The arithmetic volume of Shimura varieties of orthogonal type

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Abstract

The overall aim of this thesis is to compute arithmetic volumes of Shimura varieties of orthogonal type and natural heights of the special cycles on them. We develop a general theory of integral models of toroidal compactifications of Shimura varieties of Hodge type (and of its standard principal bundle) for the case of good reduction. This enables us, using the theory of Borcherds products, and generalizing work of Burgos, Bruinier and Kühn, to calculate the arithmetic volume of a Shimura variety associated with a lattice $L_{\mathbb{Z}}$ of discriminant D, up to $\log(p)$-contributions from primes p such that $p^2|4D$. The heights of the special cycles are calculated in the codimension 1 case up to $\log(p)$, $p|2D$, and with some additional restrictions in the codimension > 1 case. The values obtained are special derivatives of certain L-series. In the case of the special cycles they are equal to special derivatives of Fourier coefficients of certain normalized Eisenstein series (in addition, up to contributions from ∞) in accordance with conjectures of Bruinier-Kühn, Kudla, and others.

Zusammenfassung

Das Ziel dieser Arbeit ist die Berechnung der arithmetischen Volumina der Shimuravarietäten vom orthogonalen Typ und der natürlichen Höhen der speziellen Zykel auf diesen. Wir entwickeln, für den Fall guter Reduktion, eine allgemeine Theorie ganzzahliger Modelle von toroidalen Kompaktifizierungen der Shimuravarietäten vom Hodge Typ (sowie des Standardhauptfaserbündels darüber). Dies ermöglicht, unter Verwendung der Theorie der Borcherdsprodukte, das arithmetische Volumінen einer zu einem Gitter $L_{\mathbb{Z}}$ der Diskriminante D assoziierten Shimuravarietät, bis auf $\log(p)$ Beiträge zu Primzahlen p mit $p^2|4D$, zu berechnen. Dies ist eine Verallgemeinerung einer Arbeit von Burgos, Bruinier und Kühn. Die Höhen der speziellen Zykel werden im Falle von Kodimension 1 bis auf $\log(p)$-Beiträge mit $p|2D$ berechnet, sowie unter leichten zusätzlichen Einschränkungen im Falle von Kodimension > 1. The resultierenden Größen sind spezielle Ableitungswerte gewisser L-Reihen. Im Falle der speziellen Zykel stimmen diese mit speziellen Ableitungswerten gewisser normalisierter Eisensteinreihen überein (zusätzlich, bis auf Beiträge bei ∞). Dies bestätigt Vermutungen von Bruinier-Kühn, Kudla und anderen.

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Introduction

Summary

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The work consists of three parts.
In the **first part**, we develop a general theory of *canonical* integral models of toroidal compactifications of arbitrary mixed Shimura varieties of Hodge type. This relies heavily on work of Faltings/Chai, Kisin/Vasiu, Milne and Pink. We are able to prove the truth of the main statements of the theory conditionally on a missing technical result (3.3.2). The constructed models are smooth Deligne-Mumford stacks (or even smooth projective schemes, if the data satisfies the usual requirements). No moduli problem as in the approach [72] is used because we are especially interested in non-P.E.L. cases, namely Shimura varieties of orthogonal type. We emphasize that this is, by all means, restricted to the case of good reduction. We also construct a *canonical* model of the standard principal bundle on the toroidal compactification.
More precisely, for a p-integral mixed Shimura datum $\mathbf{X} = (P_{\mathbf{X}}, \mathbb{D}_{\mathbf{X}}, h_{\mathbf{X}})$, a certain compact open $K \subseteq P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ and an additional datum Δ (for an explanation of this notation see the detailed introduction to part III below), we get a model of the toroidal compactification of the associated Shimura variety $\mathrm{M}({}^K_\Delta\mathbf{X})$, a model of the 'compact' dual $\mathrm{M}^\vee(\mathbf{X})$, and a 1-morphism

$$\Xi : \mathrm{M}({}^K_\Delta\mathbf{X}) \to [\mathrm{M}^\vee(\mathbf{X})/P_{\mathbf{X}}]$$

to the quotient stack of the 'compact' dual by the group scheme $P_{\mathbf{X}}$.
A $P_{\mathbf{X}}$-equivariant locally free sheaf on the dual (sheaf on the right hand side) with a $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-invariant Hermitian metric on the image of the Borel embedding gives a

well defined Hermitian automorphic vector bundle on the model $\mathrm{M}({}^K_\Delta\mathbf{X})$. Its metric may be singular along the boundary divisor. This construction is functorial in morphisms of Shimura data. It is also compatible with (formal) boundary morphisms. This yields, in particular, an integral q-expansion principle for automorphic forms.
In the **second part**, we investigate the occurring L-series, in particular the Fourier coefficients of the Eisenstein series associated with the Weil representation and their recursive properties. An 'interpolated orbit equation' is derived.
In the **third part**, we compute arithmetic volumes (absolute heights) of the constructed models for the case of orthogonal Shimura data $\mathbf{O}(L)$, associated with a quadratic lattice L of signature $(m-2,2)$, and the heights of the special cycles on them. The extended Arakelov theory of Burgos, Kramer and Kühn is used [18, 19].

We begin with a brief, and certainly very incomplete, account on the (geometric) Siegel-Weil theory and Kudla's general (arithmetic) conjectures. Thereafter we will describe the various parts in more detail.

Outline: Siegel-Weil theory

Consider two lattices $L_\mathbb{Z} \cong \mathbb{Z}^m, M_\mathbb{Z} \cong \mathbb{Z}^n$ with integral and positive definite quadratic forms Q_L, Q_M. It is a classical problem, to which already Gauss, Euler and in particular Siegel devoted themselves, to determine the representation number, that is, the number of elements in the set of isometric embeddings

$$\mathrm{I}(M_\mathbb{Z}, L_\mathbb{Z}) = \{\alpha : M_\mathbb{Z} \hookrightarrow L_\mathbb{Z} \mid \alpha \text{ is an isometry}\}.$$

It includes (for $n=1$) questions like: "In how many ways can an integer be represented as a sum of m squares?".
If $M_\mathbb{Z} \cong \mathbb{Z}^n$ is a fixed lattice, Q_M is given by an element in $\mathrm{Sym}^2(M_\mathbb{Z}^*)$ and the *generating series*, the *theta series* of $L_\mathbb{Z}$,

$$\Theta_n(L_\mathbb{Z};\tau) = \sum_{Q \in \mathrm{Sym}^2(M_\mathbb{Z}^*)} \#\,\mathrm{I}(M_\mathbb{Z}^Q, L_\mathbb{Z}) \exp(2\pi i Q \cdot \tau), \tag{1}$$

(here τ is an element in Siegel's upper half space $\mathbb{H}_g \subset (M \otimes M)^s_\mathbb{C}$, the subset of elements with positive definite imaginary part) is a *Siegel modular form* of weight $\frac{m}{2}$ for a certain congruence subgroup of $\mathrm{Sp}'(\mathfrak{M}_\mathbb{Z})$ (the symplectic or metaplectic group, according to the parity of m). For example $\Theta_1(<1>;\tau)$ is just the classical theta function.
Under certain conditions on the dimensions, a certain weighted sum over all *classes* $L_\mathbb{Z}^{(i)}$ in the *genus* $L_{\widehat{\mathbb{Z}}}$ of these theta functions is an Eisenstein series (cf. 7.5 for details):

(0.1) Theorem (Siegel-Weil).

$$\sum_i c_i \Theta_n(L_\mathbb{Z}^{(i)};\tau) = E_n(\Phi;\tau,s_0).$$

The additional parameter s_0 indicates that this Eisenstein series is in fact the (holomorphic) special value of a non-holomorphic Eisenstein series $E_n(\Phi;\tau,s)$ at $s = s_0 := \frac{m-n+1}{2}$. The Fourier coefficients of the series are given by a product formula

$$\mu(L_{\widehat{\mathbb{Z}}}, M^Q_{\widehat{\mathbb{Z}}}; s, y) = \mu_\infty(L, M^Q; s, y) \prod_p \mu_p(L_{\mathbb{Z}_p}, M^Q_{\mathbb{Z}_p}; s). \tag{2}$$

Here y is the imaginary part of τ. Its appearance indicates that this series is non-holomorphic for general s. For almost all p, the μ_p's have a very simple shape (see e.g. 8.2.1).

Φ is a certain section in an induced representation $\mathrm{I}_P^{\mathrm{Sp}'(\mathfrak{M},R)}(|\det|^s\xi)$ (7.4), constructed via the Weil representation (7.1), depending only very slightly on $L_{\widehat{\mathbb{Z}}}$. In particular, many different quadratic lattices (even genera) may yield the same Eisenstein series and therefore the same weighted sum of representation numbers.

Essentially the Siegel-Weil formula (0.1) is valid, if and only if $m > n+1$, but if $m \le 2n+2$, the value of the Eisenstein series has to be defined via analytic continuation in s and the theta function has sometimes to be complemented by indefinite coefficients. The factors $\mu_p(L_{\mathbb{Z}_p}, M^Q_{\mathbb{Z}_p}; s_0)$ are the p-adic volumes of the varieties $\mathrm{I}(M^Q, L)(\mathbb{Z}_p)$, classically called *representation densities*. They may be computed by knowing sufficiently many representation numbers of the congruences modulo p^n.

The mere fact that the representation numbers (in an average over classes) should be given by a product over local volumes or densities can be explained easily in the adelic language:

Assume $m - n \ge 3$, for simplicity, for the rest of the discussion. On the adelic points $\mathrm{SO}(L_{\mathbb{A}})$ of the special orthogonal group of the lattice $L_{\mathbb{Z}}$, there is a canonical measure μ. It is a product over local measures μ_ν on the various $\mathrm{SO}(L_{\mathbb{Q}_\nu})$, constructed by any algebraic volume form defined over $\mathbb{Q}$ [95]. The product μ is independent of the choice of this form. The volume of $\mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathrm{SO}(L_{\mathbb{A}})$, which turns out to be finite, is called the **Tamagawa number** by Weil, and we have

(0.2) Theorem ([95]). *For $m \ge 3$*

$$\mathrm{vol}(\mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathrm{SO}(L_{\mathbb{A}})) = 2.$$

From this our fact already follows, as we will explain now (in a slightly broader context):

Let $\varphi \in S(L_{\mathbb{A}^{(\infty)}} \otimes M^*_{\mathbb{A}^{(\infty)}})$ be a Schwartz-Bruhat function (i.e. locally constant with compact support). Let $K = \prod_p K_p$ be a compact open subgroup of $\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ which stabilizes φ. For example K could be the stabilizer of the lattice $L_{\widehat{\mathbb{Z}}}$ and φ the characteristic function of $L_{\widehat{\mathbb{Z}}}$. Let K_∞ be a maximal compact subgroup of $\mathrm{SO}(L_{\mathbb{R}})$.

From (0.2) we may infer that the volume of the real analytic orbifold

$$[\mathrm{SO}(L_{\mathbb{Q}}) \backslash (\mathrm{SO}(L_{\mathbb{A}}) / K_\infty K)]\,,$$

induced by the quotient of μ_∞ and some measure on K_∞, is:

$$2\prod_\nu \mathrm{vol}_\nu^{-1}(K_\nu). \tag{3}$$

We have a finite disjoint decomposition

$$\mathrm{I}(M,L)(\mathbb{A}^{(\infty)}) \cap \mathrm{supp}(\varphi) = \bigcup_i K\alpha_i,$$

If this set is nonempty, we have by Hasse's principle an $\alpha' \in \mathrm{I}(M,L)(\mathbb{Q})$ and hence $g_i \in \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ with $g_i\alpha' = \alpha_i$. There is a lattice $L_{\mathbb{Z}}^{(i)}$ satisfying $L_{\widehat{\mathbb{Z}}}^{(i)} = g_i L_{\widehat{\mathbb{Z}}}$. We denote by $\alpha_{i,\mathbb{Z}}^\perp$ the lattice $\mathrm{im}(\alpha')^\perp \cap L^{(i)}$. We have $\alpha_{i,\mathbb{Z}}^\perp \otimes \widehat{\mathbb{Z}} \cong \mathrm{im}(\alpha_i)^\perp$. Only the genus is well defined, and all objects in this section depend only on it. To L we have the associated symmetric space

$$\mathbb{D}(L) = \{\text{maximal negative definite subspaces of } L_{\mathbb{R}}\} = \mathrm{SO}(L)/K_\infty.$$

We have an embedding $\mathbb{D}(\alpha_i^\perp) \times \mathrm{SO}((\alpha_i^\perp)_{\mathbb{A}^{(\infty)}}) \hookrightarrow \mathbb{D}(L) \times \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$, given by the natural inclusion of $\mathbb{D}(\alpha_i^\perp) \hookrightarrow \mathbb{D}(L)$ and multiplication of the adelic part by g_i^{-1} from the right. We form the *special cycle*, a formal sum (with real coefficients):

$$\mathrm{Z}(L,M,\varphi;K) := \sum_i \varphi(\alpha_i)\left[\mathrm{SO}((\alpha_i^\perp)_{\mathbb{Q}})\backslash \mathbb{D}(L) \times \mathrm{SO}((\alpha_i^\perp)_{\mathbb{A}^{(\infty)}})/(K \cap \mathrm{SO}((\alpha_i^\perp)_{\mathbb{A}^{(\infty)}}))\right],$$

which we consider, by means of the embeddings above, as a formal sum of real analytic sub-orbifolds of $[\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)]$. It does not depend on the choices made above.

The *canonical* measures (6.2.3) on $\mathrm{SO}(L)$, $\mathrm{SO}(\alpha_i^\perp)$ and $\mathrm{I}(M,L)$ over any $\mathbb{Q}_\nu$ are related by an orbit equation, which we discuss in (6.4.3) — an equation of the shape:

$$\text{`volume of space'} = \sum_{\text{orbits}} \frac{\text{`volume of group'}}{\text{`volume of stabilizer'}},$$

similar to the corresponding formula for actions of finite groups on sets.

From this and (3) above

$$\frac{\mathrm{vol}(Z(L,M,\varphi;K))}{\mathrm{vol}\left(\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathbb{D}(L) \times \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K\right)} = \frac{\mathrm{vol}(K_\infty)}{\mathrm{vol}(K_\infty')} \int_{\mathrm{I}(M,L)(\mathbb{A}^{(\infty)})} \varphi(\alpha)\mu(\alpha) \tag{4}$$

follows immediately. K_∞' is any maximal compact subgroup of any of the $\mathrm{SO}(\alpha_{i,\mathbb{R}}^\perp)$. We define $\mu_\infty(L,M)$ to be the quantity $\frac{\mathrm{vol}(K_\infty)}{\mathrm{vol}(K_\infty')}$ (computed w.r.t. the canonical measures). If L is definite, it is equal to:

$$\mathrm{vol}(\mathrm{I}(M,L)(\mathbb{R})) = \prod_{k=m-n+1}^{m} 2\frac{\pi^{k/2}}{\Gamma(k/2)}.$$

Observe that

$$[\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}(\infty)})/K)] = \bigcup_j [(\mathrm{SO}(L_{\mathbb{Q}}) \cap K^{g_j})\backslash\mathbb{D}(L)]\,,$$

with respect to a set $\{g_j\}_j$ of representatives of $\mathrm{SO}(L_{\mathbb{Q}})\backslash\,\mathrm{SO}(L_{\mathbb{A}(\infty)})/K$, i.e. of the *classes of* $\mathrm{SO}(L)$ with respect to the compact open group K. (If K is the stabilizer of a lattice $L_{\widehat{\mathbb{Z}}}$, this coincides with the classical notion of classes in the genus $L_{\widehat{\mathbb{Z}}}$.) Similarly, we have

$$\mathrm{Z}(L, M, \varphi; K) = \sum_{i,k} \varphi(\alpha_{ik}) \left[(\mathrm{SO}((\alpha_i^{\perp})_{\mathbb{Q}}) \cap K^{g_{ik}})\backslash\mathbb{D}(\alpha_i^{\perp})\right], \tag{5}$$

where $\{g_{ik}\}_k$ is a set of representatives of the classes of $\mathrm{SO}((\alpha_i^{\perp})_{\mathbb{Q}})$ w.r.t. $K^{g_i} \cap \mathrm{SO}((\alpha_i^{\perp})_{\mathbb{A}(\infty)})$.

Let now K be the stabilizer of $L_{\widehat{\mathbb{Z}}}$ and φ the characteristic function. We have the following easy

(0.3) Lemma. *There is a bijection*

$$\left\{ \begin{array}{c} \textit{class } L_{\mathbb{Z}}^{(j)} \textit{ in the genus } L_{\widehat{\mathbb{Z}}}, \\ \mathrm{SO}(L_{\mathbb{Z}}^{(j)})\textit{-orbit } \mathrm{SO}(L_{\mathbb{Z}}^{(j)})\alpha \textit{ in } \mathrm{I}(M, L^{(j)})(\mathbb{Z}) \end{array} \right\}$$
$$\xrightarrow{\sim}$$
$$\left\{ \begin{array}{c} \mathrm{SO}(L_{\widehat{\mathbb{Z}}})\textit{-orbit } \mathrm{SO}(L_{\widehat{\mathbb{Z}}})\alpha \textit{ in } \mathrm{I}(M, L)(\widehat{\mathbb{Z}}) \\ \textit{class in } \mathrm{SO}(\alpha_{\mathbb{Q}}^{\perp})\backslash\,\mathrm{SO}(\alpha_{\mathbb{A}(\infty)}^{\perp})/K \cap \mathrm{SO}(\alpha_{\mathbb{A}(\infty)}^{\perp}) \end{array} \right\}.$$

We have, of course, a similar statement for any K.

We denote the cycle in this case by $\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}})$ and it is, according to the lemma and (5), equal to:

$$\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}) = \sum_j \sum_{\mathrm{SO}(L_{\mathbb{Z}}^{(j)})\alpha \subset \mathrm{I}(M, L^{(j)})(\mathbb{Z})} \left[(\mathrm{SO}(\alpha_{\mathbb{Z}}^{\perp}) \cap \mathrm{SO}(L_{\mathbb{Z}}^{(j)}))\backslash\mathbb{D}(L)\right].$$

Now, if the form Q_L is *positive definite*, the quotient of volumes above has an interpretation as a global representation number. For this observe that now just

$$\mathrm{vol}(\mathrm{SO}(L_{\mathbb{Z}})\backslash\mathbb{D}(L)) = \frac{1}{\#\,\mathrm{SO}(L_{\mathbb{Z}})}$$

and similarly

$$\mathrm{vol}((\mathrm{SO}(\alpha_{\mathbb{Z}}^{\perp}) \cap \mathrm{SO}(L_{\mathbb{Z}}^{(j)}))\backslash\mathbb{D}(\alpha^{\perp})) = \frac{1}{\#(\mathrm{SO}(\alpha_{\mathbb{Z}}^{\perp}) \cap \mathrm{SO}(L_{\mathbb{Z}}^{(j)}))}.$$

Furthermore, we have by the set theoretical orbit equation,

$$\frac{\#\,\mathrm{I}(M,L^{(j)})(\mathbb{Z})}{\#\,\mathrm{SO}(L_{\mathbb{Z}}^{(j)})}=\sum_{\mathrm{SO}(L_{\mathbb{Z}}^{(j)})\alpha\in\mathrm{I}(M,L^{(j)})(\mathbb{Z})}\frac{1}{\#(\mathrm{SO}(\alpha_{\mathbb{Z}}^{\perp})\cap\mathrm{SO}(L_{\mathbb{Z}}^{(j)}))}.$$

Hence we get

$$\frac{\mathrm{vol}(\mathrm{Z}(L_{\mathbb{Z}},M_{\mathbb{Z}}))}{\mathrm{vol}(\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L)\times\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)}=\frac{\sum_j\frac{\#\,\mathrm{I}(L_{\mathbb{Z}}^{(j)},M)(\mathbb{Z})}{\#\,\mathrm{SO}(L_{\mathbb{Z}}^{(j)})}}{\sum_j\frac{1}{\#\,\mathrm{SO}(L_{\mathbb{Z}}^{(j)})}},$$

which is precisely a weighted sum over the representation numbers. Combined with (4), we get *Siegel's formula.* Its mathematical content here is incorporated in (0.2), of course. If the quadratic form on L is indefinite, say of signature (p,q), then these representation numbers do not make sense because there are always infinitely many isometries. However, equation (4) tells us, what is the correct analogue in the indefinite case: the quotient of volumes

$$\frac{\mathrm{vol}(\mathrm{Z}(L,M,\varphi;K))}{\mathrm{vol}\left([\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L)\times(\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)]\right)}.$$

For every cohomology theory H (in a very broad sense) one might in addition consider the *classes* $[Z(L,M^Q,\varphi;K)]^H$ of these cycles and define their generating theta series, fixing $M=M_{\mathbb{Q}}$ and varying the quadratic form $Q_M\in\mathrm{Sym}^2(M^*)$:

$$\Theta_n^H(L,\varphi;\tau)=\sum_{Q\in\mathrm{Sym}^2(M^*)}[\mathrm{Z}(L,M^Q,\varphi;K)]^H\cup e_q^{n-r(Q)}\exp(2\pi iQ\cdot\tau),$$

where e_q is a certain Euler class. One is always likely to expect modularity of this function and a relation to Eisenstein series.
Kudla and Millson [59–61] have shown (generalizing work of Hirzebruch and Zagier [45]) that the generating series

$$\Theta_n^B(L,\varphi;\tau)=\sum_{Q\in\mathrm{Sym}^2(M^*)}[\mathrm{Z}(L,M^Q,\varphi;K)]^B\cup e_q^{n-r(Q)}\exp(2\pi iQ\cdot\tau),$$

with values in the Betti cohomology groups

$$H^{(p-n)q}\left([\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L)\times(\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)],\mathbb{C}\right)$$

is a modular form itself and under certain conditions on m, n and the Witt rank of L, its 'arithmetic degree' is the special value of an Eisenstein series:

$$\langle\Theta_n^B(L,\varphi;\tau),e_q^{m-n}\rangle=\mathrm{vol}_{e_q}\left([\mathrm{SO}(L_{\mathbb{Q}})\backslash\mathbb{D}(L)\times(\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)]\right)E_n(\Phi;\tau,s_0).$$

The latter equation follows essentially again from the Siegel-Weil formula (in its full

generality) or the Tamagawa number result, respectively. If $L_{\mathbb{Q}}$ is anisotropic, the locally symmetric space is compact and the pairing on the left is the degree of the product in cohomology (Poincaré duality pairing). If $L_{\mathbb{Q}}$ is isotropic, the locally symmetric space is non-compact but the expression still makes sense, because the natural forms defining e_q^{m-n} are integrable on the special cycles. For details see also [58].
The $\Theta_n^H(L, \varphi; \tau)$'s are also always expected to satisfy a product relation like:

$$\Theta_{n_1}^H(L, \varphi_1; \tau_1) \cup \Theta_{n_2}^H(L, \varphi_2; \tau_2) = \Theta_{n_1+n_2}^H(L, \varphi_1 \otimes \varphi_2; \begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}). \tag{6}$$

An important case: Special cycles on Shimura varieties

The above is particularly interesting if the signature is $(m-2, 2)$. In this case, the locally symmetric orbifold $[\mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathbb{D}(L) \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)]$ is, in fact, the *complex* analytic orbifold associated with an algebraic Deligne-Mumford stack $\mathrm{M}({}^K\mathbf{O}(L))$, a *Shimura variety of orthogonal type*. In particular, the $\mathrm{Z}(L, M, \varphi; K)$'s may be considered as *algebraic* cycles on $\mathrm{M}({}^K\mathbf{O}(L))$ and we may form

$$\Theta_n^{\mathrm{CH}}(L, \varphi; \tau) = \sum_{Q \in \mathrm{Sym}^2(M^*)} [\mathrm{Z}(L, M, \varphi; K)]^{\mathrm{CH}} \cup \mathrm{c}_1(\Xi^* \mathcal{E})^{n-r(Q)} \exp(2\pi i Q \cdot \tau),$$

with values in $\mathrm{CH}^n(\mathrm{M}({}^K_\Delta\mathbf{O}(L))_{\mathbb{C}}) \otimes \mathbb{C}$, where $\mathrm{M}({}^K_\Delta\mathbf{O}(L))$ is a toroidal compactification of $\mathrm{M}({}^K\mathbf{O}(L))$. $\Xi^*\mathcal{E}$ is a certain ample (automorphic) line bundle on $\mathrm{M}({}^K_\Delta\mathbf{O}(L))$.
It is equipped with a Hermitian metric $\Xi^* h_{\mathcal{E}}$ (singular along ∞), whose associated Chern form is (roughly) e_2 above (see 10.4.1). The series is therefore a 'lift' of Θ_n^B with respect to the cycle class map.
The only known fact, however, in the direction of modularity *in arbitrary dimensions* is the following theorem of Borcherds [5]:

(0.4) Theorem. $\Theta_1^{\mathrm{CH}}(L, \varphi; \tau)$ *is a modular form of weight* $\frac{m}{2}$.

(In low dimensional cases more is known — see the section on Kudla's program below)
The theta functions Θ_n^B, in this case, do satisfy the relation (6) [54]. An analogue of this for Θ_n^{CH} is not known in general.

Kudla's program: A 'first derivative' of Siegel-Weil

The overall aim of Kudla's program is an *arithmetic* analogue of this. The algebraic Chow group is replaced by an *Arakelov Chow group*, whose elements are classes of algebraic cycles on *integral models* of the varieties in question, complemented by analytic data, i.e. Green's functions for the 'generic fibre' of these cycles. The presence of this analytic data is, in a sense, due to the non-properness of spec($\mathbb{Z}$). Arakelov theory provides intersection products between these cycles, too, with analogous properties as in geometrical intersection theory. Many arithmetic questions translate into problems in Arakelov geometry. For example the arithmetic complexity, called *height*, of a point on a variety (the amount of information contained in its coordinates) may be expressed as an intersection product, similar to the *degree* in algebraic geometry.

Each of the Shimura varieties in question has a smooth *canonical* integral model over $\operatorname{spec}(\mathbb{Z}[1/2D])$, where D is the discriminant of the underlying quadratic lattice. There exist toroidal compactifications of them, too. These will be constructed in part I in the generality needed here (see the introduction to part I below). We may therefore define arithmetic theta functions $\Theta_n^{\widehat{\mathrm{CH}}}$. Here natural Greens functions for the cycles $\mathrm{Z}(L, M, \varphi; K)$ are provided by work of Kudla and Millson [59, 60] (cf. also section 7.6). Like all *natural* Greens functions on noncompact Shimura varieties, they have singularities along the boundary. Burgos, Kramer and Kühn, in a huge joint project [18, 19] constructed extended Arakelov Chow groups $\widehat{\mathrm{CH}}^i(\mathrm{M}({}^K_\Delta\mathbf{O}(L)))$ suitable for dealing with Greens functions with singularities of log-log-type. However, it seems that the Greens functions of Kudla and Millson in general do not have this type of singularity. We ignore this problem for the moment.

We define the arithmetic theta functions as:

$$\Theta_n^{\widehat{\mathrm{CH}}}(L, \varphi; \tau) = \sum_{Q \in \operatorname{Sym}^2(M^*)} \widehat{\mathrm{Z}}(L, M^Q, \varphi; K, y) \cdot (\widehat{\mathrm{c}}_1\, \Xi^* \overline{\mathcal{E}})^{n-r(Q)} \exp(2\pi i Q \cdot \tau).$$

Here $\widehat{\mathrm{Z}}(L, M^Q, \varphi; K, y)$ is the corresponding arithmetic cycle (its Greens function depends on y, the imaginary part of τ, as well). $\Xi^*\overline{\mathcal{E}}$ is an integral Hermitian automorphic line bundle on $\mathrm{M}({}^K_\Delta\mathbf{O}(L))$ (see 10.4.1) coming from a canonically metrized integral bundle on the compact dual. The construction of these bundles in general uses the theory of the integral standard principal bundle constructed in part I (cf. the introduction to part I below).

Kudla's first conjecture is

(0.5) Conjecture. *After possibly modifying the $\widehat{\mathrm{Z}}$'s at primes of bad reduction of them and at ∞ (modification of the Greens functions — see above), $\Theta_n^{\widehat{\mathrm{CH}}}$ is modular, and*

$$\langle \Theta_n^{\widehat{\mathrm{CH}}}(L, \varphi; \tau), (\widehat{\mathrm{c}}_1\, \Xi^*(\overline{\mathcal{E}}))^{m-1-n} \rangle = \mathcal{E}_n'(\Phi; \tau, s_0),$$

where $\mathcal{E}$ is a suitably normalized version of the Eisenstein series $E_n(\Phi; \tau, s)$ (here $'$ means derivative with respect to s.)

Observe, that we saw already that it was necessary to 'normalize' the special *value* of the Eisenstein series by the volume $\operatorname{vol}(\mathrm{M}({}^K\mathbf{O}(L)))$. In this case, we have to 'normalize' by a function, whose value at $s = s_0$ is the volume as above, but whose derivative at $s = s_0$ is the *arithmetic* volume $\widehat{\operatorname{vol}}(\mathrm{M}({}^K\mathbf{O}(L)))$. We will explain this (and its Arakelov theoretical meaning) in detail during the discussions of the results of part III.

The occurrence of a special value of the *first derivative* of the Eisenstein series at the same point is the main mystery of the whole subject and was already crucial in Gross and Zagier's work [33] on the Birch and Swinnerton-Dyer conjecture.
Kudla's second conjecture asks for the relation (6):

(0.6) Conjecture.

$$\Theta_{n_1}^{\widehat{\mathrm{CH}}}(L,\varphi_1;\tau_1)\cdot\Theta_{n_2}^{\widehat{\mathrm{CH}}}(L,\varphi_2;\tau_2)=\Theta_{n_1+n_2}^{\widehat{\mathrm{CH}}}(L,\varphi_1\otimes\varphi_2;\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}).$$

Both conjectures together imply inner product formulæ involving special derivatives of general L-series, which are vast generalizations of the formula of Gross and Zagier [33]. This uses the doubling integral of Rallis and Piatetski-Shapiro (which is a kind of Rankin-Selberg integral). For a systematic overview of this, we refer the reader to [58].

Known results in the direction of Kudla's conjectures

For lattices of small dimensions the associated Shimura varieties are of P.E.L. type and were already subject to a variety of classical work of Heegner, Hilbert, Hirzebruch, Riemann, Shimura, Siegel, Zagier and many others. A few of them are listed in the following table:

	sign.	**Witt rk.**	**classical name**
I	(0,2)	0	Heegner points
II	(1,2)	0	Shimura curves
III	(1,2)	1	Modular curve (moduli space of elliptic curves)
IV	(2,2)	0	
V	(2,2)	1	Hilbert-Blumenthal varieties
VI	(2,2)	2	product of modular curves
VII	(3,2)	1	twisted Siegel modular threefolds
VIII	(3,2)	2	Siegel modular threefold (moduli space of Abelian surfaces)

Modularity of $\Theta_r^{\widehat{\mathrm{CH}}}$ is widely unknown, especially for higher dimensional varieties with non-empty boundary, which requires the use of extended Arakelov theories like [18, 19]. Modularity was obtained so far only for the cases II and III above — for II, by work of Kudla, Rapoport and Yang [68, 69] culminating in their recent book "Modular forms and special cycles on Shimura curves" [70]. They obtained modularity of $\Theta_1^{\widehat{\mathrm{CH}}}$ and $\Theta_2^{\widehat{\mathrm{CH}}}$, as well as their connections to the corresponding special derivatives of Eisenstein series of genus 1, respectively 2 and of weight $\frac{3}{2}$. Also a formula like in (0.6) was established, yielding inner product formulæ. This completed earlier work started by Kudla in the 90's [53, 55, 56] and [66].
A non-singular, positive definite Fourier coefficient in the expression

$$\langle\widehat{\mathrm{c}}_1(\Xi^*\overline{\mathcal{E}})^{m-1-n},\Theta_n^{\widehat{\mathrm{CH}}}\rangle,$$

which should be equal to the corresponding one of the Eisenstein series, is given by the sum of the height (w.r.t. $\Xi^*\overline{\mathcal{E}}$) of the corresponding special cycle and the integral of the chosen Greens function over $\mathrm{M}({}^K_\Delta\mathbf{O}(L))_\mathbb{C}$. (If the cycle is smooth, e.g. consists itself of Shimura varieties with good reduction, its height is equal to its 'arithmetic volume', i.e. to the arithmetic degree of $\widehat{\mathrm{c}}_1(\Xi^*\overline{\mathcal{E}})^{m-1-n}$ pulled back to it.)
On the other hand, the corresponding Fourier coefficient of the Eisenstein series (cf. (2)) may be decomposed

$$\mu(L, M^Q, \varphi; y, s) = \mu'_\infty(L, M^Q, \varphi; y, s)\mu(L, M^Q, \varphi; s)$$

where μ'_∞ is — in a certain sense — the non-holomorphic part of μ_∞. If $n = 1$, it is identically 1 for $y \to \infty$, and for arbitrary n always 1 for $s = s_0$.
Assume for the moment that $\mathrm{vol}(\mathrm{Z}(L, M^Q, \varphi; K)) \neq 0$, i.e. in particular $m - 1 - n > 0$. The integral over the Greens function (depending on a parameter y as well) should be given by the 'non-holomorphic' part

$$\mathrm{vol}(\mathrm{Z}(L, M^Q, \varphi; K)) \left.\frac{\frac{\mathrm{d}}{\mathrm{d}\,s}\mu'_\infty(L, M^Q, \varphi; y, s)}{\mu'_\infty(L, M^Q, \varphi; y, s)}\right|_{s=s_0}$$

of the derivative of the normalization and the height should be given by

$$\mathrm{vol}(\mathrm{Z}(L, M^Q, \varphi; K)) \left.\left[\frac{\frac{\mathrm{d}}{\mathrm{d}\,s}\mu(L, M^Q, \varphi; s)}{\mu(L, M^Q, \varphi; s)} + \frac{\frac{\mathrm{d}}{\mathrm{d}\,s}\lambda^{-1}(L; s)}{\lambda^{-1}(L; s)}\right]\right|_{s=s_0}$$

(recall that $\mathrm{vol}(\mathrm{Z}(L, M^Q, \varphi; K))$ is the *value* of the 'normalized' Eisenstein series at s_0). If $m - 1 - n = 0$, the full derivative is just equal to

$$4(-1)^m\lambda^{-1}(L; s_0) \left.\frac{\mathrm{d}}{\mathrm{d}\,s}\mu(L, M^Q, \varphi; y, s)\right|_{s=s_0},$$

because $\mu(L, M^Q, \varphi; y, s)$ vanishes at s_0.
To obtain results (at least) about the equality of heights with the 'non-holomorphic' part of the special derivatives of Eisenstein series, there are in principle two approaches:

i. The *first* approach is by comparison of direct calculations of the finite intersection numbers of the cycles $\widehat{\mathrm{Z}}(L, M, \varphi; K, y)$ and of the special derivative of the Eisenstein series, respectively. These lines have been followed predominantly in the above mentioned work. In these cases, the equality of the 'non-holomorphic part' of the special derivative with the integral of the corresponding Kudla-Millson Greens functions has also been verified.

   Evidence in higher dimensions had been provided so far only by work of Kudla and Rapoport, [65] for Hilbert Blumenthal varieties (V), and [67] for Siegel modular varieties (VII, VIII). These approaches rely heavily on explicit use of the underlying moduli problem. In particular, the special cycles are defined algebraically via a sub-moduli problem involving additional special endomorphisms.

ii. The *second* approach, which is used in part III of this thesis (cf. the introduction to part III below) is by an inductive method, generalized from Burgos, Bruinier and Kühn [15], who investigated a special case of (V) above. However, for cycles of codimension $n > 1$, it seems to be restricted to the case of indices Q_M, where Q_M has 'good shape' at all primes p considered, at least such that $M_{\mathbb{Z}_p}^*/M_{\mathbb{Z}_p}$ is at most cyclic (i.e. essentially the codimension one case). Otherwise it seems to require at least as much knowledge about bad reduction as a direct computation of finite intersection numbers in the first approach requires.

   It has, however, the advantage of giving results in arbitrary dimension, even for non-P.E.L. type Shimura varieties, and with boundary, too — cases, which seem out of reach for the first method. It uses modular forms living on these Shimura varieties, constructed by Borcherds [4] by purely analytic means, using ideas from physics. They have a divisor consisting precisely of the codimension one cycles $\mathrm{Z}(L, < q >, \varphi; K)$ and have integral Fourier coefficients. This approach involves a computation of an integral of their norm, accomplished before by Kudla [57] and by Bruinier and Kühn [13].

In part III, using the second approach, we compute the respective heights for *all* Shimura varieties of orthogonal type and *all* cycles $\mathrm{Z}(L, M, \varphi; K)$ on them, but only up to contributions (multiples of $\log(p)$) from primes p, where the above requirement of 'good shape' is violated[1]. In case $n = 1$ (codimension 1) the heights of the special cycles can be computed *for any* $M =< q >$, $q \neq 0$ only up to contributions from bad reduction of the surrounding Shimura variety.

In the 'simplest' case, which initiated the whole program, namely the case of the modular curve, Yang [97] verified the modularity of $\Theta_1^{\widehat{\mathrm{CH}}}$ and the identity of $\langle \Theta_1^{\widehat{\mathrm{CH}}}, \widehat{\mathrm{c}}_1(\Xi^* \overline{\mathcal{E}}) \rangle$ with the special derivative of an Eisenstein series, using Chow groups of an extended Arakelov theory as in [18, 19] which, however, for the case needed here (arithmetic surfaces) had already been constructed long before by Kühn [71] and by Bost [9], independently. It should be mentioned that the equality of $\deg(\Theta_1^B)$ with the special *value* of the same Eisenstein series in this case is more difficult because the modular curve is, in a sense, an extremal case. One has to introduce also *negative, non-holomorphic* Fourier coefficients. The positive ones here are given by the class numbers of binary quadratic forms (the $\mathrm{Z}(L_{\mathbb{Z}}, < q >_{\mathbb{Z}})$ consist of special *points* in this case, corresponding to them). This special value of the Eisenstein series, which is accordingly also non-holomorphic, is Zagier's famous Eisenstein series [98] of weight $\frac{3}{2}$. The other conjectures have not been verified so far in this special case, but Bruinier and Yang succeeded in obtaining the formula of Gross and Zagier and generalizations directly, also using Borcherds products [14].

We will now describe the various parts in more detail:

[1] and up to contributions from $p = 2$, due to the still incomplete theory of good reduction of integral models of Shimura varieties of non-P.E.L. type

Part I

Even to formulate Kudla's conjectures in higher dimensions, one needs, in some sense *canonical*, integral models of Shimura varieties, which have to be compactified as well. The (finite parts of the) arithmetic special cycles $\widehat{\mathrm{Z}}$ on them are build from models of this kind themselves. Furthermore the Hermitian line bundle $\Xi^*(\mathcal{E}, h)$, involved in the definition of the arithmetic theta function and used to compute the 'arithmetic degree' of this function, has to be defined as some kind of 'canonical integral model' of an automorphic line bundle. In addition, to be able to work with Borcherds products as sections of them, one needs some kind of 'q-expansion principle' to examine integrality. The best and broadest context for all of these considerations is a fully functorial theory of canonical integral models of toroidal compactifications of mixed Shimura varieties, of the standard principal bundle on them, and of their 'compact' dual.

Consider a p-integral mixed Shimura datum $\mathbf{X}$, consisting of a group scheme $P_{\mathbf{X}}$ over $\mathbb{Z}_{(p)}$, a generalized Hermitian symmetric space $\mathbb{D}_{\mathbf{X}}$ (a principal $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-space, where $U_{\mathbf{X}}$ is a certain subgroup of the unipotent radical of $P_{\mathbf{X}}$), and an equivariant morphism $h_{\mathbf{X}} : \mathbb{D} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{X},\mathbb{C}})$, such that roughly $(P_{\mathbf{X},\mathbb{Q}}, \mathbb{D}_{\mathbf{X}}, h_{\mathbf{X}})$ satisfies Pink's axioms for a mixed Shimura datum and $P_{\mathbf{X}}$ is a group scheme of a certain type, which we call type (P).

To understand, why analytic locally symmetric varieties (or orbifolds) of the form

$$\left[P_{\mathbf{X}}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)\right]$$

should have *canonical* algebraic models defined over number fields (or even rings of integers) at all, and where this structure is supposed to come from, one should bear in mind the following philosophy (here described 'localized at p'):

If some faithful representation (closed embedding) $\rho : P_{\mathbf{X}} \to \mathrm{GL}(L_{\mathbb{Z}_{(p)}})$ is given (fixing some polarization form), compatible with some weight filtration $W_i \subset L_{\mathbb{Q}}$, there is always a finite set of tensors $v_i \in L^{\otimes}_{\mathbb{Z}_{(p)}}$ (5.1) such that the image of ρ (in the stabilizer of the weight filtration in the similitude group of the polarization form) is precisely the stabilizer of these tensors. The complex manifold $\mathbb{D}_{\mathbf{X}}$ can be seen as an *open* $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-orbit in the parameter space of (polarized) mixed Hodge structures (w.r.t. the filtration W_i) on $L_{\mathbb{C}}$, having the property that all v_i lie in $(L^{\otimes})^{(0,0)}$. Furthermore there is a category (groupoid) of families of mixed Hodge structures on arbitrary local systems over a base analytic space B. It is convenient to take local systems of $\mathbb{Q}$-vector spaces and equip the families with a K-level structure (for a compact open $K \subset P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$). This groupoid is denoted by

$$[\ B\text{-}^{K}\mathbf{X}\text{-}L\text{-}\mathbf{loc\text{-}mhs}\ ].$$

In fact, they form a category fibered in groupoids, which is an analytic Deligne-Mumford stack (orbifold) represented by the quotient

$$\left[P_{\mathbf{X}}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)\right],$$

the analytic mixed Shimura variety associated with $\mathbf{X}$.
If the group $P_{\mathbf{X}}$ and $\mathbb{D}_{\mathbf{X}}$ form a p-integral Shimura datum (2.2.2) one expects that over any base scheme S over $\mathcal{O}$ (a reflex ring of $\mathbf{X}$), there is a category (groupoid) of mixed motives

$$[\ S\text{-}\mathbf{X}\text{-}L\text{-}\mathbf{mot}\ ],$$

which should (very roughly) be seen as the category of those polarized mixed motives M of fixed weight filtration type with morphisms $v_i' : \mathbb{Z}(0) \to M^{\otimes}$, which have the property that, etale locally, there is a trivialization (respecting weight filtration and polarization) of some realization (H^{et} or H^{dR}, say) with L mapping $H(v_i')$ to v_i for every i. This will be made precise for certain Shimura data and certain associated standard representations — corresponding to 1-motives — in chapter 4. It can be made precise for all 'P.E.L. situations' (pure weight 1 and all v_i are endomorphisms) and we refer to [52] or [72] for this. For Hodge type Shimura data, also the truth of the Hodge conjecture would allow to pose a moduli problem requiring existence of certain algebraic cycles.
Furthermore, one expects (functorial) maps

$$[\ S\text{-}{}^{K}\mathbf{X}\text{-}L\text{-}\mathbf{mot}\ ] \to [\ S^{an}\text{-}{}^{K}\mathbf{X}\text{-}L\text{-}\mathbf{loc\text{-}mhs}\ ], \tag{7}$$

if S is of finite type over $\mathbb{C}$, which are *equivalences* for $S = \mathrm{spec}(\mathbb{C})$. Here, on the left hand side, we consider now motives up to $\mathbb{Z}_{(p)}$-isogeny with a $K^{(p)}$-level structure (on the etale realization in $\mathbb{A}^{(\infty,p)}$-vector spaces), for convenience, too. (Assume that K is admissible, i.e. of the form $P_{\mathbf{X}}(\mathbb{Z}_p) \times K^{(p)}$, in particular, hyperspecial.)
$[\ S\text{-}{}^{K}\mathbf{X}\text{-}L\text{-}\mathbf{mot}\ ]$ should be (represented by) an *algebraic* smooth Deligne-Mumford stack $\mathrm{M}({}^{K}\mathbf{X})$ over $\mathrm{spec}(\mathcal{O})$, which would then be a model of the analytic Shimura variety because of (7).
It is also important to look at the categories of motives, like above, equipped with a trivialization of H^{et} (with values in $\mathbb{A}^{(\infty,p)}$ vector spaces, say), H^{dR}, and, in the analytic setting, of H_B — in each case *respecting* the $P_{\mathbf{X}}$-structure (given by the tensors, polarization and weight filtration). These groupoids should be represented by

$$\mathrm{M}^p(\mathbf{X}) := \varprojlim_{K \subset P_{\mathbf{X}}(\mathbb{A}^{(\infty)}) \text{ admissible}} \mathrm{M}({}^{K}\mathbf{X}), \tag{8}$$

in the etale case,

$$\mathrm{P}({}^{K(1)}\mathbf{X}) \tag{9}$$

in the de Rham case, which is a right $P_{\mathbf{X}}$-torsor on $\mathrm{M}({}^{K(1)}\mathbf{X})$, called standard principal bundle, and

$$\mathbb{D}_{\mathbf{X}} \tag{10}$$

itself, in the Betti case, as mentioned above.
Analytic comparison isomorphisms should give embeddings $\mathbb{D}_{\mathbf{X}} \hookrightarrow (\mathrm{M}^p(\mathbf{X})_{\mathbb{C}})^{an}$ and $\mathbb{D}_{\mathbf{X}} \hookrightarrow (\mathrm{P}({}^{K(1)}\mathbf{X})_{\mathbb{C}})^{an}$. The image under ρ of an element in $P_{\mathbf{X},\mathbb{C}}$ which translates the intersection of the image of the map $\mathbb{D}_{\mathbf{X}} \hookrightarrow (\mathrm{M}^p(\mathbf{X})_{\mathbb{C}})^{an}$ with some fibre into *an integral* point of that fibre is precisely a *period matrix.* The standard principal bundle therefore is sometimes also called 'period torsor' because it encodes (or is supposed to encode)

relations between periods.
The main point, which makes it possible to approach the theory of these models without having an appropriate theory of mixed motives, is that all objects $\mathrm{M}({}^K\mathbf{X})$, $\mathrm{P}({}^K\mathbf{X})$, $\mathbb{D}_\mathbf{X}$, etc. should be *independent of the representation* ρ. Moreover, it *is* possible to characterize models intrinsically, which we call *canonical.* These should always represent the corresponding moduli problem, if an appropriate one in terms of motives can be posed. This intrinsic characterization is as follows:

i. $\mathbb{D}_\mathbf{X}$ is seen as a certain conjugacy class of morphisms in $\mathrm{Hom}(\mathbb{S}_\mathbb{C}, P_{\mathbf{X},\mathbb{C}})$ (defined over $\mathbb{R}$ modulo $U_\mathbf{X}(\mathbb{C})$, a part of the unipotent radical). If a representation ρ is chosen, composition with it yields morphisms $\mathbb{S}_\mathbb{C} \to \mathrm{GL}(L_\mathbb{C})$, which are splittings for the corresponding mixed Hodge structures. In particular, this determines already an intrinsic complex analytic structure (via Borel embedding) to the Shimura variety.

ii. The characterization of the (projective limit of the) $\mathrm{M}({}^K\mathbf{X})_E$'s (rational models) is reduced, requiring functoriality in Shimura data, to the case where $P_\mathbf{X}$ is a torus and the analytic Shimura variety is 0 dimensional, accordingly. The characterization in that case is in terms of class field theory and is motivated by the theory of complex multiplication of Abelian varieties. This marvelous idea is due to Deligne [22, 25] and was extended to the mixed case by Pink [83]. The characterization of $\mathrm{M}({}^K\mathbf{X})$ (integral model) itself is as follows. One requires the limit $\mathrm{M}^p(\mathbf{X})$ to satisfy an extension property very similar to the Neron property (in fact this *is* the Neron property for the first step in an unipotent extension). This idea is due to Milne [74, 75].

iii. The characterization of $\mathrm{P}({}^K\mathbf{X})$ can via functoriality, at least for a wide class of (mixed) Shimura data, be reduced to the case of the symplectic Shimura data, where a moduli problem in terms of 1-motives is available. It is now possible to show well-definition directly. I do not know of a better characterization which works in the integral case, too.

If a faithful representation $\rho : P_\mathbf{X} \hookrightarrow \mathrm{GL}(L_{\mathbb{Z}_{(p)}})$ is given, the objects (8-10) yield an l-adic sheaf (for every $l \neq p$), a vector bundle with connection, and a local system (in the analytic case), respectively, on the Shimura variety. Whenever it is possible to precise the moduli problem determined by this representation, these sheaves should be equal to the corresponding realizations of the universal mixed motive.
However, it should be possible to reconstruct the filtration steps of the de Rham bundle and tensor constructions of them, too. This is seen as follows: If a moduli problem exists and $\mathrm{P}({}^K\mathbf{X})$ represents motives together with a trivialization of de Rham, the filtration of de Rham yields a filtration on L_S, compatible with the $P_\mathbf{X}$-structure (determined by the tensors, polarization and weight filtration). Filtrations of this type on L_S with varying S are represented by a quasi-projective variety (projective, if $\mathbf{X}$ is pure) $\mathrm{M}^\vee(\mathbf{X})$, called the 'compact' dual. It is defined over $\mathcal{O}$ and independent of ρ, too. Hence we get an $P_\mathbf{X}$-equivariant morphism $\mathrm{P}({}^K\mathbf{X}) \to \mathrm{M}^\vee(\mathbf{X})$ — or, in more fancy terms — a morphism

of Artin stacks

$$\Xi : \mathrm{M}({}^K\mathbf{X}) \to [\mathrm{M}^\vee(\mathbf{X})/P_\mathbf{X}]. \tag{11}$$

This allows to associate with *every* $P_\mathbf{X}$-bundle $\mathcal{E}$ on $\mathrm{M}^\vee(\mathbf{X})$ a bundle $\Xi^*\mathcal{E}$ on $\mathrm{M}({}^K\mathbf{X})$ called an (integral) automorphic vector bundle. (In particular for the bundles in the universal filtration associated with ρ.) The integral structure, however, is of course not pinned down by considering $\Xi^*\mathcal{E}$ as an abstract sheaf. The analytic comparison isomorphism, however, allows to compare this map with the Borel embedding

$$\mathbb{D}_\mathbf{X} \hookrightarrow \mathrm{M}^\vee(\mathbf{X})(\mathbb{C}).$$

Therefore, if $\mathcal{E}_\mathbb{C}|_{\mathbb{D}_\mathbf{X}}$ is equipped with a $P_\mathbf{X}(\mathbb{R})U_\mathbf{X}(\mathbb{C})$-invariant Hermitian metric h, we may define $\Xi^*(\mathcal{E}, h)$ (by slight abuse of notation). It is an Hermitian arithmetic vector bundle on $\mathrm{M}({}^K\mathbf{X})$.
For many purposes, in particular our ambitions for part III, this is not sufficient because $\mathrm{M}({}^K\mathbf{X})$ is not proper. Desirable are toroidal compactifications $\mathrm{M}({}^K_\Delta\mathbf{X})$, depending on a rational polyhedral cone decomposition Δ of the conical complex $C_\mathbf{X}$ associated with $\mathbf{X}$. Furthermore, an extension $\mathrm{P}({}^K_\Delta\mathbf{X})$ of $\mathrm{P}({}^K\mathbf{X})$, or equivalently of the morphism (11), is needed to extend automorphic vector bundles. This would yield proper varieties and Hermitian automorphic vector bundles $\Xi^*(\mathcal{E}, h)$ on them. It turns out that there is only one meaningful way to extend $\mathrm{P}({}^K\mathbf{X})$, pinned down by the structure of an Abelian unipotent extension as a torus torsor. In fact, this structure trivializes the standard principal bundle along this unipotent fibre and since the compactification along the unipotent fibre is defined by a torus embedding of the corresponding torus, the trivialization defines a 'trivial' extension of the bundle. This pins down the extensions in general, if one requires functoriality with respect to boundary maps (which are, in the algebraic setting, maps between formal completions). This functoriality also yields a 'q-expansion principle' for integral automorphic forms.
The so constructed extensions of automorphic vector bundles are the same as described before by Mumford [79] (fully decomposed bundles) and Deligne (local systems).
The state of the art towards existence of these canonical models, outlined in the following table, was the existence of many partial, nevertheless very deep, results:

theory of/over	$\mathbb{C}$	**number fields**	**rings of integers**
pure SV	Baily, Borel [3]	Shimura, Deligne [20, 21]	[80], symplectic [52], P.E.L. [49], [91], general
mixed SV		[74, 75] [83]	—
toroidal comp.	Ash, Mumford, Rapoport, Tai [2]	[83]	Faltings, Chai [27], symplectic [72], P.E.L. —, general
std. princ. bundle (period torsor)		[74, 75], pure Harris [39–41], tor. comp.	—

We construct models of toroidal compactifications by a mixture of the approach of Pink (rational mixed case) and Kisin/Vasiu (integral pure uncompactified case). For the uncompactified models we use the notion of canonicity by an extension property of Milne/Moonen. First, we extend these notions and constructions to mixed Shimura varieties. We develop a theory of p-integral mixed Shimura data by involving group schemes over $\mathrm{spec}(\mathbb{Z}_{(p)})$. This uses wide parts of [37]. There however predominantly reductive group schemes are considered. We defined a notion 'type (P)' which includes the group schemes needed for mixed Shimura data. It is a generalization of 'type (R)' and 'type (RR)' considered in [37]. (Maybe, for $p \neq 2$ there would have been an easier approach because in that case there exists an exponential function for the occurring unipotent groups). We extend all operations defined in [83] to p-integral Shimura data. The main theorems about

- integral models of mixed Shimura varieties
- integral models of toroidal compactifications
- integral models of the 'compact' duals
- integral models of the standard principal bundle

are found in section (3.2). The proofs involve several steps. Roughly:
I. Construction of integral models of uncompactified Shimura varieties. First we describe the 'well-known' construction of the integral mixed Shimura varieties associated with the symplectic Shimura data (2.5) as moduli spaces of 1-motives (section 4). The section contains also a definition of the standard principal bundle in this case and some material about 1-motives, biextensions, etc. The construction for arbitrary Hodge type pure Shimura data is done in [49] or [91]. We extend it (5.2) to the mixed case. Furthermore we show an extension property for the constructed models. This was done in [76], [78] for the pure case and the extension to the mixed case is done in (3.7). This shows, in particular, that our models are uniquely determined, furthermore it yields functoriality w.r.t. morphisms of p-integral mixed Shimura data.
II. Construction of integral models of (uncompactified) standard principal bundles. This is done in (5.3.1) for the pure case and in (5.4.1) for the mixed case. The pure case follows from a theorem of [49], see (5.1.4), and algebro-geometric arguments. The mixed case is easy (linear algebra!) but involved and technical. In (5.6.1) we show that the constructions do not depend on the chosen Hodge embedding.
III. Construction of integral models of the toroidal compactifications. This is done technically as in the uncompactified case by choosing a Hodge embedding and taking the normalization of the Zariski closure in the compactified integral Shimura variety associated with a symplectic (mixed) Shimura datum. The latter is constructed in [27]. By a formal argument, we deduce the formal isomorphism at the boundary with a mixed Shimura variety of easier type by the corresponding isomorphisms in the rational cases and in the integral 'surrounding' case. Here we have to use a technical assumption, which remains unproven (3.3.2). The formal isomorphism yields, in particular, a 'q-expansion principle' and smoothness. The standard principal bundle is constructed by an easy formal argument using the existence on the rational (or complex) level. It yields

in addition a 'q-expansion principle' for integral automorphic vector bundles. This is quite explicit, giving exact information about the integrality of any special trivialization described analytically to obtain a Fourier expansion (see 10.4.10 in part III for our main application).

The method used requires to pass to sufficiently fine rational polyhedral cone decompositions, which, however, is sufficient for most applications.

IV. Extension of all maps induced by morphisms of Shimura data to the compactifications. This, again, is completely formal using the formal isomorphism with a mixed Shimura variety compactified by means of a torus embedding. It yields also canonicity of the compactifications.

Part I is restricted to the case $p \neq 2$ because the work of [49], as well as the extension property as defined in [78] work only for $p \neq 2$. The constructions involving group schemes, however, already include the case $p = 2$.

Part II

In sections (6.1-6.4) an (interpolated) non-Archimedean local orbit equation is developed. This works for arbitrary discriminants, however is not directly applicable to Arakelov geometry, unless the discriminant is square-free. It is, however, interesting in its own right and may be used for computations. Along these lines, we obtain an easy proof of a classical formula of Kitaoka on representation densities as well, which has the spirit of an orbit equation, too. Everything follows formally from a good notion of canonical measure on the isometry sets $\mathrm{I}(M, L)(\mathbb{Q}_\nu)$ and the compatibility with composition (6.2.8).

(6.5) compares to the definition of local zeta-function by Weil. It is also used in the explicit computation of the 'arithmetic volumes' of the 0-dimensional Heegner points using Kronecker's limit formula (11.2), which are well-known.

(7.1-7.5) contains a systematic description of the Weil representation, the Siegel-Weil formula, the definition and investigation of the Eisenstein series involved and related matters. These sections are expository and contain at most sketches of proofs. (7.8) contains a brief description of Borcherds lifts in the adelic language.

(7.7) investigates the non-Archimedean Whittaker integrals and relates them to the quantities μ_p (defined via 'adding hyperbolic planes') occurring in the orbit equation.

(7.6) investigates the Archimedean Whittaker integral and (7.9) relates it to an easy (ad hoc) Archimedean version of the orbit equation. This relies on Shimura's [87] work on this subject. Here a mysterious $D(M)^{\frac{1}{2}s}$ occurs, disturbing the (naive) orbit equation (cf. also 11.2.12).

(7.10) derives the global orbit-equation, whose special value and derivative are compared in part III to relations between heights, resp. arithmetic and geometric volumes of orthogonal Shimura varieties.

In (8.2), the functions μ_p and λ_p are calculated explicitly and in (8.3) the expansion at 0 of its global product is given for some lattices, yielding values determined, respectively conjectured in [13–15, 18, 19, 53–58, 65–71, 97], etc.

For example a Siegel modular threefold (associated with a lattice described in 8.3) has geometric volume
$$\zeta(-1)\zeta(-3)\prod_{p|D}(p^2-1)$$
and arithmetic volume
$$\zeta(-1)\zeta(-3)\prod_{p|D}(p^2-1)$$
$$\cdot\left(-2\frac{\zeta'(-1)}{\zeta(-1)}-2\frac{\zeta'(-3)}{\zeta(-3)}+\frac{1}{2}\sum_{p|D}\frac{p^2+1}{p^2-1}\log(p)-\frac{17}{6}+2(\gamma+\log(2\pi))\right)$$
(or without $2(\gamma+\log(2\pi))$, if a different normalization of the Hermitian metric involved in the definition is chosen).
A 10-dimensional Shimura variety associated with a unimodular lattice has geometric volume
$$\frac{1}{4}\zeta(-1)\zeta(-3)\zeta^2(-5)\zeta(-7)\zeta(-9)$$
and arithmetic volume[2]
$$\frac{1}{4}\zeta(-1)\zeta(-3)\zeta^2(-5)\zeta(-7)\zeta(-9)$$
$$\cdot\left(-2\frac{\zeta'(-1)}{\zeta(-1)}-2\frac{\zeta'(-3)}{\zeta(-3)}-3\frac{\zeta'(-5)}{\zeta(-5)}-2\frac{\zeta'(-7)}{\zeta(-7)}-2\frac{\zeta'(-9)}{\zeta(-9)}-\frac{14717}{1260}+\frac{11}{2}(\gamma+\log(2\pi))\right)$$
(or without $\frac{11}{2}(\gamma+\log(2\pi))$, as before).

Part III

This part starts with the construction of integral Hermitian automorphic vector bundles, using the theory of the integral standard principal bundle. This is predominantly contained in (9.1). A definition of arithmetic volume for arbitrary Shimura varieties follows (9.4). The arithmetic volume depends on the choice of a $P_{\mathbf{X}}$-invariant line bundle on the integral model of the compact dual together with a $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-invariant metric on the image of the Borel embedding.
The rest of the whole part focuses on the orthogonal case. Associated with a quadratic unimodular lattice $L_{\mathbb{Z}_{(p)}}$, there is a p-integral pure Shimura datum $\mathbf{O}(L)$ (10.2.1) whose underlying (reductive) group scheme is the special orthogonal group SO of $L_{\mathbb{Z}_{(p)}}$, as well as $\mathbf{S}(L)$, whose underlying group scheme is the general spin group scheme GSpin. The symmetric space $\mathbb{D}_{\mathbf{S}}$ is the set of (oriented) negative definite subspaces of $L_{\mathbb{R}}$, in a natural way a Hermitian symmetric domain, and we have a natural map $h_{\mathbf{S}}:\mathbb{D}_{\mathbf{S}}\to\operatorname{Hom}(\mathbb{S},\operatorname{GSpin}(L_{\mathbb{R}}))$. $\mathbf{S}(L)$ is of Hodge type. This allows to use the results of part I for $\mathbf{S}(L)$ as well as for $\mathbf{O}(L)=\mathbf{S}(L)/\mathbb{G}_m$.

[2] possibly up to a rational multiple of log(2)

There is a natural $P_{\mathbf{O}}$-equivariant line bundle $\mathcal{E}$ on the compact dual, the zero quadric

$$\mathrm{M}^\vee(\mathbf{O}(L)) = \{< v >\in \mathbb{P}(L_{\mathbb{Z}_{(p)}}) \mid Q_L(v) = 0\},$$

which is the restriction of the tautological bundle on $\mathbb{P}(L_{\mathbb{Z}_{(p)}})$. It carries a natural Hermitian metric h when restricted to the image of the Borel embedding (10.4.1). The geometric and arithmetic volumes, respectively, of the orthogonal Shimura varieties are understood to be with respect to the Hermitian automorphic line bundles $\Xi^*(\mathcal{E}, h)$ for the rest of the discussion.
On the associated Shimura variety $\mathrm{M}({}^K\mathbf{O})$ for a compact open subgroup K, with

$$\mathrm{M}({}^K\mathbf{O})(\mathbb{C}) = [\mathrm{SO}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{O}} \times (\mathrm{SO}(\mathbb{A}^{(\infty)})/K)],$$

we have the family of *special cycles* $\mathrm{Z}(L, M, \varphi; K)$, parametrized by a negative definite space $M_\mathbb{Q}$ and a K-invariant Schwartz function $\varphi \in S((M^* \otimes L)_{\mathbb{A}^{(\infty)}})$ as explained above. If K is admissible, i.e. of the form $K^{(p)} \times \mathrm{SO}(\mathbb{Z}_p)$, for K^p compact open in $\mathrm{SO}(L_{\mathbb{A}^{(\infty,p)}})$, and φ is K-invariant, the Z's itself (or, more precisely, their Zariski closure in the model) consist of images of canonical models of orthogonal Shimura varieties with good reduction. Furthermore, these cycles extend to the toroidal compactifications of part I, too.
In part II, we defined 2 functions, $\lambda_p(L_{\mathbb{Z}_p}; s)$ and $\mu_p(L, M, \varphi; s)$ of $s \in \mathbb{C}$, such that

$$\lambda_p(L_{\mathbb{Z}_p}; 0) = \mathrm{vol}(\mathrm{SO}'(L_{\mathbb{Z}_p})),$$
$$\mu_p(L, M, \varphi; 0) = \int_{\mathrm{I}_{\mathbb{Q}_p}(M,L)} \varphi(x^*)\mu_Q(x^*)$$

with respect to the canonical measures determined by the quadratic form (6.2.3). Here SO' stands for discriminant kernel. We defined also a similar factor for ∞. The continuation in s for μ_p is determined by its interpretation as p-Whittaker integral corresponding to the Eisenstein series associated with the Weil representation of L. For λ it is chosen in such a way that μ and λ satisfy the local orbit equation:

$$\sum_{\mathrm{SO}'(L_{\mathbb{Z}_p})-\text{ orbits in } I_{\mathbb{Z}_p}} \frac{\lambda_p(L_{\mathbb{Z}_p}; s)}{\lambda_p(\mathrm{im}(M)^\perp; s)} = \mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s)$$

and hence, in the product over all ν, a global orbit equation which holds at least for the value at $s = 0$ and, up to multiples of certain specified $\log(p)$, for the derivative at $s = 0$. Here $\mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s)$ is as before, but with φ equal to the characteristic function of a class $\kappa \in (L^*_{\mathbb{Z}_p}/L_{\mathbb{Z}_p}) \otimes M^*_{\mathbb{Z}_p}$.
We prove in this work the following (see section 10.5):

(10.5.2) Main theorem. Let $L_\mathbb{Z}$ be a lattice with quadratic form of discriminant $D \neq 0$ and signature $(m-2, 2)$. Let K be the *discriminant kernel* of $L_{\widehat{\mathbb{Z}}}$. It is an admissible compact open subgroup for all $p \nmid D$. Let Δ be a complete and smooth K-admissible rational polyhedral cone decomposition and let $\mathrm{M} = \mathrm{M}({}^K_\Delta\mathbf{O}(L))$. Let $\overline{\mathcal{E}}$ be as before.

We have

$$\begin{array}{lrcll} \text{(i)} & \mathrm{vol}_E(\mathrm{M}) & = & 4(-1)^m\lambda^{-1}(L_{\mathbb{Z}};0) & \\ \text{(ii)} & \widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}(\mathrm{M}) & \equiv & \frac{d}{ds}4(-1)^m\lambda^{-1}(L_{\mathbb{Z}};s)|_{s=0} & \text{in } \mathbb{R}_{2D} \end{array}$$

The formula in (i) is well known, and is (more or less) equivalent to the fact that the Tamagawa number of $\mathrm{SO}(L_{\mathbb{Q}})$ is 2. We give two proofs of this, one using the Tamagawa number directly, and the other using the Siegel-Weil formula and Kudla-Millson Greens functions.

It is also possible to have an equality in $\mathbb{R}_{2N}$, where N is the product of primes p, such that $p^2|D$, when the model is taken to be any model, induced locally by an embedding of $L_{\mathbb{Z}_{(p)}}$ into a unimodular lattice (10.5.9).

The actual relations with the Eisenstein series can be seen by taking the value and derivative of the orbit equation at $s=0$:

$$\mu(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa;0)\,\mathrm{vol}_E(\mathrm{M}({}^K\mathbf{O}(L))) = \sum_{\mathrm{SO}'(L_{\widehat{\mathbb{Z}}})x\subset \mathrm{I}(M,L)\cap\kappa} \mathrm{vol}_E(\mathrm{M}({}^{K_x}\mathbf{O}(x^\perp)))$$
$$(=\mathrm{vol}(\mathrm{Z}(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa)_{\mathbb{C}}))$$

is its value, i.e. $\mu(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa;0)=\deg(\mathrm{Z}(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa))$ (relative degree) and

$$\begin{aligned} &\mu'(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa;0)\,\mathrm{vol}(\mathrm{M}({}^K\mathbf{O}(L))) + \deg(\mathrm{Z}(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa))\widehat{\mathrm{vol}}(\mathrm{M}({}^K\mathbf{O}(L))) \\ &= \sum_{\mathrm{SO}'(L_{\widehat{\mathbb{Z}}})x\subset \mathrm{I}(M,L)\cap\kappa} \widehat{\mathrm{vol}}(\mathrm{M}({}^{K_x}\mathbf{O}(x^\perp))) \qquad (=\mathrm{ht}(\mathrm{Z}(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa)) \end{aligned} \tag{12}$$

is its derivative, or in other form:

$$\frac{\mathrm{d}}{\mathrm{d}\,s}\,4(-1)^m\left(\lambda^{-1}(L_{\mathbb{Z}};s)\mu(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa;s)\right)\Big|_{s=0} = \mathrm{ht}(\mathrm{Z}(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa)). \tag{13}$$

(13) is true (under slight restrictions on dimension and or Witt rank of L) in $\mathbb{R}_{2DN}$, where N is the product of primes where $M_{\mathbb{Z}}$ is not cyclic (main theorem 10.5.5). The cyclicity restriction means that we are essentially reduced to the case $\dim(M)=1$.

For $\dim(M)=1$ the equation (12) will be interpreted in Arakelov theoretical terms and proven directly. For this, the theory of Borcherds products (as in [15]) and a computation of an integral of a Borcherds forms is used, done in [13], or [57] (see 11.6).

The truth of (12) (or equivalently 13) permits to prove main theorem (10.5.2) by induction. The induction basis (lattices of dimension 3 and Witt rank 1, 11.1.1) is provided by Kühn's thesis [71]. Heegner points are treated separately (11.2) — here also general 'bad reduction' can be examined purely analytically using Kronecker's limit formula. This is well known, of course, and already present e.g. in Gross' and Zagier's work [33] and in Yang's treatment [97].

(12) can be proven only in a certain average (11.6.2), due to obstructions in constructing

Borcherds forms. We had to solve the following additional problems:

- The multiple of $\widehat{\mathrm{vol}}(\mathrm{M}({}^{K}\mathbf{O}(L)))$ occurring in the average version of (12) should not be 0. This corresponds to the task of constructing Borcherds products of non-zero weights (11.3). Furthermore, all quantities $\widehat{\mathrm{vol}}(\mathrm{M}({}^{K_x}\mathbf{O}(x^{\perp})))$ have to be known already by induction hypothesis. This is not so easy as in the geometric volume case because the method of using Borcherds products works only if the Witt rank of L is not zero (i.e. if $\mathrm{M}({}^{K}\mathbf{O}(L))$ has cusps). Therefore, we first calculate the arithmetic volume of the surrounding variety, avoiding cycles without boundary in the divisor of the constructed Borcherds product. Then we allow certain cycles without boundary (in a controlled way) and reverse the argument to calculate the arithmetic volume of those (11.3). This requires (Serre duality) arguments from [5] (11.3.7), an investigation of the Galois action on modular forms for the Weil representation, certain theorems on lacunarity of modular forms (11.5), etc.

- A q-expansion principle for orthogonal modular forms is established in (10.4) using the formal boundary isomorphism of part I. Its applicability to Borcherds form is shown in (10.4.12) by an adelization of its product formula. Here the main additional difficulty is not to establish integrality (this is literally seen from Borcherds' product formula), but the fact that they are defined over $\mathbb{Q}$. It is not convenient to establish this in the classical context because it may require to take Borcherds lifts for different classes in the genus of $L_{\mathbb{Z}}$ into account.

- Certain boundary terms in the integrals over star products of the occurring Greens functions (log of the Hermitian norm of sections) have to be shown to vanish (11.6.3). This is especially hard for small dimensions. It would probably not be needed by all means for these extremal cases, but allows to use (11.6.2) carelessly as long as $\dim(L) \geq 5$ (any Witt rank) or $\dim(L) = 4$ and the Witt rank is 1.

Notation

	Technical notation, S a base scheme	
$\mathbb{W}(E)$	'additive group'-scheme for a coherent sheaf E on S, representing $\mathbb{W}(E)(S') = (E \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})(S')$	(1.1.1)
$\mathbb{G}_a = \mathbb{W}(\mathcal{O}_S)$	the additive group scheme over S	
$D(L)$	$= \underline{\mathrm{Hom}}(X, \mathbb{G}_m)$	(1.2.1)
$\mathbb{G}_m = D(\mathbb{Z}_S)$	multiplicative group scheme over S $\mathbb{G}_m(S') = (\mathcal{O}_{S'})(S')^*$	
$\mathrm{Lie}(G)$	Lie algebra functor for a S-group scheme G	
$\mathbb{S}$	Deligne torus	(2.1.3)
H_0		(2.1.3)
w	natural (weight) morphism $\mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}$	
$\mathbb{A}$	adeles of $\mathbb{Q} = \prod'_{\nu} \mathbb{Q}_\nu$, restricted product over all places of $\mathbb{Q}$	
$\mathbb{A}^{(S)}$	$= \prod'_{\nu \notin S} \mathbb{Q}_\nu$, e.g. $\mathbb{A}^{(\infty)}$ or $\mathbb{A}^{(\infty,p)}$, where p is a prime For group schemes G over $\mathbb{Q}$, and $S \subset S'$, we have natural embeddings $G(\mathbb{A}^{(S')}) \hookrightarrow G(\mathbb{A}^{(S)})$. We have also ring homomorphisms $\mathbb{Q} \hookrightarrow \mathbb{A}^{(S)}$ for all S, and correspondingly $G(\mathbb{Q}) \hookrightarrow G(\mathbb{A}^{(S)})$ but incompatible with the embeddings above	
$[X/G]$	algebraic or analytic quotient stack	
(P)	a certain property of group schemes	(1.8.1)
$\mathcal{PAR}$	scheme of parabolics of a reductive (or type (P)) group scheme	(1.9.4)
$\mathcal{QPAR}$	scheme of quasi-parabolics of a group scheme of type (P)	(1.9.9)
$\mathcal{TYPE}$	classifying scheme of types of quasi-parabolics	(1.9.9)
$\mathcal{FTYPE}$	classifying scheme of types of cocharacters	(1.9.9)
$\mathrm{par}, \mathrm{qpar}$	(quasi-)parabolic associated with a cocharacter	(1.9.9)
$\mathrm{type}, \mathrm{ftype}$	type of a quasi-parabolic, resp. cocharacter	(1.9.9)
$C_I(A)$	completion of a ring A with respect to the I-adic topology	
$C_Y(X)$	completion of a scheme X along a closed subscheme Y	
$\widehat{X}$	dito	
$N(X), N(A), N(\mathcal{X})$	normalization of a scheme, a ring, a formal scheme, etc.	(5.8)
$\mathbb{R}'$	$\mathbb{R}$ modulo rational multiples of $\log(N)$, $N \in \mathbb{N}$	(9.3.2)
$\mathbb{R}_N$	$\mathbb{R}$ modulo rational multiples of $\log(p)$, $p \mid N$	(9.3.2)
$\mathbb{R}^{(p)}$	$\mathbb{R}$ modulo rational multiples of $\log(q)$, $q \neq p$	(9.3.2)

	Tensor algebra, L a coherent sheaf or module	
$\mathrm{T}(L)$	tensor algebra	
$\mathrm{Sym}^n(L)$	symmetric power	
$(L \otimes \cdots \otimes L)^s$	symmetric elements of $\mathrm{T}^n(L)$	
$\Lambda^n(L)$	exterior power = alternating elements of $\mathrm{T}^n(L)$	
$\mathrm{C}(L)$	Clifford algebra of L	(10.1.1)
$\mathrm{C}^+(L), \mathrm{C}^-(L)$	even and odd part of the Clifford algebra	(10.1.1)
$L^{\otimes}$	$= \bigoplus_{i,j} L^{\otimes i} \otimes (L^*)^{\otimes j}$	

	Shimura data	
p	some prime, mostly fixed, main theorems of part I are proven only for $p \neq 2$	
$\mathbf{X}, \mathbf{Y}, \mathbf{B}$	p-integral mixed Shimura data consisting of (say) $P_{\mathbf{X}}$, $\mathbb{D}_{\mathbf{X}}$, $h_{\mathbf{X}}$	(2.2.2)
$P_{\mathbf{X}}$	group scheme over $\mathbb{Z}_{(p)}$ underlying $\mathbf{X}$	(2.2.2)
$\mathbb{D}_{\mathbf{X}}$	generalized symmetric space underlying $\mathbf{X}$	(2.2.2)
$h_{\mathbf{X}}$	$P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-equivariant morphism $\mathbb{D}_{\mathbf{X}} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{X},\mathbb{C}})$ underlying $\mathbf{X}$	(2.2.2)

$W_{\mathbf{X}}$	unipotent radical of $P_{\mathbf{X}}$	(2.2.9)
$U_{\mathbf{X}}$	certain central subgroup scheme of $U_{\mathbf{X}}$	(2.2.9)
$V_{\mathbf{X}}$	$W_{\mathbf{X}}/U_{\mathbf{X}}$	(2.2.9)
$G_{\mathbf{X}}$	$P_{\mathbf{X}}/W_{\mathbf{X}}$, the reductive part of $P_{\mathbf{X}}$	(2.2.9)
$C_{\mathbf{X}}$	the conical complex of $\mathbf{X}$	(2.4.9)
$\alpha : \mathbf{X} \to \mathbf{Y}$	morphism of p-integral Shimura data	(2.2.2)
$\iota : \mathbf{B} \Longrightarrow \mathbf{X}$	boundary morphism —	(2.2.2)
	leads to a partial order between boundary components	
	The group $P_{\mathbf{X}}$ acts by conjugation, preserving order	
$\mathrm{GSp}(L)$	group scheme of symplectic similitudes	(2.5)
$\mathrm{Sp}(L)$	symplectic group scheme	(2.5)
$\mathrm{PSp}(L_0, I)$	extension of $\mathrm{GSp}(L_0)$ by $\mathbb{W}(I \otimes L_0)$, occurs as	(2.5)
	group scheme in the unipotent extension $\mathbf{H}_g[0, L_0 \otimes I]$.	
$\mathrm{USp}(L_0, I)$	extension of $\mathrm{GSp}(L_0)$ by $\mathrm{WSp}(L_0, I)$, occurs as	(2.5)
	group scheme in the unipotent extension $\mathbf{H}_g[(I \otimes I)^s, L_0 \otimes I]$.	
$\mathrm{WSp}(L_0, I)$	a central extension of $\mathbb{W}(I \otimes L_0)$ by $\mathbb{W}((I \otimes I)^s)$	(2.5)
$\mathrm{GSpin}(L)$	general spin group scheme	(10.1.4)
$\mathrm{Spin}(L)$	spin group scheme	(10.1.4)
$\mathrm{SO}(L)$	(special) orthogonal group scheme	
$\mathbf{H}_g = \mathbf{H}_g(L_{\mathbb{Z}_{(p)}})$	Shimura datum of symplectic type	(2.5)
	$P_{\mathbf{H}_g} = \mathrm{GSp}(L_{\mathbb{Z}_{(p)}})$, the symplectic similitude group scheme of	
	a lattice $L_{\mathbb{Z}_{(p)}}$ of rank $2g$ with non-degenerate symplectic form,	
	$\mathbb{D}_{\mathbf{H}_g}$ = conjugacy class of morphisms $\mathbb{S} \to \mathrm{GSp}(L_{\mathbb{R}})$, yielding	
	polarized Hodge structures of type $(-1, 0), (0, -1)$ on $L_{\mathbb{R}}$	
L	space, mostly over $\mathbb{Z}_{(p)}$, with non-deg. symplectic form (Part I)	
	or non-degenerate quadratic form Q_L (Part II/III)	
	— more generally — any representation of $P_{\mathbf{X}}$	
I	isotropic subspace of L	
L_0	subspace of L, inducing a splitting	
	$L = I^* \oplus L \oplus I$, with natural symplectic resp. orthogonal form	
$\langle v, w \rangle$	denotes the symplectic form on L	
	— or, in the orthogonal case — $Q(v + w) - Q(v) - Q(w)$	
Q	denotes the quadratic form on L	(6.1.1)
γ_L, γ_Q	denotes the associated symmetric morphism $L \to L^*$	(6.1.1)
$d(L_{\mathbb{Z}}), d(L_{\mathbb{Z}_p}), d(L_{\mathbb{Q}})$	the discriminant	(6.1.3)
$< a_1, \ldots, a_n >$	the space $\mathbb{Z}^n$, $\mathbb{Z}_p^n$ or $\mathbb{Z}_{(p)}^n$ (according to the context) with	
	quadratic form $x \mapsto \sum_i a_i x_i^2$. It has discriminant $\prod_i 2a_i$	
$M \perp L$	orthogonal direct sum of two quadratic spaces or lattices	
$\mathbf{X}[U, V]$	unipotent extension, where U, V are representations of $P_{\mathbf{X}}$	(2.2.10)
	with symplectic form $V \times V \to U$	
	(implicit in the notation), satisfying certain axioms,	
	we always have an iso $\mathbf{X} \cong \mathbf{X}/W_{\mathbf{X}}[U_{\mathbf{X}}(\mathbb{Z}_{(p)}), V_{\mathbf{X}}(\mathbb{Z}_{(p)})]$	
$\mathbf{H}_{g_0}[(I \otimes I)^s, I \otimes L_0]$	Shimura datum of general symplectic type	(2.5)
	I is any space over $\mathbb{Z}_{(p)}$, L_0 is the symplectic space	
	associated with $\mathbf{H}_{g_0}$.	
	P is equipped with a standard rep'n on $L := I \oplus L_0 \oplus I^*$	
$\mathbf{S}(L_{\mathbb{Z}_{(p)}})$	Shimura datum of spin type	(10.2.1)
	$P_{\mathbf{S}} = \mathrm{GSpin}(L_{\mathbb{Z}_{(p)}})$, the general spin group scheme of a	
	lattice $L_{\mathbb{Z}_{(p)}}$ of rank m with non-degenerate quadratic form of	
	signature $(m - 2^+, 2^-)$	
	$\mathbb{D}_{\mathbf{S}} = \mathrm{Grass}^{\pm}(L_{\mathbb{R}})$, set of oriented pos. def. subspaces of $L_{\mathbb{R}}$	
$\mathbf{O}(L_{\mathbb{Z}_{(p)}})$	$= \mathbf{S}(L_{\mathbb{Z}_{(p)}})/\mathbb{G}_m$ Shimura datum of orthogonal type,	(10.2.1)
	$P_{\mathbf{O}} = \mathrm{SO}(L_{\mathbb{Z}_{(p)}})$, the special orthogonal group scheme	
	$\mathbb{D}_{\mathbf{O}} = \mathrm{Grass}^{\pm}(L_{\mathbb{R}})$, as above	
Δ	a (partial) rational polyhedral cone decomposition of $C_{\mathbf{X}}$	(2.4.10)
${}^K_\Delta\mathbf{X}$	extended compactified p-int. mixed Shimura datum (p-ECMSD),	(2.4.11)
	consisting of a p-integral mixed Shimura datum $\mathbf{X}$,	

	a p-admissible compact open subgroup $K \subset P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$, and a (partial) rational cone decomposition of $C_{\mathbf{X}}$	
$(\alpha,\rho) : {}^{K'}_{\Delta'}\mathbf{Y} \to {}^{K}_{\Delta}\mathbf{X}$	a morphism of p-ECMSD, consisting of a morphism α of p-integral mixed Shimura data and $\rho \in P_{\mathbf{X}}(\mathbb{A}^{(\infty,p)})$, satisfying some compatibility conditions	(2.4.11)
$(\iota,\rho) : {}^{K'}_{\Delta'}\mathbf{B} \Longrightarrow {}^{K}_{\Delta}\mathbf{X}$	boundary morphism	(2.4.13)
$\mu_{I,I'}$	certain cocharacter of an orthogonal group associated with dual isotropic lines I, I'	(10.2.13)

	mixed Hodge Structures, B an analytic base	
[B-**X**-L-**mhs**]	category of B-families of mixed Hodge structures on a fixed vector space compatible with $(P_{\mathbf{X}}, L)$-structure	(2.3.4)
[B-**X**-L-**mhs'**]	category of B-families of mixed Hodge structures on a fixed vector space compatible with $(P_{\mathbf{X}}, L)$-structure	(2.3.1)
[B-**X**-L-**filt**]	category of certain filtrations on a fixed vector space compatible with $(P_{\mathbf{X}}, L)$-structure	(2.3.2)
[B-**X**-L-**loc-mhs**]	category of mixed Hodge structures on local systems on B, locally isomorphic to one in [B-**X**-L-**mhs'**]	(2.3.5)
[B-^{K}**X**-L-**loc-mhs**]	dito, with K-level structure — represented by an analytic mixed Shimura variety	(2.3.6)
[B-^{K}**X**-L-**triv-mhs**]	dito, with K-level structure and *analytic* trivialization of the local system — represented by the analytic standard principal bundle	(4.5.2)

	1-motives, S a base scheme over $\mathbb{Z}_{(p)}$, $\mathbf{X} \in \{\mathbf{H}_{g_0}, \mathbf{H}_{g_0}[0, L_0 \otimes I], \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]\}$ K a p-admissible compact open subgroup of $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$	
[S-**1mot**]	category of 1-motives over S	(4.1.1)
[S-^{K}**X**-L-**mot**]	category of 1-motives with extra structure, yields a moduli problem defining the canonical model $\mathrm{M}({}^{K}\mathbf{X})$. For $S = \mathrm{spec}(\mathbb{C})$ it is equivalent to [S^{an}-^{K}**X**-L-**loc-mhs**]	(4.1.11)
[S-^{K}**X**-L-**triv-mot**]	algebraic version of [B-^{K}**X**-L-**triv-mhs**], defining a canonical model of the standard principal bundle $\mathrm{P}({}^{K}\mathbf{X})$	(4.5.1)

	Shimura varieties and (Hermitian) automorphic vector bundles	
$E_{\mathbf{X}}$	the reflex field $\subset \mathbb{C}$ of $\mathbf{X}$	(3.1.1)
$\mathcal{O}$	a fixed reflex ring of $\mathbf{X}$ (d.v.r. of $E_{\mathbf{X}}$ ass. with some prime above p)	(3.1.1)
$\mathrm{M}({}^{K}_{\Delta}\mathbf{X})$	canonical model of the toroidal compactification of the associated Shimura variety, a Deligne-Mumford stack (and a scheme, if K is neat) over $\mathcal{O}_{\mathbf{X}}$	(3.3.5)
D	boundary divisor on $\mathrm{M}({}^{K}_{\Delta}\mathbf{X})$	
$\mathrm{M}^{\vee}(\mathbf{X})$	associated 'compact' dual, a right $P_{\mathbf{X}}$-scheme over $\mathcal{O}_{\mathbf{X}}$	(3.4.1)
$\mathrm{P}({}^{K}_{\Delta}\mathbf{X})$	standard principal bundle, a right $P_{\mathbf{X}}$-torsor over $\mathrm{M}({}^{K}_{\Delta}\mathbf{X})$ equipped with a morphism $\Pi : \mathrm{P}({}^{K}_{\Delta}\mathbf{X}) \to \mathrm{M}^{\vee}(\mathbf{X})$	(3.5.2)
Ξ	morphism of (Artin) stacks $\mathrm{M}({}^{K}_{\Delta}\mathbf{X}) \to [\mathrm{M}^{\vee}(\mathbf{X})/P_{\mathbf{X}}]$ encoding $\mathrm{P}({}^{K}_{\Delta}\mathbf{X})$ and Π	(3.5.3)
$\mathcal{E}$	a $P_{\mathbf{X}}$-equivariant bundle on $\mathrm{M}^{\vee}(\mathbf{X})$; in part III, this is the canonical line bundle on $\mathrm{M}^{\vee}(\mathbf{O}_n) \subset \mathbb{P}^1(L_{\mathbb{Z}_{(p)}})$	(9.1) (10.4.1)
$\Xi^*\mathcal{E}$	associated automorphic vector bundle on $\mathrm{M}({}^{K}_{\Delta}\mathbf{X})$	(9.1)
$h_{\mathcal{E}}$	a $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-invariant Hermitian metric $h_{\mathcal{E}}$ defined on the image of the Borel embedding in part III, this is the metric $v, w \mapsto -\frac{1}{2}e^{-C}\langle v, \overline{w}\rangle$	(9.1) (10.4.1)
$\overline{\mathcal{E}}$	the pair $(\mathcal{E}, h_{\mathcal{E}})$	
$\Xi^*\overline{\mathcal{E}}$	associated Hermitian automorphic vector bundle on $\mathrm{M}({}^{K}_{\Delta}\mathbf{X})$ with log-singular metric along D	(9.1)
$\mathrm{Z}(L, M, \varphi; K)$	special cycle on an orthogonal Shimura variety, M a quadratic	(11.2)

	space over $\mathbb{Q}$, $\varphi \in S(M^*_{\mathbb{A}^{(\infty)}} \otimes L_{\mathbb{A}^{(\infty)}})$ and K a compact open subgroup of $\mathrm{SO}(\mathbb{A}^{(\infty)})$ fixing φ	
$\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa)$	same as before, with φ equal to the characteristic function of $\kappa \in (L^*_{\mathbb{Z}}/L_{\mathbb{Z}}) \otimes M^*_{\mathbb{Z}}$ and $K = \mathrm{SO}'(L_{\widehat{\mathbb{Z}}})$	(11.2)

	Representation densities, Weil representation Eisenstein series	
$\lambda(L_{\mathbb{Z}}; s)$	naturally defined complex analytic function encoding geometric and arithmetic volume of an orthogonal Shimura variety as special value and derivative at $s = 0$ (up to a factor 4)	(6.4.10)
$\mu(L, M^Q_{\mathbb{Z}}, \varphi; s)$	strongly related to the Fourier coefficient of the Eisenstein series associated with the Weil representation, for L; here $M_{\mathbb{Z}}$ is a lattice with quadratic form $Q \in \mathrm{Sym}^2(M^*_{\mathbb{Q}})$ and $\varphi \in S(\mathbb{A}^{(\infty)})$ is any Schwartz-Bruhat function.	(6.4.10)
$\mu(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s)$	The same as before for φ equal to the characteristic function of $\kappa \in (L^*_{\mathbb{Z}}/L_{\mathbb{Z}}) \otimes M_{\mathbb{Z}}$	
$\mathrm{I}_R(s)$	(normalized) parabolically induced representation $\mathrm{ind}_{P(R)}^{\mathrm{Sp}(\mathfrak{M}_R)}(\lvert \det \rvert^s)$	(7.1)
$E(\Psi; s)$	Eisenstein series	(7.4)
$E_{M^{*\prime}}$	(degenerate) part of the Eisenstein series associated with a sublattice $M^{*\prime} \subset M^*$	(7.4)
E_γ	for $\gamma \in (M^* \otimes M^*)^s$, Fourier coefficient of the Eisenstein series	(7.4)
$\Theta(\varphi; g)$	theta series	(7.5.1)
$W_{\nu,Q,M^{*\prime}}(\Phi_\nu; g_\nu)$	general Whittaker integral at ν where $\Phi = \Phi(s) \in \mathrm{ind}_P^{\mathrm{Sp}'(M,R)}(\lvert \det \rvert^s \xi)$	(7.4.3)
$W_{Q,M^{*\prime}}(\Phi; g)$	$\prod_\nu (W_{\nu,Q,M^{*\prime}}(\Phi_\nu, g_\nu))$	
$\zeta_p(L_{\mathbb{Z}_p}; s)$	normalized local zeta function associated with a lattice $L_{\mathbb{Z}_p}$	(6.5)
$\widetilde{\mu}_\psi$	canonical volume form w.r.t. a bilinear form ψ	(6.2.3)
μ_ψ	canonical measure w.r.t. a bilinear form ψ	(6.2.3)
$\mathrm{I}(M, L)$	variety of isometries from M to L	(6.2.4)
$\mathrm{I}^1(M, L)$	variety of injective isometries from M to L	(6.2.4)
$\beta_p(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \varphi; s)$	representation densities	(6.3)
$\mathrm{SO}'(L)$	discriminant kernel	
$S(L_{\mathbb{A}}), S(L_{\mathbb{R}}),$ $S(L_{\mathbb{Q}_p})$, ...	space of Schwartz-Bruhat functions For $\mathbb{R}$ these are rapidly decaying and smooth For $\mathbb{Q}_p$ these are locally constant with compact support	(6.2.2)
φ_ν, φ	Schwartz-Bruhat functions	(6.2)
φ_∞	predominantly the Gaussian	(7.6.5)
φ^0_∞	the Gaussian of the 'positivization' with respect to a maximal negative definite subspace N of an indefinite form	(7.6.7)
φ_{KM}	the Kudla-Millson form $= \nabla \varphi^0_\infty$	(7.6.7)
φ^+_∞	defined (for signature $(m-2, 2)$) by $\varphi_{KM}\, \mathrm{c}_1(\Xi^* \overline{\mathcal{E}})^{m-2-n} = \varphi^+_\infty\, \mathrm{c}_1(\Xi^* \overline{\mathcal{E}})^{m-2}$	(7.6.7)
Υ_R	certain character of the Witt group of $R = \mathbb{Q}_\nu, \mathbb{A}^{(\infty)}, \mathbb{A}$ denoted γ in [93]	(7.1)
$\widetilde{\Upsilon}_R$	'correction factor' in the formulæ of the Weil representation	(7.2.1)
$\mathrm{Mp}(\mathfrak{M}_R)$	metaplectic groups	(7.1.8)
$\mathrm{Sp}'(\mathfrak{M}_R)$	denotes either $\mathrm{Sp}(\mathfrak{M}_R)$ or $\mathrm{Mp}(\mathfrak{M}_R)$ according to the parity of the dimension of L	(7.2.1)
$\mathrm{Weil}(L^*_{\mathbb{Z}}/L_{\mathbb{Z}})$	denotes the Weil representation restricted to $\mathbb{C}[L^*_{\mathbb{Z}}/L_{\mathbb{Z}}] \subset S(L_{\mathbb{A}^{(\infty)}})$ equipped with natural $\mathbb{Q}$ and $\mathbb{Z}$-structures	(7.7.4)
$\mathcal{WEIL}(L^*_{\mathbb{Z}}/L_{\mathbb{Z}})$	denotes the bundle of modular forms associated with the Weil representation $\mathrm{Weil}(L^*_{\mathbb{Z}}/L_{\mathbb{Z}})$, defined over $\mathbb{Q}$	(11.3.4)
$\mathfrak{M}, \mathfrak{I}$, etc.	$= M^* \oplus M$, say, with natural symplectic form	

$P = U \cdot G_l$	certain standard maximal parabolic subgroup of $\mathrm{Sp}(\mathfrak{M})$	(7.1)
$g_l(\alpha), u(B), d(\beta)$	certain elements of G_l, U, and $\mathrm{Sp}(\mathfrak{M})$, respectively	(7.1)
$\Gamma_n(s), \Gamma_{n,m}(s)$	'higher dimensional' gamma functions	(7.6.10)
$\zeta(Z, a, b)$	confluent hypergeometric function	(7.6.10)
$FC(f, \alpha, I, I', \rho, k, \sigma)$	Fourier coefficient of a (meromorphic) orthogonal modular form, w.r.t. some tube domain realization of $\mathbb{D}_{\mathbf{O}}$, valid in the cone σ (*local notation*).	(10.4.7)
$\mathrm{ModForm}(\Gamma, V, k)$	classical weakly holomorphic modular forms for Γ, of weight k and representation V	(11.3)
$\mathrm{HolModForm}(\Gamma, V, k)$	classical holomorphic modular forms for Γ of weight k and representation V	(11.3)
$\mathrm{PowSer}(\Gamma)$, $\mathrm{Laur}(\Gamma)$, $\mathrm{Sing}(\Gamma)$	notation from Borcherds [5]	(11.3.7)

Part I.

