi_2, \ldots$ defined in Section 2.2.3 are contained in $\Psi(\mathbb{R}^N)$.

Lemma 5.2 *There exists a quasiconvex function $f \in \mathcal{C}(\mathbb{R}^2)$ so that for every function $\psi \in \Psi(\mathbb{R}^2)$ the convolution $f * \psi$ is non-quasiconvex.*

Proof. We consider the function $f \in \mathcal{C}(\mathbb{R}^2)$ (see Figure 5.1(a) below) given by

$$f(x) = f(x^{(1)}, x^{(2)}) = \begin{cases} 0 & x^{(1)} \geq 0 \,\wedge\, x^{(2)} \geq 0 \\ -x^{(1)} & x^{(1)} < 0 \\ \max\{-x^{(1)}, x^{(2)}\} & x^{(1)} \geq 0 \,\wedge\, x^{(2)} < 0. \end{cases}$$

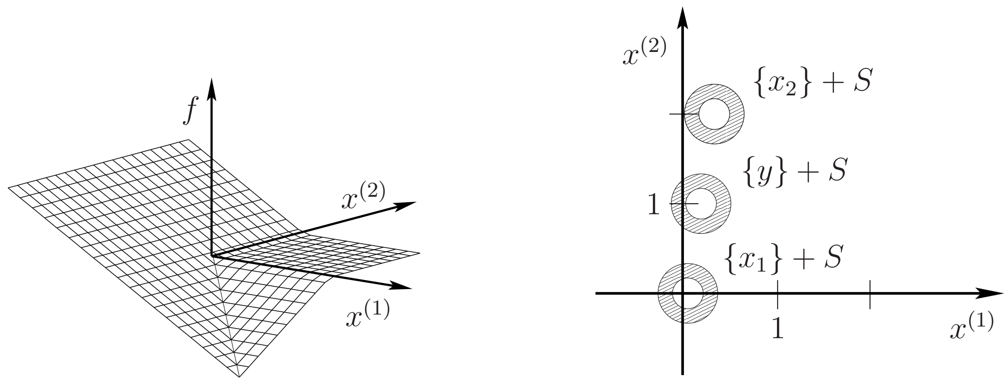
The lower level sets of f are

$$\mathcal{N}_{f,\alpha} = \begin{cases} [-\alpha, +\infty) \times [\alpha, -\infty) & \alpha < 0 \\ [-\alpha, +\infty) \times \mathbb{R} & \alpha \geq 0. \end{cases}$$

Hence the function f is quasiconvex. Fix a function $\psi \in \Psi(\mathbb{R}^2)$. We show that $f * \psi$ is not quasiconvex.

We have $(f * \psi)(-1, 0) > 0$ and $(f * \psi)(1, 0) < 0$, since ψ is non-negative and vanishes outside $\overline{\mathcal{B}}_{0,1}$. The continuity of the convolution $f * \psi$ implies that we can find a vector $x_1 \in [-1,1] \times \{0\}$ so that $(f * \psi)(x_1) = 0$. Now we choose the smallest real number $x_2^{(1)} \in \mathbb{R}$ so that $(f * \psi)(x_2) = 0$ holds for the vector $x_2 = (x_2^{(1)}, 2)$.

Set $S = \{x \in \mathbb{R}^2 \mid \psi(x) > 0\}$ and $y = \frac{1}{2}(x_1 + x_2)$. By construction, the intersection of the sets $\{y\} + S$ and $(-\infty, 0] \times \mathbb{R}$ has positive Lebesgue measure (see Figure 5.1(b)). Hence we get $(f * \psi)(\frac{1}{2}(x_1 + x_2)) > 0$. This shows that $f * \psi$ is not quasiconvex. $\square$



(a) The graph of f (b) Translations of the set S

Figure 5.1: A quasiconvex counter-example

By a simple generalization, such examples can be constructed for every $N \geq 2$. However, the case $N = 1$ is completely different. This is illustrated by the following remark:

Remark 5.3 *There exists an integrable function $\psi \in \Psi(\mathbb{R})$ so that the convolution $f * \psi$ is quasiconvex for every quasiconvex function $f \in \mathcal{C}(\mathbb{R})$.*

Proof. There are only three possible cases for a quasiconvex function f in $\mathcal{C}(\mathbb{R})$:

(1) f is monotone on $\mathbb{R}$.

(2) $-f$ is monotone on $\mathbb{R}$.

(3) $-f$ is monotone on $(-\infty, x_0]$ and f is monotone on $[x_0, +\infty)$ for some real number $x_0 \in \mathbb{R}$.

We consider the function $\psi \in \Psi(\mathbb{R})$ given by

$$\psi(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & \text{else.} \end{cases}$$

We show that the convolution $f * \psi$ is quasiconvex. An argument similar to that of Lemma 2.6 shows that nothing is left to prove for the cases (1) and (2). Let f be of the form (3). Without loss of generality, we can assume that $f \geq 0$ on $\mathbb{R}$.