Toroidal compactifications of mixed Shimura varieties

1. Preliminaries on group schemes

In this chapter, we cite a number of statements on group schemes from [37] and extend some results to certain non-reductive groups that we call of type (P) (1.8.1). They are of a slightly more general nature than the groups of type (R) and (RR) considered in [loc. cit.]. They will occur as group schemes underlying a p-integral mixed Shimura datum. Furthermore we describe a couple of results, clarifying the relations between (quasi-)parabolics, filtrations, P-structures, etc.

1.1. Group schemes of additive type

(1.1.1) Let S be a base scheme. For each coherent sheaf E on S, there is a group functor $\mathbb{W}(E)$:

$$\mathbb{W}(E)(S') = (E \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})(S'),$$

where the right hand side is equipped with its additive group structure. It is representable. $\mathbb{W}$ is additive, $\mathbb{W}(\mathcal{O}_S) = \mathbb{G}_a$, and there is a natural isomorphism $\mathrm{Lie} \circ \mathbb{W} \cong \mathrm{id}$. If E is locally free, then $\mathbb{W}(E)$ is a smooth group scheme over S.
Forms G of $\mathbb{G}_a^n$ over S (i.e. for which there is a etale covering $\{S_i \to S\}$ such that $G_{S_i} \cong \mathbb{G}_a^n$ are classified by the etale cohomology group

$$H^1_{et}(S, \mathrm{GL}(n))$$

because the automorphism functor of $\mathbb{G}_a^n$ is represented by $\mathrm{GL}(n)$.

The following is well known:

(1.1.2) Theorem ('Hilbert 90'). *Let $S = \mathrm{spec}(\mathcal{O})$, where $\mathcal{O}$ is a discrete valuation ring or a field. Then*

$$H^1_{et}(S, \mathrm{GL}(n)) = 1.$$

1.2. Group schemes of multiplicative type, Tori

(1.2.1) Let S be a base scheme. Let X be a group scheme over S. Consider the functor

$$D(X) = \underline{\mathrm{Hom}}(X, \mathbb{G}_m).$$

If M is an ordinary Abelian group, denote by M_S the **constant group scheme** associated with M over S. $D(M_S)$ is representable. Any group scheme isomorphic to some $D(M_S)$ is called a **diagonalizable** group. If G is of finite type over S, then it is equivalent to be diagonalizable locally in the etale or fpqc topology. Groups with this property are called **of multiplicative type**. They are called **tori** if they are locally isomorphic to $D(M_S)$, where M is free. A sheaf for the etale topology, which is locally isomorphic to a constant sheaf of ..., we call an **etale sheaf** of .... For all etale sheaves of finitely generated Abelian groups, D is fully faithful, exact and reflexive. Hence we have an anti-equivalence:

$$\{ \text{ etale sheaves of fin. gen. (free) Abelian groups} \} \leftrightarrow \{ \text{ groups of mult. type (tori) } \}.$$

(1.2.2) Remark ([37, X, 4.5; 5.16]). If S is *normal*, a torus, or equivalently an etale sheaf of finitely generated free Abelian groups, is locally trivial for the etfg topology (hence for the etf topology), and trivial if it is trivial over a dense open set.

(1.2.3) Theorem ([37, I, 4.7.3]). *Let $G = D(M_S)$ be a diagonalizable group scheme over S. There is an equivalence*

$$\left\{ \begin{array}{c} \textit{quasi-coherent sheaves on } S \\ \textit{with linear } G\textit{-action} \end{array} \right\} \leftrightarrow \{ \textit{ quasi-coherent } M\textit{-graduated sheaves } \}.$$

(1.2.4) A quasi-coherent sheaf E is M**-graduated**, if there is a decomposition

$$E = \bigoplus_{m \in M} E^m.$$

If E corresponds to a G-module as above, then G acts on E^m via m considered as Element in $\mathrm{Hom}(G, \mathbb{G}_m)$.

(1.2.5) A linear G-action on E can be seen either as an usual action of G on $\mathbb{W}(E)$ compatible with the action of the canonical ring scheme over S or - if G is affine - as a morphism

$$E \to E \otimes_{\mathcal{O}_S} A(G),$$

where $A(G)$ is the bigebra-sheaf corresponding to G, satisfying certain axioms expressing action and linearity. For details, see [37, I, 4].

(1.2.6) Theorem ([37, XI, 4.2]). *Let G be some smooth affine group scheme over S and H a group scheme of multiplicative type over S. Then $\underline{\mathrm{Hom}}(H, G)$ is representable and smooth over S. The morphism*

$$G \times_S \underline{\mathrm{Hom}}(H, G) \to \underline{\mathrm{Hom}}(H, G) \times_S \underline{\mathrm{Hom}}(H, G),$$

expressing the action by conjugation, is smooth.

1.3. Semi-Abelian schemes

(1.3.1) Definition ([27, I, Def. 2.3]). *Let S be a scheme. A* ***semi-Abelian scheme*** *over S is a smooth commutative group scheme $G \to S$ with (geometrically) connected fibers, such that each fibre G_s is an extension of an Abelian variety A_s by a torus*

$$0 \longrightarrow T_s \longrightarrow G_s \longrightarrow A_s \longrightarrow 0.$$

(1.3.2) Theorem ([27, I, 2.10, 2.11]). *The function* $\mathrm{rk}(T_s)$ *is upper semicontinuous on S. It is locally constant, iff G is globally an extension*

$$0 \longrightarrow T \longrightarrow G \longrightarrow A \longrightarrow 0$$

with T a torus, and A an Abelian variety.

There is a unique etale constructible sheaf $\underline{X}(G)$, such that for each geometric point s

$$\underline{X}(G) \cong X^*(T_s)$$

(+ a condition for behaviour under specializations)
$G \mapsto \underline{X}(G)$ commutes with base change.
Assume that G is globally an extension. Then $\underline{X}(G)$ is the etale sheaf of lattices $\underline{\mathrm{Hom}}(T, \mathbb{G}_m)$.

(1.3.3) Theorem. *In this case, any extension as above can be given by a homomorphism*

$$\underline{X} \to A^\vee,$$

where $\underline{X}$ is an etale sheaf with fibers isomorphic to $\mathbb{Z}^r$.

Proof. $A^\vee$ is isomorphic (as an etale sheaf) to $\underline{\mathrm{Ext}}(A, \mathbb{G}_m)$ and we have a map

$$\underline{X} \to \underline{\mathrm{Ext}}(A, \mathbb{G}_m)$$

given locally by associating $x \in \underline{X}(U)$ the pushout G_x:

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, "x"'] & G \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{G}_m \arrow[r] & G_x \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Conversely consider a etale covering $\{U_i\}$, such that $\underline{X}(U_i)$ is trivial. Take a basis e_{ij} of $\underline{X}(U_{ij})$ and construct the corresponding extensions

$$0 \longrightarrow T_{U_i} \cong \mathbb{G}_m^n \longrightarrow G_i = G_{i,1} \times_A \cdots \times_A G_{i,n} \longrightarrow A_{U_i} \longrightarrow 0,$$

then glue. □

One has $\mathrm{Hom}(T, G) = \mathrm{Hom}_S(\underline{X}, \underline{Y})$. for a torus T with character sheaf $\underline{Y}$.

1.4. Maximal tori

(1.4.1) Definition ([37, XII, 1.3]). *Let G be a group scheme over S, T a sub-group scheme. T is called a **maximal torus** of G if T is a torus and for each $s \in S$, $T_{\overline{s}}$ is a maximal torus of $G_{\overline{s}}$.*

(1.4.2) Definition ([37, XII, 1]). *Let G be a smooth affine group scheme over S. For each $s \in S$, we define the **reductive rank** $\rho_r(s)$ as the dimension of a maximal torus of $G_{\overline{s}}$, the **nilpotent rank** $\rho_r(s)$ as the dimension of the centralizer of a maximal torus of $G_{\overline{s}}$ and the **unipotent rank** $\rho_u(s) = \rho_n(s) - \rho_r(s)$ as the dimension of the unipotent radical of the centralizer of a maximal torus of $G_{\overline{s}}$.*

(1.4.3) Theorem ([37, XII, 1.7]). *Let G be a smooth affine group scheme over S. The function $\rho_r(s)$ is lower semicontinuous, $\rho_u(s)$ and $\rho_s(s)$ are upper semicontinuous. They are locally constant, if and only if G possesses a maximal tori locally in the etale topology. In that case $\rho_u(s)$ and hence $\rho_s(s)$ are also locally constant.*
If there are two maximal tori T_1 and T_2 of G, they are conjugated locally in the etale topology.

1.5. Root systems

Let G be a smooth group scheme over S with connected fibers and T a maximal torus (1.4.1).
T acts via conjugation on G and the induced action on $\mathrm{Lie}(G)$ is linear in the sense of (1.2.3). Hence the Lie algebra of each geometric fibre

$$\mathrm{Lie}(G_{\overline{s}}) = \mathrm{Lie}(G) \otimes_{\mathcal{O}_S} \mathcal{O}_{\overline{s}}$$

decomposes as

$$\mathrm{Lie}(G_{\overline{s}}) = \mathrm{Lie}(T_{\overline{s}}) \oplus \bigoplus_{r \in M_{\overline{s}}} \mathrm{Lie}(G_{\overline{s}})^r,$$

where $M = D(T_{\overline{s}})$.

(1.5.1) Definition ([37, XIX, 3.2, 3.6]). *We call $r \in D(T)(S)$ a root of G with respect to T, if $r_{\overline{s}}$ occurs for each s as a* nontrivial *root in the decomposition as above, i.e. $r_{\overline{s}} \neq 1$ and* $\mathrm{Lie}(G_{\overline{s}})^{r_{\overline{s}}} \neq 0$.
A subset $R \subset M(S)$ is called a ***system of roots*** *for G with respect to T if it consists of roots as above and the following equivalent conditions are satisfied:*

i. *For each s, $r \mapsto r_{\overline{s}}$ is a bijection of R with the set of roots of $G_{\overline{s}}$.*

ii. *For each s, the elements of $\{r_{\overline{s}}\}_{r\in R}$ are distinct and* $\dim(G_{\overline{s}}) - \dim(T_{\overline{s}}) = \#R$.

iii. $\mathrm{Lie}(G) = \mathrm{Lie}(T) \oplus \bigoplus_{r\in R} \mathrm{Lie}(G)^r$.

A system of roots exists, for example, if G is reductive (1.6.5) and T is diagonalizable (1.2.3). Then the $\mathrm{Lie}(G)^r$ are locally free of rang 1, and $-R = R$.

(1.5.2) Definition ([37, XXI, 1]). *A* ***reduced root system*** *is a quadruple*

$$(M, M^*, R, R^*),$$

where M, M^* are lattices in duality, $R \subset M$ and $R^* \subset M^*$ are finite subsets with a bijection $r \mapsto r^*$ subject to the following conditions

i. $\langle r, r^*\rangle = 2 \ \forall r \in R$,

ii. $s_r(R) = R$, $s_{r^*}(R^*) = R^*$,

iii. *if $r, \alpha r \in R$, then $\alpha = \pm 1$,*

where $s_r(x) = x - \langle x, r\rangle x$ for $x \in M^*, r \in R$. A p-morphism (where p is a prime) of reduced root systems

$$(M, M^*, R, R^*) \to (M', (M')^*, R', (R')^*)$$

is a group homomorphism $\alpha : M \to M'$, such that there exists a function $q : R \to \{p^n \mid n \in \mathbb{Z}_{\geq 0}\}$ and a bijection $u : R \to R'$, with the properties:

i. $\alpha(r) = q(r)u(r)$,

ii. $\alpha^*(u(r)^*) = q(r)r^*$.

A ***pinned root system*** *is a quintuple (M, M^*, R, R^*, R_0), where (M, M^*, R, R^*) is a root system, and $R_0 \subset R$ is a system of simple roots (i.e. every other root is expressible as a unique integral linear combination of roots from R_0 with only positive or negative coefficients). A p-morphism of pinned root systems has to respect the sets R_0.*

1.6. Reductive group schemes

(1.6.1) Definition ([37, XIX, 2.7]). *A group scheme G over S is called* ***reductive (resp. semi-simple, resp. unipotent)****, if it is affine and smooth over S, with connected and reductive (resp. semi-simple, resp. unipotent) geometric fibers.*

(1.6.2) Theorem ([37, XIX, 2.5]). *Let G be a reductive group scheme over S. The function $\rho_r(s)$ (1.4.2) is continuous and hence maximal tori (1.4.1) exist.*
Furthermore for each torus T' in G, there exists locally in the etale topology a maximal torus T containing T'.

(1.6.3) Definition. *Let G be a reductive group scheme over S. We say (G, T, M, R) is* ***split****, if T is a maximal torus of G, with isomorphism $T \cong D(M_S)$, M being a lattice and $R \subset M$ is a system of roots (1.5.1) with respect to T of G.*

(1.6.4) Definition ([37, XXIII, 1.1]). *A sextuple*

$$(G, T, M, R, R_0, \{X_r\}_{r \in R_0})$$

is called an ***pinned reductive group****, where (G, T, M, R) is a split reductive group scheme over S, $R_0 \subset R$ a system of simple roots and for each $r \in R_0$, $X_r \in \mathrm{Lie}(G)^r$ a nontrivial section.*
A isogeny of pinned reductive groups

$$(G, T, M, R, R_0, \{X_r\}_{r \in R_0}) \to (G', T', M', R', R'_0, \{X'_r\}_{r \in R'_0})$$

is a morphism of group schemes $\alpha : G \to G'$, which induces an isogeny *$T \to T'$ and which respects R_0 and the X_r.*

(1.6.5) Theorem ([37, XXIII]). *If G is a reductive group scheme over S, then locally in the etale topology there exist $T, M, R, R_0, \{X_r\}_{r \in R_0}$, such that*

$$(G, T, M, R, R_0, \{X_r\}_{r \in R_0})$$

is a pinned reductive group.

(1.6.6) Theorem ([37, XXII, XXIII]). *There is an equivalence of categories (compatible with base change and direct products)*

$$\left\{ \begin{array}{c} \text{pinned reductive groups over } S \\ \text{with isogenies} \end{array} \right\} \to \left\{ \begin{array}{c} \text{pinned root systems} \\ \text{with (p-)morphisms} \end{array} \right\}$$
$$(G, T, M, R, R_0, \{X_r\}_{r \in R_0}) \mapsto (M, M^*, R, R^*, R_0).$$

Here p is the prime such that $x \mapsto x^p$ is an endomorphism of $\mathbb{G}_{a,S}$ (if there is any). Furthermore there exists a certain open dense subscheme ('grosse cellule') Ω of G. The functor and Ω are characterized by:

i. *For each $r \in R$ there exists a closed embedding*

$$\exp_r : \mathbb{W}(\mathrm{Lie}(G)^r) \to G$$

inducing via Lie *the inclusion.*

ii. *For any ordering of R^+ (system of positive roots determined by R_0) the morphism induced by the* $\exp_r$*:*

$$\prod_{r \in R^-} \mathbb{W}(\mathrm{Lie}(G)^r) \times_S T \times_S \prod_{r \in R^+} \mathbb{W}(\mathrm{Lie}(G)^r) \to G$$

is an open immersion onto Ω.

iii. *For each $r \in R^+$ there is a unique duality $\langle \cdot, \cdot \rangle$*

$$\mathrm{Lie}(G)^r \otimes \mathrm{Lie}(G)^{-r} \to \mathcal{O}_S$$

and a $r^* \in M^* = \mathrm{Hom}(\mathbb{G}_m, T)$ such that for each $S' \to S$ and $X \in \mathrm{Lie}(G_{S'})^r(S')$, $Y \in \mathrm{Lie}(G_{S'})^{-r}(S')$:

$$\exp_r(X) \exp_{-r}(Y) \in \Omega(S') \Leftrightarrow 1 + \langle X, Y \rangle \in \mathcal{O}_{S'}^*$$

and in this case

$$\exp_r(X) \exp_{-r}(Y) = \exp_{-r}\left(\frac{Y}{1 + \langle X, Y \rangle}\right) r_*(1 + \langle X, Y \rangle) \exp_r\left(\frac{X}{1 + \langle X, Y \rangle}\right).$$

iv. *Explicitly for each $r \in R^+$ and for each $a \in \mathcal{O}_S$ and $X \in \mathrm{Lie}(G)^r$ there is a unique $X^{-1} \in \mathrm{Lie}(G)^{-r}$ such that $\langle X, X^{-1} \rangle = 1$ Then:*

$$r^*(a) = \exp_{-r}((a^{-1} - 1)X^{-1}1) \exp_r(x) \exp_{-r}((a-1)X^{-1}) \exp_r(-a^{-1}x).$$

Thus $R^* \subset M^*$ and the map $R \to R^*$ are defined.

v. $r \circ r^* = 2$, $(-r)^* = -(r^*)$.

(1.6.7) Theorem ([6, 14.10 (1)]). *Let G be a semi-simple algebraic group over $S = \mathrm{spec}(\overline{k})$, $\overline{k}$ algebraically closed. Let H be a integral normal subgroup of G. Let $H' = \mathrm{Cent}(H, G)^0$. Then*

- *H and H' are semi-simple, integral and normal,*
- *$G = H \cdot H'$ and $H \cap H'$ is contained in the finite group $C(G)$, i.e. G is isogenous to $H \times H'$.*

(1.6.8) Theorem ([37, XXII, 6.2.4]). *Let G be a reductive group scheme over S. There is a semi-simple subgroup scheme G_{der} (the derived group) of G and a central torus C (the radical) such that*

$$G_{der} \times C \to G$$

is an isogeny.

(1.6.9) Theorem. *Let S be normal integral and Q the generic point of S. Let G be a reductive group scheme over S. If $H_Q \subset S_Q$ is a geometrically integral normal subgroup of G_Q then its closure H is reductive.*

Proof. Let $\overline{s}$ be any geometric point and $\overline{Q}$ a geometric point lying over Q. There is an etale neighborhood $S' \to S$ of $\overline{s}$, such that G can be pinned: $(G, T, M, R, R_0, \{X_r\}_{r \in R_0})$. Then $H_{Q'}$ can be pinned over Q' as well. For, let $\widetilde{T}_1$ be maximal torus of $H_{Q'}$ and T_1 a maximal torus of $G_{Q'}$ containing $\widetilde{T}_1$. T_1 is conjugated over $\overline{Q}$ to T. Hence, since $H_{Q'}$ is normal, $T_{Q'} \cap H_{Q'}$ is also a maximal torus of $H_{Q'}$ (being the conjugate of $\widetilde{T}_1$). Now $T_{Q'} \cap H_{Q'}$, being a subtorus of $T_{Q'}$ is defined and split over S' as well. The previous theorems imply that there is an isogeny

$$H_{\overline{Q}} \times H'_{\overline{Q}} \to G_{\overline{Q}}$$

and $H'_{\overline{Q}}$ is reductive and normal as well, hence can be pinned with respect to $T_{\overline{Q}}$.
The isogeny hence comes from some morphism of pinned root systems (1.6.6). This in turn implies that there are semi-simple group schemes $\widetilde{H}$, H' *defined over* S' and an isogeny

$$\widetilde{H} \times H' \to G_{S'},$$

inducing the previous over Q'.
An isogeny is closed (it is faithfully flat and finite [SGA III, XXII, 4.2.10]), hence it induces a closed embedding of $\widetilde{H}$ and hence $\widetilde{H}$ is isomorphic to H, hence in particular $H_{\overline{s}}$ is connected and reductive. □

1.7. P-structures

Let P be a group scheme over S. Let P act linearly on a locally free sheaf of $\mathcal{O}_S$-modules L (1.2.3).

(1.7.1) Definition. *Let E be a locally free sheaf of $\mathcal{O}_S$-modules. A (P, L)**-structure** on E is a section of the etale quotient sheaf*

$$\alpha \in H^0(\underline{\mathrm{Iso}}(E, L)/P, S).$$

The pre-image under the projection $\underline{\mathrm{Iso}}(E, L) \to \underline{\mathrm{Iso}}(E, L)/P$ is a right P-torsor $\mathrm{tor}(\alpha)$, *which we call the **associated right P-torsor**.*

(1.7.2) Remark. If E is a locally free sheaf over S' for some scheme $S' \to S$, by abuse of notation, we call a $(P \times_S S', L \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})$-structure on E simply a (P, L)-structure.

In the analytic context, we have the following variants of a P-structure:

(1.7.3) Definition. *Let $R \subset \mathbb{Q}$ be a ring $S = \mathrm{spec}(R)$, P a group scheme over S. and L_R a vector space, with a linear action of P. Let B be an analytic manifold (or an analytic Deligne-Mumford stack[1]).*
Let E be a local system of R-modules on B. A $(P(R), L_R)$-structure α on E is a section of the quotient sheaf

$$\alpha \in H^0(\underline{\mathrm{Iso}}(E, L_R)/P(R)_B, B).$$

The preimage of α in $\underline{\mathrm{Iso}}(E, L_R)$ is a local system locally isomorphic to the constant sheaf $P(R)_B$. We call it also the associated right $P(R)$-torsor $\mathrm{tor}(\alpha)$.
Let E be a holomorphic vector bundle on B. A P-structure α on E is a section of the quotient sheaf

$$\alpha \in H^0(\underline{\mathrm{Iso}}(E, L_{\mathcal{O}_B})/P_{\mathcal{O}_B}, B).$$

Here $P_{\mathcal{O}_B}$ is the sheaf of groups on B of analytic maps to $P(\mathbb{C})$. We have also an analytic associated right $P_{\mathcal{O}_B}$-torsor $\mathrm{tor}(\alpha)$.

Clearly, a $(P(R), L_R)$-structure α on a local system E induces a (P, L_R)-structure $\alpha \otimes_R \mathcal{O}_B$ on $E \otimes_R \mathcal{O}_B$.

1.8. Group schemes of type (P)

(1.8.1) Definition. *Let P be a group scheme of finite type over S. We call P of type (P), if the following conditions are satisfied:*
There exists a closed unipotent normal subgroup scheme W (called unipotent radical).
For each point there is an etale neighborhood $S' \to S$ and

i. a closed reductive subgroup scheme G of $P_{S'}$ such that $P_{S'}$ is isomorphic to the semi-direct product of $W_{S'}$ with $G_{S'}$,

   (Then P is smooth and affine over S and there exist maximal split tori locally in the etale topology (1.4.3)),

ii. a split maximal torus $T \cong D(M_{S'})$ of $G_{S'}$, where M is some lattice,

[1] Also called an analytic orbifold.

iii. a system of roots $R = R_G \dot{\cup} R_W \subset M$, *where:*

$$\mathrm{Lie}(G_{S'}) = \mathrm{Lie}(T, S') \oplus \bigoplus_{r \in R_G} \mathrm{Lie}(G, S')^r$$
$$\mathrm{Lie}(W_{S'}) = \bigoplus_{r \in R_W} \mathrm{Lie}(W_{S'})^r,$$

iv. closed embeddings

$$\exp_r : \mathbb{W}(\mathrm{Lie}(W_{S'})^r) \to W$$

for all $r \in R_W$, *inducing via* Lie *the inclusion and satisfying for all* $S'' \to S'$, $t \in T(S'')$, $X \in \mathrm{Lie}(G_{S''})^r$*:*

$$\mathrm{int}(t) \exp_r(X) = \exp_r(r(t)X).$$

with the property that any two different $r_1 \in R$, $r_2 \in R_W$ *are linearly independent.*

This property is stable under base change.

(1.8.2) Remark. Notice that the $\mathrm{Lie}(G_{S'})^r$ are allowed to be higher dimensional, the notion 'type (P)' is therefore slightly more general than the notion 'type (RR)' considered in [37, XXII, 5.1]. The later cannot be used in our context because it is not stable under restriction to certain normal subgroups.

(1.8.3) Lemma. *If* P *is of type* (P), *then* R *can be decomposed:* $R = R^+ \dot{\cup} - R^+$ *such that* $R_W \subset R^+$.

(1.8.4) Lemma. *Let* k *be an algebraically closed field, and* $T = D(M)$ *a torus acting on* $\mathbb{G}_a$. *There exists an element* $r \in M$, *such that the action is given by*

$$(\alpha, X) \mapsto r(\alpha)X.$$

Proof. Each $\alpha \in T(k)$ has to act by an additive polynomial. An additive polynomial, which induces an isomorphism is multiplication by a scalar. □

(1.8.5) Lemma. *Let* k *be an algebraically closed field. Let* E *be a finitely generated* k*-vector space, and a* k*-torus* $T = D(M)$ *operating by a non-zero* $r \in M$ *on* $\mathbb{W}(E)$. *If* $\alpha : \mathbb{G}_a \to \mathbb{W}(E)$ *is a closed embedding which is* T*-stable, then it comes from a non-zero element* $X \in E$.

Proof. Since α is a closed embedding and T-stable, there is an induced action on $\mathbb{G}_a$, which can only be the action via r (1.8.4). Take a basis of E and consider the resulting projections of $\alpha, \alpha_i : \mathbb{G}_a \to \mathbb{G}_a$. The α_i are given by polynomials, which are additive, hence of the form $c_iY + \sum_j c_{ij}Y^{p^j}$ if $p = \mathrm{char}(k) > 0$ or c_iY if $\mathrm{char}(k) = 0$. This is compatible with the action of T only if all c_{ij} are 0. Hence the embedding comes from $X = (c_1, \dots, c_n)$, which cannot be 0. □

(1.8.6) Theorem. *The restriction*

$$\prod_{r \in R_W} \mathbb{W}(\mathrm{Lie}(W_{S'})^r) \to W$$

is an isomorphism of schemes for any ordering.

Proof. Compare [37, XXII, 4.1]. Since both objects are flat and of finite type it suffices to check this over each geometric fibre [36, I, 5.7].

This is stated in [89, 8.2.2] under additional assumptions. But we can argue analogously in the more general case. We proceed by induction on $\#R_W, \dim W$. If there is only one root, we have a closed embedding

$$u_r : \mathbb{W}(\mathrm{Lie}(W_{S'})^r) \to W,$$

which on the level of Lie algebras is an isomorphism. Since both sides are connected and reduced, it is an isomorphism. There is a T-invariant subgroup N isomorphic to $\mathbb{G}_a$ in the center [89, 6.3.4]. $\mathrm{Lie}(N)$ is therefore contained in an $\mathrm{Lie}(W)^r$. Consider the group $W' = Z_W(T_r)^0$, where $T_r = \ker(r)^0$. $\mathrm{Lie}(W') = \mathrm{Lie}(W)^r$ because the r's are linearly independent. By the previous $\mathbb{W}(\mathrm{Lie}(W)^r) \cong W'$. Since $N \subset Z_W(T_r))^0$ (T has to act via r on N by 1.8.4), by lemma (1.8.5) N occurs as a linear subspace of $\mathbb{W}(\mathrm{Lie}(W)^r)$. Therefore there is a closed embedding $\mathbb{W}(\mathrm{Lie}(W)^r / \mathrm{Lie}(N)) \to W/N$. Therefore the assumptions are true (this is verified as in the proof of [89, 8.2.2]) for the group $W'' = W/N$ and $\mathrm{Lie}(W)^r$ replaced by $\mathrm{Lie}(W'')^r = \mathrm{Lie}(W)^r / \mathrm{Lie}(N)$ (if not 0). By induction the statement is true for W'' and hence for W by the conclusion in the proof of [89, 8.2.2]. □

Over fields of characteristic 0, the situation is much more easy:

(1.8.7) Theorem. *Let k be a field of characteristic 0 and G an unipotent k-group. There exists an isomorphism of varieties*

$$\exp : \mathbb{W}(\mathrm{Lie}(G)) \to G,$$

with the property that the restriction to $\mathbb{W}$ of any commutative sub Lie algebra of $\mathrm{Lie}(G)$ *is a closed embedding of algebraic groups and* Lie *of it is the inclusion.*

(1.8.8) Lemma. *Let W be a unipotent group scheme over S, acted on by a split torus $T = D(M_S)$, satisfying property (iv) of the definition of type (P) (1.8.1). Let $X_i \in \mathrm{Lie}(W)^{r_i}, i = 1, 2$ be given. The morphism*

$$\begin{aligned} \mathbb{G}_a \times \mathbb{G}_a &\to W \\ u, v &\mapsto \exp_{r_1}(uX_1)\exp_{r_2}(vX_2) \end{aligned}$$

factors through the closed subscheme

$$\prod_{r = ir_1 + jr_2, i,j \geq 0} \mathbb{W}(\mathrm{Lie}(W)^r)$$

(which is independent of the ordering of the roots by (1.8.6)).

Proof. It suffices to show this locally on S so we may assume S affine. The morphism is then given by a polynomial

$$\sum X_{ij} u^i v^j$$

with $X_{ij} \in \mathrm{Lie}(W)(S)$. Letting the torus act by conjugation, we get

$$\mathrm{int}(t)X_{ij} = r_1(t)^i r_2(t)^j X_{ij},$$

i.e. $X_{ij} \in \mathrm{Lie}(W)^{ir_1 + jr_2}$. □

(1.8.9) Theorem. *Let S be a* reduced *scheme. Let W be a unipotent group scheme over S acted on by a split torus $T = D(M_S)$, satisfying property (iv) of the definition of type (P) (1.8.1). If H is a smooth subgroup scheme of W, stable under T, then it is closed and uniquely determined by* $\mathrm{Lie}(H)$*.*

Proof. We will show that, over every geometric point $\overline{s}$ of S, $H_{\overline{s}}$ is directly spanned by

$$\exp_r\left(\mathbb{W}(\mathrm{Lie}(W_{\overline{s}})^r \cap \mathrm{Lie}(H_{\overline{s}}))\right).$$

This shows that H is uniquely determined by $\mathrm{Lie}(H)$ (because it is smooth over S). It is closed because the restriction of the isomorphism

$$\prod_{r \in R_W} \mathbb{W}(\mathrm{Lie}(W_{S'})^r) \to W$$

to the product of the $\mathbb{W}(\mathrm{Lie}(W)^r \cap \mathrm{Lie}(H))$ is a closed embedding. Its image in W has to be equal to H because both are smooth over S. Note that $\mathrm{Lie}(W)^r \cap \mathrm{Lie}(H) \subseteq \mathrm{Lie}(W)^r$ is saturated because H is smooth.

For the required result over the geometric point $\overline{s}$, adapt the proof of [6, Prop. 14.4]. □

(1.8.10) Theorem. *If P is a group scheme of type (P) over an affine scheme S, then*

$$H^1_{et}(S, W) = 1.$$

Proof. Analogous to [37, XXVI, 2.2]. □

(1.8.11) Theorem. *If P is a group scheme of type (P) over an affine scheme S then there is a closed subgroup scheme G of P, such that $P = G \rtimes W$. Any two such are conjugated by an element in $W(S)$. For any maximal torus T of P, there is a unique such G containing T such that every closed reductive subgroup scheme of P, containing T is contained in G.*

Proof. Is deduced like in [37, XXVI, §1]. □

1.9. Filtrations and parabolic groups

Let S be a base scheme.

(1.9.1) Definition. *Let L be a locally free sheaf on S. Consider an (increasing or decreasing) filtration $F^\bullet$ of L. It is called* ***saturated****, if $\mathrm{gr}^F(L)$ is again locally free. If 2 filtrations $F^\bullet$ and $G^\bullet$ are given, we call them* ***bisaturated*** *if the bigraded $\mathrm{gr}^F \mathrm{gr}^G(L) = \mathrm{gr}^G \mathrm{gr}^F(L)$ is locally free.*
A morphism $\gamma : L \to M$ of sheaves is a morphism of filtered sheaves, if $\gamma(F^i(L)) \subseteq F^i(M)$. It is called ***strict*** *if the diagrams*

$$\begin{tikzcd} F^i(L) \arrow[r, hook] \arrow[d, "\gamma"] & F^{i+1}(L) \arrow[d, "\gamma"] \\ F^i(M) \arrow[r, hook] & F^{i+1}(M) \end{tikzcd}$$

are Cartesian.

We have the following obvious

(1.9.2) Lemma. *Let L be a locally free sheaf S, and a filtration $F^\bullet$ of L be given. The following are equivalent conditions:*

i. $F^\bullet$ is a saturated filtration.

ii. There is a splitting for $F^\bullet$:

$$L = \bigoplus_i L^i \qquad F^j = \bigoplus_{i \geq j} L^i.$$

iii. There exists a linear action of $\mathbb{G}_m$ on L inducing a splitting.

Let $G^\bullet$ be another filtration on L. The following are equivalent conditions:

i. $F^\bullet$, $G^\bullet$ form a pair of bisaturated filtrations.

ii. There is a simultaneous splitting for $F^\bullet$ and $G^\bullet$:

$$L = \bigoplus_i L^{i,i'} \qquad F^j = \bigoplus_{i \geq j, i'} L^{i,i'} \qquad G^{j'} = \bigoplus_{i, i' \geq j'} L^{i,i'}.$$

iii. There exists a linear action of $\mathbb{G}_m \times \mathbb{G}_m$ on L inducing a simultaneous splitting.

(1.9.3) Definition ([37, XV, 6.1]). *Let P be a group scheme of finite type over S. A **parabolic** subgroup scheme of P is a smooth subgroup scheme Q of G, such that for each $s \in S$, $Q_{\overline{s}}$ is a parabolic subgroup of $G_{\overline{s}}$, (i.e. such that $G_{\overline{s}}/Q_{\overline{s}}$ is proper).*

(1.9.4) Theorem ([37, XXII, 5.8.3-5.8.5]). *Let G be a reductive group scheme over S.*

i. The functor $S' \mapsto \{$parabolic subgroups of $G_{S'}\}$ is representable by a smooth projective S-scheme $\mathcal{PAR}$.

ii. There is an etale sheaf $\mathcal{TYPE}$ of finite sets over S and a morphism

$$\mathrm{type} : \mathcal{PAR} \to \mathcal{TYPE},$$

with the property Q and Q' are locally conjugated in the etale topology, if and only if $\mathrm{type}(Q) = \mathrm{type}(Q')$.

iii. If (G, T, M, R) is split

$$\mathcal{TYPE} \cong \left\{ \begin{array}{c} W(R)\text{-conjugacy classes of closed subsets of } R \\ \text{containing a set of positive roots} \end{array} \right\}_S$$
$$\cong \{ \text{ subsets of a set of simple roots } R_0 \}_S$$

iv. If P is any subgroup scheme, smooth and of finite type, the following conditions are equivalent:

a) For each $s \in S$, $P_{\overline{s}}$ is a parabolic subgroup of $G_{\overline{s}}$, i.e. $G_{\overline{s}}/P_{\overline{s}}$ is proper.

b) The quotient sheaf G/P is representable by a smooth projective S-scheme.

Under these conditions $P = \underline{\mathrm{Norm}}_G(P)$ *(it is in particular closed) and the morphism* $G/P \to \mathcal{PAR}$*,* $g \mapsto \mathrm{int}(g)P$ *is an open and closed immersion onto a connected component of* $\mathcal{PAR}$*. It is hence the fibre above* $\mathrm{type}(P)$*.*

v. There is a surjective morphism

$$\mathrm{par} : \underline{\mathrm{Hom}}(\mathbb{G}_m, G) \to \mathcal{PAR},$$

characterized by the properties:

a) Let $\alpha : \mathbb{G}_{m,S'} \to G_{S'}$ *be some morphism. For each etale* $S'' \to S'$*, where there is a splitting* $(G_{S''}, T, M, R)$ *such that* $\alpha_{S''} : \mathbb{G}_{m,S''} \to T \subset G_{S''}$*, we have*

$$\mathrm{type}(\mathrm{par})(\alpha)) = W(R)\{r \in R \mid r \circ \alpha \geq 0\}.$$

b) For each $\alpha : \mathbb{G}_{m,S'} \to G_{S'}$*,* α *factors through* $\mathrm{par}(\alpha)$*.*

Proof. The assertions are stated in [loc. cit.], however some only for Borel subgroups. See also [37, XXVI] for the case of parabolics. □

(1.9.5) Remark. Over a geometric point, we have

$$\mathrm{par}(\alpha)(\overline{s}) = \{x \in G(\overline{s}) \mid \lim_{a \to 0} \alpha(a)x\alpha(a)^{-1} \text{ exists}\}.$$

For the precise meaning of this compare [89, p.148].

(1.9.6) Theorem. *Let* P *be a group scheme of type* (P) *over* S*. Let* $G = P/W$*. The parabolic subgroup schemes of* P *are in 1:1 correspondence with the parabolic subgroup schemes of* G*. The functor of parabolic subgroups is representable by a smooth projective scheme over* S*. The parabolic subgroups are of type* (P)*.*

Proof. Every geometric fibre $P'_{\overline{s}}$ of a parabolic subgroup scheme P' of P has to contain $W_{\overline{s}}$ so P' contains W and P'/W has to be a parabolic subgroup of G. In turn if P' is a parabolic subgroup of G, $W \rtimes P'$ is a parabolic subgroup of P. The second statement follows therefore from (1.9.4).
For the third statement assume first $W = 1$. Then a parabolic subgroup Q is of type (R) [37, XXII, 5.2.1], which means that it is smooth of finite type and for each $s \in S$, $P_{\overline{s}}$ contains a Cartan subgroup of $G_{\overline{s}}$.
In this case, there is a closed[2] subset $R' \subset R$, such that

$$\mathrm{Lie}(Q) = \mathrm{Lie}(T) \oplus \bigoplus_{r \in R'} \mathrm{Lie}(G)^r.$$

[2] i.e. $r, s \in R', r + s \in R \Rightarrow r + s \in R'$

In fact, it suffices to check this over a geometric point where it follows from [89, 8.4.3. (iv)]. This means by definition that Q is of type (RC) [37, XXII, 5.11.1] Now the assertions follow from [XXII, 5.11.3, 5.11.4]SGAIII. Here of course all $\mathrm{Lie}(G)^r$ are 1-dimensional.

In the general case Q is given by the product $W \rtimes Q'$. Let W' is the unipotent radical of Q'. The multiplication

$$W' \times W \to Q$$

is a closed embedding (because we had a semi-direct product). It hence induces a closed smooth subscheme, which serves as unipotent radical of Q. (It is a subgroup scheme because W' normalizes W). The other statements are satisfied because they are satisfied for Q'. □

(1.9.7) Theorem. *Let S be a* reduced *scheme, and let (P, T, M, R) be a split group of type (P) over S. If $R' \subset R$ is a closed subset, then there exists a unique smooth subgroup scheme $P' \subset P$, containing T, such that*

$$\mathrm{Lie}(P') = \mathrm{Lie}(T) \oplus \bigoplus_{r \in R'} \mathrm{Lie}(P)^r$$

P' is of type (P) and closed in P. In particular, if P is reductive, then each parabolic is of this form.

Proof. $R' \cap R_G$ is also closed in R_G and hence there exists a unique subgroup scheme P'_G of G with $\mathrm{Lie}(P'_G) = \mathrm{Lie}(T) \oplus \bigoplus_{r \in R'} \mathrm{Lie}(G)^r$ [37, XXII, 5.3.5, 5.4.7]. It is closed in G [37, XXII, 5.11.4].

Consider the product

$$\prod_{r \in R' \cap R_W} \mathbb{W}(\mathrm{Lie}(W)^r).$$

It is closed embedded into G. Over each geometric point of S it is a subgroup scheme, so it is a closed subgroup scheme P'_W because S is reduced. Furthermore it is the unique T-stable smooth subgroup scheme of W with Lie algebra $\bigoplus_{r \in R' \cap R_W} \mathrm{Lie}(W)^r$.

Over each geometric point of S, $P' = P'_W \cdot P'_G$ is a subgroup scheme because R' is closed. Because S is reduced this implies that this is true over S. The product is a closed embedding because P is a semi-direct product of W and G. It is of type (P), again because of [37, XXII, 5.11.3, 5.11.4] and the reasoning in the proof of the last theorem. □

(1.9.8) Definition. *Let P be of type (P). A closed smooth subgroup scheme Q of P is called a **quasi-parabolic** group, if etale locally, say on $S' \to S$, there is a splitting (P, T, M, R), Q is of the form given in theorem (1.9.7) for a $R' \subset R$ which contains a set of positive roots (but not necessarily R_W!).*

(1.9.9) Theorem. *Let S be reduced and P be a group scheme over S of type (P).*

i. *The functor $S' \mapsto \{$quasi-parabolic subgroups of $P_{S'}\}$ is representable by a smooth quasi-projective S-scheme $\mathcal{QPAR}$.*

ii. *There is an etale sheaf $\mathcal{TYPE}$ of finite sets over S and a surjective morphism*

$$\mathrm{type} : \mathcal{QPAR} \to \mathcal{TYPE},$$

with the property Q and Q' are locally conjugated in the etale topology, if and only if $\mathrm{type}(Q) = \mathrm{type}(Q')$.

iii. *If (P, T, M, R) is split (in the obvious sense for type (P))*

$$\mathcal{TYPE} \cong \{\ W(R_G)\text{-orbits of closed subsets of } R \text{ containing a set of positive roots }\}_S.$$

iv. *Let Q be a quasi-parabolic of P. The morphism $P/Q \to \mathcal{QPAR}$, $g \mapsto \mathrm{int}(g)Q$ is an open and closed immersion onto a connected component of $\mathcal{QPAR}$. It is hence the fibre above* $\mathrm{type}(Q)$.

v. $\underline{\mathrm{Hom}}(\mathbb{G}_m, P)$ *is representable by a smooth affine scheme over S.*

There is an etale sheaf $\mathcal{FTYPE}$ of (infinite) sets over S and a surjective morphism

$$\mathrm{ftype} : \underline{\mathrm{Hom}}(\mathbb{G}_m, P) \to \mathcal{FTYPE},$$

with the property α and α' are locally conjugated in the etale topology, if and only if $\mathrm{ftype}(\alpha) = \mathrm{ftype}(\alpha')$.

vi. *If (P, T, M, R) is split:*

$$\mathcal{FTYPE} \cong \{\ W(R_G)\text{-orbits in } M^*\ \}_S.$$

vii. *There is a surjective morphism*

$$\mathrm{qpar} : \underline{\mathrm{Hom}}(\mathbb{G}_m, P) \to \mathcal{QPAR},$$

characterized by the properties

a) *Let $\alpha : \mathbb{G}_{m,S'} \to G_{S'}$ be some cocharacter. For each etale $S'' \to S'$, where there is a splitting $(G_{S''}, T, M, R)$ such that $\alpha_{S''} : \mathbb{G}_{m,S''} \to T \subset G_{S''}$, we have*

$$\mathrm{type}(\mathrm{qpar}(\alpha)) = W(R_G)\{r \in R \mid r \circ \alpha \geq 0\}.$$

b) *For each $\alpha : \mathbb{G}_{m,S'} \to G_{S'}$, α factors through* $\mathrm{qpar}(\alpha)$.

There is a commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\mathbb{G}_m, P) & \xrightarrow{\mathrm{qpar}} & \mathcal{QPAR} \\ \Big\downarrow{\scriptstyle \mathrm{ftype}} & & \Big\downarrow{\scriptstyle \mathrm{type}} \\ \mathcal{FTYPE} & \longrightarrow & \mathcal{TYPE} \end{array}$$

If (P, T, M, R) *is split, the morphism on the bottom is induced by*

$$M^* \ni m \mapsto \{r \in R \mid \langle r, m \rangle \geq 0\}.$$

Proof. (compare [37, XXII, 5.11.5]). The functor $\mathcal{QPAR}$ is an etale sheaf by definition of quasi-parabolic.
The functors $\mathcal{FTYPE}$, resp. $\mathcal{TYPE}$, are the etale quotient-sheaves of $\underline{\mathrm{Hom}}(\mathbb{G}_m, S)$, resp. $\mathcal{QPAR}$, by the action of conjugation. It suffices hence to show representability of them and of $\mathcal{QPAR}$ locally in the etale topology and we may hence assume, (P, T, M, R) split over S. We will construct a morphism

$$\mathcal{QPAR} \rightarrow \{\ W\text{-orbits of closed subsets } R' \subset R \text{ containing a set of positive roots}\}_S.$$

Let $Q \in \mathcal{QPAR}(S)$. There is by definition an etale cover $U_i \rightarrow S$, splittings

$$(P, T_i, M_i, R_i)$$

such that Q_{U_i} is a group like in theorem (1.9.7). Hence there is an associated closed subset $R_i' \subset R_i$. By refining the cover we may assume that T_i is conjugated to T on U_i. Transporting R_i' to R we get a well-defined W-orbit of closed subsets of R. Surjectivity follows from theorem (1.9.7). Furthermore if the images of Q_1, Q_2 are the same, they are conjugated locally in the etale topology because of the unique charactization of quasi-parabolics by their Lie algebra. The fibre of 'type' over a point contains the group Q constructed in (1.9.7) and is identified with P/Q. This shows (i)-(iv).
Representability of $\underline{\mathrm{Hom}}(\mathbb{G}_m, P)$ is (1.2.6). Next we construct a morphism

$$\underline{\mathrm{Hom}}(\mathbb{G}_m, P) \rightarrow \{\ W\text{-orbits in } M^*\ \}_S.$$

Let $\alpha \in \mathrm{Hom}(\mathbb{G}_{m,S}, P_S)$ be given. There is an etale cover $U_i \rightarrow S$, splittings

$$(P, T_i, M_i, R_i)$$

such that α_{U_i} factors through T_i. Hence we get an element $m_i \in M_i^*$. By refining the cover we may assume that T_i is conjugated to T on U_i. Transporting m_i to M^* we get a well-defined W-orbit in M^* of R. The morphism is surjective. α_1, α_2 are locally conjugated in the etale topology, if and only if their images are the same. This shows (v), (vi).
(vii) follows because we can construct the morphism qpar etale locally. $\square$

(1.9.10) Remark. Over a field, our quasi-parabolics are called preudo-parabolics by Springer [89, 15.1.1]. However, we caution the reader against the wrong statement [loc. cit., 15.1.2 (ii)].

(1.9.11) Theorem. *Let S be reduced. For any morphism of group schemes $\alpha : P_1 \to P_2$ of type (P) over S, we get an induced morphism $\alpha' : \mathcal{QPAR}_1 \to \mathcal{QPAR}_2$. If α is a closed embedding then α' is an embedding. It is closed, if α induces a morphism $W_1 \to W_2$. It is open, if P_1 is a quasi-parabolic of P_2.*

Proof. We construct the morphism again locally in the etale topology, and hence may assume that there exist $Q_1 \in \mathcal{QPAR}_1$ a morphism

$$\alpha : \mathbb{G}_m \to \mathcal{P}$$

such that $Q_1 = \mathrm{qpar}(\alpha)$. Define $Q_2 = \alpha'(Q_1) := \mathrm{qpar}(\gamma \circ \alpha)$.
We know (1.9.9) that there are connected components of $\mathcal{QPAR}_i$ isomorphic to P_i/Q_i for $i = 1, 2$.
We have $\gamma(Q) \subseteq Q'$, and therefore an induced map $\alpha' : \mathcal{QPAR}_1 \to \mathcal{QPAR}_2$. If γ is a closed embedding, then $Q = Q' \cap P$, therefore the induced map on the quotients is an embedding. Closedness in the required case follows directly from the case of $\mathcal{PAR}$. Openness can be checked on geometric fibers, where one checks that $P \in \mathcal{QPAR}_2 - \mathcal{QPAR}_1$ is a closed condition. □

(1.9.12) Definition. *Let E be a locally free sheaf of $\mathcal{O}_S$-modules with (P, L)-structure ρ. A filtration F on E is called* ***compatible with the (P, L)-structure*** *if etale locally (say on U) there is an isomorphism β in the associated G_U-torsor and a morphism $\alpha : \mathbb{G}_m \to P_U$ which splits the filtration (via the induced (by β) representation of P_U on E_U).*

(1.9.13) Definition/Theorem. *Let S be reduced. Let E be a locally free sheaf of $\mathcal{O}_S$-modules with (P, L)-structure ρ and $F^\bullet$ a filtration on E compatible with the (P, L)-structure. Let an etale neighborhood U and β be given as in the definition. Then P acts via β on E_U and there exists a splitting $\alpha : \mathbb{G}_m \to P_U$ of the filtration.*

i. The stabilizer Q_U of the filtration $F^\bullet$ of E_U is the quasi-parabolic group $\mathrm{qpar}(\alpha)$ *of theorem (1.9.9).*

ii. Any two morphisms $\mathbb{G}_{m,U} \to P_U$ splitting the filtration are conjugated in Q_U locally in the etale topology. Hence the image under ftype *of these morphisms is well defined. We call the corresponding section $t' \in \mathcal{FTYPE}(S)$ the* ***type of $F^\bullet$****. Let t be its image in $\mathcal{TYPE}(S)$.*

iii. $F^\bullet$ *defines a* P*-equivariant morphism*

$$\mathrm{tor}(\rho) \to \mathcal{QPAR} \times_{\mathcal{TYPE},\mathrm{type},t} S.$$

Proof. We may restrict to an etale cover S, where E is free and β exists. Let $F^\bullet$ be a filtration, split by $\alpha : \mathbb{G}_m \to P$. We may also assume that there is a splitting (P, T, M, R) over S such that α factors through T. Let Q be the stabilizer of $F^\bullet$. It is a closed subgroup scheme of P. Let $X \in L(S)$. Take a basis $\{e_i\}_i$ of $\mathrm{Lie}(P)^r$. The map

$$u \mapsto \exp_r(ue_i)X$$

is a morphism $\mathbb{G}_a \to \mathbb{W}(L)$ and hence given by a polynomial $\sum u^n X_n$ for $X_n \in L(S)$. Letting $\alpha : \mathbb{G}_m \to P$ act on the image, we get

$$\alpha(z)X_n = z^{n(r\circ\alpha)+i}X_n.$$

Therefore the image of $\exp_r$ fixes the filtration, if $r \circ \alpha \geq 0$. Hence for the set of roots satisfying $r \circ \alpha \geq 0$ the group Q' of theorem (1.9.7) fixes the filtration (because it is fixed by the images of the various exp and T whose generated subgroup scheme is dense). On the other hand, the Lie algebra $\mathrm{Lie}(Q_U)$ acts on E_U and stabilizes the filtration. Therefore, since $Q' \subset Q$, we have $\mathrm{Lie}(Q) = \mathrm{Lie}(Q')$. But Q is T-stable, hence by theorem (1.9.7), $Q = Q'$.
The group Q is of type (P) (1.9.7), hence maximal tori exist and are conjugated locally in the etale topology. We may hence assume that there exists a split maximal torus $T = D(M_S)$ of Q over which α factors. E is then decomposed as

$$E = \bigoplus_{r\in M} E^r.$$

Now we have

$$F^i(E) = \bigoplus_{r\in M, r\circ\alpha\leq i} E^r$$

and hence

$$\alpha \circ r = \min_{E^r \subseteq F^i(E)} i \qquad \text{if } E^r \neq 0.$$

This determines α because the set $\{r \mid L^r \neq 0\}$ contains a basis of $M_\mathbb{Q}$ because the representation is assumed to be faithful.
The morphism (iii) is constructed functorially as follows. Let $\beta \in \mathrm{tor}(\rho)(S)$ be an isomorphism $E_S \to L_S$. Via β, P acts on E as well. Define the image of β to be $\mathrm{Stab}(F^\bullet, P_S)$ which is a quasi-parabolic of type t be the foregoing. The morphism is P-equivariant. ☐

(1.9.14) Theorem. *Let* S *be reduced. Let* L *be a locally free sheaf on* S*,* P *a group scheme of type* (P) *over* S *acting linearly and faithfully on* L*, i.e. induced by a closed* *embedding* $P \hookrightarrow \mathrm{GL}(L)$*. Let* $t' \in \mathcal{FTYPE}(S)$ *and* t *be its image in* $\mathcal{TYPE}(S)$*. The fibre above* t *of* type *in* $\mathcal{QPAR}$ *is isomorphic to*

$$S' \mapsto \{ \text{ filtrations } F^\bullet \text{ on } L_{S'} \text{ compatible with } (\mathcal{P}_{S'}, L_{S'}) \text{ of type } t'\},$$

compatible with P*-action.*

Proof. We construct the isomorphism locally in the etale topology. Then there exist a morphism $\alpha : \mathbb{G}_m \to P$ of some type t'. and a corresponding filtration $F^\bullet$. The set of all filtrations of type t' is then isomorphic to $P/\operatorname{Stab}(F^\bullet)$ because filtrations of type t' are conjugated locally in the etale topology because their splittings are. But $P/\operatorname{Stab}(F^\bullet)$ is isomorphic to the fibre of type in $\mathcal{QPAR}$ canonically by (1.9.9, iv). Therefore these isomorphisms glue. □

(1.9.15) Example. *Let* E *be a locally free sheaf on* S *of dimension* n*,* $P = GL(E)$*.*

$$\mathcal{TYPE}_P = \{ \text{ subsets of } \{1, \ldots, n-1\} \}_S,$$
$$\mathcal{FTYPE}_P = \{ \text{ sequences of non-negative integers } \{d_i\}_{i\in\mathbb{Z}} \text{ with } \sum d_i = n \}_S.$$

Filtrations of type $\{d_i\}$ *are those saturated filtrations, where* $\dim(\mathrm{gr}^i_F) = d_i$*. The map* type *is given by associating with* $\{d_i\}$ *the complement of the set partial sums* $\sum_{i\le j} d_i (\neq n)$*.*
Let F^i *be a filtration of type* t' *and* Q *the stabilizer-group. We have*

$$\mathcal{TYPE}_Q = \{ \text{ subsets of } \mathrm{type}_P(Q)\}_S,$$
$$\mathcal{FTYPE}_Q = \{ \text{ sequences of non-negative integers } \{d_{i,j}\}_{i,j\in\mathbb{Z}} \text{ with } \sum_j d_{i,j} = d_i \}_S.$$

Filtration of type $\{d_{i,j}\}$ *are those filtrations* $G^\bullet$*, which together with* $F^\bullet$ *are* bisaturated *and such that* $\dim(\mathrm{gr}^j_G \, \mathrm{gr}^i_F) = d_{i,j}$*.*

2. Preliminaries on mixed Shimura data and varieties

In this chapter, we define p-integral mixed Shimura data. This is very much the same definition as in Pink's thesis [83], except that we use a stronger requirement on the center (for psychological comfort) and more importantly, the group scheme P in [loc. cit.] is replaced by a group scheme of type (P), defined over $\mathbb{Z}_{(p)}$. All compact open subgroups are then required to be of the form $P(\mathbb{Z}_p) \times K^{(p)}$, where $K^{(p)}$ is a compact open subgroup of $P(\mathbb{A}^{(\infty,p)})$. We call them admissible. We explain the connection to mixed Hodge structures, recall all relevant definitions from [83], and extend the technical results on mixed Shimura data to p-integral ones. In (2.4) boundary components are investigated. In (2.5) we define the 'symplectic (mixed) Shimura data' whose associated (mixed) Shimura varieties are moduli spaces of 1-motives. (This will be explained in detail in chapter 4). We will restrict (essentially) to mixed Shimura data of Hodge type, i.e. those which embed into the symplectic ones (2.6). In the end (2.7) we briefly explain the interpretation of mixed Shimura varieties over $\mathbb{C}$ as parameter spaces of families (or variations) of mixed Hodge structures.
We diverge from the custom to denote a (mixed) Shimura datum by (P, X), (G, X) or (P, X, h). We use an abstract letter (e.g. $\mathbf{X}$) for it and denote its constituents by $P_{\mathbf{X}}, \mathbb{D}_{\mathbf{X}}, h_{\mathbf{X}}$. This is because they always occur in fixed pairs (or triples) like for the symplectic Shimura varieties $\mathbf{H}_g$ or the orthogonal resp. spin ones $\mathbf{O}(L)$ resp. $\mathbf{S}(L)$. Furthermore often different, but related ones (boundary components, those associated with strata, unipotent extensions, etc.) are used. We find it confusing and senseless to invent special symbols for the respective symmetric domains in each case. This convention also permits to denote unipotent extensions just by a symbolic construction: $\mathbf{X}[U, V]$.

2.1. Mixed Hodge structures

(2.1.1) Definition. *Let B be an analytic Deligne-Mumford stack. We define the category*

$$[\ B\textbf{-}\boldsymbol{mhs}\ ]$$

*of families of **rational mixed Hodge structures** above B as given by the following data: A local system of $\mathbb{Q}$-vector-spaces M above B.*
*An increasing **weight filtration***

$$0 = W_i(M) \subset W_{i+1}(M) \subset \cdots \subset W_{i+n}(M) = M$$

by local systems of $\mathbb{Q}$-vector-spaces.

A decreasing ***Hodge filtration***

$$0 = F^j(M) \subset F^{j-1}(M) \subset \cdots \subset F^{j-m}(M) = W \otimes_{\mathbb{Q}} \mathcal{O}_B$$

by holomorphic subsheaves such that for the induced Hodge filtration on gr^n_W *one has point-wise*

$$\forall p + q = n + 1 \qquad M_{\mathbb{C}} \cong F^p(M) \oplus \overline{F^q(M)},$$

i.e. determines a usual pure Hodge structure.

(2.1.2) A pure Hodge structure (of weight n) is determined by $M = \bigoplus_{p+q=n} H^{p,q}(M)$ by setting

$$\begin{aligned} F^p(M) &= \bigoplus_{p' \geq p} H^{p',q}(M), \\ H^{p,q}(M) &= F^p(M) \cap \overline{F^q(M)} \quad \text{if } p + q = n. \end{aligned}$$

For a mixed Hodge structure there is a unique decomposition $M = \bigoplus_{p+q=n} H^{p,q}(M)$ with

$$\begin{aligned} F^p(M) &= \bigoplus_{p' \geq p} H^{p',q}(M), \\ W_n(M_{\mathbb{C}}) &= \bigoplus_{p+q \leq n} H^{p,q}(M), \\ H^{p,q}(M) &= \overline{H^{q,p}(M)} \qquad \mod \bigoplus_{p'<p, q'<q} H^{p',q'}(M). \end{aligned}$$

This determines a morphism $h : \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}(M_{\mathbb{C}})$.

(2.1.3) Definition.

$$\begin{aligned} \mathbb{S} &:= \mathbb{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m), \\ H_0 &:= \{(z, \alpha) \in \mathbb{S} \times \mathrm{GL}_{2,\mathbb{R}} \mid z\overline{z} = \det(\alpha)\}. \end{aligned}$$

Let $P_{\mathbf{X},\mathbb{Q}}$ be a connected linear algebraic group over $\mathbb{Q}$. Let $W_{\mathbf{X},\mathbb{Q}}$ be the unipotent radical of $P_{\mathbf{X},\mathbb{Q}}$ and

$$P_{\mathbf{X},\mathbb{Q}} \xrightarrow{\pi} G_{\mathbf{X},\mathbb{Q}} = P_{\mathbf{X},\mathbb{Q}}/W_{\mathbf{X},\mathbb{Q}}$$

be the projection. Let $\rho : P_{\mathbf{X},\mathbb{Q}} \to \mathrm{GL}(L_{\mathbb{Q}})$ be a (faithful) representation on a $\mathbb{Q}$-vector-space $L_{\mathbb{Q}}$. A morphism $h : \mathbb{S}_C \to P_{\mathbf{X},\mathbb{C}}$ determines a bigrading of $L_{\mathbb{C}}$ and hence a weight and Hodge filtration on $L_{\mathbb{Q}}$, for every representation of $P_{\mathbf{X},\mathbb{Q}}$ on $L_{\mathbb{Q}}$. To obtain the bigrading, the character group of $\mathbb{S}_{\mathbb{C}}$ is identified with $\mathbb{Z}^2$ via $p, q \mapsto (z, \overline{z} \mapsto z^{-p}\overline{z}^{-q})$.

(2.1.4) Definition. *A morphism $h : \mathbb{S}_{\mathbb{C}} \to P_{\mathbf{X},\mathbb{C}}$ is called **admissible** if*

- $\pi \circ h$ *is defined over* $\mathbb{R}$,
- $\pi \circ h \circ w$ *is a cocharacter of the center of* $P_{\mathbf{X},\mathbb{Q}}/W_{\mathbf{X},\mathbb{Q}}$ *defined over* $\mathbb{Q}$.
- *Under the weight filtration on* $\mathrm{Lie}(P_{\mathbf{X},\mathbb{Q}})$ *defined by* $\mathrm{Ad} \circ h_{\mathbf{X}}$*:* $W_{-1}(\mathrm{Lie}(P_{\mathbf{X},\mathbb{Q}})) = \mathrm{Lie}(W_{\mathbf{X},\mathbb{Q}})$.

(2.1.5) Theorem. *A morphism $h : \mathbb{S} \to \mathrm{GL}(L_{\mathbb{R}})$ is associated with a mixed Hodge structure if and only if h factors through a subgroup $P_{\mathbf{X},\mathbb{Q}} \subset \mathrm{GL}(L_{\mathbb{Q}})$, and the induced morphism is admissible.*

Proof. [83, prop. 1.5] □

2.2. p-integral mixed Shimura data

(2.2.1) Lemma. *Let T be a $\mathbb{Q}$-torus. The following are equivalent conditions:*

i. $T(\mathbb{Q})$ *is discrete in* $T(\mathbb{A}^{(\infty)})$

ii. T *is an almost direct product of a $\mathbb{Q}$-split torus with a torus T', such that $T'(\mathbb{R})$ is compact.*

Proof. [77, Theorem 5.26] □

T' in the lemma can be described by $T' = \bigcap_{\chi \in \mathrm{Hom}(T,\mathbb{G}_{m,\mathbb{Q}})} \ker(\chi)$.

(2.2.2) Definition. *Let p be a prime.*
*A **p-integral mixed Shimura datum (p-MSD)** $\mathbf{X}$ consists of*

i. *a group scheme $P_{\mathbf{X}}$ of type (P) over $S = \mathrm{spec}(\mathbb{Z}_{(p)})$.*

ii. *a homogeneous space $\mathbb{D}_{\mathbf{X}}$ under $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$ ($U_{\mathbf{X}}$ being determined by $P_{\mathbf{X}}$, see below)*

iii. *a $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-equivariant finite to one morphism $h_{\mathbf{X}} : \mathbb{D}_{\mathbf{X}} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{X},\mathbb{C}})$, such that the image consists of* admissible *morphisms (2.1.4),*

subject to the following condition: For (one, hence for all) $h_x, x \in \mathbb{D}_{\mathbf{X}}$,

i. $\mathrm{Ad}_P \circ h_x$ *induces on* $\mathrm{Lie}(P)$ *a mixed Hodge structure of type*

$$(-1,1),(0,0),(1,-1) \qquad (-1,0),(0,-1) \qquad (-1,-1),$$

ii. the weight filtration on $\mathrm{Lie}(P_{\mathbb{Q}})$ *is given by*

$$W_i(\mathrm{Lie}(P_{\mathbb{Q}})) = \begin{cases} \mathrm{Lie}(P_{\mathbb{Q}}) & \text{if } i \geq 0, \\ \mathrm{Lie}(W_{\mathbb{Q}}) & \text{if } i = -1, \\ \mathrm{Lie}(U_{\mathbb{Q}}) & \text{if } i = -2, \\ 0 & \text{if } i < -2, \end{cases}$$

where $W_{\mathbb{Q}}$ *is the unipotent radical of* $P_{\mathbb{Q}}$ *and* $U_{\mathbb{Q}}$ *is a central subgroup,*

iii. $\mathrm{int}(\pi(h_x(i)))$ *induces a Cartan involution on* $G^{ad}(\mathbb{R})$*, where* $G = P/W$*,*

iv. $G^{ad}(\mathbb{R})$ *possesses no nontrivial factors of compact type that are defined over* $\mathbb{Q}$*,*

v. the center Z *of* P *satisfies the properties of lemma (2.2.1).*

$\mathbf{X}$ *is called* ***pure****, if* $W_{\mathbf{X}} = 1$*. A* ***morphism of*** p***-MSD*** $\mathbf{X} \to \mathbf{Y}$ *is a pair of a homomorphism of group schemes* $P_{\mathbf{X}} \to P_{\mathbf{Y}}$ *and a homomorphism* $\mathbb{D}_{\mathbf{X}} \to \mathbb{D}_{\mathbf{Y}}$ *respecting the maps* h*.*
We call a morphism an ***embedding****, if the morphism of group schemes is a* closed *embedding, and the map* $\mathbb{D}_{\mathbf{X}} \to \mathbb{D}_{\mathbf{Y}}$ *is* injective.
If we have a p-MSD $\mathbf{X}$*, then an* ***admissible*** *compact open subgroup of* $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ *is a group of the form* $K^{(p)}P_{\mathbf{X}}(\mathbb{Z}_p)$*, where* $K^{(p)}$ *is a compact open subgroup of* $P_{\mathbf{X}}(\mathbb{A}^{(\infty,p)})$*.*

see [83, 2.1] for the rational case.

(2.2.3) Remark. The property (v) in the definition is stated in [loc. cit.] in a weaker form, namely it is required that the action on W is through a torus of type (2.2.1). However one checks that the operations on Shimura data performed in [loc. cit.] (boundary components, quotients, etc.) preserve this condition. Furthermore, an embedding $\mathbf{X} \hookrightarrow \mathbf{Y}$ where $\mathbf{Y}$ satisfies condition (v) immediately implies $\mathbf{X}$ satisfy condition (v). Therefore the Shimura data of Hodge type (2.6.1), predominantly considered here, satisfy condition (v) anyway.

(2.2.4) Definition. *A pair* ${}^{K}\mathbf{X}$ *is called* p***-integral extended mixed Shimura data (***p***-EMSD)****, where* $\mathbf{X}$ *is p-integral Shimura data,* K *is an admissible compact open subgroup of* $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$*.*
A ***morphism of*** p***-EMSD*** ${}^{K'}\mathbf{Y} \to {}^{K}\mathbf{X}$ *is a morphism of* p*-integral mixed Shimura data* $\gamma : \mathbf{Y} \to \mathbf{X}$ *and a* $\rho \in P_{\mathbf{X}}(\mathbb{A}^{(\infty,p)})$ *such that* $\gamma(K')^{\rho} \subset K$*,*
If γ *is an automorphism then we call the morphism a* ***Hecke operator****. If* γ *is an embedding and* $\gamma(K_1)^{\rho} = K_2 \cap P_1(\mathbb{A}^{(\infty)})$*, and such that the map*

$$[P_{\mathbf{Y}}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{Y}} \times (P_{\mathbf{Y}}(\mathbb{A}^{(\infty)})/K')] \to [P_{\mathbf{X}}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)]$$

is a closed embedding (compare also 2.2.8), then we call the morphism an ***embedding****.*

(2.2.5) Lemma. *$P_{\mathbf{X}}(\mathbb{Q}) \cap \mathrm{Stab}(x, P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C}))K$ is finite for every compact open subgroup $K \subset P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ and trivial for sufficiently small K.*

Proof. This is a consequence of properties (iii) and (v) in definition (2.2.2). □

(2.2.6) Definition. *Let M be an integer. Let $P_{\mathbf{X}}$ be a group scheme over $\mathbb{Z}[1/M]$ of type (P), and $\mathbb{D}_{\mathbf{X}}$ such that they define* rational *mixed Shimura data $\mathbf{X}$. For each $p \nmid M$, $P_{\mathbf{X}} \times_{\mathbb{Z}[1/M]} \mathbb{Z}_{(p)}$ will then define p-*integral *mixed Shimura data which we equally denote by $\mathbf{X}$. Let $L_{\mathbb{Z}}$ be a lattice, such that there is a faithful representation, i.e. a closed embedding $P_{\mathbf{X}} \hookrightarrow \mathrm{GL}(L_{\mathbb{Z}[1/M]})$.*
For each integer N, we define the following compact open subgroup of $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$:

$$K(N) := \{g \in P_{\mathbf{X}}(\mathbb{A}^{(\infty)}) \mid gV_{\widehat{\mathbb{Z}}} = V_{\widehat{\mathbb{Z}}}, g \equiv \mathrm{id} \mod N\}.$$

If $p \nmid N$ then $K(N)$ is admissible.

(2.2.7) Lemma. *For each (admissible) compact open subgroup $K \subset P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$, there is a $\gamma \in P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ (resp. $\in P_{\mathbf{X}}(\mathbb{A}^{(\infty,p)})$) such that*

$$P_{\mathbf{X}}^{\gamma} \subseteq K(1).$$

Proof. Follows from [84, Prop. 1.12]. □

(2.2.8) Lemma. *i. For each ${}^{K_1}\mathbf{X}_1$ and an embedding $\alpha : \mathbf{X}_1 \hookrightarrow \mathbf{X}_2$, there is an admissible $K_2 \subset P_{\mathbf{X}_2}(\mathbb{A}^{(\infty)})$ such that ${}^{K_1}\mathbf{X}_1 \hookrightarrow {}^{K_2}\mathbf{X}_2$ is an embedding (2.2.4).*

ii. If ${}^{K_1}\mathbf{X}_1 \hookrightarrow {}^{K_2}\mathbf{X}_2$ is an embedding, and if K_2 is neat, then for each $K_2' \subset K_2$, the map ${}^{K_2' \cap P_{\mathbf{X}_1}(\mathbb{A}^{(\infty)})}\mathbf{X}_1 \hookrightarrow {}^{K_2'}\mathbf{X}_2$ is an embedding.

iii. Let $P_{\mathbf{X}_2}$ act linearly on a $V_{\mathbb{Z}_{(p)}}$ and choose a lattice $V_{\mathbb{Z}} \subset V_{\mathbb{Z}_{(p)}}$. There is an integer N such that for all $(M,p) = 1, N|M$, ${}^{K(M)_1}\mathbf{X}_1 \hookrightarrow {}^{K(M)_2}\mathbf{X}_2$ is an embedding.

Proof. (i) Without the attribute 'admissible', this is shown in [22, Prop. 1.15] (pure case) or in [83, 3.8] (mixed case). Difficulties arise, when one weakens condition (v) of definition (2.2.2). We do not know, whether there exists an *admissible* compact open subgroup in that case with this property. With condition (v), we proceed as follows: Like in [loc. cit.] one is reduced to show that

$$\alpha : P_{\mathbf{X}_1}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}_1} \times P_{\mathbf{X}_1}(\mathbb{A}^{(\infty)})/K_1 \to \varprojlim_{\substack{K_2 \supset K_1 \\ \text{admissible}}} P_{\mathbf{X}_2}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}_2} \times (P_{\mathbf{X}_2}(\mathbb{A}^{(\infty)})/K_2)$$

is injective. Suppose that this map were not injective, i.e. there are $(x,p), (x',p')$ and sequences $p_{i,\mathbb{Q}} \in P_{\mathbf{X}_2}(\mathbb{Q})$, $p_{i,K} \in K_{2,i}$, $i \in \mathbb{N}$ where the $K_{2,i}$ are decreasing and $\bigcap_i K_{2,i} =$

$K_1 P_{\mathbf{X}_2}(\mathbb{Z}_p)$:

$$(x,p) = (p_{i,\mathbb{Q}} x', p_{i,\mathbb{Q}} p' p_{i,K}).$$

Now, the $p_{i,\mathbb{Q}}$ are in $P_{\mathbf{X}_2}(\mathbb{Q}) \cap \operatorname{Transp}(x,x')(\mathbb{R}) \cdot p' K_{2,1} p^{-1}$. Like in lemma (2.2.5) one sees that this set is finite. Hence we have some sequence like above, where all $p_{i,\mathbb{Q}}$ are equal. But then also the $p_{i,K}$ are equal and hence lie in $\bigcap_i K_{2,i} = K_1 P_{\mathbf{X}_2}(\mathbb{Z}_p)$ Looking at some $l \neq p$ we see that $(p_{i,\mathbb{Q}})_l \in P_{\mathbf{X}_1}(\mathbb{Q}_l)$, and therefore $p_{i,\mathbb{Q}} \in P_{\mathbf{X}_1}(\mathbb{Q})$ and $p_{i,K} \in K_1$.
(ii) Suppose

$$(x,p) = (p_{2,\mathbb{Q}} x', p_{2,\mathbb{Q}} p' p_{2,K}).$$

Then there exist $p_{1,\mathbb{Q}} \in P_{\mathbf{X}_1}(\mathbb{Q})$, $p_{1,K} \in K_1$, such that

$$(x,p) = (p_{1,\mathbb{Q}} x', p_{1,\mathbb{Q}} p' p_{1,K}).$$

as well. Now $p_{1,\mathbb{Q}}^{-1} p_{2,\mathbb{Q}} = p'(p_{1,K} p_{2,K}^{-1})(p')^{-1}$ and $p_{1,\mathbb{Q}}^{-1} p_{2,\mathbb{Q}}$ stabilize x'.
Hence $p' p_{1,K} p_{2,K}^{-1} (p')^{-1}$ lie in a finite subgroup of $p' K_2 (p')^{-1} \cap P_{\mathbf{X}_2}(\mathbb{Q})$, so they are equal (because K_2 is neat, hence $p' K_2 (p')^{-1}$).
(iii) For $N' \geq 3$, $p \nmid N'$, $K(N')_2$ is neat and admissible. There is by (i) an admissible compact open subgroup $K_2 \subset K(M')_2$ such that

$$^{K(M)_1}\mathbf{X}_1 \hookrightarrow {}^{K_2}\mathbf{X}_2$$

is an embedding. Now there is a $K(N) \subset K_2$, $p \nmid N$ because the $K(N)$ with $p \nmid N$ are cofinal for admissible compact open subgroups. By (ii), (iii) is satisfied with respect to this N. □

(2.2.9) Theorem. *Let* $\mathbf{X}$ *be p-MSD.*
There are smooth closed subgroup schemes of $P_{\mathbf{X}}$*:* $W_{\mathbf{X}}$ *(the unipotent radical) and* $U_{\mathbf{X}} = \mathbb{W}(W_{-2}(\operatorname{Lie}(P_{\mathbf{X}})))$*. There is a smooth group scheme* $V_{\mathbf{X}} = \mathbb{W}(\operatorname{gr}_{-1}(\operatorname{Lie}(P_{\mathbf{X}}))$ *and an exact sequence*

$$0 \longrightarrow U_{\mathbf{X}} \longrightarrow W_{\mathbf{X}} \longrightarrow V_{\mathbf{X}} \longrightarrow 0.$$

There is a closed reductive subgroup scheme $G_{\mathbf{X}}$*, such that* $P_{\mathbf{X}} = W_{\mathbf{X}} \rtimes G_{\mathbf{X}}$*. Any two such subgroup schemes are conjugated by an element in* $W_{\mathbf{X}}(\mathbb{Z}_{(p)})$*.*
For a morphism of p-MSD $\alpha : \mathbf{X} \to \mathbf{Y}$*, we have a diagram*

$$\begin{tikzcd} 0 \arrow[r] & U_{\mathbf{X}} \arrow[r] \arrow[d] & W_{\mathbf{X}} \arrow[r] \arrow[d] & V_{\mathbf{X}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & U_{\mathbf{Y}} \arrow[r] & W_{\mathbf{Y}} \arrow[r] & V_{\mathbf{Y}} \arrow[r] & 0. \end{tikzcd}$$

If α *is an embedding, the vertical maps are closed embeddings and the outer ones are given by saturated inclusions (i.e. inducing an inclusion mod p as well) of the corresponding modules over* $\mathbb{Z}_{(p)}$*. Furthermore, there are closed reductive subgroup schemes* $G_{\mathbf{X}}$ *and* $U_{\mathbf{Y}}$ *of* $P_{\mathbf{X}}$ *and* $P_{\mathbf{Y}}$*, respectively, such that* $P_{\mathbf{X}} = W_{\mathbf{X}} \rtimes G_{\mathbf{X}}$*,* $P_{\mathbf{Y}} = W_{\mathbf{Y}} \rtimes G_{\mathbf{Y}}$*, and such that* $G_{\mathbf{X}}$ *is mapped to* $G_{\mathbf{Y}}$*.*

Proof. (cf. [83, 2.15] for the rational case). By (1.8.6) we have (etale locally: $S' \to S$) an isomorphism

$$\prod_{r \in R_W} \mathbb{W}(\mathrm{Lie}(W_{S'})^r) \to W_{S'}$$

(for any ordering) induced by the $\exp_r$ and multiplication in W. Hence we can define a closed subschemes of W by

$$U := \prod_{r, row=-2} \mathbb{W}(\mathrm{Lie}(W_{S'})^r) \to W_{S'}.$$

Its generic fibre is the *Abelian* unipotent subgroup $U_{\mathbb{Q}}$ of $W_{\mathbb{Q}}$ [83, 2.15]. Hence the map is a morphism of group schemes. Now consider the map

$$\prod_{r, row=-1} \mathbb{W}(\mathrm{Lie}(W_{S'})^r) \to (W/U)_{S'}.$$

Now, since both sides are flat and of finite type over S, the map is an isomorphism, if it is an isomorphism on geometric fibers. The latter follows like in the proof of (1.8.6). Since again the generic fibre is the *Abelian* unipotent group $V_{\mathbb{Q}}$ [83, 2.15], it is an isomorphism of group schemes. Since the filtration $W_i(\mathrm{Lie}(W_{S'}))$ is already defined over S, one concludes the first statement of the theorem by etale descent, using (1.1.2). The existence and conjugacy of G's is stated in (1.8.11).
Now suppose we are given a morphism of p-MSD $\alpha : \mathbf{X} \to \mathbf{Y}$. It induces a map from $\mathbb{D}_{\mathbf{X}}$ to $\mathbb{D}_{\mathbf{Y}}$. Let $x \in \mathbb{D}_{\mathbf{X}}$ be given and $x' \in \mathbb{D}_{\mathbf{Y}}$ be its image. We have $w_x = w_{x'} \circ \alpha$ and w_x, resp. $w_{x'}$ are defined over S because they are central (follows from 1.6.8). Therefore $\mathrm{Lie}(\alpha)$ is *strict* with respect to the weight filtrations on $\mathrm{Lie}(P_{\mathbf{X}})$, resp. $\mathrm{Lie}(P_{\mathbf{Y}})$.
The diagram exists over $\mathbb{Q}$: We have already $\alpha(W_{\mathbf{X},\mathbb{Q}}) \subseteq W_{\mathbf{Y},\mathbb{Q}}$ because we are in characteristic 0 and the groups are reduced and connected. For the same reason $\alpha(U_{\mathbf{X},\mathbb{Q}}) \subseteq U_{\mathbf{Y},\mathbb{Q}}$, hence α induces also a map $V_{\mathbf{X},\mathbb{Q}} \to V_{\mathbf{Y},\mathbb{Q}}$. We now get automatically morphisms for the their closures $W_{\mathbf{X}} \to W_{\mathbf{Y}}, U_{\mathbf{X}} \to U_{\mathbf{Y}}$, hence an induced morphism $V_{\mathbf{X}} \to V_{\mathbf{Y}}$ as well.
The maps $V_{\mathbf{X},\mathbb{Q}} \to V_{\mathbf{Y},\mathbb{Q}}$, $U_{\mathbf{X},\mathbb{Q}} \to U_{\mathbf{Y},\mathbb{Q}}$ are given by linear maps, hence this is automatically true for their extensions to S. If we have a closed embedding, the restriction to the closed subgroups $W_{\mathbf{X}}$ and $U_{\mathbf{X}}$ has to be a closed embedding as well. The morphism $V_{\mathbf{X}} \to V_{\mathbf{Y}}$ is a closed embedding, too, because Lie of it has to be $\mathrm{gr}_2(\mathrm{Lie}(P_{\mathbf{X}})) \hookrightarrow \mathrm{gr}_2(\mathrm{Lie}(P_{\mathbf{Y}}))$ and it is a *saturated inclusion* because we have the *Cartesian* diagram

$$\begin{tikzcd}
W_{-2}(\mathrm{Lie}(P_{\mathbf{X}})) \arrow[r, hook] \arrow[d, hook] & W_{-1}(\mathrm{Lie}(P_{\mathbf{X}})) \arrow[d, hook] \\
W_{-2}(\mathrm{Lie}(P_{\mathbf{Y}})) \arrow[r, hook] & W_{-1}(\mathrm{Lie}(P_{\mathbf{Y}}))
\end{tikzcd}$$

of *saturated* inclusions. □

(2.2.10) Definition/Theorem. *Let* $\mathbf{X}$ *be p-MSD. We have the following converse to*

(2.2.9):
Given two $\mathbb{Z}_{(p)}$-modules V and U acted on by $P_{\mathbf{X}}$ with non-degenerate invariant *symplectic form $\Psi : V \times V \to U$ (i.e. inducing an isomorphism $V \simeq V^*$) This defines a group scheme W_0 sitting in an exact sequence:*

$$0 \longrightarrow U_0 \longrightarrow W_0 \longrightarrow V_0 \longrightarrow 0,$$

where $U_0 := \mathbb{W}(U)$ and $V_0 := \mathbb{W}(V)$. By the action of $P_{\mathbf{X}}$ we may form a semi-direct product $P_{\mathbf{X}'} := W_0 \rtimes P_{\mathbf{X}}$. Assume that every subquotient of $\mathrm{Lie}(W_{0,\mathbb{R}})$ *is of type*

$$\{(-1,0),(0,-1)\} \text{ or } \{(-1,-1)\}.$$

Define $\mathbb{D}_{\mathbf{X}'}$ as

$$\{(x,k) \in \mathbb{D}_{\mathbf{X}} \times \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P'_{\mathbb{C}}) \mid h_x = \phi \circ k; \pi' \circ k : \mathbb{S}_{\mathbb{C}} \to (P'/U')_{\mathbb{C}} \text{ is defined over } \mathbb{R}\}.$$

$\mathbf{X}'$ is p-MSD, called a ***unipotent extension*** *of $\mathbf{X}$, denoted by $\mathbf{X}[U,V]$.*
$\mathbb{D}_{\mathbf{X}'} \to \mathbb{D}_{\mathbf{X}}$ it a torsor under $W_0(\mathbb{R})(W_0 \cap U_{\mathbf{X}'})(\mathbb{C})$.
We have $\mathbf{X}[U,V]/W_0 \simeq \mathbf{X}$.

Proof. See [83, 2.16] for the rational case. That $P_{\mathbf{X}'}$ is of type (P) follows from the construction. ☐

Theorem (2.2.9) may be reformulated as follows. Every p-MSD satisfies

$$\mathbf{X} \simeq (\mathbf{X}/W_{\mathbf{X}})[\mathrm{Lie}(U_{\mathbf{X}}), \mathrm{Lie}(V_{\mathbf{X}})],$$

the symplectic form and action of $G_{\mathbf{X}}$ being determined by $\mathbf{X}$.

2.3. Mixed Hodge structures continued

Let $\mathbf{X}$ be a mixed Shimura datum.

(2.3.1) Definition. *Let B be an analytic Deligne-Mumford stack, $\rho : P_{\mathbf{X},\mathbb{Q}} \hookrightarrow \mathrm{GL}(L_{\mathbb{Q}})$ be a faithful representation of $P_{\mathbf{X},\mathbb{Q}}$. We define*

$$[\ B\text{-}\mathbf{X}\text{-}L\text{-}\textbf{\textit{mhs'}}\ ]$$

as the set of families of rational mixed Hodge structures (2.1.1) on $(L_{\mathbb{Q}})_B$ such that point-wise the associated morphism h factors via ρ and is of the form h_x, $x \in \mathbb{D}_{\mathbf{X}}$.

(2.3.2) Definition. *For u_x, $x \in \mathbb{D}_{\mathbf{X}}$ consider $t' = \mathrm{ftype}(u_x) \in \mathcal{FTYPE}(\mathbb{C})$. It is* *independent of the chosen $x \in \mathbb{D}_{\mathbf{X}}$. We define*

$$[\ \ B\text{-}\mathbf{X}\text{-}L\text{-}\boldsymbol{filt}\ \ ]$$

as the set of filtrations on $L_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathcal{O}_B$ of type t'.

(2.3.3) Theorem.

$$B \mapsto [\ \ B\text{-}\mathbf{X}\text{-}L\text{-}\boldsymbol{mhs}\ \ ]$$

is isomorphic to $h(\mathbb{D}_{\mathbf{X}})$, where the complex structure is given by the morphism (Borel embedding)

$$[\ \ B\text{-}\mathbf{X}\text{-}L\text{-}\boldsymbol{mhs'}\ \ ] \to [\ \ B\text{-}\mathbf{X}\text{-}L\text{-}\boldsymbol{filt}\ \ ].$$

It is an open embedding and an isomorphism, if $G_{\mathbf{X},\mathbb{Q}}$ is a torus.

(2.3.4) Definition. *We define also a*

$$B \mapsto [\ \ B\text{-}\mathbf{X}\text{-}L\text{-}\boldsymbol{mhs}\ \ ]$$

as above but additionally with a morphism $B \to \mathbb{D}_{\mathbf{X}}$, giving back via h the morphism determined by the previous theorem.

It is (tautologically) represented by $\mathbb{D}_{\mathbf{X}}$ equipped with the complex structure determined by the covering $h_{\mathbf{X}}$.

(2.3.5) Definition. *The groupoid*

$$[\ \ B\text{-}\mathbf{X}\text{-}L\text{-}\boldsymbol{loc\text{-}mhs}\ \ ]$$

is the category of families (over B) of rational mixed Hodge structures $M, W_\bullet(M), F^\bullet(M)$ (2.1.1), where M is equipped with a $(P_{\mathbf{X}}(\mathbb{Q}), L_{\mathbb{Q}})$-structure α, such that locally where there is an isomorphism β in the associated $P_{\mathbf{X}}(\mathbb{Q})$-torsor (1.7.1), the pullback of the family of mixed Hodge structures via β is in

$$[\ \ B\text{-}\mathbf{X}\text{-}L\text{-}\boldsymbol{mhs}\ \ ]$$

(2.3.4).
Morphisms are isomorphisms of local systems, respecting $(P_{\mathbf{X}}(\mathbb{Q}), L_{\mathbb{Q}})$-structures and the family of mixed Hodge structures.

(2.3.6) Definition. *The groupoid*

$$[\ \ B\text{-}{}^{K}\mathbf{X}\text{-}L\text{-}\boldsymbol{loc\text{-}mhs}\ \ ]$$

*is the category of data as above, where in addition we have a K-**level-structure**, i.e. a section ξ of the quotient sheaf*

$$\underline{\mathrm{Iso}}(L_{\mathbb{A}^{(\infty)}}, M \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)})/K.$$

Morphisms are isomorphisms as above, respecting K-level-structures.

(2.3.7) Theorem.

$$B \mapsto [\quad B\text{-}\mathbf{X}\text{-}L\text{-}\boldsymbol{filt}\quad]$$

is isomorphic to $(P_{\mathbf{X},\mathbb{C}}/Q_{\mathbb{C}})^{an}$, where $Q = \mathrm{qpar}(u_x)$ is the stabilizer of a Hodge filtration associated with some $h_x, x \in \mathbb{D}_{\mathbf{X}}$. $(P_{\mathbf{X},\mathbb{C}}/Q_{\mathbb{C}})^{an}$ is the analytic manifold associated with the fibre of type *over* type(Q).

Proof. This will be proven in the algebraic context in (3.2.2). The proof in the analytic case is analogous. ☐

2.4. Boundary components

(2.4.1) Definition. *Let $\mathbf{X}$ be (rational) mixed Shimura data. For $G_{\mathbf{X},\mathbb{Q}} := P_{\mathbf{X},\mathbb{Q}}/W_{\mathbf{X},\mathbb{Q}}$ every $\mathbb{Q}$-parabolic subgroup of $P_{\mathbf{X},\mathbb{Q}}$ is the inverse image of a $\mathbb{Q}$-parabolic subgroup of $G^{ad}_{\mathbf{X},\mathbb{Q}}$. Let $G^{ad}_{\mathbf{X},\mathbb{Q}} = G_{1,\mathbb{Q}} \times \cdots G_{r,\mathbb{Q}}$ be the decomposition into $\mathbb{Q}$-simple factors. Choose $\mathbb{Q}$-parabolic subgroup $Q_{i,\mathbb{Q}} \subseteq G_{i,\mathbb{Q}}$ for every i and let $Q_{\mathbb{Q}}$ be the inverse image of $Q_{1,\mathbb{Q}} \times \cdots Q_{r,\mathbb{Q}}$ in $P_{\mathbf{X},\mathbb{Q}}$. We call $Q_{\mathbb{Q}}$ an **admissible** $\mathbb{Q}$-parabolic subgroup of $P_{\mathbf{X},\mathbb{Q}}$, if every $Q_{i,\mathbb{Q}}$ is either equal to $G_{i,\mathbb{Q}}$ or a maximal proper $\mathbb{Q}$-parabolic subgroup of $G_{i,\mathbb{Q}}$.*

(2.4.2) Lemma. *If $\mathbf{X}$ is p-integral mixed Shimura data, and $Q_{\mathbb{Q}}$ a parabolic subgroup of $P_{\mathbf{X},\mathbb{Q}}$, then there is a parabolic subgroup scheme Q of $P_{\mathbf{X}}$ (1.9.3), such that $Q_{\mathbb{Q}} = Q \times_S \mathrm{spec}(\mathbb{Q})$.*

Proof. According to (1.9.6) the functor of parabolic subgroups for $P_{\mathbf{X}}$ is representable by a *projective* scheme $\mathcal{PAR}$ over S. Hence $\mathcal{PAR}(\mathbb{Z}_{(p)}) = \mathcal{PAR}(\mathbb{Q})$. ☐

(2.4.3) Let S be a maximal $\mathbb{R}$-split torus of $G_{\mathbb{R}}$. Let R be the root system of S (acting on Lie(G^{ad})). The irreducible components of R are of type (C) or (BC) [23, Corollaire 3.1.7]. In every irreducible component, the set of *long roots* [loc. cit.] $\{\alpha_1, \ldots, \alpha_g\}$ forms an ordered set of mutually orthogonal roots. A maximal proper $\mathbb{R}$-parabolic of the corresponding $\mathbb{R}$-simple factor is determined by a homomorphism $\lambda : \mathbb{G}_{m,\mathbb{R}} \hookrightarrow S$, where

$$\langle \lambda, \alpha_i \rangle = \begin{cases} 2 & i \leq s, \\ 0 & i > s. \end{cases}$$

The parabolic is defined over $\mathbb{Q}$, iff the corresponding cocharacter can be defined over $\mathbb{Q}$. It is a defining cocharacter for the parabolic $Q_{\mathbb{Q}}$ in the sense of (1.9.4), i.e. we have $Q_Q = \mathrm{par}(\lambda)$.

(2.4.4) Theorem. *Let $Q_{\mathbb{Q}}$ be a $\mathbb{Q}$-parabolic subgroup of $P_{\mathbf{X},\mathbb{Q}}$. Let π' be the projection $P_{\mathbf{X}} \to P_{\mathbf{X}}/U_{\mathbf{X}}$. The following are equivalent:*

i. $Q_{\mathbb{Q}}$ is admissible

ii. For every $x \in \mathbb{D}_{\mathbf{X}}$ there is a unique homomorphism

$$\omega_x : H_{0,\mathbb{C}} \to P_{\mathbf{X},\mathbb{C}}$$

such that

a) $\pi' \circ \omega_x : H_{0,\mathbb{C}}$ is already defined over $\mathbb{R}$,

b) $h_x = \omega_x \circ h_0$,

c) $\omega_x \circ h_\infty \circ w : \mathbb{G}_m \to Q_{\mathbb{C}}$ is of the form $\mu \cdot \lambda$, where λ is the morphism constructed in (2.4.3), $\mu = h_x \circ w$ and $\mathrm{Lie}(Q_{\mathbb{C}})$ is the direct sum of all nonnegative weight spaces in $\mathrm{Lie}(P_{\mathbf{X},\mathbb{C}})$ under $\mathrm{Ad}_P \circ \omega_x \circ h_\infty \circ w$.

iii. There exists an $x \in \mathbb{D}_{\mathbf{X}}$ and a homomorphism ω_x such that the three conditions in ii. are satisfied.