We set $c = \frac{1}{2}$. The function $f * \psi$ lies in $\mathcal{C}^1(\mathbb{R})$ and the first derivative is given by

$$\mathrm{D}(f * \psi)(x) = f\left(x + c\right) - f\left(x - c\right), \; x \in \mathbb{R}. \tag{5.1}$$

Now assume that $f * \psi$ is not quasiconvex. Then f is neither of forms (1), (2) nor (3). Hence there exist real numbers $x_1 < x_2$ so that $\mathrm{D}(f*\psi)(x_1) > 0$ and $\mathrm{D}(f * \psi)(x_2) < 0$. In view of (5.1), this implies that $f(x_1 - c) < f(x_1 + c)$ and $f(x_2 - c) > f(x_2 + c)$. Due to the properties of x_1 and x_2, only two possible cases remain:

(i) $x_1 - c < x_1 + c < x_2 - c < x_2 + c$ or

(ii) $x_1 - c < x_2 - c < x_1 + c < x_2 + c$.

Set $\alpha = \max\left\{f\left(x_1 - c\right), f\left(x_2 + c\right)\right\}$. In both cases (i) and (ii), we get that

$$\alpha < \max\left\{f\left(x_1 + c\right), f\left(x_2 - c\right)\right\}.$$

Hence the lower level set $\mathcal{N}_{f,\alpha}$ is not convex. This is a contradiction to the quasiconvexity of f. We conclude that $f * \psi$ must be quasiconvex. □

5.2 Characterization by Polynomials

Before we present our main approximation result on quasiconvex functions, we prove a lemma about convex functions with prescribed level sets.

Lemma 5.4 *Let $K, L \subseteq \mathbb{R}^N$ be convex compact sets and $\epsilon > 0$ a positive real number so that $K + \overline{\mathcal{B}}_{0,\epsilon} \subseteq L$ holds. Then there exists a convex continuous function*[1] *$d\colon L \longrightarrow \mathbb{R}$ so that $\mathcal{N}_{d,0} = K$ and $\mathcal{N}_{d,1} = L$.*

Proof. We will use the concept of convex envelopes[2]. See Rockafellar [1970] for an introduction.

We consider the lower semicontinuous function $\tilde{d} \in \mathcal{LSC}(\mathbb{R}^N)$ given by

$$\tilde{d}(x) = \begin{cases} 0 & x \in K \\ 1 & x \in \partial L \\ +\infty & \text{else.} \end{cases}$$

[1]Let $L \subseteq \mathbb{R}^N$ be a convex set and $d\colon L \longrightarrow \mathbb{R}$ a given function. Consider the function $f \in \mathcal{LSC}(\mathbb{R}^N)$ that is given by the conditions $f = d$ on L and $f = +\infty$ on $\mathbb{R}^N \setminus L$. Then d is called *convex* (on L) if f is convex in the sense of Definition 3.1.

[2]The convex envelope is sometimes called *convex hull* in the literature.

Let the function $d \in \mathcal{LSC}(\mathbb{R}^N)$ be the (lower semicontinuous) convex envelope of $\tilde{d}$. Then d is a convex function and it can be represented in the following way: The function d is the pointwise supremum of the collection of all affine functions from $\mathbb{R}^N$ to $\mathbb{R}$ majorized by $\tilde{d}$. Compare Rockafellar [1970, Corollary 12.1.1].

In view of this representation, it is not hard to see that $d(x) \in [0,1]$ holds for every $x \in L$ and that d is even continuous on L up to the boundary. In addition, we get $\mathcal{N}_{d,0} = K$ and $\mathcal{N}_{d,1} = L$ as desired. $\square$

The next theorem states our approximation result for quasiconvex lower semicontinuous functions. See Corollary 5.6 for the continuous case.

Theorem 5.5 *Every quasiconvex function $f \in \mathcal{LSC}(\mathbb{R}^N)$ can be approximated by quasiconvex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^N)$ so that $P_s \to f$ in $\mathcal{LSC}(\mathbb{R}^N)$.*

Proof. The proof consists of four steps. In Step 1, we will approximate the function f by quasiconvex continuous functions. Step 2 and Step 3 contain the difficult part of the proof. Here we show how to find approximating functions that are compositions of monotone functions and convex functions. Finally, in Step 4, we construct the desired sequence of quasiconvex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^N)$.

Step 1: Approximation by continuous functions

If $f = +\infty$ on the whole of $\mathbb{R}^N$, we set $P_s = s$ for $s = 1, 2, \ldots$. Now let $f \neq +\infty$. We translate the coordinate system (if necessary) so that $f(0) < +\infty$.
Fix a positive integer $s > 0$. We use the construction in the proof of Lemma 2.5 and define the function $f_s \in \mathcal{C}(\mathbb{R}^N)$ by specifying its lower level sets

$$\mathcal{N}_{f_s,\alpha} = \begin{cases} (\mathcal{N}_{f,\alpha} + \overline{\mathcal{B}}_{0,5^{-s}(\alpha+s)}) \cap \overline{\mathcal{B}}_{0,s+5^{-s}(\alpha+s)} & \alpha \geq -s \\ \emptyset & \alpha < -s. \end{cases} \tag{5.2}$$

Following the proof of Lemma 2.5, we get that $f_s \to f$ in $\mathcal{LSC}(\mathbb{R}^N)$ and we have the estimate

$$|f_s(x) - f_s(y)| \leq 5^s |x - y|, \; x, y \in \mathbb{R}^N. \tag{5.3}$$

Moreover, in view of (5.2), it is not difficult to check that all functions $f_1, f_2, \ldots$ are quasiconvex.

Step 2: Construction of the function g_s

Let $s > 0$ be any positive integer. We are going to construct a convex function $g_s \in \mathcal{C}(\mathbb{R}^N)$ and, afterwards, a monotone function $h_s \in \mathcal{C}(\mathbb{R})$ so that the composition fulfills the condition

$$f_s(x) - 5^{-s} \geq (h_s \circ g_s)(x) \geq f_s(x) - 4 \cdot 5^{-s},\ x \in \mathbb{R}^N. \tag{5.4}$$

The function $f_s - 2 \cdot 5^{-s}$ is continuous and has only compact lower level sets. Hence

$$\tilde{\alpha} = \min \{f_s(x) - 2 \cdot 5^{-s} \mid x \in \mathbb{R}^N\}$$

is a real number. For every $i = 0, 1, \ldots$ we set $\alpha_i = \tilde{\alpha} + i \cdot 5^{-s}$ and

$$K_i = \mathcal{N}_{f_s - 2\cdot 5^{-s}, \alpha_i},\ L_i = K_i + \overline{\mathcal{B}}_{0, 5^{-2s}/2}.$$

Then, having in mind (5.3), we know that $K_i + \overline{\mathcal{B}}_{0,5^{-2s}} \subseteq K_{i+1}$ and, hence,

$$K_i \subseteq L_i \subseteq K_{i+1},\ i = 0, 1, \ldots. \tag{5.5}$$

We initialize the construction of the function g_s by the continuous function $g_s^{(0)} \colon L_0 \longrightarrow \mathbb{R}$ given by $g_s^{(0)} = 0$. We proceed by induction. Let $i > 0$ be a positive integer and assume that the function $g_s^{(i-1)} \colon L_{i-1} \longrightarrow \mathbb{R}$ is convex as well as continuous on L_{i-1} and fulfills the inequality

$$\min \{g_s^{(i-1)}(x) \mid x \in \partial L_{i-1}\} \geq \max \{g_s^{(i-1)}(x) \mid x \in K_{i-1}\}. \tag{5.6}$$

We show how $g_s^{(i)}$ is defined on the set L_i. Lemma 5.4 implies that there exists a convex continuous function $d \colon L_i \longrightarrow \mathbb{R}$ so that $\mathcal{N}_{d,0} = K_{i-1}$ and $\mathcal{N}_{d,1} = L_i$. We can find positive real numbers $\mu, \nu > 0$ so that the following conditions are fulfilled:

(i) $\max \{\mu \cdot d(x) - \nu \mid x \in K_{i-1}\} < \min \{g_s^{(i-1)}(x) \mid x \in K_{i-1}\}$,

(ii) $\min \{\mu \cdot d(x) - \nu \mid x \in \partial L_{i-1}\} > \max \{g_s^{(i-1)}(x) \mid x \in L_{i-1}\}$,

(iii) $\min \{\mu \cdot d(x) - \nu \mid x \in \partial L_i\} \geq \max \{\mu \cdot d(x) - \nu \mid x \in K_i\}$.

In fact, (iii) is true for any $\mu, \nu > 0$. We choose ν first so that (i) holds. Note that (i) remains true for every $\mu > 0$, since $d = 0$ on K_{i-1}. Hence we can choose $\mu > 0$ large enough so that (ii) becomes true.

Step 3: Construction of the function h_s

Now we are in the position to define the function $g_s^{(i)}\colon L_i \longrightarrow \mathbb{R}$. We set

$$g_s^{(i)}(x) = \begin{cases} \max\{g_s^{(i-1)}(x), \mu \cdot d(x) - \nu\} & x \in L_{i-1} \\ d(x) & x \in L_i \setminus L_{i-1}. \end{cases}$$

Due to (i), (ii) and (iii), $g_s^{(i)}$ is continuous and $g_s^{(i)} = g_s^{(i-1)}$ holds on K_{i-1}. The function $g_s^{(i)}$ is also convex on L_i, since convexity is a local property. By (5.5) and (iii), we have

$$\min\{g_s^{(i)}(x) \mid x \in \partial L_i\} \geq \max\{g_s^{(i)}(x) \mid x \in K_i\}.$$

This induction process produces convex continuous functions $g_s^{(i)}\colon L_i \longrightarrow \mathbb{R}$, $i = 0, 1, \ldots$. Since the sets $K_0, K_1, \ldots$ cover the whole of $\mathbb{R}^N$, the expression

$$g_s(x) = \begin{cases} g_s^{(0)}(x) & x \in K_0 \\ g_s^{(1)}(x) & x \in K_1 \setminus K_0 \\ \vdots & \vdots \end{cases}$$

defines a continuous function $g_s\colon \mathbb{R}^N \longrightarrow \mathbb{R}$. Again the locality of convexity implies that g_s is convex on $\mathbb{R}^N$. We define $\beta_i = \max\{g_s(x) \mid x \in K_i\}$ for every $i \geq 0$. Then (ii) implies that we get an increasing sequence $\beta_0 < \beta_1 < \ldots$ of real numbers. In view of (5.6), we have

$$K_i \subseteq \mathcal{N}_{g_s,\beta_i} \subseteq L_i,\ i = 0, 1, \ldots. \tag{5.7}$$