Proof. [83, Prop. 4.6]. There, it is claimed that λ in (c) has to be of the form described in [loc. cit., 4.1]. This has to be corrected as above. Furthermore the morphism λ_1 in the proof of [loc. cit., 4.6], defined in [2, Theorem 2, p.205], is the morphism $\mathbb{G}_m \to H_1$, $\alpha \mapsto (1, \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix})$. λ is only conjugated within its corresponding Borel of H_1 to $h_\infty \circ w$ of [loc. cit., 4.2]. Hence both cocharacters $\omega_x \circ h_\infty \circ w$ and $\omega_x \circ \lambda_1$, when projected to G^{ad}, define the projection of $Q_{\mathbb{C}}$. The second, however, is defined over $\mathbb{Q}$. There is a lift of λ_1, i.e. λ as in 2.4.3, to G^{der} because it extends to a morphism of SL_2 (which is algebraically simply connected). $\square$

If $P_{\mathbf{X}}$ is reductive. Q corresponds to a boundary component in the sense of [2, III, p. 220, no. 2]. The morphism $\omega_x \circ h_\infty$ is independent of the choice h_0.

(2.4.5) Definition/Theorem. *Assume that $\mathbf{X}$ is p-MSD. Choose an admissible $Q_{\mathbb{Q}}$ as above. We will define mixed Shimura data $\mathbf{B}$ as follows:*
Let $P_{\mathbf{B},\mathbb{Q}}$ be the smallest normal $\mathbb{Q}$-subgroup of Q, such that $\omega_x \circ h_\infty$ factorizes through it.

Consider the map

$$\begin{aligned} \mathbb{D}_{\mathbf{X}} &\to \pi_0(\mathbb{D}_{\mathbf{X}}) \times \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{B},\mathbb{C}}) \\ x &\mapsto [x], \omega_x \circ h_\infty \end{aligned} \tag{1}$$

Choose a $P_{\mathbf{B}}(\mathbb{R})U_{\mathbf{B}}(\mathbb{C})$*-orbit* $\mathbb{D}_{\mathbf{B}}$ *containing an* $([x], \omega_x \circ h_\infty)$ *in the above image. The image is contained in the union of finitely many such. Each* $\mathbb{D}_{\mathbf{B}}$ *is a finite covering of the corresponding* $P_{\mathbf{B}}(\mathbb{R})U_{\mathbf{B}}(\mathbb{C})$*-orbit* $h(\mathbb{D}_{\mathbf{B}})$ *in* $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{B},\mathbb{C}})$*. Let* $\mathbb{D}_{\mathbf{B}\Longrightarrow\mathbf{X}}$ *be the inverse image*[1] *of* $\mathbb{D}_{\mathbf{B}}$ *by the map (1).*
The closure $P_{\mathbf{B}}$ *of* $P_{\mathbf{B},\mathbb{Q}}$ *in* $P_{\mathbf{X}}$ *is of type* (P) *and hence* $\mathbf{B}$ *is p-integral mixed Shimura data and called a* ***boundary component*** *of* $\mathbf{B}$*. It is called* ***proper****, if* Q *is a proper parabolic, otherwise* ***improper****.*
A ***boundary map*** $\mathbf{B} \Longrightarrow \mathbf{X}$ *consists of a closed embedding* $P_{\mathbf{B}} \hookrightarrow P_{\mathbf{X}}$*, whose image is one of the groups defined above, and an* $P_{\mathbf{B}}(\mathbb{R})U_{\mathbf{B}}(\mathbb{C})$*-equivariant isomorphism of* $\mathbb{D}_{\mathbf{B}}$ *with the spaces above.*

For each Q there are finitely many choices of $\mathbb{D}_{\mathbf{B}}$'s, and accordingly, finitely many rational boundary components.

Proof. According to (2.4.2) there is a parabolic subgroup scheme Q of $P_{\mathbf{X}}$, whose generic fibre is $Q_{\mathbb{Q}}$. It is closed by (1.9.4). Let $W_{\mathbf{X}} \rtimes G_{\mathbf{X}}$ a decomposition of $P_{\mathbf{X}}$ into reductive (resp. normal unipotent) closed subgroup schemes, which exists by (1.8.11). Q is of the form $W_{\mathbf{X}} \rtimes Q'$, for a parabolic Q' of $G_{\mathbf{X}}$, and of type (P) by (1.9.6). $G_{\mathbf{B},\mathbb{Q}} \cap Q'_{\mathbb{Q}}$ is normal in $G_{\mathbf{X},\mathbb{Q}}$, hence its closure $G_{\mathbf{B}}$ is reductive (1.6.9). The unipotent radical of $P_{\mathbf{B},\mathbb{Q}}$ is $W_{\mathbf{B},\mathbb{Q}} = P_{\mathbf{B},\mathbb{Q}} \cap W_{Q,\mathbb{Q}} = \exp(W_{-1}(\mathrm{Lie}(W_{Q,\mathbb{Q}})))$ [83, proof of lemma 4.8, p. 77], $W_{-1}(\mathrm{Lie}(W_{Q,\mathbb{Q}}))$ is the union of the $\mathrm{Lie}(W_Q)^r$ such that $r \circ w = -1$. There exist closed embeddings $\exp_r : \mathbb{W}(\mathrm{Lie}(W_Q))^r \to W_Q$, by definition of type (P). Since the product over *all* closed embeddings $\exp_r$, $r \in R_{W_Q}$ is an isomorphism onto W_Q (in any order — 1.8.6), the product map

$$\prod_{r \in R_{W_Q}, r\circ w=-1} \mathbb{W}(\mathrm{Lie}(W_Q)^r) \to W_Q$$

is a closed embedding. The generic fibre of this embedded subscheme $W_{\mathbf{B}}$ is the sub*group* $W_{\mathbf{B},\mathbb{Q}}$, hence $W_{\mathbf{B}}$ which must be the closure of $W_{\mathbf{B},\mathbb{Q}}$ is a subgroup scheme. Since $P_{\mathbf{B},\mathbb{Q}}$ is the semi-direct product of $P_{\mathbf{B},\mathbb{Q}} \cap Q'_{\mathbb{Q}}$ and $P_{\mathbf{B},\mathbb{Q}} \cap W_{\mathbf{X},\mathbb{Q}} = \exp(W_{-1}(\mathrm{Lie}(Q_{\mathbb{Q}})))$ [loc. cit.], the semi-direct product $W_{\mathbf{B}} \rtimes G_{\mathbf{B}}$ is the closure of $P_{\mathbf{B},\mathbb{Q}}$. Claim: It is of type (P). It remains to show on the one hand that we can 'group together' the closed embeddings $\exp_r$ for roots that become the same root r' for $P_{\mathbf{B},\mathbb{Q}}$. But for them we have already a closed embedding

$$\prod_{r, r\mapsto r'} \mathbb{W}(\mathrm{Lie}(W_{\mathbf{B}})^r) \to W_{\mathbf{B}}.$$

[1]in [83], this is called X^+

The generic fibre of the embedded subscheme is a *commutative* group [loc. cit.], so the morphism has to be a morphism of group schemes. On the other hand, we have to show that there is a decomposition $R_{P_{\mathbf{B},\mathbb{Q}}} = R_{W_{\mathbf{B},\mathbb{Q}}} \dot\cup R_{G_{\mathbf{B},\mathbb{Q}}}$, and pairs of roots in $R_{P_{\mathbf{B},\mathbb{Q}}} \times R_{W_{\mathbf{B},\mathbb{Q}}}$ are pairwise linearly independent, but it suffices to check this over $\mathbb{C}$, where we have some morphism $h_x : \mathbb{S} \to T \subset P_{\mathbf{B},\mathbb{Q}}$ and $h_x^*(r_i) \in \{(-1,0),(0,-1),(-2,-2)\}$ for $r \in R_{W_{\mathbf{B},\mathbb{Q}}}$ and $h_x^*(r_i) \in \{(-1,1),(0,0),(1,-1)\}$ for $r \in R_{G_{\mathbf{B},\mathbb{Q}}}$. Hence the two set are disjoint and two different r_i can only be linearly dependent if they are both in $R_{G_{\mathbf{B},\mathbb{Q}}}$. □

(2.4.6) Definition. *There is a functorial map, called* ***projection on the imaginary part****,*

$$\mathrm{im} : \mathbb{D}_{\mathbf{X}} \to U_{\mathbf{X}}(\mathbb{R})(-1) \qquad x \mapsto u_x,$$

where u_x is the unique element, such that $\mathrm{int}(u_x^{-1}) \circ h_x$ is defined over $\mathbb{R}$.

cf. [83, 4.14].

(2.4.7) Theorem. *If $\mathbf{B}$ is a p-integral boundary component of $\mathbf{X}$, and $\alpha : \mathbf{X} \to \mathbf{X}'$ is a morphism, there is a unique p-integral boundary component $\mathbf{B}'$ of $\mathbf{X}'$ and a corresponding map $\widetilde{\alpha} : \mathbf{B} \to \mathbf{B}'$. If α is an embedding, $\widetilde{\alpha}$ either.*

Proof. [83, 4.16] □

(2.4.8) Theorem. *i. Let $\mathbb{D}_{\mathbf{B}\Longrightarrow\mathbf{X}}$ be as in definition (2.4.5). Let $\mathbb{D}_{\mathbf{X}}^0$ be a connected component of $\mathbb{D}_{\mathbf{B}\Longrightarrow\mathbf{X}}$ and $\mathbb{D}_{\mathbf{B}}^0$ be corresponding component of $\mathbb{D}_{\mathbf{B}}$. Then*

a) The map $\mathbb{D}_{\mathbf{B}\Longrightarrow\mathbf{X}} \to \mathbb{D}_{\mathbf{B}}$ is an open embedding.

b) The image of $\mathbb{D}_{\mathbf{X}}^0$ in $\mathbb{D}_{\mathbf{B}}^0$ is the inverse image of an open complex cone $C := C(\mathbb{D}_{\mathbf{X}}^0, P_{\mathbf{B}}) \in U_{\mathbf{B}}(\mathbb{R})(-1)$ under the map $\mathrm{im}\,|_{\mathbb{D}_{\mathbf{B}}^0}$

c) The cone is an orbit in $U_{\mathbf{B}}(\mathbb{R})(-1)$ under translation by $U_{\mathbf{X}}(\mathbb{R})(-1)$ and conjugation by $Q(\mathbb{R})^\circ$.

It is also invariant under translation by $(U_{\mathbf{B}} \cap W_{\mathbf{X}})(\mathbb{R})(-1)$.

d) Modulo $(U_{\mathbf{B}} \cap W_{\mathbf{X}})(\mathbb{R})(-1)$ the cone C is a non-degenerate homogeneous self adjoint cone (in the sense of [2, II, p. 57, §1.1]).

ii. Consider a morphism of Shimura data $\iota : \mathbf{X} \to \mathbf{X}'$. For each rational boundary component $\mathbf{B}$ of $\mathbf{X}$ there is a unique boundary component $\mathbf{B}'$ of $\mathbf{X}'$ and a morphism $\widetilde{\iota} : \mathbf{B} \to \mathbf{B}'$ such that

$$\begin{tikzcd} \mathbb{D}_{\mathbf{B}\Longrightarrow\mathbf{X}} \arrow[r, "\iota"] \arrow[d, hook] & \mathbb{D}_{\mathbf{B}'\Longrightarrow\mathbf{X}'} \arrow[d, hook] \\ \mathbb{D}_{\mathbf{B}} \arrow[r, "\widetilde{\iota}"] & \mathbb{D}_{\mathbf{B}'} \end{tikzcd}$$

commutes.

iii. *Each boundary component* $\mathbf{B}'$ *of* $\mathbf{B}$ *is naturally a boundary component of* $\mathbf{X}$*. This defines a partial order on the set of boundary components of* $\mathbf{X}$*.*

Proof. [83, 4.15, 4.16] □

(2.4.9) Definition. *Let* $\mathbf{B}_1$ *be a rational boundary component of* $\mathbf{X}$ *and* $\mathbb{D}^0_{\mathbf{X}}$ *be a connected component of* $\mathbb{D}_{\mathbf{B}_1 \Longrightarrow \mathbf{X}}$*. Let* $C^*(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}_1}) \subset U_{\mathbf{B}_1}(\mathbb{R})(-1)$ *denote the union of the cones* $C(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}_2})$ *for all rational boundary components* $\mathbf{B}_2$ *such that* $\mathbf{B}_1 \Longrightarrow \mathbf{B}_2 \Longrightarrow \mathbf{X}$*. It is a convex cone. The following quotient*

$$C_{\mathbf{X}} := \coprod_{(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}_1})} C^*(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}_1}) / \sim$$

by the equivalence relation generated by the graph of all embeddings $C^*(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}_2}) \hookrightarrow C^*(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}_1})$ *for* $\mathbf{B}_1 \Longrightarrow \mathbf{B}_2 \Longrightarrow \mathbf{X}$*. It is called the* ***conical complex*** *associated with* $\mathbf{X}$*.*

cf. [83, 4.24].
Consider the set

$$C_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)}).$$

$P_{\mathbf{X}}(\mathbb{Q})$ acts on this from the left by conjugation of boundary components [83, 4.23] and on $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ by left multiplication. $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ acts via multiplication on the right on the second factor. These actions are denoted by $p\cdot$ and $\cdot p$ respectively. Furthermore $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ acts on the second factor through left multiplication. This action is denoted by $p_f\cdot$.
Let a set Δ of subsets of $C_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ be given, such that every $\sigma \in \Delta$ is contained in some $C(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}}) \times \rho$. We denote by $\Delta(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}}, \rho)$ the subset of $\sigma \in \Delta$, such that $\sigma \subset \overline{C(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}}) \times \rho}$.

(2.4.10) Definition. Δ *is called (finite)* K***-admissible (partial) rational polyhedral cone decomposition*** *for* $\mathbf{X}$*, if*

i. *For each* $\mathbb{D}^0_{\mathbf{X}}, \mathbf{B}$ *and* ρ*,* $\Delta(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}}, \rho)$ *is a (partial) rational polyhedral cone decomposition of the closure of* $C(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}}) \times \rho$*.* ***We understand a cone as open in its closure***[2]*.*

ii. Δ *is invariant under right multiplication by* K *and under left multiplication by* $P_{\mathbf{X}}(\mathbb{Q})$ *(with finite quotient* $P_{\mathbf{X}}(\mathbb{Q}) \backslash \Delta / K$*).*

[2]Hence our cones correspond to the *interiors of cones* in [83]

iii. *For each* $\mathbf{B}$ *the set* $\bigcup_{\rho \in P_{\mathbf{X}}(\mathbb{A}^{(\infty)})} \Delta(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}}, \rho)$ *is invariant under left multiplication by* $P_{\mathbf{B}}(\mathbb{A}^{(\infty)})$.

It is called ***complete****, if in (i)* $\Delta(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}}, \rho)$ *is a complete rational polyhedral cone decomposition.*
It is called ***projective****, if on each* $\Delta(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}}, \rho)$ *there exists a polarization function ([2, IV, §2.1], cf. [27, IV, 2.4]).*

The condition (iii) is called the arithmeticity condition. Without it, the compactification exists over $\mathbb{C}$ but may not descend to the reflex field or a reflex ring.
Let K be an (admissible) compact open subgroup of $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$. For some fixed $\mathbb{D}^0_{\mathbf{X}}$, $\mathbf{B}$ and ρ, we let $\Gamma_U \subset U_{\mathbf{B}}$ be the image of

$$(\{z \in Z(P_{\mathbb{X}}(\mathbb{Q})) \mid z|_{\mathbb{D}_{\mathbf{X}}} = \mathrm{id}\} U_{\mathbf{B}}(\mathbb{Q})) \cap {}^{\rho}K).$$

We call Δ **smooth** with respect to K, if for all $\mathbb{D}^0_{\mathbf{X}}$, $\mathbf{B}$ and ρ, as above, $\Delta(\mathbb{D}^0, P_{\mathbf{B}}, \rho)$ is smooth with respect to the lattice Γ_U.

(2.4.11) Definition. *A triple* ${}^K_\Delta\mathbf{X}$ *is called* ***p-integral extended compactified mixed Shimura data (p-ECMSD)****, where everything is as in definition (2.2.4), but* Δ *is in addition a* K*-admissible (partial) rational polyhedral cone decomposition.*
Morphisms of p-ECMSD *have to satisfy the property that for each* $\sigma_1 \in \Delta_1$ *there is a* $\sigma_2 \in \Delta_2$ *with* $\gamma(\sigma_1)^\rho \subset \sigma_2$.
Let ${}^K_\Delta\mathbf{X}$ *be p-ECMSD, and* $[\alpha, \rho] : {}^{K'}\mathbf{Y} \to {}^K\mathbf{X}$ *be a morphism of p-EMSD, such that* α *is a* closed embedding.
Set $[\alpha, \rho]^*\Delta$ *to be the set of all cones* $\{(u, \rho') \mid (\alpha(u), \alpha(\rho')\rho) \in \sigma\}$ *for all* $\sigma \in \Delta$*. This is a* K'*-admissible rational partial cone decomposition for* $\mathbf{Y}$*. It is finite, resp. complete, resp. projective if* Δ *is finite, resp. complete, resp. projective.*
This association is functorial. If $[\alpha, \rho]$ *was an embedding (this includes a condition on* K, K'*, see 2.2.4), we call* ${}^{K'}_{\Delta'}\mathbf{Y} \to {}^K_\Delta\mathbf{X}$ *an* ***embedding****.*

cf. [83, 6.5] for the special case of an automorphism of $\mathbf{X}$ or a Hecke operator.
Be aware that, in general, smoothness is not inherited by $[\alpha, \rho]^*\Delta$.

(2.4.12) Theorem. *Let* $\mathbf{X}$ *be p-MSD. Let* K *be an admissible compact open.*

i. K*-admissible (complete) rational polyhedral cone decompositions for* $\mathbf{X}$ *exist.*

ii. *If* Δ *is a* K*-admissible rational polyhedral cone decomposition for* $\mathbf{X}$*, there is a smooth and projective refinement* Δ'*.* Any *refinement of* Δ' *will be projective again.*

iii. *If* Δ_i*,* $i = 1, 2$ *are 2 rational polyhedral cone decompositions for* $\mathbf{X}$*, there is a common refinement* Δ *(supported on the intersection of their supports).*

iv. *If $\alpha : \mathbf{X} \to \mathbf{X}'$ is an embedding and $\rho \in P_{\mathbf{X}'}(\mathbb{A}^{(\infty,p)})$ is given, there is K' such that we have an embedding $[\alpha, \rho] : {}^K\mathbf{X} \to {}^{K'}\mathbf{X}'$, a K'-admissible rational polyhedral smooth and projective cone decomposition Δ' for $\mathbf{X}'$, with the property that $\Delta := [\alpha, \rho]^*\Delta'$ is smooth and projective. For* every *smooth refinement of $\widetilde{\Delta}$ of Δ, there is a smooth refinement $\widetilde{\Delta}'$ of Δ' with $\widetilde{\Delta} := [\alpha, \rho]^*\widetilde{\Delta}'$.*

Proof. (i)-(iii) is shown in [83], proofs can also be found in [48], [2] (cf. also [27, p. 97] for the case $\mathbf{X} = \mathbf{H}_g$). We give a sketch of a proof of (iii): K' can be chosen small enough, such that $[\alpha, \rho]$ induces an embedding

$$P_{\mathbf{X}}(\mathbb{Q})\backslash C_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K \to P_{\mathbf{X}'}(\mathbb{Q})\backslash C_{\mathbf{X}'} \times P_{\mathbf{X}'}(\mathbb{A}^{(\infty)})/K'.$$

Locally the map looks like an embedding $U_{\mathbf{B},\mathbb{R}} \subset U_{\mathbf{B}',\mathbb{R}}$, for compatible boundary components $\mathbf{B} \Longrightarrow \mathbf{X}$ and $\mathbf{B}' \Longrightarrow \mathbf{X}'$, and we may cut all cones of some smooth projective K'-admissible rational polyhedral cone decomposition Δ with these linear subspaces. This creates a new K'-admissible rational polyhedral cone decomposition Δ' containing the remaining pieces, as well as the cutted ones, supported on $C_{\mathbf{X}}$. Δ' will not be smooth, of course, but there will be a smooth refinement Δ''. Because Δ'' is a refinement of Δ', $\Delta''' := [\alpha, \rho]^*\Delta''$ will also be smooth. By induction on the codimension of $C_{\mathbf{X}}$ in $C_{\mathbf{X}'}$, we see that any (smooth) refinement of Δ''' can be extended to a (smooth) refinement of Δ'', by cutting out appropriate simplices. □

(2.4.13) Definition. *Let ${}^K_\Delta\mathbf{X}$ be p-ECMSD, and $\iota : \mathbf{B} \Longrightarrow \mathbf{X}$ a boundary map. For any $\rho \in P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$, we define $K' := P_{\mathbf{B}}(\mathbb{A}^{(\infty)}) \cap {}^\rho K$, write*

$$(\iota, \rho) : {}^{K'}_{\Delta'}\mathbf{B} \Longrightarrow {}^K_\Delta\mathbf{X}$$

and call this a boundary component of (or boundary morphism to) the p-ECMSD ${}^K_\Delta\mathbf{X}$. Δ' is defined as $([\iota, \rho]^*\Delta)|_{\mathbf{B}}$, where restriction is characterized by

$$\Delta|_{\mathbf{B}_1}(\mathbb{D}^0_{\mathbf{B}_1}, P_{\mathbf{B}_2}, \rho_1) = \Delta(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}_2}, \rho_1)$$

for all $\rho_1 \in P_{\mathbf{B}_1}(\mathbb{A}^{(\infty)})$, every boundary map $\mathbf{B}_2 \Longrightarrow \mathbf{B}_1$ and every pair of connected components $\mathbb{D}^0_{\mathbf{X}}$ and $\mathbb{D}^0_{\mathbf{B}_1}$ such that $\mathbb{D}^0_{\mathbf{X}} \hookrightarrow \mathbb{D}^0_{\mathbf{B}_1} \hookrightarrow \mathbb{D}_{\mathbf{B}_2}$. Δ' in general inherits neither completeness nor finiteness. It is K'-admissible.
We call two boundary components

$$(\iota', \rho') : {}^{K'}_{\Delta'}\mathbf{B}' \Longrightarrow {}^K_\Delta\mathbf{X}$$

and

$$(\iota'', \rho'') : {}^{K''}_{\Delta''}\mathbf{B}'' \Longrightarrow {}^K_\Delta\mathbf{X}$$

***equivalent**, if (the images of) $\mathbf{B}'$ and $\mathbf{B}''$ are conjugated via $\alpha \in P_{\mathbf{X}}(\mathbb{Q})$ and*

$$\alpha\rho' \in \mathrm{Stab}_{Q(\mathbb{Q})}(\mathbb{D}_{\mathbf{B}})P_{\mathbf{B}}(\mathbb{A}^{(\infty)})\rho''K.$$

(Here Q is the parabolic defining $\mathbf{B}$.) In an equivalence class, we may assume $\rho \in P_{\mathbf{X}}(\mathbb{A}^{(\infty,p)})$.

cf. [83, 6.5].

(2.4.14) Definition. *Let ${}^K_\Delta\mathbf{X}$ be p-ECMSD. Define Δ^0 as the set of all $\sigma \in \Delta$, such that $\sigma \subset C(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}}) \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ for some* improper *rational boundary component $\mathbf{B} \Longrightarrow \mathbf{X}$. We say that Δ is* ***concentrated in the unipotent fibre****, if $\Delta = \Delta^\circ$.*

2.5. The symplectic mixed Shimura data

(2.5.1) Let S be a scheme. Let L be a locally free sheaf on S with non-degenerate alternating form $\langle v, w\rangle = \Psi(v, w)$, i.e. satisfying $\langle v, v\rangle = 0$ and such that the induced homomorphism $L \to L^*$ is an isomorphism.
Let us assume first that L is *nonzero*. Define the following group functors

$$\begin{aligned} \mathrm{Sp}(L)(S') &:= \{\gamma \in \mathrm{GL}(L \otimes \mathcal{O}_{S'}) \mid \langle \gamma v, \gamma v\rangle = \langle v, v\rangle\}, \\ \mathrm{GSp}(L)(S') &:= \{\gamma \in \mathrm{GL}(L \otimes \mathcal{O}_{S'}) \mid \langle \gamma v, \gamma v\rangle = \lambda(\gamma)\langle v, v\rangle \text{ for a } \lambda(\gamma) \in H^0(S', \mathcal{O}^*_{S'})\}. \end{aligned}$$

They are representable, reductive and called the **symplectic group**, resp. the **group of symplectic similitudes** of L. λ defines a homomorphism $\mathrm{GSp}(L) \to \mathbb{G}_{m,S}$.
There is an exact sequence

$$0 \longrightarrow \mathrm{Sp}(L) \longrightarrow \mathrm{GSp}(L) \xrightarrow{\lambda} \mathbb{G}_m \longrightarrow 0.$$

Let ρ denote the standard representation of $\mathrm{GSp}(L)$ on L.
The Lie algebra of $\mathrm{Sp}(L)$ is identified with

$$\mathrm{Lie}(\mathrm{Sp}(L)) = (L \otimes L)^s$$

in the way that $X = v \otimes w$ acts as

$$v \mapsto X\Psi z := \langle v, z\rangle w.$$

We define p-integral pure Shimura data $\mathbf{H}_g$ associated with L (it depends, up to isomorphism, only on the rank $2g$ of L) by $P_{\mathbf{H}_g} := \mathrm{GSp}(L)$ and by $\mathbb{D}_{\mathbf{H}_g}$ to be *the* conjugacy class of morphisms $h : \mathbb{S} \to \mathrm{GSp}(L_{\mathbb{R}})$, such that they give pure Hodge structures of type $(-1, 0), (0, -1)$ on $L_{\mathbb{C}}$ and which are **polarized**, i.e. such that the form $\langle \cdot, h(i)\cdot\rangle$ is symmetric and (positive or negative) definite.
If L is the *zero* sheaf on S we simply define $\mathrm{Sp}(L) := 1$ and $\mathrm{GSp}(L) := \mathbb{G}_{m,S}$. And we let $\mathbb{D}_{\mathbf{H}_0}$ be the 2 point set of isomorphisms $\mathbb{Z} \to \mathbb{Z}(1)$ with the nontrivial action of $\mathbb{G}_m(\mathbb{R})$. This defines a (p-integral) Shimura datum $\mathbf{H}_0$. Let us understand the morphism $\lambda : \mathrm{GSp}(0) \to \mathbb{G}_m$ be the identity.

(2.5.2) Let L_0 be a locally free sheaf on S as before (possibly 0).
Let I be another locally free sheaf on S (also possibly 0) with actions of $\mathrm{GSp}(L_0)$ given by λ acting by scalars. Let I^* be the dual with *trivial* action of $\mathrm{GSp}(L_0)$. We define the semi-direct product

$$\mathrm{PSp}(L_0, I) := \mathbb{W}(L_0 \otimes I) \rtimes \mathrm{GSp}(L_0).$$

It is of type (P).
It acts on $L := L_0 \oplus I^*$ as follows: The action of GSp is given by the standard representation on L_0 and trivial action on I^*, $X = v' \otimes u' \in L_0 \otimes I$ acts considered as Lie algebra via

$$X(v, u^*) = ((u^* u)v', 0).$$

$\mathbb{W}(L_0 \otimes I)$ acts then via the exponential $\exp(X)(v, u^*) = (v, u^*) + X(v, u^*)$. In this case this works even over $\mathrm{spec}(\mathbb{Z})$. This is compatible with the structure of semi-direct product.
The action fixes a weight filtration

$$W_i(L_0 \oplus I^*) := \begin{cases} 0 & i \le -2 \\ L_0 & i = -1 \\ L_0 \oplus I^* & i \ge 0. \end{cases}$$

We have the unipotent extension $\mathbf{H}_{g_0}[0, I \otimes L_0]$ of the p-integral pure Shimura data $\mathbf{H}_{g_0}$. Its underlying P is $\mathrm{PSp}(L_0, I)$. Its underlying $\mathbb{D}$ (if $g \neq 0$ may be identified with *the* conjugacy class of morphisms $h : \mathbb{S} \to \mathrm{PSp}(L_{0,\mathbb{R}}, I_{\mathbb{R}})$, such that they give mixed Hodge structures of type $(-1, 0), (0, -1), (0, 0)$ with respect to the weight filtration W, such that on W_{-1} they are polarized.
(Of course $\mathrm{PSp}(0, I) = \mathrm{GSp}(0)$ and $\mathrm{PSp}(L_0, 0) = \mathrm{GSp}(L_0)$).

(2.5.3) Let I, L_0 be as before (possibly 0).
Consider the following extension of Abelian unipotent groups

$$0 \longrightarrow \mathbb{W}((I \otimes I)^s) \longrightarrow \mathrm{WSp} \longrightarrow \mathbb{W}(L_0 \otimes I) \longrightarrow 0$$

defined (if 2 is invertible in S) by the following group law (on $\mathbb{W}((I \otimes I)^s \oplus L_0 \otimes I)$):

$$(u_1 u_2, v u_3)(u_1' u_2', v' u_3') = (u_1 u_2 + u_1' u_2' + \mathbf{\frac{1}{2}} \langle \mathbf{v}, \mathbf{v'} \rangle (\mathbf{u_3 u_3'} + \mathbf{u_3' u_3}), v u_3 + v' u_3').$$

There is an action of $\mathrm{GSp}(L_0)$ on WSp given by λ acting on $(I \otimes I)^s$ by scalars and standard representation tensored with the trivial one on $L_0 \otimes I$.
We define the semi-direct product

$$\mathrm{USp}(L_0, I) := \mathrm{WSp} \rtimes \mathrm{GSp}(L_0).$$

It is again of type (P).

Denote $\mathfrak{I} := I \oplus I^*$. Choose on $L_0 \oplus \mathfrak{I}$ the symplectic form

$$\langle v_1, u_1, u_1^*; v_2, u_2, u_2^* \rangle := \langle v_1, v_2 \rangle + u_2^* u_1 - u_1^* u_2.$$

We define an action of $\mathrm{USp}(L_0, I)$ on $L := L_0 \oplus \mathfrak{I}$ as follows: The action of $\mathrm{GSp}(L_0)$ is given by the standard representation on L_0, trivial representation on I^* and λ acting by scalars on I. $X = v' \otimes u' \in L_0 \otimes I)$ acts as Lie algebra via

$$X(v, u, u^*) = ((u^* u') v', \langle v', v \rangle u', 0)$$

and $X = u_1 \otimes u_2 \in I \otimes I$ acts by

$$X(v, u, u^*) = (0, (u^* u_1) u_2, 0)$$

and $\mathbb{W}(\cdots)$ acts via the exponential $\exp(X)(v, u^*) = (v, u^*) + X(v, u^*) + \frac{1}{2} X^2(v, u^*)$. This is compatible with the group structure given above[3].
If 2 is not invertible in S we assume that there is an isomorphism $L_0 \cong L_{00} \oplus L_{00}^*$, such that the alternating form is given by the standard one. In this case we let the groups $\mathbb{W}(L_{00} \otimes I)$, $\mathbb{W}(L_{00}^* \otimes I)$ and $\mathbb{W}((I \otimes I)^s)$ acts as above via exponential (it terminates after the second step). One checks that

$$\mathbb{W}((I \otimes I)^s) \times \mathbb{W}(L_{00} \otimes I) \times \mathbb{W}(L_{00}^* \otimes I) \to \mathrm{GSp}(L)$$

is a closed embedding onto a subgroup scheme. Explicitly the group law is given by

$$(X_1, X_2, X_3)(X_1', X_2', X_3') = (X_1 + X_1' + \langle \mathbf{X_3}, \mathbf{X_2'} \rangle - \langle \mathbf{X_2'}, \mathbf{X_3} \rangle, X_2 + X_2', X_3 + X_3').$$

(here $\langle X_2', X_3 \rangle = -(\langle X_3, X_2' \rangle)^s$, hence the expression is symmetric). If $\dim(I) = 1$ and $2 = 0$ in S, then this group is commutative.
This is in any case compatible with the structure of semi-direct product. The action fixes a weight filtration

$$W_i(L) := \begin{cases} 0 & i \le -3 \\ I & i = -2 \\ L_0 \oplus I & i = -1 \\ L & i \ge 0 \end{cases}$$

We have the unipotent extension $\mathbf{H}_{g_0}[(I \otimes I)^s, I \otimes L_0]$ of the p-integral pure Shimura data $\mathbf{H}_{g_0}$. Its underlying P is $\mathrm{USp}(L_0, I)$. Its underlying $\mathbb{D}$ (if $g \neq 0$ may be identified with

[3] The commutator of all elements except two from $L_0 \otimes I$ is zero, there it is

$$[u_1 v_1, u_2 v_2] = \langle v_1, v_2 \rangle (u_1 u_2 + u_2 u_1)$$

From

$$\exp(A) \exp(B) \equiv \exp(A + B + \frac{1}{2}[A, B]) \quad \mod \deg 2$$

follows that the given representation is compatible with the foregoing definition of multiplication in WSp.

the conjugacy class of morphisms $h : \mathbb{S} \to \mathrm{USp}(L_{0,\mathbb{R}}, I_{\mathbb{R}})$, such that they give (polarized) mixed Hodge structures on $L_{\mathbb{R}}$ of type $(-1,-1), (-1,0), (0,-1), (0,0)$ with respect to the weight filtration above.
The action of $\mathrm{USp}(L_0, I)$ on $L = L_0 \oplus \mathfrak{I}$ defined above induces a closed embedding

$$\mathrm{USp}(L_0, I) \hookrightarrow \mathrm{GSp}(L).$$

The image is precisely the subgroup scheme fixing the weight filtration and $I^* = \mathrm{gr}_0(L_0 \oplus \mathfrak{I})$ point-wise.

For any saturated submodule $U' \subseteq (I \otimes I)^s$ we may define in addition the p-integral mixed Shimura data

$$\mathbf{H}_{g_0}[(I \otimes I)^s, I \otimes L_0]/\mathbb{W}(U').$$

Note that any of the p-integral mixed Shimura data in this section is of this form, which we call Shimura data of **symplectic type**.

(2.5.4) Theorem. *For each $0 \le g_0 \le g$, isotropic $I \subset L$ of dimension $g - g_0$, and choice of splitting $L \cong L_0 \oplus \mathfrak{I}$ (where $\mathfrak{I} = I^* \oplus I$, as usual, with natural symplectic form), there is a boundary morphism*

$$\mathbf{H}_{g_0}[(I \otimes I)^s, I \otimes L_0] \Longrightarrow \mathbf{H}_g.$$

These exhaust the (equivalence classes of) p-integral boundary components of $\mathbf{H}_g$.

The boundary map, however, depends on the choice of a splitting $L = L_0 \oplus I \oplus I^*$.

Proof. If $g = 0$ there is nothing to show, otherwise every boundary component (i.e. every proper maximal parabolic subgroup Q) is the stabilizer of a proper isotropic subspace $I \subset L$. This is seen as follows:
It suffices to see this for the symplectic group (because there is a 1:1 correspondence). Choose a decomposition $L = L_+ \oplus L_+^*$ and a basis $v_1, \dots, v_g$ of L_+ such that a maximal torus of P, which is split, acts diagonally with respect to this basis. The torus is then a $\mathbb{G}_m^g$ acting by $v_i \mapsto \lambda_i v_i$ and $v_i^* \mapsto \lambda_i^{-1} v_i^*$. A set of roots is given by

$r_{i,i}:$	λ_i^2	$< v_i \otimes v_i >$		$\# = g$
$r_{i,j}:$	$\lambda_i \lambda_j$	$< v_i \otimes v_j + v_j \otimes v_i >$	$i < j$	$\# = g(g-1)/2$
$r_i^j:$	$\lambda_i \lambda_j^{-1}$	$< v_i \otimes v_j^* + v_j^* \otimes v_i >$	$i < j$	$\# = g(g-1)/2$
$-r_{i,i}:$	λ_i^{-2}	$< v_i^* \otimes v_i^* >$		$\# = g$
$-r_{i,j}:$	$(\lambda_i \lambda_j)^{-1}$	$< v_i^* \otimes v_j^* + v_j^* \otimes v_i^* >$	$i < j$	$\# = g(g-1)/2$
$-r_i^j:$	$\lambda_i^{-1} \lambda_j$	$< v_i^* \otimes v_j + v_j \otimes v_i^* >$	$i < j$	$\# = g(g-1)/2$

These are $2g^2$ roots in total. A set of simple roots is given e.g. by $r_i^{i+1}, i = 1, \dots, g-1$ and $r_{g,g}$. This set induces the decomposition into negative and positive roots as in the table.

A maximal parabolic proper subgroup scheme P, up to conjugation (etale locally) is given by a maximal proper subset of this (see 1.9.4) set of simple roots. An associated filtration is given by

$$0 \subset I \subset L,$$

where $I = L_+$, if $r_{g,g}$ is the missing root, and

$$0 \subset I \subset I^\perp \subset L$$

for $I =< v_1, \dots, v_i >$, if r_i^{i+1} is the missing root.
The unipotent radical of P in this case shifts the filtration by at least 1 and it is given by the group extension

$$0 \longrightarrow \mathbb{W}((I \otimes I)^s) \longrightarrow \mathrm{WSp} \longrightarrow \mathbb{W}(I^\perp / I \otimes I) \longrightarrow 0,$$

as above. $G := P/W$ acts faithfully on $\mathrm{gr}^W(L)$ and is an almost direct product

$$G = \mathrm{GL}(I) \cdot \mathrm{GSp}(I^\perp / I)$$

(only a $\mu_{2,S}$ is contained in both groups). Here $\mathrm{GSp}(I^\perp/I)$ acts on I by λ and trivial on I^* (this choice is arbitrary at this point, otherwise one gets a different decomposition as almost direct product). This is also true for the case $I = L_+$ with our convention $\mathrm{GSp}(0) = \mathbb{G}_m$.
Here $\mathrm{GL}(I)$ acts on I^* by the representation contragredient to the standard one. Any splitting $L = L_0 \oplus I \oplus I^*$ defines a subgroup of P of this kind. Choose such.
Let $\mathbf{B}$ be the boundary component associated with I. The subgroup $\pi^{-1}(\mathrm{GSp}(I^\perp/I))$ is normal and it is the smallest such that (over $\mathbb{R}$) all morphisms $\omega_x \circ h_\infty$ for $x \in \mathbb{D}_{\mathbf{B}}$ factor through it. Hence $P_{\mathbf{B}} \simeq \mathrm{USp}(L_0, I)$.
The group $\mathrm{GSp}(L_0)$ acts transitively on $\pi_0(\mathbb{D}_{\mathbf{H}_g}) = \mathbb{D}_{\mathbf{H}_0}$. However, if $I \neq L_+$, there are 2 orbits in

$$\pi_0(\mathbb{D}_{\mathbf{H}_g}) \times \mathrm{Hom}(\mathbb{S}_\mathbb{C}, P_{\mathbf{B},\mathbb{C}})$$

containing $([x], \omega_x \circ h_\infty)$ for an $x \in \mathbb{D}_{\mathbf{H}_g}$. $\mathbb{D}_{\mathbf{B}}$ in this case is the $\mathbb{D}_{\mathbf{H}_{g_0}[(I \otimes I)^s, I \otimes L_0]} \subset \mathrm{Hom}(\mathbb{S}_\mathbb{C}, P_{\mathbf{B},\mathbb{C}})$ constructed above.
In the case $I = L_+$, there is only one orbit in

$$\pi_0(\mathbb{D}_{\mathbf{H}_g}) \times \mathrm{Hom}(\mathbb{S}_\mathbb{C}, P_{\mathbf{B},\mathbb{C}})$$

because $\mathbb{G}_m(\mathbb{R})$ acts (by conjugation) trivial on the second factor.
$h(\mathbb{D}_{\mathbf{B}}) \subset \mathrm{Hom}(\mathbb{S}_\mathbb{C}, P_{\mathbf{B},\mathbb{C}})$ in this case is isomorphic to $(I \otimes I)^s_\mathbb{C}$ (via choice of a base point $I^* \in \mathrm{M}^\vee(\mathbf{B})(\mathbb{R})$ and the image of $\mathbb{D}_{\mathbf{H}_g}$ consists of those elements whose imaginary part is definite. This defines an isomorphism of $\pi_0(\mathbb{D}_{\mathbf{H}_g})$ with the set of isomorphisms $\alpha : \mathbb{Z} \to \mathbb{Z}(1)$, i.e. with $\mathbb{D}_{\mathbf{H}_0}$. (Note that the sign of definiteness does not depend on the choice of I' nor I, but only on $\langle \cdot, \cdot \rangle$). Other viewpoint: $\alpha(1)$ is such that $\langle \cdot ; h(\alpha(1)) \cdot \rangle$ is *positive* definite.
Hence $\mathbb{D}_{\mathbf{B}} \cong \mathbb{D}_{\mathbf{H}_{g_0}[(I \otimes I)^s, I \otimes L_0]}$ as constructed above.

□

(2.5.5) Corollary. *Every boundary component of (p-integral) mixed Shimura data of symplectic type is again of symplectic type.*

Proof. Follows from the foregoing by the hereditary nature of boundary components. □

(2.5.6) We have quotient maps of mixed Shimura data:

$$\mathbf{H}_{g_0}[(I \otimes I)^s, I \otimes L_0] \to \mathbf{H}_{g_0}[0, I \otimes L_0] \to \mathbf{H}_{g_0}.$$

The first gives an isomorphism

$$\mathbf{H}_{g_0}[(I \otimes I)^s, I \otimes L_0]/U \simeq \mathbf{H}_{g_0}[0, I \otimes L_0]$$

(where $U = \mathbb{W}((I \otimes I)^s)$), and the second gives an isomorphism

$$\mathbf{H}_{g_0}[0, I \otimes L_0]/W \simeq \mathbf{H}_{g_0},$$

(where $W = V = \mathbb{W}(I \otimes L_0)$).

(2.5.7) Lemma. *Let S be a scheme and M a locally free sheaf on S.*

i. A $(\mathrm{GSp}(L), L)$-structure on M is a non-degenerate symplectic form on M (i.e. inducing $M \cong M^*$) up to multiplication by $H^0(S, \mathcal{O}_S^*)$

ii. A $(\mathrm{PSp}(L_0, I), L)$-structure on M is a saturated filtration

$$W_0 = M \supset W_{-1} \supset W_{-2} = 0$$

with non-degenerate symplectic form on W_{-1} up to multiplication by $H^0(S, \mathcal{O}_S^*)$ and an isomorphism $\alpha : W/W_{-1} \cong I^*$.

iii. A $(\mathrm{USp}(L_0, I), L)$-structure on M is a non-degenerate symplectic form on M up to multiplication by $H^0(S, \mathcal{O}_S^*)$, a saturated filtration

$$W_0 = M \supset W_{-1} \supset W_{-2} \supset W_{-3} = 0$$

such that W_{-2} is isotropic and $W_{-1} = (W_{-2})^\perp$, and an isomorphism $\alpha : W/W_{-1} \cong I^*$.

Proof. i. is easy.
ii. Obviously $L = L_0 \oplus I^*$ carriers such a structure. Conversely let a module M with filtration and structure as above be given. Then there is an isomorphism extending α:

$\mu : M \to L_0 \oplus I^*$, such that μ maps (W_{-1}) onto L_0 and is a symplectic similitude. The isomorphisms of such a structure (transported via μ) are given by $\mathrm{PSp}(L_0, I)$ acting on L.
iii. Obviously $L = L_0 \oplus \mathfrak{I}$ carriers such a structure. Conversely let a symplectic module M with filtration be given. One finds a symplectic similitude $\mu : M \to L$, extending α, such that μ maps (W_{-1}) onto $L_0 \oplus I$. W_{-2} has to be mapped automatically to I and the isomorphism is determined by μ. The isomorphisms of such a structure (transported via μ) are given by $\mathrm{USp}(L_0, I)$ acting on L. $\square$

Similar statements are true for a $(\mathrm{GSp}(L)(R), L_R)$ (resp. ...) structure on a local system.

(2.5.8) Assume, we have an embedding as in (2.2.9), i.e. a group scheme $P_{\mathbf{X}} = W_{\mathbf{X}} \rtimes G_{\mathbf{X}}$ of type (P), a closed embedding $P_{\mathbf{X}} \hookrightarrow \mathrm{USp}(L_{0,\mathbb{Z}_{(p)}}, I_{\mathbb{Z}_{(p)}})$ which induces a closed embedding $G \hookrightarrow \mathrm{GSp}(L_{0,\mathbb{Z}_{(p)}})$ and a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{W}(U) = U_{\mathbf{X}} \arrow[r] \arrow[d, hook] & W_{\mathbf{X}} \arrow[r] \arrow[d, hook] & V_{\mathbf{X}} = \mathbb{W}(V) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathbb{W}((I \otimes I)^s_{\mathbb{Z}_{(p)}}) \arrow[r] & \mathrm{WSp} \arrow[r] & \mathbb{W}((I \otimes L_0)_{\mathbb{Z}_{(p)}}) \arrow[r] & 0
\end{tikzcd} \tag{2}$$

The subgroup scheme $G_{\mathbf{X}}$ is mapped to a conjugate of GSp by an element of $\mathrm{WSp}(\mathbb{Z}_{(p)})$. In our application we may conjugate the whole embedding by this element and hence assume that $G_{\mathbf{X}}$ is embedded into $\mathrm{GSp}(L_0)$ (1.8.11). We consider the group scheme WSp as closed subscheme of $\mathrm{GSp}(L) \subseteq \mathrm{GL}(L)$, where $L = L_0 \oplus \mathfrak{I}$, as usual.
Claim: Under these circumstances the image of $W_{\mathbf{X}}$ is uniquely determined by the sublattices $U \subset (I \otimes I)^s$ and $V \subset L_0 \otimes I$. Indeed, it suffices to check this over $\mathbb{C}$: Let $h_x, x \in \mathbb{D}_{\mathbf{X}}$ be a morphism factorizing through $G_{\mathbf{X}}(\mathbb{R})$. Now $W_{\mathbf{X}}(\mathbb{C})$ is generated by the following 3 groups:
1. $F^0(W_{\mathbf{X}}(\mathbb{C})) := \mathrm{Stab}(h, W_{\mathbf{X}}(\mathbb{C}))$. We call its projection, which is isomorphic to it, $F^0(V_{\mathbf{X}}(\mathbb{C}))$.
2. the complex conjugate $\overline{F^0(W_{\mathbf{X}}(\mathbb{C}))}$.
3. $U_{\mathbf{X}}(\mathbb{C})$.
However $F^0(W_{\mathbf{X}}(\mathbb{C})) \subset F^0(\mathrm{WSp}(\mathbb{C}))$, hence it is uniquely determined
(as lift of $F^0(V_{\mathbf{X}}(\mathbb{C}))$).
In other words, the exponentials of $V_{\mathbb{C}}$ considered as subset of $(L_0 \otimes I)_{\mathbb{C}} = (I \otimes L_0 \oplus L_0 \otimes I)^s_{\mathbb{C}}$ lie in $W_{\mathbf{X}}(\mathbb{C})$.
We will construct a diagram of group schemes over $\mathbb{Z}_{(p)}$ of the form:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{W}(\widetilde{U}) = \widetilde{U}_{\mathbf{X}} \arrow[r] \arrow[d, hook] & \widetilde{W}_{\mathbf{X}} \arrow[r] \arrow[d, hook] & V_{\mathbf{X}} \oplus V_{\mathbf{X}} = \mathbb{W}(V \oplus V) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathbb{W}((I \otimes I)_{\mathbb{Z}_{(p)}}) \arrow[r] & \widetilde{\mathrm{WSp}} \arrow[r] & \mathbb{W}((I \otimes L_0)_{\mathbb{Z}_{(p)}}) \oplus \mathbb{W}((L_0 \otimes I)_{\mathbb{Z}_{(p)}}) \arrow[r] & 0
\end{tikzcd} \tag{3}$$

where again the outer inclusions are given by saturated inclusions of the corresponding modules. $\widetilde{\mathrm{WSp}}$ is constructed as a group scheme over $\mathrm{spec}(\mathbb{Z})$ such that its $\mathbb{Z}$-points — if there is a Hodge structure on L_0 — give an extension with corresponds via (4.2.3) to the $I \otimes I$-Poincaré biextension (see also 4.2.6). It can be constructed as a subgroup scheme of $\mathrm{GL}(L)$ by exponentials as well, if $I \otimes L_0$, $L_0 \otimes I$ and $I \otimes I$ are considered as subsets of $L \otimes L$ identified with $\mathrm{End}(L) = \mathrm{Lie}(\mathrm{GL}(L))$ via contraction with the symplectic forms. However, these exponentials are not symplectomorphisms (of course neither symplectic similitudes), i.e. $\widetilde{\mathrm{WSp}} \not\subseteq \mathrm{GSp}(L)$. There exists an automorphism s of $\mathrm{GL}(L)$ which is given by contraction with the symplectic form followed by inversion, i.e. $\mathrm{Sp}(L) = \mathrm{GL}(L)^s$ and we have $\mathrm{WSp} = \widetilde{\mathrm{WSp}}^s$.

$\widetilde{\mathrm{WSp}}$ can be given also explicitly (even over $\mathrm{spec}(\mathbb{Z})$) by the group law on

$$(X_1, X_2, X_3)(X_1', X_2', X_3') = (X_1 + X_1' + \langle \mathbf{X_3}, \mathbf{X_2'} \rangle, X_2 + X_2', X_3 + X_3')$$

on

$$\mathbb{W}(I \otimes I) \times \mathbb{W}(I \otimes L_0) \times \mathbb{W}(L_0 \otimes I).$$

Be careful: In this description the embedding of WSp and the description of s are more complicated, i.e. not via diagonal embedding, resp. switching factors! — see also (4.2.4))

Consider the subvectorspace $\widetilde{U}_{\mathbb{Q}} \subseteq (I \otimes I)_{\mathbb{Q}}$ generated by $\langle V_{\mathbb{Q}}, V_{\mathbb{Q}} \rangle$ and $U_{\mathbb{Q}}$ and define $\widetilde{U} = \widetilde{U}_{\mathbb{Q}} \cap (I \otimes I)_{\mathbb{Z}_{(p)}}$. Define $\widetilde{W}_{\mathbf{X}}$ as the subgroup scheme of $\widetilde{\mathrm{WSp}}$, directly generated by $\mathbb{W}(V) \subset \mathbb{W}(I \otimes L_0)$, $\mathbb{W}(V) \subset \mathbb{W}(L_0 \otimes I)$, and $\mathbb{W}(\widetilde{U})$. This group fits obviously in a diagram as (3).

We claim: $W_{\mathbf{X}} = (\widetilde{W}_{\mathbf{X}})^s$. First, $\widetilde{W}_{\mathbf{X}}$ is stable under s because s acts on the generators. All maps in the diagram are compatible with s. Now we have obviously for the projections $V_{\mathbf{X}} = (V_{\mathbf{X}} \times V_{\mathbf{X}})^s$.

Furthermore we claim $\mathbb{W}(U) = \mathbb{W}(\widetilde{U})^s$. For this, it suffices to show that $U_{\mathbb{Q}}$ is the set of elements of the form $v + s(v), v \in \widetilde{U}_{\mathbb{Q}}$. Now $U_{\mathbb{Q}}$ is point-wise stable under s, so it is generated by elements of this form (over $\mathbb{Q}$!). But on the other hand, every element of the form $v + s(v), v \in \widetilde{U}_{\mathbb{Q}}$ lies in $U_{\mathbb{Q}}$ because the commutator of (any lift of) 2 elements $v_1, v_2 \in V$ is $\langle v_1, v_2 \rangle - \langle v_2, v_1 \rangle = \langle v_1, v_2 \rangle + s(\langle v_1, v_2 \rangle)$ and lies in U. Since everything is saturated we are done.

Hence, $(\widetilde{W}_{\mathbf{X}})^s$ is a subgroup scheme of WSp, fitting in the same diagram (2) as above. However over $\mathbb{C}$, it contains $F^0(W_{\mathbf{X}}(\mathbb{C}))$ (and hence $\overline{F^0(W_{\mathbf{X}}(\mathbb{C}))}$) because $F^0(\mathrm{WSp}(\mathbb{C}))$ is the exponential of $F^0(\mathbb{W}(L_0 \otimes I)(\mathbb{C}))$ embedded diagonally, so $F^0(W_{\mathbf{X}}(\mathbb{C}))$ is the exponential of $F^0(V_{\mathbb{C}})$ (embedded diagonally), which is contained in $\widetilde{W}_{\mathbf{X}}$ by construction. Therefore $(\widetilde{W}_{\mathbf{X}})^s = W_{\mathbf{X}}$.

2.6. Mixed Shimura data of Hodge type

(2.6.1) Definition. *Let* $\mathbf{X}$ *be p-integral mixed Shimura data (p-MSD). We call it of* ***Hodge type****, if there is an embedding into the p-integral mixed Shimura datum (2.5)*

$$\mathbf{X} \hookrightarrow \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]/\mathbb{W}(U'),$$

for $\mathbb{Z}_{(p)}$ *lattices* L_0 *and* I*, where* L_0 *is either 0 or carries a primitive symplectic form and a saturated sublattice* $U' \subset (I \otimes I)^s$*.*
(this includes in particular the cases $\mathbf{H}_g$*,* $\mathbf{H}_{g_0}[0, L_0 \otimes I]$ *and embeddings into them).*
In this case, we call also all ${}^K\mathbf{X}$ *and* ${}^K_\Delta\mathbf{X}$ *of Hodge type.*

(2.6.2) Remark. In [83, 2.26] it is shown that *in any case* for the *rational* Shimura data, there is a embedding

$$\mathbf{X}' \hookrightarrow \mathbf{X}/W \times \prod_i \mathbf{H}_{g_{0,i}}[(I_i \otimes I_i)^s, I_i \otimes L_{0,i}],$$

for some $\mathbb{Q}$-vector spaces $I_i, L_{0,i}$, where the $L_{0,i}$ are either zero or carry non degenerate symplectic forms, and where $\mathbf{X}/W$ is *pure* Shimura data (the pure quotient of $\mathbf{X}$).
Here $\mathbf{X}'$ may be an extension of $\mathbf{X}$.
It is therefore probable that it is possible to adapt our construction of compactified mixed Shimura varieties, such that one needs merely to assume $\mathbf{X}/W$ be of Hodge type. Maybe in this case it is necessary to strengthen the notion 'type (P)' slightly to ensure the existence of the above embedding over $\mathrm{spec}(\mathbb{Z}_{(p)})$.

(2.6.3) Theorem. *Let* $\mathbf{X}$ *be p-integral mixed Shimura data, and* $\mathbf{B}$ *a boundary component of* $\mathbf{X}$*. If* $\mathbf{X}$ *is of Hodge type then* $\mathbf{B}$ *is either.*

Proof. (2.5.4) and (2.4.7). □

2.7. Properties of mixed Shimura varieties over $\mathbb{C}$

(2.7.1) Definition/Theorem. *Let* ${}^K\mathbf{X}$ *be rational EMSD.*

$$B \mapsto [\quad B\text{-}{}^K\mathbf{X}\text{-}L_{\mathbb{Q}}\text{-}\boldsymbol{loc\text{-}mhs}\quad]$$

(2.3.5) is represented by[4] *the quotient stack*

$$[P_{\mathbf{X}}(\mathbb{Q})\backslash \mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)].$$

[4] All our groupoids (with parameter B, resp. S) are actually fibers over B, resp. S of a *category fibered in groupoids* with obvious pullbacks. For better readability, we do not mention this structure explicitly.

*It is a smooth analytic Deligne-Mumford stack and called the analytic **mixed Shimura variety** associated with* $\mathbf{X}$.
It is obviously independent of the chosen representation $L_{\mathbb{Q}}$.

(2.7.2) Remark. The previous therem would not be true, if we had not insisted in property (2.2.2, v) because then (2.2.5) would not be true.

(2.7.3) Let $(\gamma, \rho) : {}^{K_1}\mathbf{X}_1 \to {}^{K_2}\mathbf{X}_2$ be a morphism of EMSD. There is an induced map

$$[P_{\mathbf{X}_1}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{X}_1} \times (P_{\mathbf{X}_1}(\mathbb{A}^{(\infty)})/K_1)] \to [P_{\mathbf{X}_2}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{X}_1} \times (P_{\mathbf{X}_2}(\mathbb{A}^{(\infty)})/K_2)],$$

given by

$$[x, \xi] \mapsto [\gamma(x), \gamma(\xi) \cdot \rho].$$

These maps are compatible with composition.

Let $L_{\mathbb{Q}}$ be a representation of $P_{\mathbf{X},\mathbb{Q}}$. Let $M_{\mathbb{Q}}$ be the associated universal local system over $[P_{\mathbf{X}_1}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{X}_1} \times (P_{\mathbf{X}_1}(\mathbb{A}^{(\infty)})/K_1)]$ with $(P_{\mathbf{X},\mathbb{Q}}, L_{\mathbb{Q}})$-structure α. We have the associated right $P_{\mathbf{X}}(\mathbb{Q})_B$-torsor $\mathrm{tor}(\alpha)$ and the analytic right $P_{\mathbf{X},\mathcal{O}_B}$-torsor $\mathrm{tor}(\alpha \otimes_{\mathbb{Q}} \mathcal{O}_B)$.

(2.7.4) Definition/Theorem. $\mathrm{tor}(\alpha)$ *is canonically isomorphic to*

$$\mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)$$

*and is called the **standard local system***
($P_{\mathbf{X}}(\mathbb{Q})$ acts on the right by $[x, \rho]q = [q^{-1}x, q^{-1}\rho]$).
The right $P_{\mathcal{O}_B}$-torsor $\mathrm{tor}(\alpha \otimes_{\mathbb{Q}} \mathcal{O}_B)$ *is canonically isomorphic to*

$$[P_{\mathbf{X}}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{C}) \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)]$$

*and is called the analytic **standard principal bundle**. ($P_{\mathbf{X}}(\mathbb{C})$ acts on the right via $[x, p, \rho]g = [x, pg, \rho]$)*
Obviously both are again independent of the chosen representation $L_{\mathbb{Q}}$.

(2.7.5) There is a $P_{\mathbf{X}}(\mathbb{C})$-equivariant map

$$\Pi_{\mathbb{C}} : \mathrm{tor}(\alpha \otimes_{\mathbb{Q}} \mathcal{O}_B) \to (Q_{\mathbb{C}}\backslash P_{\mathbf{X},\mathbb{C}})^{an} \subseteq (\mathcal{QPAR}_{\mathbb{C}})^{an},$$

which is given on the level of double quotients by

$$\begin{aligned}[P_{\mathbf{X}}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{C}) \times (P(\mathbb{A}^{(\infty)})/K)] &\to (\mathcal{QPAR}_{\mathbb{C}})^{an} \\ [x, p, \xi] &\mapsto \mathrm{qpar}(u_x)^p.\end{aligned}$$

Here Q is any parabolic of the form $\mathrm{qpar}(u_x)$ for some $x \in \mathbb{D}_{\mathbf{X}}$.
If a representation $\rho : P_{\mathbf{X}} \to \mathrm{GL}(L)$ is given, $\mathrm{qpar}(u_x)$ corresponds to (i.e. fixes, see 1.9.14) the Hodge filtration determined by the Hodge structure determined by h_x for $x \in \mathbb{D}_{\mathbf{X}}$.

(2.7.6) The analytic standard principal bundle, together with the map $\Pi_{\mathbb{C}}$ determine a morphism of (analytic) Artin stacks:

$$\Xi_{\mathbb{C}} : [P_{\mathbf{X}}(\mathbb{Q})\backslash \mathbb{D}_{\mathbf{X}} \times (P(\mathbb{A}^{(\infty)})/K)] \to [\mathrm{M}^{\vee}(\mathbf{X})_{\mathbb{C}}/P_{\mathbf{X},\mathbb{C}}].$$

Here $\mathrm{M}^{\vee}(\mathbf{X})_{\mathbb{C}}$ is $Q_{\mathbb{C}}\backslash P_{\mathbf{X},\mathbb{C}}$ and will later be called the (analytic) compact dual associated with $\mathbf{X}$ (cf. 3.5.2). The stack on the right hand side is isomorphic to $[\cdot/Q_{\mathbb{C}}]$

(2.7.7) Lemma. *The standard principal bundle is trivial along the U-fibre.*

Proof. Let $\mathbf{X}'$ be $\mathbf{X}/U_{\mathbf{X}}$. The fibre of $\mathrm{M}({}^{K'}\mathbf{X})$ above a point $[x, \rho] \in [P_{\mathbf{X}'}(\mathbb{Q})\backslash \mathbb{D}_{\mathbf{X}'} \times (P_{\mathbf{X}'}(\mathbb{A}^{(\infty)})/(K/U))]$ is given by

$$U_{\mathbf{X}}(\mathbb{Q})\backslash U_{\mathbf{X}}(\mathbb{C})x \times U_{\mathbf{X}}(\mathbb{A}^{(\infty)})\rho/(K \cap U(\mathbb{A}^{(\infty)}))$$

for some K.
The standard principal bundle over this fibre is equal to

$$U_{\mathbf{X}}(\mathbb{Q})\backslash U_{\mathbf{X}}(\mathbb{C})x \times P_{\mathbf{X}}(\mathbb{C}) \times U_{\mathbf{X}}(\mathbb{A}^{(\infty)})\rho/(K \cap U(\mathbb{A}^{(\infty)})).$$

We have a trivialization, mapping a point of the form $[ux, \rho]$ to $[u, ux, \rho]$. This allows (one may do this in families, i.e. varying with $[x, \rho] \in \mathrm{M}({}^{K}\mathbf{X}'))$ to extend the standard principal bundle to the partial compactification along the U-fibre (cf. 3.5.1). □

3. Integral models (good reduction)

3.1. Reflex rings

(3.1.1) Definition/Theorem. *Let $\mathbf{X}$ be p-integral mixed Shimura data. With $x \in \mathbb{D}_{\mathbf{X}}$ there is associated $h_x : \mathbb{S}_{\mathbb{C}} \to P_{\mathbf{X},\mathbb{C}}$, hence a morphism $u_x : \mathbb{G}_{m,\mathbb{C}} \to P_{\mathbf{X},\mathbb{C}}$ via composition with the inclusion of the first factor into $\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$. The conjugacy class M of these morphisms is defined over a number field $E_{\mathbf{X}}$, called* ***the reflex field****, unramified over p. Let $\wp$ be a prime above p in $E_{\mathbf{X}}$ and $\mathcal{O}$ the corresponding d.v.r. We call $\mathcal{O}$ a* ***reflex ring*** *of $\mathbf{X}$. The closure $\overline{M}$ in $\underline{\mathrm{Hom}}(\mathbb{G}_{m,\mathcal{O}}, P_{\mathbf{X},\mathcal{O}})$ maps surjectively onto* $\mathrm{spec}(\mathcal{O})$.

Proof. Let $S = \mathrm{spec}(\mathbb{Z}_{(p)})$. First of all, there is a splitting $(P_{\mathbf{X}}, T, M, R)$ (1.6.3) over some etale surjective extension $S' \to S$. Over $\mathbb{C}$, T is conjugated to a torus which contains u_x. Hence we may assume w.l.o.g. that u_x factors via $T_{\mathbb{C}}$. This implies that u_x is defined over S'. Hence the conjugacy class M is defined over a field E, which is contained in the function field of S' which is a number field, unramified at p. The fibre above $\wp$ of $\overline{M}$ is non-empty because over S', it contains the section $u_x : S' \to \overline{M}$. □

(3.1.2) Remark. Observe that the reflex field or the reflex rings are given as subrings of $\mathbb{C}$, not only as abstract rings.

(3.1.3) Lemma. *The reflex field of the symplectic mixed Shimura data (2.5) is $\mathbb{Q}$.*

3.2. Integral models of mixed Shimura varieties

(3.2.1) We begin by describing the functorial theory of integral models of mixed Shimura varieties in the case of good reduction. The following theorem is essentially due to Kisin or Vasiu in the pure case.

(3.2.2) Main theorem. *Let $p \neq 2$ be a prime.*
There is a unique *(up to unique isomorphism) map associating to each p-EMSD ${}^K\mathbf{X}$ of Hodge type and reflex ring $\mathcal{O} \subset \mathbb{C}$ a smooth Deligne-Mumford stack* $\mathrm{M}({}^K\mathbf{X})$ *over $S = \mathrm{spec}(\mathcal{O})$ with an isomorphism of analytic Deligne-Mumford stacks*

$$[P_{\mathbf{X}}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)] \cong \left(\mathrm{M}({}^K\mathbf{X}) \times_S \mathrm{spec}(\mathbb{C})\right)^{an}$$

(2.7.1), satisfying the following properties

i. *For each morphism of p-EMSD* $[\gamma,\rho] : {}^{K_1}\mathbf{X}_1 \to {}^{K_2}\mathbf{X}_2$ *there is an associated morphism*

$$\mathrm{M}(\gamma,\rho) : \mathrm{M}({}^{K_1}\mathbf{X}_1) \to \mathrm{M}({}^{K_2}\mathbf{X}_2) \times_{S_2} S_1.$$

Here $S_i = \mathrm{spec}(\mathcal{O}_i)$*, where* $\mathcal{O}_i$ *are reflex rings of* $\mathbf{X}_i$ *such that* $\mathcal{O}_2 \subset \mathcal{O}_1$*. Over* $\mathbb{C}$ *these maps are given by (2.7.3).*

If γ *is an embedding,* $\mathrm{M}(\gamma,\rho)$ *is a normalization map followed by a closed embedding. If* γ *is a Hecke operator, then* $\mathrm{M}(\gamma,\rho)$ *is etale and finite. The maps are compatible with composition.*

ii. *(rational canonicity) For a pure Shimura datum* $\mathbf{Y}$*, where* $P_{\mathbf{Y}}$ *is a torus, the composition of the reciprocity isomorphism (normalized as in [83, 11.3])*

$$\mathrm{Gal}(\overline{E}|E)^{ab} \cong \pi_0(T_E(\mathbb{Q})\backslash T_E(\mathbb{A})),$$

(where $T_E = \mathrm{res}^E_{\mathbb{Q}}(\mathbb{G}_m)$*) with* π_0 *of the reflex norm [83, 11.4]*

$$\pi_0(T_E(\mathbb{Q})\backslash T_E(\mathbb{A})) \to \pi_0(P_{\mathbf{Y}}(\mathbb{Q})\backslash P_{\mathbf{Y}}(\mathbb{A}))$$

defines by means of the natural action of $\pi_0(P_{\mathbf{Y}}(\mathbb{Q})\backslash P_{\mathbf{Y}}(\mathbb{A}))$ *on* $\mathrm{M}({}^K\mathbf{Y})(\mathbb{C})$ *the rational model* $\mathrm{M}({}^K\mathbf{Y})_E$*.*

iii. *(integral canonicity) The projective limit*

$$\mathrm{M}^p(\mathbf{X}) := \varprojlim_K \mathrm{M}({}^K\mathbf{X}),$$

where the limit is taken over all admissible *compact open subgroups, satisfies the following property:*

For each test scheme *(3.7.1)* T *over* S *and a given morphism*

$$\alpha_E : T \times_S Q \to \mathrm{M}^p(\mathbf{X}) \times_S Q,$$

there exists a uniquely determined morphism

$$\alpha : T \to \mathrm{M}^p(\mathbf{X})$$

such that $\alpha_E = \alpha \times_S Q$*.*

iv. *M*$({}^K_\Delta\mathbf{X})$ *is a quasi-projective scheme, if* K *is neat.*

Proof. By [83, 11.18] the assertion of the theorem is true, over the reflex *fields*. To extend it to reflex rings, we proceed in three steps:
STEP 1: Construction for the symplectic Shimura data (2.5). By (4.3.3), we have a $\mathrm{M}({}^K\mathbf{X})$ for each of the symplectic Shimura data. It is a model of the one considered

in [83, 11.18] because our moduli problem, restricted to the generic fibre, is the same moduli problem used in [loc. cit.].
They satisfy (i) for every Hecke operator, and (iii) by (3.7.9) because every $\mathrm{M}({}^K\mathbf{X})$ is smooth.
For construction of the model associated with quotients $\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]/\mathbb{W}(U')$ we observe that the models of $\mathrm{M}({}^K\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I])$ are torsors for the split torus with cocharacter group, (in general non canonically) isomorphic to $(I \otimes I)^s \cap K$ over $\mathrm{M}({}^K\mathbf{H}_{g_0})$. We hence may factor out by the action of the split subtorus with cocharacter group $U' \cap K$ to get a smooth model $\mathrm{M}({}^K\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]/\mathbb{W}(U'))$.
STEP 2: Construction for arbitrary Hodge type p-EMSD ${}^K\mathbf{X}$ (2.6.1). We choose an embedding ${}^K\mathbf{X} \hookrightarrow {}^{K'}\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]/\mathbb{W}(U')$ in a datum of symplectic type. This exists because of (2.2.8), and define $M({}^K\mathbf{X})$ as the normalization of the Zariski closure in $\mathrm{M}({}^{K'}\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]/\mathbb{W}(U'))$. Hecke operators automatically extend to this family (the extension property is inherited: 3.7.8, ii, iii) and satisfy the properties of (3.6.1). Hence to show smoothness, it suffices to show this for a family of compact open subgroups like in (3.6.1, iii). We may take, according to (2.2.8, iii), the family $K_U(M^2)K_V(M)K_G(N)$, which is however conjugated (already by an element in $P_{\mathbf{X}}(\mathbb{A}^{(\infty,p)})$) to $K_W(1)K_G(N)$ (see proof of 4.3.2). In (5.2.1) it is shown that the normalization of the closure is *smooth.* The projective limit satisfies (iii) by (3.7.8).
STEP 3: All models constructed in STEP 1 and 2 are smooth and in the limit satisfy (iii), hence from the existence of the morphisms in (i) over the reflex fields, we deduce their extendibility to the reflex rings. In particular these models are uniquely determined (up to a unique isomorphism). □

(3.2.3) Conjecture. *The previous therem is true for arbitrary p (including $p = 2$) and for arbitrary mixed Shimura data.*

For $p = 2$, it may be necessary to change the notion of 'integral canonicity' as formulated above, cf. [78, 3.4].

3.3. Toroidal compactifications

(3.3.1) Unfortunately, so far, we are not able to derive *all* desired main properties of integral models of toroidal compatifications of mixed Shimura varieties from work of Kisin, Vasiu, Faltings and Chai, Pink and others. We need to assume the truth of a conjecture stated below that will assure that the Zariski closure of the rational canonical model in the toroidal compactifications of [27] will intersect the boundary divisor (of the latter) properly also in the special fibre (the conjecture is even slightly weaker). For most (pure) Shimura varieties of Hodge type that are (analytically) compact, this has been shown in [92]. For Shimura varieties of P.E.L. type, it follows from [72].

(3.3.2) Conjecture. *Let $\mathbf{X}$ be pure p-integral Shimura data. Let any embedding ${}^K_\Delta\mathbf{X} \hookrightarrow {}^{K'}_{\Delta'}\mathbf{X}'$ in a datum of symplectic type $\mathbf{X}' = \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]/\mathbb{W}(U')$ be given, such that* Δ and Δ' *are smooth projective, and K, K' are neat.*
Consider the toroidal compactification $\mathrm{M}({}^{K'}_{\Delta'}\mathbf{X}')$ over $\mathrm{spec}(\mathbb{Z}_{(p)})$, constructed in [27][1], and let D be its boundary divisor. Let $S = \mathrm{spec}(\mathcal{O})$ for a reflex ring $\mathcal{O} \subset E(\mathbf{X})$ of $\mathbf{X}$ with maximal ideal $\wp|(p)$. Let $\mathcal{M}'$ be the Zariski closure of (the rational canonical model of) $P_\mathbf{X}(\mathbb{Q})\backslash\mathbb{D}_\mathbf{X} \times (P_\mathbf{X}(\mathbb{A}^{(\infty)})/K)$ in $\mathrm{M}({}^{K'}_{\Delta'}\mathbf{X}')_S$.
No connected *component of $N(\mathcal{M}') \cap D_S$ lies entirely in the fibre above $\wp$.*

(3.3.3) The toroidal compactifications will have a stratification, which consists of mixed Shimura varieties itself. For this, we have to set up some notation (cf. [83, 7.3ff]). Let $(\rho, \iota) : {}^{K'}_{\Delta'}\mathbf{B} \Longrightarrow {}^K_\Delta\mathbf{X}$ be a boundary component. For each cone $\sigma \in \Delta'$ with $\sigma \subset C(\mathbb{D}^0_\mathbf{X}, P_\mathbf{B}) \times \rho'$ for some ρ' in the equivalence class of ρ, we define a p-ECMSD ${}^{K_{[\sigma]}}_{\Delta_{[\sigma]}}\mathbf{B}_{[\sigma]}$ as follows:
$\mathbf{B}_{[\sigma]}$ is defined by

$$\begin{aligned} P_{\mathbf{B}_{[\sigma]}} &:= P_\mathbf{B}/ < \sigma >, \\ \mathbb{D}_{\mathbf{B}_{[\sigma]}} &:= P_{\mathbf{B}_{[\sigma]}}(\mathbb{R})U_{\mathbf{B}_{[\sigma]}}(\mathbb{C}) - \text{orbit generated by } (\mathbb{D}^0_\mathbf{X}/ < \sigma > (\mathbb{C})) \times \{\sigma P_\mathbf{X}(\mathbb{A}^{(\infty)})\} \\ &\quad \text{in } (\mathbb{D}^0_\mathbf{X}/ < \sigma > (\mathbb{C})) \times \{[\sigma]/P_\mathbf{X}(\mathbb{A}^{(\infty)})\} \text{ for } \sigma \in [\sigma] \cap \Delta(\mathbb{D}^0_\mathbf{X}, P_\mathbf{B}, 1). \end{aligned}$$

$K_{[\sigma]}$ is the image of K' under the projection $P_\mathbf{B} \to P_{\mathbf{B}_{[\sigma]}}$. $\Delta_{[\sigma]}$ is defined by

$$\Delta_{[\sigma]}(\mathbb{D}^0_{\mathbf{B}_{[\sigma]}}, P_{\mathbf{B}_{[\sigma]}}, \overline{\rho}) := \{\tau \bmod < \sigma > \ | \ \tau \in \Delta(\mathbb{D}^0_\mathbf{X}, P_\mathbf{B}, \rho) \text{ such that } \sigma \text{ is a face of } \tau\}.$$

It is $K'_{[\sigma]}$-admissible.
Furthermore, we define

$$\Gamma := (\mathrm{Stab}_{Q(\mathbb{Q})}(\mathbb{D}_\mathbf{B}) \cap (P_\mathbf{B}(\mathbb{A}^{(\infty)})({}^\rho K)))/P_\mathbf{B}(\mathbb{Q}),$$

where Q is the parabolic describing $\mathbf{B}$.
$\Sigma(\iota, \rho)$ is defined to be the set of equivalence classes of the action of Γ (defined above) on those $[\sigma] \in P_\mathbf{B}(\mathbb{Q})\backslash(\Delta')^0/P_\mathbf{B}(\mathbb{A}^{(\infty)})$, which satisfy $\sigma \in C(\mathbb{D}^0_\mathbf{X}, P_\mathbf{B}) \times \rho'$ for some $\mathbb{D}^0_\mathbf{X}$.

(3.3.4) Consider an Abelian unipotent extension

$$\mathbf{X}[U, 0] \to \mathbf{X}.$$

Recall from (2.2.9): Any morphism $\mathbf{Y} \to \mathbf{Y}/U_\mathbf{Y}$ is of this form. It is a torsor under the *group object*

$$\mathbf{X}/W_\mathbf{X}[U, 0] \to \mathbf{X}/W_\mathbf{X}$$

[1] An explicit comparison of our language to the one used in [27] will be provided in forthcoming work

(cf. also [83, 6.6-6.8]).
The corresponding map of Shimura varieties (assume all K's in the sequel to be neat):

$$\mathrm{M}({}^K(\mathbf{X}/W_{\mathbf{X}})[U,0]) \to \mathrm{M}({}^{K/U}\mathbf{X}/W_{\mathbf{X}})$$

is a split torus with character group (in general non canonically) isomorphic to $U_{\mathbb{Q}} \cap K$. If Δ is a K-admissible polyhedral cone decomposition for $\mathbf{X}[U,0]$, concentrated in the fibre U, we can define an associated torus embedding. Recall: The rational polyhedral cone decomposition is determined by its restriction to any $(U_{\mathbb{R}})(-1) \times \rho = C(\mathbb{D}^0, P_{\mathbf{X}}) \times \rho$ by the arithmeticity condition (2.4.10, iii). Therefore this determines a torus embedding (see [27, IV, §2] for the integral case):

$$\mathrm{M}({}^K\mathbf{X}/W_{\mathbf{X}}[U,0]) \hookrightarrow \mathrm{M}({}^K_\Delta\mathbf{X}/W_{\mathbf{X}}[U,0]),$$

and a corresponding toroidal embedding of the torsor:

$$\mathrm{M}({}^{K'}\mathbf{X}[U,0]) \hookrightarrow \mathrm{M}({}^{K'}_\Delta\mathbf{X}[U,0]).$$

(here $K = K'/W$).

(3.3.5) Main theorem. *Let $p \neq 2$ be a prime.*
There is at most a unique *(up to unique isomorphism) map associating to each p-ECMSD ${}^K_\Delta\mathbf{X}$ of Hodge type and reflex ring $\mathcal{O} \subset \mathbb{C}$ a Deligne-Mumford stack $M({}^K_\Delta\mathbf{X})$ over $S =$* $\mathrm{spec}(\mathcal{O})$ *with open dense embedding*

$$\mathrm{M}({}^K\mathbf{X}) \hookrightarrow \mathrm{M}({}^K_\Delta\mathbf{X}),$$

satisfying the properties:

i. *For each morphism of p-ECMSD $[\gamma,\rho] : {}^{K_1}_{\Delta_1}\mathbf{X}_1 \to {}^{K_2}_{\Delta_2}\mathbf{X}_2$ there is an associated morphism*
$$\mathrm{M}(\gamma,\rho) : \mathrm{M}({}^{K_1}_{\Delta_1}\mathbf{X}_1) \to \mathrm{M}({}^{K_2}_{\Delta_2}\mathbf{X}_2) \times_{S_2} S_1,$$
extending the map in (3.2.2, i.).

Here $S_i = \mathrm{spec}(\mathcal{O}_i)$, where $\mathcal{O}_i$ are reflex rings of $\mathbf{X}_i$ such that $\mathcal{O}_2 \subset \mathcal{O}_1$. If $[\gamma,\rho]$ is an embedding, $\mathrm{M}(\gamma,\rho)$ *is a normalization map followed by a closed embedding.*

ii. *If $\Delta = \Delta^0$ (i.e. Δ is concentrated in the unipotent fibre),* $\mathrm{M}({}^K_\Delta\mathbf{X})$ *is the toroidal embedding described in (3.3.4).*

iii. $\mathrm{M}({}^K_\Delta\mathbf{X})$ *possesses a stratification into mixed Shimura varieties (stacks)*

$$\coprod_{\substack{[(\iota,\rho): {}^{K'}_{\Delta'}\mathbf{B} \Longrightarrow {}^K_\Delta\mathbf{X}] \\ [\sigma] \in \Sigma(\iota,\rho)}} [\mathrm{Stab}_\Gamma([\sigma]) \backslash \mathrm{M}({}^{K_{[\sigma]}}\mathbf{B}_{[\sigma]})].$$

Here $[(\iota, \rho) : {}^{K'}_{\Delta'}\mathbf{B} \Longrightarrow {}^{K}_{\Delta}\mathbf{X}]$ *in the first line means equivalence classes of boundary components of* ${}^{K}_{\Delta}\mathbf{X}$ *(2.4.13).*

iv. *For each boundary map (2.4.13)*

$$(\iota, \rho) : {}^{K'}_{\Delta'}\mathbf{B} \Longrightarrow {}^{K}_{\Delta}\mathbf{X},$$

and $\sigma \in \Delta'$ *such that* $\sigma \subset C(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}}) \times \rho'$ *for some* $\mathbb{D}^0_{\mathbf{X}}$ *and* ρ'*, there is a boundary isomorphism*

$$\mathrm{M}(\iota, \rho) : \left[\mathrm{Stab}_\Gamma([\sigma])\backslash\widehat{\mathrm{M}({}^{K'}_{\Delta'}\mathbf{B})}\right] \to \widehat{\mathrm{M}({}^{K}_{\Delta}\mathbf{X})}$$

of the formal completions along the boundary stratum

$$\left[\mathrm{Stab}_\Gamma([\sigma])\backslash\, \mathrm{M}({}^{K_{[\sigma]}}_{\Delta_{[\sigma]}}\mathbf{B}_{[\sigma]})\right].$$

The complexification of $\mathrm{M}(\iota, \rho)$ *converges in a neighborhood of the boundary stratum and is in the interior, via the identification with the complex analytic mixed Shimura varieties given by a quotient of the map*

$$\mathbb{D}_{\mathbf{B} \Longrightarrow \mathbf{X}} \times (P_{\mathbf{B}}(\mathbb{A}^{(\infty)})/K') \to \mathbb{D}_{\mathbf{B} \Longrightarrow \mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)$$

induced by the closed embedding $P_{\mathbf{B}} \hookrightarrow P_{\mathbf{X}}$*, where we considered* $\mathbb{D}_{\mathbf{B} \Longrightarrow \mathbf{X}} \subseteq \mathbb{D}_{\mathbf{X}}$ *as a subset of* $\mathbb{D}_{\mathbf{B}}$ *via the analytic boundary map (2.4.4).*

The stratification is compatible with the morphisms in (i), i.e. they induce morphisms of the strata of the type considered in (i) again.

If main conjecture (3.3.2) is assumed*, the compactifications exist at least for those p-ECMSD* ${}^{K}_{\Delta}\mathbf{X}$ *with sufficiently small smooth and projective* Δ*.*
We have then in addition: $\mathrm{M}({}^{K}_{\Delta}\mathbf{X})$ *is smooth.* $\mathrm{M}({}^{K}_{\Delta}\mathbf{X})$ *is proper, if* Δ *is complete. If* K *is, in addition, neat then* $\mathrm{M}({}^{K}_{\Delta}\mathbf{X})$ *is a projective scheme.*

(3.3.6) Remark. The stratification in (iii) is indexed by pairs of an equivalence class of boundary components $[(\iota, \rho) : {}^{K'}_{\Delta'}\mathbf{B} \Longrightarrow {}^{K}_{\Delta}\mathbf{X}]$ and a class $[\sigma] \in \Sigma(\iota, \rho)$. The set of these pairs is just isomorphic to the set of double cosets $P_{\mathbf{X}}(\mathbb{Q})\backslash\Delta/K$. The bijection is as follows. Each $\sigma \in \Delta$ is supported on a $C(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}}) \times \rho$. $\mathbf{B}$ and ρ determine a boundary component $(\iota, \rho) : {}^{K'}_{\Delta'}\mathbf{B} \Longrightarrow {}^{K}_{\Delta}\mathbf{X}$ and the class of the image of σ under restriction to Δ' lies in $\Sigma(\iota, \rho)$.
The stratum $\mathrm{M}({}^{K_{[\sigma]}}\mathbf{B}_{[\sigma]})$ is contained in the closure of the stratum $\mathrm{M}({}^{K_{[\tau]}}\mathbf{B}_{[\tau]})$, if and only if (up to a change of representatives in $P_{\mathbf{X}}(\mathbb{Q})\backslash\Delta/K$) τ is a face of σ.

Proof of theorem 3.3.5. By [83, 12.4] the assertion of the theorem is true over the reflex *fields*. To extend it to reflex rings, we proceed again in several steps:
STEP 0: We first show that it is sufficient to consider the case in which K is neat. This is not so easy as in the uncompactified case, where we have (3.6.1). Let K be arbitrary

admissible. Suppose that a K-admissible Δ is given. Take $K' \lhd K$ a neat normal subgroup. The problem is that (as we formulated the theory) $\mathrm{M}({}^{K}_{\Delta}\mathbf{X})$ cannot be defined as the stack quotient of $\mathrm{M}({}^{K'}_{\Delta}\mathbf{X})$ by $K'\backslash K$. This would lead to additional (unipotent) stabilizer groups along the boundary strata. Hence we have to define the stack by a mixture of forming a quotient and glueing. Consider the finite set of (equivalence classes of) boundary components $(\iota_i, \rho_i) : {}^{K_i}_{\Delta_i}\mathbf{B}_i \Longrightarrow {}^{K}_{\Delta}\mathbf{X}, i \in I$. We may present the uncompactified $\mathrm{M}({}^{K}\mathbf{X})$ in a very redundant way by the following groupoid object:

$$\begin{array}{c} \coprod_{i,j\in I} \mathrm{M}({}^{K'}\mathbf{X}) \times_S (K'\backslash K)_S \\ {\scriptstyle s}\Big\downarrow\Big\downarrow{\scriptstyle d} \\ \coprod_{i\in I} \mathrm{M}({}^{K'}\mathbf{X}), \end{array}$$

where s is given by the projection and d is given by the action of $K'\backslash K$ on the right, mapping however something indexed by (i,j) to the i-th copy (s) and to j-th copy (d), respectively. The composition is given (functorially) by $(x, g; i, j)(x', g'; k, l) = (x, gg'; i, l)$, as long as $x' = xg$ and $j = k$ (trivial generalization of the usual groupoid induced by a group action).

Now the groups $K_i = ({}^{\rho}U_{\mathbf{B}_i}(\mathbb{A}^{(\infty)}) \cap K)K'$ are neat again, because the $U_{\mathbf{B}_i}$ are unipotent. Denote $U_i := K_i/K'$. Now we let U_i operate (on the left) on the i-th factor in the bottom (via g^{-1} acting from the right) and on each (i,j)-th factor on the top by via g^{-1} acting from the right on $\mathrm{M}({}^{K'}\mathbf{X})$ and multiplication by g from the left on $(K'\backslash K)_S$. We let U_j operate on the (i,j)-th factor on the top, too, by multiplication by g from the right on $(K'\backslash K)$. We form the quotient

$$\begin{array}{c} \coprod_{i,j\in I} U_i\backslash \mathrm{M}({}^{K'}\mathbf{X}) \times_S (K'\backslash K)_S/U_j \\ {\scriptstyle s}\Big\downarrow\Big\downarrow{\scriptstyle d} \\ \coprod_{i\in I} \mathrm{M}({}^{K_i}\mathbf{X}). \end{array}$$

Note that all groups act freely and the composition map is invariant. We compactify

$$\begin{array}{c} \coprod_{i,j\in I, k\in K/K_j} \mathrm{M}({}^{K_i}_{(P_{\mathbf{X}}(\mathbb{Q})\Delta_i K')\cap(P_{\mathbf{X}}(\mathbb{Q})\Delta_j K'k^{-1})}\mathbf{X}) \\ {\scriptstyle s}\Big\downarrow\Big\downarrow{\scriptstyle d} \\ \coprod_{i\in I} \mathrm{M}({}^{K_i}_{P_{\mathbf{X}}(\mathbb{Q})\Delta_i K'}\mathbf{X}), \end{array}$$

where we identified $U_i\backslash \mathrm{M}({}^{K'}\mathbf{X}) \times_S (K'\backslash K)_S/U_j$ with $\coprod_{k\in K/K_j} \mathrm{M}({}^{K_i}\mathbf{X})$ in a non canonical way. $(P_{\mathbf{X}}(\mathbb{Q})\Delta_i K') \cap (P_{\mathbf{X}}(\mathbb{Q})\Delta_j K'k^{-1}) = P_{\mathbf{X}}(\mathbb{Q})\Delta_k K'$ does not depend on this choice. Here k is the (equivalence class of) the biggest boundary component, such that (a representative in the equivalence classes of) (ι_i, ρ_i) and $(\iota_j, \rho_j k^{-1})$, respectively, are both itself boundary components of (ι_k, ρ_k). The composition map and projections extend by functoriality (i of the theorem). We define $\mathrm{M}({}^K_\Delta\mathbf{X})$ as the Deligne-Mumford stack described by this groupoid of schemes. It is equipped with a morphism $\mathrm{M}({}^{K'}_\Delta\mathbf{X}) \to \mathrm{M}({}^K_\Delta\mathbf{X})$ which is not etale at the boundary. The stack hence is proper, if Δ was complete. It satisfies the other properties stated in the theorem. Note: It would not have been strictly necessary to consider *all* boundary components in the above procedure. Minimal ones are sufficient.

STEP 1: From now on, we may assume that all occurring K's are neat. (ii) pins down the compactification 'along the unipotent fibre'. These satisfy all functorialities as required in (i) as long as only Δ's are involved which are concentrated in the unipotent fibre. This is literally shown as in [83, 6.7] (use only the case of neat groups!)

STEP 2: By [27, IV, Theorem 6.7] and [27, VI, Theorem 1.13] compactifications $\mathrm{M}({}^K_\Delta\mathbf{X})$ exist for all symplectic Shimura data $\mathbf{X} = \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]/\mathbb{W}(U')$, a cofinal system of K's and all smooth Δ. They satisfy properties (iii)-(iv) and are smooth Deligne-Mumford stacks. They are smooth projective schemes, if the K's are neat and Δ is projective [27, V, §5]. To apply the results of [27], it may already be necessary to pass to a refinement of Δ, which is compatible with the maps (2.5.6)[2].

STEP 3: Construction of arbitrary toroidal compactifications. We choose an embedding ${}^K_\Delta\mathbf{X} \hookrightarrow {}^{K'}_{\Delta'}\mathbf{X}'$ in a datum of symplectic type $\mathbf{X}' = \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]/\mathbb{W}(U')$. This exists because of (2.2.8). We may also refine Δ, such that it *and* Δ' are smooth projective (2.4.12). Define $\mathrm{M}({}^K_\Delta\mathbf{X})$ as the normalization of the Zariski closure in $\mathrm{M}({}^{K'}_{\Delta'}\mathbf{X}')_S$, where $S = \mathrm{spec}(\mathcal{O})$ for the chosen reflex ring of $\mathbf{X}$.

Let $(\iota, \rho) : {}^{K_1}_{\Delta_1}\mathbf{B} \Longrightarrow {}^K_\Delta\mathbf{X}$ a boundary component. There is a boundary component $(\iota', \rho') : {}^{K'_1}_{\Delta'_1}\mathbf{B}' \Longrightarrow {}^K_\Delta\mathbf{X}'$ such that we have a commutative diagram:

$$\begin{tikzcd} {}^{K_1}_{\Delta_1}\mathbf{B} \arrow[r, Rightarrow] \arrow[d, hook] & {}^K_\Delta\mathbf{X} \arrow[d, hook] \\ {}^{K'_1}_{\Delta'_1}\mathbf{B}' \arrow[r, Rightarrow] & {}^{K'}_{\Delta'}\mathbf{X}'. \end{tikzcd}$$

Let σ be as in (iv) of the theorem. Denote $\mathcal{M} := \mathrm{M}({}^{K'}_{\Delta'}\mathbf{X}')_S$, $\widetilde{\mathcal{M}} := \mathrm{M}({}^{K'_1}_{\Delta'_1}\mathbf{B}')_S$, $M' := \mathrm{M}({}^K_\Delta\mathbf{X})_E$, $\widetilde{M}' := \mathrm{M}({}^{K_1}_{\Delta_1}\mathbf{B})_E$. Let C and C' be the respective rational (closed) boundary strata associated with $[\sigma]$ and $\mathcal{C}, \mathcal{C}'$ be their closures. Then (iv), for (closed) codimension 1 boundary strata, is true by lemma (5.10.2), using main conjecture (3.3.2) which we have to assume at this point.

The fibre above $\wp$ hence is smooth by induction on the dimension of the reductive part. We have also that the Zariski closure of C' in the normalization of $\mathcal{M}'$ is already normal. Reason: This is true in $\widetilde{\mathcal{M}}'$ because the latter is by construction (etale) locally a product of $\mathcal{C}'$ with a (torus-embedding) compactification of $\mathbb{G}_m^n$ (which is normal).

[2] An explicit comparison of our language to the one used in [27] will be provided in forthcoming work. In [27] the compactification is literally worked out only if $\mathbf{X} = \mathbf{H}_{g_0}[0, L_0 \otimes I]$, however the 'second unipotent step' is much easier

Hence normalization of $\mathcal{M}'$ also normalizes $\mathcal{C}$.
The stratification is obtained by induction on the dimension (the union of all C, obtained by the above procedure, contains all other strata by 3.3.2).
For each of the induction steps, we may have to refine the original Δ, but only for each equivalence class of boundary components, so a finite number of times in each induction step.
It remains to see that (iv) holds for codimension ≥ 1 strata as well. For this, we may assume that Δ is fine enough, such that each $\sigma \in \Delta$, $\dim \sigma > 1$ has the property that there exists a 1 dimensional face τ, such that $\sigma, \tau \subset C(\mathbb{D}^0, P_{\mathbf{B}}) \times \rho$. Then the statement is obvious because it holds for $[\tau]$ and we may just complete again both mixed Shimura varieties at the boundary stratum corresponding to $[\sigma]$.
STEP 4: We have to show that all maps of p-ECMSD induce maps between toroidal compactifications (i). We proceed again by induction on the dimension of the reductive part using (iv) and (5.11.1). Note that maps extend along the unipotent fibre by STEP 1. The existence of these extensions shows, in particular, uniqueness of our models. □

(3.3.7) Remark. By the methods of [22, 25] one can construct models also for a (pure) Shimura datum $\mathbf{X}$ for which there is $\mathbf{Y}$ is of Hodge type and an isomorphism $(P_{\mathbf{X}})^{ab} \cong (P_{\mathbf{Y}})^{ab}$. We have to assume furthermore that the morphisms $P_{\mathbf{X}} \to (P_{\mathbf{X}})^{ab}$, resp. $P_{\mathbf{Y}} \to (P_{\mathbf{Y}})^{ab}$ are smooth.
The main theorems (3.2.2) and (3.3.5) of this section also hold true for Shimura data of this kind (cf. [78, 3.23] for the uncompactified case)[3].

(3.3.8) Theorem. *Let ${}^K_\Delta\mathbf{X}$ be irreducible p-ECMSD. If Δ is smooth, then the complement D of the open stratum $\mathrm{M}({}^K\mathbf{X})$ in $\mathrm{M}({}^K_\Delta\mathbf{X})$ — the union of all lower dimensional strata in (3.2.2, v) — is a smooth divisor with normal crossings on $\mathrm{M}({}^K_\Delta\mathbf{X})$.*

Proof. Via the formal isomorphisms (3.3.5, iv) this follows from the corresponding property of torus embeddings [27, , IV, §2]. □

3.4. Integral duals

Let p be a prime and $\mathbf{X}$ be p-integral mixed Shimura data. Let $\mathcal{O}$ be a reflex ring of it at p and $S = \mathrm{spec}(\mathcal{O})$.
The closure $\overline{M}$ of the conjugacy class of the morphisms $u_x, x \in \mathbb{X}$ is defined over $\mathcal{O}$ by definition (3.1.1). It corresponds to a section $t' \in \mathcal{FTYPE}(S)$.
The image of $\overline{M}$ via qpar is a fibre of the morphism 'type' above a section $t: S \to \mathcal{TYPE}$. We define $\mathrm{M}^\vee(\mathbf{X})$ to be this fibre. We understand here the action of $\mathcal{P}$ on $\mathcal{QPAR}$ by

[3] This argument lacks details. It will be stated more precisely in forthcoming work. See also M. Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. 23 (2010), 967-1012 for the uncompactified case

conjugation on the right (contrary to 1.9.9). The action fixes the morphism 'type', hence we have an induced action on $\mathrm{M}^\vee(\mathbf{X})$. For this we have

(3.4.1) Main theorem. $\mathrm{M}^\vee(\mathbf{X})$ *is a smooth connected quasi-projective right $P_\mathbf{X}$-homogeneous scheme over S called **the dual** of $\mathbf{X}$ (depending only on $P_\mathbf{X}$ and $h(\mathbb{D}_\mathbf{X})$) with the following properties:*

i. If $\mathbf{X}$ is pure, $\mathrm{M}^\vee(\mathbf{X})$ is projective.

ii. Let L be a free $\mathbb{Z}_{(p)}$-module of finite dimension, acted on faithfully by $P_\mathbf{X}$. $\mathrm{M}^\vee(\mathbf{X})$ represents

$$S' \mapsto \{ \text{ filtrations } \{F^\bullet\} \text{ on } L_{S'} \text{ compatible with } (P_{\mathbf{X},S'}, L_{S'}) \text{ of type } t' \ \},$$

compatible with $P_\mathbf{X}$-action (defined on L as $v \cdot g := g^{-1}v$).

iii. For each map of p-integral mixed Shimura data $\gamma : \mathbf{X} \to \mathbf{Y}$ let $\mathcal{O}'$ be a reflex ring of the second data such that $\mathcal{O}' \subset \mathcal{O}$, and $S' = \mathrm{spec}(\mathcal{O}')$. There is a morphism

$$\mathrm{M}^\vee(\gamma) : \mathrm{M}^\vee(\mathbf{X}) \to \mathrm{M}^\vee(\mathbf{Y}) \times_{S'} S.$$

It is a closed embedding, if γ is an embedding. These maps are homogeneous and compatible with composition.

iv. For each p-integral boundary map $\iota : \mathbf{B} \Longrightarrow \mathbf{X}$, there is a $P_\mathbf{B}$-equivariant open embedding

$$\mathrm{M}^\vee(\iota) : \mathrm{M}^\vee(\mathbf{B}) \hookrightarrow \mathrm{M}^\vee(\mathbf{X}).$$

v. There is an $P_\mathbf{X}(\mathbb{R})W_\mathbf{X}(\mathbb{C})$-equivariant open embedding (Borel embedding)

$$h(\mathbb{D}_\mathbf{X}) \hookrightarrow (\mathrm{M}^\vee(\mathbf{X})_\mathbb{C})^{an}.$$

It is an isomorphism, if $G_\mathbf{X}$ (the reductive part of $P_\mathbf{X}$) is a torus. These embeddings are compatible with the embeddings in (iv) resp. (2.4.4, ii. (a))[4]. Furthermore they are compatible with morphisms of Shimura data.

Proof. (i) follows from (1.9.4, (i)), since $P_\mathbf{X}$ is reductive in this case and hence parabolics and quasi-parabolics are the same.
(ii) is (1.9.14).
(iii) is (1.9.11).
(iv) (1.9.11). It is open because the image has to be dominant (it suffices that this is true over $\mathbb{C}$ and this follows from (v)).
(v) is well known.

□

[4] going in different directions!

(3.4.2) Remark. For a boundary component $\mathbf{B} \Longrightarrow \mathbf{X}$, there is also an embedding

$$\mathrm{M}^\vee(\mathbf{B}/W_\mathbf{B}) \hookrightarrow \mathrm{M}^\vee(\mathbf{X})$$

into (in general not onto) the complement of the image of $\mathrm{M}^\vee(\mathbf{B})$. The image of the composition with the Borel embedding $h(\mathbb{D}_\mathbf{B}/W_\mathbf{B}) \hookrightarrow \mathrm{M}^\vee(\mathbf{B}/W_\mathbf{B})(\mathbb{C})$ is the boundary component of $h(\mathbb{D}_\mathbf{X})$ associated with Q (the parabolic associated with $\mathbf{B}$) in the sense of [2].

3.5. Integral standard principal bundle

(3.5.1) Consider the situation of (3.3.4) again — an arbitrary *Abelian* unipotent extension

$$\mathbf{X}[U,0] \to \mathbf{X},$$

a torsor under the *group object*

$$\mathbf{X}/W_\mathbf{X}[U,0] \to \mathbf{X}/W_\mathbf{X}.$$

The corresponding map of Shimura varieties (assume all K's in the sequel to be neat):

$$\mathrm{M}({}^{K'}(\mathbf{X}/W_\mathbf{X})[U,0]) \to \mathrm{M}({}^{K'/U}\mathbf{X}/W_\mathbf{X})$$

is a split torus with character group (in general non canonically) isomorphic to $U_\mathbb{Q} \cap K$. Assume that a functorial theory of models of the standard principal bundle exists (as described in theorem 3.5.2 below) on *uncompactified* mixed Shimura varieties. We have then automatically a compatible action of the torus $T := \mathrm{M}({}^{K}(\mathbf{X}/W_\mathbf{X})[U,0])$ on $\mathrm{P}({}^{K'}\mathbf{X}[U,0])$ as well. Hence the standard principal bundle may, etale locally (say, on $S \to \mathrm{M}({}^{K'/U}\mathbf{X}/W_\mathbf{X})$) be trivialized T-invariantly.
Let Δ be a K-admissible polyhedral cone decomposition for $\mathbf{X}[U,0]$, concentrated in the fibre U. There is a unique extension of $\mathrm{P}({}^{K}\mathbf{X}[U,0])$ to the torus embedding $\mathrm{M}({}^K_\Delta\mathbf{X}[U,0])$ by means of extending any T-invariant trivialization described above. We denote it by $\mathrm{P}({}^K_\Delta\mathbf{X}[U,0])$.
These extensions are compatible with all maps between p-ECMSD, which involve only rational polyhedral cone decompositions along the unipotent fibre.
From the functoriality also follows that Π is constant on these trivializations. Its image is a single S-valued point of the compact dual. Therefore the map Π also extends (cf. also 2.7.7).

(3.5.2) Main theorem. *There is a* unique *(up to unique isomorphism) map associating with every p-ECMSD* ${}^K_\Delta\mathbf{X}$ *of Hodge type and reflex ring* $\mathcal{O}$ *(such that* $\mathrm{M}({}^K_\Delta\mathbf{X})$ *exists with the properties of 3.3.5) a right* $P_\mathbf{X}$*-torsor* $\mathrm{P}({}^K_\Delta\mathbf{X}) \to \mathrm{M}({}^K_\Delta\mathbf{X})$ *and a* $P_\mathbf{X}$*-equivariant morphism*

$$\Pi : \mathrm{P}({}^K_\Delta\mathbf{X}) \to \mathrm{M}^\vee(\mathbf{X})$$

and an $P_{\mathbf{X}}(\mathbb{C})$*-equivariant isomorphism*

$$(\mathrm{P}({}^{K}\mathbf{X}) \times_S \mathrm{spec}(\mathbb{C}))^{an} \cong P_{\mathbf{X}}(\mathbb{Q})\backslash P_{\mathbf{X}}(\mathbb{C}) \times \mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)$$

see (2.7.4), such that $\Pi_{\mathbb{C}}$ *is the one given in (2.7.4), with the following properties:*

i. For each morphism of p-ECMSD $[\gamma, \rho] : {}^{K_1}_{\Delta_1}\mathbf{X}_1 \to {}^{K_2}_{\Delta_2}\mathbf{X}_2$ *there is an induced* $P_{\mathbf{X}_1}$*-equivariant morphism*

$$\mathrm{P}(\gamma, \rho) : \mathrm{P}({}^{K_1}_{\Delta_1}\mathbf{X}_1) \to \mathrm{P}({}^{K_2}_{\Delta_2}\mathbf{X}_2) \times_{S_2} S_1.$$

Here $S_i = \mathrm{spec}(\mathcal{O}_i)$*, where* $\mathcal{O}_i$ *are reflex rings of* $\mathbf{X}_i$ *such that* $\mathcal{O}_2 \subset \mathcal{O}_1$*.*

The diagram

$$\begin{tikzcd}
\mathrm{M}({}^{K_1}_{\Delta_1}\mathbf{X}_1) \arrow[d, "\mathrm{M}(\gamma,\rho)"'] & \mathrm{P}({}^{K_1}_{\Delta_1}\mathbf{X}_1) \arrow[l] \arrow[r] \arrow[d, "\mathrm{P}(\gamma,\rho)"] & \mathrm{M}^{\vee}(\mathbf{X}_1) \arrow[d, "\mathrm{M}^{\vee}(\gamma)"] \\
\mathrm{M}({}^{K_2}_{\Delta_2}\mathbf{X}_2) \times_{S_2} S_1 & \mathrm{P}({}^{K_2}_{\Delta_2}\mathbf{X}_2) \times_{S_2} S_1 \arrow[l] \arrow[r] & \mathrm{M}^{\vee}(\mathbf{X}_2) \times_{S_2} S_1
\end{tikzcd}$$

is commutative. The 1st vertical map is the map (3.2.2, iv), the 3rd is (3.4.1, ii). $\mathrm{P}(\gamma, \rho)_{\mathbb{C}}$ *is equal to the obvious map on double quotients (2.7.4).*

ii. Let $(\iota, \rho) : {}^{K'}_{\Delta'}\mathbf{B} \Longrightarrow {}^{K}_{\Delta}\mathbf{X}$ *be a boundary map and* $\sigma \in \Delta'$ *such that* $\sigma \in C(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}})$*, as in (3.3.3).*

There is a $P_{\mathbf{X}}$*-equivariant map* $\mathrm{P}(\iota, \rho)$ *fitting into the commutative diagram*

$$\begin{tikzcd}
\widehat{[\mathrm{Stab}_{\Gamma}([\sigma])\backslash \mathrm{M}({}^{K'}_{\Delta'}\mathbf{B})]} \arrow[d, "\sim"', "\mathrm{M}(\iota,\rho)"] & \widehat{[\mathrm{Stab}_{\Gamma}([\sigma])\backslash \mathrm{P}({}^{K'}_{\Delta'}\mathbf{B})]} \arrow[l] \arrow[r] \arrow[d, "\mathrm{P}(\iota,\rho)"] & \mathrm{M}^{\vee}(\mathbf{B}), \arrow[d, hook, "\mathrm{M}^{\vee}(\iota)"] \\
\widehat{\mathrm{M}({}^{K}_{\Delta}\mathbf{X})} & \widehat{\mathrm{P}({}^{K}_{\Delta}\mathbf{X})} \arrow[l] \arrow[r] & \mathrm{M}^{\vee}(\mathbf{X})
\end{tikzcd}$$

where the formal completions are taken along

$$\left[\mathrm{Stab}_{\Gamma}([\sigma])\backslash \mathrm{M}({}^{K'_{[\sigma]}}_{\Delta'_{[\sigma]}}\mathbf{B}_{[\sigma]})\right]$$

respectively along their pre-images in the standard principal bundles.

The complexification of $\mathrm{P}(\iota, \rho)$ *converges in a neighborhood of the boundary stratum and is in the interior, via the identification with the complex analytic mixed Shimura varieties, given by a quotient of the map*

$$\mathbb{D}_{\mathbf{B}\Longrightarrow\mathbf{X}} \times P_{\mathbf{B}}(\mathbb{C}) \times (P_{\mathbf{B}}(\mathbb{A}^{(\infty)})/K') \to \mathbb{D}_{\mathbf{B}\Longrightarrow\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{C}) \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K)$$

induced by the closed embedding $P_{\mathbf{B}} \hookrightarrow P_{\mathbf{X}}$*, where we considered* $\mathbb{D}_{\mathbf{B}\Longrightarrow\mathbf{X}} \subseteq \mathbb{D}_{\mathbf{X}}$ *as* *a subset of $\mathbb{D}_{\mathbf{B}}$ via the analytic boundary map (2.4.4).*

iii. (canonicity) For $\mathbf{X} = \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]$ (and $\Delta = 0$), $\mathrm{P}({}^K\mathbf{X})$ represents

$$[\ {}^K\mathbf{X}\text{-}L\text{-}\boldsymbol{triv\text{-}1mot}\ ]$$

defined in (4.5.2), in such a way that the map Π is identified with the map Π given there.

iv. If Δ is concentrated in the unipotent fibre, $\mathrm{P}({}^K_\Delta\mathbf{X})$ is the extension defined in (3.5.1).

Proof. The truth of this theorem for the rational models follows from [39–41] (cf. also [42, §3]).
STEP 0: Again, it suffices to consider the case K neat. This is shown by the same procedure as in STEP 0 of the proof of (3.3.5).
STEP 1: The construction in the uncompactified pure and mixed case is done in (5.3.1) and (5.4.1) respectively, using a Hodge embedding.
STEP 2: Functoriality (i) and independence of the Hodge embedding is shown in (5.6.1). The maps to the compact dual are extended in (5.5) to the integral models.
STEP 3: For the extension to the whole compactification (neat case) we proceed as follows.
We choose an embedding ${}^K_\Delta\mathbf{X} \hookrightarrow {}^{K'}_{\Delta'}\mathbf{X}'$ in a datum of symplectic type $\mathbf{X}' = \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]/\mathbb{W}(U')$. This exists because of (2.2.8). We may also refine Δ, such that it *and* Δ' are smooth projective (2.4.12). By assumption $\mathrm{M}({}^K_\Delta\mathbf{X})$ exists.
Consider the corresponding morphism of models

$$\mathrm{M}({}^K_\Delta\mathbf{X}) \to \mathrm{M}({}^{K'}_{\Delta'}\mathbf{X}') \times_{\mathbb{Z}_{(p)}} \mathcal{O}.$$

It is a normalization followed by a closed embedding. Consider the pullback of the de Rham bundle H on $\mathrm{M}({}^{K'}_{\Delta'}\mathbf{X}')$ to $\mathrm{M}({}^K_\Delta\mathbf{X})$. The de Rham bundle is the unique extension to $\mathrm{M}({}^{K'}_{\Delta'}\mathbf{X}')$ of the de Rham realization (4.1.5) of the universal 1-motive over $\mathrm{M}({}^{K'}\mathbf{X}')$, with the property that the Gauss-Manin connection has logarithmic singularities along D (cf. [27]).
Choose an open cover $\{U_i\}$ of $\mathrm{M}({}^K_\Delta\mathbf{X})$ trivializing H by $P_{\mathbf{X}'}$-equivariant isomorphisms with $L_{\mathbb{Z}_{(p)}}$. We have to give a $P_{\mathbf{X}}$-structure extending the one given on the generic fibre (by the rational theory). The required structure can be given by morphisms $U_i \to P_{\mathbf{X}'}/P_{\mathbf{X}}$. We know that they extend to the complement of the boundary divisor (STEP 1). It also extends to the compactification along the unipotent fibre, by means of a trivialization using the torsor property as in (3.5.1). Now consider any stratum C in the stratification (3.3.5, iii) which does not belong to the unipotent fibre (in particular is not dense). We have a corresponding formal isomorphism (3.3.5, iv):

$$\widehat{\mathrm{M}({}^{K_1}_{\Delta_1}\mathbf{B})} \simeq \widehat{\mathrm{M}({}^K_\Delta\mathbf{X})},$$

where the completion is taken along the closure of C and ${}^{K_1}_{\Delta_1}\mathbf{B} \Longrightarrow {}^{K}_{\Delta}\mathbf{X}$ is the corresponding boundary component.
By induction on the dimension of $G_{\mathbf{X}}$, we may assume that there exists a $P_{\mathbf{B}}$-structure on the pullback of H to the left, extending the one given over $\mathbb{C}$. We hence have a morphism

$$\widehat{U_i} \to P_{\mathbf{X}'}/P_{\mathbf{B}} \to P_{\mathbf{X}'}/P_{\mathbf{X}},$$

extending the one given over $\mathbb{C}$, for every irreducible component of the boundary divisor. By (5.11.1) we get the required $P_{\mathbf{X}}$ structure. (ii) holds by construction and (5.5) for maximal boundary components.
STEP 4: The maps required in (i), especially yielding uniqueness, extend like in the proof of (3.3.5). □

We have the following translation of theorem (3.5.2) into the language of (Artin) stacks:

(3.5.3) Theorem. *There is a* unique *(up to unique isomorphism) map associating with every p-ECMSD ${}^{K}_{\Delta}\mathbf{X}$ of Hodge type and reflex ring $\mathcal{O}$ (such that* $\mathrm{M}({}^{K}_{\Delta}\mathbf{X})$ *exists with the properties of 3.3.5) a 1-morphism*

$$\Xi({}^{K}_{\Delta}\mathbf{X}) : \mathrm{M}({}^{K}_{\Delta}\mathbf{X}) \to \left[\mathrm{M}^\vee(\mathbf{X})/P_{\mathbf{X}}\right],$$

and a 2-isomorphism

$$\Xi({}^{K}\mathbf{X}) \times_{\mathrm{spec}(\mathcal{O})} \mathrm{spec}(\mathbb{C}) \to \Xi_{\mathbb{C}}({}^{K}\mathbf{X}),$$

where $\Xi_{\mathbb{C}}({}^{K}\mathbf{X})$ is the complex analytic morphism described in (2.7.4), satisfying:

i. For each morphism of p-ECMSD $[\gamma, \rho] : {}^{K_1}_{\Delta_1}\mathbf{X}_1 \to {}^{K_2}_{\Delta_2}\mathbf{X}_2$ there is a 2-isomorphism $[\gamma, \rho]_\Xi$ fitting into the diagram

$$\begin{tikzcd}
\mathrm{M}({}^{K_1}_{\Delta_1}\mathbf{X}_1) \arrow[r, "\Xi({}^{K_1}_{\Delta_1}\mathbf{X}_1)"] \arrow[d, "\mathrm{M}(\gamma,\rho)"'] \arrow[dr, bend right] \arrow[dr, bend left] \arrow[dr, phantom, "{[\gamma,\rho]_\Xi}\Uparrow"] & \left[\mathrm{M}^\vee(\mathbf{X}_1)/P_{\mathbf{X}_1}\right] \arrow[d, "\mathrm{M}^\vee(\gamma)"] \\
\mathrm{M}({}^{K_2}_{\Delta_2}\mathbf{X}_2) \times_{S_2} S_1 \arrow[r, "\Xi({}^{K_2}_{\Delta_2}\mathbf{X}_2)"'] & \left[\mathrm{M}^\vee(\mathbf{X}_2)/P_{\mathbf{X}_2}\right] \times_{S_2} S_1
\end{tikzcd}$$

These 2-isomorphisms are compatible with composition. Here $S_i = \mathrm{spec}(\mathcal{O}_i)$, where $\mathcal{O}_i$ are reflex rings of $\mathbf{X}_i$ such that $\mathcal{O}_2 \subset \mathcal{O}_1$.

$[\gamma, \rho]_{\Xi,\mathbb{C}}$ is the obvious 2-morphism over $\mathbb{C}$.

ii. Let $[\iota, \rho] : {}^{K'}_{\Delta'}\mathbf{B} \Longrightarrow {}^{K}_{\Delta}\mathbf{X}$ be a boundary map and $\sigma \in \Delta'$ such that $\sigma \in C(\mathbb{D}^0_{\mathbf{X}}, P_{\mathbf{B}})$, as in (3.3.3).

There is a 2-isomorphism $[\iota, \rho]_\Xi$ fitting into the diagram

$$\begin{tikzcd}
{[\widehat{\mathrm{Stab}_\Gamma([\sigma])\backslash \mathrm{M}({}^{K'}_{\Delta'}\mathbf{B})}]} \arrow[r, "\Xi({}^{K'}_{\Delta'}\mathbf{B})"] \arrow[d, "\mathrm{M}(\iota,\rho)"'] \arrow[dr, bend right, ""{name=L}] \arrow[dr, bend left, ""'{name=U}] & {[\mathrm{M}^\vee(\mathbf{B})/P_\mathbf{B}]} \arrow[d, "\mathrm{M}^\vee(\iota)"] \\
\widehat{\mathrm{M}({}^{K}_{\Delta}\mathbf{X})} \arrow[r, "\Xi({}^{K}_{\Delta}\mathbf{X})"'] & {[\mathrm{M}^\vee(\mathbf{X})/P_\mathbf{X}]}
\arrow[Rightarrow, from=L, to=U, "{[\iota,\rho]_\Xi}"]
\end{tikzcd}$$

where the formal completion is taken along

$$\left[\mathrm{Stab}_\Gamma([\sigma])\backslash \mathrm{M}({}^{K'_{[\sigma]}}_{\Delta'_{[\sigma]}}\mathbf{B}_{[\sigma]})\right].$$

$[\iota, \rho]_{\Xi,\mathbb{C}}$ *is the obvious 2-morphism over* $\mathbb{C}$.

iii. *(canonicity) For* $\mathbf{X} = \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]$ *(and* $\Delta = 0$*),* $\Xi({}^K\mathbf{X})$ *is given as described in (4.5.2).*

iv. *For* Δ *concentrated in the unipotent fibre,* Ξ *is given by via the extension described in (3.5.1).*

(3.5.4) Conjecture. *The previous theorems are true for arbitrary* p *(including* $p = 2$*) and for arbitrary mixed Shimura data.*

Maybe for this to be true, it is necessary to find a better condition of canonicity.

3.6. Generalities on models and the adelic action

(3.6.1) Theorem. *Let* $\mathbf{X}$ *be* p*-integral mixed Shimura data and* $\mathcal{O}$ *a reflex ring. It is equivalent to give*

i. *a scheme* $\mathrm{M}^p(\mathbf{X})$ *over* $\mathrm{spec}(\mathcal{O})$ *with a continuous right* $P_\mathbf{X}(\mathbb{A}^{(\infty,p)})$*-action,*

ii. *for each admissible compact open subgroup* K *a Deligne-Mumford stack*

$$\mathrm{M}({}^K\mathbf{X})$$

over $\mathrm{spec}(\mathcal{O})$ *and maps* $J_{L,K}(x)$ *for any admissible* K, L *such that* $xKx^{-1} \subset L$ *with the properties*

a) $J_{M,L}(y)J_{L,K}(x) = J_{M,K}(yx)$,

b) $J_{K,K}(x) = \mathrm{id}$ *if* $x \in K$, *c) If $K \trianglelefteq L$, $J_{K,K}$ defines an action of $K \backslash L$ on* $\mathrm{M}({}^K\mathbf{X})$ *and* $J_{K,L}(1)$ *defines the quotient* $[\mathrm{M}({}^K\mathbf{X})/(K\backslash L)] \cong \mathrm{M}({}^L\mathbf{X})$.

iii. for some admissible maximal *compact open subgroup K_0 and some cofinal system $\mathcal{K}$ of neat and admissible normal subgroups $K \subset K_0$ and for each $K \in \mathcal{K}$ a scheme*

$$\mathrm{M}({}^K\mathbf{X})$$

over $\mathrm{spec}(\mathcal{O})$ *and maps $J_{L,K}(x)$ for each $K, L \in \mathcal{K}$ such that $xKx^{-1} \subset L$ with the properties in (ii) above.*

Proof. (i) $\Leftrightarrow$ (ii) is indicated in [Deligne5], (iii) $\Rightarrow$ (i) is proven the same way (using conjugacy of maximal compact open subgroups), and (ii) $\Rightarrow$ (iii) is trivial. □

3.7. The extension property

(3.7.1) Definition. *Let $\mathcal{O}$ be a discrete valuation ring with fraction field F. Let a* ***test scheme*** *S over $\mathcal{O}$ be as in [78, Def. 3.5], i.e. S has a cover by open affines* $\mathrm{spec}(A)$, *such that there exist rings $\mathcal{O} \subseteq \mathcal{O}' \subseteq A_0 \subseteq A$, where*

- *$\mathcal{O} \subseteq \mathcal{O}'$ is a faithfully flat and unramified extension of d.v.r. with $\mathcal{O}'/(\pi)$ separable over $\mathcal{O}/(\pi)$.*
- *A_0 is a smooth $\mathcal{O}'$-algebra.*
- *$A_0 \subseteq A_1 \subseteq \cdots \subseteq A$ is a countable union of etale extensions.*

As is explained in [loc. cit.], this has to be seen only as a working definition. The arguments below work, for example, only for $p \neq 2$. Note that the projective limit $\mathrm{M}^p(\mathbf{X})$ of *smooth* models of Shimura varieties $\mathrm{M}({}^K\mathbf{X})$ over all admissible K is a test scheme itself.

(3.7.2) Theorem. *Let S be a test scheme over $\mathcal{O}$. For every closed subscheme $Z \hookrightarrow S$, disjoint from S_F and of codimension at least 2 in S, every Abelian scheme or 1-motive over $U = S \setminus Z$ extends to an Abelian scheme over S.*

Proof. [78, 3.6] for the case of Abelian schemes. A semi-Abelian scheme extends to by S [27, V, 6.7], which we may apply for this class of test schemes [78, 3.6 ff.]. The Abelian part extends to an Abelian scheme, hence the semi-Abelian extension is globally an extension. The morphism $\underline{Y} \to A$ extends as well because it gives a semi-Abelian scheme extending $A^\vee$. The rest can, as in the proof of (3.7.6), be reduced to the case of a morphism $U \to \mathbb{G}_m$, i.e. a unit in $\mathcal{O}_U$. It extends because S is normal. □

(3.7.3) Theorem. *Let S be a normal scheme, and $U \hookrightarrow S$ be a dense open subscheme. If $M_i, i = 1, 2$ are 1-motives over S, every homomorphism $\phi : M_1|U \to M_2|U$ extends uniquely to a homomorphism $\widetilde{\phi} : M_1 \to M_2$.*

Proof. [76, Prop. 2.12] for the case of Abelian schemes. Morphisms between etale sheaves of free $\mathbb{Z}$-modules $\underline{X}_i$, resp. $\underline{Y}_i$ clearly extend uniquely. □

In particular an extension of A over U is unique, if it exists.

(3.7.4) Theorem. *Let S be a Dedekind scheme (e.g. the spectrum of a d.v.r.) with function field K. Let A be an Abelian scheme over S, then it is the Neron model of A_K, which means: for any smooth S-scheme Y we have*

$$\mathrm{Hom}_K(Y_K, X_K) = \mathrm{Hom}_S(Y, X).$$

Proof. [8, Definition 1.2/1] and [8, Proposition 1.2/8] □

(3.7.5) Theorem. *Let R be a discrete valuation ring, with field of fractions K. Let A_K be an Abelian scheme over* $\mathrm{spec}(K)$, *with Neron model A over* $\mathrm{spec}(R)$. *The following is equivalent*

i. A is an Abelian scheme

ii. The inertia subgroup of $\mathrm{Gal}(K^s|K)$ *operates trivially on $H_1^{et}(A, \mathbb{Z}_l)$, where l is invertible in $R/\mathfrak{m}$.*

Proof. [8, 7.4, Theorem 5] □

(3.7.6) Theorem. *Let S be a test scheme over $\mathbb{Z}_{(p)}$, with generic point η, and $l \neq p$ a prime. Let $U \hookrightarrow S$ be a dense open subscheme, $Y = S - U$ and A an Abelian scheme (resp. 1-motive) on U. If the monodromy representation $\pi_1(U, \eta) \to \mathrm{Aut}(H_1^{et}(A_\eta, \mathbb{Z}_l))$ factors through $\pi_1(Y, \eta)$, then A extends to an Abelian scheme (resp. 1-motive) on S.*

Proof. [76, Prop. 2.13] for the case of an Abelian scheme: Because of (3.7.3), there is a largest open subscheme U of S such that A extends to U. Suppose $U \neq S$, and let $y \in S \setminus U$. If $\mathcal{O}_y$ has dimension 1, then it is a discrete valuation ring. Its field of fractions is the function field k of S and of characteristic zero, since S is a test scheme over $\mathbb{Z}_{(p)}$. The assumption means that the action of $\mathrm{Gal}(\overline{k}, k)$ factors through the inertia group at y. Then for the case of Abelian schemes (3.7.5) implies that A extends to $\mathcal{O}_y$. Hence A extends to an open neighborhood of y, contradicting the choice of y. So $\mathcal{O}_y$ has dimension ≥ 2 and hence $S \setminus U$ has codimension ≥ 2. (3.7.2) now implies $S = U$.

A semi-Abelian scheme can be given by a morphism $\alpha^\vee : \underline{X} \to A^\vee$. It suffices to assume that S is of the form spec(B), as in the definition of test scheme. So B is the union of etale rings over B_0, which are of finite type and smooth over $\mathcal{O}'$. A is defined on some spec(B_j), since the moduli space of Abelian schemes is of finite type. Hence $\alpha : \underline{X} \to A^\vee$ is also defined over some spec$(B_i)_U$, since A over B_j and B_j itself are of finite type. Now spec$(B_i)_U$ is smooth so the morphism α to A over k extends to a morphism spec$(B_i) \to A$ over $\mathcal{O}_y$ by the Neron property. The conclusion is the same.
A one-motive is given in addition by a morphism $\alpha : \underline{Y} \to G$. Take etale locally a basis of X, Y. This restricts to the case where G is an extension of A by $\mathbb{G}_m$ and we have to extend *a point* $\xi \in G$. The projection $\pi(\xi)$ of ξ onto A extends to some point in G because a $\mathbb{G}_m$-torsor (fibre over $\pi(\xi)$) over $\mathcal{O}_y$ is trivial. Subtracting this point we are reduced to the case, where we have a k-point $\xi \in \mathbb{G}_m$. We must show that it extends to a point over $\mathcal{O}_y$. The given information translates to the fact that the action of Gal$(\overline{k}, k)$ for each l, $(l, p) = 1$ on the sequence

$$0 \longrightarrow \mu_n \longrightarrow H^{et}(M, \mathbb{Z}/n\mathbb{Z}) \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

factors through the inertia group I at y. Here recall that

$$H^{et}(M, \mathbb{Z}/n\mathbb{Z}) = \{(y, g) \in (\mathbb{Z} \times \mathbb{G}_m) \mid yp = ng\}/\{(ny, yp) \mid y \in \mathbb{Z}\}$$

the inclusion on the left is given by $(0, \zeta)$ for $\zeta \in \mu_n$ and the projection by $(y, g) \mapsto (y \bmod n)$. This means that we can choose a I-invariant lift for the above sequence, hence I operates trivial on the pre-image on ξ under $[n] : \mathbb{G}_m \to \mathbb{G}_m \bmod \mathbb{Z}\xi$. This implies that y extends to an $\mathcal{O}_y$-point because this means that the extension generated by the pre-image is unramified, hence the additive valuation of y has to be divisible by l for all $(l, p) = 1$, so it has to be 0. $\square$

(3.7.7) Definition. *Let $\mathcal{O}$ be a discrete valuation ring, faithfully flat and unramified over $\mathbb{Z}_{(p)}$ with fraction field F, and let X an $\mathcal{O}$-scheme. It has the **extension property**, if for every test scheme T above $\mathcal{O}$, every morpism $T_F \to X_F$ extends uniquely to a morphism $T \to X$.*

(3.7.8) Lemma. *i. If a model X over $\mathcal{O}$ of a scheme X_F over F has the extension property and is itself a test scheme, then this model is unique (up to unique isomorphism).*

ii. If X over $\mathcal{O}$ has the extension property, then for any subscheme $Y \subset X$, Y has the extension property.

iii. If X over $\mathcal{O}$ has the extension property and if X_F is normal, then its normalization $\widetilde{X}$ has the extension property.

iv. Let $X \to S$ and $Y \to S$ two morphisms of $\mathcal{O}$-schemes, if X, Y and S have the extension property, then $X \times_S Y$ has the extension property.

v. *Let $\mathcal{O} \subseteq \mathcal{O}'$ be a faithfully flat and unramified extension of d.v.r.. If T is a test-scheme over $\mathcal{O}$, then $T \times_{\mathrm{spec}(\mathcal{O})} \mathrm{spec}(\mathcal{O}')$ is a test scheme over $\mathcal{O}'$. If T is a test scheme over $\mathcal{O}'$, then considered as a scheme over $\mathcal{O}$ it is a test scheme as well.*

vi. *Let $\mathcal{O} \subseteq \mathcal{O}'$ be as before. If X on $\mathcal{O}$ has the extension property, then $X \times_{\mathrm{spec}(\mathcal{O})} \mathrm{spec}(\mathcal{O}')$ has the extension property.*

Proof. We show property (iii), compare [78, 3.19]: Let S be a test scheme. If $S_F \to \widetilde{X}_F = X_F$ is a morphism, then it extends to a morphism $S \to X$. The projection $\mathcal{S} = S \times_X \widetilde{X} \to S$ has to be an isomorphism since over F it is one, it has to be finite, since $\mathcal{X} \to X$ is finite and S is normal. □

(3.7.9) Theorem. *For* $\mathbf{X} = \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]$,

$$\mathrm{M}^p(\mathbf{X}) := \varprojlim\nolimits_{K \textit{ admissible c. o.}} \mathrm{M}({}^K\mathbf{X})$$

considered over $\mathrm{spec}(\mathbb{Z}_{(p)})$ *has the extension property.*

Proof. (compare [76, 2.10] for the case of Abelian schemes, i.e. $\mathbf{X} = \mathbf{H}_g$.) Consider a test scheme T over $\mathrm{spec}(\mathbb{Z}_{(p)})$. and a morphism $\phi : T_{\mathbb{Q}} \to \mathrm{M}^p(\mathbf{X})$. By the functorial description, this corresponds to the following data (up to $\mathbb{Z}_{(p)}$-isomorphism): an 1-motive $M_{\mathbb{Q}}$ over $T_{\mathbb{Q}}$, a $\mathbb{Z}_{(p)}$-polarization, a trivialization $\underline{Y} \to (U_{\mathbb{Z}})$ and a section of the etale sheaf (on $T_{\mathbb{Q}}$)

$$s \in \mathcal{ISO}(H^{et}(M_{\mathbb{Q}}, \mathbb{A}^{(\infty,p)}), L_{\mathbb{A}^{(\infty,p)}}).$$

i.e. $H^{et}(M_{\mathbb{Q}}, \mathbb{A}^{(\infty,p)})$ is constant, in particular, $\pi_1(T_{\mathbb{Q}}, \eta)$, where η is the generic point of $T_{\mathbb{Q}}$ acts trivial on $H^{et}(M_{\mathbb{Q}}, \mathbb{Z}_l)$. Therefore $M_{\mathbb{Q}}$ extents to M on T by (3.7.6). The polarization extends uniquely by (3.7.3) and [27, p. 6, 1.10b] (for the Abelian part being really a *polarization*). The level structure extends as well uniquely, since $H^{et}(M, \mathbb{A}^{(\infty,p)})$ has to be constant as well. Also the trivialization of $\underline{Y}$ extends. Therefore, we get a unique extension

$$\widetilde{\phi} : T \to \mathrm{M}^p(\mathbf{X}).$$

□

4. One motives

4.1. Definition and realizations

We recall the definition of 1-motives and their most important covariant 'realizations', see e.g. [24] or [16] for proofs and details.

(4.1.1) Definition. *Let S be a scheme. A* ***1-motive*** $M = [\, \underline{Y} \xrightarrow{\alpha} G \,]$ *over S is a complex, where*

i. G is a semi-Abelian scheme with constant rank,

ii. $\underline{Y}$ is an etale sheaf of lattices (of constant rank),

iii. $\alpha : \underline{Y} \to G$ is a homomorphism.

We consider $\underline{Y}$ as being in degree 0 and G as being in degree 1.
Each 1-motive has a filtration where

$$W_i = \begin{cases} M & i \geq 0, \\ [\, 0 \longrightarrow G \,] & i = -1, \\ [\, 0 \longrightarrow T \,] & i = -2, \\ 0 & i < -2. \end{cases}$$

This defines a category

$$[\ S\text{-}\boldsymbol{1mot}\ ]$$

with morphisms being morphisms of complexes.
A morphism of 1-motives

$$\begin{array}{ccc} \underline{Y}_1 & \xrightarrow{\alpha_1} & G_1 \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ \underline{Y}_2 & \xrightarrow{\alpha_2} & G_2 \end{array}$$

is called an ***isogeny****, if for every geometric point $\overline{s}$, $a_{\overline{s}}$ is surjective with finite kernel and $b_{\overline{s}} : \mathbb{Z}^r \hookrightarrow \mathbb{Z}^r$ is injective with finite cokernel. It is called an p-isogeny, if for every $\overline{s}$ the respective kernel and cokernel are of rank prime to p. If S is connected, these ranks are constant and their product is called the* ***degree*** *of the isogeny.*

(4.1.2) Definition. *We define a* ***torsion 1-motive*** *as a finite etale group scheme E over S with a filtration*

$$0 = W_{-3} \subset W_{-2} \subset W_{-1} \subset W_0 = E$$

morphisms being strict *morphisms.*

For an isogeny

$$\psi : [\, \underline{Y}_1 \xrightarrow{\alpha_1} G_1 \,] \to [\, \underline{Y}_2 \xrightarrow{\alpha_2} G_2 \,]$$

of degree N, suppose that N is invertible in S. We define the torsion 1-motive $\ker(\psi)$ *as the cohomology of the double complex*

$$\begin{tikzcd} \underline{Y}_1 \arrow[r, "\alpha_1"] \arrow[d, "\psi_Y"] & G_1 \arrow[d, "\psi_G"] \\ \underline{Y}_2 \arrow[r, "\alpha_2"] & G_2 \end{tikzcd}$$

(there is only one non-trivial cohomology group). It has a filtration coming from the vertical filtration, which defines the subgroup scheme

$$W_{-1}(\ker(\psi)) := \ker(\psi_G).$$

It has a further filtration

$$W_{-2}(\ker(\psi)) := \ker(\psi_G|_T).$$

(4.1.3) Definition. *Let $M = (G, \underline{Y}, \alpha)$ be given. We define the* ***dual motive*** *as $M^\vee = (G', \underline{X}, \alpha')$, where $\underline{X}$ is the etale sheaf $\underline{X}(G)$ as above, G' is the extension*

$$\begin{tikzcd} 0 \arrow[r] & T' \arrow[r] \arrow[d, equal] & G' \arrow[r] \arrow[d, equal] & A^\vee \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Hom}(\underline{Y}, \mathbb{G}_m) \arrow[r] & \mathrm{Ext}(M/W_{-2}M, \mathbb{G}_m) \arrow[r] & \mathrm{Ext}(A, \mathbb{G}_m) \arrow[r] & 0 \end{tikzcd}$$

(it is determined by $\alpha : \underline{Y} \to A = (A^\vee)^\vee$), α' is given by the following construction: Locally each $x \in \underline{X}$ gives a pushout M_x,

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[d, "x"'] & M \arrow[r] \arrow[d] & M/W_{-2}M \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{G}_m \arrow[r] & G_x \arrow[r] & M/W_{-2}M \arrow[r] & 0, \end{tikzcd}$$

and therefore an element of $\mathrm{Ext}(M/W_{-2}M, \mathbb{G}_m)$.

A ***polarization*** *of 1-motives is a morphism $\psi : M \to M^\vee$, inducing a polarization* $\mathrm{gr}_{-1} M \to (\mathrm{gr}_{-1} M)^\vee = \mathrm{gr}_{-1}(M^\vee)$ *and such that* $D(\mathrm{gr}_0 \psi) = \mathrm{gr}_{-2} \psi$.

(4.1.4) There is a symmetric description of the above construction: A 1-motive $M = (G, \underline{Y}, \alpha)$ is equivalent to the following data $(A, A^\vee, \underline{X}, \underline{Y}, \alpha, \alpha^\vee, \nu)$, where

i. A and $A^\vee$ are an Abelian scheme, resp. its dual,

ii. $\underline{X}$ and $\underline{Y}$ are 2 etale sheaves with fibres isomorphic to $\mathbb{Z}^r$, $\mathbb{Z}^q$,

iii. $\underline{Y} \xrightarrow{\alpha} A$ and $\underline{X} \xrightarrow{\alpha^\vee} A^\vee$ are 2 morphism, and

iv.
$$\nu : (\alpha, \alpha^\vee)^* \mathcal{P} \cong (\mathbb{G}_m)_{\underline{Y} \times \underline{X}}$$
is a trivialization of the pullback of the universal (Poincaré) biextension on $A \times A^\vee$.

A morphism of a 1-motives

$$\psi : (A_1, A_1^\vee, \underline{X}_1, \underline{Y}_1, \alpha_1, \alpha_1^\vee, \nu_1) \to (A_2, A_2^\vee, \underline{X}_2, \underline{Y}_2, \alpha_2, \alpha_2^\vee, \nu_2)$$

is, in this description, given by morphisms

$$\psi_A : A_1 \to A_2 \tag{1}$$
$$\psi_Y : \underline{Y}_1 \to \underline{Y}_2 \tag{2}$$
$$\psi_X : \underline{X}_2 \to \underline{X}_1 \tag{3}$$

compatible in the sense that

$$\begin{tikzcd} \underline{Y}_1 \arrow[r, "\alpha_1"] \arrow[d, "\psi_Y"] & A_1 \arrow[d, "\psi_A"] \\ \underline{Y}_2 \arrow[r, "\alpha_2"] & A_2 \end{tikzcd} \qquad \begin{tikzcd} \underline{X}_1 \arrow[r, "\alpha_1"] & A_1^\vee \\ \underline{X}_2 \arrow[r, "\alpha_2"] \arrow[u, "\psi_X"] & A_2^\vee \arrow[u, "\psi_A^\vee"] \end{tikzcd}$$

commute and via the isomorphism $(1, \psi_A^\vee)^* \mathcal{P}_1 \cong (\psi_A, 1)^* \mathcal{P}_2$ on $A_1 \times A_2^\vee$ one has

$$\nu_1(x_1, \psi_X(y_2)) = \nu_2(\psi_Y(x_1), y_2).$$

The assignment is as follows: $\alpha^\vee$ is the map $\underline{X} \to A^\vee$ that describes G, and ν is given via the following construction: The extension of α from a map to A to a map to G can be interpreted as a trivialization of the pullback of the extension G via α

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[d, equal] & G' \arrow[r] \arrow[d] & \underline{Y} \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & T \arrow[r] & G \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

This is the same as giving a trivialization of the universal extension $\mathcal{P}$ on $A \times A^\vee$ pulled back via $(\alpha, \alpha^\vee)$ to $\underline{Y} \times \underline{X}$.
The dual of a 1-motive $(A, A^\vee, \underline{X}, \underline{Y}, \alpha, \alpha^\vee, \nu)$ is then the 1-motive $(A^\vee, A, \underline{Y}, \underline{X}, \alpha^\vee, \alpha, \nu)$.

(4.1.5) Definition/Theorem (COVARIANT REALIZATIONS). *Let S be a connected scheme (or a Deligne-Mumford stack). To each 1-motive $M = [\,\underline{Y} \xrightarrow{\alpha} G\,]$ over S we associate, ...*

*i. if S is a DM-stack of finite type over $\mathbb{C}$, the **Betti realization** $H^B(M)$, a local system on S^{an} defined by the following commutative diagram with exact lines*

$$\begin{tikzcd}
0 \arrow[r] & H_1(G,\mathbb{Z}) \arrow[r] & \mathrm{Lie}(G) \arrow[r] & G \arrow[r] & 0 \\
0 \arrow[r] & H_1(G,\mathbb{Z}) \arrow[r] \arrow[u, equal] & H^B(M) \arrow[r] \arrow[u] & \underline{Y} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Set

$$W_i(H(M)) = \begin{cases} H^B(M) & i \geq 0, \\ \ker(\beta) = H_1(G,\mathbb{Z}) & i = -1, \\ H_1(T,\mathbb{Z}) = \ker(H_1(G,\mathbb{Z}) \to H_1(A,\mathbb{Z})) & i = -2, \\ 0 & i < -2. \end{cases}$$

There are isomorphisms

$$\mathrm{gr}_i(H^B(M)) = \begin{cases} \underline{Y} & i = 0, \\ H_1(A,\mathbb{Z}) & i = -1, \\ H_1(T,\mathbb{Z}) = \underline{X}^*(1) & i = -2. \end{cases}$$

The Betti realization is a covariant functor. It extends to $\mathbb{Z}_{(p)}$-morphisms which give morphisms between $(H^B(M) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})$'s.

*ii. if l is invertible in S, an etale sheaf of free $\mathbb{Z}/l^n\mathbb{Z}$-modules, the **etale realization***

$$\begin{aligned} H^{et}(M,\mathbb{Z}/l^n\mathbb{Z}) &= \ker([l^n]) \\ &= \{(y,g) \in \underline{Y} \times G \mid \alpha(y) = l^n g\}/\{(l^n y, \alpha(y)) \mid y \in \underline{Y}\} \end{aligned}$$

with filtration

$$W_i(H^{et}(M,\mathbb{Z}/l^n\mathbb{Z})) = \begin{cases} H^{et}(M,\mathbb{Z}/l^n\mathbb{Z}) & i \geq 0, \\ \{(0,g) \mid ng = 0\} = H_1^{et}(G,\mathbb{Z}/l^n\mathbb{Z}) & i = -1, \\ \{(0,t) \mid nt = 0, t \in T\} = H_1^{et}(T,\mathbb{Z}/l^n\mathbb{Z}) & i = -2, \\ 0 & i < -2. \end{cases}$$

It is determined by a representation of $\pi_1^{et}(S,\overline{s})$ on a fibre $H^{et}(M_{\overline{s}},\mathbb{Z}/l^n\mathbb{Z})$ (because S is connected), where $\overline{s}$ is some geometric point in S.

We may form the projective limit

$$H^{et}(M_{\overline{s}},\mathbb{Z}_l) := \varprojlim_n H^{et}(M_{\overline{s}},\mathbb{Z}/l^n\mathbb{Z}).$$

It is a continuous *representation of* $\pi_1^{et}(S,\overline{s})$ *where* $H^{et}(M_{\overline{s}},\mathbb{Z}_l)$ *carries the profinite topology. It is topologically isomorphic to some* $\mathbb{Z}_l^n$ *equipped with the* l*-adic topology.*

The limes is constructed for the maps induced by $[l^i]$ *(they are strict):*

$$H^{et}(M,\mathbb{Z}/l^{n+i}\mathbb{Z}) \to H^{et}(M,\mathbb{Z}/l^n\mathbb{Z})$$
$$(x,g) \mapsto (x,l^i g)$$

which are compatible with the filtration and induce a similar and saturated *filtration on the limit.*

For a p*-isogeny* ψ *and if* S *is a scheme over* $\operatorname{spec}(\mathbb{Z}_{(p)})$*, we have*

$$\ker(\psi) \cong \operatorname{coker}(H^{et}(\psi))$$

for the induced map $H^{et}(\psi) : H^{et}(M_1,\widehat{\mathbb{Z}}^{(p)}) \hookrightarrow H^{et}(M_2,\widehat{\mathbb{Z}}^{(p)})$*.*

If S *is a smooth DM-stack over* $\mathbb{C}$ *there is the comparison isomorphism*

$$\rho_{et,B} : H^{et}(M,\mathbb{Z}_p) \to H^B(M) \otimes_{\mathbb{Z}} \mathbb{Z}_p,$$

where here $H^{et}(M,\mathbb{Z}_p)$ *is considered as local system on* S^{an} *as well. This isomorphism respects the weight filtration.*

The etale realization is a covariant functor. It extends to $\mathbb{Z}_{(p)}$*-morphisms which give morphisms between* $(H^{et}(M,\mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l)$*'s.*

iii. *if* S *is arbitrary, the* ***de Rham*** *realization (cf. [24, 10.1.7])*

$$H^{dR}(M),$$

which is a locally free sheaf on S*, with a flat connection*

$$\nabla : H^{dR}(M) \to H^{dR}(M) \otimes \Omega_{S|\operatorname{spec}(\mathbb{Z})},$$

weight filtration $W_\bullet(H^{dR})$ *and Hodge filtration* $F^\bullet(H^{dR})$ *such that they are bisaturated, and for each* S *of finite type over* $\mathbb{C}$ *the requirements of a family of mixed Hodge structures are fulfilled.*

$H^{dR}(M)$ *is defined to be* $\operatorname{Lie}(G')$*, where* G' *is the group underlying the universal vector group extension of* M *[loc. cit.]. One has* $F^0(H^{dR}(M)) = \ker(\operatorname{Lie}(G') \to \operatorname{Lie}(G)) \cong (\operatorname{Ext}^1(M,\mathbb{G}_a))^*$*.*

Furthermore:

$$\operatorname{gr}_i(H^{dR}(M)) = \begin{cases} \underline{Y} \otimes \mathcal{O}_S & i=0, \\ H_1^{dR}(A) & i=-1, \\ \operatorname{Lie}(T) & i=-2, \end{cases}$$

where $\mathcal{O}_S$*, resp.* $\operatorname{Lie}(T)$ *are equipped with trivial Hodgestructure of type* $(0,0)$ *and*

$(-1,-1)$, *respectively, and the induced Hodge structure on* $H_1^{dR}(A)$ *is the usual one.*

If S *is a smooth Deligne-Mumford stack over* $\mathbb{C}$, *there is the comparison isomorphism [loc. cit.]*

$$\rho_{dR,B} : H^{dR}(M) \otimes_{\mathcal{O}_S} \mathcal{O}_{S^{an}} \to H^B(M) \otimes_{\mathbb{Z}} \mathcal{O}_{S^{an}},$$

compatible with weight filtration and such that sections of $H^B(M)$ *are precisely those which are flat for the Gauss-Manin connection on the left.*

The de Rham realization is a covariant functor. If S *is a scheme over* $\operatorname{spec}(\mathbb{Z}_{(p)})$, *it extends to* $\mathbb{Z}_{(p)}$*-morphisms.*

(4.1.6) Theorem. *One has*

$$H^B(M^\vee) = H^B(M)^*(1)$$
$$H^{dR}(M^\vee) = H^{dR}(M)^*(1)$$
$$H^{et}(M^\vee) = H^{et}(M)^*(1)$$

where each is compatible with weight filtrations, and the second with Hodge-filtration. Everything is compatible with comparison isomorphisms.
A polarisation induces symplectic *forms*

$$H^B(M) \times H^B(M) \to \mathbb{Z}^B(1)$$
$$H^{dR}(M) \times H^{dR}(M) \to \mathbb{Z}^{dR}(1)$$
$$H^{et}(M) \times H^{et}(M) \to \mathbb{Z}^{et}(1)$$

$W_{-2}(\cdots)$ *is an isotropic (primitive) sublattice and* $W_{-1}(\cdots) = W_{-2}(\cdots)^\perp$ *for each object* $\cdots$ *above. Furthermore, the only nontrivial filtration step* $F^0(H^{dR}(M))$ *is isotropic. In the case of a* $\mathbb{Z}_p$*-polarization everything hold when tensored with* $\mathbb{Z}_{(p)}$.

(4.1.7) Theorem. *Let* M_1, M_2 *be two 1-motives over an algebraically closed field.* $\operatorname{Hom}(M_1, M_2)$ *is a free* $\mathbb{Z}$ *module of finite rank. The map*

$$\operatorname{Hom}(M_1, M_2) \otimes_{\mathbb{Z}} \mathbb{Z}_l \hookrightarrow \operatorname{Hom}(H^{et}(M_1, \mathbb{Z}_l), H^{et}(M_2, \mathbb{Z}_l))$$

is injective.

(4.1.8) Theorem. *Let* $N \in \mathbb{Z}$ *be an integer invertible in* S, M *a 1-motive over* S *and let* $E \subset \ker([N])$ *be a subscheme with induced filtration (i.e.* $E \to \ker([N])$ *is strict). Then there is an isogeny* $\psi : M \to M'$ *with* $\ker(\psi) = E$.

(4.1.9) Definition. *A $\mathbb{Z}_{(p)}$**-morphism** of 1-motives is given by a $\mathbb{Z}_{(p)}$-morphism $\psi_A : A_1 \to A_2$ and by*

$$\psi_X : \underline{X}_1 \otimes \mathbb{Z}_{(p)} \to \underline{X}_2 \otimes \mathbb{Z}_{(p)}$$
$$\psi_Y : \underline{Y}_2 \otimes \mathbb{Z}_{(p)} \to \underline{Y}_1 \otimes \mathbb{Z}_{(p)}$$

such that there exists an $n \in \mathbb{N}, p \nmid n$, such that $n\psi$ is an ordinary morphism and satisfies the compatibility requirements above. Observe, that it may be also the case that ψ_A, ψ_X, ψ_Y are already given by ordinary morphisms but satisfy for example the compatibility with the ν's only after multiplication with some n. Denote the group of $\mathbb{Z}_{(p)}$-morphisms of 1-motives by $\mathrm{Hom}^p(M_1, M_2)$. These groups define a category of 1-motives 'up to p-isogeny'. Every p-isogeny becomes an isomorphism here because for them the morphisms ψ_X, ψ_Y above are isomorphisms.
This has an equivalent description (like in the case of Abelian varieties): $\mathbb{Z}_{(p)}$-morphisms of 1-motives are given by pairs $[n, \alpha]$, where $n \in \mathbb{Z} \setminus 0, p \nmid n$ and α is an ordinary morphism, subject to the equivalence relation

$$[n, \alpha] \sim [m, \beta] \quad \Leftrightarrow \quad m\alpha = n\beta.$$

The map

$$\mathrm{Hom}(M_1, M_2) \to \mathrm{Hom}^p(M_1, M_2)$$
$$\alpha \mapsto [1, \alpha]$$

is injective. This description is equivalent to the description above.

(4.1.10) Definition. *Let all $l \nmid p$ be invertible in S.*
*A p**-polarization** of a 1-motive M over S is a $\mathbb{Z}_{(p)}$-morphism*

$$\psi : M \to M^\vee,$$

such that ψ_A is a p-polarization $A \to A^\vee$ and such that $\psi_X = \psi_Y$.
In the second, symmetric description this means that there is

i. a p-polarization $\psi_1 : A \to A^\vee$ (remember that it is automatically symmetric $\psi_1^\vee = \psi_1$ via the canonical identification $(A^\vee)^\vee = A$).

ii. an isomorphism $\psi_2 : \underline{X} \otimes \mathbb{Z}_{(p)} \to \underline{Y} \otimes \mathbb{Z}_{(p)}$ such that there is an $n \in \mathbb{N}$, $p \nmid n$ such that $n\psi_1$ and $n\psi_2$ are morphisms, $n\psi_1\alpha = n\alpha^\vee\psi_2$ and ν is symmetric via $n\psi_2$, i.e. $\nu(x_1, n\psi_2(x_2)) = \nu(n\psi_2(x_1), x_2)$ on $(1, n\psi_1)^*\mathcal{P} = (n\psi_1, 1)^*\mathcal{P}^\vee$ on $A \times A$.

(4.1.11) Definition. *Let S be a base scheme, with morphism $S \to \mathrm{spec}(\mathbb{Z}_{(p)})$. Let $L_{\mathbb{Z}_{(p)}} \neq 0$ be a lattice of dimension $2g$ with non-degenerate (perfect) symplectic form.*

We define the following groupoid (adelic case):

$$[\ S\text{-}^K\mathbf{H}_g\text{-}L\text{-}\boldsymbol{mot}\ ],$$

where $K \subset \mathrm{GSp}(V_{\mathbb{A}^{(\infty)}})$ *is an admissible compact open subgroup, as the category of the following data*

i. *An Abelian scheme A of dimension g with p-polarization*

$$\psi : A \to A^\vee.$$

ii. *A $K^{(p)}$-level structure*

$$\xi \in \underline{\mathrm{Iso}}_{(\mathrm{GSp}(L_{\mathbb{Z}_{(p)}}),L_{\mathbb{Z}_{(p)}})}(L_{\mathbb{A}^{(\infty,p)}}, H_1^{et}(A, \mathbb{A}^{(\infty,p)}))/K^{(p)},$$

where the isomorphisms have to be compatible with the $(\mathrm{GSp}(L_{\mathbb{A}^{(\infty,p)}}), L_{\mathbb{A}^{(\infty,p)}})$-*structure (see below) on both parameters.*

This means, more precisely, that ξ can be given as a class mod $K^{(p)}$ of isomorphisms

$$\xi_{\overline{s}}K^{(p)} : (L \oplus I)_{\mathbb{A}^{(\infty,p)}} \to H^{et}(M_{\overline{s}}, \mathbb{A}^{(\infty,p)})$$

(respecting the $(\mathrm{GSp}(V_{\mathbb{A}^{(\infty,p)}}), L_{\mathbb{A}^{(\infty,p)}})$-*structure) for some geometric point $\overline{s}$ in each connected component, and such that the class is invariant under the action of the etale fundamental group $\pi_1^{et}(S, \overline{s})$.*

The $(\mathrm{GSp}(L_{\mathbb{A}^{(\infty,p)}}), L_{\mathbb{A}^{(\infty,p)}})$-*structure is given as follows: The p-polarization induces an isomorphism*

$$H_1^{et}(A, \mathbb{A}^{(\infty,p)}) \to H_1^{et}(A, \mathbb{A}^{(\infty,p)})^*(1).$$

Choosing some isomorphism $\mathbb{Z}_{(p)}^{et}(1) \cong \mathbb{A}^{(\infty,p)}$, we get an alternating form on $H_1^{et}(A, \mathbb{A}^{(\infty,p)})$ up to a scalar (i.e. a GSp-*structure, see 2.5.7).*

Isomorphisms are $\mathbb{Z}_{(p)}$-morphisms, compatible with polarization (up to scalar) and level structures.

(4.1.12) Definition. *Let S be a base scheme, with morphism $S \to \mathrm{spec}(\mathbb{Z}_{(p)})$. Let $L_{0,\mathbb{Z}_{(p)}} \neq 0$ be a lattice of dimension $2g_0$ with non-degenerate (perfect) symplectic form. Let $I_{\mathbb{Z}_{(p)}} \neq 0$ be a lattice. We define the following groupoid (adelic case):*

$$[\ S\text{-}^K\mathbf{H}_{g_0}[0, I \otimes L_0]\text{-}L\text{-}\boldsymbol{mot}\ ],$$

where $L = L_0 \oplus I^*$ and $K \subset \mathrm{PSp}(L_{\mathbb{A}^{(\infty)}})$ is an admissible compact open subgroup, as the category of the following data

i. *A 1-motive $M = (A, A^\vee, 0, \underline{Y}, 0, \alpha^\vee, 0)$ over S, where A is of dimension g_0.*

ii. A p-polarization

$$\psi_1 : A \to A^\vee.$$

iii. An isomorphism $\rho : (I^*_{\mathbb{Z}_{(p)}})_S \cong \underline{Y} \otimes \mathbb{Z}_{(p)}$.

iv. A $K^{(p)}$*-level structure (as above)*

$$\xi \in \underline{\mathrm{Iso}}_{(\mathrm{PSp}(L_{0,\mathbb{Z}_{(p)}}, I_{\mathbb{Z}_{(p)}}), L_{\mathbb{Z}_{(p)}})}(L_{\mathbb{A}^{(\infty,p)}}, H^{et}(M, \mathbb{A}^{(\infty,p)}))/K^{(p)},$$

where the isomorphisms on the right have to be compatible with the $\mathrm{PSp}(L)$*-structure (see below) on both parameters. (note: in particular* ψ_X *has to be the identity for every isomorphism!)*

The $(\mathrm{PSp}(L_{0,\mathbb{A}^{(\infty,p)}}, I_{\mathbb{A}^{(\infty,p)}}), L_{\mathbb{A}^{(\infty,p)}})$*-structure is given as follows: The p-polarization induces an isomorphism*

$$W_{-1}(H^{et}(M, \mathbb{A}^{(\infty,p)})) \to W_{-1}(H^{et}(M, \mathbb{A}^{(\infty,p)}))^*(1).$$

Choosing some isomorphism $\mathbb{Z}^{et}_{(p)}(1) \cong \mathbb{A}^{(\infty,p)}$*, we get an alternating form on* $W_{-1}(H^{et}(M, \mathbb{A}^{(\infty,p)}))$ *up to a scalar. Furthermore, we have (via* ρ*) an isomorphism*

$$\mathrm{gr}^0(H^{et}(M, \mathbb{A}^{(\infty,p)})) = \underline{Y} \otimes_{\mathbb{Z}} \mathbb{A}^{(\infty,p)} = I^*_{\mathbb{A}^{(\infty,p)}}$$

(i.e. a PSp*-structure, see 2.5.7).*

Isomorphisms are $\mathbb{Z}_{(p)}$*-morphisms of the 1-motives, compatible with polarization (on A, up to a scalar),* ρ*'s and level structure.*

(4.1.13) Definition. *Let* S *be a base scheme, with morphism* $S \to \mathrm{spec}(\mathbb{Z}_{(p)})$*. Let* $L_{0,\mathbb{Z}_{(p)}}$ *a lattice of dimension* $2g_0$*, possible 0, otherwise with non-degenerate perfect symplectic form. Let* $I_{\mathbb{Z}_{(p)}} \neq 0$ *be a lattice. We define the following groupoid (adelic case):*

$$[\ S\text{-}{}^K\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]\text{-}L\text{-}\boldsymbol{mot}\ ],$$

where $L = I^* \oplus L_0 \oplus I$ *and* $K \subset \mathrm{PSp}(L_{\mathbb{A}^{(\infty)}})$ *is an admissible compact open subgroup, as the category of the following data*

i. A 1-motive $M = (A, A^\vee, \underline{X}, \underline{Y}, \alpha, \alpha^\vee, \nu)$ *over* S*, with* $\dim(A) = g_0$*.*

ii. A p-polarization of some degree $d \in {\mathbb{Z}_{(p)}}^*$

$$\psi_1 : A \to A^\vee$$

and an isomorphism

$$\psi_2 : \underline{X} \otimes \mathbb{Z}_{(p)} \to \underline{Y} \otimes \mathbb{Z}_{(p)}$$

of the same degree d*, such that they give a p-polarization of* M*.*

iii. *An isomorphism* $\rho : (I^*_{\mathbb{Z}_{(p)}})_S \cong \underline{Y} \otimes \mathbb{Z}_{(p)}$.

iv. *A* $K^{(p)}$*-level structure (as above)*

$$\xi \in \underline{\mathrm{Iso}}_{(\mathrm{USp}(L_{0,\mathbb{Z}_{(p)}}, I_{\mathbb{Z}_{(p)}}), L_{\mathbb{Z}_{(p)}})}(L_{\mathbb{A}^{(\infty,p)}}, H^{et}(M, \mathbb{A}^{(\infty,p)}))/K^{(p)},$$

where the isomorphisms in the etale sheaf on the right have to be compatible with the $(\mathrm{USp}(L_{0,\mathbb{Z}_{(p)}}, I_{\mathbb{Z}_{(p)}}), L_{\mathbb{Z}_{(p)}})$*-structure (see below) on both parameters.*

The $(\mathrm{USp}(L_{0,\mathbb{A}^{(\infty,p)}}, I_{\mathbb{A}^{(\infty,p)}}), L_{\mathbb{A}^{(\infty,p)}})$*-structure is given as follows: The p-polarization induces an isomorphism*

$$H^{et}(M, \mathbb{A}^{(\infty,p)}) \to H^{et}(M, \mathbb{A}^{(\infty,p)})^*(1).$$

Choosing some isomorphism $\mathbb{Z}^{et}_{(p)}(1) \cong \mathbb{A}^{(\infty,p)}$*, we get an alternating form on* $H^{et}(M, \mathbb{A}^{(\infty,p)})$ *up to a scalar. The weight filtration satisfies* W_{-2} *totally isotropic and* $W_{-1} = (W_{-2})^{\perp}$*. Furthermore, we have (via* ρ*) an isomorphism*

$$\mathrm{gr}^0(H^{et}(M, \mathbb{A}^{(\infty,p)})) = \underline{Y} \otimes_{\mathbb{Z}} \mathbb{A}^{(\infty,p)} = I^*_{\mathbb{A}^{(\infty,p)}}$$

(i.e. a USp*-structure, see 2.5.7).*

Isomorphisms are $\mathbb{Z}_{(p)}$*-morphisms of 1-motives, compatible with polarization up to scalar,* ρ*'s and level structures.*

(4.1.14) Definition. *Let* S *be a base scheme, with morphism* $S \to \mathrm{spec}(\mathbb{Z}_{(p)})$*. We define a variant of the foregoing category for the case* $L_{0,\mathbb{Z}_{(p)}} = 0$*. Let* $U_{\mathbb{Z}_{(p)}}$ *be a lattice of dimension* k *(it could be* $(I \otimes I)^s$*). Consider* $\mathbf{X} = \mathbf{H}_0[U_{\mathbb{Z}_{(p)}}, 0]$*, with* $P_{\mathbf{X}} = \mathbb{W}(U_{\mathbb{Z}_{(p)}}) \rtimes \mathbb{G}_m$*.* $P_{\mathbf{X}}$ *acts on* $L_{\mathbb{Z}_{(p)}} = \mathbb{Z}_{(p)} \oplus U^*_{\mathbb{Z}_{(p)}}$ *as follows:* $\mathbb{G}_m$ *acts as scalar multiplication on* $\mathbb{Z}_{(p)}$ *and* $\mathbb{W}(U_{\mathbb{Z}_{(p)}})$ *acts via* $u(x, u^*) = (x + u^*u, u^*)$*. We define the following groupoid (adelic case):*

$$[\ S\text{-}{}^K\mathbf{H}_0[U_{\mathbb{Z}_{(p)}}, 0]\text{-}L\text{-}\boldsymbol{mot}\ ],$$

where $K \subset \mathrm{PSp}(L_{\mathbb{A}^{(\infty)}})$ *is an admissible compact open subgroup, as the category of the following data*

i. *A 1-motive of the form* $M = [\ \underline{Y} \longrightarrow \mathbb{G}_m\ ]$.

ii. *An isomorphism* $\rho : (U^*_{\mathbb{Z}_{(p)}})_S \cong \underline{Y} \otimes \mathbb{Z}_{(p)}$.

iii. *A* $K^{(p)}$*-level structure (as above)*

$$\xi \in \underline{\mathrm{Iso}}_{(P_{\mathbf{X}}, L_{\mathbb{Z}_{(p)}})}(L_{\mathbb{A}^{(\infty,p)}}, H^{et}(M, \mathbb{A}^{(\infty,p)}))/K^{(p)},$$

where the isomorphisms in the etale sheaf on the right have to be compatible with the obvious $(P_{\mathbf{X}}, L_{\mathbb{Z}_{(p)}})$*-structure on both parameters.*

Isomorphisms are $\mathbb{Z}_{(p)}$-morphisms of 1-motives, which are the identity on $\mathbb{G}_m$, compatible with ρ's and level structure.

(4.1.15) Remark. For $U_{\mathbb{Z}_{(p)}} = (I \otimes I)^s_{\mathbb{Z}_{(p)}} \neq 0$, there are easy functorial equivalences:

$$[\ S\text{-}^K\mathbf{H}_0[U_{\mathbb{Z}_{(p)}}, 0]\text{-}\mathbb{Z}_{(p)} \oplus U^*_{\mathbb{Z}_{(p)}}\text{-}\mathbf{mot}\ ] \to [\ S\text{-}^K\mathbf{H}_0[(I \otimes I)^s_{\mathbb{Z}_{(p)}}, 0]\text{-}I^* \oplus I\text{-}\mathbf{mot}\ ].$$

In the case $U_{\mathbb{Z}_{(p)}} = 0$, we have isomorphisms

$$[\ S\text{-}^{K(N)}\mathbf{H}_0\text{-}\mathbb{Z}_{(p)}\text{-}\mathbf{mot}\ ] \to \mathrm{Hom}(S, \mu_{N,S})$$

given by the level structure. We will examine these examples further in (5.7).

We have the following 'integral' variant of the foregoing groupoids: We will describe this only for the case (4.1.13) because the others are degenerating analogous cases of this construction.

(4.1.16) Definition. *Let S be a base scheme. Let $L_{0,\mathbb{Z}}$ be a with non-degenerate (perfect) symplectic form of dimension $2g$. Let $I_{\mathbb{Z}}$ be another lattice. We define the following groupoids for $N \in \mathbb{N}$.*

- $$[\ S\text{-}^N\mathbf{H}_{g_0}[(I_{\mathbb{Z}} \otimes I_{\mathbb{Z}})^s, L_{0,\mathbb{Z}} \otimes I_{\mathbb{Z}}]\text{-}L_{\mathbb{Z}}\text{-}\boldsymbol{mot}\ ]$$

  as the category of the following data

  i. *A 1-motive $M = (A, A^\vee, \underline{X}, \underline{Y}, \alpha, \alpha^\vee, \nu)$ over S, where* $\dim(A) = g$.

  ii. *A principal polarization*
  $$\psi_1 : A \to A^\vee$$
  and an isomorphism
  $$\psi_2 : \underline{X} \to \underline{Y}$$
  such that they induce a polarization of M.

  iii. *An isomorphism $\rho : (I^*_{\mathbb{Z}})_S \cong \underline{Y}$.*

  iv. *A level-N-structure*
  $$\xi \in \mathrm{Iso}_{(\mathrm{USp}(L_{0,\mathbb{Z}/N\mathbb{Z}}, I_{\mathbb{Z}/N\mathbb{Z}}), L_{\mathbb{Z}/N\mathbb{Z}})}(L_{\mathbb{Z}/N\mathbb{Z}}, H^{et}(M, \mathbb{Z}/N\mathbb{Z})).$$
  This means that the isomorphisms have to be compatible with the
  $(\mathrm{USp}(L_{0,\mathbb{Z}/N\mathbb{Z}}, I_{\mathbb{Z}/N\mathbb{Z}}), L_{\mathbb{Z}/N\mathbb{Z}})$*-structure (see the adelic case) on both parameters.*

  Isomorphisms are morphisms of the 1-motives, compatible with polarization and level structures. (Note: In particular ψ_Y is determined by the ρ's and ψ_X by them *and the polarization.)*

(4.1.17) Remark. Without the isomorphism $\rho : I_{\mathbb{Z}}^* \to \underline{Y}$, we had to take another group, which is an almost semi-direct product of $\mathrm{USp}(L_0, I)$ and $\mathrm{GL}(I)$. It is precisely the maximal parabolic subgroup of $\mathrm{GSp}(L)$, fixing the weight filtration. It does however not define a valid mixed Shimura datum and it is only the subgroup USp, which is associated with boundary components of GSp (see 2.5.4).

(4.1.18) Definition. *Let* $\mathbf{X}$ *be one of the symplectic mixed Shimura data above group schemes above, with natural representation of* $P_{\mathbf{X}}$ *on* $L_{\mathbb{Z}_{(p)}}$*. Furthermore, for each* $\rho \in P_{\mathbf{X}}(\mathbb{A}^{(\infty,p)})$*, and admissible compact open subgroups* K_1, K_2*, such that* $K_1^\rho \subseteq K_2$ *we have a map*

$$[\ S\text{-}^{K_1}\mathbf{X}\text{-}L\text{-}\boldsymbol{mot}\ ] \to [\ S\text{-}^{K_2}\mathbf{X}\text{-}L\text{-}\boldsymbol{mot}\ ]$$

by multiplication of the level structure by ρ *from the right. These maps satisfy the axioms of (3.6.1).*

(4.1.19) Theorem. *Let* S *be a scheme over* $\mathrm{spec}(\mathbb{Z}_{(p)})$*. For each of the* p*-integral mixed symplectic Shimura data as above, we have equivalences*

$$[\ S\text{-}^{N}\mathbf{X}\text{-}L_{\mathbb{Z}}\text{-}\boldsymbol{mot}\ ] \to [\ S\text{-}^{K(N)}\mathbf{X}\text{-}L\text{-}\boldsymbol{mot}\ ].$$

Proof. We will prove this for the case $\mathbf{H}_{g_0}[(I_{\mathbb{Z}} \otimes I_{\mathbb{Z}})^s, L_{0,\mathbb{Z}} \otimes I_{\mathbb{Z}}]$, where $I_{\mathbb{Z}} \neq 0$, $L_{0,\mathbb{Z}} \neq 0$ — the other cases are degenerate special cases of this construction. We may assume that S is connected.

We first describe the functor. Let

$$[A, \underline{X}, \underline{Y}, \alpha, \alpha^\vee, \nu, \psi, \rho, \xi]$$

be an object of $[\ S\text{-}^{N}\mathbf{X}\text{-}L_{\mathbb{Z}}\text{-}\mathbf{mot}\ ]$. Choose a geometric point $\overline{s}$. ξ can be considered as an isomorphism

$$\xi : L_{\mathbb{Z}/N\mathbb{Z}} \to H^{et}(M_{\overline{s}}, \mathbb{Z}/N\mathbb{Z})$$

invariant under the action of $\pi_1^{et}(S, \overline{s})$ (i.e. $H^{et}(M, \mathbb{Z}/N\mathbb{Z})$ has to be constant).

Choose some isomorphism

$$\delta : H^{et}(M_{\overline{s}}, \widehat{\mathbb{Z}}^{(p)}) \to L_{\widehat{\mathbb{Z}}^{(p)}}$$

compatible with USp-structures. Composing the reduction mod N with ξ, we get an element of $\mathrm{USp}(L_{0,\mathbb{Z}/N\mathbb{Z}}, I_{\mathbb{Z}/N\mathbb{Z}})$. Since USp is a smooth group scheme over $\mathbb{Z}$, by Hensel's lemma we get a lift to $\mathrm{USp}(L_{\widehat{\mathbb{Z}}^{(p)}})$. Taking composition again with the inverse of the chosen isomorphism we get a

$$\xi' : L_{\widehat{\mathbb{Z}}^{(p)}} \to H^{et}(M_{\overline{s}}, \widehat{\mathbb{Z}}^{(p)}),$$

reducing mod N to ξ. It is well-defined mod $K(N)$ and the class is, by construction, invariant under $\pi_1^{et}(S,\overline{s})$.
This functor is faithful. It is full because a p-morphism of 1-motives, which induces an isomorphism $H^{et}(M_{\overline{s}},\widehat{\mathbb{Z}}^{(p)}) \cong H^{et}(M_{\overline{s}},\widehat{\mathbb{Z}}^{(p)})$, must be a morphism. This follows from (4.1.7).
Let on the other hand $[A,\underline{X},\underline{Y},\alpha,\alpha^\vee,\nu,\psi,\rho,\xi']$ be an object of [S-${}^{K(N)}$**X**-L-**mot**]. Choose a geometric point $\overline{s}$.
If ξ' is represented by an isomorphism

$$L_{\widehat{\mathbb{Z}}^{(p)}} \to H^{et}(M_{\overline{s}},\widehat{\mathbb{Z}}^{(p)})$$

then the object is in the image of the functor because of the following:

i. ξ' can be given as a lift of a ξ as above.

ii. There is a principal polarization in the class of ψ. For, there is a $d \in \mathbb{A}^{(\infty,p)}$, such that ψ induces an isomorphism

$$H^{et}(M_{\overline{s}},\widehat{\mathbb{Z}}^{(p)}) \to dH^{et}(M_{\overline{s}}^\vee,\widehat{\mathbb{Z}}^{(p)})$$

because ξ' is a morphism of USp-structures, hence a symplectic similitude. Hence we get a principal polarization by changing ψ by $+d$ or $-d$, which w.l.o.g. lies in $\mathbb{Z}_{(p)}^*$. Only one sign leads to a *polarization*.

iii. ν itself has to satisfy the compatibility condition.

iv. ρ maps $I_{\mathbb{Z}}^*$ to $\underline{Y}$.

If ξ' is not represented by an isomorphism as above, we have to show that there exists an isogenous object with this property.
Composing with a p-isogeny $\in \mathbb{Z}_{(p)} \setminus \{0\}$, we may assume that there is an $c, p \nmid c$, and a diagram

$$\begin{tikzcd}
0 \arrow[r] & H^{et}(M_{\overline{s}},\widehat{\mathbb{Z}}^{(p)}) \arrow[r, hook, "\xi^{-1}"] \arrow[d, equal] & L_{\widehat{\mathbb{Z}}^{(p)}} \arrow[r] \arrow[d, hook] & K \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & H^{et}(M_{\overline{s}},\widehat{\mathbb{Z}}^{(p)}) \arrow[r, "{[c]}"] & H^{et}(M_{\overline{s}},\widehat{\mathbb{Z}}^{(p)}) \arrow[r] & \ker([c]) \arrow[r] & 0
\end{tikzcd}$$

Furthermore, the operation of $\pi_1^{et}(S,\overline{s})$ induces one on $K_{\overline{s}}$, hence there is a finite etale group scheme $K \subset \ker([c])$ with fibre $K_{\overline{s}}$ and we have (4.1.8) an isogeny $\psi : M \to M'$ with $\ker(\psi) = K$, hence a diagram

$$\begin{tikzcd}
0 \arrow[r] & H^{et}(M_{\overline{s}},\widehat{\mathbb{Z}}^{(p)}) \arrow[r, hook, "\xi^{-1}"] \arrow[d, equal] & L_{\widehat{\mathbb{Z}}^{(p)}} \arrow[r] & K \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & H^{et}(M_{\overline{s}},\widehat{\mathbb{Z}}^{(p)}) \arrow[r, "\psi"] & H^{et}(M'_{\overline{s}},\widehat{\mathbb{Z}}^{(p)}) \arrow[r] \arrow[u, "(\xi')^{-1}"'] & K \arrow[r] & 0
\end{tikzcd}$$

There is an ρ' because ξ is a morphism of USp-structures and hence Ψ_Y has to be an isomorphism of $\mathbb{Z}$-lattices! Therefore we get an isogenous object with the property that ξ' is an isomorphism

$$L_{\widehat{\mathbb{Z}}^{(p)}} \to H^{et}(M_{\overline{s}}, \widehat{\mathbb{Z}}^{(p)}).$$

□

(4.1.20) Remark. In particular, transporting Hecke operators via these equivalences, we get a action of them on ordinary 1-motives with level-N-structures. For a more explicit description of this action see [83, 10.11]. There $\dim I = 1$.

(4.1.21) Remark. We will need later an integral description of the groupoid

$$[\ S\text{-}^{K_W(1)\rtimes K_G(N)}\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]\text{-}L\text{-}\mathbf{mot}\ ],$$

too. It is given as in (4.1.19), but with level structure

$$\xi : L_{0,\mathbb{Z}/N\mathbb{Z}} \to H^{et}(\operatorname{gr}^{-1} M, \mathbb{Z}/N\mathbb{Z})$$

only.

4.2. Biextensions

(4.2.1) Definition ([38, VII, 3]). *Let G_1, G_2, C be commutative groups. A* ***biextension*** *of $G_1 \times G_2$ by C is a set B with a free C operation and an invariant map*

$$\pi : B \to G_1 \times G_2,$$

which identifies $G_1 \times G_2$ with the quotient B/C, together with 2 partial multiplication maps

$$+_1 : B \times_{G_2} B \to B \qquad +_2 : B \times_{G_1} B \to B$$

such that $+_i$ defines the law of a (relative) Abelian group on $B \to G_{i'}$[1], and operation resp. π induce an exact sequence of groups over $G_{i'}$:

$$0 \longrightarrow C_{G_{i'}} \longrightarrow B \longrightarrow G_1 \times G_2 \longrightarrow 0$$

and

$$(x +_1 y) +_2 (u +_1 v) = (x +_2 u) +_1 (y +_2 v)$$

for all elements, where this is defined.
For the category of schemes or sheaves on schemes, we define a biextension to be a biextension object in the respective category.

[1] where $1' = 2$ and $2' = 1$.

Denote by $\mathrm{Biext}^0(G_1, G_2, H)$ *the group of isomorphisms of the trivial biextension and by* $\mathrm{Biext}^1(G_1, G_2, H)$ *the group of isomorphism classes of biextensions of* G_1 *and* G_2 *by* C. *(There is a unique group law on isomorphism classes of biextensions, such that for the associated extension of groups over* $G_i, i = 1, 2$ *it gives back the usual group law).*

(4.2.2) Theorem ([38, VII, 3]). *If* G_1, G_2 *and* C *are sheaves of Abelian groups on some scheme* S, *there are canonical isomorphisms*

$$\begin{aligned}\mathrm{Biext}^1(G_1, G_2, H) &= \mathrm{Ext}^1(G_1 \overset{L}{\otimes} G_2, H)\\ = \mathrm{Ext}^1(G_1, R\underline{\mathrm{Hom}}(G_2, C)) &= \mathrm{Ext}^1(G_2, R\underline{\mathrm{Hom}}(G_1, C))\end{aligned}$$

and hence an exact sequence

$$\begin{aligned}0 \longrightarrow \mathrm{Ext}^1(G_1, \underline{\mathrm{Hom}}(G_2, C)) \longrightarrow \mathrm{Biext}^1(G_1, G_2, C) \longrightarrow\\ \mathrm{Hom}(G_1, \mathrm{Ext}^1(G_2, C)) \longrightarrow \mathrm{Ext}^2(G_1, \underline{\mathrm{Hom}}(G_2, C)) \longrightarrow \mathrm{Ext}^2(G_1, R\underline{\mathrm{Hom}}(G_2, C)).\end{aligned}$$

We are mainly interested in the case where $G_i = A_i$ are Abelian schemes and C is a torus. In the case, $\mathrm{Hom}(A_2, C) = 0$ and we get an isomorphism

$$\mathrm{Biext}^1(A_1, A_2, C) \cong \mathrm{Hom}(A_1, \mathrm{Ext}^1(A_2, C))$$

and $\mathrm{Ext}^1(A_2, C)$ is isomorphic to $A_2^\vee \otimes \underline{\mathrm{Hom}}(\mathbb{G}_m, C)$.

By the theorem of the square a birigidified C-torsor over $A_1 \times A_2$ is already a biextension.

(4.2.3) Theorem. *If* $S = \mathrm{spec}(\mathbb{C})$, *and* $C = D_S(M)$ *is a torus we have a bijection*

$$\left\{ \begin{array}{c}\textit{(usual) group ext. } E \textit{ of } H_1(A_1 \times A_2, \mathbb{Z}) \textit{ by } M(1), \textit{ with compatible sections}\\ s_i : H_1(A_i, \mathbb{Z}) \to E \textit{ and } s : F^0(H_1(A_1 \times A_2, \mathbb{C})) \to E(\mathbb{C})\end{array} \right\} \cong \mathrm{Biext}^1(A_1, A_2, C)$$

(here $E(\mathbb{C})$ *is the induced extension of* $H_1(A_1 \times A_2, \mathbb{C})$ *by* $M(1)_\mathbb{C} = M_\mathbb{C}$*).*

Proof. The bijection is given analytically by:

$$(E, s) \mapsto E \backslash E(\mathbb{C})/F^0(E),$$

where $F^0(E)$ is the image of the section s. Here, on the right hand side the group operations $+_1$ and $+_2$ are both induced by the group structure on E by $x +_1 y = xs_1(p_1(x))^{-1}y$, if $p_1(x) = p_1(y)$ and similarly for $+_2$.

The right hand side is algebraic because

$$\mathrm{Ext}^1(A_2^{an}, C^{an}) = \mathrm{Ext}(A_2, C)$$

and

$$\mathrm{Hom}(A^{an}, B^{an}) = \mathrm{Hom}(A, B)$$

for all Abelian varieties. The inverse is gives as follows. The map $A_1 \to A_2^\vee \otimes M$ describing the extension is given by a bilinear form

$$\psi : H_1(A_1, \mathbb{Z}) \times H_1(A_2, \mathbb{Z}) \to M(1).$$

Since it is induced by a morphism of Hodge structures the form is trivial on

$$F^0(H_1(A_1, \mathbb{Z})) \times F^0(H_1(A_2, \mathbb{Z})).$$

It defines the group structure

$$(u', v_1', v_2')(u, v_1, v_2) \to (u' + u - \psi(v_1, v_2'), v_1' + v_1, v_2' + v_2)$$

on $E = M(1) \oplus H_1(A_1, \mathbb{Z}) \oplus H_1(A_2, \mathbb{Z})$. One checks that these constructions are inverse to each other (up to isomorphism). □

(4.2.4) Remark. If $A := A_1 = A_2$ and we have a polarization $\psi : A \to A^\vee$, it describes an *alternating* form

$$\psi : H_1(A, \mathbb{Z}) \times H_1(A, \mathbb{Z}) \to \mathbb{Z}(1)$$

and the associated biextension is the universal Poincaré biextension. In this case, a more symmetric description of $E(\mathbb{C})$ is convenient, transforming the previous description via the map

$$(u, v_1, v_2) \mapsto (u + \frac{1}{2}\psi(v_1, v_2), v_1, v_2).$$

Then the morphism of symmetry becomes just exchanging the rightmost factors, but the lattice $E(\mathbb{Z})$ is moved. In the original description s acts by $(u, v_1, v_2) \mapsto (u + \psi(v_1, v_2), v_2, v_1)$ preserving the lattice.
Consider the (obvious) extension

$$\psi : I \otimes H_1(A, \mathbb{Z}) \times H_1(A, \mathbb{Z}) \otimes I \to (I \otimes I)(1),$$

which is not alternating anymore, but satisfies $(\psi(v, w) = -\psi(w, v)^s)$. The associated biextension is an I-Poincaré biextension described in (4.2.6). The map

$$(u, v_1, v_2) \mapsto (u + \frac{1}{2}\psi(v_1, v_2), v_1, v_2)$$

is defined as well, and the morphism of symmetry becomes exchanging the rightmost factors and s on the leftmost one. In the original description s acts by $(u, v_1, v_2) \mapsto ((u + \psi(v_1, v_2))^s, v_2, v_1)$, preserving the lattice.

(4.2.5) Theorem. *Let S be a normal scheme, and $U \hookrightarrow S$ be a dense open subscheme.*

If $B_i, i = 1, 2$ are biextensions of Abelian schemes by tori over S, every homomorphism $\phi : B_1|U \to B_2|U$ extends uniquely to a homomorphism $\widetilde{\phi} : B_1 \to B_2$.

Proof. It suffices to show the extendibility of the induced map on the Abelian schemes resp. the torus . The map on Abelian schemes extends uniquely by (3.7.3) and the map on tori is given by a homomorphism of etale sheaves of lattices, hence extends as well. □

(4.2.6) Some tautologies about biextensions and 1-motives:
Let A be an Abelian scheme. There is the universal (Poincaré) biextension of $A \times A^\vee$ by $\mathbb{G}_m$.

$$\mathbb{G}_m \longrightarrow \mathcal{P} \longrightarrow A \times A^\vee$$

given by the canonical isomorphism $A \to (A^\vee)^\vee$.
If $\psi : A \to A^\vee$ is a polarization, then we get via pullback along $\mathrm{id} \times \psi$ a biextension

$$\mathbb{G}_m \longrightarrow \mathcal{P}^\psi \longrightarrow A \times A\ .$$

The symmetry morphism $s : A \times A \to A \times A$ lifts to $\mathcal{P}^\psi$ (because ψ is a *symmetric* morphism) and the invariants under s are a rigidified $\mathbb{G}_m$-torsor, which is the same as the pullback of the biextension (considered as rigidified $\mathbb{G}_m$-torsor) along the diagonal:

$$\mathbb{G}_m \longrightarrow (\mathcal{P}^\psi)^s \longrightarrow A$$

(over geometric points we have $\mathcal{L}_{\overline{s}}^{\otimes 2} \cong (\mathcal{P}^\psi)^s_{\overline{s}}$, where $\mathcal{L}_{\overline{s}}$ is the ample line bundle associated with the polarization).
Let I be a lattice (or an etale sheaf of lattices). We get also a biextension

$$I \otimes \mathbb{G}_m \longrightarrow \mathcal{P}^{\psi,I} \longrightarrow I \otimes A \times A$$

given by the morphism $\psi \otimes \mathrm{id} : I \otimes A \to I \otimes A$, a biextension

$$I \otimes \mathbb{G}_m \otimes I \longrightarrow \mathcal{P}^{\psi,I\otimes I} \longrightarrow I \otimes A \times A \otimes I$$

given by the morphism $I \otimes A \to I \otimes I \otimes A \otimes I^*$ (contraction). The morphism $s : I \otimes A \times A \otimes I$ exchanging the factors extends again to $\mathcal{P}^{\psi,I\otimes I}$, and the invariants are a rigidified $(I \otimes I)^s \otimes \mathbb{G}_m$-torsor:

$$\mathbb{G}_m \otimes (I \otimes I)^s \longrightarrow \mathcal{P}^{\psi,(I\otimes I)^s} \longrightarrow I \otimes A.$$

A section m in the biextension $\mathcal{P}^{\psi,I\otimes I}(S)$ determines a 1-motive over S as follows: It is the same as giving a morphism $I^* \to \mathcal{P}^{\psi,I}$ whose projection to $I \otimes A$ is constant. The fibre over the image is a semi-Abelian scheme $0 \longrightarrow I \otimes \mathbb{G}_m \longrightarrow G \longrightarrow A \longrightarrow 0$ (by $+_2$). The restriction $I^* \to G$ is a homomorphism, hence a 1-motive. It is symmetric

(w.r.t $\psi, I \to (I^*)^*$) if and only if m was invariant under s.
Giving m is as well the same as giving a morphism $I \times I \to \mathcal{P}$ such that the projections $I \to A$ and $I \to A^\vee$ are homomorphisms (this is the second symmetric description).

4.3. Representability

(4.3.1) Remember the quotient maps $\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I] \to \mathbf{H}_{g_0}[0, L_0 \otimes I] \to \mathbf{H}_{g_0}$ (2.5.6). (Assume $L_0 \neq 0$.) They are compatible via the standard representations with the quotient maps $L_0 \otimes \mathfrak{I} \to L_0 \oplus I^* \to L_0$. If a compact open subgroup is such that $\pi(K') \subset K$, then there are forgetful maps

$$[\ {}^{K'}\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]\text{-}L_0 \oplus \mathfrak{I}\text{-}\mathbf{mot}\ ] \to [\ {}^{K}\mathbf{H}_{g_0}[0, L_0 \otimes I]\text{-}L_0 \oplus I^*\text{-}\mathbf{mot}\ ]$$

and

$$[\ {}^{K'}\mathbf{H}_{g_0}[0, L_0 \otimes I]\text{-}L_0 \oplus I^*\text{-}\mathbf{mot}\ ] \to [\ {}^{K}\mathbf{H}_{g_0}\text{-}L_0\text{-}\mathbf{mot}\ ],$$

respectively.