Now we define the monotone function $h_s \in \mathcal{C}(\mathbb{R})$ by

$$h_s(x) = \begin{cases} \alpha_0 & x \leq \beta_0 \\ \frac{\beta_1 - x}{\beta_1 - \beta_0}\alpha_0 + \frac{x - \beta_0}{\beta_1 - \beta_0}\alpha_1 & \beta_0 < x \leq \beta_1 \\ \vdots & \vdots \end{cases}$$

so that $h_s(\beta_i) = \alpha_i$ for every $i \geq 0$. Let $\alpha \in \mathbb{R}$ be any real number so that $\alpha_i \leq \alpha \leq \alpha_{i+1}$ holds for some $i \geq 0$. By (5.5) and (5.7), we know that

$$K_i \subseteq \mathcal{N}_{h_s \circ g_s, \alpha_i} \subseteq \mathcal{N}_{h_s \circ g_s, \alpha} \subseteq \mathcal{N}_{h_s \circ g_s, \alpha_{i+1}} \subseteq L_{i+1} \subseteq K_{i+2}.$$

If we recall how the sets K_i and K_{i+2} were defined, we get

$$\mathcal{N}_{f_s - 5^{-s}, \alpha} \subseteq \mathcal{N}_{f_s - 5^{-s}, \alpha_{i+1}} = \mathcal{N}_{f_s - 2 \cdot 5^{-s}, \alpha_i} = K_i$$

on the left-hand side and

$$K_{i+2} = \mathcal{N}_{f_s - 2\cdot 5^{-s}, \alpha_{i+2}} = \mathcal{N}_{f_s - 4 \cdot 5^{-s}, \alpha_i} \subseteq \mathcal{N}_{f_s - 4\cdot 5^{-s}, \alpha}$$

on the right-hand side. Hence we have shown that

$$\mathcal{N}_{f_s - 5^{-s}, \alpha} \subseteq \mathcal{N}_{h_s \circ g_s, \alpha} \subseteq \mathcal{N}_{f_s - 4 \cdot 5^{-s}, \alpha}.$$

This implies (5.4), since we can take any $\alpha \geq \alpha_0$.

Step 4: Conclusion

Following Theorem 3.11 and Theorem 4.8, we can approximate g_s in $\mathcal{C}(\mathbb{R}^N)$ by convex polynomials $P_1', P_2', \ldots \in \mathcal{P}(\mathbb{R}^N)$ and h_s in $\mathcal{C}(\mathbb{R})$ by monotone polynomials $P_1'', P_2'', \ldots \in \mathcal{P}(\mathbb{R})$. Locally uniform convergence is carried over to the sequence of compositions. Hence the functions $P_1'' \circ P_1', P_2'' \circ P_2', \ldots$ are quasiconvex polynomials and converge to $h_s \circ g_s$ in $\mathcal{C}(\mathbb{R}^N)$. This holds true for every fixed $s > 0$.

We let s tend to infinity. Recall that $f_s \to f$ in $\mathcal{LSC}(\mathbb{R}^N)$. In view of (5.4), this implies that $h_s \circ g_s \to f$ in $\mathcal{LSC}(\mathbb{R}^N)$. Now we are in a situation where Lemma 2.2 can be applied. As a consequence, we can choose quasiconvex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^N)$ so that $P_s \to f$ holds in $\mathcal{LSC}(\mathbb{R}^N)$. $\square$

We get the following corollary due to Lemma 2.2:

Corollary 5.6 *Every quasiconvex function $f \in \mathcal{C}(\mathbb{R}^N)$ can be approximated by quasiconvex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^N)$ so that $P_s \to f$ in $\mathcal{C}(\mathbb{R}^N)$.*

5.3 A Quasiconvex Programming Problem

We study the minimization problem $QOP(g; g_1, \ldots, g_l)$ of the form

$$\inf \left\{ g(x) \mid x \in \mathbb{R}^N, g_1(x) \leq 0, \ldots, g_l \leq 0 \right\} \tag{5.8}$$

where $g, g_1, \ldots, g_l \in \mathcal{LSC}(\mathbb{R}^N)$ are quasiconvex lower semicontinuous functions and $l > 0$ a fixed positive integer.

Note that the infimum in (5.8) can be $+\infty$, $-\infty$ or a real number, in particular, $QOP(g; g_1, \ldots, g_l)$ might be insolvable. In any case, the problem $QOP(g; g_1, \ldots, g_l)$ is equivalent to

$$\inf \left\{ f(x) \mid x \in \mathbb{R}^N \right\}$$

where we define the function $f \colon \mathbb{R}^N \longrightarrow \mathbb{R} \cup \{+\infty\}$ by

$$f(x) = \begin{cases} g(x) & g_1(x) \leq 0, \ldots, g_l(x) \leq 0 \\ +\infty & \text{else.} \end{cases} \tag{5.9}$$

It is immediate that f is lower semicontinuous and, hence, an element in $\mathcal{LSC}(\mathbb{R}^N)$.

5.4 Approximation via Γ-Convergence

In order to show that the convergence in $\mathcal{LSC}(\mathbb{R}^N)$ fits nicely in the context of Γ-convergence, we first prove a lemma. It can be seen as a direct consequence of an abstract result given by Attouch [1984, Theorem 2.40].