(4.3.2) Theorem. *These maps are representable and smooth and the fibers are projective schemes, resp. quasi-projective schemes.*

Proof. Since these morphisms are compatible with the operation of $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ we may restrict to the case of the groups $K_U(M^2)K_V(M)K_G(N)$ (they form a confinal system of normal subgroups of $K(1)$). These in turn are conjugated to $K_W(1)K_G(N)$ by $w(M)$, where w is the weight morphism, acting by 1 on I^*, by M^{-1} on L_0 and by M^{-2} on I. Hence we are restricted to the integral versions of the category described in (4.1.21).
Let $[A, \psi, \xi]$ be an object of $[\ S\text{-}{}^{N}\mathbf{H}_{g_0}\text{-}L_0\text{-}\mathbf{mot}\ ]$. Fibers of the first morphism consist of a morphism $\alpha : \underline{Y} \to A$, an isomorphism $\rho : \underline{Y} \cong U^*_{\mathbb{Z}}$. This is represented by

$$A \otimes_{\mathbb{Z}} I_{\mathbb{Z}},$$

which is projective.
In the second case, the fibre consists of the following. The polarization determines $\underline{X}$, $\alpha^\vee$, so it remains the variability of a symmetric trivialization of the pullback of the Poincaré biextension along $\alpha \times \alpha^\vee$, or equivalently (considering α as a point in $A \otimes I_{\mathbb{Z}}$ and $\alpha^\vee$ as a point in $A^\vee \otimes I_{\mathbb{Z}}$) of a *point* (=trivialization of one fibre) in $\mathcal{P}^{\psi,(I\otimes I)^s}$ (4.2.6). It is a $\mathbb{G}_m \otimes (I_{\mathbb{Z}} \otimes I_{\mathbb{Z}})^s$-torsor above $A \otimes I_{\mathbb{Z}}$.
Fixing (A, Ψ), there are no further automorphisms because everything is fixed by ρ. Hence the second fibre is represented by the fibre of $\mathcal{P}^{(I\otimes I)^s} \to A \otimes I_{\mathbb{Z}}$ over α. It is a quasi-projective scheme.

□

(4.3.3) Theorem (Mumford, Artin, Deligne). *For each of the p-integral mixed*

Shimura data of symplectic type and each K, an admissible compact open subgroup

$$[\ {}^K\mathbf{X}\text{-}L\text{-}\boldsymbol{mot}\ ] \to \operatorname{spec}(\mathbb{Z}_{(p)})$$

is representable and a smooth Deligne-Mumford stack of finite type. It is a quasi-projective scheme, if K is neat.

Proof. It suffices to restrict to the fundamental system of normal subgroups $K(N) \subset K(1), p \nmid N$. For $\mathbf{X} = \mathbf{H}_g$ we have an equivalence

$$[\ \mathbf{S}\text{-}{}^{K(N)}\mathbf{H}_g\text{-}L\text{-}\mathbf{mot}\ ] \to [\ \mathbf{S}\text{-}{}^{N}\mathbf{H}_g\text{-}L_{\mathbb{Z}}\text{-}\mathbf{mot}\ ]$$

and representability of $[\ \mathbf{S}\text{-}{}^{N}\mathbf{H}_g\text{-}L_{\mathbb{Z}}\text{-}\mathbf{mot}\ ]$ over $\operatorname{spec}(\mathbb{Z}[1/N])$ is well-known, and follows for example from [80], where symplectic level structures (instead of similitudes) are used. (If $g = 0$, we saw this in 4.1.15.) For the other groups, the statement follows from (4.3.2). The last statement follows because, if K is neat, the objects in $[\ S\text{-}{}^K\mathbf{X}\text{-}L\text{-}\mathbf{mot}\ ]$ have no nontrivial automorphisms for any S. □

The maps of (4.1.18) yield *etale and finite* maps of Deligne-Mumford stacks.

4.4. Comparison with mixed Hodge structures

The notation is justified by the following

(4.4.1) Theorem (Riemann, Deligne). *Let S be a smooth DM-stack of finite type over $\mathbb{C}$. Then there are functors*

$$[\ S\text{-}{}^K\mathbf{X}\text{-}L\text{-}\boldsymbol{mot}\ ] \to [\ S^{an}\text{-}{}^K\mathbf{X}\text{-}L\text{-}\boldsymbol{mhs}\ ]$$

for $\mathbf{X} = \mathbf{H}_{g_0}$, $\mathbf{H}_{g_0}[0, L_0 \otimes I]$ and $\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]$, respectively and the standard representation spaces L of the various $P_{\mathbf{X}}$ which for $S = \operatorname{spec}(\mathbb{C})$ are equivalences of categories. The functors are compatible with the forgetful functors (4.3.1), respectively the maps induced by the maps of Shimura data (2.5.6). They are compatible with the $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$-action and functors associated with $K' \subseteq K$, too.

Proof. We will show this only for the case $\mathbf{X} = \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]$ — the others are (degenerate) special cases of this construction.
The functors are defined via the comparison isomorphisms. Let $M = (A, A^\vee, \underline{X}, \underline{Y}, \nu)$ be a 1-motive together with a polarization Ψ, an isomorphism $\rho : (I_{\mathbb{Z}}^*) \cong \underline{Y}$ and level structure, i.e. an object of $[\ S\text{-}{}^K\mathbf{X}\text{-}L\text{-}\mathbf{mot}\ ]$. We have $H^B(M, \mathbb{Q})$, $H^{dR}(M)$ and $H^{et}(M, \mathbb{A}^{(\infty)})$. All equipped with symplectic pairing (determined up to scalar)

weight filtration, isomorphism $\mathrm{gr}^0(H^{et}) \cong I^*_{\mathbb{A}^{(\infty)}}$, resp. $\mathrm{gr}^0(H^{dR}) \cong I^*_{\mathbb{Z}} \otimes \mathcal{O}_S$, resp. $\mathrm{gr}^0(H^B) = (I_{\mathbb{Z}})^*_{S^{an}}$. induced by ρ. This determines a USp-structure on them.
The functor associates to this the local system $H^B(M,\mathbb{Q})$, equipped with the USp-structure described above, the mixed Hodge structure transported to $H^B(M,\mathbb{Q}) \otimes \mathcal{O}_S^{an}$ by means of $\rho_{B,dR}$ and K-level structure transported to $H^B(M,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{A}^{(\infty)}$ by means of $\rho_{B,et}$. Since the Hodge-filtration is isotropic and *point-wise* polarized on $\mathrm{gr}^{-1} H^B(M,\mathbb{R})$ and by construction the weight filtration is the filtration determined by the USp-structure, the associated morphism lies in $\mathbb{D}_{\mathbf{X}}$. I.e. we get an object of

$$[\ S^{an}\text{-}{}^K\mathbf{X}\text{-}L\text{-}\mathbf{mhs}\ ].$$

In the case $S = \mathrm{spec}(\mathbb{C})$, we construct an object as above out of a vector space $L_{\mathbb{Q}}$ with USp-structure (as local system) with mixed Hodge structure on $L_{\mathbb{C}}$ of type $(-1,-1)$, $(0,-1)$, $(-1,0)$, $(0,0)$ and K-level structure as follows: Choose any lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Q}}$. Form the quotient

$$G = W_{-1}L(\mathbb{Z})\backslash W_{-1}(L_{\mathbb{C}})/F^0(L_{-1}V_{\mathbb{C}}).$$

It is a semi-Abelian variety over $\mathbb{C}$ because the complex torus quotient is polarized.

Now look at the exact diagram:

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & I^* \arrow[d, equal] & \\
0 \arrow[r] & W_{-1}(L_{\mathbb{Z}}) \arrow[r] \arrow[d] & L_{\mathbb{Z}} \arrow[r] \arrow[d] & \mathrm{gr}^0(L_{\mathbb{Z}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & W_{-1}(L_{\mathbb{C}})/F^0(W_{-1}L_{\mathbb{C}}) \arrow[r] \arrow[d] & L_{\mathbb{C}}/F^0(L_{\mathbb{C}}) \arrow[r] \arrow[d] & \mathrm{gr}_0^W(L_{\mathbb{C}})/F^0(\mathrm{gr}_0^W(L_{\mathbb{C}})) = 0 \arrow[r] \arrow[d, equal] & 0 \\
 & G \arrow[r] \arrow[d] & \widetilde{G} \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}
$$

By the snake lemma, we get a map

$$\alpha : I^*_{\mathbb{Z}} \to G,$$

hence a 1-motive $M = [\, I^*_{\mathbb{Z}} \xrightarrow{\alpha} G \,]$. This 1-motive has a polarization induced by the map $L \to L^*$ given by the symplectic form.

One checks that the two functors are inverse to each other (up to isomorphism). $\square$

(4.4.2) Remark. If the map α above is sufficiently general — more precisely: if $F^0 \cap \overline{F^0} = 0$ — $\widetilde{G}$, which is according to the diagram above either the quotient $G/\alpha(I^*_{\mathbb{Z}})$ or $L_{\mathbb{Z}} \backslash L_{\mathbb{C}}/F^0(L_{\mathbb{C}})$ is an Abelian variety. Its Hodge structure emerges from the mixed Hodge structure of M by simply forgetting the weight filtration. The moduli points of the two are related by the boundary map (2.4.5).

(4.4.3) Remark. Let a Hodge structure of type $\mathbb{D}_{\mathbf{H}_{g_0}}$ on L_0 be given and let A be the corresponding Abelian variety. The above equivalence induces an isomorphism

$$\mathrm{WSp}(L_0, I)(\mathbb{Z}) \backslash \mathrm{WSp}(L_0, I)(\mathbb{C})/F^0(W) \cong \mathcal{P}^{(I \otimes I)^s}$$

(here $F^0(W)$ is the stabilizer of the (trivial) filtration $F^0(L_{0,\mathbb{C}}) \oplus I^*_{\mathbb{C}}$). The left hand side a fibre of the analytic map of Shimura varieties

$$P_{\mathbf{X}} \backslash \mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K(1) \to P_{\mathbf{H}_{g_0}} \backslash \mathbb{D}_{\mathbf{H}_{g_0}} \times P_{\mathbf{H}_{g_0}}(\mathbb{A}^{(\infty)})/K(1),$$

which is, according to (2.7.1), a moduli space for the categories (fibre)

$$[\ S\text{-}^{K(1)}\mathbf{X}\text{-}L\text{-}\mathbf{mhs}\ ]$$

and $\mathcal{P}^{(I\otimes I)^s}$ is the $\mathbb{G}_m \otimes (I \otimes I)^s$-torsor described in (4.2.6) which is according to (4.3.2) a moduli space for the categories (fibre) $[\ S\text{-}^{K(1)}\mathbf{X}\text{-}L\text{-}\mathbf{mot}\ ]$.
The isomorphism may also be described this way: In (2.5.8) we defined the group $\widetilde{\mathrm{WSp}}(L_0, I)$. Its $\mathbb{Z}$-points are a group extension

$$0 \longrightarrow I \otimes I \longrightarrow \mathrm{WSp}(L_0, I) \longrightarrow L_0 \otimes I \oplus I \otimes L_0 \longrightarrow 0$$

with sections $s_1 : L_0 \otimes I \to \widetilde{\mathrm{WSp}}(L_0, I)$, and $s_2 : I \otimes L_0 \to \widetilde{\mathrm{WSp}}(L_0, I)$. We have also the stabilizer F^0 of the filtration $F^0(L_{0,\mathbb{C}}) \oplus I_{\mathbb{C}}^*$ in $\mathrm{WSp}(L_0, I)(\mathbb{C})$. By the procedure (4.2.3) this defines an analytical biextension of $A \otimes I$, $I \otimes A$ by $\mathbb{G}_m \otimes (I \otimes I)$ (the isomorphism $I \cong I(1)$ being determined by the connected component of $\mathbb{D}_{\mathbf{H}_g}$ determined by F^0). It has an operation of s which is the involution transpose w.r.t. the symplectic form on L followed by inversion. The biextension is canonically isomorphic to $\mathcal{P}^{U\otimes U}$. and this isomorphism is compatible with s. The reduction of this isomorphism to the invariants under s gives another description of the isomorphism above. The horizontal fibers of the biextensions (analytically or algebraically) at a moduli point are related to the 1-motive described by the moduli point, by $M \mapsto (\mathrm{id}_{I^*})^*(M \otimes I_{\mathbb{Z}})$ (cf. also 5.4.1).

4.5. Standard principal bundle

(4.5.1) Definition. *Let S be a scheme over* $\mathrm{spec}(\mathbb{Z}_{(p)})$*, and $\mathbf{X}$ one of the symplectic p-integral mixed Shimura data. Define the following groupoid:*

$$[\ S\text{-}^{K}\mathbf{X}\text{-}L\text{-}\boldsymbol{triv\text{-}mot}\ ],$$

where $K \subset P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ is an admissible compact open subgroup, as the category of the following data: An object $(M, \cdots)$ of $[\ S\text{-}^{K}\mathbf{X}\text{-}L\text{-}\boldsymbol{mot}\ ]$ together with a trivialization (i.e. an isomorphism of locally free sheaves with $(P_{\mathbf{X}}, L)$-structures)

$$H^{dR}(M) \to L \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_S.$$

The morphisms are isomorphisms of $[\ S\text{-}^{K}\mathbf{X}\text{-}L\text{-}\boldsymbol{mot}\ ]$, respecting these trivializations. Note, that a p-(iso-)morphism induces an (iso-)morphism on de Rham realization because S is a scheme over $\mathrm{spec}(\mathbb{Z}_{(p)})$.

(4.5.2) Theorem. *Let $\mathbf{X}$ be one of the symplectic p-integral mixed Shimura data.*

$$S \mapsto [\ S\text{-}^{K}\mathbf{X}\text{-}L\text{-}\boldsymbol{triv\text{-}mot}\ ]$$

is representable by the associated torsor of the $(P_{\mathbf{X}}, L)$-structure on $H^{dR}(M)$, where M

is a universal 1-motive over $[\ S\text{-}{}^{K}\mathbf{X}\text{-}L\text{-}\boldsymbol{mot}\ ]$.
If S *is a smooth Deligne-Mumford stack over* $\mathbb{C}$, $\rho_{B,dR}$ *induces a functor*

$$[\ S\text{-}{}^{K}\mathbf{X}\text{-}L\text{-}\boldsymbol{triv\text{-}mot}\ ] \cong [\ S^{an}\text{-}{}^{K}\mathbf{X}\text{-}L\text{-}\boldsymbol{triv\text{-}mhs}\ ],$$

i.e. to the analytic standard principal bundle (2.7.4). For $S = \mathrm{spec}(\mathbb{C})$ *this is an equivalence.*
The Hodge filtration on $H^{dR}(M)$ *induces a* $P_{\mathbf{X}}$*-equivariant morphism*

$$\Pi : [\ {}^{K}\mathbf{X}\text{-}L\text{-}\boldsymbol{triv\text{-}mot}\ ] \to \mathrm{M}^{\vee}(\mathbf{X}),$$

which is compatible with the morphism $\Pi_{\mathbb{C}}$ *already defined (2.7.5).*

Proof. The first statement follows directly from the definition. Let Y be

$$[\ {}^{K}\mathbf{X}\text{-}L\text{-}\mathbf{triv\text{-}mot}\ ]$$

and let X be

$$[\ {}^{K}\mathbf{X}\text{-}L\text{-}\mathbf{mot}\ ].$$

X is a smooth Deligne-Mumford stack over $\mathrm{spec}(\mathbb{Z}_{(p)})$, by (4.3.3). The second statement follows from (4.4.1), and the third follows from (1.9.13), once we have shown that the Hodge filtration on $H^{dR}(M)$ is compatible with the $(P_{\mathbf{X}}, L_X)$-structure, but this is true over $X_{\mathbb{C}}$ and we get a morphism

$$\Pi_{\mathbb{C}} : Y_{\mathbb{C}} \to \mathrm{M}^{\vee}(\mathbf{X}).$$

Now there is a *closed* embedding $\mathrm{M}^{\vee}(\mathbf{X}) \hookrightarrow \mathcal{QPAR}_Q$ (1.9.11), where Q is the parabolic of $\mathrm{GL}(L)$ fixing the weight filtration. Π extends to a morphism (1.9.15):

$$\Pi : Y \to \mathcal{QPAR}_Q$$

because the weight and Hodge filtration are a bisaturated pair. Hence we get an induced

$$\Pi : Y \to M^{\vee}(\mathbf{X}).$$

□

5. Constructions for mixed Shimura varieties of Hodge type

5.1. Hodge tensors

Let G be a reductive group of $\mathrm{spec}(\mathbb{Z}_{(p)})$ and L a lattice with non-degenerate primitive symplectic form. Suppose we are given a closed embedding $G \hookrightarrow \mathrm{GSp}$, for example, coming from a Hodge embedding $\mathbf{X} \hookrightarrow \mathbf{H}_g$.
Denote

$$L^{\otimes} := \bigoplus_{i-j=2n} L^{\otimes i} \otimes (L^*)^{\otimes j} \otimes \mathbb{Z}_{(p)}(n)$$

and similarly for any kind of object, for which this makes sense. (If we are just considering lattices, put $\mathbb{Z}_{(p)}(n) = \mathbb{Z}_{(p)}$).
G operates also linearly on $L^{\otimes}$ (with weight action of GSp on $\mathbb{Z}_{(p)}(n)$).

(5.1.1) Theorem. *There is a finite family $\{s_i\} \subset L^{\otimes}$, such that*

$$G = \mathrm{Stab}(\{s_i\}, \mathrm{GSp}(L)).$$

Proof. Analogous to [26, Prop. 3.1] (cf. also [49]). □

(5.1.2) Definition. *Let A be an Abelian variety over a number field F and let*

$$(s_{dR}, s_{et}) \in F^0(H_1^{dR}(A)^{\otimes}) \times H_1^{et}(A, \mathbb{Z}_p)^{\otimes}$$

be given.
Let $\sigma : F \hookrightarrow \mathbb{C}$ be an embedding. It determines an Abelian variety $A^{\sigma} = A \times_{\mathrm{spec}(F),\sigma} \mathrm{spec}(\mathbb{C})$ over $\mathrm{spec}(\mathbb{C})$, and hence elements

$$s_{dR}^{\sigma} \in F^0(H_1^{dR}(A^{\sigma})^{\otimes}),$$
$$s_{et}^{\sigma} \in H_1^{et}(A^{\sigma}, \mathbb{Z}_p)^{\otimes}.$$

We call (s_{dR}, s_{et}) a ***Hodge tensor*** *with respect to σ, if there is an $s_{B,\sigma} \in H_1^B(A^{\sigma}, \mathbb{Z}_{(p)})^{\otimes}$ such that $\rho_{B,dR}(s_{B,\sigma}) = s_{dR}^{\sigma}$ and $\rho_{B,et}(s_{B,\sigma}) = s_{et}^{\sigma}$.*
We call (s_{dR}, s_{et}) an ***absolute Hodge tensor*** *if this is true for all embeddings.*

(5.1.3) Theorem. *With the notation in the previous definition*

$$(s_{dR}, s_{et}) \text{ Hodge (for one embedding) } \Leftrightarrow (s_{dR}, s_{et}) \text{ absolute Hodge.}$$

Proof. This is [26, Main Theorem 2.11], cf. also [26, p. 29/30]. Note that if $s_{et} \in H_1^{et}(A^\sigma, \mathbb{Z}_p) \subset H_1^{et}(A^\sigma, \mathbb{A}^{(\infty)})$ then all $s_{B,\sigma}$ have to be integral a priori. □

(5.1.4) Theorem. *Let $\mathcal{O}$ be d.v.r with fraction field F a number field, $\mathfrak{m}|(p)$ and let A be an Abelian scheme over $\mathcal{O}$, There is a ring $B \supset \widehat{\mathcal{O}}$ faithfully flat over $\widehat{\mathcal{O}}$ and an isomorphism*

$$\gamma : H^1_{et}(A_F, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B \to H^1_{dR}(A|\mathcal{O}) \otimes_{\mathcal{O}} B$$

with the property that, if $(s_{dR}, s_{et})_i \in F^0(H^1_{dR}(A_F)^{\otimes}) \times H^1_{et}(A_F, \mathbb{Z}_p)^{\otimes}$ is a family of absolute Hodge cycles on A defining a connected reductive subgroup of $\mathrm{GSp}(H^1_{et}(A_F, \mathbb{Z}_p))$ (in particular with integral s_{et}) then

$$\gamma(s_{i,et} \otimes 1) = s_{i,dR} \otimes 1 \quad \forall i.$$

In particular $s_{i,dR}$ is defined over $\mathcal{O}$.

Proof. Follows from the proposition and corollary [49, 4.2][1]. □

5.2. Smoothness

(5.2.1) Theorem. *Let $\mathbf{X}$ be p-MSD of Hodge type. If $N, p \nmid N$ is big, there is an embedding*

$${}^{K \cap P_{\mathbf{X}}(\mathbb{A}^{(\infty)})}\mathbf{X} \hookrightarrow {}^K\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I],$$

for $K = K_{\mathrm{WSp}}(1) K_{\mathrm{GSp}}(N)$. Furthermore the normalization of the closure of

$$P_{\mathbf{X}}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)}) / K \cap P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$$

in $\mathrm{M}({}^K\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I])$ is smooth.

Proof. If $\mathbf{X}$ is pure, we have $I = 0$, and the theorem is shown in [49].
Now suppose $W_{\mathbf{X}} \neq 1$. Let $\mathbf{X} \hookrightarrow {}^K\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]$ be an embedding of p-integral mixed Shimura data, which exists by the Hodge type property, and let F be the reflex field of $\mathbf{X}$ and $\mathcal{O}$ be a reflex ring.
Choose a lattice $L_{\mathbb{Z}} \subseteq L_{\mathbb{Z}_{(p)}}$ with compatible splitting $L_{\mathbb{Z}} = L_{0,\mathbb{Z}} \oplus I_{\mathbb{Z}} \oplus I^*_{Z}$. It suffices to do the construction for any conjugate of the above embedding. We may hence assume

[1] The reduction to the mentioned reference lacks details. It will be stated more precisely in forthcoming work. See also M. Kisin, *Integral models for Shimura varieties of abelian type*, J. Amer. Math. Soc. 23 (2010), 967-1012

that there is a reductive subgroup scheme $G_{\mathbf{X}} \subset P_{\mathbf{X}}$ which factors through $\mathrm{GSp}(L_0)$ under the above embedding. We will use the smooth model $\mathrm{M}({}^{K(N)}\mathbf{X}/W_{\mathbf{X}})$ already constructed (pure case). Consider the Cartesian diagram of $\mathcal{O}$-schemes

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & \mathrm{M}({}^{K(N)}\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I])_{\mathcal{O}} \arrow[d] \\
A \arrow[r] \arrow[d] & \mathrm{M}({}^{K(N)}\mathbf{H}_{g_0}[0, L_0 \otimes I])_{\mathcal{O}} \arrow[d] \\
\mathrm{M}({}^{K(N)}\mathbf{X}/W_{\mathbf{X}}) \arrow[r] & \mathrm{M}({}^{K(N)}\mathbf{H}_{g_0})_{\mathcal{O}},
\end{tikzcd}$$

where the lower horizontal arrow is a normalization followed by an embedding.

Now consider the diagram of (2.5.8). Looking at elements fixing $L_{\mathbb{Z}}$ we get a diagram of ordinary groups:

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{U}_P \arrow[r] \arrow[d, hook] & \widetilde{W}_P \arrow[r] \arrow[d, hook] & V_P \oplus V_P \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & (I \otimes I)_{\mathbb{Z}} \arrow[r] & \widetilde{\mathrm{WSp}} \arrow[r] & (I \otimes L_0)_{\mathbb{Z}} \oplus (L_0 \otimes I)_{\mathbb{Z}} \arrow[r] & 0
\end{tikzcd}$$

where the outer arrows are saturated inclusions of lattices.

The sequences above define (together with their families of Hodge structures on $(L_0 \otimes I)_{\mathbb{C}}$ resp. $V_{P,\mathbb{C}}$) analytically biextensions on $(\mathrm{M}({}^{K(N)}\mathbf{X}/W_{\mathbf{X}})_{\mathbb{C}})^{an}$ by the procedure (4.2.3).

They are in fact algebraic (as all analytic biextensions are) and extend to our integral models $\mathrm{M}({}^{K(N)}\mathbf{X}/W_{\mathbf{X}})$ because they can be described by maps between Abelian varieties and the Abelian varieties do extend: The Abelian scheme associated with the family of Hodge structures on $L_0 \otimes I$ is A in the diagram above. The Abelian scheme associated with V_P is the closure of the rational model [83] $\mathrm{M}({}^{K(N)}\mathbf{X}/U)_E$ in A. It is automatically an Abelian subscheme. The first biextension therefore is also defined over the model $\mathrm{M}({}^{K(N)}\mathbf{X}/W)$. The inclusion map of biextensions, as well as the involution s, extend to $\mathrm{M}({}^{K(N)}\mathbf{X}/W_{\mathbf{X}})$ as well because everything is normal (analog 3.7.3).

The second biextension is, by construction the pullback of $\mathcal{P}^{I \otimes I}$ to $\mathrm{M}({}^{K(N)}\mathbf{X}/W_{\mathbf{X}})$ (4.2.6). Its invariants under s is the pullback of $\mathcal{P}^{(I \otimes I)^s}$ i.e. E in the diagram above.

We now define $\mathrm{M}({}^{K(N)}\mathbf{X})$ as the invariants under s in the first biextension. That this yields a model of

$$P_{\mathbf{X}}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K(N))$$

follows from the fact that $W_{\mathbf{X}} = (\widetilde{W}_{\mathbf{X}})^s$ (2.5.8). $\square$

5.3. Construction of the standard principal bundle, pure case

Let $\mathbf{X}$ be p-integral pure Shimura data of Hodge type. Let $\mathbf{X} \hookrightarrow \mathbf{H}_g$ be an embedding and let E be the reflex field of $\mathbf{X}$ and $\mathcal{O}$ be a reflex ring.

Consider a compact open subgroup $K(N) \subset \mathrm{GSp}(L_{\mathbb{A}^{(\infty)}})$ (formed w.r.t. the standard representation on L), $p \nmid N$, N sufficiently large. (We denote its inverse image in $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$ by the same symbol because no confusion will result.) Actually the level structure does not play any role whatsoever in the construction of the standard principal bundle, as it should be.

(5.3.1) Theorem. *In the following situation*

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & \mathrm{P}({}^{K(N)}\mathbf{H}_g)_{\mathcal{O}} \arrow[d] \\ M({}^{K(N)}\mathbf{X}) \arrow[r] & \mathrm{M}({}^{K(N)}\mathbf{H}_g)_{\mathcal{O}} \end{tikzcd}$$

there is a unique model of the analytic standard principal bundle over $M({}^{K(N)}\mathbf{X})$, fitting into the diagram, such that the morphism to $\mathrm{P}({}^{K(N)}\mathbf{H}_g)_{\mathcal{O}}$ gives over $\mathbb{C}$ the obvious one on double quotients.

Proof. In the diagram, the horizontal arrow in the bottom line is a closed embedding, followed by a normalization. $\mathrm{P}({}^{K(N)}\mathbf{H}_g)$ is the associated right torsor for the (GSp, L)-structure on $H_1^{dR}(A^0)$, where A^0 is the universal Abelian scheme over $\mathrm{M}({}^{K(N)}\mathbf{H})$. We have the pullback A to $M({}^{K(N)}\mathbf{X})$ of A_0. And the pullback of $\mathrm{P}({}^{K(N)\mathbf{H}_g})$ is the associated right torsor for the (GSp, L)-structure on $H_1^{dR}(A)$.

Now we have an analytic $G_{\mathbf{X}}$-structure, i.e. locally, say on U, isomorphisms:

$$H_1^B(A^{an}|_U, \mathbb{Z}_{(p)}) \cong (L_{\mathbb{Z}_{(p)}})_U$$

such that their $G_{\mathbf{X}}(\mathbb{Z}_{(p)})$-orbit is equal on overlaps. The pullback of the tensors $s_i \in L^{\otimes}$ defining the group scheme $G_{\mathbf{X}}$ are therefore well defined tensors

$$s_{i,B} \in H_1^B(A^{an}, \mathbb{Z}_{(p)})^{\otimes}.$$

Via the comparison isomorphisms we get tensors

$$s_{i,dR} \in F^0(H_1^{dR}(A^{an})^{\otimes})$$

$$s_{i,et} \in H_1^{et}(A^{an}, \mathbb{Z}_p)^{\otimes} = H_1^{et}(A_{\mathbb{C}}, \mathbb{Z}_p)^{\otimes},$$

which are absolutely Hodge (5.1.3) on points defined over number fields. One shows (e.g.

[75, III, §4]) that they are algebraic and actually

$$s_{i,dR} \in H_1^{dR}(A_E)^{\otimes},$$

where E is the reflex field of $\mathbf{X}$.

Consider some $X = \bigoplus_i \bigoplus_{(n,m,k)\in I_i} L^{\otimes n} \otimes (L^*)^{\otimes m} \otimes \mathbb{Z}_{(p)}(k)$ (finite sums!), such that s_i lives in $\bigoplus_{(n,m,k)\in I_i} L^{\otimes n} \otimes (L^*)^{\otimes m} \otimes \mathbb{Z}_{(p)}(k)$. The group scheme GSp over $\operatorname{spec}(\mathbb{Z}_{(p)})$ acts on $\mathbb{W}(X)$. The sum of the s_i, considered as a single morphism,

$$\operatorname{spec}(\mathbb{Z}_{(p)}) \to \mathbb{W}(X)$$

define an *embedding*

$$\operatorname{GSp}/G_{\mathbf{X}} \to \mathbb{W}(X),$$

where the left hand side is a smooth quasi-projective variety. Denote by Y the image of the above embedding.

Now take etale locally $U \to \mathrm{M}({}^{K(N)}\mathbf{X})$ an isomorphism in the (GSp, L)-structure:

$$\gamma_U : H_1^{dR}(A_U) \to L \otimes \mathcal{O}_U.$$

The tensors $s_{i,dR}$ can (via this identification) be seen as a single map

$$U_E \to \mathbb{W}(X) \times_{\operatorname{spec}(\mathcal{O})} \operatorname{spec}(E). \qquad (1)$$

We claim that this map is the base change of a map $U \to \mathbb{W}(X)$ and that the image lies in Y.

It follows that, modulo a refinement of the etale cover, the $s_{i,dR}$ (considered as a map into $\mathbb{W}(X)$) lift to a morphism

$$U \to \mathrm{GSp}.$$

Changing γ_U by this automorphism of $L \otimes \mathcal{O}_U$, we get an isomorphism γ_U' mapping $s_{i,dR}$ to s_i for all i, hence the former lie in $H_1^{dR}(A_U)$. The γ_U' hence define a $G_{\mathbf{X}}$-structure on $H_1^{dR}(A)$, which is over A^{an} the $G \otimes \mathcal{O}$-structure on $H_1^{dR}(A^{an})$ defined via the comparison $\rho_{B,dR}$. The associated torsor for this $G_{\mathbf{X}}$-structure on $H_1^{dR}(A)$ therefore defines a model $\mathrm{P}({}^{K}\mathbf{X})$ of

$$G_{\mathbf{X}}(\mathbb{Q})\backslash \mathbb{D}_{\mathbf{X}} \times G_{\mathbf{X}}(\mathbb{C}) \times (G_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K(N)).$$

However, we are left to show that (1) comes from a map $U \to \mathbb{W}(X)$ and that the image lies in Y. For this is suffices to show (5.11.2) that for each point $s \in U$, lying in the special fibre, there is a d.v.r. $\mathcal{O}'$, with embedding $\operatorname{spec}(\mathcal{O}') \to U$ and special point s such that the map extends to $\mathcal{O}'$. However, it suffices to show this over a faithfully flat extension of $\mathcal{O}'$. Hence it follows from (5.1.4). □

5.4. Construction of the standard principal bundle, mixed case

Let $\mathbf{X} \hookrightarrow \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I]$ be an embedding of p-integral mixed Shimura data, and let E be the reflex field of $\mathbf{X}$ and $\mathcal{O}$ be a reflex ring.

Consider a compact open subgroup $K(N) \subset \mathrm{USp}(L_{0,\mathbb{A}^{(\infty)}}, I_{\mathbb{A}^{(\infty)}})$, $p \nmid N$, N sufficiently large. (We denote its inverse image in $P_{\mathbf{X}}(\mathbb{A}^{(\infty)})$, quotients mod various W's, etc. by the same symbol because no confusion will result — all groups are of the type $K(N)$ w.r.t. some standard representation as defined in 2.5.) We may conjugate the above embedding and may hence assume w.l.o.g. that $G_{\mathbf{X}}$ for a splitting as in (2.2.9, cf. also 2.5.8) factors through $\mathrm{GSp}(L)$. Actually the level structure does not play any role whatsoever in the construction of the standard principal bundle, as it should be.

(5.4.1) Theorem. *In the following situation*

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & \mathrm{P}({}^{K(N)}\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I])_{\mathcal{O}} \arrow[d] \\ \mathrm{M}({}^{K(N)}\mathbf{X}) \arrow[r] & \mathrm{M}({}^{K(N)}\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I])_{\mathcal{O}} \end{tikzcd}$$

there is a unique model of the analytic standard principal bundle over $M({}^{K(N)}\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I])_{\mathcal{O}}$, fitting into the diagram, such that the morphism to $\mathrm{P}({}^{K(N)}\mathbf{H}_g)_{\mathcal{O}}$ gives over $\mathbb{C}$ the obvious one on double quotients.

Proof. Here the horizontal arrow in the bottom line is a closed embedding, followed by a normalization. $\mathrm{P}({}^{K(N)}\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I])$ is, by definition, the associated right torsor for the (USp, L)-structure on $H^{dR}(M_0)$, where M_0 is the universal 1-motive over $\mathrm{M}({}^{K(N)}\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I])$. We have the pullback M to $\mathrm{M}({}^{K(N)}\mathbf{X})$ of $M_{0,\mathcal{O}}$. And the pullback of $\mathrm{P}({}^{K(N)}\mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I])_{\mathcal{O}}$ is equally the associated right torsor for the (USp, L)-structure on $H^{dR}(M)$.

Over $\mathbb{C}$, in the analytic category, we have inside it the right $P_{\mathbf{X},\mathbb{C}}$-torsor which is associated with the analytic $P_{\mathbf{X}}$-structure on $H^{dR}(M_{\mathbb{C}}{}^{an})$ transported via $\rho_{B,dR}$. We have to show that it is algebraic, defined over E, and its closure is a right $P_{\mathbf{X}}$-torsor. We have already constructed a $G_{\mathbf{X}}$-structure on $\mathrm{gr}^W_{-1}(H^{dR}(M))$, which over $\mathbb{C}$ gives the one coming from this analytic $P_{\mathbf{X}}$-structure.

Now consider the diagram of (2.5.8). Looking at elements fixing $L_{\mathbb{Z}}$ we get a diagram of ordinary groups:

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{U}_P \arrow[r] \arrow[d, hook] & \widetilde{W}_P \arrow[r] \arrow[d, hook] & V_P \oplus V_P \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & (I \otimes I)_{\mathbb{Z}} \arrow[r] & \widetilde{\mathrm{WSp}} \arrow[r] & (I \otimes L_0)_{\mathbb{Z}} \oplus (L_0 \otimes I)_{\mathbb{Z}} \arrow[r] & 0, \end{tikzcd} \tag{2}$$

where the outer arrows are saturated inclusions of lattices.
We recall from the proof of (5.2): The bottom line with the induced Hodge structure on $((L_0 \otimes I)_{\mathbb{C}})_{\mathrm{M}({}^{K(N)}\mathbf{X}/W_{\mathbf{X}})}$ defines a biextension of $A := \mathrm{gr}_{-1} M$, which is algebraic and defined over $\mathrm{M}({}^{K(N)}\mathbf{X}/W_{\mathbf{X}})$. It is the pullback of $\mathcal{P}^{I\otimes I}$ (4.2.6) to $M({}^{K(N)}\mathbf{X}/W_{\mathbf{X}})$. Its invariants under s is the already constructed model $\mathrm{M}({}^{K(N)}\mathbf{X}/W_{\mathbf{X}} \times_{\mathbf{H}_{g_0}} \mathbf{H}_{g_0}[(I \otimes I)^s, L_0 \otimes I])$.
The upper line defines a biextension, too, which extends to $\mathrm{M}({}^{K(N)}\mathbf{X}/W_{\mathbf{X}})$ such that its invariants under s are $\mathrm{M}({}^{K(N)}\mathbf{X})$. Hence for each morphism $\alpha : S \to \mathrm{M}({}^{K(N)}\mathbf{X})$ we get a 1-motive M_α over S, which is the horizontal fibre at α of $\mathcal{P}^{I\otimes I}$ together with the section α.
It is related to the 1-motive $\alpha^* M$ parametrized by α via:

$$M_\alpha = (\mathrm{id}_{I^*})^*(\alpha^* M \otimes I_{\mathbb{Z}})$$

(tensor $\alpha^* M$ with $I_{\mathbb{Z}}$ and 'pullback'[2] via $\mathrm{id}_{I^*} : \mathbb{Z} \to I_{\mathbb{Z}} \otimes I^*_{\mathbb{Z}}$).
We now have a sub-1-motive M'_α of M_α which is the horizontal fibre at α of the biextension associated to the first line. We know that $H^{dR}(\alpha^* M)$ carries a $(\mathrm{USp}(L_0, I), L)$-structure and we have

$$H^{dR}(M_\alpha) = (H^{dR}(\alpha^* M) \otimes_{\mathbb{Z}} I_{\mathbb{Z}})^{\mathrm{id}_{I^*}},$$

where $(\cdots)^{\mathrm{id}_{I^*}}$ means restriction to those elements whose projection to gr_0 lies in $\mathcal{O}_S\,\mathrm{id}_{I^*}$. Note that an isomorphism $\mathrm{gr}_0(\alpha^* M) \cong I^* \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_S$ is part of the $(\mathrm{USp}(L_0, I), L)$-structure.
Consider the embedding

$$H^{dR}(M'_\alpha) \hookrightarrow H^{dR}(M_\alpha).$$

We have

$$H^{dR}(M'_\alpha)/W_{-1}(H^{dR}(M'_\alpha)) = H^{dR}(M_\alpha)/W_{-1} H^{dR}(M_\alpha) = \mathcal{O}_S\,\mathrm{id}_{I^*}\,.$$

We may take a lift e_{I^*} of id_{I^*} to $F^0(H^{dR}(M'_\alpha))$. Since

$$F^0(H^{dR}(M_\alpha)) = (F^0(H^{dR}(\alpha^* M)) \otimes_{\mathbb{Z}} I_{\mathbb{Z}})^{id}$$

and $F^0(H^{dR}(\alpha^* M))$ is isotropic, e_{I^*}, considered as map $I^* \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_S \to H^{dR}(\alpha^* M)$ has isotropic image. Since mod W_{-1} it is the identity, we get a unique splitting $H^{dR}(\alpha^* M) \cong L \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_S = (I^* \oplus L_0 \oplus I) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_S$ (within the USp-structure) such that e_{I^*} composed with it, is just the inclusion of $I^* \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_S$.
The $P_{\mathbf{X},S}$-orbit of this splitting is a $P_{\mathbf{X}}$-structure, for which it is not clear so far, that it is uniquely determined, i.e. independent of the choice of the lift e_{I^*}.
Doing this construction for the universal 1-motive M over $S = \mathrm{M}({}^{K(N)}P_{\mathbf{X}})$, we get

[2]For each 1-motive $M = [\,\underline{Y} \xrightarrow{\alpha} G\,]$ and morphism $\gamma : \underline{Y}' \to \underline{Y}$ there is the 'pullback': $\gamma^* M = [\,\underline{Y}' \xrightarrow{\alpha\circ\gamma} G\,]$

a $P_{\mathbf{X}}$-structure on $H^{dR}(M)$. If we can show that for each $\mathbb{C}$-valued point of S, this $P_{\mathbf{X}}$-structure induces the $P_{\mathbf{X},\mathbb{C}}$-structure determined via $\rho_{B,dR}$ by the canonical $P_{\mathbf{X}}(\mathbb{Z})$-structure on $H^B((M_{\mathbb{C}})^{an})$ over the mixed Shimura variety over $\mathbb{C}$, then we have found the required model $\mathrm{P}({}^{K(N)}\mathbf{X})$ of the analytic standard principal bundle.
So take $S = \mathrm{spec}(\mathbb{C})$ and any point $\alpha : \mathrm{spec}(\mathbb{C}) \to \mathrm{M}({}^{K(N)}\mathbf{X})$ and show that any lift e_{I^*}, as above, defines the $P_{\mathbf{X},\mathbb{C}}$-structure determined by $\rho_{B,dR}$. We may identify $H^{dR}(\alpha^* M)$, resp. $H^B(\alpha^* M)$ with $L_{\mathbb{C}}$, resp. $L_{\mathbb{Z}}$. This induces an identification of $H^{dR}(M_\alpha)$ with

$$< \mathrm{id}_{I^*} >_{\mathbb{C}} \oplus (L_0 \otimes I)_{\mathbb{C}} \oplus (I \otimes I)_{\mathbb{C}} \tag{3}$$

The Hodge structure on $H^{dR}(M)$ is given by $F^0(H^{dR})(M) = g(I^*_{\mathbb{C}} \oplus F^0(L_{0,\mathbb{C}}))$, where g is some element of $W_{\mathbf{X}}(\mathbb{C})$. M_α is therefore determined by the Hodge structure

$$F^0(H^{dR})(M_\alpha) = g(\mathrm{id}_{I^*} \oplus F^0(L_{0,\mathbb{C}}) \otimes_{\mathbb{Z}} I_{\mathbb{Z}}),$$

where the action of $\mathrm{WSp}(\mathbb{C})$, on the space (3) is determined by the action of the Lie algebra of $\widetilde{\mathrm{WSp}}$ given as follows: an element $(i_1 \otimes i_2) \in I \otimes I$ acts as $(\alpha\,\mathrm{id}, l_0 \otimes i_3, i_4 \otimes i_5) \mapsto (0, 0, \alpha i_1 \otimes i_2)$ and an element $(i_1 \otimes l_{0,1})$ acts as $(\alpha\,\mathrm{id}, l_{0,2} \otimes i_2, i_3 \otimes i_4) \mapsto (0, \alpha i_1 \otimes l_{0,1}, 0)$ and an element $(l_{0,1} \otimes i_1)$ acts as $(\alpha\,\mathrm{id}, l_{0,2} \otimes i_2, i_3 \otimes i_4) \mapsto (0, 0, \langle l_{0,1}, l_{0,2}\rangle i_1 \otimes i_2)$.
The identification may be chosen in such a way that $H^{dR}(M'_\alpha)$ is given by the submodules $< \mathrm{id}_{I^*} > \oplus V_{P,\mathbb{C}} \oplus \widetilde{U}_{P,\mathbb{C}}$, similarly for $H^B(M'_\alpha)$. This is read off from diagram (2) and the construction of biextensions (4.2.3). In particular this identification lies in the analytic $P_{\mathbf{X}}$-structure.
Now, take any element $e_{I^*} \in F^0(H^{dR})(M'_\alpha)$ congruent to id_{I^*} mod $W_{-1}H^{dR}(M'_\alpha)$ as in the construction above, We have to show that there exists $h \in W_P(\mathbb{C})$ such that $he_{I^*} = \mathrm{id}_{I^*}$. but there is such $h \in W_{\mathrm{USp}}(\mathbb{C})$ because e_{I^*} determines another splitting $H^{dR} \cong L_{\mathbb{C}}$ compatible with weight filtration and symplectic structure because it had isotropic image (see above). Now look at $e_{I^*} - \mathrm{id}_{I^*} = (v_P, u_P)$ in $W_{-1}H^{dR}(M'_\alpha) = V_P \oplus \widetilde{U}_P$. Changing e_{I^*} w.l.o.g by an element of $W_P(\mathbb{C})$, whose projection is v_P, we may assume that $v_P = 0$. However, since $h \in \mathrm{WSp}(\mathbb{C})$, we read off from the induced action on (3), described above that v_P =projection of h to $(I \otimes L_0)_{\mathbb{C}}$ in the defining sequence for $\mathrm{WSp}(\mathbb{C})$. Then, for the modified h, we have $h \in (I \otimes I)^s_{\mathbb{C}} \subset \mathrm{WSp}(\mathbb{C})$, i.e. $h = u_P$. Now u_P lies also in $\widetilde{U}_P$ hence in $U_P \subset W_P$. □

5.5. Maps to the compact dual

(5.5.1) Lemma. *Assume we have an* embedding *of Shimura data $\mathbf{X} \hookrightarrow \mathbf{Y}$ and a diagram*

$$\begin{tikzcd}
\mathrm{P}({}^{K}\mathbf{X}) \arrow[r] \arrow[d] & \mathrm{P}({}^{K'}\mathbf{Y}) \arrow[d] \\
\mathrm{M}({}^{K}\mathbf{X}) \arrow[r] & \mathrm{M}({}^{K'}\mathbf{Y})
\end{tikzcd}$$

compatible with the obvious maps on double quotients over $\mathbb{C}$.

Assume that the map to the compact dual (given analytically) extends to a morphism $\mathrm{P}({}^{K'}\mathbf{Y}) \to \mathrm{M}^\vee(\mathbf{Y})$ *of models. Then the* $P_{\mathbf{X},\mathbb{C}}$*-equivariant map* $\mathrm{P}({}^{K}\mathbf{X})_\mathbb{C} \to \mathrm{M}^\vee(\mathbf{X})_\mathbb{C}$ *(given analytically) extends in a compatible way to a morphism*

$$\mathrm{P}({}^{K}\mathbf{X}) \to \mathrm{M}^\vee(\mathbf{X}).$$

Proof. There is a *closed* embedding (3.2.2)

$$\mathrm{M}^\vee(\mathbf{X}) \to \mathrm{M}^\vee(\mathbf{Y}).$$

The composition $\mathrm{P}({}^{K}\mathbf{X}) \to \mathrm{P}({}^{K'}\mathbf{Y}) \to \mathrm{M}^\vee(\mathbf{Y})$ has the property that $\mathrm{P}({}^{K}\mathbf{X})_\mathbb{C}$ is mapped to $\mathrm{M}^\vee(\mathbf{X})_\mathbb{C}$. Since the above embedding is closed, we get automatically that the composition factors through $\mathrm{M}^\vee(\mathbf{X})$ and is $P_\mathbf{X}$-equivariant. □

5.6. Independence of the Hodge embedding

(5.6.1) Theorem. *The constructions of (5.3.1) and (5.4.1) are independent of the chosen Hodge embedding. More generally, for any morphism* $\mathbf{X} \to \mathbf{H}_g[(I \otimes I)^s, I \otimes L_0]$ *of p-MSD (not necessarily an embedding) the analytic morphism on standard principal bundles is induced by a morphism of models* $\mathrm{P}({}^{K'}\mathbf{X}) \to \mathrm{P}({}^{K}\mathbf{H}_g[(I \otimes I)^s, I \otimes L_0])$*, where the left model is constructed by means of any (other) Hodge embedding.*

Proof. Let $\iota_i : \mathbf{X} \to \mathbf{H}_{g_i}[(I_i \otimes I_i)^s, I_i \otimes L_{0,i}]$, $i = 1, 2$ be two morphisms of integral Shimura data, where ι_1 an embedding. Denote $g = g_1 + g_2$, $I = I_1 \oplus I_2$ and $L_0 = L_{0,1} \oplus L_{0,2}$. There is an embedding

$$\mathbf{H}_{g_1}[(I_1 \otimes I_1)^s, I_1 \otimes L_{0,1}] \times_{\mathbf{H}_0} \mathbf{H}_{g_2}[(I_2 \otimes I_2)^s, I_2 \otimes L_{0,2}] \hookrightarrow \mathbf{H}_g[(I \otimes I)^s, I \otimes L_0],$$

given, via the modular description, by direct sum of 1-motives, and the ι_i induce an embedding

$$\mathbf{X} \hookrightarrow \mathbf{H}_{g_1}[(I_1 \otimes I_1)^s, I_1 \otimes L_{0,1}] \times_{\mathbf{H}_0} \mathbf{H}_{g_2}[(I_2 \otimes I_2)^s, I_2 \otimes L_{0,2}]$$

(the maps $\lambda \circ \iota_i : \mathbf{X} \to \mathbf{H}_0$ have to be the same because of weight reasons).
Let $L_1, L_2, L = L_1 \oplus L_2$ denote the standard representations.
Choose admissible compact open subgroups. Let M_i, resp. M be the 1-motives over $\mathrm{M}({}^{K}\mathbf{X})$, pullback of the universal ones (base changed to $\mathrm{spec}(\mathcal{O})$) along the various morphisms.
By construction $M = M_1 \oplus M_2$, hence $H^{dR}(M) = H^{dR}(M_1) \oplus H^{dR}(M_2)$.
We have 3 torsors T, T_1, T_2 of isomorphisms $L_S \to H^{dR}(M_S)$, $L_{i,S} \to H^{dR}(M_{i,S})$ defined over $\mathcal{O}$ (chosen reflex ring of $\mathbf{X}$). They are torsors for the groups $P_\mathbf{X}$, $P_\mathbf{X}$ and $\iota_2(P_\mathbf{X})$. (The last one may be a quotient of $P_\mathbf{X}$.)
Let S vary over an etale cover of $\mathrm{M}({}^{K}\mathbf{X})$ which trivializes all torsors. The isomorphisms $H^{dR}(M_S) \to L_S$ in the first right torsor have to respect the decomposition, because this

is true over $\mathbb{C}$. Therefore restriction constitutes a morphism of right torsors. We get an isomorphism (because this is true over $\mathbb{C}$) of right torsors $T \to T_1$ and a quotient map $T \to T_2$, whence a quotient map $T_1 \to T_2$. In fact, this has to be an isomorphism by the same reason, if ι_2 was an embedding as well. $\square$

5.7. Simple boundary points

If a rational boundary component ${}^{K'}_{\Delta'}\mathbf{B}$ of a Shimura datum ${}^{K}_{\Delta}\mathbf{X}$ has the property that $\mathrm{M}({}^{K'_{[\sigma]}}_{\Delta'_{[\sigma]}}\mathbf{B}_{[\sigma]})$, the corresponding boundary stratum (cf. 3.2.2) for a top-dimensional σ, i.e. $\mathrm{M}({}^{K'}_{\Delta'}\mathbf{B}/U_{\mathbf{B}})$ consists (over $\mathbb{C}$) only of a bunch of points, $P_{\mathbf{B}}/U_{\mathbf{B}}$ necessarily has to be a torus T. We will consider in this section the simplest case $T = \mathbb{G}_m$, and will also consider only special $\mathbb{D}_{\mathbf{B}}$ and $h_{\mathbf{B}}$. These boundary components occur as smallest ones in the case of symplectic Shimura data, as well as for the case of orthogonal Shimura data. See (10.4) for the main application of this section.

(5.7.1) Let $P_{\mathbf{X}}$ be a group scheme over $\mathrm{spec}(\mathbb{Z}_{(p)})$ with an isomorphism

$$\alpha_0 : U \rtimes \mathbb{G}_m \to P,$$

where $\mathbb{G}_m$ operates via its natural action on $U = \mathbb{W}(M)$ for a $\mathbb{Z}_{(p)}$-lattice M.
Let $\mathbb{D}_{\mathbf{X}} := \mathbb{D}_{\mathbf{H}_0} \times \mathbb{D}^0$, where $\mathbb{D}^0 \subset \mathrm{Hom}(\mathbb{S}, P_{\mathbb{C}})$ is the $M_{\mathbb{C}} \rtimes \mathbb{R}^*$-conjugacy class of morphisms containing $h_0 : z \mapsto \alpha_0(0, z\overline{z})$. Hence for any splitting (even only defined over $\mathbb{R}$), $\mathbb{D}^0$ will contain the so constructed morphism. There is a non-canonical isomorphism $\mathbb{D}^0 \cong M_{\mathbb{C}}$.
In this section, we will describe explicitly the canonical integral models associated with the data ${}^{K}\mathbf{X}$ for a compact open subgroup $K = K_U \rtimes K(m)$, where $K_U = M_{\widehat{\mathbb{Z}}}$, corresponding to some $\mathbb{Z}$-Lattice $M_{\mathbb{Z}}$ in $M_{\mathbb{Z}_{(p)}}$ and $K(m) = \{a \in \widehat{\mathbb{Z}}^* \mid a \equiv 1 \bmod m\}$. It is admissible, iff $p \nmid m$.
This is an explicit description of the unipotent extension $\mathbf{H}_0[M, 0]$ and we have an obvious morphism

$$\mathbf{X} \to \mathbf{H}_0.$$

According to (4.1.15), $\mathrm{M}({}^{K(m)}\mathbf{H}_0) \cong \mathrm{spec}(\mathbb{Z}_{(p)}[\zeta_m])$, where ζ_m is an (abstract) m-th root of unity.

(5.7.2) Theorem. *Consider the morphism*

$${}^{K}\mathbf{X} \to {}^{K(m)}\mathbf{H}_0.$$

The corresponding morphism of mixed Shimura varieties is $(\mathbb{G}_m \otimes M_{\mathbb{Z}})_{\mathrm{M}({}^{K(m)}\mathbf{H}_0)}$ *over* $\mathrm{M}({}^{K(m)}\mathbf{H}_0)$*, and the isomorphism*

$$\{(\mathbb{C}^* \otimes M_{\mathbb{Z}})_{\zeta_{m,\mathbb{C}}}\} \to P_{\mathbf{X}}(\mathbb{Q})\backslash \mathbb{D}_{\mathbf{H}_0} \times \mathbb{X}^0 \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K,$$

is given as follows:
Choose a representative $[\alpha, \rho] \in \mathbb{Q}^*\backslash \mathbb{D}_{\mathbf{H}_0} \times {\mathbb{A}^{(\infty)}}^*/K(m)$, *with* $\rho \in K(1)$. α *and* $\rho K(m)$ *are therefore determined (simultaneously) up to* ± 1. *Let* $\zeta := \exp(\alpha(\rho/N))$ *be the corresponding root of unity* $\in \mathbb{C}$ *(we consider* α *as isomorphism* $\mathbb{Z} \to \mathbb{Z}(1)$*, as usual). We map* $z \in \mathbb{C}^* \otimes M_{\mathbb{Z}}$ *in the fibre over* ζ *to* $[\alpha, \alpha^{-1}(\log(z)), 0, \rho] \in P(\mathbb{Q})\backslash \mathbb{D}_{\mathbf{H}_0} \times M_{\mathbb{C}} \times M_{\mathbb{A}^{(\infty)}} \rtimes {\mathbb{A}^{(\infty)}}^*/K$.

This is a well defined class, and these maps together yield an isomorphism as required.

Proof. Follows directly from the description (4.1.14) of

$$[\ S\text{-}{}^K\mathbf{H}_0[M_{\mathbb{Z}_{(p)}}, 0]\text{-}\mathbb{Z}_{(p)} \oplus M^*_{\mathbb{Z}_{(p)}}\text{-}\mathbf{mot}\ ].$$

(Because of (4.1.15) this stack defines the canonical model, too.) An element is w.l.o.g. represented by (cf. proof of 4.1.19) a motive of the form $[\ (M^*_{\mathbb{Z}})_S \xrightarrow{\beta} \mathbb{G}_m\ ]$, where $M_{\mathbb{Z}} = K_U \cap M_{\mathbb{Q}}$. □

(5.7.3) According to (3.3.4), the toroidal compactification (which is, in any case, a 'partial' one) of the above mixed Shimura variety is given by the torus embedding constructed from the torus $\mathbb{G}_m \otimes M_{\mathbb{Z}}$ over $\mathrm{M}({}^{K(m)}\mathbf{H}_0)$. It amounts to the same as a rational polyhedral cone decomposition on $U_{\mathbf{X}}(\mathbb{R}) = M_{\mathbb{R}}$. It is smooth (with respect to K), if every $\sigma \subset M_{\mathbb{R}}$ is generated by part of a basis of $M_{\mathbb{Z}}$, where $M_{\mathbb{Z}} = K_U \cap M_{\mathbb{Q}}$.
Therefore for any σ, the completion at the corresponding boundary point is given by

$$\mathrm{spf}(\mathbb{Z}_{(p)}[\zeta_m][\![\sigma^\vee \cap L^*_{\mathbb{Z}}]\!]).$$

Choose an integral section v of $\mathrm{M}^\vee(\mathbf{X})$. We have a map $l : \mathbb{D}_{\mathbf{X}} \to P(\mathbb{C})$ given via the splitting determined by v as

$$\begin{aligned} \mathbb{D}^\circ \times \mathbb{D}_{\mathbf{H}_0} &\to M_{\mathbb{C}} \rtimes \mathbb{C}^* \\ (\beta v, \alpha) &\mapsto \beta\alpha(1). \end{aligned}$$

(Recall: $\alpha(1) = \pm 2\pi i$, periods of the Tate motive).
It sits in a commutative diagram

$$\begin{tikzcd} & P_{\mathbf{X}}(\mathbb{C}) \arrow[d, "\beta\mapsto\beta v"] \\ \mathbb{D}_{\mathbf{X}} \arrow[ur, "l"] \arrow[r, "\mathrm{Borel}\circ h"'] & \mathrm{M}^\vee(\mathbf{X})(\mathbb{C}) \end{tikzcd}$$

The reason why considering this is that l trivializes $\mathrm{P}({}^K\mathbf{X})$ *integrally*:

(5.7.4) Theorem. *The standard principal bundle* $\mathrm{P}({}^K\mathbf{X})$ *is trivial. An integral trivializing section is given for example (over* $\mathbb{C}$*) as:*

$$s : P_{\mathbf{X}}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K \to P_1(\mathbb{Q})\backslash P_{\mathbf{X}}(\mathbb{C}) \times \mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K$$
$$[x,p] \mapsto [l(x)^{-1}, x, p] \qquad \text{for } p \in K_U K(1)!$$

It is extended to any $\mathrm{M}({}^K_\Delta\mathbf{X})$ *by means of this trivialization. The composition with the map* Π *to the compact dual is projection to* v *(considered as scheme* $\cong \mathrm{spec}(\mathbb{Z}_{(p)})$*).*

Proof. This follows from the explicit description of

$$[\ S\text{-}{}^K\mathbf{H}_0[M_{\mathbb{Z}_{(p)}}, 0]\text{-}\mathbb{Z}_{(p)} \oplus M^*_{\mathbb{Z}_{(p)}}\text{-}\mathbf{mot}\ ],$$

too. An element is w.l.o.g. represented by a motive of the form $[\ (M^*_{\mathbb{Z}})_S \xrightarrow{\beta} \mathbb{G}_m\ ]$, where $M_{\mathbb{Z}} = K_U \cap M_{\mathbb{Q}}$. The trivializing section is defined by the splitting $H^{dR} \cong \mathcal{O}_S(\frac{\mathrm{d}\, z}{z})^* \oplus M^*_S$ (isomorphic to $(\mathbb{Z}_{(p)} \oplus M^*)_S$ canonically), with respect to which $F^0(H^{dR}) = \mathcal{O}_S(\frac{\mathrm{d}\, z}{z})^* = \mathrm{Lie}(\mathbb{G}_m)$. If $S = \mathrm{spec}(\mathbb{C})$ the lattice H^B included via $\rho_{B,dR}$ is given by

$$\{(\log(\beta(m)), m) \mid m \in M_{\mathbb{Z}}\} + (\mathbb{Z}(1), 0).$$

Comparing this with the analytic description of the standard principal bundle, we get the result. $\square$

Let $\mathcal{E}$ be an equivariant vector bundle on $\mathrm{M}^\vee(\mathbf{X})$. It is trivialized $U_{\mathbf{X}} = \mathbb{W}(M)$-equivariantly by choosing a basis $\mathfrak{B}$ of the lattice $\mathcal{E}_v$.

By (9.1.2), we know that $H^0(\mathrm{M}({}^K\mathbf{X})_{\mathbb{C}}, \Xi^*(\mathcal{E}))$ is canonically identified with the set of $P_{\mathbf{X}}(\mathbb{Q})$-invariant sections $\mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K \to \mathcal{E}$, where here, $\mathcal{E}$ is pulled back along $\mathrm{Borel} \circ h$.

We want to describe analytically the *integral* trivialization of $\Xi^*(\mathcal{E})$ given by pullback along

$$\mathrm{M}({}^K\mathbf{X}) \overset{s}{\hookrightarrow} \mathrm{P}({}^K\mathbf{X}) \overset{\Pi}{\to} \mathrm{M}^\vee(\mathbf{X}),$$

observing that the image is just v and the fibre above v was trivialized by $\mathfrak{B}$.

(5.7.5) Lemma. *The trivialization is given analytically by the sections*

$$s_i : \mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K \to \mathcal{E}$$
$$x, [p] \mapsto l(x)v_i \qquad \text{for } p \in K_U K(1),$$

where $\mathfrak{B} = \{v_i\}$ *and the* v_i *are considered as points in the fibre over* v*. (Extend it to other* p *by* $P_{\mathbf{X}}(\mathbb{Q})$*-invariance).*

Proof. Consider the diagram (everything split via v):

$$\begin{tikzcd}
 & M_{\mathbb{C}} \times \mathbb{D}_{\mathbf{H}_0} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K \arrow[d, "\iota"] & \\
P_{\mathbf{X}}(\mathbb{Q})\backslash M_{\mathbb{C}} \times \mathbb{D}_{\mathbf{H}_0} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K \arrow[r, bend left, "s"] & P_{\mathbf{X}}(\mathbb{Q})\backslash P_{\mathbf{X}}(\mathbb{C}) \times \underbrace{M_{\mathbb{C}} \times \mathbb{D}_{\mathbf{H}_0}}_{\mathbb{D}_{\mathbf{X}}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K \arrow[l] \arrow[r, "\Pi"] & M_{\mathbb{C}}
\end{tikzcd}$$

where the maps are given as follows:

$$\begin{aligned}
\iota &: [Z, \alpha, p] \mapsto [1, Z, \alpha, p] = [l(Z, \alpha)^{-1}, Z, \alpha, p] \circ l(Z, \alpha) \\
s &: [Z, \alpha, p] \mapsto [l(Z, \alpha)^{-1}, Z, \alpha, p] \\
\Pi &: [g, Z, \alpha, p] \mapsto gZ
\end{aligned}$$

(here $\circ$ denotes the group operation on the standard principal bundle). From this, the statement follows. □

5.8. Normalization of formal schemes

Let $\mathcal{X}$ be a noetherian formal scheme. Locally it is of the form $\mathrm{spf}(A)$, where A is a topological ring, with an ideal of definition I. A is a noetherian ring, hence its normalization $N(A)$ is finitely generated as a A-module. The induced topology (for which it is hence complete) is the same as its $IN(A)$-adic topology.
Let S be a multiplicative subset of A. We may form the ring $A\{S^{-1}\}$, which is the completion of $A[S^{-1}]$ by the ideal $S^{-1}I$. It is flat over A. It satisfies a similar universal property like localization for usual rings.
For example, the ring $\mathcal{O}_{\mathrm{spf}(A)}(D_s)$ for the fundamental open set D_s is just $A\{s^{-1}\}$. Local rings $\mathcal{O}_{\mathrm{spf}(A),p}$ for a point p corresponding to an open ideal J are not of the form $A\{S^{-1}\}$ but its completion w.r.t. $\mathcal{O}_{\mathrm{spf}(A),p}I$ is of that form for $S = A - J$. The corresponding inclusion is faithfully flat. See e.g. [34, 1, 7.6].

(5.8.1) Theorem. *Let A be a completion of an excellent ring.*
We have

$$\begin{aligned}
N(A)\{s^{-1}\} &= N(A\{s^{-1}\}), \\
N(A_p) &= N(A)_p, \\
N(C_{IA_p}(A_p)) &= C_{IN(A)_p}(N(A)_p), \\
N(C_{m_p A_p}(A_p)) &= C_{m_p N(A)_p}(N(A)_p).
\end{aligned}$$

Here N denotes normalization and $C_I(A)$ denotes completion of A with respect to the I-adic topology.

Proof. First of all, we have a map

$$N(A)\{s^{-1}\} \to N(A\{s^{-1}\})$$

by the universal property of $N(A)\{s^{-1}\}$.
Both sides are finitely generated modules over $A\{s^{-1}\}$ since A and $A\{s^{-1}\}$ are noetherian. Kernel and cokernel of the map have the property that there are non-zero if and only if there is a point $p \in \mathrm{spf}(A\{s^{-1}\})$ for which they are non-zero over the respective local ring. Hence it suffices to show that over any local ring the above morphism is an isomorphism because these rings are flat over $A\{s^{-1}\}$.
Hence we have to show that the map

$$N(A)_p \to N(A\{s^{-1}\})_p$$

is an isomorphism. It suffices to do this after completion w.r.t. m_p because this is faithfully flat over the local ring [34, 1, 7.3.4]. We get the map

$$C_{m_p}(N(A)) \to C_{m_p}(N(A\{s^{-1}\})).$$

There is another map

$$C_{m_p}(N(A\{s^{-1}\})) \to N(C_{m_p}(A))$$

obtained by the universal property of completion. The composition of these maps is, however, an isomorphism by [35, 2, 7.8.3] because A is the completion of an excellent ring.
For all other cases one argues in a similar fashion. □

In particular the ringed spaces $\mathrm{spf}(N(A))$ can be glued canonically to a formal scheme, which we call the normalization $N(\mathcal{X})$ of $\mathcal{X}$. A formal scheme is called normal, if it is equal to its normalization.

(5.8.2) Theorem. *Let X be an integral excellent scheme and Y a closed subscheme. Form the formal completion $\mathcal{X} := C_Y(X)$.*

i. We have

$$C_{\iota^{-1}Y}(N(X)) = N(C_Y(X)),$$

where ι is the morphism $N(X) \to X$.

ii. Let $\mathcal{X}$ be normal (satisfied e.g. if X is normal) and U be an open subset of $\mathcal{X}$.

$\mathcal{O}_{\mathcal{X}}(U)$ is an integral domain, if and only if U is connected.

iii. Let $\mathcal{X}$ be normal and p be any point of $\mathcal{X}$.

$\mathcal{O}_{\mathcal{X},p}$ and its completions w.r.t. I_Y and m_p are integral domains.

Proof. The first statement is proven as in the previous theorem, using the local complete case.
That the completion of $A = \mathcal{O}_{\mathcal{X}}(U)$ at some prime ideal m_p is integral follows from normality. Since the maps from the any stalk $\mathcal{O}_{\mathcal{X},p}$ into it are injective, we get integrality of the local rings.
It remains to show that A is integral, too. It suffices to show that the map of A to any localization is injective. Let f be an element. Let $\mathrm{supp}(f)$ be set of points on which f does not vanish in the stalk $\mathcal{O}_{\mathcal{X},p}$. It is closed, as for any sheaf. We have to show that also $V := U - \mathrm{supp}(f)$, i.e. the set of points, where f does vanish in $\mathcal{O}_{\mathcal{X},p}$ is closed as well. We will show that it is closed under specialization. Let p be a point of V and p' any specialization. We know that $\mathcal{O}_{\mathcal{X},p}$ is a noetherian integral domain and $\mathcal{O}_{\mathcal{X},p'}$ is build from it by a process of localization and completion. Hence the map

$$\mathcal{O}_{\mathcal{X},p} \to \mathcal{O}_{\mathcal{X},p'}$$

is injective. Hence f does not vanish in $\mathcal{O}_{\mathcal{X},p'}$ as well. If U, in turn, is not connected, lifting of the existing nontrivial idempotents in $\mathcal{O}(U)/I_Y(U)$ yields zero divisors.

□

In other words for integral normal excellent domains, lifting of idempotents is the only source for zero divisors in *any* completion.

5.9. Abstract 'q-expansion'

Let R be an excellent normal integral domain and I an ideal, such that $\mathrm{spf}(C_I(R))$ is connected (or equivalently, such that $R/\sqrt{I}$ has no nontrivial idempotents).
Let s be a prime element of R, neither a zero divisor nor a unit of R/I. Let M be a projective R-module.

(5.9.1) Lemma. *The diagram*

$$\begin{tikzcd}
M \arrow[r, hook] \arrow[d, hook] & M[s^{-1}] \arrow[d, hook] \\
C_I(M) \arrow[r, hook] & C_I(M)\{s^{-1}\} = C_{I[s^{-1}]}(M[s^{-1}])
\end{tikzcd}$$

is Cartesian.

Proof. We show it for the case $M = R$, the general case is completely analogous because M is assumed to be projective.
The top horizontal map is injective because R is integral. The vertical maps are injective by Krull's theorem. Next, the ring $C_I(R)$ is an integral domain by (5.8.2). Hence the bottom horizontal arrow is injective. Therefore also the map $C_I(M)[s^{-1}] \to C_I(M)\{s^{-1}\}$

is injective, and we are left to show that for arbitrary n, the left square in the following diagram with exact lines is Cartesian.

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d, hook] & \frac{1}{s^n}R \arrow[r, "\cdot s^n"] \arrow[d, hook] & R/s^nR \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \widehat{R} \arrow[r] & \widehat{\frac{1}{s^n}R} \arrow[r] & \widehat{R/s^nR} \arrow[r] & 0 \end{tikzcd}$$

where in the bottom line we mean the I-adic completions of the respective f. g. R-modules. By a diagram chase, the square is Cartesian if and only if the right vertical map is injective. (The rows are exact because exactness of the completion functor (on f. g. modules). This is, however, not used in the sequel.) The right vertical map is injective, if $0 \in R/s^nR$ is closed w.r.t. the I-adic topology. Its closure is (by an extension of Krull's theorem [10, Chap. III, §3, 2., Prop. 5]) the set of $x \in R/s^nR$ for which there exists an $m \in I$, such that $(1-m)x = 0$ holds true. Therefore the above map is injective, if no $m \in I$ and $x \in R$ exist, such that $s^n|(1-m)x$ and $s^n \nmid x$. Since s is prime, this is the case, if no $m \in I$ exists, such that $s|1-m$. Since by assumption s is not a unit modulo I, this is indeed impossible. □

5.10. Formal Zariski closure

Let $\mathcal{M}$, $\widetilde{\mathcal{M}}$, $\mathcal{C}$ be integral schemes ($\mathcal{M}$ and $\widetilde{\mathcal{M}}$ normal), of finite type over $S = \mathrm{spec}(R)$, where R is an excellent discrete valuation ring and $q = \mathrm{spec}(Q)$ is the generic point, where Q is the quotient field of R. Give two closed embeddings $\mathcal{C} \hookrightarrow \mathcal{M}$ and $\mathcal{C} \hookrightarrow \widetilde{\mathcal{M}}$ such that the images are divisors.
and let $M = \mathcal{M} \times_S q$, $\widetilde{M} = \widetilde{\mathcal{M}} \times_S q$, $C = \mathcal{C} \times_S q$. Suppose, we are given closed reduced subschemes $M' \subset M$ and $\widetilde{M}' \subset \widetilde{M}$. Assume that they are all smooth. Denote their closures, by $\mathcal{M}'$, $\widetilde{\mathcal{M}}'$. Denote $\mathcal{C}' = \mathcal{C} \times_{\mathcal{M}} \mathcal{M}'$.
We assume that the following condition is satisfied:

(5.10.1) $$\pi_0(\mathcal{C} \times_{\mathcal{M}} N(\mathcal{M}')) \simeq \pi_0(C').$$

This is for example satisfied, if every irreducible component of $\mathcal{C}'$ contained in the special fibre (which is hence also an irreducible component of the special fibre of $\mathcal{M}'$) has at least one point, lying on the image of a section $\mathrm{spec}(R') \to \mathcal{M}'$ for some (possibly ramified) extension R' of R, such that its generic point is contained in C.

(5.10.2) Lemma. *If there is an isomorphism*

$$C_{\mathcal{C}}(\mathcal{M}) \to C_{\mathcal{C}}(\widetilde{\mathcal{M}}),$$

inducing an isomorphism

$$C_{C'}(M') \to C_{C'}(\widetilde{M}')$$

with the property that $C' := C \times_M M' = C \times_{\widetilde{M}} \widetilde{M}'$ *is reduced then this induces an isomorphism*

$$C_{\mathcal{C}'}(\mathcal{M}') \to C_{\mathcal{C}'}(\widetilde{\mathcal{M}}')$$

and similarly for the normalizations.

It follows also that

$$\mathcal{C} \times_{\mathcal{M}} N(\mathcal{M}') \simeq \mathcal{C} \times_{\widetilde{\mathcal{M}}} N(\widetilde{\mathcal{M}}').$$

Proof. The question is local on $\mathcal{M}$, resp. $\widetilde{\mathcal{M}}$, hence we may assume that everything is affine and denote coordinate rings of the M's with A's and of the C's by B's. We denote the ideals of the C's by I's. We choose the covering such that on an open set in the cover the generic fibre, C and C' are connected. Because of assumption (5.10.1) we may assume also, that $\mathcal{C} \times_{\mathcal{M}} N(\mathcal{M}')$ is connected.

The above situation induces a cube:

$$\begin{tikzcd}
& C_{\mathcal{I}}(\mathcal{A}) \arrow[rr, "\sim"] \arrow[dl, hook] \arrow[dd, two heads] & & C_{\widetilde{\mathcal{I}}}(\widetilde{\mathcal{A}}) \arrow[dl, hook] \arrow[dd, two heads] \\
C_I(A) \arrow[rr, "\sim"] \arrow[dd, two heads] & & C_{\widetilde{I}}(\widetilde{A}) \arrow[dd, two heads] & \\
& C_{\mathcal{I}'}(\mathcal{A}') \arrow[rr, dotted, "?"] \arrow[dl, hook] & & C_{\widetilde{\mathcal{I}}'}(\widetilde{\mathcal{A}}') \arrow[dl, hook] \\
C_{I'}(A') \arrow[rr, "\sim"] & & C_{\widetilde{I}'}(\widetilde{A}') &
\end{tikzcd}$$

If we can justify the injectivity and surjectivity properties of the maps in the cube, the required map can be constructed by a 'diagram chase' in the cube. From the construction follows that it is an isomorphism.

The maps going from back to front are injective, if the corresponding ring in the back has no zero-divisors. For the rings on the top, this follows from (5.8.2) because C, hence $\mathcal{C}$, is connected here. For the normalization of the rings at the bottom, this follows from (5.8.2) because according to assumption (5.10.1) also $\mathcal{C} \times_{\mathcal{M}} N(\mathcal{M}')$ is connected. Since the map from a ring into its normalization is injective by construction, the rings in bottom back are integral domains as well.

The vertical maps are surjective because they are if there is an n such that, for example, $\forall x \in I'$ there exists an $y \in I$ such that $y \cong x^n \bmod J$, where J is the ideal of M', i.e. if $I' \subset \sqrt{I+J}$. But we have even $I' = I + J$ by definition.

Once established, the formal isomorphism yields in particular

$$\mathcal{C} \times_{\mathcal{M}} N(\mathcal{M}') \simeq \mathcal{C} \times_{\widetilde{\mathcal{M}}} N(\widetilde{\mathcal{M}}')$$

because $\mathcal{M}$ is normal and C' is reduced. □

5.11. Extension of morphisms

(5.11.1) Lemma. *Let $\mathcal{M}$ be an integral scheme of finite type smooth over* spec(R)*, where R is an excellent d.v.r. with maximal ideal $\mathfrak{m}$ and quotient field Q. Assume that the special fibre of $\mathcal{M}$ is irreducible (by Zariski's connectedness theorem this follows automatically, if $\mathcal{M}$ is projective over R).*
Let $\mathcal{C}$ be an integral closed subscheme of $\mathcal{M}$ such that $\mathcal{C}_{\mathfrak{m}}$ (special fibre) is integral as well. Let there be a Q-morphism $\alpha : \mathcal{M}_Q \to \mathcal{N}$ into any scheme of finite type over spec(R)*. Assume that α extends to $\mathcal{M} - \mathcal{C}$ and to $C_{\mathcal{C}}(\mathcal{M})$. Then α extends to $\mathcal{M}$.*

Proof. This follows immediately from (5.9.1) for s being a uniformizer of R. s is prime because of the assumption that the special fibre be irreducible. (If it is empty the statement is either). □

(5.11.2) Lemma. *Let $\mathcal{M}$ be an integral normal scheme of finite type over* spec(R)*, where R is a d.v.r. with quotient field Q. Let $M := \mathcal{M}_Q$ and $\alpha : M \to \mathcal{N}$ be a morphism to any* affine *scheme of finite type over* spec(R)*, If every point in the special fibre of $\mathcal{M}$ lies on a section* spec$(R') \to M$ *for a finite extension R' of R, such that the induced map* spec$(Q') \to \mathcal{N}$ *extends to* spec(R')*, then α extends to $\mathcal{M}$.*

Proof. Since $\mathcal{N}$ is affine and of finite type, this boils down to extending regular functions. Whether the latter extend can be checked in the way described because $\mathcal{M}$ is normal. □

Part II.

Quadratic L-functions, representation densities

6. Quadratic forms and representation densities

6.1. Quadratic forms and symmetric bilinear forms

Mainly to fix notation, we begin by recalling the definition of quadratic form and symmetric bilinear form and their relation.
Let S be a base scheme.

(6.1.1) Definition. *Let L_S be a locally free sheaf on S.*
*A **quadratic form** on L_S is a function*

$$Q_L : L_S \to \mathcal{O}_S,$$

satisfying

i. $Q_L(\alpha v) = \alpha^2 v$ *locally for all sections* $v \in L_S$ *and* $\alpha \in \mathcal{O}_S$*,*

ii. the form $\langle v, w\rangle_Q := Q_L(v+w) - Q_L(v) - Q_L(w)$ *is bilinear.*

If 2 is invertible in S, it is possible to reconstruct Q_L from $\langle \cdot, \cdot \rangle_Q$ by

$$v \mapsto \frac{1}{2}\langle v, v\rangle_Q.$$

We will sometimes denote the associated morphism $L_S \to L_S^*$ by γ_{Q_L} or γ_L.
If R is a ring and $q_i \in R$, we denote by $< q_1, \dots, q_n >$ the space R^n with quadratic form

$$x \mapsto \sum_{i=1}^n q_i x_i^2.$$

(6.1.2) Lemma.

$$\begin{array}{rcccl}
\mathrm{Sym}^2(L^*) & = & \{\textit{quadratic forms on } L\} & = & ((L \otimes L)^s)^* \\
\mathrm{Sym}^2(L)^* & = & \{\textit{symm. bilinear forms on } L\} & = & (L^* \otimes L^*)^s \\
\mathrm{Sym}^2(L) & = & \{\textit{quadratic forms on } L^*\} & = & ((L^* \otimes L^*)^s)^* \\
\mathrm{Sym}^2(L^*)^* & = & \{\textit{symm. bilinear forms on } L^*\} & = & (L \otimes L)^s,
\end{array}$$

here $(\cdots)^s$ denotes symmetric elements, i. e. invariants under the automorphism switching factors.

Proof. A tensor $f \otimes g \in \mathrm{Sym}^2(L_R^*)$ is identified with the quadratic form $v \mapsto f(v)g(v)$. Similarly a tensor $f \otimes g \in L^* \otimes L^*$ is identified with the bilinear form $v, w \mapsto f(v)g(w)$, and the properties of symmetry correspond. Furthermore there is a non-degenerate bilinear map

$$\mathrm{Sym}^2(L_R^*) \times (L_R \otimes L_R)^s \to R,$$

induced by the contraction $(f \otimes g), (x \otimes y) \mapsto f(x)g(y)$. Similarly for the other cases. $\square$

There is a natural symmetrization map

$$\begin{aligned} \mathrm{Sym}^2(L^*) &\longrightarrow \mathrm{Sym}^2(L)^* \\ f_1 \otimes f_2 &\mapsto \{v_1 \otimes v_2 \mapsto f_1(v_1)f_2(v_2) + f_1(v_2)f_2(v_1)\} \end{aligned} \tag{1}$$

(and similarly $\mathrm{Sym}^2(L) \longrightarrow \mathrm{Sym}^2(L^*)^*$). which is an isomorphism, if 2 is invertible in S. If L_R is free with basis $\{e_i\}$, a basis $e_{ij} = e_i^* \otimes e_j^*$ of $\mathrm{Sym}^2(L_R^*)$ is mapped to $\{2e_{ii}^*\}_i \cup \{e_{ij}^*\}_{i<j}$. The association $Q_L \mapsto \langle \cdot, \cdot \rangle_Q$ in (6.1.1, ii.) just corresponds to the symmetrization map (1).

(6.1.3) Definition. *We denote the determinant of the matrix $(\langle e_i, e_j \rangle_Q)_{ij}$ by*

$$d(e_1, \ldots, e_n)$$

*and call it the **discriminant of** L with respect to $\{e_i\}$.*
For $R = \mathbb{Z}$ or $\mathbb{Z}_p$, we also write $d(L_\mathbb{Z})$ or $d(L_{\mathbb{Z}_p})$ for the discriminants using any basis. In the second case, it is determined only up to $(\mathbb{Z}_p^*)^2$. In particular their valuations $|\cdot|_\infty$ and $|\cdot|_p$, respectively, are well defined.

(6.1.4) Remark. In contrast to the considerations above, we have a canonical isomorphism

$$\Lambda^i L^* \xrightarrow{\sim} (\Lambda^i L)^*.$$

6.2. Canonical measures

(6.2.1) In the following, we will work predominantly with the following natural (up to a choice of $i \in \mathbb{C}$) characters on $R = \cdots$:

$\mathbb{R}$: $\chi_\infty(x) := e^{2\pi i x}$.

$\mathbb{Q}_p$: $\chi_p(x) := e^{-2\pi i [x]}$, where $[x] = \sum_{i<0} x_i p^{-i}$ is the principal part.
It has level (or conductor) 1.

$\mathbb{A}^S$: $\chi = \prod_{\nu \notin S} \chi_\nu$.

The corresponding self-dual additive Haar measures are the Lebesque measure on $\mathbb{R}$, the standard measures on $\mathbb{Q}_p$, giving $\mathbb{Z}_p$ the volume 1, and their product, respectively.

(6.2.2) Let R be one of the rings of (6.2.1). Let X be an algebraic variety over R and $\widetilde{\mu}$ an algebraic volume form on X. As is explained in [95] (cf. also [84, §3.5]), this defines a well-defined measure μ on $X(R)$, which depends on the choice of χ (resp. the additive Haar measure). For the special case of a lattice L of dimension r, and $\widetilde{\mu} \in \Lambda^r L^*$, there is $\widetilde{\mu}^* \in \Lambda^r L$ satisfying $\widetilde{\mu}^* \widetilde{\mu} = 1$. In this case, the measures μ and μ^* are dual to each other with respect to the bicharacter $v, v^* \mapsto \chi(v^* v)$, i.e. for

$$F_\Psi(w^*) = \int_L \Psi(w)\chi(w^* w)\mu(w) \qquad \Psi \in S(L)$$
$$F_\Psi(w) = \int_{L^*} \Psi(w^*)\chi(w^* w)\mu^*(w^*) \qquad \Psi \in S(L^*)$$

(where $S(\cdots)$ denotes space of Schwartz-Bruhat functions), we have $F_{F_\Psi}(w^*) = \Psi(-w^*)$.

(6.2.3) Definition. *Let L be an R-lattice with non-degenerate quadratic form Q_L. Then there is a canonical (translation invariant) measure μ_L with $\mu_L^* = \mu_L$ under the identification $\gamma_Q : L \xrightarrow{\sim} L^*$. Let $e_1, \dots, e_m$ be a basis of L, $e_1^*, \dots, e_m^*$ the dual basis and $\widetilde{\mu} = e_1^* \wedge \cdots \wedge e_m^*$. Let A be the matrix of $\langle , \rangle_Q$ in this basis. The measure μ_L is then given by*

$$\mu_L = |A|^{1/2}\mu,$$

*where $|A|$ is the modulus of the determinant. We call it the **canonical measure** on L with respect to Q_L.*

Let M be another R-lattice, equipped with a non-degenerate quadratic form Q_M.

*Choose a basis $f_1, \dots, f_n$ of M, too, and denote $\widetilde{\mu} := \bigwedge_{i,j} e_i^* \otimes f_j^* \in \bigwedge^{nm} L^* \otimes M^*$. We call $\mu_{L,M} = |A|^{n/2}|B|^{m/2}\mu$ the **canonical measure** on $L \otimes M$, where A are B the matrices of the associated bilinear forms, $m = \dim(L)$ and $n = \dim(M)$.*

*$\{e_i \otimes e_j\}_{i \le j}$ is a basis of $\mathrm{Sym}^2(L)$. We denote the corresponding dual basis by $\{(e_i \otimes e_j)^*\}_{i \le j}$. Let $\widetilde{\mu} = \bigwedge_{i<j} (e_i \otimes e_j)^* \in \bigwedge^{\frac{m(m+1)}{2}} \mathrm{Sym}^2(L)^*$. In this case we call $\mu_L = |A|^{\frac{m+1}{2}}\mu$ the **canonical measure** on $\mathrm{Sym}^2(L)$.*

*Let $\widetilde{\mu} = \bigwedge_i (e_i \otimes e_i)^* \wedge \bigwedge_{i<j} (e_i \otimes e_j + e_j \otimes e_i)^*$. In this case, we call $\mu_L = |A|^{\frac{m+1}{2}}\mu$ the **canonical measure** on $(L \otimes L)^s$.*

*Similarly, we get a **canonical measure** $\mu_L = |A|^{\frac{m-1}{2}}\mu$ on $\Lambda^2 L$.*

According to these definitions, the measures μ_L on $\mathrm{Sym}^2(L)$ and μ_L on $(L^* \otimes L^*)^s = \mathrm{Sym}^2(L)^*$ are dual. However, the symmetrization map (1) $\mathrm{Sym}^2(L^*) \xrightarrow{\sim} \mathrm{Sym}^2(L)^*$ sends the canonical measure μ_L of the left hand side to the $|2|^{-m}$-multiple of μ_L on the right hand side.

Let $L_\mathbb{Q}, M_\mathbb{Q}$ be vector spaces of dimensions m resp. n with quadratic forms Q_L and Q_M, respectively. Assume Q_L non-degenerate.

(6.2.4) Definition. *For each $\mathbb{Q}$-algebra R define the set*

$$\mathrm{I}(M,L)_R = \{\alpha : M_R \to L_R \mid \alpha \textit{ is an isometry}\}.$$

(If lattices $L_\mathbb{Z}$ and $M_\mathbb{Z}$ are chosen, this makes sense for each $\mathbb{Z}$-algebra.)
$\mathrm{I}(M,L)$ is an affine algebraic variety over $\mathbb{Q}$.
If Q_M is degenerate, define in addition

$$\mathrm{I}^1(M,L)_R = \{\alpha : M_R \to L_R \mid \alpha \textit{ is an injective isometry }\}.$$

(6.2.5) Assume $m \geq n \geq 0$. We identify $M^* \otimes L$ with $\mathrm{Hom}(M,L)$ in what follows. There is a fibration

$$\mathrm{I}(M^Q,L) \hookrightarrow M^* \otimes L \xrightarrow{\alpha \mapsto \alpha^! Q_L} \mathrm{Sym}^2(M^*), \tag{2}$$

where $\mathrm{I}(M^Q,L)$ is the pre-image of $Q = Q_M$ and $\alpha^! Q_L$ denotes pullback of Q_L to M via α. As soon as we choose (translation invariant) measures $\mu_{1,2}$ on $M^* \otimes L$ and $\mathrm{Sym}^2(M^*)$ respectively, this defines a measure $\frac{\mu_1}{\mu_2}$ on the fibers, restricted to the submersive set $(M_R^* \otimes L_R)^{reg}$ of the map $\alpha \mapsto \alpha^! Q_L$. This set coincides with the locus of maps with maximal rank n. This means in particular that the following integral formula holds true for these measures:

$$\int_{\mathrm{Sym}^2(M_R^*)} \mu_2(Q) \int_{\mathrm{I}(M^Q,L)(R)} \frac{\mu_1}{\mu_2}(\alpha)\varphi(\alpha) = \int_{M_R^* \otimes L_R} \mu_1(\alpha)\varphi(\alpha), \tag{3}$$

where φ is continuous with compact support on $(M_R^* \otimes L_R)^{reg}$.

(6.2.6) Lemma. *For $m \geq 2n+1$ and R local, $\varphi \in S(M_R^* \otimes L_R)$ (space of Schwartz-Bruhat functions) is integrable with respect to the measure $\frac{\mu_1}{\mu_2}$, too.*

Proof. [94, §34] □

In the case $m \geq 2n+1$, the integrals $\int_{\mathrm{I}(M^Q,L)} \varphi(\alpha) \frac{\mu_1}{\mu_2}(\alpha)$ may be computed by Fourier analysis (cf. 7.2.3).
If Q_M is non-degenerate, the canonical measures on $M_R^* \otimes L_R$ and $\mathrm{Sym}^2(M_R^*)$, introduced in the last section, in particular define a *canonical* measure on the fibre $\mathrm{I}(M,L)_R$ (which is the fibre above $Q = Q_M$) by means of this fibration.
For $R = \mathbb{Q}_\nu$ and $n = m = 1$, $\mathrm{I}(M,L)_R$ consists of 2 points, each of which has volume 1.

(6.2.7) Let R be a $\mathbb{Q}_p$ or $\mathbb{R}$. Let L, M and N be R-vector spaces with non-degenerate quadratic forms Q_L, Q_M, Q_N respectively.

Consider the composition map:

$$\mathrm{I}(N, M) \times \mathrm{I}(M, L) \to \mathrm{I}(N, L).$$

Fixing an α in $\mathrm{I}(N, M)$, we may identify the fibre of the resulting map

$$\begin{aligned} \mathrm{I}(M, L) &\to \mathrm{I}(N, L) \\ \delta &\mapsto \delta \circ \alpha \end{aligned} \tag{4}$$

over $\beta \in \mathrm{I}(N, L)$ with $\mathrm{I}(\mathrm{im}(\alpha)^{\perp}, \mathrm{im}(\beta)^{\perp})$.

(6.2.8) Theorem. *The resulting fibration is compatible with the canonical measures. If* $\dim(M) = \dim(N)$ *this means that the map (4) preserves volume.*

Proof. Let Q_M be the chosen form on M. By assumption, $\alpha^! Q_M$ is the chosen form Q_N. We decompose M in $M_1 = \alpha(N)$ and $M_2 = M_1^{\perp}$ (orthogonal with respect to Q_M). Decompose $\mathrm{Sym}^2(M^*)$ with respect to Q_M:

$$\mathrm{Sym}^2(M^*) = \mathrm{Sym}^2(M_1^*) \oplus (M_1^* \otimes M_2^*) \oplus \mathrm{Sym}^2(M_2^*),$$

where $\mathrm{Sym}^2(M_1^*) = \mathrm{Sym}^2(N^*)$ via α.
We get a commutative diagram of fibrations:

$$\begin{tikzcd}
\mathrm{I}(\alpha(N)^{\perp\gamma}, \beta(N)^{\perp}) \arrow[r, hook, dotted] \arrow[d, hook] & M_2^* \otimes L \arrow[r, "\delta \mapsto \begin{pmatrix} \delta^! Q_L \\ \langle \beta, \delta \rangle_L \end{pmatrix}"] \arrow[d, hook, "incl. + \beta \circ \mathrm{pr}_1"] & \mathrm{Sym}^2(M_2^*) \oplus (M_1^* \otimes M_2^*) \arrow[d, hook, "incl. + \beta^! Q_L"] \\
\mathrm{I}(M^{\gamma}, L) \arrow[r, hook] \arrow[d, "\circ\alpha"] & M^* \otimes L \arrow[r, "\delta \mapsto \delta^! Q_L"] \arrow[d, two heads, "\circ\alpha"] & \mathrm{Sym}^2(M^*) \arrow[d, "\alpha^!"] \\
\mathrm{I}(N^{\alpha^! \gamma}, L) \arrow[r, hook] & N^* \otimes L \arrow[r, "\delta \mapsto \delta^! Q_L"] & \mathrm{Sym}^2(N^*)
\end{tikzcd}$$

where β varies in $N^* \otimes L$ such that $\beta^! Q_L$ varies in a neighborhood of Q_N and γ varies in $\mathrm{Sym}^2(M^*)$ such that $\beta^! Q_L = \alpha^! \gamma$. $\mathrm{I}(M^{\gamma}, L)$ is the fibre of the map $\delta \mapsto \delta^! Q_L$ in the middle row over γ and $\mathrm{I}(N^{\alpha^! \gamma}, L)$ is the fibre of the map $\delta \mapsto \delta^! Q_L$ in the bottom row above $\alpha^! \gamma$. $\mathrm{I}(\alpha(N)^{\perp\gamma}, \beta(N)^{\perp})$ in the top-left corner is the fibre of the composition with α above β.
In the diagram, the vertical middle and rightmost fibrations come from (splitting) exact sequences of vector spaces.
The dotted map is defined by commutativity of the diagram. First observe that the (underlying) vertical exact sequences of vector spaces are exact with respect to canonical measures on the various spaces associated with Q_M, Q_L and $\alpha^! Q_M = Q_N$ and the

restrictions of Q_M to M_1, M_2 respectively. The induced measure on $\mathrm{I}(\alpha(M)^{\perp\gamma}, \beta(N)^\perp)$ hence is described by the topmost *horizontal* fibration as well.
Decompose $M_2^* \otimes L = M_2^* \otimes (\beta(N) \oplus \beta(N)^\perp)$. The map $\delta \mapsto \langle \beta, \delta \rangle_{Q_L}$ is an isomorphism $(M_2^* \otimes \beta(N)) \cong (M_1^* \otimes M_2^*)$ and 0 on the other factor. This isomorphism preserves the canonical volume, if $\beta^! Q_L = Q_N$.
Letting γ vary only in $\mathrm{Sym}^2(M_2^*)$ fixing the other projections to 0 in $M_1^* \otimes M_2^*$ and to Q_N in $\mathrm{Sym}(M_1^*)$ (i.e. having also $\beta^! Q_L = Q_N$), we get an equivalent topmost horizontal fibration:

$$\mathrm{I}(\alpha(N)^{\perp\gamma}, \beta(N)^\perp) \hookrightarrow M_2^* \otimes \beta(N)^\perp \xrightarrow{\delta \mapsto \delta^! Q_L} \mathrm{Sym}^2(M_2^*)$$

and the dotted map is equal to the canonical inclusion into $\alpha(N)^{\perp *} \otimes \beta(N)^\perp$, noting $\alpha(N)^{\perp *} = M_2^*$. The induced measure on $\mathrm{I}(\alpha(N)^{\perp\gamma}, \beta(N)^\perp)$, however, is *by definition* the canonical measure. □

(6.2.9) Lemma. *For positive definite spaces $M_\mathbb{R}, L_\mathbb{R}$, we get*

$$\mathrm{vol}(\mathrm{I}(M,L)_\mathbb{R}) = \prod_{k=m-n+1}^{m} 2\frac{\pi^{k/2}}{\Gamma(k/2)}.$$

Proof. By (6.2.8), we are reduced to the case $n = 1$. We choose bases on M and L and get the following canonical measures:

$$\begin{array}{lll} L = \mathbb{R}^m & Q_L : \frac{1}{2}{}^t x A x & |A|^{\frac{1}{2}}\,\mathrm{d}\,x \\ M = \mathbb{R} & Q_M : \frac{1}{2}{}^t x B x & |B|^{\frac{1}{2}}\,\mathrm{d}\,x \\ M^* = \mathbb{R}^m & Q_{M^*} : \frac{1}{2}{}^t x B^{-1} x & |B|^{-\frac{1}{2}}\,\mathrm{d}\,x \\ \mathrm{Sym}^2(M^*) = \mathbb{R} & & |B|^{-\frac{n+1}{2}}\,\mathrm{d}\,x \end{array}$$

Therefore, using the integral formula (3):

$$\begin{aligned} |B|^{-1}\,\mathrm{vol} &= \frac{\mathrm{d}}{\mathrm{d}\,r} \int_{\frac{1}{2}{}^t x A x \le r} |B|^{-\frac{m}{2}} |A|^{\frac{1}{2}} dx \bigg|_{r=\frac{1}{2}B} \\ &= \frac{\mathrm{d}}{\mathrm{d}\,r} |B|^{-\frac{m}{2}} \frac{(2\pi r)^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)} \bigg|_{r=\frac{1}{2}B} \\ \mathrm{vol} &= 2\frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2})}. \end{aligned}$$

□

We especially get a canonical and also invariant measure on every $\mathrm{SO}(L_R)$, coming (up to a real factor) from an algebraic volume form. On the other hand, algebraic invariant volume forms on $\mathrm{SO}(L)$ are canonically identified with $\Lambda^{\frac{m(m-1)}{2}}\mathrm{Lie}(\mathrm{SO}(L))^*$. Hence every invariant measure on $\mathrm{SO}(L_R)$ is given by a translational invariant measure on $\mathrm{Lie}(\mathrm{SO}(L))^*$. We have the following

(6.2.10) Lemma. *The associated element in* $\Lambda^{\frac{m(m-1)}{2}}\mathrm{Lie}(\mathrm{SO}(L_R))$ *is the canonical volume form (6.2.3) on* $\Lambda^2 L_R$ *under the natural identification* $\mathrm{Lie}(\mathrm{SO}(L_R))) \cong \Lambda^2 L_R$ *given by contraction with the bilinear form associated with* Q_L*.*

(6.2.11) Lemma. *Let* $M_{\mathbb{Q}}$*,* $L_{\mathbb{Q}}$ *be vector spaces of dimensions* n*,* m *respectively, where* $n \le m-1$*, with quadratic forms* Q_M *and* Q_L*. If* $m \ge n+3$*, the product of the canonical measures on* $I(M_{\mathbb{Q}_\nu}, L_{\mathbb{Q}_\nu})$ *converges absolutely (in the sense of [95]) and yields the canonical measure on* $I(M_{\mathbb{A}}, L_{\mathbb{A}})$*.*
In the case $n = m$*, the product of the canonical measures on* $\mathrm{SO}(L_{\mathbb{Q}_\nu})$ *converges absolutely and yields the the canonical measure on* $\mathrm{SO}(L_{\mathbb{A}})$ *provided* $m \ge 3$*. It is the Tamagawa measure.*

Proof. Follows directly from the explicit volume formulæ (8.2.1) and standard facts about absolute convergence of the occurring infinite products. One just obtains the Tamagawa number in the second case because of the 'product formula' $|x|_{\mathbb{A}} = 1$ for $x \in \mathbb{Q}^*$ for the adelic modulus, because the discriminant factors cancel in the product. □

6.3. Relation with classical representation densities

(6.3.1) Consider again the case $R = \mathbb{Q}_p$ and let $\varphi \in S(M^*_{\mathbb{Q}_p} \otimes L_{\mathbb{Q}_p})$ be a Schwartz function, i.e. locally constant with compact support. Choose lattices $L_{\mathbb{Z}_p}$ and $M_{\mathbb{Z}_p}$ and bases $\{f_i\}$ of $M_{\mathbb{Z}_p}$ and $\{e_i\}$ of $L_{\mathbb{Z}_p}$, respectively.
Assume the conditions of (6.2.6). The integral of φ over $\mathrm{I}(M,L)_{\mathbb{Q}_p}$ may be computed explicitly as follows:

$$\begin{aligned}
&\lim_{l\to\infty} \frac{\int_{\alpha^* Q_L - Q_M \in \wp^l \mathrm{Sym}^2(M^*_{\mathbb{Z}_p})} \mathrm{d}\,\alpha\ \varphi(\alpha)}{\mathrm{vol}(\wp^l \mathrm{Sym}^2(M^*_{\mathbb{Z}_p}))} \\
&= \lim_{l\to\infty} \frac{\mathrm{vol}(\wp^l L_{\mathbb{Z}_p} \otimes M^*_{\mathbb{Z}_p})}{\mathrm{vol}(\wp^l \mathrm{Sym}^2(M^*_{\mathbb{Z}_p}))} \sum_{\substack{\{\delta_i\} \subset \wp^{-r} L_{\mathbb{Z}_p}/\wp^l L_{\mathbb{Z}_p} \\ Q_L(\delta_i) \equiv Q_M(f_i), \langle \delta_i, \delta_j\rangle_L \equiv \langle f_i, f_j\rangle_M \mod p^l}} \varphi(\sum_i f_i^* \otimes \delta_i) \\
&= D(M_{\mathbb{Z}_p}, L_{\mathbb{Z}_p}) \lim_{l\to\infty} p^{l(n(n+1)/2 - mn)} \sum \cdots =: D(M_{\mathbb{Z}_p}, L_{\mathbb{Z}_p}) \beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \varphi)
\end{aligned}$$

for sufficiently large r (φ has compact support). Here, the limit eventually becomes stationary. We denote $\beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \varphi)$ also simply by $\beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p})$ if φ is the characteristic function of $M^*_{\mathbb{Z}_p} \otimes L_{\mathbb{Z}_p}$.

Here we denoted

$$D(M_{\mathbb{Z}_p}, L_{\mathbb{Z}_p}) := |d(e_1, \dots, e_m)|_p^{n/2} |d(f_1, \dots, f_n)|_p^{(n-m+1)/2}.$$

A characteristic special case is given as follows: Choose a class $\kappa \in (L^*_{\mathbb{Z}_p}/L_{\mathbb{Z}_p}) \otimes M^*_{\mathbb{Z}_p}$ Consider the characteristic function φ of κ. Let $\{f_i\}$ be a basis of $M_{\mathbb{Z}_p}$ We get:

$$\begin{aligned}\beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \varphi) = \lim_{l \to \infty} & p^{-nl(m-n+1)/2} \\ & \#\left\{ x_i \in L^*_{\mathbb{Z}_p}/\wp^l L_{\mathbb{Z}_p} \;\middle|\; {}_{Q_L(x_i) \equiv Q_M(f_i), \langle x_i, x_j \rangle_L \equiv \langle f_i, f_j \rangle_M \quad \mathrm{mod}\ p^l}^{x_i \equiv \kappa f_i \quad \mathrm{mod}\ L_{\mathbb{Z}_p},} \right\}\end{aligned}$$

We will write as well $\beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa)$ for this.

The quantities β_p occurring in this section are historically called **representation densities**, cf. [50].

(6.3.2) We now use the compatibility of canonical measures with the composition fibration (6.2.8) to derive relations among representation densities. As a warming up, we will derive an inductive formula of Kitaoka's from it.

Consider the situation of (6.2.7) for $R = \mathbb{Q}_p$. Consider $N_{\mathbb{Q}_p}$ via $\alpha \in N^*_{\mathbb{Q}_p} \otimes M_{\mathbb{Q}_p}$ as subspace of $M_{\mathbb{Q}_p}$. Choose lattices $N_{\mathbb{Z}_p} \subset M_{\mathbb{Z}_p}$ such that $M_{\mathbb{Z}_p} = N_{\mathbb{Z}_p} \oplus N^\perp_{\mathbb{Z}_p}$. Choose a third lattice $L_{\mathbb{Z}_p}$ as well. Assume that $Q_N \in \mathrm{Sym}^2(N^*_{\mathbb{Z}_p}), \dots$ for these lattices. Integrate the characteristic function of

$$(M^*_{\mathbb{Z}_p} \otimes L_{\mathbb{Z}_p}) \cap \mathrm{I}(M, L)(\mathbb{Q}_p)$$

with respect to the canonical volume on $\mathrm{I}(M, L)(\mathbb{Q}_p)$.

The intersections with the fibers of the map $\mathrm{I}(M, L)(\mathbb{Q}_p) \to \mathrm{I}(N, L)(\mathbb{Q}_p)$ can be identified with those isometries in $\mathrm{I}(N^\perp, \beta(N)^\perp)(\mathbb{Q}_p)$ which map $N^\perp_{\mathbb{Z}_p}$ to $\beta(N_{\mathbb{Z}_p})^\perp$.

The volume of these sets is constant, for conjugated $\beta(N_{\mathbb{Q}_p}) \cap L_{\mathbb{Z}_p}$, hence let K_i be representatives of these conjugacy classes and collect the fibres over those β with $\beta(N_{\mathbb{Z}_p}) \cong K_i$. Denote the corresponding density by $\beta_p(L_{\mathbb{Z}_p}, N_{\mathbb{Z}_p}; K_i)$.

This yields the formula

$$\begin{aligned}& d_p(M_{\mathbb{Z}_p})^{\frac{n-m+1}{2}} d_p(L_{\mathbb{Z}_p})^{\frac{n}{2}} \beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}) \\ &= \sum_i d_p(N_{\mathbb{Z}_p})^{\frac{k-m+1}{2}} d_p(L_{\mathbb{Z}_p})^{\frac{k}{2}} \beta_p(L_{\mathbb{Z}_p}, N_{\mathbb{Z}_p}; K_i) \\ &\cdot d_p(N^\perp_{\mathbb{Z}_p})^{\frac{n-m+1}{2}} d_p(K_i^\perp)^{\frac{n-k}{2}} \beta_p(K_i^\perp, N^\perp_{\mathbb{Z}_p})\end{aligned}$$

and after a reordering of discriminant factors:

(6.3.3) Corollary.

$$\beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}) = \sum_i \left(\frac{d_p(K_i^\perp)}{d_p(N_{\mathbb{Z}_p})d_p(L_{\mathbb{Z}_p})} \right)^{\frac{n-k}{2}} \beta(L_{\mathbb{Z}_p}, N_{\mathbb{Z}_p}; K_i)\beta(K_i^\perp, N_{\mathbb{Z}_p}^\perp).$$

This is the aforementioned formula due to Kitaoka [50, Theorem 5.6.2][1]. However, we will need a variant of this formula, which is in a sense the *orbit equation* of the operation of the orthogonal group $\mathrm{SO}(M)$ on the vector space M.
(There is also a version including a $\kappa \in (L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}) \otimes M_{\mathbb{Z}_p}^*$, if above we instead integrate over the characteristic function of κ.)

6.4. The non-Archimedean orbit equation

(6.4.1) Let $L_{\mathbb{Q}_p}$, $M_{\mathbb{Q}_p}$ be vector spaces over $\mathbb{Q}_p$ with non-degenerate quadratic forms Q_L, Q_M respectively.
Consider again an $\alpha \in \mathrm{I}(M_{\mathbb{Q}_p}, L_{\mathbb{Q}_p})$ and the resulting map:

$$\begin{aligned} \mathrm{I}(L_{\mathbb{Q}_p}, L_{\mathbb{Q}_p}) &\to \mathrm{I}(M_{\mathbb{Q}_p}, L_{\mathbb{Q}_p}) \\ \delta &\mapsto \delta \circ \alpha \end{aligned}$$

The fibre of this map over $\beta \in \mathrm{I}(M_{\mathbb{Q}_p}, L_{\mathbb{Q}_p})$ can again be identified with

$$\mathrm{I}(\alpha(L)_{\mathbb{Q}_p}^\perp, \beta(L)_{\mathbb{Q}_p}^\perp).$$

In particular the fibre over α is an orthogonal group again. We denote by $\mathrm{SO}(L_{\mathbb{Q}_p})$, respectively $\mathrm{SO}(\alpha(L_{\mathbb{Q}_p})^\perp)$ the corresponding *special* orthogonal groups.
Choose lattices $L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}$ such that $Q_L \in \mathrm{Sym}^2(L_{\mathbb{Z}_p})$. Let κ be a class in $(L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}) \otimes M_{\mathbb{Z}_p}^*$.
Consider the discriminant kernel $\mathrm{SO}'(L_{\mathbb{Z}_p}) \subseteq \mathrm{SO}(L_{\mathbb{Q}_p})$ of $L_{\mathbb{Z}_p}$, i.e. the kernel of the induced homomorphism $\mathrm{SO}(L_{\mathbb{Z}_p}) \to \mathrm{Aut}(L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p})$. It is a compact open subgroup. Choose representatives α_i of the orbits of $\mathrm{SO}'(L_{\mathbb{Z}_p})$ in $\mathrm{I}(M_{\mathbb{Q}_p}, L_{\mathbb{Q}_p}) \cap \kappa$. The fibres over $\mathrm{SO}'(L_{\mathbb{Z}_p})\alpha_i$ all can be identified with $\mathrm{SO}'(\alpha_i(M_{\mathbb{Q}_p})^\perp \cap L_{\mathbb{Z}_p})$. This follows from the following lemma:

(6.4.2) Lemma. *Let $X \subset L_{\mathbb{Z}_p}^*$ be a subset and* $\mathrm{Stab}(X) \subset \mathrm{SO}'(L_{\mathbb{Z}_p})$ *the* point-wise *stabilizer. Assume that $X_{\mathbb{Q}_p}$ is non-degenerate. Then we have*

$$\mathrm{Stab}(X) = \mathrm{SO}'(X_{\mathbb{Z}_p}^\perp),$$

where $X_{\mathbb{Z}_p}^\perp = \{v \in L_{\mathbb{Z}_p} \mid \langle v, w \rangle = 0 \ \forall w \in X\}$.

[1] Observe: $\#(N_{\mathbb{Z}_p}^*/N_{\mathbb{Z}_p}) = d_p(N_{\mathbb{Z}_p})^{-1}$

Proof. Let $\beta \in \mathrm{SO}'(L_{\mathbb{Z}_p})$, i. e. $\beta v - v \in L_{\mathbb{Z}_p}$ for all $v \in L^*_{\mathbb{Z}_p}$. If we have $\beta w = w$ for all $w \in X$, one can consider β via restriction as an element of $\mathrm{SO}(X^\perp_{\mathbb{Z}_p})$. If moreover $w^\perp \in (X^\perp_{\mathbb{Z}_p})^*$, we find a $v \in L^*_{\mathbb{Z}_p}$ satisfying $v = w + w^\perp$ with some $w \in < X >_{\mathbb{Q}_p}$ ($X^\perp_{\mathbb{Z}_p}$ is a primitive sublattice). This yields $\beta w^\perp - w^\perp = \beta v - v \in L_{\mathbb{Z}_p}$. Hence $\mathrm{Stab}(X) \subseteq \mathrm{SO}'(X^\perp_{\mathbb{Z}_p})$.
On the other hand, let $\beta \in \mathrm{SO}'(X^\perp_{\mathbb{Z}_p})$. It can be extended uniquely to an element $\beta \in \mathrm{SO}(L_{\mathbb{Q}_p})$, fixing $< X >_{\mathbb{Q}_p}$ point-wise. We claim that this extension lies in fact in $\mathrm{SO}(L_{\mathbb{Z}_p})$. For, let $v \in L_{\mathbb{Z}_p}$ be given and write $v = w^\perp + w$. Then we have $w^\perp \in (X^\perp_{\mathbb{Z}_p})^*$. hence $\beta v - v = \beta w^\perp - w^\perp \in L_{\mathbb{Z}_p}$.
Now suppose $v \in L^*_{\mathbb{Z}_p}$. Then we have still $w^\perp \in (X^\perp_{\mathbb{Z}_p})^*$ because $\langle v, w^{\perp\prime}\rangle = \langle w^\perp, w^{\perp\prime}\rangle \in \mathbb{Z}_p$ for all $w^{\perp\prime} \in X^\perp_{\mathbb{Z}_p}$. Hence $\beta v - v \in L_{\mathbb{Z}_p}$ as well, which means $\beta \in \mathrm{SO}'(L_{\mathbb{Z}_p})$ □

(6.4.3) It follows that

$$\mathrm{vol}(\mathrm{SO}'(\alpha_i(M)^\perp_{\mathbb{Z}_p})|d(L_{\mathbb{Z}_p})|^{n/2}|d(M_{\mathbb{Z}_p})|^{(n-m+1)/2}\beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; \alpha_i) = \mathrm{vol}(\mathrm{SO}'(L_{\mathbb{Z}_p})),$$

where in $\beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; \alpha_i)$ only representations are counted, which lie in the orbit of α_i.
Summed up over all orbits, we get

$$\mathrm{vol}(\mathrm{SO}'(L_{\mathbb{Z}_p}))^{-1}|d(L_{\mathbb{Z}_p})|^{n/2}|d(M_{\mathbb{Z}_p})|^{(n-m+1)/2}\beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa) = \sum_i \mathrm{vol}(\mathrm{SO}'(\alpha_i(M)^\perp_{\mathbb{Z}_p})^{-1} \tag{5}$$

(6.4.4) Let $H_{\mathbb{Z}_p} = \mathbb{Z}_p^2$ be a hyperbolic plane, $\varphi \in S(L_{\mathbb{Q}_p} \otimes M^*_{\mathbb{Q}_p})$ and form

$$\varphi^{(s)} := \varphi \otimes \chi_{H^s_{\mathbb{Z}_p} \otimes M^*_{\mathbb{Z}_p}} \in S((L_{\mathbb{Q}_p} \oplus H^s_{\mathbb{Q}_p}) \otimes M^*_{\mathbb{Q}_p}).$$

We are interested in the function

$$s \mapsto \beta_p(L \oplus H^s, M_{\mathbb{Z}_p}, \varphi^{(s)}),$$

for $s \in \mathbb{Z}_{\geq 0}$ (see 7.5.1). The so constructed 'continuation' of $\beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa)$ we will also denote by $\beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s)$. This construction is motivated by the natural continuation of Fourier coefficients of Eisenstein series, see (7.7.5).

We begin with an investigation of $\beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s)$ in the case $n = 1$:

(6.4.5) Lemma. *We have the following relation to representation numbers:*

$$\beta_p(L_{\mathbb{Z}_p}, < q >, \kappa; s) = \#\Omega_{\kappa,q}(w)p^{w(1-m-s)} + (1 - p^{-s}) \sum_{j=0}^{w-1} \#\Omega_{\kappa,q}(j)p^{j(1-m-s)} \tag{6}$$

Here $\Omega_{\kappa,q}(j) = \{v \in L_{\mathbb{Z}/p^j\mathbb{Z}} \mid Q_M(v+\kappa) \equiv q \ (p^j)\}$ *and* w *is a sufficiently big integer. (Explicitely:* $w \geq 1 + 2\nu_p(2q\,\mathrm{ord}(\kappa))$ *— the formula then does not depend on* w*.)*

This can be written as follows ($\Re s > 1$)*:*

$$\sum_l \frac{\#\Omega_{\kappa,q}(l)}{p^{l(m-1+s)}} = \frac{\beta_p(L_{\mathbb{Z}_p}, <q>, \kappa; s)}{1-p^{-s}}. \tag{7}$$

(6.4.6) Remark. We will not use the formula (6) in an essential way, except for the computation of the zeta function of a 2-dimensional lattice. It is, however, interesting to see that continuation in s by means of adding hyperbolic planes (forced by the connection to Whittaker integrals) recovers the exact number of representations modulo prime powers (in the case $n = 1$). In addition the formula allows comparison with results of [13].

Proof. Formula (6) is obviously true for $s = 0$. Under the substitution $L \rightsquigarrow L \oplus H$ the left hand side becomes $\beta_p(L_{\mathbb{Z}_p}, <q>, \kappa; s+1)$ and the right hand side becomes the same expression for $s+1$, if we use the relation:

$$\#\Omega_{\kappa,q}(L \oplus H, r) = \sum_{\nu=0}^{r-1} p^{(r-\nu)m}(p^r - p^{r-1})\#\Omega_{\kappa,q}(L,\nu) + \#\Omega_{\kappa,q}(L,r)p^r. \tag{8}$$

Proof of the relation: An explicit calculation shows:

$$\#\Omega_n(H,l) = \begin{cases} (\nu_p(n)+1)(p^l - p^{l-1}) & \nu_p(n) < l, \\ l(p^l - p^{l-1}) + p^l & \nu_p(n) \geq l. \end{cases} \tag{9}$$

Hence:

$$\begin{aligned}
&\#\Omega_{\kappa,q}(L+H,l) \\
&= \sum_{n \in \mathbb{Z}/p^l\mathbb{Z}} \#\Omega_{\kappa,q-n}(L,l)\#\Omega_n(H,l) \\
&= \sum_{\nu=0}^{l} \#\Omega_{p^\nu}(H,l) \sum_{\substack{n\in\mathbb{Z}/p^l\mathbb{Z} \\ \nu_p(n)=\nu}} \#\Omega_{\kappa,q-n}(L,l) \\
&= \sum_{\nu=0}^{l} \#\Omega_{p^\nu}(H,l) \left(\sum_{\substack{n\in\mathbb{Z}/p^l\mathbb{Z} \\ \nu_p(n)\geq\nu}} \#\Omega_{\kappa,q-n}(L,l) - \sum_{\substack{n\in\mathbb{Z}/p^l\mathbb{Z} \\ \nu_p(n)\geq\nu+1}} \#\Omega_{\kappa,q-n}(L,l) \right) \\
&= \sum_{\nu=0}^{l} \#\Omega_{p^\nu}(H,l) \left(p^{(l-\nu)m}\#\Omega_{\kappa,q}(L,\nu) - p^{(l-\nu-1)m}\#\Omega_{\kappa,q}(L,\nu+1) \right).
\end{aligned}$$

From this the relation (8) follows. (7) is obtained by letting $w \to \infty$ since (6) does not depend on w. □

(6.4.7) Now assume $p \neq 2$ and that $p^{-1}Q_L$ is not integral.

Diagonalize Q_L, i.e. $L_{\mathbb{Z}_p} = \mathbb{Z}_p^m$,

$$Q_L(x) = \sum_i \varepsilon_i p^{l_i} x_i^2,$$

where $\varepsilon_i \in \mathbb{Z}_p^*$ and $l_i \in \mathbb{Z}_{\geq 0}, l_1 \leq \cdots \leq l_m$. According to the assumption, we have $l_1 = 0$. This is possible by (6.4.15). Denote:

$$\begin{aligned} L(k,1) &:= \{1 \leq i \leq m \mid l_i - k < 0 \text{ is odd}\} \\ l(k,1) &:= \# L(k,1) \\ d(k) &:= k + \frac{1}{2} \sum_{l_i < k} (l_i - k) \\ v(k) &:= \left(\frac{-1}{p}\right)^{\lfloor \frac{l(k,1)}{2} \rfloor} \prod_{i \in L(k,1)} \left(\frac{\varepsilon_i}{p}\right) \end{aligned}$$

(6.4.8) Theorem. *With this notation, we have*

$$\beta_p(L_{\mathbb{Z}_p}, < \alpha p^a >, L_{\mathbb{Z}_p}; s) = 1 + R(\alpha p^a; p^{-s}),$$

where $\alpha \in \mathbb{Z}_p^*$, $a \in \mathbb{Z}_{\geq 0}$ *and*

$$\begin{aligned} R(\alpha p^a; X) =& (1 - p^{-1}) \sum_{\substack{0 < k \leq a \\ l(k,1) \text{ is even}}} v(k) p^{d(k)} X^k \\ &+ v(a+1) p^{d(a+1)} X^{a+1} \cdot \begin{cases} -\frac{1}{p} & \text{if } l(a+1,1) \text{ is even,} \\ \left(\frac{\alpha}{p}\right) \frac{1}{\sqrt{p}} & \text{if } l(a+1,1) \text{ is odd.} \end{cases} \end{aligned}$$

We have:

$$\beta_p(L_{\mathbb{Z}_p}, < 0 >, L_{\mathbb{Z}_p}; s) = 1 + R(0; p^{-s}),$$

where

$$R(0; X) = (1 - p^{-1}) \sum_{\substack{k > 0 \\ l(k,1) \text{ is even}}} v(k) p^{d(k)} X^k.$$

Proof. [96, Theorem 3.1] □

(6.4.9) Because it is a polynomial in p^{-s}, we see in particular that there is a natural 'continuation' of $\beta_p(L_{\mathbb{Z}_p}, < q >, \kappa; s)$ to arbitrary $s \in \mathbb{C}$ if $n = 1$.
We will now construct a 'continuation' of the volume of the orthogonal group, or, more precisely, of its discriminant kernel such that the above orbit equation remains true as an identity of functions in s.
Let $L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}$ be lattices with non-degenerate quadratic forms $Q_L \in \mathrm{Sym}^2(L_{\mathbb{Z}_p}^*)$. Let κ be an element in $(L_{\mathbb{Z}_p}^*/L_{\mathbb{Z}_p}) \otimes M_{\mathbb{Z}_p}^*$.
As a first step, we have

$$\begin{aligned}\mathrm{vol}(\mathrm{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p}^s))^{-1} |d(L_{\mathbb{Z}_p})|^{\frac{n}{2}} |d(M)|^{-s+\frac{n-m+1}{2}} \beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s)\\ = \sum_i \mathrm{vol}(\mathrm{SO}'(\alpha_i(M_{\mathbb{Z}_p})^\perp)^{-1}\end{aligned}$$

where the α_i are representatives of the $\mathrm{SO}'(L_{\mathbb{Z}_p})$-orbits in $\mathrm{I}(M, L)(\mathbb{Q}_p) \cap \kappa$. The stability of these orbits for arbitrary s in this formula will be shown (at least for $p \neq 2$) in lemma (6.4.20) below. This occurs at least if L splits n hyperbolic planes, in particular for $s \geq n$ for any lattice. We call these orbits **stable**.

(6.4.10) Definition. *We introduce the following notation:*

$$\begin{aligned}\lambda_p(L_{\mathbb{Z}_p}; s) &:= \frac{\mathrm{vol}(\mathrm{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p}^s))}{\prod_{i=1}^s (1 - p^{-2i})},\\ \mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s) &:= \mathrm{vol}(\mathrm{I}(M, L \oplus H_{\mathbb{Z}_p}^s)(\mathbb{Q}_p) \cap \kappa \oplus H_{\mathbb{Z}_p}^s \otimes M_{\mathbb{Z}_p}^*)\\ &= |d(L_{\mathbb{Z}_p})|^{\frac{n}{2}} |d(M_{\mathbb{Z}_p})|^{-s+\frac{1+n-m}{2}} \beta_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s),\end{aligned}$$

cf. (6.3). Here all volumes are understood to be calculated w.r.t. the canonical measures (6.2.3).

We have the following fundamental **orbit equation**:

(6.4.11) Theorem. *Let $p \neq 2$. For s sufficiently big (explicitly, if $n = 1$: $s \geq 1$, in general, or $s \geq 0$, if L splits an hyperbolic plane), we have*

$$\lambda_p(L_{\mathbb{Z}_p}; s)^{-1} \cdot \mu_p(L_{\mathbb{Z}_p}, M_{\mathbb{Z}_p}, \kappa; s) = \sum_i \lambda_p(\alpha_i(M_{\mathbb{Z}_p})^\perp; s)^{-1},$$

where the sum is taken over stable orbits.

(6.4.12) Remark. In (8.2.1) we will establish that $\lambda_p(L_{\mathbb{Z}_p}; s)$ is a polynomial in p^{-s} for $s \geq 1$ (in general) hence the orbits equation makes sense as an identity of polynomials and so for all $s \in \mathbb{C}$.
Observe: $\prod_{i=1}^s (1 - p^{-2i}) = (1 + p^{-s}) \,\mathrm{vol}(\mathrm{SO}(H^s))$ for $s \in \mathbb{N}$ (cf. 8.2.1).

We emphasize that this naive equation can be related (in this form) to Arakelov geometry only if there is only one orbit. For computational reasons, and to understand the properties of the representation densities/volumes, it is interesting in its own right, however. See (11.2.12) for a discussion of its failure.
In (8.2.1) the occurring functions will be calculated explicitly in a lot of cases. We now will turn anyway to the problem of stability of orbits, for which we need some lemmas (some are stated for $\mathbb{Z}_{(p)}$ instead of $\mathbb{Z}_p$ for later need):

(6.4.13) Lemma. *Let $M_{\mathbb{Z}_p}$ an* unimodular *sublattice of $L_{\mathbb{Z}_p}$. Then we have*

$$L_{\mathbb{Z}_p} = M_{\mathbb{Z}_p} \perp M_{\mathbb{Z}_p}^{\perp}.$$

Proof. Follows from [50, Prop. 5.2.2]. □

(6.4.14) Lemma. *If $L_{\mathbb{Z}_p}$ is unimodular, we have*

$$L_{\mathbb{Z}_p} \simeq H_{\mathbb{Z}_p}^r \perp L_{\mathbb{Z}_p}^0,$$

with $L_{\mathbb{Z}_p}^0$ anisotropic. Here $H_{\mathbb{Z}_p}$ is an hyperbolic plane.

Proof. [50, Theorem 5.2.2] □

(6.4.15) Lemma. *If $p \neq 2$, there is a basis $e_1, \dots, e_m$ of $L_{\mathbb{Z}_{(p)}}$, with respect to which the Q_L is given by*

$$Q_L : x \mapsto \sum_i \varepsilon_i p^{\nu_i} x_i^2,$$

where $\varepsilon_i \in {\mathbb{Z}_{(p)}}^*, \nu_i \in \mathbb{Z}_{\geq 0}$ and $\nu_1 \leq \dots \leq \nu_m$.

Proof. This works for any discrete valuation ring in which $|2| = 1$. It exists a vector v of maximal length in $L_{\mathbb{Z}_{(p)}}$ since $|Q_L(\cdot)|$ is certainly bounded and the valuation is discrete. We claim that $|\langle v, w\rangle| \leq |\langle v, v\rangle|$ for every $w \in L_{\mathbb{Z}_{(p)}}$. For, if this is not the case, we would have

$$|\langle v + w, w + w\rangle| \leq \max(|\langle v, v\rangle|, |2\langle v, w\rangle|, |\langle w, w\rangle|)$$

and equality, if one of the terms is strictly bigger than the other two. We have $|\langle w, w\rangle| \leq |\langle v, v\rangle|$. Hence, if $|\langle v, w\rangle|$ would be strictly bigger than $|\langle v, v\rangle|$, $|\langle v + w, w + w\rangle|$ would be strictly bigger than $|\langle v, v\rangle|$, too, a contradiction. Therefore the projectors

$$w \mapsto \frac{\langle v, w\rangle}{\langle v, v\rangle} v \qquad w \mapsto w - \frac{\langle v, w\rangle}{\langle v, v\rangle} v$$

are defined and yield $L_{\mathbb{Z}_{(p)}} =< Q_L(v) >\perp L'_{\mathbb{Z}_{(p)}}$. The statement follows by induction. □

(6.4.16) Lemma. *Let $p \neq 2$. Assume $L^*_{\mathbb{Z}_p}/L_{\mathbb{Z}_p}$ is cyclic. Then*

$$\mathrm{SO}(L_{\mathbb{Z}_p})/\,\mathrm{SO}'(L_{\mathbb{Z}_p}) = \begin{cases} 1 & \text{if } \nu = 0 \text{ or } m = 1, \\ \mathbb{Z}/2\mathbb{Z} & \text{otherwise,} \end{cases}$$

where $p^\nu = |D(L_{\mathbb{Z}_p})|^{-1}$ is the order of $L^*_{\mathbb{Z}_p}/L_{\mathbb{Z}_p}$.

Proof. In the representation given by (6.4.15) ν_m is equal to ν and all other ν_i vanish. Hence $v := p^{-\nu}e_m$ is a generator of $L^*_{\mathbb{Z}_p}/L_{\mathbb{Z}_p}$ and $Q(v) = \varepsilon_m p^{-\nu}$. Let v' be its image under an arbitrary isometry. v' has a representation

$$v' = \sum_{i<m} \alpha_i e_i + p^{-\nu}\alpha_m e_m \qquad \alpha_i \in \mathbb{Z}_p$$

and

$$Q(v') = \sum_{i<m} \varepsilon_i \alpha_i^2 + \varepsilon_m p^{-\nu}\alpha_m^2 = \varepsilon_m p^{-\nu}.$$

From this it follows

$$\alpha_m^2 \equiv 1 \mod p^\nu,$$

hence ($p \neq 2$)

$$\alpha \equiv \pm 1 \mod p^\nu.$$

The occurring sign defines a character of the orthogonal group. An element is in its kernel, precisely if it is in the discriminant kernel. Moreover, if $m > 1$ there are elements in SO, which yield sign -1, for example composition of reflection along e_m and any $e_i, i < m$. □

(6.4.17) Lemma. *Assume $p \neq 2$ and let*

$$L_{\mathbb{Z}_p} = M_{\mathbb{Z}_p} \perp M'_{\mathbb{Z}_p} = N_{\mathbb{Z}_p} \perp N'_{\mathbb{Z}_p},$$

with $\beta : M_{\mathbb{Z}_p} \cong N_{\mathbb{Z}_p}$. Then we have

$$M'_{\mathbb{Z}_p} \cong N'_{\mathbb{Z}_p}.$$

In particular, there exists an isometry $\alpha \in \mathrm{SO}(L_{\mathbb{Z}_p})$ with $\alpha(M_{\mathbb{Z}_p}) = N_{\mathbb{Z}_p}$. If $M_{\mathbb{Z}_p}$ is unimodular, we may choose $\alpha \in \mathrm{SO}'(L_{\mathbb{Z}_p})$. If $M'_{\mathbb{Z}_p}$ has a vector of unit length, we may assume in addition, that $\alpha|_{M_{\mathbb{Z}_p}} = \beta$.

Proof. The first part of the assertion is shown in [50, Corollary 5.3.1]. It remains to see that we may choose the isometry in the discriminant kernel, if $M_{\mathbb{Z}_p}$ is unimodular: For this we proceed by induction on the dimension on $M_{\mathbb{Z}_p}$. If $M_{\mathbb{Z}_p}$ is one dimensional, let v be a generating vector of unit length and v' its image under β. One of the vectors $v + v'$

or $v - v'$ has unit length, call it w. The reflection along w lies obviously in $O'(L_{\mathbb{Z}_p})$ and interchanges $< v >$ and $< v' >$. By composition with the reflection along v', we may assume, that it lies in $\mathrm{SO}'(L_{\mathbb{Z}_p})$.
Assume now $\dim(M_{\mathbb{Z}_p}) > 1$. Let v be a vector of unit length in $M_{\mathbb{Z}_p}$ and v' its image. We have

$$L_{\mathbb{Z}_p} =< v >\perp v^{\perp} =< v' >\perp v'^{\perp},$$

(Lemma 6.4.13) and

$$v^{\perp L_{\mathbb{Z}_p}} = v^{\perp M_{\mathbb{Z}_p}} \perp M'_{\mathbb{Z}_p} \qquad v'^{\perp L_{\mathbb{Z}_p}} = v'^{\perp N_{\mathbb{Z}_p}} \perp N'_{\mathbb{Z}_p},$$

and we have an isometry (case above) in the discriminant kernel, which maps $< v >$ to $< v' >$, and hence $v^{\perp}$ to $v'^{\perp}$. $v^{\perp M_{\mathbb{Z}_p}}$ and $v'^{\perp N_{\mathbb{Z}_p}}$ are now isomorphic (again by the case above) hence (induction hypothesis), there is an isometry in $\mathrm{SO}'(v'^{\perp})$ mapping the image of $v^{\perp M_{\mathbb{Z}_p}}$ to $v'^{\perp N_{\mathbb{Z}_p}}$. It lifts to $\mathrm{SO}'(L_{\mathbb{Z}_p})$. Composition with the first isometry gives the induction step. The proof shows, that we may arrange $\alpha|_{M_{\mathbb{Z}_p}} = \beta$ if there is a reflection in $\mathrm{SO}'(M'_{\mathbb{Z}_p})$. $\square$

(6.4.18) Lemma. *Let* $L_{\mathbb{Z}_p}$ $(\dim(L) \geq 3)$ *be a unimodular lattice,* $p \neq 2$*, and* $q \in \mathbb{Z}_p \setminus \{0\}$*.*
Then $\mathrm{SO}(L_{\mathbb{Z}_p})$ *acts transitively on* $\{\alpha \in \mathrm{I}(< q >, L_{\mathbb{Z}_p}) \mid \mathrm{im}(\alpha)$ *is saturated*$\}$*. In particular, it acts transitively on* $\mathrm{I}(< q >, L_{\mathbb{Z}_p})$ *with* $|q|_p = \frac{1}{p}$*.*

Proof. Take any v with $Q_L(v) = q$. Diagonalize the form Lemma (6.4.15) and take the reflection v' of v at any basis vector e_i with the property that $v_i \in \mathbb{Z}_p^*$ (this must exist, since otherwise the vector would not be primitive). We have $p \nmid \langle v, v' \rangle$. Therefore the form on $\mathbb{Z}_p v \oplus \mathbb{Z}_p v'$ is unimodular, hence $\mathbb{Z}_p v \oplus \mathbb{Z}_p v'$ is primitive and a direct summand by Lemma (6.4.13). It is necessarily a hyperbolic plane, since modulo p it represents zero. We have shown that any primitive vector in $\mathrm{I}(< q >, L)$ lies in a hyperbolic plane. Now use Lemma (6.4.17) and the fact, that $\mathrm{O}(H_{\mathbb{Z}_p})$ (not SO!) acts transitively on primitive vectors of length q on $H_{\mathbb{Z}_p}$. $\square$

We see that for $p \neq 2$, $j > 0$, and an unimodular lattice $L_{\mathbb{Z}_{(p)}}$, there are precisely $\lfloor \frac{j}{2} \rfloor + 1$ orbits of vectors of length p^j, indexed according to their 'saturatedness'.

(6.4.19) Lemma. *Consider* $L_{\mathbb{Z}_p} \oplus H^2 = L_{\mathbb{Z}_p} \oplus \mathbb{Z}_p^4$ *(i.e. a space with quadratic form* $Q(x_L, x_0, \ldots, x_3) = Q_L(x_L) + x_0 x_1 + x_2 x_3$*). Let* $w = (w_L, w_0, \ldots, w_3) \in L_{\mathbb{Z}_p} \oplus H^2$ *be a vector with* $Q(w) \neq 0$*. It follows*

$$< w >^{\perp} \equiv H \perp \Lambda.$$

Proof. We may assume w.l.o.g. that $\nu_p(w_0)$ is minimal among the $\nu_p(w_i)$. $< w >^{\perp}$ is described by the equation $\langle w_L, x_L \rangle + w_0 x_1 + w_1 x_0 + w_2 x_3 + w_3 x_2 = 0$. The map $(x_2, x_3) \mapsto$

$(0_L, 0, -\frac{w_2x_3+w_3x_2}{w_0}, x_2, x_3)$ therefore is an isometric embedding of an hyperbolic plane into $< w >^\perp$. The assertion now follows from (6.4.13). $\square$

We are now able to prove **stability of orbits**:
Let $\kappa \in (L^*_{\mathbb{Z}_p}/L_{\mathbb{Z}_p}) \otimes M^*_{\mathbb{Z}_p}$ and $\{\alpha_i\}$ be a set of representatives of the orbits under $\mathrm{SO}'(L_{\mathbb{Z}_p})$ acting on $\mathrm{I}(M,L)(\mathbb{Q}_p) \cap \kappa$.

(6.4.20) Lemma. *Assume $M_{\mathbb{Z}_p}$ has dimension n, $p \neq 2$ and $L_{\mathbb{Z}_p}$ splits n hyperbolic planes.*
Then $\{\alpha_i\}$ is a set of representatives of the $\mathrm{SO}'(L_{\mathbb{Z}_p} \oplus H^s_{\mathbb{Z}_p})$-orbits in $\mathrm{I}(M, L \perp H^s)(\mathbb{Q}_p) \cap \kappa \oplus (H^s_{\mathbb{Z}_p} \otimes M^*_{\mathbb{Z}_p})$ for all s.

Proof. We begin by showing that, if $\alpha_i = g\alpha_j$ for some $g \in \mathrm{SO}'(L_{\mathbb{Z}_p} \oplus H^s_{\mathbb{Z}_p})$, then we have $\alpha_i = g'\alpha_j$ for some $g' \in \mathrm{SO}'(L_{\mathbb{Z}_p})$ as well. We have

$$\alpha_i(M_{\mathbb{Z}_p})^\perp = H^s_{\mathbb{Z}_p} \perp \alpha_i(M_{\mathbb{Z}_p})^{\perp L_{\mathbb{Z}_p}}.$$

Since the form is integral in $L_{\mathbb{Z}_p}$, we have according to Lemma (6.4.14):

$$\alpha_i(M_{\mathbb{Z}_p})^\perp = g(H^s_{\mathbb{Z}_p}) \perp \Lambda_{\mathbb{Z}_p}$$

(because $H^s_{\mathbb{Z}_p} \perp \alpha_j(M_{\mathbb{Z}_p})$.) Hence (Lemma 6.4.14 - here $p \neq 2$ is used), there is an isometry $\alpha_i(M_{\mathbb{Z}_p})^\perp$, which maps $g(H^s_{\mathbb{Z}_p})$ to $H^s_{\mathbb{Z}_p}$ and lies in $\mathrm{SO}'(\alpha_i(M_{\mathbb{Z}_p})^\perp)$. Hence its lifts to an isometry in $\mathrm{SO}'(L_{\mathbb{Z}_p} \oplus H^s_{\mathbb{Z}_p})$, which fixes $M_{\mathbb{Z}_p}$ point-wise (Lemma 6.4.2). Composition with g yields the required g'.
Secondly, let an isometry $\alpha : M_{\mathbb{Z}_p} \to L_{\mathbb{Z}_p} \perp H^s_{\mathbb{Z}_p}$ be given. We have to show that it is mapped by an element in $\mathrm{SO}'(L_{\mathbb{Z}_p} \oplus H^s_{\mathbb{Z}_p})$ to any of the α_i. It clearly suffices (induction on s) to consider the case $s = 1$.
We proceed by induction on n and first prove the case $n = 1$: By Lemma (6.4.19) $\alpha_i(M_{\mathbb{Z}_p})^\perp$ ($\perp$ with respect to $L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p}$) splits an hyperbolic plane because by assumption $L_{\mathbb{Z}_p}$ splits already one. Then apply Lemma (6.4.14). We get

$$\alpha(M_{\mathbb{Z}_p})^\perp = \Lambda_{\mathbb{Z}_p} \perp \Lambda'_{\mathbb{Z}_p},$$

with $\Lambda_{\mathbb{Z}_p} \cong H_{\mathbb{Z}_p}$.
In addition, we have (again Lemma 6.4.14)

$$L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p} = \Lambda_{\mathbb{Z}_p} \perp \Lambda^\perp_{\mathbb{Z}_p},$$

hence (Lemma 6.4.17) $\Lambda^\perp_{\mathbb{Z}_p} \cong L_{\mathbb{Z}_p}$ and the image of α of course lies in $\Lambda^\perp_{\mathbb{Q}_p}$ because $\Lambda_{\mathbb{Q}_p} \perp \alpha(M_{\mathbb{Q}_p})$. Now there is an isometry g, which maps $\Lambda^\perp_{\mathbb{Z}_p}$ to $L_{\mathbb{Z}_p}$ and $\Lambda_{\mathbb{Z}_p}$ to $H_{\mathbb{Z}_p}$ (even in $\mathrm{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p})$ - Lemma (6.4.17)). The image of $g \circ \alpha$ then lies in $L_{\mathbb{Q}_p}$ and the

element of $\mathrm{SO}'(L_{\mathbb{Z}_p})$, which maps $g \circ \alpha$ to any α_i, lifts to an isometry $g' \in \mathrm{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p})$. Hence α is conjugated to α_i under $\mathrm{SO}'(L_{\mathbb{Z}_p} \oplus H_{\mathbb{Z}_p})$.
We now assume, that the statement has been proven for M up to dimension $n-1$. We choose some splitting $M = N \oplus N^\perp$ with $\dim(N) = n-1$ and accordingly decomposition $\kappa = \kappa_N + \kappa_{N^\perp}$. Let $\alpha : M_{\mathbb{Z}_p} \to L_{\mathbb{Z}_p} \perp H^s_{\mathbb{Z}_p}$ be given. Induction hypothesis shows, that w.l.o.g. $\alpha(N) \subset L_{\mathbb{Q}_p}$. Since $L_{\mathbb{Z}_p}$ splits an unrequired hyperbolic plane, we may even assume, that $\alpha(N)^\perp \cap L_{\mathbb{Z}_p}$ splits a hyperbolic plane, too. Hence we may apply the $n=1$ case to $\alpha(N)^\perp \oplus H^s_{\mathbb{Z}_p}$ and $M^\perp$ and observe that the constructed isometries in this step all lift by Lemma (6.4.2). □

6.5. Connection with the local zeta function

Let $L_{\mathbb{Z}_p}$ be a lattice with non-degenerate quadratic form $Q_L \in \mathrm{Sym}^2(L^*_{\mathbb{Z}_p})$, as before. In the case $n = 1$ (i.e. $M_{\mathbb{Z}_p} =< q >$), we will give a connection of the above representation densities with the local zeta function of $L_{\mathbb{Z}_p}$. We recover the classical zeta function [95, 4.3] for $q = 0$. Assume that Q_L is integral and primitive (i.e. $p^{-1}Q_L$ not integral) on $L_{\mathbb{Z}_p}$.

(6.5.1) Theorem. *Let $s \in \mathbb{Z}_{\geq 0}$, $q \in \mathbb{Z}_p$ arbitrary, $\kappa \in L^*_{\mathbb{Z}_p}/L_{\mathbb{Z}_p}$ and $m \geq 2$.*

$$\int_\kappa |Q_L(v) - q|^s \,\mathrm{d}\, v = p^s + \beta(L_{\mathbb{Z}_p}, < q >, \kappa; s+1) \frac{1-p^s}{1-p^{-s-1}},$$

where $\mathrm{d}\, v$ *is the translation invariant measure with* $\mathrm{vol}(L_{\mathbb{Z}_p}) = 1$.

Proof. Observe that

$$\begin{aligned}
\int_\kappa |Q_L(v) - q|^s \,\mathrm{d}\, v &= \sum_{i=0}^\infty \left(\mathrm{vol}\, \kappa \cap \{|Q_L - Q| \leq \frac{1}{p^i}\} - \mathrm{vol}\, \kappa \cap \{|Q_L - Q| \leq \frac{1}{p^{i+1}}\} \right) \frac{1}{p^{is}} \\
&= 1 + \sum_{i=1}^\infty \mathrm{vol}\, \kappa \cap \{|Q_L - Q| \leq \frac{1}{p^i}\} (1-p^s) \frac{1}{p^{is}} \\
&= 1 + \sum_{i=1}^\infty \frac{\#\Omega_{\kappa,q}(i)}{p^{i(s+n)}} (1-p^s) \\
&= 1 - (1-p^s) + \sum_{i=0}^\infty \frac{\#\Omega_{\kappa,q}(i)}{p^{i(s+n)}} (1-p^s).
\end{aligned}$$

The formula now follows using identity (7). □

It is convenient to write

$$\beta(L_{\mathbb{Z}_p}, 0; s) = 1 + (1-p^{-1}) \delta(p^{-s}),$$

where δ is a polynomial, which is, according to Yang's formula (6.4.8), given by:

$$\delta(X) = \sum_{\substack{k>1 \\ l(k,1)\equiv 0 \ (2)}} \nu(k) p^{d(k)} X^k$$

(with the local notation from 6.4.8). Here $\delta(0) = 0$.

Therefore:

$$\begin{aligned} E := \mathrm{vol}\{x \in L_{\mathbb{Z}_p} | Q_L(v) \in \mathbb{Z}_p^*\} &= \lim_{s\to\infty} \int_{L_{\mathbb{Z}_p}} |Q_L(v)|^s \,\mathrm{d}\, v \\ &= \lim_{X\to 0} (1 - \frac{1 + (1-p^{-1})\delta(p^{-1}X)}{1 - p^{-1}X}) \\ &= (1-p^{-1})(1 - \delta'(0)p^{-1}). \end{aligned}$$

with

$$\delta'(0) = \begin{cases} \nu(1)p^{d(1)} & l(1,1) \equiv 0 \ (2), \\ 0 & l(1,1) \equiv 1 \ (2). \end{cases}$$

(6.5.2) Definition. *We define the* ***normalized local zeta function*** *associated with L by*

$$\zeta_p(L_{\mathbb{Z}_p}, s) := \frac{1}{E} \int_{L_{\mathbb{Z}_p}} |Q_L(v)|^{s-1} \,\mathrm{d}\, v.$$

For two dimensional lattices, this coincides for example with the usual zeta function of the associated order in the associated quadratic field ($\frac{\mathrm{d}\, v}{E|Q_L|}$ is the multiplicatively invariant measure for which $(\mathcal{O} \otimes \mathbb{Z}_p)^*$ has volume 1).

(6.5.3) Here, we explicitly compute the zeta function for an arbitrary two dimensional lattice. This will be used to compute the arithmetic volume of Shimura varieties associated with 2-dimensional lattices in (8.1), including also information from bad primes.

Let $L_{\mathbb{Z}_p}$, $p \neq 2$ be a quadratic lattice, w.l.o.g. of the form $(x_1)^2 + \varepsilon p^l (x_2)^2$. The zeta function of a 2 dimensional lattices depends only on the discrimiant, because it is invariant under $\mathrm{SL}_2(\mathbb{Z}_p)$ and under multiplication of the form by a scalar.

With the notation of Yang (6.4.8), we have:

$$L(k;1) = \begin{array}{ccc} l \text{ even} & & l \text{ odd} \\ \{\} & 0 < k \le l \text{ even} & \{\} \\ \{1\} & 0 < k \le l \text{ odd} & \{1\} \\ \{\} & l < k \text{ even} & \{2\} \\ \{1,2\} & l < k \text{ odd} & \{1\} \end{array}$$

$$d(k) = \begin{cases} \frac{1}{2}k & k \le l, \\ \frac{1}{2}l & l < k, \end{cases}$$

$$\nu(k) = \begin{cases} (\frac{-\varepsilon}{p})^k & l < k, l \text{ even}, \\ 1 & k \le l \text{ or } l \text{ odd} . \end{cases}$$

Assume first $l = 0$, then (as expected):

$$\delta(X) = \sum_{k>1} (\frac{-\varepsilon}{p})^k X^k = \frac{(\frac{-\varepsilon}{p})X}{1 - (\frac{-\varepsilon}{p})X},$$

$$\zeta_p(L_{\mathbb{Z}_p}, s) = \frac{(1-p^{-1})(1 + (1-(pX)^{-1})\delta(X))}{(1-X)E} = \frac{1}{(1-X)(1-(\frac{-\varepsilon}{p})X)}.$$

For l odd, the above yields:

$$\delta(X) = \sum_{k'=1}^{\frac{l-1}{2}} p^{k'} X^{2k'} = (pX)^2 \frac{1 - (pX^2)^{\frac{l-1}{2}}}{1 - (pX)^2},$$

$$\zeta_p(L_{\mathbb{Z}_p}, s) = \frac{1 - (pX^2)^{\frac{l+1}{2}} - X + X(pX^2)^{\frac{l-1}{2}}}{(1-X)(1-pX^2)}.$$

For $l \ge 2$, even, it yields:

$$\delta(X) = \sum_{k=l}^{\infty} (\frac{-\varepsilon}{p})^k p^{\frac{1}{2}l} X^k + \sum_{k'=1}^{\frac{l}{2}-1} p^{k'} X^{2k'}$$

$$= p^{\frac{1}{2}l} X^l \frac{1}{1 - (\frac{-\varepsilon}{p})X} + pX^2 \frac{1 - (pX^2)^{\frac{l}{2}-1}}{1 - pX^2},$$

$$\zeta_p(L_{\mathbb{Z}_p}, s) = \frac{p^{\frac{l}{2}} X^l - p^{\frac{l}{2}-1} X^{l-1}}{(1-X)(1-(\frac{-\varepsilon}{p})X)} + \frac{1 - (pX^2)^{\frac{l}{2}} - X + X(pX^2)^{\frac{l}{2}-1}}{(1-X)(1-pX^2)}.$$

7. The Weil representation

7.1. General definition

(7.1.1) The Weil representation [93] can be defined for any Abelian locally compact group G. We will restrict here to the case of a finite free module M over R equal to a $\mathbb{Q}_p, \mathbb{R}, \mathbb{A}$ or $\mathbb{A}^{(\infty)}$ respectively.

Let M be an R-vector space. $\mathfrak{M} = M \oplus M^*$ becomes a symplectic vector space in a canonical way by

$$\langle \begin{pmatrix} w_1 \\ w_1^* \end{pmatrix}, \begin{pmatrix} w_2 \\ w_2^* \end{pmatrix} \rangle \mapsto w_1^*(w_2) - w_2^*(w_1).$$

Associated with M there is a **Heisenberg group**:

$$H = R \times M \times M^*,$$

defined by the group law

$$(r_1, x_1, x_1^*)(r_2, x_2, x_2^*) = (r_1 + r_2 + x_1^* x_2, x_1 + x_2, x_1^* + x_2^*).$$

Choose any non-trivial additive character χ on R. We get an action of H on $S(M^*)$ by

$$g\varphi : x^* \mapsto \chi(r_1 + x^* x_1)\varphi(x^* + x_1^*)$$

for $g = (r_1, x_1, x_1^*)$.

This is essentially (up to topological issues) the unique irreducible representation of H, where R acts through χ.

The unicity yields a projective representation of the automorphism group of H. This group is (in these cases) the symplectic group:

$$\mathrm{Sp}(\mathfrak{M}) = \{\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid {}^*\sigma\sigma = 1\},$$

with $\alpha \in \mathrm{End}(M)$, $\beta \in M \otimes M$, $\gamma \in M^* \otimes M^*$ and $\delta \in \mathrm{End}(M^*)$. Here ${}^*\sigma$ is the transpose with respect to the symplectic form:

$${}^*\sigma = \begin{pmatrix} {}^t\delta & -{}^t\gamma \\ -{}^t\beta & {}^t\alpha \end{pmatrix}.$$

It acts by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (r, m, m^*) = (f(m, m^*) + r, \alpha m + \beta m^*, \gamma m + \delta m^*),$$

with

$$f(m, m^*) = \frac{1}{2}(\langle \gamma m + \delta m^*, \alpha m + \beta m^* \rangle - \langle m^*, m \rangle). \tag{1}$$

This projective representation can of course be considered as an honest representation of an extension

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \widetilde{\mathrm{Sp}} \longrightarrow \mathrm{Sp} \longrightarrow 0.$$

It is called the **Weil representation**.

(7.1.2) The Weil representation can be described explicitly. We will do this first for the following subgroups of $\mathrm{Sp}(\mathfrak{M})$:
Denote:

$$g_l(\alpha) := \begin{pmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{pmatrix} \tag{2}$$

$$u(B) := \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \tag{3}$$

$$d(\gamma) := \begin{pmatrix} 0 & -{}^t\gamma^{-1} \\ \gamma & 0 \end{pmatrix} \tag{4}$$

for $\alpha \in \mathrm{Aut}(M)$, $B \in \mathrm{Sym}^2(M)$ and $\beta \in \mathrm{Hom}(M^*, M)$ is the bilinear form on M^* associated with B (or the associated morphism) and $\gamma \in \mathrm{Iso}(M, M^*)$.
We denote the image of g_l and u by G_l and U respectively, and their product by P, which is a maximal parabolic of $\mathrm{Sp}(\mathfrak{M})$.

(7.1.3) For each decomposition $M^* = M^{*\prime} \oplus M^{*\prime\prime}$ there is an embedding

$$\iota : \mathrm{Sp}(\mathfrak{M}') \hookrightarrow \mathrm{Sp}(\mathfrak{M}).$$

Let $\gamma_0 : M' \to M^{*\prime}$ be a symmetric isomorphism.
Let $\Omega_{M^{*\prime}} = UG_l\iota(d(\gamma_0)U_{\mathfrak{M}'})$. Every element in $\Omega_{M^{*\prime}}$ possesses a unique representation in this way. The set $\Omega_{M^{*\prime}}$ does not depend on the choice of complement $M^{*\prime\prime}$ and on γ_0, and consists precisely of those elements, for which $\mathrm{im}({}^t\gamma_0) = M^{*\prime}$.
We have the following decomposition:

$$\mathrm{Sp}(\mathfrak{M}) = \bigcup_{M^{*\prime} \subset M^*} \Omega_{M^{*\prime}} \tag{5}$$

(7.1.4) The groups defined in (7.1.2) have lifts to $\widetilde{\mathrm{Sp}}$ given by the following action on $S(M^*_{\mathbb{R}})$

$$\begin{aligned} r(g_l(\alpha))\varphi :& x^* \mapsto |\alpha|^{\frac{1}{2}}\varphi({}^t\alpha x^*) \\ r(u(B))\varphi :& x^* \mapsto \chi(B(x^*))\varphi(x^*) \\ r(d(\gamma))\varphi :& x^* \mapsto |\gamma|^{-\frac{1}{2}}\int_M \varphi({}^t\gamma x)\chi(-\langle x^*, x\rangle)\,\mathrm{d}\,x \end{aligned}$$

for $\alpha \in \mathrm{GL}(M^*), \beta \in M \otimes M$ $(m = \dim L)$.
Here $\mathrm{d}\,x$ is any measure on M^* and $|\gamma|$ are the comparison factor between the image under γ of the chosen measure on M and the dual of the chosen measure. Observe, that the third formula does not depend on this choice.
As every element in the 'maximal' set Ω_{M^*} in the decomposition (5) has a unique expression of the form $u(B_1)d(\gamma)u(B_2)$, r therefore extends to a lift defined on Ω_{M^*}. it is described by the formula

$$r(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix})\varphi : x^* \mapsto |\gamma|^{-\frac{1}{2}}\int_M \varphi({}^t\alpha x^* + {}^t\gamma x)\chi(f(x, x^*))\,\mathrm{d}\,x,$$

where f is defined in (1). Remember, the condition for an element of being in Ω_{M^*} was just γ being an isomorphism.

(7.1.5) With respect to the decomposition $M^* = M^{*\prime} \oplus M^{*\prime\prime}$ and dual decomposition $M = M' \oplus M''$ (see 7.1.3), we have lifts of the corresponding subgroups of $\mathrm{Sp}(\mathfrak{M}')$ operating as:

$$\begin{aligned} r'(\iota(g_l(\alpha)))\varphi :& x^* \mapsto |\alpha|^{1/2}\varphi({}^t\alpha x^*) \\ r'(\iota(u(B)))\varphi :& x^* \mapsto \chi(B((x^*)'))\varphi(x^*) \\ r'(\iota(d(\gamma)))\varphi :& x^* \mapsto |\gamma|^{-1/2}\int_{M'} \varphi({}^t\gamma x' + (x^*)'')\chi(-\langle x^*, x'\rangle)\,\mathrm{d}\,x' \end{aligned}$$

for $\alpha \in GL(M'), \beta \in (M' \otimes M')^s$, where $(x^*)'$, resp. $(x^*)''$ are the projections with respect to the chosen decomposition.

(7.1.6) On the 'maximal' set Ω_{M^*} in the decomposition (5) above, the deviance for r from being part of a group homomorphism is measured by a character $\Upsilon : W(R) \to \mathbb{C}^*$, where $W(R)$ is the Witt group[1]. It is determined as follows: Let B be a quadratic form on M^* and $\beta : M \to M^*$ is the symmetric morphism associated with B. We have [93, Théorème 2]:

$$F_{f*\chi\circ B}(x) = \Upsilon(B)|\beta|^{-\frac{1}{2}}F_f(x^*)\chi(B(-\beta^{-1}x)),$$

[1] If R is a field, $W(R)$ is the set of isomorphism classes of quadratic forms over R modulo splitting hyperbolic planes under the operation of $\perp$. If $R = \mathbb{A}^{(S)}$, we understand by $W(R)$ the product over the local Witt groups over all places not in S.

for any $f \in S(R)$.
If R is a local field, it can be calculated by the formula [93, Proposition 4]

$$\Upsilon(<1, a, b, ab>) = (a|b),$$

where $(a|b)$ is the Hilbert symbol. It follows

$$\left(\frac{\Upsilon(<a>)}{\Upsilon(<1>)}\right)^2 = (a|-1)$$

and

(7.1.7) Lemma.
$$\Upsilon(Q) = \Upsilon(<1>)^{m-1}\Upsilon(<\widetilde{D}>)\varepsilon(Q),$$

where $\varepsilon(Q)$ is the Hasse invariant. Here $\widetilde{D}$ is the discriminant of Q if m is even and 2 times the discriminant if m is odd (some authors define the discriminant like this).

Proof. Easy induction on $\dim L$, using that Υ is a character of the Witt group. $\square$

It determines the deviance as follows. Let

$$s'' = ss'$$

be an equation, where

$$s = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, s' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}, s'' = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix}$$

are elements in Ω_{M^*}.
Let Q be the quadratic form associated with the *symmetric* morphism $\gamma^{-1}\gamma''(\gamma')^{-1}$. We have then [93, Théorème 3]:

$$\Upsilon(Q)r(s'') = r(s)r(s') \tag{6}$$

(7.1.8) This allows to construct a modified lifting r', see also [93, §43], determined by the formulas:

$$\begin{aligned}
r'(g_l(\alpha)) &:= \frac{\Upsilon(<1>)}{\Upsilon(<\det(\alpha)>)} r(g_l(\alpha)) \\
r'(u(B)) &:= r(u(B)) \\
r'(d(\gamma)) &:= \Upsilon(<1>)^{m-1}\Upsilon(<\det(\frac{1}{2}\gamma)>)r(d(\gamma))
\end{aligned}$$

(7.1.9) Lemma. *This defines a twofold covering* $\mathrm{Mp}(\mathfrak{M})$ *of* $\mathrm{Sp}(\mathfrak{M})$*, called the* ***metaplectic group****.*

Proof. It suffices to see (compare [93, §43]) that the deviation in equation (6) becomes just a sign ± 1, which follows from the formulæ for Υ given above. □

7.2. A dual reductive pair

(7.2.1) Let L be a quadratic space over R of dimension m. There is an injection

$$\mathrm{Sp}(\mathfrak{M}) \times \mathrm{SO}(L) \hookrightarrow \mathrm{Mp}((M \otimes L^*) \oplus (M^* \otimes L)),$$

if m is even and an injection

$$\mathrm{Mp}(\mathfrak{M}) \times \mathrm{SO}(L) \hookrightarrow \mathrm{Mp}((M \otimes L^*) \oplus (M^* \otimes L)),$$

if m is odd. The image is called a **dual reductive** pair. We are interested in the restriction of the Weil representation.

The injection is a lift of the morphism mapping an element $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ of $\mathrm{Sp}(\mathfrak{M})$ to $\begin{pmatrix} \alpha \otimes \mathrm{id} & \beta \otimes \gamma_L \\ \gamma \otimes \gamma_L^{-1} & \delta \otimes \mathrm{id} \end{pmatrix}$, where γ_L is the morphism $L \to L^*$ associated with Q_L. An element $\lambda \in \mathrm{SO}(L)$ acts as λ, resp. ${}^t\lambda^{-1}$ on L resp. L^*. Via the Weil representation it acts by

$$\lambda\varphi : x^* \mapsto \varphi(\lambda^{-1}x^*).$$

An element $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Omega_{M^*}$ has a lift $r''(\sigma)$, which may differ from the pullback of r' by a sign. It determines the metaplactic group again, if m is odd, and extends to a homomorphism of $\mathrm{Sp}(\mathfrak{M})$ if m is even. The correcting factor of the action of $d(\gamma)$ we will denote by $\widetilde{\Upsilon}(\gamma)$. For $n = 1$ it is just given by $\widetilde{\Upsilon}(\gamma) = \Upsilon(\gamma \otimes Q_L)$.

The Weil representation on the restriction (of r) may be described as follows:

$$\begin{aligned} r(g_l(\alpha))\varphi : & x^* \mapsto |\alpha|^{\frac{m}{2}} \varphi({}^t\alpha x^*) \\ r(u(B))\varphi : & x^* \mapsto \chi((x^*)^! Q_L \cdot \beta)\varphi(x^*) \\ r(d(\gamma))\varphi : & x^* \mapsto |\gamma|^{-\frac{m}{2}} \int_{M \otimes L^*} \varphi({}^t\gamma x)\chi(-\langle x^*, x\rangle)\, \mathrm{d}\, x \end{aligned}$$

Here we choose a measure on $\mathrm{d}\, x$ which is the tensor product of the canonical (with respect to Q_L) on L_R^* with some measure $\mathrm{d}\, x'$ on M_R. $|\gamma|$ in the third formula is computed with respect to the latter and its dual on M_R and M_R^* respectively.

It is convenient to introduce the following notation:

$$\mathrm{Sp}' = \begin{cases} \mathrm{Sp} & n \text{ even,} \\ \mathrm{Mp} & n \text{ odd.} \end{cases}$$

(7.2.2) The following theorem is well known and central for the connection between Eisenstein series (or Whittaker integrals) and volumes. It is analogous to the connection between Gauss sums and representation numbers over finite fields. Analogously it can be used to compute volumes (cf. 6.4.8).

(7.2.3) Theorem ([94, Proposition 6]). *Let R be local and $m \geq 2n+1$. Let $\varphi \in S(M^* \otimes L)$. The function*

$$\Psi(Q) = \int_{\mathrm{I}(M^Q, L)} \varphi(\alpha) \frac{\mu_1}{\mu_2}(\alpha) \qquad Q \in \mathrm{Sym}^2(M^*)$$

has Fourier transform

$$\Psi'(\beta) = \int_{M^* \otimes L} \varphi(x^*) \chi((x^*)^! Q_L \cdot \beta) \mu_1(x^*) = \widetilde{\Upsilon}(\gamma_0)^{-1} |\gamma_0|^{1/2} (d(\gamma_0) u(\beta) \varphi)(0),$$

$\beta \in (M \otimes M)^s$, with respect to a measure μ_2 on $\mathrm{Sym}^2(M^*)$ (in the second formula γ_0 is an arbitrary isomorphism $M \to M^*$). $|\gamma_0|$ is computed via μ_1. Here μ_1, μ_2 and $\frac{\mu_1}{\mu_2}$ are connected via the fibration (6.2.5).

7.3. The Weil representation and automorphic forms

(7.3.1) For $R = \mathbb{A}$, the factor $\Upsilon_{\mathbb{A}}(Q)$ is the product over all local $\Upsilon_\nu(Q)$s. It is 1 if Q is a rational form. (This is a generalization of the law of quadratic reciprocity). Therefore the formulas (7.2.1) for r'' determine a canonical homomorphism $\mathrm{Sp}(L_{\mathbb{Q}}) \to \mathrm{Mp}(L_{\mathbb{A}})$.
We have embeddings $\mathrm{Mp}(\mathbb{Q}_\nu) \hookrightarrow \mathrm{Mp}(\mathbb{A})$.
An automorphic form for the metaplectic group is a function on on $\mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}}) \backslash \mathrm{Mp}(\mathfrak{M}_{\mathbb{A}})$, fulfilling the same properties than for Sp, in particular, it is right invariant under a compact open subgroup of $\mathrm{Mp}(\mathfrak{M}_{\mathbb{A}^{(\infty)}})$ and finite under a maximal compact subgroup of $\mathrm{Mp}(\mathfrak{M}_{\mathbb{R}})$. We denote the corresponding space by $\mathcal{A}(\mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}}) \backslash \mathrm{Mp}(\mathfrak{M}_{\mathbb{A}}))$.
For sufficiently small $K \subset \mathrm{Sp}(\mathfrak{M}_{\widehat{\mathbb{Z}}})$ there is a lift $K \hookrightarrow \mathrm{Mp}(\mathbb{A}^{(\infty)})$. The automorphic functions invariant under K are functions on

$$X = \mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}}) \backslash \mathrm{Mp}(\mathfrak{M}_{\mathbb{A}})/K.$$

By strong approximation we have

$$\mathrm{Mp}(\mathfrak{M}_{\mathbb{A}}) = \mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}}) \, \mathrm{Mp}(\mathfrak{M}_{\mathbb{R}}) K$$

and therefore

$$X = \Gamma \backslash \operatorname{Mp}(\mathfrak{M}_{\mathbb{R}}),$$

where Γ is the lift of $K \cap \operatorname{Sp}(\mathfrak{M}_{\mathbb{Q}})$ to $\operatorname{Mp}(M)_{\mathbb{R}}$.

(7.3.2) A classical Siegel modular form f of (half-integral) weight k on $(M \otimes M)^s$ corresponds to an automorphic form which is given by

$$F : \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \times 1_f \mapsto \det(\gamma\gamma_0^{-1}i + \delta)^k f((\alpha\gamma_0^{-1}i + \beta)(\gamma\gamma_0^{-1}i + \delta)^{-1}),$$

depending on the 'base point' $i\gamma_0^{-1}$, transforming by $\chi_k : u' \mapsto \det(u)^k$, where u' is an element of K_∞ (determined by γ_0, see 7.6.3 below) mapping down to $u \in U(M^*)$. If k is half-integral, the sign of $\det(u)^k$ is determined by the pre-image u' which in turn is just the corresponding component in an Iwasawa decomposition of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $\operatorname{Mp}(\mathbb{R})$.

In the other direction, an automorphic form F transforming by χ_k under K_∞ corresponds to the classical Siegel modular form

$$f : \tau \mapsto \det(\Im(\tau)\gamma_0)^{\frac{k}{2}} F(g_\tau \times 1_f),$$

where $g_\tau = r(g_l((\Im(\tau)\gamma_0)^{\frac{1}{2}})u(\gamma_0^{-1}\Im(\tau)^{-1}\Re(\tau)))$ (r = canonical lift to Sp').

7.4. The Φ-operator and Eisenstein series

(7.4.1) For each ν (including ∞) choose a maximal compact subgroup K'_ν of $\operatorname{Sp}'(\mathbb{Q}_\nu)$. We have an **Iwasawa decomposition**:

$$\operatorname{Sp}'(\mathfrak{M}, \mathbb{Q}_\nu) = P'K'_\nu.$$

where P' is the pre-image of $P = UG_l$.

Let ξ' denote a character of $P'(\mathbb{A})$, which is trivial on $U(\mathbb{A})$ and on $P(\mathbb{Q})$, but nontrivial on the metaplectic kernel (if m is odd). If m is even, ξ comes from a character ξ of $\mathbb{A}^*/\mathbb{Q}^*$ lifted to G_l via $g_l(\alpha) \mapsto \xi(\det(\alpha))$.

Let $\mathrm{I}_R(s, \xi'_R)$ the (normalized) parabolically induced representation

$$\mathrm{I}_R(s, \xi'_R) := \mathrm{I}_{P'}^{\operatorname{Sp}'(\mathfrak{M}, R)}(|\det|^s \xi'),$$

which is the space of smooth K_R-finite functions Ψ, satisfying

$$\Psi(pg) = \xi'_R(p)|\det(\alpha(p))|^{s+(n+1)/2}\Psi(g)$$

(remember $n = \dim M$).

In the space $\mathrm{I}_{\mathbb{R}}(s, \xi_R)$, we find the following orthogonal system of functions

$$\Psi_{l,\mathbb{R}}(pk, s) = \xi'_{\mathbb{R}}(p)|\alpha(p)|^{s+(n+1)/2}\chi_l(k),$$

where $\chi_l(k)$ is the character $\det^l$ of $K_\infty \cong U'_m$. If $\mathrm{Sp}' = \mathrm{Mp}$, l has to be half-integral, and the definition then does not depend on the decomposition pk, which is unique only up to the metaplectic kernel. It is a (Hilbert-space) basis if $n = 1$.
Choose a global lattice $M_{\mathbb{Z}}$ and let K_p be the stabilizer of $M_{\mathbb{Z}_p}$. The K_p are maximal compact. Thus, for almost all p, in $\mathrm{I}_{\mathbb{Q}_p}(s, \xi_p)$ we find the vector

$$\Psi_{0,p}(pk, s) = \xi'_p(p)|\alpha(p)|^{s+(n+1)/2}\,\mathrm{sign}(k),$$

where $\mathrm{sign}(k) = \pm 1$ according to whether k lies in the canonical lift of $\mathrm{Sp}(M_{\mathbb{Z}_p})$ (7.7.1) or not.
With this defined, we have

$$\mathrm{I}_{\mathbb{A}}(s, \xi) = {\bigotimes_\nu}' \, \mathrm{I}_{\mathbb{Q}_\nu}(s, \xi_\nu),$$

where ${\bigotimes_\nu}'$ is the restricted tensor product with respect to the vectors $\Psi_{0,p}$. Note that if n is odd, this gives a representation of $\mathrm{Mp}(\mathbb{A})$, since the kernel of $\mathrm{Mp}(\mathfrak{M}_{\mathbb{Q}_\nu}) \to \mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}_\nu})$ acts in the same way.
In particular, for any R as above, the Weil representation defines the following $\mathrm{Sp}'(\mathfrak{M}_R)$-equivariant operator:

$$\begin{aligned} \Phi : S(M^* \otimes L) &\to \mathrm{I}(s_0 = \tfrac{m-n-1}{2}, \xi) \\ \varphi &\mapsto \{g \mapsto (g\varphi)(0)\} \end{aligned}$$

Here ξ is the character (up to sign, if m is odd) $\alpha \mapsto \widetilde{\Upsilon}(\gamma_0\alpha)/\widetilde{\Upsilon}(\gamma_0)$ for some γ_0.

(7.4.2) Let $\Psi(s_0) \in \mathrm{I}(s_0, \xi)$ be given ($s_0 = \frac{m-n-1}{2}$). Such a function can be extended uniquely to a 'section' parameterized by $s \in \mathbb{C}$, with the property that the restriction to K is independent of s. Using the Iwasawa composition we see that it is given by the formula

$$\Psi(u(\beta)g_l(\alpha)k, s) = |\alpha|^{s-s_0}\Psi(g_l(\alpha)k, s_0).$$

To any such 'section', there is an associated **Eisenstein series**. This association is a $\mathrm{Sp}'(\mathbb{A})$-equivariant map

$$E : \mathrm{I}_{\mathbb{A}}(s, \xi) \to \mathcal{A}(\mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}}) \backslash \mathrm{Sp}'(\mathfrak{M}_{\mathbb{A}}))$$

$$\Psi(s) \mapsto \sum_{\gamma \in P(\mathbb{Q}) \backslash \mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}})} \Psi(s)(\gamma g),$$

where $\mathcal{A}$ is the space of automorphic functions. This series converges absolutely if $\Re(s) > \frac{n+1}{2}$ and possesses a meromorphic continuation in s to all of $\mathbb{C}$. Note, that ξ is trivial on $\mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}})$.

The Eisenstein series decomposes according to the 'Bruhat' decomposition (5) as follows:

$$E(\Psi, s) = \sum_{M^{*\prime} \subset M^*} E_{M^{*\prime}}(\Psi, s),$$

with

$$E_{M^{*\prime}}(\Psi, g, s) = \sum_{\beta \in (M' \otimes M')^s_{\mathbb{Q}}} \Psi(s)(\iota(d(\gamma_0)u(\beta))g).$$

At $s = s_0$, with $m > 2n + 2$ (this assures convergence), we get:

$$E_{M^{*\prime}}(\Psi, g, s_0) = \sum_{\beta \in (M' \otimes M')^s_{\mathbb{Q}}} \left(\iota(d(\gamma_0)u(\beta))g\varphi\right)(0).$$

Using (7.2.3) and Poisson summation this yields:

$$E_{M^{*\prime}}(\Psi, g, s_0) = \sum_{Q \in \mathrm{Sym}^2(M^{*\prime})_{\mathbb{Q}}} \int_{\mathrm{I}_{\mathbb{A}}(M'^Q, L)} (g\varphi)(x^*) \,\mathrm{d}\, x^*.$$

Here we interpret $g\varphi$ by composition with the embedding $M^{*\prime} \otimes L \hookrightarrow M^* \otimes L$ as a function on $(M^{*\prime} \otimes L)_{\mathbb{A}}$. The chosen decomposition does not play any role here (and $E_{M^{*\prime}}$ is a priori independent of it). $\mathrm{d}\, x^*$ is the Tamagawa measure.

This 'is' essentially the Fourier expansion of the Eisenstein series. To see this, we calculate its Fourier coefficient with index $Q \in \mathrm{Sym}(M^*))$ explicitly (all measures are Tamagawa measures):

$$\begin{aligned}
E_Q(\Psi, g, s_0) &= \int_{(M \otimes M)^s \backslash (M \otimes M)^s_{\mathbb{A}}} \sum_{M^{*\prime} \subset M^*} E_{M^{*\prime}}(\Psi, u(\beta)g, s_0)\chi(\beta Q) \,\mathrm{d}\, \beta \\
&= \sum_{M^{*\prime} \subset M^*} \int_{(M \otimes M)^s \backslash (M \otimes M)^s_{\mathbb{A}}} E_{M'}(\Psi, u(\beta)g, s_0)\chi(\beta Q) \,\mathrm{d}\, \beta \\
&= \sum_{M^{*\prime} \subset M^*} \int_{(M' \otimes M')^s \backslash (M' \otimes M')^s_{\mathbb{A}}} \sum_{\overline{\beta} \in (M' \otimes M')^s} \Psi\left(\iota(d(\gamma_0)u(\overline{\beta}))u(\beta)g\right) \chi(\beta Q) \,\mathrm{d}\, \beta \\
&= \sum_{M^{*\prime} \subset M^*} \int_{(M' \otimes M')^s_{\mathbb{A}}} \left(\iota(d(\gamma_0))u(\beta)g\varphi\right)(0)\chi(\beta Q) \,\mathrm{d}\, \beta \\
&= \sum_{\{M^{*\prime} \subset M^* \ \mid \ Q \in \mathrm{Sym}^2(M^{*\prime})\}} \int_{\mathrm{I}_{\mathbb{A}}(M'^Q, L)} (g\varphi)(x) \,\mathrm{d}\, x
\end{aligned}$$

(observe $\widetilde{\Upsilon}_{\mathbb{A}}\gamma_0 = 1$). In the second step, we divided the integral in an integral over (classes in) $(M' \otimes M')^s_{\mathbb{A}}$, $(M' \otimes M'')_{\mathbb{A}}$ and $(M'' \otimes M'')^s_{\mathbb{A}}$. Then use

$$E_{M^{*\prime}}(\Psi, u(\beta_1)u(\beta_2)u(\beta_3)g, s_0) = E_{M^{*\prime}}(\Psi, u(\beta_1)g, s_0)$$

for this decomposition and finally (e.g.)

$$\int_{(M''\otimes M'')^s\backslash(M''\otimes M'')^s_{\mathbb{A}}} \chi(\beta_3, Q)\,\mathrm{d}\,\beta_3$$

is 0 if Q has any nonzero projection to $\mathrm{Sym}^2(M^{*\prime\prime})$ and is 1 otherwise. In the last step, we used (7.2.3).

It is therefore convenient to make the following

(7.4.3) Definition. *The **Whittaker integral** is defined as*

$$W_{\nu,Q,M^{*\prime}}(\Psi_\nu, g_\nu) := \int_{(M'\otimes M')^s_{\mathbb{Q}_\nu}} \Psi_\nu(d(\gamma_0)u(\beta)g_\nu)\chi(\beta Q)\mu_{\gamma_0}(\beta_\nu),$$

where we now chose the canonical measure μ_{γ_0} with respect to some fixed γ_0 on M and hence M' for convenience.

Hence we have

$$E_Q(\Psi, g; s) = \sum_{\{M^{*\prime}\subset M^* \ \mid\ Q\in\mathrm{Sym}^2(M^{*\prime})\}} \prod_\nu W_{\nu,Q,M^{*\prime}}(\Psi_\nu(s), g_\nu)$$

and again by (7.2.3):

$$W_{\nu,Q,M^{*\prime}}(\Phi_\nu(\varphi_\nu; s_0), g) = \widetilde{\Upsilon}_\nu(\gamma_0)\int_{\mathrm{I}_{\mathbb{Q}_\nu}(M'^Q, L)} (g\varphi_\nu)(x_\nu^*)\mu_{Q_L,\gamma_0}(x_\nu^*),$$

where μ_{Q_L,γ_0} is the measure on $\mathrm{I}_{\mathbb{Q}_\nu}(M'^Q, L)$ induced by the canonical ones on $\mathrm{Sym}^2(M^*)$ and $M^*\otimes L$ with respect to γ_0 and Q_L via (6.2.5).

(7.4.4) Lemma. *The Whittaker integrals (non-degenerate case) satisfy the following general transformation laws:*

i. $W_{Q,\nu}(\Psi_\nu(s), u(\beta')g) = \chi(-\beta' Q)W_{Q,\nu}(\Psi_\nu(s), g)$

ii. $W_{Q,\nu}(\Psi_\nu(s), g_l(\alpha)g) = |\alpha|^{-s+\frac{n+1}{2}} W_{\alpha^! Q,\nu}(\Psi_\nu(s), g)$

Proof. (i) Substitute $\beta'' = \beta + \beta'$ in the definition of Whittaker integral. (ii) Substitute $\beta'' = \alpha^{-1}\beta^t\alpha^{-1}$ and use $d(\gamma_0)u(\beta)g_l(\alpha) = g_l(\gamma_0^{-1t}\alpha^{-1}\gamma_0)d(\gamma_0)u(\alpha^{-1}\beta^t\alpha^{-1})$. □

7.5. Theta series and the Siegel-Weil formula

(7.5.1) Assume $m > 2n+2$ again. To any given $\varphi \in S(M^* \otimes L)_{\mathbb{A}}$ one may define an associated **theta function**

$$\Theta(\varphi; g) = \sum_{x^* \in M^*_{\mathbb{Q}} \otimes L_{\mathbb{Q}}} (g\varphi)(x^*).$$

This association is a $\mathrm{SO}(L_{\mathbb{A}}) \times \mathrm{Mp}(\mathfrak{M}_{\mathbb{A}})$-equivariant map[2]

$$S(M^* \otimes L)_{\mathbb{A}} \to \mathcal{A}(\mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathrm{SO}(L_{\mathbb{A}}) \times \mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}}) \backslash \mathrm{Sp}'(\mathfrak{M}_{\mathbb{A}})).$$

The theta function decomposes as follows:

$$\begin{aligned}
\Theta(\varphi; g) &= \sum_{Q \in \mathrm{Sym}^2(M^*)} \Theta_Q(\varphi, g) \\
\Theta_Q(\varphi; g) &= \sum_{x^* \in \mathrm{I}_{\mathbb{Q}}(M^Q, L)} (g\varphi)(x^*) = \sum_{\{M^{*\prime} \subset M^* \ \mid \ Q \in \mathrm{Sym}^2(M^{*\prime})\}} \Theta^1_{Q,M'}(\varphi, g) \\
\Theta^1_{Q,M^{*\prime}}(\varphi; g) &= \sum_{x^* \in \mathrm{I}^1_{\mathbb{Q}}(M', L)} (g\varphi)(x^*)
\end{aligned}$$

Because $\mathrm{I}^1_{\mathbb{Q}}(M', L)$ consists of one orbit under $\mathrm{SO}(L_{\mathbb{Q}})$, we get

$$\int_{\mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathrm{SO}(L_{\mathbb{A}})} \Theta^1_Q(\varphi; gh) \,\mathrm{d}\, h = \tau(\mathrm{Stab}(\alpha)) \int_{\mathrm{I}_{\mathbb{A}}(M'^Q, L)} (g\varphi)(x^*) \,\mathrm{d}\, x^*,$$

where $\tau(\cdots)$ denotes the Tamagawa number and $\mathrm{d}\, h$, resp. $\mathrm{d}\, x^*$ are the Tamagawa measures.
In [94], Weil proves the **Siegel-Weil formula**:

(7.5.2) Theorem.

$$E(\Phi(\varphi); g) = \tau(\mathrm{SO}(L_{\mathbb{Q}})) \int_{\mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathrm{SO}(L_{\mathbb{A}})} \Theta(\varphi, gh) \,\mathrm{d}\, h.$$

This can be extended in various ways beyond the range $m > 2n+2$ (cf. e.g. [64]).
According to the above, comparing Fourier coefficients, one obtains in particular

$$\tau(\mathrm{SO}(\alpha^\perp)) = \tau(\mathrm{SO}(L))$$

for any $\alpha \in \mathrm{I}_{\mathbb{Q}}(M^Q, L)$ for Q non-degenerate, which may be used to prove Weil's theorem that $\tau(\mathrm{SO}(L)) = 2$ for all orthogonal groups with $\dim(L) \geq 3$.

[2] equivariance under $\mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}})$ is essentially Poisson summation for $M^*_{\mathbb{Q}} \otimes L_{\mathbb{Q}}$

7.6. The Weil representation over $\mathbb{R}$

(7.6.1) Much about the Weil representation over $\mathbb{R}$ can be found in [7] or [73]. Choose a root i of -1, any real parameter $h \in \mathbb{R}$ and the character $x \mapsto \exp(2\pi i \frac{x}{h})$ of $\mathbb{R}$. We may identify the Lie algebra $\mathrm{Lie}(H)$ with $R \oplus M \oplus M^*$ acting as follows

$$\begin{aligned}[r]f(x^*) &= r\frac{h}{2\pi i}f(x^*)\\ [m^*]f(x^*) &= \frac{h}{2\pi i}\partial_{m^*}f(x^*)\\ [m]f(x^*) &= \langle x^*, m\rangle f(x^*)\end{aligned}$$

This defines also $[x]$ for $x \in \mathfrak{M}_{\mathbb{R}}$, with the Heisenberg commutation relation

$$[[x_1],[x_2]] = [\langle x_1, x_2\rangle] = \frac{h}{2\pi i}\langle x_1, x_2\rangle.$$

Lie(Sp) may be identified with $(\mathfrak{M}\otimes\mathfrak{M})^s$ operating via contraction with the symplectic form. Here an element $x_1 \otimes x_2$ in $\mathfrak{M}$ operates by

$$\frac{1}{2}[x_1][x_2]$$

on $S(\mathbb{R})$. This can be seen from the commutation rule

$$\left[\frac{1}{2}[x_1][x_2],[x]\right] = \frac{1}{2}([\langle x_1, x\rangle][x_2] + [\langle x_2, x\rangle][x_1]).$$

Embed Lie(Sp) in $\mathrm{End}(\mathfrak{M})$. $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ lies in Lie(Sp) if and only if $\delta = -{}^t\alpha$ and γ and β are symmetric. It is identified with the symmetric element $\alpha + \beta - \gamma - \delta$ of $(\mathfrak{M}\otimes\mathfrak{M})^s$.

(7.6.2) We have the simple formula

$$\Upsilon(Q) = \exp(2\pi i \tfrac{p-q}{8}),$$

where (p, q) is the signature of Q.

(7.6.3) Let γ_0 be a symmetric and positive definite form on M. It defines an isomorphism:

$$\begin{aligned} M_{\mathbb{C}}^* &\to \mathfrak{M}\\ w_1 + iw_2 &\mapsto w_1 - {}^t\gamma_0^{-1}w_2 = w_1 - \gamma_0^{-1}w_2\end{aligned}$$

and a corresponding map

$$\begin{aligned} k : \mathrm{End}(M_{\mathbb{C}}^*) &\to \mathrm{End}(\mathfrak{M}_{\mathbb{R}}) \\ \alpha_1 + i\alpha_2 &\mapsto \begin{pmatrix} \gamma_0^{-1}\alpha_1\gamma_0 & -\gamma_0^{-1}\alpha_2 \\ \alpha_2\gamma_0 & \alpha_1 \end{pmatrix}. \end{aligned}$$

This identifies the unitary group of the hermitian form given by γ_0 on $M_{\mathbb{C}}$ with the stabilizer of $d(\gamma_0)$ in $\mathrm{Sp}(\mathfrak{M})$, which is a maximal compact subgroup.

(7.6.4) Let K_∞ be the stabilizer of $d(\gamma_0)$ for some symmetric and positive definite $\gamma_0 \in M \otimes M$ as above (image of $\mathrm{End}(M_{\mathbb{C}}^*)$).
Let $\mathrm{Lie}(\mathrm{Sp}(\mathfrak{M}_{\mathbb{R}})) = \mathfrak{p}^+ \oplus \mathrm{Lie}(K_\infty) \oplus \mathfrak{p}^-$ the decomposition induced by the complex structure on $\mathbb{D}_{\mathbf{H}(\mathfrak{M})}$. These spaces are generated by the following elements:

$$\mathrm{Lie}(K_\infty) = \{\begin{pmatrix} \gamma_0\alpha_1\gamma_0^{-1} & -\gamma_0\alpha_2 \\ \beta_0^{-1}\alpha_2 & \alpha_1 \end{pmatrix}\} \tag{7}$$

$$\mathfrak{p}^+ = \{\begin{pmatrix} \gamma_0\alpha\gamma_0^{-1} & -i\gamma_0\alpha \\ -i\alpha\gamma_0^{-1} & -\alpha \end{pmatrix}\} \tag{8}$$

$$\mathfrak{p}^- = \{\begin{pmatrix} \gamma_0\alpha\gamma_0^{-1} & i\gamma_0\alpha \\ i\alpha\gamma_0^{-1} & -\alpha \end{pmatrix}\}\} \tag{9}$$

where α and α_2 are symmetric with respect to γ_0 and α_1 is skew-symmetric with respect to γ_0.

Proof. These are the eigen-spaces for $\mathrm{ad}(\begin{pmatrix} 0 & -\gamma_0^{-1} \\ \gamma_0 & 0 \end{pmatrix})$ of eigenvalue $(-2i, 0, 2i)$ respectively. □

An element $i\alpha_2 \in \mathrm{Lie}(U)$ with α_2 symmetric with respect to γ_0 acts on $S(M^*)$ by the 'Hamiltonian':

$$[\gamma_0\alpha_2] + [\alpha_2\gamma_0^{-1}].$$

(7.6.5) Assume that we are given a positive definite space $L_{\mathbb{R}}$ as well and are in the situation of (7.2.1).
There is one natural special element in $S(M_{\mathbb{R}}^* \otimes L_{\mathbb{R}})$, which corresponds to the 'lowest energy level' with respect to the Hamiltonian $k(i)$, the **Gaussian**:

$$\varphi_\infty := e^{-2\pi((x^*)^! Q_L)\cdot\gamma_0^{-1}}.$$

We have

$$u(\alpha)\varphi_\infty = \tfrac{\mathrm{tr}(\alpha)}{2}\varphi_\infty,$$

where $\alpha = \alpha_1 + \alpha_2 i$, where α_1 is skew-symmetric and α_2 is symmetric with respect to γ_0. (hence $\operatorname{tr}(\alpha) = i \operatorname{tr} \alpha_2$)
Integrated, we get

$$\exp(u(\alpha))\varphi_\infty = \det(\exp(\alpha))^{\frac{1}{2}}\varphi_\infty.$$

Since the unitary group is generated by such exponentials, this determines the action.
From the commutation relation

$$[\begin{pmatrix} 0 & -\gamma_0^{-1} \\ \gamma_0 & 0 \end{pmatrix}, p] = -2ip,$$

for any $p \in \mathfrak{p}^-$ and the fact, that φ_∞ is of lowest possible energy, we get $\mathfrak{p}^-\varphi_\infty = 0$. From this, it follows that the associated theta series is holomorphic (see 7.5).