Lemma 5.7 *Let $f, f_1, f_2, \ldots \in \mathcal{LSC}(\mathbb{R}^N)$ be lower semicontinuous functions so that $f_s \to f$ holds in $\mathcal{LSC}(\mathbb{R}^N)$. Then the sequence $f_1, f_2, \ldots$ Γ-converges to f with respect to the Euclidean topology in $\mathbb{R}^N$.*

Proof. In this proof, we make use of the conditions (Γ1) and (Γ2) given in Definition 3.25 as well as conditions (L1) and (L2) in Section 2.2.4.

Fix a vector $x \in \mathbb{R}^N$. We show (Γ2). Therefore it is sufficient to construct a sequence $x_1, x_2, \ldots \in \mathbb{R}^N$ so that $x_s \to x$ in $\mathbb{R}^N$ and $f_s(x_s) \to f(x)$ in $\mathbb{R} \cup \{+\infty\}$. We know that $f_s \to f$ holds in $\mathcal{LSC}(\mathbb{R}^N)$. Then the condition (L2) implies that $f_s \to f$ pointwise on the whole of $\mathbb{R}^N$. Hence we can set $x_s = x$ for every $s = 1, 2, \ldots$ and get a sequence as desired.

We show (Γ1) by contradiction. Fix a sequence $x_1, x_2, \ldots \in \mathbb{R}^N$ so that $x_s \to x$ holds in $\mathbb{R}^N$ and

$$f(x) > \lim_{s \to +\infty} f_s(x_s) + \epsilon \tag{5.10}$$

for some positive real number $\epsilon > 0$. Then there exists a positive integer $s_1 > 0$ so that $f(x) > f_s(x_s) + \epsilon/2$ for every $s \geq s_1$. The set of vectors $x_1, x_2, \ldots \in \mathbb{R}^N$ is relatively compact in $\mathbb{R}^N$. In view of the condition (L1), we conclude that there exists a positive integer $s_2 \geq s_1$ so that we have

$$f(x) > f_t(x_s) + \epsilon/2, \ s \geq t \geq s_2.$$

By the lower semicontinuity of the functions $f_1, f_2, \ldots$, we get $f(x) > f_t(x) + \epsilon/2$ for every $t \geq s_2$. This is a contradiction to the pointwise convergence of the sequence $f_1, f_2, \ldots$. □

With the help of Lemma 5.7 and Theorem 5.5, we can easily prove the following theorem:

Theorem 5.8 *Let $l > 0$ be a positive integer and $g, g_1, \ldots, g_l \in \mathcal{LSC}(\mathbb{R}^N)$ quasiconvex lower semicontinuous functions. Then there exist quasiconvex polynomials $P_i, P_{j,i} \in \mathcal{P}(\mathbb{R}^N)$, $i = 1, 2, \ldots$, $j = 1, \ldots, l$, so that the following conditions are fulfilled:*

(i) We have $P_s \to g$ and $P_{j,s} \to g_j$, $j = 1, \ldots, l$, in $\mathcal{LSC}(\mathbb{R}^N)$.

(ii) Let $x_1, x_2, \ldots \in \mathbb{R}^N$ be a bounded sequence so that x_s is a solution to $QOP(P_s; P_{1,s}, \ldots, P_{l,s})$ for every $s = 1, 2, \ldots$. Then any cluster point of this sequence solves the problem $QOP(g; g_1, \ldots, g_l)$.

Proof. By Theorem 5.5, we can choose quasiconvex polynomials $P_i, P_{j,i} \in \mathcal{P}(\mathbb{R}^N)$, $i = 1, 2, \ldots$, $j = 1, \ldots, l$, so that (i) holds true. We show that (i) implies (ii). Therefore we make use of a Γ-convergence argument.

We consider the lower semicontinuous function $f \in \mathcal{LSC}(\mathbb{R}^N)$ given by (5.9). In the same manner, we define functions $f_1, f_2, \ldots \in \mathcal{LSC}(\mathbb{R}^N)$ by

$$f_s(x) = \begin{cases} P_s(x) & P_{1,s}(x) \leq 0, \ldots, P_{l,s}(x) \leq 0 \\ +\infty & \text{else} \end{cases}, \; s = 1, 2, \ldots.$$

If the sequence $f_1, f_2, \ldots$ Γ-converges to f, we have (ii). In view of Lemma 5.7, it suffices to show that $f_s \to f$ in $\mathcal{LSC}(\mathbb{R}^N)$. However, this is a direct consequence of (i). The condition (L1) in Section 2.2.4 is immediate. We show (L2). Fix a vector $x \in \mathbb{R}^N$. Assume first that $g_j(x) \leq 0$ for every $j = 1, \ldots, l$. Then $P_s \to g$ in $\mathcal{LSC}(\mathbb{R}^N)$ implies that $f_s(x) \to f(x)$ in $\mathbb{R} \cup \{+\infty\}$. Now assume that $g_j(x) > 0$ for some $j \in \{1, \ldots, l\}$. Then $f(x) = +\infty$ holds. The fact that $P_{j,s} \to g_j$ in $\mathcal{LSC}(\mathbb{R}^N)$ implies that $P_{j,s}(x) > 0$ for $s > 0$ large enough. Hence we have $f_s(x) \to +\infty$. This finishes the proof. $\square$

5.5 Remark on the Fenchel Problem of Level Sets

The Fenchel problem of level sets[3] in dimension 2 reads: “In einem konvexen Bereich C der x_1x_2-Ebene sei eine Schar (geschlossener oder offener) konvexer Kurven gegeben, die C einfach und lückenlos überdecken. Gefragt wird, unter welchen Bedingungen eine stetige konvexe Funktion $f(x_1, x_2)$ existiert, deren Niveaukurven diese Kurven sind.” Fenchel [1956, p. 1].

This problem is closely related to the question whether or not a given quasiconvex function can be written as the composition of a convex and a monotone function. We do not study this question in this thesis. However, in view of the proof of Theorem 5.5, we get the following corollary:

Corollary 5.9 *Every quasiconvex function $f \in \mathcal{LSC}(\mathbb{R}^N)$ can be approximated by quasiconvex polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^N)$ so that $P_s \to f$ in $\mathcal{LSC}(\mathbb{R}^N)$. In particular, the polynomials $P_1, P_2, \ldots \in \mathcal{P}(\mathbb{R}^N)$ can be chosen in such a way that $P_s = P''_s \circ P'_s$, $s = 1, 2, \ldots$, holds for some convex polynomials $P'_1, P'_2, \ldots \in \mathcal{P}(\mathbb{R}^N)$ and monotone polynomials $P''_1, P''_2, \ldots \in \mathcal{P}(\mathbb{R})$.*

[3] Rapcsák [2005] uses the name *Fenchel problem of level sets*. However, the problem was studied for the first time by de Finetti [1949].

Chapter 6

Conclusion and Outlook

In this thesis, we have studied three different classes of non-linear problems. Particularly, we are concerned with minimization in the calculus of variations (Problem 1), non-linear partial differential equations (Problem 2) and non-linear programming (Problem 3). We have seen that, under certain assumptions on the parameters, the considered non-linear problems can be approximated by polynomial problems both in the parameter space and in the solution space. In order to achieve these approximation results, we have divided up our analysis into the following two parts: the approximation on the level of functions and the approximation on the level of problems.

On the level of functions, the main task was to preserve the following properties during the approximation procedure: quasiconvexity in the calculus of variations, quasimonotonicity in the context of partial differential equations and quasiconvexity in non-linear programming. The first two of them were treated in almost the same way. Therefore we presented an abstract framework that mostly relies on tools from convex analysis. Although related to this framework, the quasiconvexity in non-linear programming required a different ansatz.

The advantage of our approach lies in its generality. The techniques that we have used can be applied to various notions of convexity and monotonicity. Only a selection of these notions has been studied here, since a complete discussion would have lain beyond the scope of this thesis. However, it is certainly possible to apply our ideas to other properties of functions as well, for example, to k-convexity, a property that has been studied by Trudinger and Wang [1999].

One of the open questions in the calculus of variations is whether rank-one convexity and quasiconvexity coincide for real-valued functions over $\mathbb{R}^{2\times n}$, $n \geq 2$. We have shown that it is sufficient to investigate all polynomial functions in order to answer this question. The same is true in the presence of objectivity and isotropy.

Another interesting question in the calculus of variations is whether quasiconvex lower semicontinuous functions of the form $f\colon \mathbb{R}^{m\times n} \longrightarrow \mathbb{R}\cup\{+\infty\}$ can be approximated by quasiconvex continuous functions (and, hence, also by quasiconvex polynomials). Special cases have been studied by Müller [1999a] and Wagner [forthcoming], while the complete answer is still unknown. Moreover, in the context of quasimonotonicity, it remains unclear whether polynomial approximation is possible without growth conditions on the limit function.

On the level of non-linear problems, we have used the concepts of Γ-convergence and G-convergence, which both guarantee a certain convergence in the solution space. The solution space is an infinite dimensional Banach space for Problems 1 and 2. That is why we could not expect more than the weak convergence of solutions.

The regularity of solutions has not been studied in this thesis. Apart from the existence of solutions, their regularity plays an equally important role in the calculus of variations. The known regularity results for solutions to Problem 1 in the presence of quasiconvexity are promising. See Acerbi and Fusco [1987] as well as Kristensen and Mingione [2007]. However, these results hold only for prescribed polynomial coercivity and growth conditions on the integrand function. In order to prove more general regularity results, one starting point could be to investigate in more detail the polynomial approximation of the integrand.

Bibliography

Emilio Acerbi and Nicola Fusco. Semicontinuity problems in the calculus of variations. *Arch. Ration. Mech. Anal.*, 89:125–145, 1984.

Emilio Acerbi and Nicola Fusco. A regularity theorem for minimizers of quasiconvex integrals. *Arch. Ration. Mech. Anal.*, 99:261–281, 1987.

Robert A. Adams. *Sobolev Spaces*. Pure and Applied Mathematics. Academic Press, 1978.

Ilka Agricola and Thomas Friedrich. *Globale Analysis: Differentialformen in Analysis, Geometrie und Physik*. Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden, 2001.