(7.6.6) Recall the Φ-Operator from (7.4.2). From the discussion in (7.6.5) follows

$$\Phi(\varphi_\infty) = \Psi_{\frac{m}{2}}(\cdot, \tfrac{m-n-1}{2}).$$

If Q_L has signature $(m-2, 2)$ to each negative definite subspace N , we can consider the element $\alpha_N \in \operatorname{End}(M)$ operating as -1 on N and as +1 on the orthogonal complement (note: $\operatorname{tr}(N) = m - 4$). We form the Gaussian

$$\varphi_\infty^0(N) := e^{-2\pi (x^*)^! Q_L \cdot (\alpha_N \gamma_0^{-1})},$$

which is now in the Schwartz space. We have obviously by the above:

$$H(\gamma_0)\varphi_\infty^0(N) = \frac{\operatorname{tr}(N^{-1})}{2} i \varphi_\infty^0(N) = (\frac{m}{2} - 2) i \varphi_\infty^0(N).$$

Hence:

$$\Phi(\varphi_\infty^0(N)) = \Psi_{\frac{m}{2}-2}(\cdot, \tfrac{m-n-1}{2})$$

in this case.
φ_∞^0 may be considered as element in $(S(M^* \otimes L)_\mathbb{R} \otimes \mathcal{C}(\mathbb{D}_\mathbf{O}))^{\mathrm{SO}(L_\mathbb{R})}$ with the notation $\mathbb{D}_\mathbf{O}$ of (10.2.1).
In addition we have

(7.6.7) Theorem. *There exist forms*

$$\widetilde{\varphi}_{KM} := \nabla \varphi_\infty^0$$

in $(S(M^* \otimes L)_\mathbb{R} \otimes \mathcal{E}^{n,n}(\mathbb{D}_\mathbf{O}))^{\mathrm{SO}(L_\mathbb{R})}$, *where* ∇ *is a Howe operator [59], having the following properties*

i. $k\varphi_{KM} = \chi_{m/2}(k)\varphi_{KM}$ *for* $k \in K_\infty$.

ii. $\mathrm{d}\,\varphi_{KM}(x^*) = 0$.

iii. There is a current $\xi \in (S(M^* \otimes L)_{\mathbb{R}} \otimes \mathcal{D}^{n-1,n-1})^{\mathrm{SO}(L_{\mathbb{R}})}$ *with*

$$\mathrm{d}\,\mathrm{d}^c\,\xi(x^*) + \exp(-2\pi (x^*)^! Q_L \cdot \gamma_0^{-1})\delta_{\mathbb{D}_{x^*}} = [\varphi_{KM}(x^*)].$$

Here $\delta_{\mathbb{D}_{x^*}}$ *is the current of integration along* $\mathbb{D}_{x^*} = \{N \in \mathbb{D}_{\mathbf{O}(L)} \mid \langle x^*(M), N\rangle = 0\}$.

iv. $\varphi_{KM}(0) = \mathrm{c}_1(\Xi^* \overline{\mathcal{E}})^n$ *(10.4.1).*

Proof. [59, 60]. See also [54]. □

Now define $\varphi_\infty^+(x^*)$ by

$$\varphi_\infty^+(x^*)\,\mathrm{c}_1(\Xi^*\overline{\mathcal{E}})^{m-2-n} = \varphi_{KM}(x^*)\,\mathrm{c}_1(\Xi^*\overline{\mathcal{E}})^{m-2}.$$

From properties i. and iv. of (7.6.7) follows immediately:

$$\Phi(\varphi_\infty^+(N)) = \Psi_{m/2}(\cdot, \tfrac{m-n-1}{2}) \qquad \forall N \in \mathbb{D}_{\mathbf{O}(L)}.$$

(7.6.8) Theorem. *Assume* $m > 2n+2$ *and* Q *non-degenerate. Let* $\varphi = \varphi_\infty^+ \otimes \varphi_f$ *for a* $\varphi_f \in S(L_{\mathbb{A}^{(\infty)}})$. *Then we have for* $Q \in \mathrm{Sym}^2(M)$*:*

$$\int_{\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathrm{SO}(L_{\mathbb{A}})} \Theta_Q(\varphi(N))(g)\,\mathrm{d}\,g =$$
$$\tau(\mathrm{SO}(L_{\mathbb{Q}}))\exp(-2\pi Q\cdot\gamma_0^{-1})\frac{\int_{\mathrm{Z}(L,M^Q,\varphi_f;K)} \mathrm{c}_1(\Xi^*\overline{\mathcal{E}})^{m-2-n}}{\int_{[\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathrm{SO}(L_{\mathbb{A}})/K_\infty K]} \mathrm{c}_1(\Xi^*\overline{\mathcal{E}})^{m-2}},$$

with K *such that* $\varphi \in S((M^*\otimes L)_{\mathbb{A}^{(\infty)}})^K$. *Here* $\mathrm{Z}(L, M^Q, \varphi_f; K)$ *is the* special cycle *associated with* Q *and* φ_f, *defined in (10.3) in part III.*

Proof. We give a formal sketch only, compare e.g. [57, 4.17, 4.20, 4.21] for details about convergence.
Decompose $\mathrm{I}_{\mathbb{A}^{(\infty)}}(M^\gamma, L) \cap \mathrm{supp}(\varphi_f) = \bigcup_i K x_i$ as in (10.3.2) with $x_i \in M_{\mathbb{Q}}^* \otimes L_{\mathbb{Q}}$, $x_i = g_i x_0$.

$$\begin{aligned}&\int_{\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathrm{SO}(L_{\mathbb{A}})} \Theta_Q(\varphi(N))(g)\,\mathrm{d}\,g\\ =&\int_{\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathrm{SO}(L_{\mathbb{A}})} \sum_{x^*\in \mathrm{I}(M^Q,L)_{\mathbb{Q}}} (\varphi_\infty^+(g_\infty^{-1}x^*, N)\varphi_f(g_f^{-1}x))\,\mathrm{d}\,g\end{aligned}$$

$$= \sum_{g_f \in \mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K} \sum_i \varphi_f(g_f^{-1} x_i^*) \operatorname{vol}(K) \cdot$$

$$\cdot \int_{\mathrm{SO}(L_{\mathbb{Q}}) \cap K^{g_f} \backslash \mathrm{SO}(\mathbb{R})} \sum_{x^* \in \mathrm{I}(M^Q, L)_{\mathbb{Q}} \cap K^{g_f} x_i^*} \varphi_\infty^+(g_\infty^{-1} x_i^*, N) \,\mathrm{d}\, g_\infty$$

$$= \sum_{g_f \in \mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K} \sum_i \varphi_f(g_f^{-1} x_i^*) \operatorname{vol}(K) \cdot$$

$$\cdot \int_{\mathrm{SO}(L_{\mathbb{Q}}) \cap K^{g_f} \backslash \mathbb{D}_{\mathbf{O}}} \sum_{x^* \in \mathrm{I}(M^Q, L)_{\mathbb{Q}} \cap K^{g_f} x_i^*} \varphi_\infty^+(x_i^*, g_\infty N) \,\mathrm{d}\, g_\infty'$$

$$= \operatorname{vol}(K) \Lambda \sum_{g_f \in \mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K} \sum_i \varphi_f(g_f^{-1} x_i^*) \cdot$$

$$\cdot \int_{\mathrm{SO}(L_{\mathbb{Q}}) \cap K^{g_f} \backslash \mathbb{D}_{\mathbf{O}}} \sum_{x \in \mathrm{I}(M^Q, L)_{\mathbb{Q}} \cap K^{g_f} x_i^*} \varphi_\infty^+(x_i^*, g_\infty N) \, \mathrm{c}_1(\Xi^* \overline{\mathcal{E}})^{m-2}.$$

By property (iii) of (7.6.7), the integral is equal to:

$$\exp(-2\pi Q_L \cdot \gamma_0^{-1}) \sum_{x^* \in (\mathrm{SO}(L_{\mathbb{Q}}) \cap K^{g_f}) \backslash \mathrm{I}(M^Q, L)_{\mathbb{Q}} \cap K^{g_f} x_i^*} \int_{\mathrm{SO}(L_{\mathbb{Q}}) \cap K^{g_f} \backslash \mathbb{D}_x} \mathrm{c}_1(\Xi^* \overline{\mathcal{E}})^{m-2-n},$$

hence the result. Observe the relation:

$$\tau(\mathrm{SO}(L_{\mathbb{Q}})) = \operatorname{vol}(K) \Lambda \int_{\mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathrm{SO}(L_{\mathbb{A}})/K_\infty K} \mathrm{c}_1(\Xi^* \overline{\mathcal{E}})^{m-2},$$

where the volume $\operatorname{vol}(K)$ is computed with respect to the finite part of the chosen Tamagawa measure and Λ is the comparison factor between the infinite part $\mathrm{d}\, g_\infty$ of the Tamagawa measure and $\mathrm{c}_1(\Xi^* \overline{\mathcal{E}})^{m-2}$ product with the measure on $K_\infty \subset \mathrm{SO}(\mathbb{R})$, the stabilizer of N, giving it volume 1. $\square$

(7.6.9) Combining (7.6.8) with the Siegel-Weil formula (7.5.2), we get the Fourier coefficient of the classical holomorphic Eisenstein series:

$$\det(\Im(\tau)\gamma_0)^{\frac{m}{2}} E_Q(\Phi(\varphi), g_\tau g_f) = \exp(2\pi i \tau \cdot Q) \frac{\int_{\mathrm{Z}(M^Q, g_f \varphi_f, K)} \mathrm{c}_1(\Xi^* \overline{\mathcal{E}})^{m-2-n}}{\int_{\mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathrm{SO}(L_{\mathbb{A}})/K} \mathrm{c}_1(\Xi^* \overline{\mathcal{E}})^{m-2}}$$

for g_τ as in (7.3.2). The whole expression does not depend on γ_0 anymore. It can be used to inductively compute the geometric volumes of the Shimura varieties

$$\mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathrm{SO}(L_{\mathbb{A}})/K.$$

Another calculation by direct derivation from the Tamagawa number, which in turn confirms the formula above, is given in (10.5.2).

Here $\Phi(\varphi)$ depends only slightly on L. One may find a positive definite space L' and a

$\varphi'\varphi_\infty \otimes \varphi_f \in S(L'_{\mathbb{A}(\infty)})$, where φ_∞ is the Gaussian, such that $\Phi(\varphi) = \Phi(\varphi')$. The series above then is the usual theta function associated with that space and φ'_f. Kudla denotes such spaces 'matching'.

(7.6.10) We would like to understand the whole Whittaker integral $W_{Q,\infty,M}(\Psi(s), 1)$ of the Eisenstein series as a function of s. Assume, that Q is non-degenerate on M.

(7.6.11) Definition. *We define the following functions*

$$\Gamma_n(s) = \pi^{\frac{n(n-1)}{4}} \prod_{k=0}^{n-1} \Gamma(s - \frac{k}{2}). \tag{10}$$

This is standard notation for the higher dimensional gamma function (compare [87]). Define in addition:

$$\Gamma_{n,m}(s) = 2^n \frac{\pi^{\frac{n}{2}(s+m)}}{\Gamma_n(\frac{1}{2}(s+m))}. \tag{11}$$

Another, more meaningful form of expression (11) is

$$\prod_{k=m-n+1}^{m} 2\frac{\pi^{\frac{1}{2}(s+k)}}{\Gamma(\frac{1}{2}(s+k))}.$$

The Iwasawa decomposition of the argument of $\Psi_{\infty,l}$ in the Whittaker integral can be expressed as (using the notation of 7.6.3)

$$d(\gamma_0)u(\beta) = u(\Delta^2\beta)g_l(\Delta)k(\gamma_0\Delta\beta + i\gamma_0\Delta\gamma_0^{-1}),$$

where $\Delta = (\sqrt{1 + (\beta\gamma_0)^2})^{-1}$. It satisfies ${}^t\Delta = \gamma_0\Delta\gamma_0^{-1}$.
Hence we get:

$$W_{Q,\infty}(\Psi_\infty(s), 1) = \int_{(M\otimes M)^s_{\mathbb{R}}} |\Delta|^{s+\frac{n+1}{2}} \chi_l(\gamma_0\Delta\beta + i^t\Delta)\chi(Q\cdot\beta)\mu_{\gamma_0}(\beta)$$

and after choosing an orthonormal basis for γ_0:

$$W_{Q,\infty}(\Psi_\infty(s), 1) = \int_{(\mathbb{R}^n\otimes\mathbb{R}^n)^s} \det(X+i)^{-a} \det(X-i)^{-b} e^{-2\pi i \operatorname{tr}(\frac{1}{2}X\gamma)} \, \mathrm{d}\, X,$$

where γ is the bilinear form associated with Q (expressed in the chosen basis in the above formula).
with $a = \frac{1}{2}(s+\frac{n+1}{2}+l)$ and $b = \frac{1}{2}(s+\frac{n+1}{2}-l)$. Here $\det(X+i)^{-a} = e^{-a(\frac{n}{2}\pi i + \log(\det(1-iX)))}$ and $\det(X-i)^{-b} = e^{-b(-\frac{n}{2}\pi i + \log(\det(1+iX)))}$, where log is the main branch of logarithm. $\mathrm{d}\, X$ is the measure defined in (6.2.3) for the standard basis, without the determinant

factor. It is the same as used in [87].

Shimura denotes this function in analogy to the one dimensional case a **confluent hypergeometric function**. Furthermore [87, 1.29, 3.1K, 3.3], if Q is positive definite, the RHS equals

$$e^{-\pi \operatorname{tr} \gamma + i\pi \frac{n}{2}(b-a)} \pi^{na+nb} 2^n \det(\gamma)^{a+b-\frac{n+1}{2}} \Gamma_n(a)^{-1} \Gamma_n(b)^{-1} \zeta(2\pi\gamma; a, b),$$

where

$$\zeta(Z; a, b) = \int_{X>0} e^{-\operatorname{tr} ZX} \det(X+1)^{a-\frac{n+1}{2}} \det(X)^{b-\frac{n+1}{2}} \,\mathrm{d}\, X.$$

(7.6.12) In the 1 dimensional case, this gives

$$\zeta(Z, a, b) = \Gamma(b) U(b, a+b, Z),$$

where $U(k, l, Z)$ is a solution of the classical hypergeometric differential equation

$$Zf''(Z) + (l - Z)f'(Z) - kf(Z) = 0,$$

see [1, §13]. We have: $U(b, a+b, Z) = Z^{-b}(1 + O(|Z|^{-1}))$ [loc. cit.].

Therefore the 'value at ∞' for $l = \frac{m}{2}$ is computed as (this will later allow to use the computation of [57]):

$$\begin{aligned}
&\lim_{\alpha\to\infty} |\alpha|^{-\frac{m}{2}} e^{2\pi\alpha^2 Q} W_{Q,\infty}(\Psi_\infty(s), g_l(\alpha)) \\
&= \lim_{\alpha\to\infty} |\alpha|^{-s+1-\frac{m}{2}} e^{\pi\alpha^2\gamma} W_{\alpha^! Q,\infty}(\Psi_\infty(s), 1) \\
&= \lim_{\alpha\to\infty} |\alpha|^{-s+1-\frac{m}{2}} e^{-i\pi\frac{m}{4}} \pi^{s+1} 2|\alpha^2\gamma|^s \Gamma_1(a)^{-1}\Gamma_1(b)^{-1}\zeta(2\pi\alpha^2\gamma, a, b) \\
&= e^{-i\pi\frac{m}{4}} \pi^{s+1} 2|\gamma|^{\frac{1}{2}(s-1+\frac{m}{2})} \Gamma_1(\frac{1}{2}(s+1+\frac{m}{2}))^{-1} (2\pi)^{-\frac{1}{2}(s+1-\frac{m}{2})} \\
&= \widetilde{\Upsilon}_\infty(\gamma_0)\Gamma_{1,m}(s-s_0)|\gamma|^{\frac{1}{2}(s+s_0)} 2^{-\frac{1}{2}(s-s_0)},
\end{aligned}$$

where $s_0 = \frac{m}{2} - 1$, cf. also [13, Prop. 3.1] — there we have $\kappa_{BK} = \frac{m}{2}$ and $s_{BK} = \frac{1}{2}(s + 1 - \frac{m}{2})$.

(7.6.13) In the higher dimensional case, but for $s = \frac{m-n-1}{2}$ and $l = \frac{m}{2}$, we have $a = \frac{1}{2}m$

and $b=0$ (holomorphic Eisenstein series) and we get for $m>2n$:

$$\begin{aligned}
&\det(\alpha)^{-\frac{m}{2}} e^{\pi\operatorname{tr}({}^t\alpha\gamma\alpha)} W_{Q,\infty}(\Psi_\infty(s), g_l(\alpha)) = \\
&\det(\alpha)^{-\frac{m}{2}} e^{\pi\operatorname{tr}({}^t\alpha\gamma\alpha)} \det(\alpha)^{\frac{m}{2}} \int_{(\mathbb{R}^n\otimes\mathbb{R}^n)^s} \det(X+\alpha^t\alpha i)^{-\frac{1}{2}m} e^{-\pi i\operatorname{tr}(X\gamma)}\,\mathrm{d}\,X \\
&= e^{\pi\operatorname{tr}({}^t\alpha\gamma\alpha)} e^{-i\pi\frac{nm}{4}} (2\pi)^{\frac{1}{2}nm} \int_{(\mathbb{R}^n\otimes\mathbb{R}^n)^s} \det(2\pi iX+2\pi\alpha^t\alpha)^{-\frac{1}{2}m} e^{\pi i\operatorname{tr}(X\gamma)}\,\mathrm{d}\,X \\
&= \begin{cases} e^{-i\pi\frac{nm}{4}}\Gamma_{n,m}(0)\det(\gamma)^{\frac{m-n-1}{2}} & \gamma>0, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

using [87, p. 174, (1.23)].
In particular, given a positive definite form Q_L on $L_\mathbb{R}$, we have:

$$W_{Q,\infty}(\Psi_\infty(s),1) = e^{-2\pi Q^t\gamma_0^{-1}}\widetilde{\Upsilon}^{-1}(\gamma_0)\Gamma_{n,m}(0)|\gamma|^{\frac{m-n-1}{2}}.$$

This is expected from (7.2.3), since we have by (6.2.9):

$$\int_{\mathrm{I}(M^Q,L)_\mathbb{R}} \varphi_\infty(\alpha)\mu_{Q_L,\gamma_0}(\alpha) = |\gamma|^{\frac{m-n-1}{2}} e^{-2\pi Q\cdot\gamma_0^{-1}}\Gamma_{n,m}(0),$$

where φ_∞ is the Gaussian (7.6.5). Observe, that the integrand is constant and equal to $e^{-2\pi Q^t\gamma_0^{-1}}$. Here $|\gamma|$ is computed by means of the canonical measure μ_{γ_0}.

7.7. The Weil representation over p-adic fields

(7.7.1) For $R=\mathbb{Q}_p$, the factor $\Upsilon(Q)$ is 1 if there exists a unimodular lattice in $L_{\mathbb{Q}_p}$. In particular it is 1 for hyperbolic planes.
Otherwise, it can be computed by the classical theory of quadratic Gauss sums. For example for $p\neq 2$, we have

$$\Upsilon(paz^2) = \frac{1}{\sqrt{p}}\sum_{x\in\mathbb{Z}/p\mathbb{Z}}\exp(2\pi iax^2) = \begin{cases} \left(\frac{a}{p}\right) & p\equiv 1\ (4), \\ i\left(\frac{a}{p}\right) & p\equiv 3\ (4). \end{cases}$$

In particular, if $p\neq 2$ and Q has a unimodular lattice $L_{\mathbb{Z}_p}$, r'' determines a canonical lift $\mathrm{Sp}(\mathfrak{M}_{\mathbb{Z}_p})\to\mathrm{Sp}'(\mathfrak{M}_{\mathbb{Q}_p})$ as well, see also [93, §20].

(7.7.2) Choose lattices $L_{\mathbb{Z}_p}$ and $M_{\mathbb{Z}_p}$. Assume $Q_L\in\mathrm{Sym}^2(L^*_{\mathbb{Z}_p})$. Consider the characteristic functions χ_κ of the cosets $\kappa\in(L^*_{\mathbb{Z}_p}/L_{\mathbb{Z}_p})\otimes M^*_{\mathbb{Z}_p}$. This defines an embedding of $\mathbb{C}[(L^*_{\mathbb{Z}_p}/L_{\mathbb{Z}_p})\otimes M^*_{\mathbb{Z}_p}]$ into $S((M^*\otimes L)_{\mathbb{A}^{(\infty)}})$.

(7.7.3) Lemma. *$\mathrm{Sp}'(\mathfrak{M}_{\mathbb{Z}_p})$ and $\mathrm{SO}(\mathfrak{M}_{\mathbb{Z}_p})$ induce an action on $\mathbb{C}[(L^*_{\mathbb{Z}_p}/L_{\mathbb{Z}_p})\otimes M^*_{\mathbb{Z}_p}]$. Here $\mathrm{Sp}'(\mathfrak{M}_{\mathbb{Z}_p})$ is the pre-image of $\mathrm{Sp}(\mathfrak{M}_{\mathbb{Z}_p})$ in $\mathrm{Sp}'(\mathfrak{M}_{\mathbb{Q}_p})$.*

In the induced action $K(p^n) \subset \mathrm{Sp}(\mathfrak{M}_{\mathbb{Z}_p})$, lifted by the canonical lift (7.7.1), and $\mathrm{SO}'(L_{\mathbb{Z}_p})$ (discriminant kernel) act trivial on $\mathbb{C}[(L^*_{\mathbb{Z}_p}/L_{\mathbb{Z}_p}) \otimes M^*_{\mathbb{Z}_p}]$. Here $-n$ is the minimal valuation of Q on $L^*_{\mathbb{Z}_p}$.

(7.7.4) If a global lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Q}}$ is given, with $Q_L \in \mathrm{Sym}^2(L^*_{\mathbb{Z}})$, we get an action of $\mathrm{Sp}'(\mathfrak{M}_{\widehat{\mathbb{Z}}}) \subset \mathrm{Sp}'(\mathfrak{M}_{\mathbb{A}^{(\infty)}})$ on $\mathbb{C}[(L^*_{\mathbb{Z}}/L_{\mathbb{Z}}) \otimes M^*_{\mathbb{Z}_p}]$ and therefore also an action of $\mathrm{Sp}'(\mathfrak{M}_{\mathbb{Z}})$. if m is odd, this group itself is a non-trival double cover of $\mathrm{Sp}(\mathfrak{M}_{\mathbb{Z}})$. It can equivalently be defined as the pre-image of $\mathrm{Sp}(\mathfrak{M}_{\mathbb{Z}})$ in $\mathrm{Sp}'(\mathfrak{M}_{\mathbb{R}})$.

If $M = \mathbb{Q}$, it is generated by 2 elements S and T, satisfying $Z = S^2 = (ST)^3$ and ($Z^4 = 1$ or $Z^2 = 1$ according to whether m is odd or even), mapping down to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, respectively in $\mathrm{Sp}_2(\mathfrak{M}_{\mathbb{Z}})$, and operating (via the lift r'' — cf. 7.2.1) explicitly as:

$$\begin{aligned} T\chi_\kappa &= \exp(2\pi i Q_L(\kappa))\chi_\kappa, \\ S\chi_\kappa &= \frac{\Upsilon_\infty(Q_L)^{-1}}{\sqrt{D}} \sum_{\delta \in L^*_{\mathbb{Z}}/L_{\mathbb{Z}}} \exp(-2\pi i \langle \delta, \kappa \rangle \chi_\delta). \end{aligned}$$

Note that the occurring correction factor $\widetilde{\Upsilon}_f(1) = \Upsilon_f(Q_L)$ is the product over the local $\Upsilon_p(Q_L)$'s which is, however, the same as $\Upsilon_\infty(Q_L)^{-1}$ and we have $\Upsilon_\infty(Q_L) = \exp(2\pi i(p-q))$, if the signature of Q is (p,q).

We will denote the image of $\mathbb{C}[L^*_{\mathbb{Z}}/L_{\mathbb{Z}}]$ in $S(L_{\mathbb{A}^{(\infty)}})$ sometimes as $\mathrm{Weil}(L^*_{\mathbb{Z}}/L_{\mathbb{Z}})$ and understand it as a representation of either $\mathrm{Sp}'_2(\mathbb{Z})$, $\mathrm{Sp}'_2(\widehat{\mathbb{Z}})$ or a suitable $\mathrm{Sp}'_2(\mathbb{Z}/N\mathbb{Z})$. We will also remember its canonical $\mathbb{Z}$-structure coming from $\mathbb{Z}[L^*_{\mathbb{Z}}/L_{\mathbb{Z}}]$.

(7.7.5) Define $L^{(r)} := L \oplus H^r$ as in section (6.4). The Weil representation on $S(L^{(r)} \otimes M^*)$ is the tensor product of the respective Weil representation on $S(L \otimes M^*)$ and $S(H^{(r)} \otimes M^*)$. We defined $\varphi^{(r)}$ by tensoring φ with the characteristic function $\chi^{(r)}$ of $H^r_{\mathbb{Z}_p} \otimes M^*_{\mathbb{Z}_p}$ (depends on the choice of $M_{\mathbb{Z}_p}$). We have

$$(w_{H^r}(u(B)g_l(\alpha)k)\chi^{(r)})(0) = |\alpha|^r,$$

where $k \in K$ and K is the maximal compact open subgroup associated with the lattice $\mathfrak{M}_{\mathbb{Z}_p}$.

Assume, that γ_0 and $M_{\mathbb{Z}_p}$ are chosen such that γ_0 induces an isomorphism $M_{\mathbb{Z}_p} \to M^*_{\mathbb{Z}_p}$.

The considerations above show that the following holds true:

(7.7.6) Theorem. *Let $Q \in \mathrm{Sym}^2(M^*)$ be non-degenerate and $r \in \mathbb{Z}_{\geq 0}$.*

$$W_{Q,p}(\Phi_p(\varphi_p; s), g) = \widetilde{\Upsilon}_p^{-1}(\gamma_0) \int_{\mathrm{I}_{\mathbb{Q}_p}(M^Q, L \oplus H^r)} (g\varphi^{(r)})(x^*) \mu_{\gamma_0, Q_L^{(r)}}(x^*), \qquad s = s_0 + r$$
$$= \widetilde{\Upsilon}_p^{-1}(\gamma_0) |\gamma|_p^s \mu_p(L, M_{\mathbb{Z}_p}^Q, g\varphi; s - s_0).$$

Furthermore the left hand side is a polynomial in p^{-s} and therefore determined by the above values. Here $|\gamma|$ is computed with respect to the measure μ_{γ_0} on M.

Proof. Only the assertion, that the left hand side is a polynomial in p^{-s} remains unproven. This follows from the arguments given in [85, p. 101]. □

7.8. Borcherds lifts

(7.8.1) Consider a one-dimensional space $M_R = R$. We canonically identify $M^* \otimes L = L$. Recall from (7.1) the definition of the function $\varphi_\infty^0 \in S(L_\mathbb{R}) \otimes \mathcal{C}(\mathbb{D}_\mathbf{O})$. It satisfies

$$\varphi_\infty^0(hx, hN) = \varphi_\infty^0(x, N) \qquad \forall h \in \mathrm{SO}(L_\mathbb{R}).$$

For any $\varphi_f \in S(L_{\mathbb{A}^{(\infty)}})$ define the theta function

$$\Theta(g', N, h; \varphi_f) = \sum_{v \in L(\mathbb{Q})} \sum \omega(g')(\varphi_\infty(\cdot, N) \otimes \omega(h)\varphi)(x)$$

as a function of $g' \in \mathrm{Sp}'(\mathfrak{M}_\mathbb{A}), N \in \mathbb{D}_\mathbf{O}, h \in \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$. Here ω is the Weil representation (7.1) defined by the lift r'' (7.2.1).
We have (7.5) for any $\gamma \in \mathrm{SO}(L_\mathbb{Q})$ and $\gamma' \in \mathrm{Sp}'(\mathfrak{M}_\mathbb{Q})$,

$$\Theta(\gamma' g', \gamma N, \gamma h; \varphi_f) = \Theta(g', N, h; \varphi_f)$$

and for $g_1' \in \mathrm{Sp}'(\mathfrak{M}_{\mathbb{A}^{(\infty)}})$ and $h_1 \in \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ we have

$$\Theta(g' g_1', N, hh_1; \varphi_f) = \Theta(g', N, h; \omega(g_1')\omega(h_1)\varphi_f).$$

By these invariance properties, we may consider Θ as a function

$$\mathrm{Sp}'(\mathfrak{M}_\mathbb{Q}) \backslash \mathrm{Sp}'(\mathfrak{M}_{\mathbb{A}^{(\infty)}}) \times \mathrm{M}({}^K\mathbf{O}(L))_\mathbb{C} \to \left(S(L_{\mathbb{A}^{(\infty)}})^K\right)^*.$$

In (7.6) we saw that

$$\omega(k_\infty')\varphi_\infty^0(\cdot, N) = \chi_l(k_\infty')\varphi_\infty^0(\cdot, N)$$

for $l = \frac{n}{2} - 1$.
It follows that

$$\Theta(g' k_\infty' k', N, h) = \chi_l(k_\infty')({}^t\omega(k'))^{-1}\Theta(g', N, h)$$

for all $k'_\infty \in K'_\infty$ and $k' \in K'$.
Now consider a automorphic function

$$F : \mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}})\backslash \mathrm{Sp}'(\mathfrak{M}_{\mathbb{A}^{(\infty)}}) \to S(L_{\mathbb{A}^{(\infty)}}),$$

satisfying

$$F(g'k'_\infty k') = \chi_{-l}(k'_\infty)\omega(k')^{-1}F(g')$$

and invariant under some compact open subgroup $K \subset \mathrm{SO}(\mathbb{A}^{(\infty)})$. We have seen that this corresponds to a usual modular form of weight $-l$.
Define the **Borcherds lift** of F as

$$\Phi(N, h; F) = \int^{\bullet}_{G'_{\mathbb{Q}}\backslash G'_{\mathbb{A}}} \Theta(g', N, h; F(g'))\,\mathrm{d}\,g',$$

where $\bullet$ denotes Borcherds regularization [4], described in [54, p. 302ff] in the adelic language. $\mathrm{d}\,g'$ is the Tamagawa measure.
Note, that this formation does not depend on K.
By definition, we have

$$\Phi(N, h\rho; F) = \Phi(N, h; (\omega(\rho))^{-1}F). \tag{12}$$

It compares to the language of Borcherds as follows: If $K \subset \mathrm{SO}(\mathbb{A}^{(\infty)})$ is the discriminant kernel with respect to a lattice $L_{\mathbb{Z}}$, we may consider Θ as valued in $\mathbb{C}[(L^*_{\mathbb{Z}}/L_{\mathbb{Z}})]^*$ and F, valued in $\mathbb{C}[(L^*_{\mathbb{Z}}/L_{\mathbb{Z}})]$, relates to a usual modular form (if l is odd, of half integer weight) by

$$y^{l/2}F(g'_\tau) = f(\tau) \qquad \tau = x + iy \in \mathbb{H}.$$

(7.8.2) Lemma. *We have*

$$\Phi(\iota(N), 1; F) = \Phi_{classic}(\iota, L_{\mathbb{Z}}, N, f)$$

for $N \in \mathrm{Grass}^-(L_{\mathbb{R}})$ and according to $\iota : \mathrm{Grass}^-(L_{\mathbb{R}}) \hookrightarrow \mathbb{D}_{\mathbf{O}}$. ι is a parameter in Borcherds theory (cf. [4, p. 46]).

Proof. In (7.7.4) we saw that the Weil representation composed with the inclusion $\mathrm{Sp}'_2(\mathbb{Z}) \to \mathrm{Sp}'_2(\widehat{\mathbb{Z}})$ acts on the subspace $\mathbb{C}[L^*_{\mathbb{Z}}/L_{\mathbb{Z}}] \subset S(L_{\mathbb{A}^{(\infty)}})$ by the same formulas used in [4]. □

We call f, resp. F, nearly holomorphic, if F is holomorphic on $\mathbb{H}$ and has a pole along the cusp ∞. Let

$$F(\cdot, \tau) = \sum_{\gamma \in \mathbb{Q}^*} c(\gamma, \cdot)\exp(2\pi i\gamma\tau)$$

be the Fourier expansion of F. Assume that all $c(\gamma, \cdot)$ are integer valued on L^*/L. In this case the Borcherds lift $\Phi(N, 1; F)$ has nice properties, and is a meromorphic modular form on the analytic Shimura variety

$$(\mathrm{M}({}^K\mathbf{O}(L))_{\mathbb{C}})^{an} = [\mathrm{SO}(L_{\mathbb{Q}})\backslash \mathbb{D}_{\mathbf{O}} \times (\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K)]\,.$$

7.9. The Archimedean orbit equation

(7.9.1) Definition. *We define factors at ∞ analogously to λ_p and μ_p (6.4.10).*

$$\lambda_\infty(L;s) := \Gamma_{m-1,m}(s),$$
$$\mu_\infty(L,M;s) := \Gamma_{n,m}(s),$$

where, as usual, $n = \dim(M)$ and $m = \dim(L)$.

With this notation, we have also a (rather trivial) Archimedean analogue of the orbit equation (with only one orbit):

(7.9.2) Theorem. *If Q_M is positive definite, we have*

$$\mu_\infty(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s) \cdot \lambda_\infty(L_{\mathbb{Z}}; s)^{-1} = \lambda_\infty(\alpha(L)^\perp; s)^{-1}.$$

Here α is any real embedding $M_{\mathbb{R}} \to L_{\mathbb{R}}$, with $\alpha^! Q_L = Q_M$. The above depends only on the respective dimensions and is formulated in dependence of L and M only in order to have the same shape than the non-Archimedean orbit equation (6.4.11).

Furthermore in the positive definite case, we have, analogously to the non-Archimendian case:

$$\mathrm{vol}(\mathrm{I}(M,L)_{\mathbb{R}}) = \Gamma_{n,m}(0) \qquad n \ge m$$

and in particular

$$\mathrm{vol}(\mathrm{SO}(M_{\mathbb{R}})) = \Gamma_{m-1,m}(0)$$

(for the canonical volumes 6.2.3).

7.10. The global orbit equation

(7.10.1) Let $L_{\mathbb{Z}}$ be a lattice of dimension m with $Q_L \in \mathrm{Sym}^2(L_{\mathbb{Z}}^*)$ of signature $(m-2,2)$. For 1-dimensional $M \cong \mathbb{Z}$, $(n = 1)$, we have $Q \in \mathbb{Q}$ describing the quadratic form

$x \mapsto Qx^2$ with associated 'symmetric morphism' $\gamma = 2Q$. Take $\gamma_0 = 1$. We get:

$$\begin{aligned}
&\lim_{\alpha\to\infty} |\alpha|^{-\frac{m}{2}} e^{2\pi\alpha^2 Q} E_Q(\Psi_{\infty,\frac{m}{2}}\Phi(\chi_\kappa); g_l(\alpha), s) \\
&= \lim_{\alpha\to\infty} |\alpha|^{-\frac{m}{2}} e^{2\pi\alpha^2 Q} W_{Q,\infty}(\Psi_{\infty,\frac{m}{2}}(s), g_l(\alpha)) \prod_p W_{Q,p}(\Phi_p(\chi_\kappa; s), 1) \\
&= |\gamma|_\infty^s |2\gamma|_\infty^{\frac{1}{2}(s-s_0)} \mu_\infty(L_{\mathbb{Z}_p}, < Q >, \kappa; s - s_0) \prod_p |\gamma|_p^s \mu_p(L_{\mathbb{Z}_p}, < Q >, \kappa; s - s_0).
\end{aligned}$$

see (7.7.5) and (7.6.12).
This is the quantity, which will be related to Arakelov geometry in Part III, using the result of [57], see (10.4.12) and (11.6).
We therefore define

$$\mu(L_\mathbb{Z}, M_\mathbb{Z}, \kappa; s) = \prod_\nu \mu_\nu(L_\mathbb{Z}, M_\mathbb{Z}, \kappa; s)$$

and

$$\widetilde{\mu}(L_\mathbb{Z}, M_\mathbb{Z}, \kappa; s) = |2d(M_\mathbb{Z})|_\infty^{-\frac{1}{2}s} \mu(L_\mathbb{Z}, M_\mathbb{Z}, \kappa; s).$$

With this definition we have in the 1-dimensional case:

$$\lim_{\alpha\to\infty} |\alpha|^{-\frac{m}{2}} e^{2\pi\alpha^2 Q} E_Q(\Psi_\infty\Phi(\chi_\kappa); g_l(\alpha), s - s_0) = \widetilde{\mu}(L_{\mathbb{Z}_p}, < Q >, \kappa; s). \tag{13}$$

For $n > 1$ we do not now, if the global $\widetilde{\mu}$ occurs as a limit in this fashion as well.

(7.10.2) We would like to have a global orbit equation. Let D be the discriminant of L. We define

$$\lambda_p(L; s) := \mathrm{vol}(\mathrm{SO}'(L_{\mathbb{Z}_p})),$$

with respect to the canonical volume for $p^2|4D$. Otherwise λ_p has been defined in (6.4.10) and is a polynomial in p^{-s} (8.2.1). Define now

$$\lambda(L_\mathbb{Z}; s) := \prod_\nu \lambda_\nu(L_\mathbb{Z}; s)$$

and

$$\widetilde{\lambda}(L; s) := |D|_\infty^{\frac{1}{2}s} \lambda(L_\mathbb{Z}; s),$$

where D is the discriminant of $L_\mathbb{Z}$. Sometimes we will use also $\widetilde{\lambda}_p(\dots)$ for $|D|_p^{-\frac{1}{2}s}\lambda_p(\dots)$, and similarly $\widetilde{\mu}_p(\dots) := |2d(M_\mathbb{Z})|_p^{\frac{1}{2}s} \mu_p(\dots)$.

(7.10.3) Lemma. *If $L_\mathbb{Z}$ is of signature $(m-2, 2)$. Let $M_\mathbb{Z}$ be a positive definite lattice. Take a $\kappa \in (L^*_{\widehat{\mathbb{Z}}}/L_{\widehat{\mathbb{Z}}}) \otimes M^*_{\widehat{\mathbb{Z}}}$. Assume $m - n \geq 1$ and $\mathrm{I}(M, L)(\mathbb{A}^{(\infty)}) \cap \kappa \neq \emptyset$.*
$\mu(L_\mathbb{Z}, M_\mathbb{Z}, \kappa; s), \lambda(L_\mathbb{Z}; s)$ have meromorphic continuations to the entire complex plane and are holomorphic and nonzero in a neighborhood of $s = 0$. Similarly for $\widetilde{\mu}, \widetilde{\lambda}$. They

depend only on the genera of $L_{\mathbb{Z}}, M_{\mathbb{Z}}$.

Proof. This follows directly from (8.2.1) and using standard facts about the occurring quadratic L-series. □

In particular, for the meromorphic continuation of the Eisenstein series (13) above remains true also in exceptional cases.

We get for $\kappa \in (L^*_{\widehat{\mathbb{Z}}}/L_{\widehat{\mathbb{Z}}}) \otimes M^*_{\widehat{\mathbb{Z}}}$ at least:

(7.10.4) Theorem. *Assume* $m \geq 3$, $m - n \geq 1$. *Let* D *be the discriminant of* $L_{\mathbb{Z}}$ *and* D' *be the* D*-primary part of the discriminant of* $M_{\mathbb{Z}}$.

$$\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};0)\widetilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0) = \sum_{\alpha\,\mathrm{SO}'(L_{\widehat{\mathbb{Z}}}) \subset \mathrm{I}(M,L)(\mathbb{A}^{(\infty)}) \cap \kappa} \widetilde{\lambda}^{-1}(\alpha^{\perp}; 0)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\,s}\left(\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};s)\widetilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s)\right)\Big|_{s=0} \equiv \sum_{\alpha\,\mathrm{SO}'(L_{\widehat{\mathbb{Z}}}) \subset \mathrm{I}(M,L)(\mathbb{A}^{(\infty)}) \cap \kappa} \frac{\mathrm{d}}{\mathrm{d}\,s}\,\widetilde{\lambda}^{-1}(\alpha^{\perp}; s)\Big|_{s=0}$$

in $\mathbb{R}^{(p)}$ *(9.3.2) for all* p *such that* $p \nmid 2D$ *and* $p^2 \nmid D'$.

Proof. Follows directly from the definitions, the local orbit equations (6.4.11) and (7.9.2) as well as (by induction) from the fact that for $p \nmid D$ and q square-free at p there is only 1 orbit (6.4.18) in $\mathrm{I}(< q >, L)(\mathbb{Z}_p)$ and $\alpha^{\perp}$ has discriminant p at p. □

We will need also a global version of Kitaoka's formula (6.3.3):

(7.10.5) Theorem. *Assume* $m \geq 3$, $m - n \geq 1$. *Let* D *be the discriminant of* $L_{\mathbb{Z}}$ *and* $M_{\mathbb{Z}} = M'_{\mathbb{Z}} \perp M''_{\mathbb{Z}}$. *Let* D' *be the* D*-primary part of the discriminant of* $M'_{\mathbb{Z}}$ *(not* $M_{\mathbb{Z}}$ *!). Let* $\kappa \in (L^*_{\mathbb{Z}}/L_{\mathbb{Z}}) \otimes M^*_{\mathbb{Z}}$ *with a corresponding decomposition* $\kappa = \kappa' \oplus \kappa''$. *We have*

$$\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};0)\widetilde{\mu}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0) = \sum_{\substack{\alpha\,\mathrm{SO}'(L_{\widehat{\mathbb{Z}}}) \subset \mathrm{I}(M',L)(\mathbb{A}^{(\infty)}) \cap \kappa' \\ \kappa'' \cap \alpha^{\perp}_{\mathbb{A}^{(\infty)}} \otimes (M''_{\mathbb{A}^{(\infty)}})^* \neq \emptyset}} \widetilde{\lambda}^{-1}(\alpha^{\perp}_{\mathbb{Z}}; 0)\widetilde{\mu}(\alpha^{\perp}_{\mathbb{Z}}, M_{\widehat{\mathbb{Z}}}, \kappa''; 0)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\,s}\left(\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};s)\widetilde{\mu}(L_{\mathbb{Z}},M_{\mathbb{Z}},\kappa;s)\right)\Big|_{s=0}$$
$$=\sum_{\substack{\alpha\,\mathrm{SO}'(L_{\widehat{\mathbb{Z}}})\subset \mathrm{I}(M',L)(\mathbb{A}^{(\infty)})\cap\kappa' \\ \kappa''\cap\alpha^{\perp}_{\mathbb{A}^{(\infty)}}\otimes(M''_{\mathbb{A}^{(\infty)}})^*\neq\emptyset}}\frac{\mathrm{d}}{\mathrm{d}\,s}\left(\widetilde{\lambda}^{-1}(\alpha^{\perp}_{\mathbb{Z}};s)\widetilde{\mu}(\alpha^{\perp}_{\mathbb{Z}},M'',\kappa'';s)\right)\Big|_{s=0}$$

in $\mathbb{R}_{2DD'}$. *Here* $\kappa''\in L^*_{\widehat{\mathbb{Z}}}/L_{\widehat{\mathbb{Z}}}$ *is considered as an element of* $(\alpha^{\perp}_{\widehat{\mathbb{Z}}})^*/\alpha^{\perp}_{\widehat{\mathbb{Z}}}\otimes(M''_{\widehat{\mathbb{Z}}})^*$ *via* $\kappa''\mapsto\kappa''\cap\alpha^{\perp}_{\mathbb{A}^{(\infty)}}\otimes(M''_{\mathbb{A}^{(\infty)}})^*$.

Proof. Let first $s\in\mathbb{Z}_{\geq 0}$. For all $p\nmid 2DD'$, there is only one orbit (generated for $s=0$ by α, say) in $\mathrm{I}(M',L\oplus H^s)(\mathbb{Z}_p)$. Hence we have, using (a variant of) Kitaoka's formula (6.3.3):

$$\lambda_p^{-1}(L_{\mathbb{Z}_p};s)\widetilde{\mu}_p(L_{\mathbb{Z}_p},M_{\mathbb{Z}_p},L_{\mathbb{Z}_p};s)=\lambda_p^{-1}(\alpha^{\perp};s)\widetilde{\mu}_p(\alpha^{\perp}_{\mathbb{Z}_p},M''_{\mathbb{Z}_p},\kappa'';s).$$

For all other p, we have the equation

$$\lambda_p^{-1}(L_{\mathbb{Z}_p};0)\mu_p(L_{\mathbb{Z}_p},M_{\mathbb{Z}_p},\kappa;0)=\sum_{\substack{\alpha\,\mathrm{SO}'(L_{\mathbb{Z}_p})\subset\mathrm{I}(M',L)(\mathbb{Q}_p)\cap\kappa' \\ \kappa''\cap\alpha^{\perp}_{\mathbb{Q}_p}\otimes(M''_{\mathbb{Q}_p})^*\neq\emptyset}}\lambda_p^{-1}(\alpha^{\perp}_{\mathbb{Z}_p};0)\mu_p(\alpha^{\perp}_{\mathbb{Z}_p},M'',\kappa'';0).$$

and since the quantities $\widetilde{\mu}_p$ are polynomials in p^{-s} in this case (cf. 7.7.5), the assertion is true. $\square$

8. Explicit calculations

8.1. Kronecker limit formula

Let

$$E(\tau,s)=\sum_{g\in P(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})}\Psi(gg_\tau)(s)=\frac{1}{2}\sum_{g\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})}\Im(g\circ\tau)^s=\frac{1}{2}\sum_{\substack{(m,n)\in\mathbb{Z}\\ \gcd(n,m)=1}}\frac{y^s}{|m\tau+n|^{2s}}$$

be the standard Eisenstein series of weight 0. Here $\Psi=\prod_\nu\Psi_\nu$, the Ψ_νs are the standard sections:

$$\Psi_\nu(g_l(\alpha)u(\beta)k)=|\alpha|_\nu^{2s}.$$

They satisfy, however, a different normalization than in (7.4)! Let $Z(s)=\zeta(s)\Gamma(\frac{1}{2}s)\pi^{-\frac{1}{2}s}$ be the normalized Riemann zeta function, satisfying $Z(s)=Z(1-s)$. The Eisenstein series (in this normalization) satisfies the functional equation [17, Theorem 1.6.1]:

$$Z(2s)E(\tau,s)=Z(2-2s)E(\tau,1-s).$$

In part III we will need the **Kronecker limit formula** for the computation of the arithmetic volume of Heegner points:

(8.1.1) Theorem.

$$\begin{aligned}
E(\tau,s)=&1+\frac{1}{12}\log(|\Delta(\tau)|^2\Im(\tau)^{12})s+O(s^2)\\
Z(2s)E(\tau,s)=&-\frac{1}{2s}+\frac{1}{2}(\gamma-\log(2\pi)-\log(2))-\frac{1}{24}\log(|\Delta(\tau)|^2\Im(\tau)^{12})\\
&+O(s)\\
Z(2s)E(\tau,s)=&\frac{1}{2(s-1)}+\frac{1}{2}(\gamma-\log(2\pi)-\log(2))-\frac{1}{24}\log(|\Delta(\tau)|^2\Im(\tau)^{12})\\
&+O(s-1)\\
\frac{Z(2s)}{Z(s)}E(\tau,s)=&\frac{1}{2}+\left(\frac{1}{4}(\gamma-\log(2\pi)-\log(2))-\frac{1}{24}\log(|\Delta(\tau)|^2\Im(\tau)^{12})\right)(s-1)\\
&+O((s-1)^2)
\end{aligned}$$

Proof. [71, Prop. 1.8.3] states:

$$\log(|\Delta(\tau)|^2\Im(\tau)^{12}) = -4\pi \lim_{s\to 1}(E(\tau,s) - \frac{Z(2s-2)}{Z(2s)}).$$

Application of the functional equation yields:

$$\log(|\Delta(\tau)|^2\Im(\tau)^{12}) = -4\pi \lim_{s\to 0}\frac{Z(2s)}{Z(2-2s)}(E(\tau,s)-1).$$

In addition we have:

$$\frac{Z(2-2s)}{Z(2s)} = \frac{3}{\pi}(s-1)^{-1} + O(1),$$

hence $E(\tau,0)=1$ and

$$\log(|\Delta(\tau)|^2\Im(\tau)^{12}) = 12E'(\tau,0).$$

This proves the first version of the formula.
We have:

$$Z(2s) = -\frac{1}{2s} + \frac{1}{2}(\gamma - \log(2\pi) - \log(2)) + O(s)$$

and hence the second and third version (they are equivalent according to the functional equation). Last:

$$Z(s) = \frac{1}{s-1} + \frac{1}{2}(\gamma - \log(2\pi) - \log(2)) + O(s-1)$$

and hence

$$Z(s)^{-1} = (s-1) - \frac{1}{2}(\gamma - \log(2\pi) - \log(2))(s-1)^2 + O((s-1)^3).$$

□

8.2. Explicit calculation of μ and λ

In this section, we will compute the functions μ and λ (6.4.10) explicitly in special cases. The expression for λ is quite general and can in principle be used to compute it for all lattices.

(8.2.1) Theorem. *Assume $p \neq 2$.*

i. Let $s \in \mathbb{N}$. $\mathrm{vol}(\mathrm{SO}(H^s)) = (1-p^{-s}) \cdot \prod_{i=1}^{s-1}(1-p^{-2i})$.

ii. Let $s \in \mathbb{Z}_{\geq 0}$. $\lambda(L_{\mathbb{Z}_p} \perp H; s) = (1-p^{-2s-2}) \cdot \lambda(L_{\mathbb{Z}_p}; s+1)$.

iii. Let $s \in \mathbb{Z}_{\geq 0}$. Let $L_{\mathbb{Z}_p} =< \varepsilon_1, \ldots, \varepsilon_k >\perp L'_{\mathbb{Z}_p}$, where $\varepsilon_i \in \mathbb{Z}_p^*$ and $p^{-1}Q_L$ is integral *on $L'_{\mathbb{Z}_p}$. Assume $k>1$. Let $\varepsilon := (-1)^{\frac{k}{2}} \prod_{i=1}^{k} \varepsilon_i$, if k is even. Then we have*

$$\frac{\lambda(L_{\mathbb{Z}_p};s)}{\lambda(L_{\mathbb{Z}_p};0)} = |D|_p^s \prod_{i=1}^{\lfloor\frac{k-1}{2}\rfloor} \frac{1-p^{-2i-2s}}{1-p^{-2i}} \begin{cases} 1 & k\equiv 1 \ (2), \\ \frac{1-(\frac{\varepsilon}{p})p^{-\frac{k}{2}-s}}{1-(\frac{\varepsilon}{p})p^{-\frac{k}{2}}} & k \equiv 0 \ (2). \end{cases}$$

In particular $\lambda(L_{\mathbb{Z}_p};s)$ is a (quite simple) polynomial in p^{-s}.

iv. *Let $s\in\mathbb{Z}_{\geq 0}$. For a unimodular lattice of discriminant $2^m\varepsilon$ and $\varepsilon' \in {\mathbb{Z}_p}^*$ we have:*

$$\mu(L_{\mathbb{Z}_p}, <\varepsilon'>; s) = \begin{cases} (1-(\frac{(-1)^{\frac{m}{2}}\varepsilon}{p})p^{-\frac{m}{2}-s}) & m\equiv 0 \ (2), \\ (1+(\frac{(-1)^{\frac{m-1}{2}}\varepsilon\varepsilon'}{p})p^{-\frac{m-1}{2}-s}) & m \equiv 1 \ (2). \end{cases}$$

v. *Let $s\in\mathbb{Z}_{\geq 0}$. For a unimodular lattice of discriminant $2^m\varepsilon$ we have:*

$$\lambda(L_{\mathbb{Z}_p};s) = \prod_{i=1}^{\lfloor\frac{m-1}{2}\rfloor} (1-p^{-2i-2s}) \begin{cases} (1-(\frac{(-1)^{\frac{m}{2}}\varepsilon}{p})p^{-\frac{m}{2}-s}) & m\equiv 0 \ (2), \\ 1 & m\equiv 1 \ (2). \end{cases}$$

vi. *Let $s \in \mathbb{Z}_{\geq 0}$. For a lattice with $L^*_{\mathbb{Z}_p}/L_{\mathbb{Z}_p}$ cyclic of order $p^\nu \neq 1$ and dimension $m\geq 2$, we may assume $L=\mathbb{Z}_p^m$, $Q_L(x) = \sum_{j=1}^{m-1}\varepsilon_j x_j^2 + p^\nu \varepsilon_m x_m^2$. Denote $\varepsilon = (-1)^{\frac{m-1}{2}}\prod_{j=1}^{m-1}\varepsilon_j$ if m is odd. We have*

$$\lambda(L_{\mathbb{Z}_p};s) = |p^\nu|_p^{s+\frac{m-1}{2}} \prod_{i=1}^{\lfloor\frac{m}{2}\rfloor-1} (1-p^{-2i-2s}) \begin{cases} 1 & m\equiv 0 \ (2), \\ 1-(\frac{\varepsilon}{p})p^{-\frac{m-1}{2}-s} & m\equiv 1 \ (2). \end{cases}$$

Proof. i. According to Kitaoka's formula (cf. Theorem 6.3.3), we have

$$\mathrm{vol}(\mathrm{SO}(H^s)) = \beta_p(H^s,1)\beta_p(H^{s-1}\perp<-1>,-1)\,\mathrm{vol}(\mathrm{SO}(H^{s-1})).$$

Theorem (6.4.8) yields $\beta_p(H^s,1) = 1-p^{-s}$ and $\beta_p(H^{s-1}+<-1>,1) = 1+p^{-s+1}$. Furthermore, we have $\mathrm{vol}(\mathrm{SO}(H)) = \beta_p(H,1) = 1-p^{-1}$.
ii. follows immediately from the definition of λ and i.
iii. Let $L_{\mathbb{Z}_p}$ be a lattice, and $S=<\alpha_1,\alpha_2>$ a unimodular plane (e.g. a hyperbolic one). Using the (elementary) orbit equation (6.4.3) and Theorem (6.3.3), we get

$$\begin{aligned}\mathrm{vol}(\mathrm{SO}'(L_{\mathbb{Z}_p}\perp S)) = |D|^{\frac{1}{2}}\beta_p(L_{\mathbb{Z}_p}\perp S,\alpha_2)|D|^{\frac{1}{2}}\beta_p(L_{\mathbb{Z}_p}\perp<\alpha_1>,\alpha_1)\\ \cdot\,\mathrm{vol}(\mathrm{SO}'(L_{\mathbb{Z}_p})).\end{aligned} \tag{1}$$

Hence, we have to apply Theorem (6.4.8) to forms of the shape

$$L'_{\mathbb{Z}_p} = <\varepsilon_1,\ldots,\varepsilon_{k'}, p^{\nu_{k'+1}}\varepsilon_{k'+1},\ldots,p^{\nu_m}\varepsilon_m> .$$

We get

$$\beta_p(L'_{\mathbb{Z}_p},\varepsilon';s) = 1 + v(1)p^{d(1)}p^{-s}\begin{cases} -p^{-1} & l(1,1)\equiv 0 \ (2), \\ (\frac{-\varepsilon'}{p})p^{-\frac{1}{2}} & l(1,1)\equiv 1 \ (2). \end{cases}$$

We have

$$\begin{aligned} l(1,1) &= k' \\ d(1) &= 1-\frac{1}{2}k' \\ v(1) &= (\frac{-1}{p})^{\lfloor \frac{k'}{2}\rfloor}\prod_{i=1}^{k'}(\frac{\varepsilon_i}{p}) \end{aligned}$$

Hence

$$\beta_p(L'_{\mathbb{Z}_p},\varepsilon';s) = \begin{cases} 1-(\frac{(-1)^{\frac{k'}{2}}\prod_{i=1}^{k'}\varepsilon_i}{p})p^{-\frac{k'}{2}-s} & k'\equiv 0 \ (2), \\ 1+(\frac{(-1)^{\frac{k'-1}{2}}\prod_{i=1}^{k'}\varepsilon_i\varepsilon'}{p})p^{-\frac{k'-1}{2}-s} & k'\equiv 1 \ (2). \end{cases}$$

Applying this to $L'_{\mathbb{Z}_p} = L_{\mathbb{Z}_p}\perp S$ and $L'_{\mathbb{Z}_p}\perp<\alpha>$, we get the result for k odd. For k even write $L_{\mathbb{Z}_p} = L'_{\mathbb{Z}_p}\perp<\alpha>$, use

$$\mathrm{vol}(\mathrm{SO}'(L'_{\mathbb{Z}_p})) = |D|^{\frac{1}{2}}\beta_p(L'_{\mathbb{Z}_p},\alpha)\cdot\mathrm{vol}(\mathrm{SO}'(L_{\mathbb{Z}_p})) \tag{2}$$

twice and the k odd part. Recall (Lemma 6.4.17) that vectors of length $\alpha\in\mathbb{Z}_p^*$ form one orbit under SO', as long as the lattice in question splits a unimodular plane, otherwise there are 2 orbits of equal volume.
iv. This is Siegel's formula, a special case of Theorem (6.4.8).
v. Follows from iv. and the orbit equation, Theorem (6.4.11).
vi. Follows from iii. and the following calculation for $m\geq 2$ (which follows easily from equations (1) and (2) and the fact $\mathrm{vol}(\mathrm{SO}'(<x>)) = 1)$). Observe, that there are 2 orbits (of equal volume) of vectors of length β in a lattice $<\alpha p^\nu,\beta>$.

$$\lambda(L_{\mathbb{Z}_p};0) = |p^\nu|^{\frac{m-1}{2}}\prod_{i=1}^{\lfloor\frac{m}{2}\rfloor-1}(1-p^{-2i})\begin{cases} 1 & m\equiv 0 \ (2), \\ 1-\left(\frac{\varepsilon}{p}\right)p^{-\frac{m-1}{2}} & m\equiv 1 \ (2). \end{cases}$$

Here $\varepsilon = (-1)^{\frac{m-1}{2}}\prod_{i=1}^{m-1}\varepsilon_i$. □

Without proof, we give here come calculations in the case $p=2$.

(8.2.2) Theorem.

i. Let $s\in\mathbb{N}$. $\mathrm{vol}(\mathrm{SO}(H^s)) = (1-2^{-s})\cdot\prod_{i=1}^{s-1}(1-2^{-2i})$.

ii. Let $s\in\mathbb{Z}_{\geq 0}$. $\lambda(L_{\mathbb{Z}_2}\perp H;s) = (1-2^{-2s-2})\cdot\lambda(L_{\mathbb{Z}_2};s+1)$.

iii. Let $s \in \mathbb{Z}_{\geq 0}$. For a unimodular lattice of even dimension of discriminant ε we have:

$$\lambda(L_{\mathbb{Z}_2}; s) = \prod_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} (1 - 2^{-2i-2s})(1 - (\frac{\varepsilon}{2}) 2^{-\frac{m}{2}-s}).$$

Here $(\frac{\varepsilon}{2}) = (-1)^{\frac{\varepsilon^2-1}{8}}$ is the Kronecker symbol.

iv. Let $s \in \mathbb{Z}_{\geq 0}$. For a lattice $L_{\mathbb{Z}_2}$ of the form $L' \perp < \varepsilon' >$, where $L'_{\mathbb{Z}_2}$ is unimodular of discriminant ε, and $\varepsilon, \varepsilon' \in \mathbb{Z}_2^*$, we have:

$$\lambda(L_{\mathbb{Z}_2}; s) = |2|_2^{s + \frac{m-1}{2}} \prod_{i=1}^{\lfloor \frac{m-1}{2} \rfloor} (1 - 2^{-2i-2s}).$$

v. Let $L_{\mathbb{Z}_2}$ be a lattice of the form $< \varepsilon_1, \varepsilon_2 >, \varepsilon_i \in \mathbb{Z}_2^*$. We have

$$\lambda(L_{\mathbb{Z}_2}; 0) = \frac{1}{2}.$$

8.3. Examples

(8.3.1) Recall the definitions of $\lambda(L_{\mathbb{Z}}; s)$ and its modification $\widetilde{\lambda}(L_{\mathbb{Z}}; s)$ from (7.10.4). They will be related (for square-free discriminants) to arithmetic and geometric volumes of Shimura varieties of orthogonal type in the Main Theorems (10.5.2), (10.5.7) and (10.5.9) of part III.

We compute them for a couple of lattices using the explicit computation (8.2.1), (8.2.2) and bring them to a form involving derivatives of L-series at negative integers as it is common in the literature. See for example [15, 71] and [57, §5–6] for similar calculations. The summand $-\frac{m-1}{2}C$, where $C = \gamma + \log(2\pi)$, in the logarithmic derivative of $\widetilde{\lambda}^{-1}(L_{\mathbb{Z}}; s)$ can be avoided by just erasing e^{-C} from the definition of $h_{\mathcal{E}}$ in (10.4.1). In turn the nice connection with λ and especially μ (coefficient of the Eisenstein series) becomes slightly more complicated.

We will use the following functional equations for quadratic L-series:

(8.3.2) Theorem ([81, Kap. VII]).

$$L(s,\chi_D) = (\frac{\pi}{D})^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)} L(1-s,\chi_D) \qquad D>0, D \textit{ square free,}$$

$$L(s,\chi_D) = (\frac{\pi}{D})^{s-\frac{1}{2}} \frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s+\frac{1}{2})} L(1-s,\chi_D) \qquad D<0, D \textit{ square free,}$$

$$\zeta(s) = \frac{\pi^{s-\frac{1}{2}}\Gamma(\frac{1}{2}-\frac{1}{2}s)}{\Gamma(\frac{1}{2}s)}\zeta(1-s).$$

(8.3.3) Example (Heegner points). *Let* $L_{\mathbb{Z}}$ *be a* two dimensional *negative definite lattice with square-free discriminant* $D>0$ *(in particular* $2\nmid D$ *and* $-D$ *is automatically fundamental). We get, using (8.2.1, v, vi), (8.2.2, iii) and (7.9.1):*

$$\begin{aligned}
\lambda^{-1}(L_{\mathbb{Z}};s) =& \Gamma_{1,2}^{-1}(s)\prod_p |D|_p^{-\frac{1}{2}-s}\frac{1}{1-(\frac{-D}{p})p^{-s-1}} \\
=& \frac{1}{2}\frac{\Gamma(\frac{1}{2}s+1)}{\pi^{\frac{1}{2}s+1}} L(\chi_{-D},s+1)|D|_\infty^{\frac{1}{2}+s} \\
=& \frac{1}{2}\frac{\Gamma(\frac{1}{2}-\frac{s}{2})}{\pi^{\frac{1}{2}-\frac{s}{2}}} L(\chi_{-D},-s) \\
=& \frac{1}{2}L(\chi_{-D},0) \\
&+ \frac{1}{2}L(\chi_{-D},0)\cdot\left(-\frac{L'(\chi_{-D},0)}{L(\chi_{-D},0)}+\frac{1}{2}C+\frac{1}{2}\log(2)\right)s \\
&+ O(s^2)
\end{aligned}$$

and hence:

$$\begin{aligned}
\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};s) =& L(\chi_{-D},0) \\
&+ L(\chi_{-D},0)\cdot\left(-\frac{L'(\chi_{-D},0)}{L(\chi_{-D},0)}+\frac{1}{2}C-\frac{1}{2}\log(D)+\frac{1}{2}\log(2)\right)s \\
&+ O(s^2)
\end{aligned}$$

(8.3.4) Example (modular curves). *Let* $L_{\mathbb{Z}}$ *be a* three dimensional *lattice with form* $x_1x_2-\varepsilon x_3^2$*,* ε *square free,* $2\nmid\varepsilon$*. The discriminant* D *is* 2ε*. We get, using (8.2.1, v, vi),* *(8.2.2, iv) and (7.9.1):*

$$
\begin{aligned}
\lambda^{-1}(L_{\mathbb{Z}};s) =& \Gamma_{2,3}^{-1}(s)\prod_{p\nmid D}\frac{1}{1-p^{-2-2s}}|D|_\infty^{1+s}\prod_{p|\varepsilon}\frac{1}{1-p^{-s-1}} \\
=& \frac{1}{4}\frac{\Gamma(\frac{1}{2}s+1)\Gamma(\frac{1}{2}s+\frac{3}{2})}{\pi^{s+\frac{5}{2}}}\zeta(2s+2)|D|_\infty^{1+s}\prod_{p|\varepsilon}(1+p^{-s-1}) \\
=& \frac{1}{4}\frac{\Gamma(\frac{1}{2}s+1)\Gamma(\frac{1}{2}s+\frac{3}{2})\Gamma(-s-\frac{1}{2})}{\Gamma(s+1)\pi^{-s+1}}\zeta(-1-2s)|D|_\infty^{1+s}\prod_{p|\varepsilon}(1+p^{-s-1}) \\
=& -\frac{1}{2}\zeta(-1)\prod_{p|\varepsilon}(p+1) \\
& -\frac{1}{2}\zeta(-1)\prod_{p|\varepsilon}(p+1)\left(-2\frac{\zeta'(-1)}{\zeta(-1)}+\sum_{p|\varepsilon}\frac{p\log(p)}{p+1}-1+C+\log(2)\right)s \\
& +O(s^2)
\end{aligned}
$$

and hence:

$$
\begin{aligned}
\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};s) =& -\frac{1}{2}\zeta(-1)\prod_{p|\varepsilon}(p+1)-\frac{1}{2}\zeta(-1)\prod_{p|\varepsilon}(p+1)\cdot \\
& \cdot\left(-2\frac{\zeta'(-1)}{\zeta(-1)}+\frac{1}{2}\sum_{p|\varepsilon}\frac{p-1}{p+1}\log(p)-1+C+\frac{1}{2}\log(2)\right)s \\
& +O(s^2)
\end{aligned}
$$

(8.3.5) Example (Shimura curves). *Let $L_{\mathbb{Z}}$ be a* three dimensional *lattice again with discriminant $D=2\varepsilon$, $2\nmid\varepsilon$, ε square-free, and assume that the form is* anisotropic at all *$p|\varepsilon$. We get, using (8.2.1, v, vi), (8.2.2, iv) and (7.9.1):*

$$
\begin{aligned}
\lambda^{-1}(L_{\mathbb{Z}};s) =& \frac{1}{4}\frac{\Gamma(\frac{1}{2}s+1)\Gamma(\frac{1}{2}s+\frac{3}{2})\Gamma(-s-\frac{1}{2})}{\Gamma(s+1)\pi^{-s+1}}\zeta(-1-2s)|D|_\infty^{1+s}\prod_{p|\varepsilon}(1-p^{-s-1}) \\
=& -\frac{1}{2}\zeta(-1)\prod_{p|D}(p-1) \\
& +-\frac{1}{2}\zeta(-1)\prod_{p|D}(p-1)\left(-2\frac{\zeta'(-1)}{\zeta(-1)}+\sum_{p|\varepsilon}\frac{p\log(p)}{p-1}-1+C+\log(2)\right)s \\
& +O(s^2)
\end{aligned}
$$

and hence:

$$\begin{aligned}\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};s) = & -\frac{1}{2}\zeta(-1)\prod_{p|\varepsilon}(p-1) + -\frac{1}{2}\zeta(-1)\prod_{p|\varepsilon}(p-1)\cdot \\ & \cdot\left(-2\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}\sum_{p|\varepsilon}\frac{p+1}{p-1}\log(p) - 1 + C + \frac{1}{2}\log(2)\right)s \\ & + O(s^2)\end{aligned}$$

(8.3.6) Example (Hilbert modular surfaces). *Let* $L_{\mathbb{Z}}$ *a* four dimensional *lattice, being an orthogonal direct sum of a two dimensional indefinite of discriminant* $D<0$*, square free (for simplicity) and a hyperbolic plane. We get, using (8.2.1, v, vi), (8.2.2, iii) and (7.9.1):*

$$\begin{aligned}\lambda^{-1}(L_{\mathbb{Z}};s) = & \Gamma_{3,4}^{-1}(s)\prod_p |D|_p^{-\frac{3}{2}-s}\frac{1}{1-p^{-2-2s}}\frac{1}{1-(\frac{-D}{p})p^{-s-2}} \\ = & \frac{1}{8}\frac{\Gamma(\frac{1}{2}s+1)\Gamma(\frac{1}{2}s+\frac{3}{2})\Gamma(\frac{1}{2}s+2)}{\pi^{\frac{3}{2}s+\frac{9}{2}}}\zeta(2s+2)L(\chi_{-D},s+2)|D|_\infty^{\frac{3}{2}+s} \\ = & \frac{1}{8}\frac{\Gamma(\frac{1}{2}s+\frac{3}{2})\Gamma(\frac{1}{2}s+2)\Gamma(-s-\frac{1}{2})\Gamma(-\frac{1}{2}s-\frac{1}{2})}{\Gamma(s+1)\pi^{-\frac{3}{2}s+\frac{3}{2}}}\zeta(-1-2s)L(\chi_{-D},-1-s) \\ = & \frac{1}{4}\zeta(-1)L(\chi_{-D},-1) \\ & + \frac{1}{4}\zeta(-1)L(\chi_{-D},-1)\left(-2\frac{\zeta'(-1)}{\zeta(-1)} - \frac{L'(\chi_{-D},-1)}{L(\chi_{-D},-1)} - \frac{3}{2} + \frac{3}{2}C + \frac{1}{2}\log(2)\right)s \\ & + O(s^2)\end{aligned}$$

and hence:

$$\begin{aligned}\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};s) = & \frac{1}{4}\zeta(-1)L(\chi_{-D},-1) + \frac{1}{4}\zeta(-1)L(\chi_{-D},-1)\cdot \\ & \cdot\frac{1}{4}\left(-2\frac{\zeta'(-1)}{\zeta(-1)} - \frac{L'(\chi_{-D},-1)}{L(\chi_{-D},-1)} - \frac{3}{2} - \frac{1}{2}\log(D) + \frac{3}{2}C + \frac{1}{2}\log(2)\right)s \\ & + O(s^2)\end{aligned}$$

(8.3.7) Example (Siegel threefolds). *Let* $L_{\mathbb{Z}}$ *be a* five dimensional *lattice, which is the orthogonal sum of the negative of a three dimensional as in example (8.3.5) and a hyperbolic plane. Let* $D=2\varepsilon>0$ *be the discriminant of* $L_{\mathbb{Z}}$ *We get, using (8.2.1, v,* *vi), (8.2.2, iv) and (7.9.1):*

$$\begin{aligned}
\lambda^{-1}(L_{\mathbb{Z}};s) =&\Gamma_{4,5}^{-1}(s)\prod_p \frac{1}{(1-p^{-2-2s})(1-p^{-2-4s})}|D|_\infty^{2+s}\prod_{p|\varepsilon}(1-p^{-s-2})\\
=&\frac{1}{16}\frac{(\Gamma(\frac12 s+1)\Gamma(\frac12 s+\frac32)\Gamma(\frac12 s+2)\Gamma(\frac12 s+\frac52)}{\pi^{2s+7}}\cdot\\
&\cdot\zeta(2s+2)\zeta(2s+4)2^{2+s}\prod_{p|\varepsilon}(p^{s+2}-1)\\
=&\frac{1}{16}\frac{\Gamma(\frac12 s+1)\cdots\Gamma(\frac12 s+\frac52)\Gamma(-s-\frac12)\Gamma(-\frac32-s)}{\Gamma(s+1)\Gamma(s+2)\pi^{-2s+2}}\cdot\\
&\cdot\zeta(-1-2s)\zeta(-3-2s)2^{2+s}\prod_{p|\varepsilon}(p^{s+2}-1)\\
=&-\frac14\zeta(-1)\zeta(-3)\prod_{p|\varepsilon}(p^2-1)-\frac14\zeta(-1)\zeta(-3)\prod_{p|\varepsilon}(p^2-1)\cdot\\
&\cdot\left(-2\frac{\zeta'(-1)}{\zeta(-1)}-2\frac{\zeta'(-3)}{\zeta(-3)}+\sum_{p|\varepsilon}\frac{p^2\log(p)}{p^2-1}-\frac{17}{6}+2C+\log(2)\right)s\\
&+O(s^2)
\end{aligned}$$

and hence:

$$\begin{aligned}
\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};s) =&-\frac14\zeta(-1)\zeta(-3)\prod_{p|\varepsilon}(p^2-1)-\frac14\zeta(-1)\zeta(-3)\prod_{p|D}(p^2-1)\cdot\\
&\cdot\left(-2\frac{\zeta'(-1)}{\zeta(-1)}-2\frac{\zeta'(-3)}{\zeta(-3)}+\frac12\sum_{p|D}\frac{p^2+1}{p^2-1}\log(p)-\frac{17}{6}+2C+\frac12\log(2)\right)s\\
&+O(s^2)
\end{aligned}$$

(8.3.8) Example (A 10DIMENSIONAL SHIMURA VARIETY). *Especially simple is the situation for a Shimura variety of orthogonal type associated with an unimodular lattice (this has good reduction everywhere, except possibly $p=2$). Let for example $L_{\mathbb{Z}}$ be the orthogonal direct sum of a positive definite E_8-lattice and 2 hyperbolic planes. We get, using (8.2.1, v), (8.2.2, iii) and (7.9.1):*

$$\begin{aligned}
\lambda^{-1}(L_{\mathbb{Z}};s) =&\frac{1}{16}\zeta(-1)\zeta(-3)\zeta^2(-5)\zeta(-7)\zeta(-9)+\quad\frac{1}{16}\zeta(-1)\zeta(-3)\zeta^2(-5)\zeta(-7)\zeta(-9)\cdot\\
\cdot&\left(-2\frac{\zeta'(-1)}{\zeta(-1)}-2\frac{\zeta'(-3)}{\zeta(-3)}-3\frac{\zeta'(-5)}{\zeta(-5)}-2\frac{\zeta'(-7)}{\zeta(-7)}-2\frac{\zeta'(-9)}{\zeta(-9)}-\frac{14717}{1260}+\frac{11}{2}C\right)s+O(s^2)
\end{aligned}$$

(and here this coincides with $\widetilde{\lambda}^{-1}$).

Part III.

Hermitian automorphic vector bundles and Arakelov geometry

9. Hermitian automorphic vector bundles

In the first section of this chapter, we will define (integral) automorphic vector bundles, using the theory of part I, especially the theory of the (integral) standard principal bundle (3.5.3). We will define Hermitian metrics on their complexification, giving rise to Arakelov vector bundles. In particular we will later (10.4.1) define a canonical Hermitian line bundle $\Xi^*\mathcal{E}$ on the canonical model of a Shimura variety of orthogonal type and on their compactifications. Then we will compute the 'arithmetic volumes' of these Shimura varieties with respect to the first arithmetic Chern class of this bundle (10.5.2). The natural Hermitian metrics are, however, *singular* along the boundary of the toroidal compactifications. More precisely, they have singularities of log-type, giving rise to singularities of log-log-type for the corresponding Greens functions. Burgos, Kramer and Kühn [18, 19] constructed an extended Arakelov theory, which is able to deal with these more general objects. It is a general theory of cohomological arithmetic Chow rings, which takes an auxiliary complex of differential forms (with singularities of a prescribed type) as a parameter. The properties of these Chow rings very much depend on the properties of the complex used. In their papers [loc. cit.], they develop a theory of pre-log-log-forms as well as a theory of log-log-forms. While the first allows more general Greens functions and metrics, the second has a nicer behavior and the properties of the corresponding arithmetic Chow rings are closer to those of the classical arithmetic Chow rings as defined, for instance, in [88]. The arithmetic volumes computed with either theory agree [18, p. 624]. Since we will work only with Arakelov cycles that are intersections of the first Chern class of an Hermitian automorphic line bundle, we use here the theory of log-log-forms as developed in [18]. It will be presented in the next section. In the third section of this chapter, we recall the definition and properties of the theory of the corresponding arithmetic Chow rings. In the last section, we will define finally the arithmetic and geometric volume of arbitrary Shimura varieties.

9.1. Hermitian automorphic vector bundles

(9.1.1) Let ${}^K_\Delta\mathbf{X}$ be p-ECMSD (2.4.11) with smooth and complete Δ, E be the reflex field and $\mathcal{O}$ a reflex ring of it. For each $\sigma \in \mathrm{Gal}(\widetilde{E}|\mathbb{Q})$, where $\widetilde{E}$ is the Galois hull of E, there is a (p-integral) Shimura datum $\mathbf{X}^\sigma$ with the same group scheme $P_\mathbf{X}$ and $\mathbb{D}_\mathbf{X}$ but *different* $h_\mathbf{X}$, with reflex field E^σ and (conveniently) ring $\mathcal{O}^\sigma$, and the property $\mathrm{M}({}^K_\Delta\mathbf{X}^\sigma) \cong \mathrm{M}({}^K_\Delta\mathbf{X})^\sigma$. The complex structure induced on $\mathbb{D}_\mathbf{X}$ may vary!
The compact duals $\mathrm{M}^\vee(\mathbf{X}^\sigma)$ can all be identified with geometric components of the scheme $\mathcal{QPAR}$ (1.9.4) associated with $P_\mathbf{X}$ (cf. 3.4.1). The union $\bigcup_\sigma \mathrm{M}^\vee(\mathbf{X}^\sigma)$ is a subscheme of $\mathcal{QPAR}$ defined over $\mathbb{Z}_{(p)}$ and we may glue all these constructions for all

reflex rings $\mathcal{O} \subset E$ (corresponding to primes $\wp|(p)$ of E). Denote the result by $\mathrm{M}^\vee(\overline{\mathbf{X}})$. Similarly $\bigcup_\sigma \mathrm{M}({}^K_\Delta\mathbf{X}^\sigma)$ descends to a scheme (we assumed K to be neat) defined over $\mathbb{Z}_{(p)}$, and the similarly glued object we denote by $\mathrm{M}({}^K_\Delta\overline{\mathbf{X}})$. For a morphism of p-ECMSD $(\alpha, g) : {}^K_\Delta\mathbf{X} \to {}^{K'}_{\Delta'}\mathbf{Y}$, we have a morphism

$$\mathrm{M}(\overline{\alpha}, g) : \mathrm{M}({}^K_\Delta\overline{\mathbf{X}}) \to \mathrm{M}({}^{K'}_{\Delta'}\overline{\mathbf{Y}})$$

and similarly for $\mathrm{M}^\vee$. We denote by $\mathbb{D}_{\overline{\mathbf{X}}}$ the union of copies of $\mathbb{D}_{\mathbf{X}}$ for each σ.
Let $\mathcal{E}$ a locally free sheaf on

$$\left[\mathrm{M}^\vee(\overline{\mathbf{X}})/P_{\mathbf{X}}\right]$$

(i.e. a $P_{\mathbf{X}}$-equivariant sheaf on $\mathrm{M}^\vee(\overline{\mathbf{X}})$) and h_E be a $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-invariant Hermitian metric on $E_{\mathbb{C}}|_{h(\mathbb{D}_{\overline{\mathbf{X}}})}$ (where $h(\mathbb{D}_{\overline{\mathbf{X}}})$ is embedded into $\mathrm{M}^\vee(\overline{\mathbf{X}})(\mathbb{C})$ via the Borel embedding — 3.4.1).
Recall from (3.5.3) the 1-morphism (here extended to $\mathrm{M}({}^K_\Delta\overline{\mathbf{X}})$)

$$\Xi : \mathrm{M}({}^K_\Delta\overline{\mathbf{X}}) \to \left[\mathrm{M}^\vee(\overline{\mathbf{X}})/P_{\mathbf{X}}\right].$$

Let D be the boundary divisor (3.3.8). It is a divisor with normal crossings.

(9.1.2) We will define a Hermitian metric $\Xi^* h_E$ on $(\Xi^*\mathcal{E})_{\mathbb{C}}$, singular along $D_{\mathbb{C}}$, as follows[1]: We do this for each $\mathbf{X}^\sigma$ separately, so let $\mathbf{X}$ be one of them. By definition $\Xi^* E_{\mathbb{C}}(U)$, for any open $U \subset \mathrm{M}({}^K_\Delta\mathbf{X})(\mathbb{C})$, is given by the $P(\mathbb{Q})$-invariant sections of the pullback of $E_{\mathbb{C}}$ to the standard principal bundle $\mathrm{P}({}^K_\Delta\mathbf{X})$ via the compatibility isomorphism of Ξ over $\mathbb{C}$ with $\Xi_{\mathbb{C}}$ (3.5.3). Now consider the diagram of analytic manifolds (we assumed K to be neat):

$$\begin{tikzcd}
P_{\mathbf{X}}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K) \arrow[d, equal] & \mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K) \arrow[l] \arrow[r, "p"] \arrow[d, hook] & h(\mathbb{D}_{\mathbf{X}}) \arrow[d, hook] \\
P_{\mathbf{X}}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{X}} \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K) & P_{\mathbf{X}}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{C}) \times (P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K) \arrow[l] \arrow[r] & \mathrm{M}^\vee(\mathbf{X})(\mathbb{C})
\end{tikzcd}$$

Let $U \cong \mathrm{M}({}^K\mathbf{X})(\mathbb{C})$ be the complement of $D_{\mathbb{C}}$. The diagram shows that $\Xi^* E_{\mathbb{C}}|_U$ is canonically identified with the $P_{\mathbf{X}}(\mathbb{Q})$-invariant sections of the pullback $p^*(E_{\mathbb{C}}|_{h(\mathbb{D}_{\mathbf{X}})})$ to the standard local system $\mathbb{D}_{\mathbf{X}} \times P_{\mathbf{X}}(\mathbb{A}^{(\infty)})/K$. But this pullback carries the pullback of the Hermitian metric h_E. For the induced $P_{\mathbf{X}}(\mathbb{Q})$-action on $p^*(E_{\mathbb{C}}|_{h(\mathbb{D}_{\mathbf{X}})})$ compatible with the usual action on the standard local system, h_E is invariant as well and descends to a smooth Hermitian metric on U.

(9.1.3) Theorem. *Let ${}^K_\Delta\mathbf{X}$ be pure p-ECSD, let $(\mathcal{E}, h_E)$ as in (9.1.1) and assume that the unipotent radical of a stabilizer group under the action of $P_{\mathbf{X}}(\mathbb{C})$ on $\mathrm{M}^\vee(\mathbf{X})(\mathbb{C})$ acts*

[1] This is a slight abuse of notation, since the construction seems to depend on the particular presentation of the target stack as a quotient of $\mathrm{M}^\vee$

trivially on $\mathcal{E}$.
The vector bundles $\Xi^*\mathcal{E}_{\mathbb{C}}$ are the unique extensions to $\mathrm{M}({}^K_\Delta\overline{\mathbf{X}})$ *of these bundles on* $\mathrm{M}({}^K\overline{\mathbf{X}})$ *such that the metric $\Xi^* h_{\mathcal{E}}$ is good along the boundary divisor D_∞ in the sense of [79].*

Proof. cf. e.g. [42, §3.8][2]. □

See also (9.2.9).

(9.1.4) Definition. *Let $(\mathcal{E}, h_E)$ be as in (9.1.1). The locally free sheaf $\Xi^*\mathcal{E}$, together with the singular (along $D_{\mathbb{C}}$) Hermitian metric $\Xi^* h_E$ is called the* ***Hermitian automorphic vector bundle*** *associated to $(\mathcal{E}, h_E)$ on* $\mathrm{M}({}^K_\Delta\overline{\mathbf{X}})$*.*

Observe that this refines the notation in [18, Definition 6.1]. There merely a rational automorphic bundle with an arbitrary integral model is considered, the canonical integral structure (which relies on a chosen integral model of the input bundle) is not taken into account. We will later see (9.2.9) that these metrics have log-singularities in the sense of [18, Section 2] along the exceptional divisor, if we assume a pure Shimura datum and the automorphic vector bundle to be 'fully decomposed'.

(9.1.5) Remark. It follows directly from the construction of metrics in (9.1.2) that for a morphism of p-ECMSD

$$(\alpha, \rho) : {}^{K_1}_{\Delta_1}\mathbf{X}_1 \to {}^{K_2}_{\Delta_2}\mathbf{X}_2$$

and given $(\mathcal{E}, h_E)$, with $\mathcal{E}$ a locally free sheaf on $\left[\mathrm{M}^\vee(\mathbf{X}_1)/P_{\mathbf{X}}\right]$ and h_E a $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-invariant Hermitian metric on $E_{\mathbb{C}}|_{h(\mathbb{D}_{\mathbf{X}})}$, we have an isomorphism

$$\mathrm{M}(\alpha, \rho)^*(\Xi^*\mathcal{E}, \Xi^* h_E) \cong (\Xi^* \mathrm{M}^\vee(\alpha)^*\mathcal{E}, \Xi^* \mathrm{M}^\vee(\alpha)^* h_E).$$

Explanation: In (3.5.2, i.) the vertical maps are extensions to the models of the obvious (analytic) maps on double quotients.
Similarly, in the case of a boundary component

$$(\iota, \rho) : {}^{K_1}_{\Delta_1}\mathbf{B} \Longrightarrow {}^K_\Delta\mathbf{X},$$

if h_E is the restriction of a $P_{\mathbf{B}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-invariant metric $h_{E,1}$ on $\mathbb{D}_{\mathbf{B}}$ to $\mathbb{D}_{\mathbf{B}\Longrightarrow\mathbf{X}} \subset \mathbb{D}_{\mathbf{B}}$, we have on the formal completion

$$\mathrm{M}(\iota, \rho)^*(\Xi^*\mathcal{E}, \Xi^* h_E) \cong (\Xi^*\mathcal{E}|_{\mathrm{M}^\vee(\mathbf{B})}, \Xi^* h_{E,1})$$

taking into account that the formal isomorphism converges over $\mathbb{C}$. However, in general, the metrics h_E do not come from $h_{E,1}$'s as above (they are not even defined everywhere

[2]The reduction to the mentioned reference lacks details. It will be stated more precisely in forthcoming work

on $\mathbb{D}_{\mathbf{B}}$), but of course an $\Xi^* h_E$ is always defined in a neighborhood of $D_{\mathbb{C}}$ on $\mathrm{M}({}^{K_1}_{\Delta_1}\mathbf{B})(\mathbb{C})$ and the above statement makes sense.

9.2. The complexes of log-log-forms

(9.2.1) We set up the notation from [18, Section 2.1]:
Let X be a complex algebraic manifold of dimension d and let D a divisor with normal crossings on X. Set $U := X \setminus D$ and denote the embedding by $i : U \hookrightarrow X$.
Let V be an open coordinate subset of X with coordinates $z_1, \dots, z_n$. Denote $r_i = |z_i|$. We will say that V is **adapted to** D if the divisor $D \cap V$ on V is given by the equation $z_1 \cdots z_k = 0$. Assume that V is small, i.e. such that $\log|\log(r_i)| > 1$ on V.

(9.2.2) Definition ([18, Def. 2.2]). *Let V be a coordinate neighborhood adapted to D. For every integer $K \geq 0$, we say that a smooth complex function f on $V \setminus D$ has **logarithmic growth along** D **of order** K, if there exists an integer N_K, such that for every pair of multi-indices $\alpha, \beta \in \mathbb{Z}^d_{\geq 0}$ with $\alpha + \beta \leq K$, the following inequality holds true:*

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha}\frac{\partial^{|\beta|}}{\partial \overline{z}^\beta} f(z_1, \dots, z_d) \prec \frac{\left|\prod_{i=1}^k \log(1/r_i)\right|^{N_K}}{|z^{\alpha^{\leq k}}\overline{z}^{\beta^{\leq k}}|}.$$

*We say that f has **logarithmic growth of infinite order** along D, if it has logarithmic growth along D of order K for all $K \geq 0$. The sheaf of differential forms on X, with logarithmic growth of infinite order along D, denoted by $\mathcal{E}^*_X\langle D\rangle$ is the subalgebra of $i_*\mathcal{E}^*_U$ generated, in each coordinate neighborhood adapted to D, by the functions with logarithmic growth of infinite order along D and the differentials*

$$\frac{\mathrm{d}\, z_i}{z_i}, \frac{\mathrm{d}\, \overline{z}_i}{\overline{z}_i}, i = 1, \dots, k$$

$$\mathrm{d}\, z_i, \mathrm{d}\, \overline{z}_i, i = k+1, \dots, d.$$

Its elements are also called simply log***-forms*** *along D.*

(9.2.3) Definition ([18, Def 2.17, 2.22]). *Let V be a coordinate neighborhood adapted to D. For every integer $K \geq 0$, we say that a smooth complex function f on $V \setminus D$ has* log-log ***growth along*** D ***of order*** K*, if there exists an integer N_K, such that for every pair of multi-indices $\alpha, \beta \in \mathbb{Z}^d_{\geq 0}$ with $\alpha + \beta \leq K$, the following inequality holds true:*

$$\frac{\partial^{|\alpha|}}{\partial z^\alpha}\frac{\partial^{|\beta|}}{\partial \overline{z}^\beta} f(z_1, \dots, z_d) \prec \frac{\left|\prod_{i=1}^k \log(\log(1/r_i))\right|^{N_K}}{|z^{\alpha^{\leq k}}\overline{z}^{\beta^{\leq k}}|}.$$

We say that d has log-log ***growth of infinite order*** *along D, if it has* log-log *growth along D of order K for all $K \geq 0$. The sheaf of differential forms on X, with* log-

log *growth of infinite order along* D*, denoted by* $\mathcal{E}^*_X\langle\langle D\rangle\rangle_{gth}$ *is the subalgebra of* $i_*\mathcal{E}^*_U$ *generated, in each coordinate neighborhood adapted to* D*, by the functions with* log-log *growth of infinite order along* D *and the differentials*

$$\frac{\mathrm{d}\,z_i}{\log(r_i)z_i}, \frac{\mathrm{d}\,\overline{z}_i}{\log(r_i)\overline{z}_i}, i = 1, \ldots, k$$

$$\mathrm{d}\,z_i, \mathrm{d}\,\overline{z}_i, i = k+1, \ldots, d.$$

Its elements are also called simply log-log***-growth-forms*** *along* D*.*
The sheaf of log-log*-forms along* D*,* $\mathcal{E}^*_X\langle\langle D\rangle\rangle$ *is defined as the sheaf of forms* ω *with the property that* $\omega, \partial\omega, \overline{\partial}\omega$ *and* $\partial\overline{\partial}\omega$ *are* log-log *growth forms along* D *(this is, in this case, not implied by the requirement for* ω *itself). Its elements are also called simply* log-log***-forms*** *along* D*.*

(9.2.4) Definition ([18, Def. 4.29]). *Let* E *be a rank* n *vector bundle on* X*, let* E_0 *be the restriction to* U*. A smooth metric on* E_0 *is said to be* ***log-singular*** *along* D*, if for every* $x \in D$*, there exists a trivializing open coordinate neighborhood* V *adapted to* D *and a holomorphic frame* $\xi = \{e_1, \ldots, e_n\}$ *such that, writing* $h(\xi)_{i,j} = h(e_i, e_j)$*, we have*

i. The functions $h(\xi)_{ij}$*,* $\det\{h(\xi)\}^{-1}$ *are* log*.*

ii. The differential forms $\partial h(\xi) \cdot h(\xi)^{-1})_{ij}$ *are* log-log*.*

(9.2.5) Theorem ([18, Prop. 4.31]). *Let* E *and* F *be vector bundles on* X*, and let* E_0 *and* F_0 *their restrictions to* U*. Let* h_E *and* h_F *be smooth Hermitian metrics on* E_0 *and* F_0*. Write* $\overline{E} = (E, h_E)$ *and* $\overline{F} = (F, h_F)$*.*

i. The Hermitian vector bundle $\overline{E} \perp \overline{F}$ *is log-singular along* D*, if and only if* E *and* F *are log-singular along* D*.*

ii. If $\overline{E}$ *and* $\overline{F}$ *are log-singular along* D*, then* $\overline{E} \otimes \overline{F}$*,* $\Lambda^n\overline{E}$*,* $S^n\overline{E}$*,* $\overline{E}^*$*,* $\mathrm{Hom}(\overline{E}, \overline{F})$*, with induced metrics (see [18] or [88]), are log-singular along* D*.*

Let U be a Zariski open subset of X. There is a compactification $\pi : U \hookrightarrow \overline{U}$ of U, such that for $Y = \overline{U} \setminus \pi(U)$,

$$\overline{\pi(D)}, Y \text{ and } \overline{\pi(D)} \cup Y$$

are divisors with normal crossings on $\overline{U}$.

(9.2.6) Definition ([18, Def. 3.9]). *Define the complex of sheaves* $E^*_{l,ll}$ *on* X *by*

$$E^*_{l,ll}(U') = \lim_{\rightarrow} \Gamma(\overline{U'}, \mathcal{E}^*_{\overline{U'}}\langle B_{\overline{U'}}\rangle\langle\langle D\rangle\rangle),$$

where the limes is taken over all normal crossing compactifications $(\overline{U'}, B_{\overline{U'}})$ *of* U'*. Here* $\mathcal{E}^*_{\overline{U'}}\langle B_{\overline{U'}}\rangle\langle\langle D\rangle\rangle$ *is the sheaf of forms, which are* log *along* $B_{\overline{U'}}$ *and* log-log *along* D*.*

For each $U \subseteq X$ the complex $E^*_{l,ll}(U)$ is a Dolbeault-algebra with respect to wedge product.