Jean-Jacques Alibert and Bernard Dacorogna. An example of a quasiconvex function that is not polyconvex in two dimensions. *Arch. Ration. Mech. Anal.*, 117(2):155–166, 1992.

Hedy Attouch. *Variational Convergence for Functions and Operators*. Applicable Mathematics Series. Pitman Publishing, 1984.

Hedy Attouch, Giuseppe Buttazzo, and Gérard Michaille. *Variational Analysis in Sobolev and BV Spaces: Applications to PDEs and Optimization*. MPS-SIAM Series on Optimization. SIAM and MPS, 2006.

Mordecai Avriel, Walter E. Diewert, Siegfried Schaible, and Israel Zang. *Generalized Concavity*, volume 36 of *Mathematical Concepts and Methods in Science and Engineering*. Plenum Publishing Corporation, 1998.

John M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Archive for Rational Mechanics and Analysis*, 63(4):337–403, 1977.

John M. Ball, Bernd Kirchheim, and Jan Kristensen. Regularity of quasiconvex envelopes. *Calculus of Variations and Partial Differential Equations*, 11(4):333–359, 2000.

Bernd Bank and Reinhard Mandel. *Parametric Integer Optimization.* Akademie-Verlag, Berlin, 1988.

Bernd Bank, Jürgen Guddat, Diethard Klatte, Bernd Kummer, and Klaus Tammer. *Non-Linear Parametric Optimization*, volume 58 of *Mathematische Monographien.* Akademie-Verlag, Berlin, 1982.

Bernd Bank, Joos Heintz, Teresa Krick, Reinhard Mandel, and Pablo Solernó. Une borne géométrique pour la programmation entière à contraintes polynomiales. *Comptes Rendus de l'Académie des Sciences, series I*, 310 (6):475–478, 1990.

Sören Bartels, Carsten Carstensen, Klaus Hackl, and Ulrich Hoppe. Effective relaxation for microstructure simulations: algorithms and applications. *Computer Methods in Applied Mechanics and Engineering*, 193: 5143–5175, 2004.

Jonathan Bevan. An example of a $c^{1,1}$ polyconvex function with no differentiable convex representative. *Comptes Rendus Mathématique. Académie des Sciences. Paris*, 336(1):11–14, 2003.

Nicolas Bourbaki. *Elements of Mathematics: Topological Vector Spaces, ch. 1-5.* Springer-Verlag, 1987.

Andrea Braides. *Γ-convergence for Beginners*, volume 22 of *Oxford Lecture Series in Mathematics and its Apllications.* Oxford University Press, 2002.

Haïm Brézis. Équations et inéquations non linéaires dans les espaces vectorials en dualité. *Annales de l'institut Fourier*, 18(1):115–175, 1968.

Philippe G. Ciarlet. *Mathematical Elasticity I: Three-dimensional Elasticity*, volume 20 of *Studies in Mathematics and its Applications.* Elsevier Science Publishers, 1988.

John B. Conway. *A Course in Functional Analysis*, volume 96 of *Graduate Texts in Mathematics.* Springer-Verlag, second edition, 1990.

Bernard Dacorogna. *Direct Methods in the Calculus of Variations*, volume 78 of *Applied Mathematical Sciences.* Springer-Verlag, 1989.

Bruno de Finetti. Sulla stratificazioni convesse. *Annali di Matematica Pura ed Applicata*, 30:173–183, 1949.

Enno De Giorgi. Γ-convergenza e G-convergenza. *Boll. Un. Mat. Ital.*, 14-A (5):213–220, 1977.

Enno De Giorgi. Convergence problems for functionals and operators. In *Proc. of the Internat. Meeting on "Recent Methods in Nonlinear Analysis" (Roma, 1978)*, pages 131–188, Bologna, 1979. Pitagora.

Ronald A. DeVore and Xiang Ming Yu. Pointwise estimates for monotone polynomial approximation. *Constructive Approximation*, 1:323–331, 1985.

Georg Dolzmann. *Variational Methods for Crystalline Microstructure - Analysis and Computation*, volume 1803 of *Lecture Notes in Mathematics*. Springer-Verlag, 2003.

Lawrence C. Evans. Quasiconvexity and partial regularity in the calculus of variations. *Arch. Ration. Mech. Anal.*, 95:227–252, 1986.

Daniel Faraco and László Székelyhidi. Tartar's conjecture and localization of the quasiconvex hull in $\mathbb{R}^{2\times 2}$. *Max Planck Institute for Mathematics in the Sciences Leipzig - Preprint*, 60, 2006.

Werner Fenchel. Über konvexe Funktionen mit vorgeschriebenen Niveaumannigfaltigkeiten. *Mathematische Zeitschrift*, 63:496–506, 1956.

Johannes B.G. Frenk and Gábor Kassay. Introduction to convex and quasiconvex analysis. In *Handbook of Generalized Convexity and Generalized Monotonicity*, volume 76 of *Nonconvex Optimization and Its Aplications*, pages 3–87. Springer-Verlag, 2005.

Andreas Griewank and Patrick J. Rabier. On the smoothness of convex envelopes. *Transactions of the American Mathematical Society*, 322(2): 691–709, 1990.

Alexander Grothendieck. *Topological vector spaces.* Notes on mathematics and its applications. Gordon and Breach Science Publishers, 1973.

Sebastian Heinz. Complexity of integer quasiconvex polynomial optimization. *Journal of Complexity*, 21:543–556, 2005.

Sebastian Heinz. Quasiconvex functions can be approximated by quasiconvex polynomials. *ESAIM: COCV*, Published online: January 30, 2008.

Lars Hörmander. *Notions of Convexity*, volume 127 of *Progress in Mathematics.* Birkhäuser, 1994.

Yingkang Hu, Dany Leviatan, and Xiang Ming Yu. Convex polynomial and spline approximation in C[-1, 1]. *Constructive Approximation*, 10:31–64, 1994.

Robert G. Jeroslow. There cannot be any algorithm for integer programming with quadratic constaints. *Oper. Res.*, 21:221–224, 1973.

Konrad Königsberger. *Analysis 2*. Springer-Verlag, second extended edition, 1997.

Jan Kristensen. On the non-locality of quasiconvexity. *Ann. Inst. Henri Poincaré - Analyse non linéaire*, 16(1):1–13, 1999a.

Jan Kristensen. On conditions for polyconvexity. *Proceedings of the American Mathematical Society*, 128(6):1793–1797, 1999b.

Jan Kristensen and Giuseppe Mingione. The singular set of Lipschitzian minima of multiple integrals. *Arch. Ration. Mech. Anal.*, 184:341–369, 2007.

Rüdiger Landes. Quasimonotone versus pseudomonotone. *Proceedings of the Royal Society of Edinburgh (A)*, 126:705–717, 1996.

Dany Leviatan. Shape-preserving approximation by polynomials. *Journal of Computational and Applied Mathematics*, 121:73–94, 2000.

Mitchell Luskin. On the computation of crystalline microstructure. *Acta Numerica*, 5:191–257, 1996.

Paolo Marcellini. Approximation of quasiconvex functions, and lower semicontinuity of multiple integrals. *Manuscripta Math.*, 51:1–28, 1985.

Charles B. Morrey. Quasi-convexity and the lower semicontinuity of multiple integrals. *Pacific Journal of Mathematics*, 2:25–53, 1952.

Stefan Müller. A sharp version of Zhang's theorem on truncating sequences of gradients. *Transactions of the American Mathematical Society*, 351(11): 4585–4597, 1999a.

Stefan Müller. Rank-one convexity implies quasiconvexity on diagonal matrices. *International Mathematics Research Notices*, 1999:1087–1095, 1999b.

Stefan Müller. Variational models for microstructure and phase transitions. In *Calculus of Variations and Geometric Evolution Problems (Cetaro, Italy, June 15-22, 1996)*, volume 1713 of *Lecture Notes in Mathematics*, pages 85–210. Springer-Verlag, 1999c.

Alexander Pankov. *G-Convergence and Homogenization of Nonlinear Partial Differential Operators*, volume 422 of *Mathematics and its Applications*. Kluwer Academic Publishers, 1997.

Pablo Pedregal and Vladimír Šverák. A note on quasiconvexity and rank-one convexity for 2×2 matrices. *Journal of Convex Analysis*, 5(1):107–117, 1998.

Tamás Rapcsák. Fenchel problem of level sets. *Journal of Optimization Theory and Applications*, 127(1):177–191, 2005.

Jean-Pierre Raymond. Lipschitz regularity of solutions of some asymptotically convex problems. *Proceedings of the Royal Society of Edinburgh (A)*, 117:59–73, 1991.

R. Tyrell Rockafellar. *Convex Analysis*. Princeton Landmarks in Mathematics and Physics. Princeton University Press, 1970.

Thomàš Roubíček. *Nonlinear Partial Differential Equations with Applications*, volume 153 of *International Series of Numerical Mathematics*. Birkhäuser, 2005.

Friedrich Sauvigny. *Partial Differential Equations 1: Foundations and Integral Representations*. Springer-Verlag, 2006.

Vladimir Scheffer. *Regularity and irregularity of solutions to nonlinear second order elliptic systems of partial differential equations and inequalities*. PhD thesis, Princeton University, 1974.

Josef Stoer and Christoph Witzgall. *Convexity and Optimization in Finite Dimension I*, volume 163 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, 1970.

A. S. Švedov. Approximation to subharmonic functions by subharmonic polynomials. *Mathematical Notes*, 37(6):443–447, 1985.

Vladimír Šverák. Rank-one convexity does not imply quasiconvexity. *Proceedings of the Royal Society of Edinburgh (A)*, 120:185–189, 1992.

Neil S. Trudinger and Xu-Jia Wang. Hessian measures II. *Annals of Mathematics*, 150:579–604, 1999.

Johann von Neumann. Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen*, 100, 1928.

Marcus Wagner. On the lower semicontinuous quasiconvex envelope for unbounded integrands (I). *ESAIM: COCV*, forthcoming.

Kewei Zhang. On the Dirichlet problem for a class of quasilinear elliptic systems of partial differential equations in divergence form. In Shiing-Shen Chern, editor, *Partial Differential Equations (Tianjin, 1986)*, volume 1306 of *Lecture Notes in Mathematics*, pages 262–277. Springer-Verlag, 1988.

Selbständigkeitserklärung

Ich versichere, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 04. April 2008

Sebastian Heinz