(9.2.7) Definition ([18, Def. 3.10]). *For any Zariski open subset $U \subset X$ we put:*

$$\mathcal{D}^*_{l,ll,X,D}(U,p) = (\mathcal{D}^*_{l,ll,X,D}(U,p), \mathrm{d}_{\mathcal{D}}) = (\mathcal{D}^*(E_{l,ll}(U, D \cap U), p)^{\sigma}, \mathrm{d}_{\mathcal{D}}),$$

*where $\mathcal{D}^*(E_{l,ll}(U, D \cap U), p)$ is the Deligne algebra associated with the Dolbeault algebra $E^*_{l,ll}(U, D \cap U)$ and σ is the antilinear involution $\omega \mapsto \overline{F_\infty(\omega)}$ [19, Definition 7.18]. The complex $\mathcal{D}^*_{l,ll,X,D}$ is called the $\mathcal{D}_{\log}$**-complex of** log-log **forms**.*

Let $U \subset X$ be a Zariski open subset and $Y = X \setminus U$. Write

$$\widehat{H}^n_{\mathcal{D}^*_{l,ll},Y}(X,p) = \widehat{H}^n(\mathcal{D}^*_{l,ll,X,D}(X,p), \mathcal{D}^*_{l,ll,X,D}(U,p)),$$

where $\widehat{H}^n(\cdot,\cdot)$ means truncated relative cohomology ([19, Def. 2.55]). Similarly for the non-truncated version:

$$H^n_{\mathcal{D}^*_{l,ll},Y}(X,p) = H^n(\mathcal{D}^*_{l,ll,X,D}(X,p), \mathcal{D}^*_{l,ll,X,D}(U,p)).$$

A class $\mathfrak{g} \in \widehat{H}^n(\mathcal{D}^*_{l,ll}, Y)$ hence is the class of a pair $\mathfrak{g} = (\omega, \widetilde{g})$, where

$$\omega \in Z(\mathcal{D}^n_{l,ll,X,D}(X,p))$$

is a cocycle and

$$\widetilde{g} \in \mathcal{D}^*_{l,ll,X,D}(U,p)/\operatorname{im} \mathrm{d}_{\mathcal{D}}$$

is such that $\mathrm{d}_{\mathcal{D}}\,\widetilde{g} = \omega$. Explicitly we have:

$$\begin{aligned} Z(\mathcal{D}^n_{l,ll,X,D}(X,p)) &= \{\omega \in E^{p,p}_{l,ll}(X,D) \cap E^{2p}_{l,ll,\mathbb{R}}(X,D,p) \mid \mathrm{d}\,\omega = 0\}, \\ \frac{\mathcal{D}^*_{l,ll,X,D}(U,p)}{\operatorname{im} \mathrm{d}_{\mathcal{D}}} &= \frac{\{g \in E^{p-1,p-1}_{l,ll}(U, D \cap U) \cap E^{2p-2}_{l,ll,\mathbb{R}}(U, U \cap D, p-1)\}}{\operatorname{im} \partial + \operatorname{im} \overline{\partial}}. \end{aligned}$$

There are morphisms

$$\omega : \widehat{H}^n_{\mathcal{D}^*_{l,ll},Y}(X,p) \to Z(\mathcal{D}^n_{l,ll,X,D}(X,p)),$$

given by $\mathfrak{g} = (\widetilde{g}, \omega) \mapsto \omega$ and

$$\omega : \widehat{H}^n_{\mathcal{D}^*_{l,ll},Y}(X,p) \to H^n_{\mathcal{D}^*_{l,ll},Y}(X,p),$$

given by sending $\mathfrak{g} = (\widetilde{g}, \omega)$ to its class in the cohomology groups.

(9.2.8) Definition ([19, Def. 3.18-3.20]). *Let y be a p-codimensional algebraic cycle* *on X with* $\operatorname{supp} y \subset Y$. *A* ***weak*** log-log ***Green object for*** y *(with support in Y) is an element* $\mathfrak{g}_y \in \widehat{H}^{2p}_{\mathcal{D}_{l,ll},Y}(X,p)$ *such that*

$$\mathrm{cl}(\mathfrak{g}_y) = \mathrm{cl}(y) \in H^{2p}_{\mathcal{D}_{l,ll},Y}(X,p),$$

where $\mathrm{cl}(y)$ *is given by the image of the class of* y *in the real Deligne-Beilinson cohomology transported by the natural morphism* $H^{2p}_{\mathcal{D},Y}(X,\mathbb{R}(p)) \to H^{2p}_{\mathcal{D}_{l,ll},Y}(X,p)$. *If* $Y = \operatorname{supp} y$ *then* $\mathfrak{g}_y$ *is called a* log-log ***Green object for*** y.

(9.2.9) Theorem. *Let* ${}^K_\Delta\mathbf{X}$ *be* pure *p-ECSD, let* $(\mathcal{E}, h_E)$ *as in (9.1.1). and assume that the unipotent radical of a stabilizer group under the action of* $P_{\mathbf{X}}(\mathbb{C})$ *on* $\mathrm{M}^\vee(\mathbf{X})(\mathbb{C})$ *acts trivially on* $\mathcal{E}$.

The Hermitian metric $\Xi^* h_E$ *on the automorphic vector bundle* $\Xi^*\mathcal{E}$ *is* log-*singular along the boundary* D_∞ *of* $\mathrm{M}({}^K_\Delta\overline{\mathbf{X}})(\mathbb{C})$ *in the sense of (9.2.4).*

Proof. The assumption means that $\Xi^*\mathcal{E}$ is *fully decomposed* (see e.g. [79] or [18, section 6.1]). Recall (9.1.3) that the canonical extension of $\Xi^*\mathcal{E}$ defined via the canonical extension of the standard principal bundle coincide with the unique extensions of [79, Theorem 3.1]. Hence the statement is proven in [18, Theorem 6.3]. □

9.3. Cohomological arithmetic Chow groups

(9.3.1) Let R be a ring isomorphic to a subring of $\mathbb{C}$. There is a anti-linear involution

$$F_\infty : \bigoplus_{\sigma: R \hookrightarrow \mathbb{C}} \mathbb{C} \to \bigoplus_{\sigma: R \hookrightarrow \mathbb{C}} \mathbb{C}$$

given by $F_\infty(v)_\sigma = \overline{v_{\overline{\sigma}}}$.

For a scheme $\mathcal{X}$ over $\operatorname{spec}(R)$ we denote by

$$X_\infty = \coprod_{\sigma: R \hookrightarrow \mathbb{C}} X_\sigma(\mathbb{C}).$$

If $\mathcal{X}$ is regular and flat and quasi-projective over $\operatorname{spec}(R)$ with a divisor $\mathcal{D}$, such that $D_\infty \subset X_\infty$ is a divisor with normal crossings, then

$$(\mathcal{X}, \mathcal{D}^*_{l,ll,X_\infty,D_\infty})$$

is a $\mathcal{D}_{\log}$**-arithmetic variety** over $\operatorname{spec}(R)$ in the sense of [19, Def. 4.4].

Like in [19, Section 4] these induce **arithmetic Chow groups**:

$$\widehat{\mathrm{CH}}^*(\mathcal{X}, \mathcal{D}^*_{l,ll}) := \widehat{\mathrm{Z}}^p(\mathcal{X}, \mathcal{D}_{l,ll}) / \widehat{\mathrm{Rat}}^p(\mathcal{X}, \mathcal{D}_{l,ll}),$$

where

$$\widehat{\mathrm{Z}}^p(\mathcal{X}, \mathcal{D}_{l,ll}) = \{(y, \mathfrak{g}_y) \in \mathrm{Z}^p(\mathcal{X}) \oplus \widehat{H}^{2p}_{\mathcal{D}_{l,ll},\mathcal{Z}^p}(\mathcal{X}, p) \ \mid \ \mathrm{cl}(\mathfrak{g}_y) = \mathrm{cl}(y_\infty)\},$$

Here $\widehat{H}^{2p}_{\mathcal{D}_{l,ll},\mathcal{Z}^p}(\mathcal{X}, p) := \lim_{Y \in \mathcal{Z}_p} \widehat{H}^{2p}_{\mathcal{D}_{l,ll},Y}(X_\infty, p)^{F_\infty}$, the limit being taken over the set $\mathcal{Z}_p$ of all cycles on X_∞ of codimension $\geq p$ ordered by inclusion. $\widehat{\mathrm{Rat}}^p(\mathcal{X}, \mathcal{D}_{l,ll})$ is the subgroup of $\widehat{\mathrm{Z}}^p(\mathcal{X}, \mathcal{D}_{l,ll})$ generated by rational cycles [loc. cit.].

(9.3.2) Let $\mathbb{R}'$ be $\mathbb{R}$ modulo rational multiples of finitely many $\log(N)$, $N \in \mathbb{N}$. Define $\mathbb{R}^{(p)}$ as $\mathbb{R}$ modulo rational multiples of $\log(q)$, $q \neq p$ prime. Define $\mathbb{R}_N$ as $\mathbb{R}$ modulo rational multiples of $\log(q)$, $q|N$. Obviously $\mathbb{R}_N$ *injects* into the fibre product of all $\mathbb{R}^{(p)}$ $p \nmid N$ over $\mathbb{R}'$.
There is an obvious map $\widehat{\mathrm{CH}}^1(\mathrm{spec}(\mathbb{Z}_{(p)}))_{\mathbb{Q}} \to \mathbb{R}^{(p)}$ and in the sequel we will always consider the image in $\mathbb{R}^{(p)}$ of elements in $\widehat{\mathrm{CH}}^1(\mathrm{spec}(\mathbb{Z}_{(p)}))_{\mathbb{Q}}$.

(9.3.3) Any morphism $f : \mathcal{X}_1 \to \mathcal{X}_2$ such that $f^{-1}(\mathcal{D}_2) \subseteq \mathcal{D}_1$ induces a morphism between these $\mathcal{D}_{\log}$**-arithmetic varieties**, compare [18, section 3.4], and it induces a **pullback** morphism [18, section 5.4., iv, v]:

$$f^* : \widehat{\mathrm{CH}}^*(\mathcal{X}_2, \mathcal{D}^*_{l,ll,X_{2,\infty},D_{2,\infty}}) \to \widehat{\mathrm{CH}}^*(\mathcal{X}_1, \mathcal{D}^*_{l,ll,X_{1,\infty},D_{1,\infty}})$$

which is a ring homomorphism after tensoring with $\mathbb{Q}$, and which is compatible with the formation of Chern classes

(9.3.4) If $\mathcal{X}$ is *projective* over $\mathrm{spec}(R)$, there is a **direct image** morphism of groups [18, section 5.4, ii]:

$$\pi_* : \widehat{\mathrm{CH}}^{d+1}(\mathcal{X}, \mathcal{D}^*_{l,ll,X_\infty,D_\infty}) \to \widehat{\mathrm{CH}}^1(\mathrm{spec}(R)),$$

where d is the relative dimension of $\mathcal{X}$.

(9.3.5) Furthermore, there is a degree map $\widehat{\deg} : \widehat{\mathrm{CH}}^1(\mathrm{spec}(R)) \to \mathbb{R}_N$ for some N (see 9.3.2). Explicitly the composition $\widehat{\deg} \circ \pi_*$, which will be used for the definition of arithmetic volume, is given by:

$$\left(\sum_{P \in \mathrm{Z}^{d+1}(\mathcal{X})} n_P P, \mathfrak{g} \right) \mapsto \sum_{P \in \mathrm{Z}^{d+1}(\mathcal{X})} n_P \log(\#k(P)) + \frac{1}{(2\pi i)^d} \int_{\mathcal{X}_\infty} \mathfrak{g}$$

(similarly to the case of classical Arakelov theory).

(9.3.6) There is an **intersection pairing**:

$$\widehat{\mathrm{CH}}^p(\mathcal{X},\mathcal{D}^*_{l,ll})\otimes\widehat{\mathrm{CH}}^q(\mathcal{X},\mathcal{D}^*_{l,ll})\to\widehat{\mathrm{CH}}^{p+q}(\mathcal{X},\mathcal{D}^*_{l,ll})_{\mathbb{Q}},$$

which turns

$$\bigoplus_{p\ge 0}\widehat{\mathrm{CH}}^p(\mathcal{X},\mathcal{D}^*_{l,ll})_{\mathbb{Q}}$$

into a commutative ring [18, section 5.4, i].

If $(y,\mathfrak{g}_y)$ and $(z,\mathfrak{g}_z)$ are such that y_∞ and z_∞ intersect properly, the product is given by

$$(y,\mathfrak{g}_y)\cdot(z,\mathfrak{g}_z)=([y\cdot z],\mathfrak{g}_y*\mathfrak{g}_z).$$

Here $[y\cdot z]\in \mathrm{Z}^{p+q}(\mathcal{X})$ is already determined only up to *finite* rational equivalence, however, $[y\cdot z]_\infty$ is just the usual intersection cycle of y_∞ and z_∞. See [19, Theorem 4.19] for details.

(9.3.7) Let $U=X_\infty\setminus D_\infty$. Let $\mathrm{Z}^q_U(\mathcal{X})$ the group of codimension q cycles y on $\mathcal{X}$ such that y_∞ intersects D_∞ properly. There is a height pairing

$$(\cdot|\cdot):\widehat{\mathrm{CH}}^p(\mathcal{X},\mathcal{D}_{l,ll})\otimes\mathrm{Z}^p_U(\mathcal{X})\to\widehat{\mathrm{CH}}^{p+q-d}(\mathrm{spec}(R),\mathcal{D}_{l,ll})_{\mathbb{Q}},$$

defined in [19, Definition 7.54].

Let R now be finite over $\mathbb{Z}_{(p)}$. If we are given any log-log Green object $\mathfrak{g}_y=(\widetilde{g}_y,\omega_y)$ for y and are in the case $p+q=d+1$, the (degree of the) height may be computed by the formula:

$$\widehat{\deg}(\alpha|y)=\widehat{\deg}\circ\pi_*(\alpha\cdot[y,\mathfrak{g}_y])-\frac{1}{(2\pi i)^{p-1}}\int_{X_\infty}\widetilde{g}_y\omega(\alpha)$$

in $\mathbb{R}^{(p)}$ and does not depend on the choice of $\mathfrak{g}_y$ (it suffices even a pre-log-log Green object, but we don't need this here).

In case that y is the image of a morphism $\iota:\mathcal{Y}\to\mathcal{X}$ with $\iota(\mathcal{D}_{\mathcal{Y}})\subset\mathcal{D}_{\mathcal{X}}$ over spec(R), we get by the projection formula (follows from the abstract [19, Prop. 4.37])

$$(\alpha|y)=\pi_{Y*}\iota^*\alpha.$$

(9.3.8) Definition ([18, Definiton 5.4]). *We define the ring $\widehat{K_0}(\mathcal{X},\mathcal{D}_{l,ll})$ as the group generated by pairs $(\overline{E},\eta)$, where $\overline{E}$ is a vector bundle with log-singular Hermitian metric on $\mathcal{X}$ and*

$$\eta\in\bigoplus_p\mathcal{D}^{2p-1}_{l,ll}(X,p)/d_{\mathcal{D}}\mathcal{D}^{2p-2}_{l,ll}(X,p)$$

subject to the relations

$$(\overline{S},\eta)+(\overline{Q},\eta')=(\overline{E},\eta+\eta'+\widetilde{ch})$$

for each sequence

$$0 \longrightarrow \overline{S} \longrightarrow \overline{E} \longrightarrow \overline{Q} \longrightarrow 0,$$

where $\widetilde{ch}$ is the Bott-Chern secondary characteristic class of the sequence.
Multiplication in the ring is given by

$$(\overline{E}, \eta) \otimes (\overline{E}', \eta') = (\overline{E} \otimes \overline{E}', (\mathrm{ch}(\overline{E} + d_{\mathcal{D}}\eta) \bullet \eta' + \eta \bullet \mathrm{ch}(\overline{E}')).$$

(9.3.9) Theorem ([18, Theorem 5.5, i., v.]). *The arithmetic Chern character is a ring homomorphism:*

$$\widehat{\mathrm{ch}} : \widehat{K}_0(X, \mathcal{D}_{l,ll}) \to \bigoplus_p \widehat{\mathrm{CH}}^p(X, \mathcal{D}_{l,ll})_{\mathbb{Q}}.$$

9.4. Geometric and arithmetic volume of Shimura varieties

(9.4.1) Assume first in this section that all occurring K's are neat, see (9.4.6) for the general case. For a pure Shimura variety (version over $\mathbb{Z}_{(p)}$ as in 9.1.1) $\mathrm{M}({}^K_\Delta\overline{\mathbf{X}})$ with complete and smooth Δ, the exceptional divisor D has normal crossings (3.3.8) and we define $\widehat{\mathrm{CH}}(\mathrm{M}({}^K_\Delta\overline{\mathbf{X}}))$ as $\widehat{\mathrm{CH}}^p(\mathrm{M}({}^K_\Delta\overline{\mathbf{X}}), \mathcal{D}_{l,ll})$, where $\mathcal{D}_{l,ll}$ is the complex of forms with log-log-singularities along $D_{\mathbb{C}}$ (9.2.7). The underlying arithmetical ring is $\mathbb{Z}_{(p)}$ with its embedding into $\mathbb{C}$. We assume that $\mathrm{M}({}^K_\Delta\overline{\mathbf{X}})$ exists — this requires that Δ is sufficiently fine and that conjecture (3.3.2) is true (cf. 3.3.5).

(9.4.2) Definition. *Let ${}^K_\Delta\mathbf{X}$ be p-ECMSD with complete and smooth Δ, and $\overline{\mathcal{E}} = (\mathcal{E}, h_E)$ be an* invertible *sheaf $\mathcal{E}$ on $\left[\mathrm{M}^\vee(\overline{\mathbf{X}})/P_{\mathbf{X}}\right]$ and h_E a $P_{\mathbf{X}}(\mathbb{R})U_{\mathbf{X}}(\mathbb{C})$-invariant Hermitian metric on $E_{\mathbb{C}}|_{h(\mathbb{D}_{\overline{\mathbf{X}}})}$, respectively.*
*The **geometric volume***

$$\mathrm{vol}_E(\mathrm{M}({}^K_\Delta\mathbf{X}))$$

is defined as the volume of $\mathrm{M}({}^K_\Delta\mathbf{X})(\mathbb{C})$ with respect to the volume form $(\mathrm{c}_1(\Xi^*\overline{E}_{\mathbb{C}}))^d$.
*The **arithmetic volume** at p*

$$\widehat{\mathrm{vol}}_{\overline{\mathcal{E}},p}(\mathrm{M}({}^K_\Delta\mathbf{X}))$$

is defined as

$$\frac{1}{\deg(\widetilde{E} : \mathbb{Q})} \pi_*(\hat{\mathrm{c}}_1(\Xi^*\mathcal{E}, \Xi^* h_E)^d)$$

in $\widehat{\mathrm{CH}}^1(\mathrm{spec}(\mathbb{Z}_{(p)})) = \mathbb{R}^{(p)}$, where $\pi : \mathrm{M}({}^K_\Delta\overline{\mathbf{X}}) \to \mathrm{spec}(\mathbb{Z}_{(p)})$ is the structural morphism If $P_{\mathbf{X}}$ is a group scheme of type (P) over $\mathrm{spec}(\mathbb{Z}[1/N])$, *and K is admissible* for all $p \nmid N$, *then we define the global **arithmetic volume***

$$\widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}(\mathrm{M}({}^K_\Delta\mathbf{X}))$$

in $\mathbb{R}_N$ *as the value determined by the arithmetic volumes at* p *(compare 9.3.2, see also 9.4.7).*
If $(\alpha, \rho) : {}^{K'}_{\Delta'}\mathbf{Y}_{\mathbb{Q}} \hookrightarrow {}^{K}_{\Delta}\mathbf{X}_{\mathbb{Q}}$ *is an embedding of* rational *Shimura data, we define the* ***height*** *at* p*:*

$$\mathrm{ht}_{\overline{\mathcal{E}},p}(\mathrm{M}({}^{K'}_{\Delta'}\mathbf{Y})) = \frac{1}{\deg(\widetilde{E} : \mathbb{Q})}(\overline{\mathrm{M}(\overline{\alpha}, \rho)(\mathrm{M}({}^{K'}_{\Delta'}\overline{\mathbf{Y}}))}^{Zar} | (\widehat{\mathrm{c}}_1(\Xi^*\mathcal{E}, \Xi^* h_E))^{d-q}),$$

where $(\cdot|\cdot)$ *is the height pairing (9.3.7),* q *is the codimension, and* $\mathrm{M}(\overline{\alpha}, \rho)$ *is the* $\mathbb{Z}_{(p)}$*-morphism (9.1.1). Similarly for the (global) height.*

(9.4.3) Question. *We mention that there is a* ***proportionality principle*** *[79] in the geometric case. This means that* all *polynomials of degree* $\dim(\mathrm{M}({}^{K}_{\Delta}\mathbf{X})_{\mathbb{C}})$ *in the Chern classes of an automorphic vector bundle* $(\Xi^*\mathcal{E})_{\mathbb{C}}$*, considered as a number, are proportional to the same expression in the Chern classes of the original bundle* $\mathcal{E}$*, computed on* $\mathrm{M}^{\vee}(\mathbf{X})$*. Is there an analogue in the arithmetic case? — cf. also [51].*

(9.4.4) Remark. If (α, ρ) was in fact an embedding of *p-integral* data: ${}^{K'}_{\Delta'}\mathbf{Y} \hookrightarrow {}^{K}_{\Delta}\mathbf{X}$, we have, by the projection formula (cf. 9.3.7) and (9.1.5):

$$\mathrm{ht}_{\overline{\mathcal{E}},p}(\mathrm{M}({}^{K'}_{\Delta'}\mathbf{Y})) = \widehat{\mathrm{vol}}_{\mathrm{M}^{\vee}(\overline{\alpha},\rho)^*\overline{\mathcal{E}}}(\mathrm{M}({}^{K'}_{\Delta'}\mathbf{Y})).$$

(9.4.5) Lemma. *The geometric and arithmetic volumes do not depend on the rational polyhedral cone decomposition* Δ*, i.e. not on the chosen toroidal compactification.*

Proof. For each pair Δ_i, $i = 1, 2$, there is a common refinement Δ, cf. (2.4.12), and we have two projections (3.2.2):

$$\begin{tikzcd} & \mathrm{M}({}^{K}_{\Delta}\overline{\mathbf{X}}) \arrow[dl] \arrow[dr] & \\ \mathrm{M}({}^{K}_{\Delta_1}\overline{\mathbf{X}}) & & \mathrm{M}({}^{K}_{\Delta_2}\overline{\mathbf{X}}) \end{tikzcd}$$

Furthermore this diagram is compatible with formation of $\Xi^*\mathcal{E}, \Xi^* h_E$ by (9.1.5). Its Chern forms are therefore transported into each other by pullback along the arrows in the diagram. Since the forms are integrable on every $\mathrm{M}({}^{K}_{\Delta}\overline{\mathbf{X}})$ (see [79], cf. also 9.1.3), the geometric volume agrees.
Now, there is pullback map

$$p^* : \widehat{\mathrm{CH}}^*(\mathrm{M}({}^{K}_{\Delta_i}\overline{\mathbf{X}}))_{\mathbb{Q}} \to \widehat{\mathrm{CH}}^*(\mathrm{M}({}^{K}_{\Delta}\overline{\mathbf{X}}))_{\mathbb{Q}}$$

as well, which is a ring homomorphism and compatible with Chern classes (9.3.3).

Therefore the assertion for the arithmetic volume boils down to the fact that for a class $x = (z, \mathfrak{g}) \in \widehat{\mathrm{CH}}^{n+1}({}^K_\Delta\mathbf{X})$, where z is a zero-cycle in the fibre above p and $\mathfrak{g}$ is a top-degree Green element, we have $\widehat{\deg}(\pi_* p^*(x)) = \widehat{\deg}(\pi_* x)$, where the π are the respective structural morphisms. This is true, since we may assume w.l.o.g. that z is supported outside of D and the push-forward of $\mathfrak{g}$ which is computed by an integral. These integrals are equal by the same reason as above. □

(9.4.6) Remark. If K is not neat, M is only a Deligne-Mumford stack. Instead of extending the theory of [18, 19] to stacks like e.g. in [29], we define the arithmetic volume in this case as follows: We take some neat admissible $K' \subset K$ and define

$$\widehat{\mathrm{vol}}_{\overline{\mathcal{E}},p}(\mathrm{M}({}^K_\Delta\overline{\mathbf{X}})) := \frac{1}{[K : K']}\widehat{\mathrm{vol}}_{\overline{\mathcal{E}},p}(\mathrm{M}({}^{K'}_\Delta\overline{\mathbf{X}}))$$

as value in $\mathbb{R}^{(p)}$. Similarly for the global arithmetic volume in $\mathbb{R}_N$, assuming that both K and K' are admissible for all $p \nmid N$. It follows from (9.1.5), (9.3.3) and (9.3.5) that this is independent of the choice of K'. In particular, if K is neat — or using any reasonable extension of the theory of [18, 19] to stacks — this definition agrees with the previous one.

In this sense the geometric and arithmetic volume do not depend essentially on K, as they are both multiplied by the index, if K is changed. For the arithmetic case this is of course only true for admissible K and the more primes are considered the less admissible groups there are.

(9.4.7) Remark. From the construction process in part I of this work, since we defined the models as the normalization of a Zariski closure in a variety defined over some $\mathbb{Z}[1/N]$, it follows that there is even a model of $\mathrm{M}({}^K_\Delta\overline{\mathbf{X}})$ defined over some $\mathbb{Z}[1/N]$, yielding the models coming from the canonical over the various reflex rings $\mathcal{O}$, such that the arithmetic volume of this model is the value determined above. This justifies a priori that our value lies in $\mathbb{R}_N$. Otherwise we will not care about this (global) model.

10. Shimura varieties of orthogonal type

10.1. The spin groups

Let S be a scheme, L a locally free sheaf and $Q_L : L \to \mathcal{O}_S$ be a quadratic form (6.1.1).

(10.1.1) Definition. *We call the **Clifford algebra** $\mathrm{C}(L)$ of L an algebra with the following universal property*

$$\mathrm{Hom}_{alg}(\mathrm{C}(L), A) = \{f : \mathrm{Hom}_{\mathcal{O}_S}(L, A) \mid f^2 = e \circ Q_L\}$$

for any algebra A over S with unit $e : \mathcal{O}_S \to A$.

(10.1.2) Lemma. *The Clifford algebra of L exists, and is obtained from the tensor algebra $\mathrm{T}(L)$ by factoring out the relation $v^2 - Q_L(v)$. Since the relations are homogeneous if the degree is considered mod 2, a 2-grading*

$$\mathrm{C}(L) = \mathrm{C}(L)^+ \oplus \mathrm{C}(L)^-$$

survives. It is locally free of finite dimension.

From $v^2 = Q_L(v)$ one derives $vw + wv = \langle v, w \rangle_Q$.

(10.1.3) Definition. *We define the **main involution** on $\mathrm{C}(L)$ by*

$$(v_1 \cdots v_n)' = (-v_n) \cdots (-v_1).$$

(10.1.4) Definition/Theorem. *There are group schemes $\mathrm{Spin}(L)$ and $\mathrm{GSpin}(L)$ over S, characterized by the following functors of points:*

$$\mathrm{Spin}(L)(R) = \{g \in \mathrm{C}(L)^+_R \mid gg' = 1, gLg^{-1} = L\}$$

$$\mathrm{GSpin}(L)(R) = \{g \in \mathrm{C}(L)^+_R \mid gg' = \lambda(g), gLg^{-1} = L\}$$

where $\lambda(g) \in \mathcal{O}^*_{S'}(R)$ and R is any ring with morphism $\mathrm{spec}(R) \to S$.

(10.1.5) They sit in an exact diagram of group schemes

$$\begin{tikzcd} 1 \arrow[r] & \mu_2 \arrow[r] \arrow[d, hook] & \mathrm{Spin}(L) \arrow[r] \arrow[d, hook] & \mathrm{SO}(L) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, "2"] & \mathrm{GSpin}(L) \arrow[r] \arrow[d, "\lambda"] & \mathrm{SO}(L) \arrow[r] & 1 \\ & \mathbb{G}_m \arrow[r, equal] & \mathbb{G}_m & & \end{tikzcd} \tag{1}$$

We let $\mathrm{SO}(L)$ act on L by its natural operation and $\mathrm{GSpin}(L)$ by $gv \mapsto gvg'$. Note that these actions are *not* compatible with the projection $\mathrm{GSpin}(L) \to \mathrm{SO}(L)$.
From now on, we assume that $S = \mathrm{spec}(\mathbb{Z}_{(p)})$, $p \neq 2$. *Actually most of the statements in part III should be true for* $p = 2$, *but since the theory of part I and also several pieces of part II are not available yet for* $p = 2$, *we have not rigorously checked this.*

(10.1.6) Remark. The middle row remains exact on sections over any discrete valuation ring or field (1.1.2). Here μ_2 (resp. $\mathbb{G}_m$) are *not* equal to the center of $\mathrm{Spin}(L)$ (resp. $\mathrm{GSpin}(L)$). This is only true for n even. For n odd, the center of $\mathrm{Spin}(L_{\mathbb{Q}})$ is of the form $\mathbb{Z}/2 \times \mathbb{Z}/2$ or $\mathbb{Z}/4$ (over fields of char 0), according to parity of $n/2$.

(10.1.7) Theorem. GSpin, Spin *and* SO *are reductive, if* $\langle \cdot, \cdot \rangle_Q$ *induces an isomorphism* $L^* \to L$.

Proof. The definition is compatible with base change and over an algebraically closed field k the statement is well-known.
Hence GSpin, Spin and SO are reductive group schemes over S (cf. 1.6.1). ☐

There exists an orthogonal basis $\{v_i\}$ of L (6.4.15), such that all $Q_L(v_i)$ are units. We have the basis

$$v_{i_1} \cdots v_{i_j}, i_1 < \cdots < i_j, j \text{ even}$$

of $\mathrm{C}^+(L)$ and the basis

$$v_{i_1} \cdots v_{i_j}, i_1 < \cdots < i_j, j \text{ odd}$$

of $\mathrm{C}^-(L)$. The trace of any basis element (acting by left multiplication on $\mathrm{C}^+(L)$ or $\mathrm{C}^-(L)$) except 1 is 0. The trace of 1 is 2^{m-1} in any case.

(10.1.8) Lemma. *For an element* $\delta \in \mathrm{C}^+(L)^*$ *with* $\delta' = -\delta$, *the form*

$$\langle x, y \rangle_\delta \mapsto \mathrm{tr}(x\delta y')$$

on $\mathrm{C}^+(L)$ *is symplectic, non-degenerate and* $\mathrm{Spin}(Q)$*-invariant (respectively* $\mathrm{GSpin}(Q)$*-invariant up to scalar given by* λ*), where these groups act by left multiplication.*

Proof. The form is invariant because $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Since δ is invertible, non-degeneracy is equivalent to that of $\mathrm{tr}(xy')$ which is given by an invertible diagonal matrix with respect to the basis chosen above. The form is symplectic because tr is invariant under $'$ and $\delta' = -\delta$. □

10.2. Hermitian symmetric domains of orthogonal type

In this section we will describe the Shimura data *of orthogonal type*, in particular, their associated Hermitian symmetric domains.

(10.2.1) Suppose again, $L_{\mathbb{Z}_{(p)}}$ is a lattice with non-degenerate, unimodular quadratic form. Suppose that $L_\mathbb{R}$ has signature $(m-2,2)$, $m \geq 3$. A polarized Hodge structure of type $(-2,0),(-1,-1),(0,-2)$ on $L_\mathbb{C}$ with $\dim(L^{-2,0}) = 1$ is determined by an isotropic subspace $L^{-2,0}$, satisfying $\langle z, \overline{z} \rangle < 0$ for nonzero $z \in L^{-2,0}$. Then the other spaces in the Hodge decomposition are determined by $L^{0,-2} = \overline{L^{-2,0}}$ and $L^{-1,-1} = (L^{-2,0} + L^{0,-2})^\perp$. We define $\mathbb{D}_{\mathbf{S}(L)} = \mathbb{D}_{\mathbf{O}(L)}$ to be the set of these Hodge structures. It is identified with

$$\{< z > \in \mathbb{P}(L_\mathbb{C}) \mid \langle z, z \rangle = 0, \langle z, \overline{z} \rangle < 0\}.$$

Hence it is equipped with a natural complex structure.
$\mathbb{D}_\mathbf{S}$ has 2 connected components. There is a 2:1 map

$$\begin{aligned} \mathbb{D}_{\mathbf{S}(L)} &\to \mathrm{Grass}^-(L_\mathbb{R}) \\ < z > &\mapsto D_{<z>} := (< z, \overline{z} >)^{\mathrm{Gal}(\mathbb{C}|\mathbb{R})} \end{aligned}$$

onto the Grassmannian Grass^- of negative definite subspaces of $L_\mathbb{R}$. It has a section determined by a choice of $\mathbb{C}$-orientation[1] on any of the subspaces D. The images of these sections are the 2 connected components of $\mathbb{D}_\mathbf{S}$. It will be convenient to fix an orientation on some (hence on all) of the negative definite subspaces D. This identifies $\pi_0(\mathbb{D}_{\mathbf{S}(L)})$ with $\mathbb{D}_{\mathbf{H}_0} = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}(1))$ and induces a morphism of p-MSD

$$\mathbf{S}(L) \to \mathbf{H}_0.$$

The space $D_{<z>}$ has an orthonormal (with respect to Q_L) basis given by

$$x_1 = \frac{z + \overline{z}}{\sqrt{-\langle z, \overline{z} \rangle}}, \tag{2}$$

$$x_2 = \pm \frac{z - \overline{z}}{\sqrt{\langle z, \overline{z} \rangle}}. \tag{3}$$

Let $P_{\mathbf{S}(L)} := \mathrm{GSpin}(L_{\mathbb{Z}_{(p)}})$ act on $L_{\mathbb{Z}_{(p)}}$ via $g \cdot v = gvg'$.

[1] i.e. the choice of a linear isomorphism $D \cong \mathbb{C}$

(10.2.2) Lemma. $\mathbf{S}(L)$ *is p-integral pure Shimura data.*

In the case that $L_{\mathbb{Z}_{(p)}}$ *is as above, with* $L_{\mathbb{R}}$ *of signature* $(0,2)$, GSpin *does not operate at all on* $\mathbb{D}_{\mathbf{S}}$*, if the latter is defined as above (i.e. consisting of 2 points). In this case we get 2 different p-integral Shimura data* $\mathbf{S}(L)_{\pm}$*. If* L' *is a saturated sublattice of* L *of signature* $(m-n-2,2)$*,* $0 \leq n \leq m-2$*, it induces an embedding of p-integral Shimura data*

$$\mathbf{S}(L') \hookrightarrow \mathbf{S}(L).$$

There is an embedding

$$\mathbf{S}(L) \hookrightarrow \mathbf{H}(\mathrm{C}^{+}(L), \langle\cdot,\cdot\rangle_{\delta})$$

and accordingly $\mathbf{S}(L)$ *is of Hodge type.*

Proof. To each $< z > \in \mathbb{D}_{\mathbf{S}(L)}$ the associated morphism h factors through $\mathrm{GSpin}(L_{\mathbb{R}})$. It is explicitly given by

$$\mathbb{S} \to \mathrm{C}^{+}(L, \mathbb{R})$$
$$w = a + bi \mapsto a + bi\frac{z\overline{z} - \overline{z}z}{\langle z, \overline{z}\rangle} = a + bx_1x_2$$

for any choice of $i \in \mathbb{C}$ (note: x_2 depends on the choice of $i \in \mathbb{C}$ as well).

It even defines a field isomorphism $\mathbb{C} \cong \mathrm{C}^{+}(D_{<z>}, \mathbb{R})$. It operates on $L^{-2,0}$ by w^2, on $L^{0,-2}$ by $\overline{w}^2$ and on $L^{-1,-1}$ by $w\overline{w}$. The second statement is true because the inclusion $\mathrm{C}^{+}(L') \subset \mathrm{C}^{+}(L)$ induces a closed embedding $\mathrm{GSpin}(L') \hookrightarrow \mathrm{GSpin}(L)$ and any morphism h of the first datum defines a morphism h for the second datum of the same type.

For the property of Hodge type, fix a positive definite $\mathbb{Z}_{(p)}$-sublattice D_0 which is a direct summand (exists because the form is assumed to be unimodular — 6.4.15) and an orthonormal basis x_1, x_2 (hence an induced orientation of $D_{0,\mathbb{R}}$). Consider the element $\delta := x_1x_2$. It satisfies the requirements of lemma (10.1.8).

Let $< z >$ be a point in $\mathbb{D}_{\mathbf{S}(L)}$. $P_z^{-1,0} := \frac{z\overline{z}}{\langle z, \overline{z}\rangle}$ resp. its complex conjugate, satisfy $P_z^{-1,0} + P_z^{0,-1} = \mathrm{id}$, $(P_z^{i,j})^2 = P_z^{i,j}$, and on $P_z^{i,j}\,\mathrm{C}^{+}(L_{\mathbb{C}})$ the morphism h operates as $w^{-i}\overline{w}^{-j}$. Furthermore, for z chosen, such that $D_z = D_0$, the form $\langle\cdot, x_1x_2\cdot\rangle_{\delta} = \langle\cdot, h(i)\cdot\rangle_{\delta} = \mathrm{tr}(x\delta y'\delta') = -\mathrm{tr}(x\delta y'\delta)$ is symmetric and definite. (Here $i \in \mathbb{C}$ it the root of -1 determined by the orientation $x_1 \wedge x_2$ of $D_0 = D_z$, see (2)). The substitution $x \mapsto gx$, $y \mapsto gy$ for any g in the spin group changes x_1x_2 in $g(x_1)g(x_2)$ but does not affect the properties of symmetry and definiteness. The sign of definiteness, however, is reflected by the chosen orientation of D_0.

Hence the map $< z > \mapsto P_z^{-1,0}\,\mathrm{C}^{+}(L_{\mathbb{C}})$ may be seen as a map $\mathbb{D}_{\mathbf{S}(L)} \to \mathbb{D}_{\mathbf{H}(\mathrm{C}^{+}(L),\langle\cdot,\cdot\rangle_{\delta})}$. Together with the closed embedding

$$\mathrm{GSpin}(L) \hookrightarrow \mathrm{GSp}(\mathrm{C}^{+}(L), \langle\cdot,\cdot\rangle_{\delta}),$$

it induces an embedding $\mathbf{S}(L) \hookrightarrow \mathbf{H}(\mathrm{C}^{+}(L), \langle\cdot,\cdot\rangle_{\delta})$. $\square$

(10.2.3) Theorem. *If* $\dim(L) \geq 3$*, the reflex field of* $\mathbf{S}(L)$ *is* $\mathbb{Q}$*. The compact dual is the zero quadric, i.e.*

$$\mathrm{M}^\vee(\mathbf{S}(L))(R) = \{< z > \in \mathbb{P}(L_R) \mid \langle z, z\rangle = 0\}$$

for any ring R *over* $\mathbb{Z}_{(p)}$*.*

If $\dim(L) = 2$*, the reflex field of* $\mathbf{S}(L)_+$*, resp.* $\mathbf{S}(L)_-$ *is the imaginary quadratic field (in* $\mathbb{C}$*) associated with the (negative) definite binary quadratic form* L*. The compact dual is a point given by the isotropic vector corresponding to* the *filtration in* $\mathbb{D}_{\mathbf{S}(L)_+}$*, respectively* $\mathbb{D}_{\mathbf{S}(L)_-}$*.*

Proof. There is a negative definite subspace D defined over $\mathbb{Z}_{(p)}$ (6.4.15). Over an imaginary quadratic field F, unramified at p, there is a morphism

$$\mu : \mathbb{G}_m \hookrightarrow \mathrm{GSpin}(L),$$

acting with exponent 2 on one isotropic vector z, with exponent 0 on $\overline{z}$ and with exponent 1 on the orthogonal complement. Over $\mathbb{C}$ it yields the associated morphism u_h (3.1.1), where h is one of the 2 morphisms corresponding to D. The morphism $\overline{u_h}$ is also of the form $u_{h'}$ and hence conjugated to u_h if $n \geq 3$. The conjugacy class of these morphisms is therefore defined over $\mathbb{Q}$. If $n = 2$, $\mathrm{GSpin}(L)$ is a torus and hence the reflex field is F. Furthermore, the morphism u defines the filtration $0 \subset < z > \subset < z >^\perp \subset L$ which is completely determined by $< z >$. $\square$

(10.2.4) Remark. The Hodge embedding from (10.2.2) induces a map (3.4.1)

$$\mathrm{M}^\vee(\mathbf{S}(L)) \hookrightarrow \mathrm{M}^\vee(\mathbf{H}(\mathrm{C}^+(L), \langle\cdot,\cdot\rangle_\delta)).$$

Here $\mathrm{M}^\vee(\mathbf{S}(L))$ is the space of isotropic lines in L and $\mathrm{M}^\vee(\mathbf{H}(\mathrm{C}^+(L), \langle\cdot,\cdot\rangle_\delta))$ is the space of Lagrangian subspaces of $\mathrm{C}^+(L)$ with respect to the form $\langle\cdot,\cdot\rangle_\delta$. One can read of from the proof of (10.2.2) that this map is given (for any R) by sending a $< z >$ in L_R, with z assumed to be primitive to the subspace $zw\,\mathrm{C}^+(L_R)$, where w is a primitive vector with $\langle z, w\rangle = 1$ (this subspace actually does not depend on w). One also immediately sees that this is compatible with the Borel embedding because $w = \frac{\overline{z}}{\langle z, \overline{z}\rangle}$ leads to the projector considered in the proof of (10.2.2).

(10.2.5) Lemma. *Let* $L_{\mathbb{Q}}$ *be a vector space with non-degenerate quadratic form. Let* K *be a compact open subgroup of* $\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$*. There is a compact open subgroup* $K' \subset$

$\mathrm{GSpin}(L_{\mathbb{A}^{(\infty)}})$ *and a surjection*

$$\begin{array}{c} \mathrm{GSpin}(L_{\mathbb{Q}})\backslash(\mathbb{D}_{\mathbf{S}(L)} \times \mathrm{GSpin}(L_{\mathbb{A}^{(\infty)}})/K') \\ \big\downarrow \\ \mathrm{SO}(L_{\mathbb{Q}})\backslash(\mathbb{D}_{\mathbf{O}(L)} \times \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K). \end{array}$$

If $L_{\mathbb{Z}_{(p)}}$ is a lattice in $L_{\mathbb{Q}}$ such that the induced form in non-degenerate (i.e. induces an isomorphism $L_{\mathbb{Z}_{(p)}} \to L^*_{\mathbb{Z}_{(p)}}$ and if K' is admissible, K can be chosen admissible. The induced morphism $\mathrm{GSpin}(L_{\mathbb{Z}_{(p)}})(\mathbb{Z}_{(p)}) \to \mathrm{SO}(L_{\mathbb{Z}_{(p)}})(\mathbb{Z}_{(p)})$ is surjective in this case as well.

Proof. First, some subgroup K' with $p(K') \subset K$ exists. From (10.1.5) and Hilbert 90 we see that we have a surjection $\mathrm{GSpin}(L_{\mathbb{Q}_p}) \to \mathrm{SO}(L_{\mathbb{Q}_p})$. Since the application

$$\mathrm{GSpin}(L_{\mathbb{Z}_p}) \to \mathrm{SO}(L_{\mathbb{Z}_p})$$

is surjective for almost all p (the group are defined by smooth group schemes), we get a surjection $\mathrm{GSpin}(L_{\mathbb{A}^{(\infty)}}) \to \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$. The last statements follow because we have a map of *reductive* group schemes $\mathrm{GSpin}(L_{\mathbb{Z}_{(p)}}) \to \mathrm{SO}(L_{\mathbb{Z}_{(p)}})$. It is surjective by (1.1.2). □

(10.2.6) Remark. The Shimura datum $\mathbf{O}(L) = \mathbf{S}(L)/\mathbb{G}_m$ itself is *not* of Hodge type but satisfies the requirement of (3.3.7)[2], hence we have also integral models $\mathrm{M}({}^K\mathbf{O}(L))$ with all the good properties required. We will work predominantly with those.

(10.2.7) Lemma. *Let K be an admissible compact open subgroup of $\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ and Δ a K-admissible rational polyhedral cone decomposition. The (geometric) connected components of $\mathrm{M}({}^K_\Delta\mathbf{O})_{\overline{\mathbb{Q}}}$ and $\mathrm{M}({}^K_\Delta\mathbf{O})_{\overline{\mathbb{F}_p}}$ are in bijection and defined over $\mathrm{spec}(\mathbb{Z}_{(p)}[\zeta_N])$, where ζ_N, $p \nmid N$ is an N-th root of unity.*

Proof. This follows directly from (3.2.2, 3.3.5) using the previous (10.2.5) and [77, Theorem 5.17]. □

(10.2.8) Lemma. *Let $L_{\mathbb{Z}}$ be a lattice with quadratic form of discriminant $D \neq 0$. Let K be a compact open subgroup of $\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ and pick a set of representatives*

$$g_i \in \mathrm{SO}(\mathbb{Q})\backslash \mathrm{SO}(\mathbb{A}^{(\infty)})/K.$$

[2] (3.3.7) lacks details. This will be remedied in forthcoming work

For each g_i there is the lattice $L^{(i)}_{\mathbb{Z}}$ characterized by $L^{(i)}_{\mathbb{Z}_p} = g_i L_{\mathbb{Z}_p}$. We have

$$\left[\mathrm{SO}(\mathbb{Q})\backslash\mathbb{D}_{\mathbf{O}} \times (\mathrm{SO}(\mathbb{A}^{(\infty)})/K)\right] = \bigcup_i [\Gamma_i\backslash\mathbb{D}_{\mathbf{O}}],$$

where Γ_i is the subgroup of $\mathrm{SO}(L^i_{\mathbb{Z}})$ defined by $\mathrm{SO}(\mathbb{Q}) \cap g_i K g_i^{-1}$.
If K is the discriminant kernel, each Γ_i is the discriminant kernel of the respective $L^{(i)}$.

Proof. [46, 1.10] or [77]. □

(10.2.9) Remark. The embedding of (3.4.1, v) is just the restriction to those $< z >$ satisfying $\langle z, \overline{z}\rangle < 0$. Note that although $\mathrm{M}^\vee(\mathbf{S}(L))$ is defined over $\mathbb{Z}_{(p)}$, there are cases where $\mathrm{M}^\vee(\mathbf{S}(L))(\mathbb{Z}_{(p)})$ is empty (Witt rank 0). These cannot occur if $n \geq 5$ by Meier's theorem.

The rest of this section is devoted to an investigation of all boundary components (2.4.5) of $\mathbf{O}(L)$, or $\mathbf{S}(L)$, respectively.
According to (2.4.4) we have to classify admissible parabolic subgroups of $\mathrm{SO}(L)$ or $\mathrm{GSpin}(L)$ (this amounts to the same). We may also restrict ourselves to $\mathrm{SO}(L_{\mathbb{Q}})$ because of the projectivity of $\mathcal{PAR}$ (1.9.4).
We first examine all parabolics of $\mathrm{SO}(L_{\mathbb{C}})$. We identify the Lie algebra of $\mathrm{SO}(L)$ with $\Lambda^2 L$ acting by $(v \wedge v')w = \langle v', w\rangle v - \langle v, w\rangle v'$.
We will examine the cases $m = 4, 6, 8, \ldots$ and $m = 3, 5, 7, \ldots$ separately.

(10.2.10) Let $m = 2g \geq 4$ be even. Choose a decomposition $L_{\mathbb{C}} = L_0 \oplus L_0^*$ where L_0, L_0^* are maximal isotropic with bilinear form given by $\langle v_1 + v_1^*, v_2 + v_2^*\rangle = v_2^* v_1 + v_1^* v_2$, and a basis $v_1, \ldots, v_g$ of L_0. Denote the dual basis by $v_1^*, \ldots, v_g^*$. There is a maximal torus, a $\mathbb{G}_m^g$ acting by $v_i \mapsto \lambda_i v_i$ and $v_i^* \mapsto \lambda_i^{-1} v_i^*$. A set of roots together with their root spaces in $\mathrm{Lie}(\mathrm{SO}(L)) = \Lambda^2 L$ is given by

$r_{i,j}:$	$\lambda_i\lambda_j$	$< v_i \wedge v_j >$	$i < j$	$\# = g(g-1)/2$
$r_i^j:$	$\lambda_i\lambda_j^{-1}$	$< v_i \wedge v_j^* >$	$i < j$	$\# = g(g-1)/2$
$-r_{i,j}:$	$(\lambda_i\lambda_j)^{-1}$	$< v_i^* \wedge v_j^* >$	$i < j$	$\# = g(g-1)/2$
$-r_i^j:$	$\lambda_i^{-1}\lambda_j$	$< v_i^* \wedge v_j >$	$i < j$	$\# = g(g-1)/2$

These are $2g(g-1)$ roots in total. A set of simple roots Δ_0 is given e.g. by $r_i^{i+1}, i = 1, \ldots, g-1$ and $r_{g-1,g}$. This set induces the decomposition into negative and positive roots as in the table. A parabolic subgroup is, up to conjugation, given by a subset of this set of simple roots (1.9.4).
The roots $r_{g-1,g}$ and r_{g-1}^g play a special role. Let L_0 be the isotropic space generated by $v_1, \cdots, v_g$ as above and L_1 be the isotropic space generated by $v_1, \cdots, v_{g-1}, v_g^*$. These two do not lie in the same orbit under $\mathrm{SO}(L_{\mathbb{C}})$ or $\mathrm{GSpin}(L_{\mathbb{C}})$.

Associated filtrations for the occurring subsets fall into 4 classes according to which of the roots $r_{g-1,g}$ and r_{g-1}^g lie in the corresponding subsets:

$$\left\{\begin{matrix}\text{parabolics of type } S \subseteq \Delta_0 \\ \text{with } r_{g-1,g} \in S,\ r_{g-1}^g \in S\end{matrix}\right\} \cong \left\{\begin{matrix}\text{filtrations } 0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_i \text{ of isotropic subspaces,} \\ \text{with } \dim I_i \leq g-2\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{parabolics of type } S \subset \Delta_0 \\ \text{with } r_{g-1,g} \in S,\ r_{g-1}^g \notin S\end{matrix}\right\} \cong \left\{\begin{matrix}\text{filtrations } 0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_i \text{ of isotropic subspaces,} \\ \text{with } I_i \sim L_0,\ \dim I_{i-1} \leq g-2\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{parabolics of type } S \subset \Delta_0 \\ \text{with } r_{g-1,g} \notin S,\ r_{g-1}^g \in S\end{matrix}\right\} \cong \left\{\begin{matrix}\text{filtrations } 0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_i \text{ of isotropic subspaces,} \\ \text{with } I_i \sim L_1,\ \dim I_{i-1} \leq g-2\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{parabolics of type } S \subset \Delta_0 \\ \text{with } r_{g-1,g} \notin S,\ r_{g-1}^g \notin S\end{matrix}\right\} \cong \left\{\begin{matrix}\text{filtrations } 0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_i \text{ of isotropic subspaces,} \\ \text{with } \dim I_i = g-1\end{matrix}\right\}$$

Here in the curled brackets, we mean everything up to conjugation or action respectively. This exceptional behavior in even dimensions, which is also reflected in the branching of the Dynkin diagram, is due to the fact that an isotropic subspace I of dimension $g-1$ already determines 2 isotropic subspaces of dimension g, namely the pre-images of the 2 isotropic lines in the 2-dimensional quadratic space $I^\perp/I$.

A parabolic is defined over $\mathbb{Q}$, if and only if the associated filtration

$$0 \subseteq I \subseteq I^\perp \subseteq L,$$

is defined over $\mathbb{Q}$. Hence we read off that for signature $(m-2,2)$, maximal $\mathbb{Q}$-parabolics are given by isotropic subspaces if $n \geq 6$ and SO is simple in this case, hence admissible parabolics (2.4.4) are maximal parabolics. If $m=4$, we have either only isotropic lines defined over $\mathbb{Q}$, in which case the maximal $\mathbb{Q}$-parabolics correspond to these lines, or there exists a isotropic plane, in which case $\mathrm{SO}(L_\mathbb{Q})$ is split and isomorphic to $\mathrm{PGL}(2) \times \mathrm{PGL}(2)$, so each proper parabolic is admissible and again *admissible* parabolics are given by maximal isotropic subspaces up to conjugation. We have only one element in each of the 4 sets above. The first (point-)set do not correspond to a proper parabolic. However, we have two kinds (horizontal and vertical, if you wish) of boundary components of dimension 1.

(10.2.11) Let now $m = 2g+1 \geq 3$ be odd. Choose a decomposition $L_\mathbb{C} = L_0 \oplus L_0^* \perp < v_0 >$ where L_0, L_0^* are maximal isotropic, $\langle v_0, v_0 \rangle = 1$, and a basis $v_1, \ldots, v_g$ of L_0. Denote the dual basis by $v_1^*, \ldots, v_g^*$. There is then a maximal torus, a $\mathbb{G}_m^g$ acting by $\lambda \cdot v_i = \lambda_i v_i$ and $\lambda \cdot v_i^* = \lambda_i^{-1} v_i^*$.

A set of roots together with their root spaces in $\mathrm{Lie}(\mathrm{SO}(L)) = \Lambda^2 L$ is given by

$$\begin{array}{rlll|c}
r_i: & \lambda_i & < v_i \wedge v_0 > & & \# = g \\
r_{i,j}: & \lambda_i\lambda_j & < v_i \wedge v_j > & i<j & \# = g(g-1)/2 \\
r_i^j: & \lambda_i\lambda_j^{-1} & < v_i \wedge v_j^* > & i<j & \# = g(g-1)/2 \\
\hline
-r_i: & (\lambda_i)^{-1} & < v_i^* \wedge v_0 > & & \# = g \\
-r_{i,j}: & (\lambda_i\lambda_j)^{-1} & < v_i^* \wedge v_j^* > & i<j & \# = g(g-1)/2 \\
-r_i^j: & \lambda_i^{-1}\lambda_j & < v_i^* \wedge v_j > & i<j & \# = g(g-1)/2
\end{array}$$

These are $2g^2$ roots in total. A set of simple roots is given e.g. by $r_i^{i+1}, i = 1, \dots, g-1$ and r_g. This set induces the decomposition into negative and positive roots as in the table. It is now easier to see that in any case:

$$\{ \text{ parabolics } \} \cong \{\text{filtrations } 0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_i \text{ of isotropic subspaces } \}.$$

Since SO is simple in these cases, proper admissible parabolics are $\mathbb{Q}$-maximal and correspond to single nonzero isotropic subspaces over $\mathbb{Q}$.

Now let Q be any proper admissible parabolic. We have seen that Q corresponds to a filtration

$$0 \subset I \subseteq I^\perp \subset L,$$

whose stabilizer it is. We will now determine the boundary components in the sense of (2.4.5) associated with them:

(10.2.12) Case: I 1 dimensional (boundary point).
Choose a 1-dimensional lattice $I' \cong I^*$. This is possible because the discriminant is a unit at p.
We have the group

$$\mathrm{GSpin}(I' \oplus I) = \{\alpha z z' + \beta z' z\} \subset \mathrm{GSpin}(L).$$

Here z, z' are generators of I, I' respectively. We get a Levi decomposition

$$Q = \mathbb{W}(I \otimes I^\perp / I) \rtimes G.$$

We interpreted the set $\mathbb{D}_{\mathbf{S}(L)}$ as the set of Hodge structures of weight 2 on $L_\mathbb{R}$ with $L_{-2,0}$ isotropic. $h(\mathbb{D}_\mathbf{B})$ can be interpreted as an orbit in the set of mixed Hodge structures with respect to the weight filtration

$$W_i(L) = \begin{cases} 0 & i <= -5 \\ I & i = -4, -3 \\ I^\perp & i = -2, -1 \\ L & i \geq 0 \end{cases}$$

This follows from (2.4.4) because a splitting cocharacter for this splitting is given by the λ of (2.4.3) times the original weight ($\lambda \mapsto (\lambda^{-2}, \dots, \lambda^{-2})$ for $\mathbb{D}_\mathbf{S}$). The real torus is for $n \geq 5$ given by $S = \{\lambda_1, \lambda_2, 1, \cdots 1, \lambda^1, \lambda^2, 1, \cdots, 1 | \lambda_1 \lambda^1 = \lambda_2 \lambda^2 = 1\}$, where we assumed that v_1, v_1^*, resp. v_2, v_2^* are spanning the 2 real hyperbolic planes. The *long roots* (2.4.3) are in this case: r_1^2 and r_{12}.
The associated λ hence is the given by

$$\lambda \mapsto (\lambda^2, \lambda^0, \dots, \lambda^0; \lambda^{-2}, \lambda^0, \dots, \lambda^0)$$

because it is the only morphism satisfying:

$$\langle \lambda, r_{12} \rangle = 2 \qquad \langle \lambda, r_1^2 \rangle = 2.$$

The conditions on an isotropic line $F^0 \subset L_{\mathbb{C}}$ to yield a mixed Hodge structure are, $W_{-2} \subset (F^0 + \overline{F^0})^\perp + W_{-3}$, $F_0 \neq \overline{F_0}$. The induced Hodge structures on $\mathrm{gr}_W(L)$ have to be trivial.
Choose $F^0 = I'$ with the notations above.
Therefore, by weight-reasons, a morphism h_x, composed with the projection above has to be

$$w \mapsto (w\overline{w})^{-1}, 1, (w\overline{w})^{-1} \in \mathbb{G}_m \times \mathbb{G}_m \times \mathrm{GSpin}(L_0).$$

The semi-direct product of this image ($\cong \mathbb{G}_m$) with the unipotent radical is hence isomorphic to the smallest normal subgroup over which the morphisms factor. It is

$$P_{\mathbf{B}} = \mathbb{W}(I \otimes I^\perp / I) \rtimes \mathbb{G}_m,$$

where $\alpha \in \mathbb{G}_m$ is mapped to $\alpha z z' + z' z \in \mathrm{GSpin}(L)$. It operates, however, via the natural action of $\mathbb{G}_m$ on $\mathbb{W}(I \otimes I^\perp / I)$. Its intersection with $\mathbb{G}_m \subset \mathrm{GSpin}(L)$ is trivial, hence it is isomorphic to the corresponding boundary component of $\mathrm{SO}(L)$. α here goes to the element $z \mapsto \alpha z, z' \mapsto \alpha^{-1} z'$.

(10.2.13) Definition. *We will denote the corresponding morphism* $\mathbb{G}_m \to \mathrm{SO}$ *by* $\mu_{I,I'}$.

Recall the chosen representations for SO and GSpin (10.1.4).
Consider

$$\pi_0(\mathbb{D}_{\mathbf{S}}) \times \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbf{B},\mathbb{C}}).$$

Any $\mathbb{G}_m$, pre-image of the group determined above, acts by the obvious nontrivial operation on the set $\pi_0(\mathbb{D}_{\mathbf{S}})$. $\pi_0(\mathbb{D}_{\mathbf{S}})$ can be identified (after choice of a common orientation of the negative definite subspaces) with the set of isomorphisms $\mathbb{Z} \cong \mathbb{Z}(1)$ (choice of root of -1).
Hence the $P_{\mathbf{B}}(\mathbb{R})U_{\mathbf{B}}(\mathbb{C})$-orbits contained in the image are canonically isomorphic to

$$\mathbb{D}_{\mathbf{H}_0[I \otimes I^\perp / I, 0]},$$

and we get an isomorphism:

$$\mathbf{B} \cong \mathbf{H}_0[I \otimes I^\perp / I, 0],$$

cf. (2.2.10) for the definition.
Suppose I is spanned by the vector z, and consider the mixed Hodge structure determined by $F^0 = H^{-2,0} =< z' >$. Assume $\langle z, z' \rangle \neq 0$.
Consider the isomorphism $L_0 \cong I \otimes I^\perp / I, k \mapsto z \otimes k$, where L_0 is the orthogonal complement of z and z'. Letting it act on z', we get an isomorphism $\mathbb{D}_{\mathbf{B}} \cong \pi_0(\mathbb{D}_{\mathbf{O}}) \times L_0(\mathbb{C})$. Explicitly a vector in $k \in L_0(\mathbb{C})$ on the right hand side corresponds to the mixed

Hodge structure determined by

$$F^0 =< z' + \langle z, z'\rangle k - \left(\langle k, z'\rangle + \frac{1}{2}\langle z, z'\rangle\langle k, k\rangle\right) z > .$$

To any isotropic line $< z >$ there corresponds precisely 1 boundary component, and an embedding

$$\mathbb{D}_{\mathbf{O}} \hookrightarrow \mathbb{D}_{\mathbf{B}}.$$

The compact dual of $\mathrm{M}^{\vee}(\mathbf{S}(L))$ is given by the zero quadric and the 'compact' dual $\mathrm{M}^{\vee}(\mathbf{B})$ is the (scheme-theoretic) complement of $< z >^{\perp}$.

It is (non canonically) isomorphic to $\mathbb{W}(I \otimes I^{\perp}/I)$, depending on the choice of a *primitive* vector z' with $\langle z, z'\rangle = 1$ (this exists because the discriminant is a unit at p).

Let $K \subseteq \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ be the discriminant kernel of a lattice $L_{\mathbb{Z}}$, Consider $K_1 := K \cap P_{\mathbf{B}}(\mathbb{A}^{(\infty)})$.

(10.2.14) Claim: $K_1 \cap U_{\mathbf{B}}(\mathbb{A}^{(\infty)}) = L_{0,\widehat{\mathbb{Z}}} :=< z >_{\widehat{\mathbb{Z}}} \otimes < z >^{\perp}_{\widehat{\mathbb{Z}}} / < z >_{\widehat{\mathbb{Z}}}$.

Proof. First of all K_1 is a product of $K_{1,l}$'s because K is. For an element $(z \otimes v) \rtimes g \in I_{\mathbb{Q}_l} \otimes I^{\perp}_{\mathbb{Q}_l}/I_{\mathbb{Q}_l}$ to be in the discriminant kernel, we need to have: $\langle z, x\rangle v - \langle v, x\rangle z \in L_{\mathbb{Z}_l}$ for all $x \in L^*_{\mathbb{Z}_l}$, hence $v \in I^{\perp}_{\mathbb{Z}_l}/I_{\mathbb{Z}_l}$ (We have $< z >^{\perp} \cap L^*_{\mathbb{Z}_l} = (L_{\mathbb{Z}_l}/ < z >)^*$. Now test for all $x \in< z >^{\perp} \cap L^*_{\mathbb{Z}_l}$.

(10.2.15) If $z' \in L_{\mathbb{Z}_{(p)}}$ is any primitive isotropic vector not perpendicular to z, K_1 will then definitely contain a group $K_{\widehat{\mathbb{Z}}} \rtimes K(N) \subset U_{\mathbf{B}}(\mathbb{A}^{(\infty)}) \rtimes {\mathbb{A}^{(\infty)}}^*$, where the splitting is chosen according to $< z' >$, as explained above. Of course, N will depend on the choice of z'. Furthermore K_1 itself may not be of this form.

(10.2.16) Case: I 2 dimensional (boundary curve). If $I^{\perp} \neq I$, there is a surjective morphism

$$P \to \mathrm{GL}(I) \times \mathrm{GSpin}(I^{\perp}/I),$$

with kernel the unipotent radical. The unipotent radical consists of the exponential of elements in $I \wedge I^{\perp}$. We have an exact sequence (in the Lie algebra)

$$0 \longrightarrow \Lambda^2(I) \longrightarrow I \wedge I^{\perp} \longrightarrow I \otimes (I^{\perp}/I) \longrightarrow 0$$

where $\Lambda^2(I)$ are the elements shifting the filtration by 2 and a corresponding central extension

$$0 \longrightarrow U_{\mathbf{B}} := \mathbb{W}(\Lambda^2(I)) \longrightarrow W_{\mathbf{B}} \longrightarrow V_{\mathbf{B}} := \mathbb{W}(I \otimes (I^{\perp}/I)) \longrightarrow 0$$

We interpreted the set $\mathbb{D}_{\mathbf{O}}$ as the set of Hodge structures of weight 2 on L, such that $L_{-2,0}$ is isotropic. $h(\mathbb{D}_{\mathbf{B}})$ can be interpreted as the set of mixed Hodge structures with respect to the weight filtration

$$W_i(L) = \begin{cases} 0 & i <= -4 \\ I & i = -3 \\ I^\perp & i = -2 \\ L & i \geq -1. \end{cases}$$

The associated λ hence is given by

$$\lambda \mapsto (\lambda^1, \lambda^1, \lambda^0, \dots, \lambda^0; \lambda^{-1}, \lambda^{-1}, \lambda^0, \dots, \lambda^0)$$

because its the only morphism of correct weight, satisfying

$$\langle \lambda, r_{12} \rangle = 0 \qquad \langle \lambda, r_1^2 \rangle = 2.$$

The conditions on an isotropic line $F^0 \subset L_\mathbb{C}$ to yield a mixed Hodge structure are, if $\dim W_{-2} = 2$: $F^0 \cap W_{-2} = 0$, $W_{-2} \subset (F^0 + \overline{F^0})^\perp + W_{-3}$, $F_0 \neq \overline{F_0}$. The Hodge structure on $\mathrm{gr}_{-1}(L) = \mathrm{gr}_{-3}(L)^*(2)$ can be arbitrary. Choose an I^* which is the dual of I with respect to the bilinear form. This determines $L = L_0 \perp I \oplus I^*$ and a splitting

$$\mathrm{GL}(I) \times \mathrm{GSpin}(L') \to Q,$$

as well as an embedding

$$\mathbb{D}_{\mathbf{H}(I)} \hookrightarrow \mathbb{D}_{\mathbf{B}}.$$

The composition of any $\omega_x \circ h_\infty$ with the projection above, has to look like

$$z \mapsto (h'(z), z\overline{z}),$$

(for some $h' \in h(\mathbb{D}_{\mathbf{H}(I)})$) for weight reasons. The corresponding smallest normal subgroup, such that all morphisms h factor through it, is hence isomorphic to GL_2. We see that the group $P_\mathbf{B}$ has to be isomorphic to the group $W_\mathbf{B} \rtimes \mathrm{GL}_2$, $W_\mathbf{B}$ as above. Further investigation as in the case $\dim I = 1$ shows that in this case

$$\mathbf{B} \cong \mathbf{H}(I)[\Lambda^2(I), I \otimes (I^\perp/I)]$$

(2.2.10).

We may summarize the discussion of this section:

(10.2.17) Theorem. *There is (similar to 2.5.4) a bijection*

$$\{ \textit{ isotropic subspaces } I_\mathbb{Q} \textit{ of } L_\mathbb{Q} \} \cong \{ \textit{ boundary components } \mathbf{B} \textit{ of } \mathbf{S}(L) \textit{ or } \mathbf{O}(L) \}.$$

Let $I_\mathbb{Q}$ be an isotropic subspace, $I = I_{\mathbb{Z}_{(p)}}$ the corresponding saturated sublattice and $\mathbf{B}$

the corresponding boundary component.

We have

$$\mathbf{B} \cong \begin{cases} \mathbf{H}_0[I \otimes (I^\perp/I), 0] & \text{if } \dim(I) = 1, \\ \mathbf{H}_1(I)[\Lambda^2 I, I \otimes (I^\perp/I)] & \text{if } \dim(I) = 2, \end{cases},$$

where in the first case $\mathbb{G}_m$ *acts on* $I \otimes I^\perp/I$ *by scalars and* $\mathrm{GL}(I)$ *in the second case acts trivial on* $I^\perp/I$ *(see the definition of unipotent extension (2.2.10)).*

The isomorphisms depend on a common choice of orientation on the maximal negative definite subspaces of $L_\mathbb{R}$*.*

10.3. Special cycles

Let $L_{\mathbb{Z}_{(p)}}$ be a lattice with non-degenerate unimodular quadratic form, i.e. inducing an isomorphism $L_{\mathbb{Z}_{(p)}} \to L^*_{\mathbb{Z}_{(p)}}$, and $L'_{\mathbb{Z}_{(p)}}$ be a saturated sublattice with non-degenerate unimodular form, of signature $(m-2,2)$ and $(m-2-n,2)$, respectively, where $0 < n \leq m-2$.

Recall the embedding (10.2.2, here used mod $\mathbb{G}_m$):

$$\iota : \mathbf{O}(L') \hookrightarrow \mathbf{O}(L),$$

in the case $m-n > 2$ and the 2 embeddings

$$\iota_\pm : \mathbf{O}(L')_\pm \hookrightarrow \mathbf{O}(L),$$

in the case $m-n=2$.

Let an admissible compact open subgroup K be given. For each $g \in \mathrm{SO}(L_{\mathbb{A}^{(\infty,p)}})$ we get a conjugated embedding:

$$(\iota, g) : {}^{K'_g}\mathbf{O}(L') \hookrightarrow {}^{K}\mathbf{O}(L)$$

for $K'_g = K \cap \iota(K)^g$.

In addition, we may find smooth, complete and projective Δ, and Δ'_g for each g (2.4.12) such that models exist (3.3.5) and we have, in the end, embeddings of p-ECMSD:

$$(\iota, g) : {}^{K'_g}_{\Delta'_g}\mathbf{O}(L') \hookrightarrow {}^{K}_{\Delta}\mathbf{O}(L)$$

(resp. with $\pm$).

Let $M_{\mathbb{Z}_{(p)}}$ be another lattice with non-degenerate quadratic form and recall from (6.2.4) the set

$$I(M, L)_R = \{\alpha : M_R \to L_R \mid \alpha \text{ is an isometry}\}.$$

We have $I(M,L)_\mathbb{Q} \neq \emptyset \Leftrightarrow I(M,L)_{\mathbb{A}^{(\infty)}} \neq \emptyset$ by Hasse's principle. We assume that these sets are nonempty throughout. We may then find even an $x \in I(M,L)_{\mathbb{Z}_{(p)}}$.

For a K-invariant admissible[3] Schwartz function $\varphi \in S((M^* \otimes L)_{\mathbb{A}^{(\infty)}})$

$$I(M, L)_{\mathbb{A}^{(\infty)}} \cap \mathrm{supp}(\varphi) = \coprod_j K g_j^{-1} x$$

with finitely many $g_j \in \mathrm{SO}(L_{\mathbb{A}^{(\infty,p)}})$. We assume that Δ has been refined such that Δ'_{g_j} exists, with the properties claimed above for all j.

(10.3.1) Definition. *We define the p-integral* ***special cycle*** *associated with $M_{\mathbb{Z}_{(p)}}$ and φ on $\mathrm{M}({}^K_\Delta\mathbf{O}(L))$ as*

$$\mathrm{Z}(L, M, \varphi; K) := \sum_j \varphi(g_j^{-1} x) \, \mathrm{im}(\mathrm{M}(\iota, g_j))$$

$(m - n > 2)$ resp.

$$\mathrm{Z}(L, M, \varphi; K) := \sum_{j,\pm} \varphi(g_j^{-1} x) \, \mathrm{im}(\mathrm{M}(\iota_\pm, g_j))$$

$(m - n = 2)$.
($x^\perp$ is of course saturated, non-degenerate and unimodular).

This is a finite sum and independent of the choice of the g_i. The formation is compatible with Hecke operators (2.4.11) in the obvious sense. Observe also that in the case $m-n = 2$, $\sum_\pm \mathrm{im}(\mathrm{M}(\iota_\pm, g_j)) = \mathrm{im}(\mathrm{M}(\overline{\iota}, g_j))$ for the morphism

$$\mathrm{M}(\overline{\iota}, g_j) : \mathrm{M}({}^{K'}\overline{\mathbf{O}}(L')) \to \mathrm{M}({}^{K'}_\Delta\mathbf{O}(L))$$

of (9.1.1).
If $M_\mathbb{Q}$ is only a $\mathbb{Q}$-vector space with non-degenerate quadratic form and φ any Schwartz function, we write

$$I(M, L)_{\mathbb{A}^{(\infty)}} \cap \mathrm{supp}(\varphi) = \coprod_j K g_j^{-1} x$$

for any $x \in I(M, L)_\mathbb{Q}$, assumed again to exist. We define

(10.3.2) Definition. *We define the rational* ***special cycle*** *associated with $M_\mathbb{Q}$ and φ on $\mathrm{M}({}^K_\Delta\mathbf{O}(L))$ as (assume $m - 2 > 2$)*

$$\mathrm{Z}(L, M, \varphi; K) := \sum_j \varphi(g_j^{-1} x) \overline{\mathrm{im}(\mathrm{M}(\iota, g_j))}^{Zar}.$$

Here ι is associated with the embedding $x^\perp \hookrightarrow L_\mathbb{Q}$ and $\mathrm{M}(\iota, g_j)$ is only a morphism of

[3] by this we mean that φ is the product of a Schwartz function $\varphi \in S((M^* \otimes L)_{\mathbb{A}^{(\infty,p)}})$ with the characteristic function of $(M^* \otimes L)_{\mathbb{Z}_p}$

rational Shimura varieties. It is defined similarly for $m-2=2$, too.

From the construction of models by Zariski closure (3.3.5) follows that this definition coincides with the previous one, if there exists a lattice $M_{\mathbb{Z}_{(p)}}$, such that the restriction of the quadratic form is non-degenerate and φ is admissible.

10.4. Orthogonal modular forms

Let L be a lattice over $\mathbb{Z}_{(p)}$ with non-degenerate unimodular quadratic form of signature $(m-2,2)$, i.e. such that it induces an isomorphism $L \to L^*$. $\mathrm{SO}(L)$ is then reductive over $\mathbb{Z}_{(p)}$.
Denote $\mathbf{O} = \mathbf{O}(L)$ the corresponding p-integral orthogonal Shimura datum (resp. one of them, if $m=2$, 10.2.1).

(10.4.1) Definition. *Let $\mathcal{E}$ be the restriction of the canonical line bundle on $\mathbb{P}(L)$ to the zero-quadric $\mathrm{M}^\vee(\mathbf{O})$. It carries a $\mathrm{SO}(L)$ action. $E_{\mathbb{C}}|_{h(\mathbb{D}_{\mathbf{O}})}$ carries a natural Hermitian metric, too, which we normalize as follows:*

$$h_E : v, w \mapsto -\frac{1}{2}e^{-C}\langle v, \overline{w}\rangle,$$

where $C = \gamma + \log(2\pi)$ and $\gamma = -\Gamma'(1)$ is Euler's constant.
Denote $\overline{\mathcal{E}} = (\mathcal{E}, h_E)$.

We are going to deal with the Hermitian automorphic line bundles (9.1) $\Xi^*(\overline{\mathcal{E}})$ on the various $\mathrm{M}({}^K_\Delta\mathbf{O})$ in this section.

(10.4.2) Remark. The normalization factor e^{-C} is dictated by (10.4.12) and (11.2). In contrast, the explicit formulæ for the arithmetic volumes in terms of derivatives of L-series at negative integers (8.3) would look more appealing if we omitted it.

(10.4.3) Lemma. *With respect to the Hodge embedding*

$$\mathbf{S}(L) \hookrightarrow \mathbf{H}(\mathrm{C}^+(L), \langle,\rangle_\delta),$$

the pullback of the canonical bundle $\Lambda^g\mathcal{L}$ via the 'dual' embedding

$$\mathrm{M}^\vee(\mathbf{S}(L)) \to \mathrm{M}^\vee(\mathbf{H}(\mathrm{C}^+(L), \langle,\rangle_\delta))$$

is, up to a bundle coming from a character $P \to \mathbb{G}_m$, and up to a negative tensor-power, the bundle $\mathcal{E}$ above.

Proof. Recall that the induced map on compact duals (10.2.4) is given by $< z > \mapsto zw\,\mathrm{C}^+(L)$, where z is assumed to be primitive in $L_{\mathbb{Z}_{(p)}}$ and w is *any* primitive vector with $\langle w, z\rangle = 1$. The fibre of $\mathcal{L}$ at the image of $< z >$ is given by the space $zw\,\mathrm{C}^+(L)$ and the fibre of $\mathcal{E}$ is given by $< z >$. By an explicit determination of the action of the stabilizer group in both cases the result follows. □

(10.4.4) Corollary. *The pullback of the bundle of Siegel modular forms $\Xi^*(\Lambda^g\mathcal{L})^{\otimes n}$ of some weight n is equal to some negative power of the bundle of orthogonal modular forms $\Xi^*\mathcal{E}^{\otimes n'}$. In particular $(\Xi^*\mathcal{E})^{-1}$ is ample on every $\mathrm{M}({}^K\mathbf{S}(L))$ and $\mathrm{M}({}^K\mathbf{O}(L))$ and a suitable power has no base points on $\mathrm{M}({}^K_\Delta\mathbf{O}(L))$.*

Proof. Follows from (10.4.3) and (9.1.5). The bundle of Siegel modular forms $\Xi^*(\Lambda^g\mathcal{L})^{\otimes n}$ is ample [27][4]. The ampleness of $\Xi^*\mathcal{E}$ follows from the fact that the Hodge embedding induces an embedding (up to a normzlization) of the corresponding Shimura varieties (3.2.2) for suitable levels and the fact that all $\mathrm{M}({}^K\mathbf{S}(L))$ and $\mathrm{M}({}^K\mathbf{O}(L))$ are related by finite etale maps that pullback the $\Xi^*\mathcal{E}$'s into each other. The assertion on base points is true, because a suitable power of $\Xi^*(\Lambda^g\mathcal{L})$ defines a morphism to the minimal compactification on every $\mathrm{M}({}^{K'}_{\Delta'}\mathbf{H}_g)$ [27, Theorem 2.3, (1)]. □

We will compare the measure given by the volume form $c_1(\Xi^*\overline{\mathcal{E}})^{m-2}$, the highest power of the Chern form of the canonical Hermitian line bundle defined above with the quotient μ of the canonical volume form (6.2.3) on $\mathrm{SO}(L_\mathbb{R})$ by the one giving $K_\infty =$ stabilizer of some positive definite subspace N in $L_\mathbb{R}$, the volume 1.

(10.4.5) Lemma.

$$2\lambda_\infty^{-1}(L;0)\mu = (-c_1(\Xi^*\mathcal{E}, \Xi^*h))^{m-2}.$$

Proof. We will ignore henceforth all signs of volume forms, whenever they have no influence because we know that the 2 volume forms are positive and invariant. Choose an orthogonal basis of $L_\mathbb{R}$ with $\langle e_i, e_i\rangle = -1, i = 1, 2$ and $\langle e_i, e_i\rangle = 1, i > 2$. Set $z = \frac{1}{2}(e_3 - e_2), z' = \frac{1}{2}(e_3 + e_2)$. We have $\langle z, z\rangle = 0$ and $\langle z, z'\rangle = \frac{1}{2}$. Choose a base point $< \Xi > \in \mathbb{D}_\mathbf{O} \subset \mathbb{P}^1(L_\mathbb{C})$, $\Xi = ie_1 + e_2$. It induces an isomorphism $\mathrm{SO}(L_\mathbb{R})/K_\infty \cong \mathbb{D}_\mathbf{O}$. The tangent space at Ξ in $\mathbb{D}_\mathbf{O} \subset \mathrm{M}^\vee(\mathbf{O}(L))(\mathbb{C})$ is canonically identified with

$$\mathrm{Hom}(< \Xi >_\mathbb{C}, < \Xi >_\mathbb{C}^\perp / < \Xi >).$$

The induced map

$$\mathrm{Lie}(\mathrm{SO}(L_\mathbb{R})) \to \mathrm{Hom}(< \Xi >_\mathbb{C}, < \Xi >_\mathbb{C}^\perp / < \Xi >_\mathbb{C})$$

[4] An explicit comparison of our language to the one used in [27] will be provided in forthcoming work

is given by
$$\Lambda \mapsto \{\Xi \mapsto \Lambda\Xi \mod <\Xi>_{\mathbb{C}}\}.$$
On $\mathrm{Lie}(\mathrm{SO}(L_{\mathbb{R}}))$ we have the basis:
$$L_{ij} = \{e_i \wedge e_j\},$$
whose associated volume form is the canonical μ (6.2.10). The basis elements L_{12}, L_{ij}, $i, j \geq 3$ are mapped to 0. Its associated volume form is the canonical (6.2.3) on K_∞ (6.2.10). The induced volume form on $\mathbb{D}_{\mathbf{O}}$ hence is:
$$\bigwedge_{i=3}^{m} \widetilde{L}_{1i}^* \wedge \bigwedge_{i=3}^{m} \widetilde{L}_{2i}^* \in \bigwedge^{2(m-2)} \mathrm{Hom}(<\Xi>_{\mathbb{C}}, <\Xi>_{\mathbb{C}}^{\perp} / <\Xi>_{\mathbb{C}})^*,$$
where $\widetilde{L}_{1i} = \{\Xi \mapsto ie_i\}$ and $\widetilde{L}_{2i} = \{\Xi \mapsto e_i\}$ (wedge product over $\mathbb{R}$!).
Now let $K = <e_1, e_4, \dots e_m>_{\mathbb{C}} = <z, z'>_{\mathbb{C}}^{\perp}$. We calculate
$$\begin{aligned} 4\partial\overline{\partial} \log Y^2 &= \sum_{i,j \in I} \frac{\partial^2}{\partial y_i \partial y_j} \log(Y^2) \, \mathrm{d}\, z_i \wedge \mathrm{d}\, \overline{z}_j \\ &= \sum_{i,j \in I} -\frac{4\delta_i\delta_j y_i y_j}{Y^4} \, \mathrm{d}\, z_i \wedge \mathrm{d}\, \overline{z}_j + \sum_i \frac{2\delta_i}{Y^2} \, \mathrm{d}\, z_i \wedge \mathrm{d}\, \overline{z}_i \end{aligned}$$
where $K \ni Z = X + iY$, $I = \{1, 4, \dots, m\}$ and $\delta_1 = -1$ and $\delta_i = 1$ otherwise.
At the point ie_1 this yields:
$$(\mathrm{d}\,\mathrm{d}^c \log Y^2)^{m-2} = \left(\frac{1}{2\pi}\right)^{m-2} (m-2)! \, \mathrm{d}\, x_1 \wedge \mathrm{d}\, y_1 \wedge \cdots \wedge \mathrm{d}\, x_m \wedge \mathrm{d}\, y_m$$
(observe $\mathrm{d}\,\mathrm{d}^c = -\frac{1}{2\pi i}\partial\overline{\partial}$ and $\mathrm{d}\, z_i \wedge \mathrm{d}\, \overline{z}_i = 2i \, \mathrm{d}\, x_i \wedge \mathrm{d}\, y_i$).
Under the parametrization
$$\begin{aligned} K_{\mathbb{C}} &\to \mathrm{M}^{\vee}(\mathbf{O}(L))(\mathbb{C}) \\ Z &\mapsto Z + z' - \langle Z, Z \rangle z \end{aligned}$$
ie_1 is mapped to Ξ. The tangent map T at this point is
$$Z \mapsto \{\Xi \mapsto Z - 2i\langle Z, e_1 \rangle z \mod <\Xi>_{\mathbb{C}}\}$$
and we have: $T(e_1) = \widetilde{L}_{23}, T(ie_1) = -\widetilde{L}_{13}$ and $T(e_i) = \widetilde{L}_{1i}, T(ie_i) = \widetilde{L}_{2i}$ for $i \geq 4$.
This yields the volume form
$$(m-2)! \left(\frac{1}{2\pi}\right)^{m-2} \bigwedge_{i=3}^{m} \widetilde{L}_{1i}^* \wedge \bigwedge_{i=3}^{m} \widetilde{L}_{2i}^* \in \bigwedge^{2(m-2)} \mathrm{Hom}(<\Xi>_{\mathbb{C}}, <\Xi>_{\mathbb{C}}^{\perp} / <\Xi>_{\mathbb{C}})^*.$$
Now observe that $K_\infty = \mathrm{SO}(N) \times \mathrm{SO}(N^{\perp})$ because K_∞ is not allowed to change the

orientation of N (the $\mathbb{C}$-oriented negative definite plane corresponding to $< \Xi >$). Hence the volume of K_∞ is $2\pi \cdot \frac{1}{2} \prod_{j=1}^{m-2} 2\frac{\pi^{j/2}}{\Gamma(j/2)}$ (6.2.9). This gives the factor $2\prod_{j=2}^{m} \frac{1}{2}\frac{\Gamma(\frac{j}{2})}{\pi^{\frac{j}{2}}}$ which is $2\lambda_\infty^{-1}(L;0)$. □

(10.4.6) Remark. This will be used in the proof of (10.5.2, i) to give an alternative calculation of the geometric volume of these Shimura varieties using the Tamagawa number directly and avoiding the use of Kudla-Millson Greens functions (7.6.9).

(10.4.7) In (10.2.17) we proved that isotropic lines $I \subset L$ are in 1:1 correspondence with the set of boundary components of $\mathbf{O}(L)$ of the form considered in (5.7). Let I be an isotropic line and $\mathbf{B}$ be the corresponding boundary component. So $U_\mathbf{B}$ is isomorphic to $\mathbb{W}(U)$, with $U := I \otimes (I^\perp/I))$. For every choice of a line I^* which is dual to I with respect to the *integral* bilinear form, we get a decomposition $L = (I \oplus I^*) \perp L_0$, and a splitting

$$P_\mathbf{B} = U_\mathbf{B} \rtimes \mathbb{G}_m,$$

where $\mathbb{G}_m$ acts via the morphism $\mu_{z,z'}$ of (10.2.13) acting via the natural representation of SO (10.1.4) by $\lambda \cdot z = \lambda z$ for $z \in I$ and $\lambda \cdot z' = \lambda^{-1} z'$, for $z' \in I'$. Whenever I^* has been chosen, we write the corresponding element of $P_\mathbf{B}(\mathbb{Z}_{(p)})$ as (Z, λ), where $Z \in U_{\mathbb{Z}_{(p)}}$, $\lambda \in \mathbb{Z}_{(p)}^*$.

We have $\mathbb{D}_\mathbf{B} = \mathbb{D}_{\mathbf{H}_0} \times \mathbb{D}^\circ$, for $\mathbb{D}^\circ \simeq U_\mathbb{C}$ non canonically ($\mathbb{D}^\circ$ can be identified canonically with the set of isotropic complex lines, not perpendicular to $I_\mathbb{C}$).

Abbreviate $\mathbf{O} = \mathbf{O}(L)$.

Let K be an admissible compact open subgroup and Let Δ be a K-admissible complete rational polyhedral cone decomposition for $\mathbf{O}(L)$ and let $(\iota, \rho) : {}^{K_1}_{\Delta_1}\mathbf{B} \Longrightarrow {}^{K}_{\Delta}\mathbf{O}$ be a boundary map (2.4.11).

We have a corresponding boundary map (3.3.5):

$$\mathrm{M}(\iota,\rho) : \widehat{\mathrm{M}({}^{K_1}_{\Delta_1}\mathbf{B})} \to \widehat{\mathrm{M}({}^{K}_{\Delta}\mathbf{O})},$$

where the completion is taken with respect to a boundary stratum corresponding to a rational polyhedral cone $\sigma \in \Delta_1$, $\sigma \subset K_\mathbb{R}(-1)$. Over $\mathbb{C}$ the map converges, and on the complement of the boundary, the image of $\mathbb{D}_\mathbf{O}$ in $\mathbb{D}_\mathbf{B}$ is contained in the region of convergence and there it is just given by the map

$$P_\mathbf{B}(\mathbb{Q})\backslash\mathbb{D}_\mathbf{O} \times P_\mathbf{B}(\mathbb{A}^{(\infty)})/K_1 \to \mathrm{SO}(L_\mathbb{Q})\backslash\mathbb{D}_\mathbf{O} \times \mathrm{SO}(L_{\mathbb{A}^{(\infty)}})/K$$
$$[x,p] \mapsto [x, p\rho].$$

Consider a rational function f on $\mathrm{M}({}^{K}_{\Delta}\mathbf{O})$. There will be a $\sigma \in \Delta_1$, $\sigma \subset U_\mathbb{R}(-1)$ *of maximal dimension*, such that on each connected component $\alpha \in \mathbb{D}_{\mathbf{H}_0}$ of $\mathbb{D}_\mathbf{B} \times \rho \subseteq \mathbb{D}_\mathbf{B} \times P_\mathbf{B}$, there is a Fourier expansion of f defined on the set $U_\mathbb{R} + x + \alpha(\sigma)$, for some

$x \in \sigma^5$:
$$f : Z \mapsto \sum_{k \in U^*_{\mathbb{Q}}(1)} FC(f, \alpha, I, I', \rho, k, \sigma) \exp(kZ) \qquad Z \in U_{\mathbb{C}},$$
with bounded denominators (all k will actually lie in $(K_1 \cap U_{\mathbf{B}}(\mathbb{Q}))^*(1)$).
K_1 will contain a group of the form $K_U \times K(N)$ (cf. the end of 10.2.12). The pullback of the corresponding formal function to $\mathrm{M}(\widehat{{}^{K_U \times K(N)}_{\Delta_1}\mathbf{B}})$ via projection followed by the boundary map will be just
$$\sum_{k \in U^*_{\mathbb{Q}}} FC(f, \alpha, I, I', \rho\beta, \alpha(k), \sigma)[k]$$
in the fibre over some $\zeta_N = \exp(\alpha(\beta/N))$ for $\beta \in K(1)$. Recall from (5.7.3) that
$$\mathrm{M}(\widehat{{}^{K_U \times K(N)}_{\Delta_1}\mathbf{B}}) = \mathrm{spf}(\mathbb{Z}_{(p)}[\zeta_N][\![\sigma^\vee \cap U^*_{\mathbb{Z}}]\!]).$$
Observe furthermore that changing the boundary map from ρ to $\rho\beta$ or permuting the fibers of $\mathrm{M}({}^{K_U K(N)}\mathbf{B})$ over $\mathrm{M}({}^{K(N)}\mathbf{H}_0)$ by the operation of β is equivalent.

(10.4.8) Recall from (9.1.2) the definition of automorphic vector bundle. Corresponding to $\mathcal{E}, h_E$ above, we have a line bundle $\Xi^*(\mathcal{E})$ with Hermitian metric $\Xi^*(h_E)$, singular along the boundary, on any $\mathrm{M}({}^K_\Delta\mathbf{O})$. Its complex sections on $\mathrm{M}({}^K\mathbf{O})(\mathbb{C})$ are identified with $\mathrm{SO}(L_{\mathbb{Q}})$-invariant functions
$$\mathbb{D}_{\mathbf{O}} \times \mathrm{SO}(\mathbb{A}^{(\infty)})/K \to \mathcal{E}$$
(compatible with the Borel embedding in, respectively projection to $\mathrm{M}^\vee(\mathbf{O})$).
We may pullback $\mathcal{E}, h_E$ by the 'dual' boundary map $\mathrm{M}^\vee(\iota)$ and consider the automorphic vector bundle $\Xi_1^* \mathrm{M}^\vee(\iota)^*(\mathcal{E}, h_E)$ on any $\mathrm{M}({}^{K_1}_{\Delta_1}\mathbf{B})$. By the commutative diagram
$$\begin{tikzcd}
\Gamma \backslash \widehat{\mathrm{M}({}^{K_1}_{\Delta_1}\mathbf{B})} \arrow[d, "\sim"] & \Gamma \backslash \widehat{\mathrm{P}({}^{K_1}_{\Delta_1}\mathbf{B})} \arrow[l] \arrow[l, bend right, leftarrow, "s"'] \arrow[r, "\Pi"] \arrow[d] & \mathrm{M}^\vee(\mathbf{B}) \arrow[d, hook] \\
\widehat{\mathrm{M}({}^K_\Delta\mathbf{O})} & \widehat{\mathrm{P}({}^K_\Delta\mathbf{O})} \arrow[l] \arrow[r, "\Pi"] & \mathrm{M}^\vee(\mathbf{O}),
\end{tikzcd}$$
we get a canonical identification of $\Xi_1^* \mathrm{M}^\vee(\iota)^*(\mathcal{E}, h_E)$ with $\mathrm{M}(\iota, \rho)^* \Xi^*(\mathcal{E}, h_E)$ (cf. also 9.1.5). If a (meromorphic) section f of $\Xi^*(\mathcal{E})$ on $\mathrm{M}({}^K_\Delta\mathbf{O})$ is given, we may pullback the corresponding formal section to $\widehat{\mathrm{M}({}^{K_1}_{\Delta_1}\mathbf{B})}$. The corresponding series over $\mathbb{C}$ will converge

[5] Reason: We can pullback f to a rational function on $\mathrm{M}({}^K_{\Delta'}\mathbf{O})$ for refinements Δ' of Δ defined by subdividing a cone and making Δ K-invariant again. This corresponds to blowing up of the corresponding boundary point. After finitely many steps no pole of f around some boundary point corresponding to a shiver of the cone will be not on the boundary divisor.

in a set of the form $U_{\mathbb{R}} + x + \alpha(\sigma)$, $\sigma \in \Delta_1$ maximal dimensional (assuming Δ is appropriately chosen, see above) and compare to $f_{\mathbb{C}}$ just by restricting functions along the analytic boundary map:

$$\begin{tikzcd} \mathbb{D}_{\mathbf{O}} \times P_{\mathbf{B}}(\mathbb{A}^{(\infty)})/K_1 \arrow[r] \arrow[d] & \mathcal{E} \\ \mathbb{D}_{\mathbf{O}} \times P_{\mathbf{O}}(\mathbb{A}^{(\infty)})/K \arrow[ur] & \end{tikzcd}$$

where, as usual, in the first line, we imagine $\mathbb{D}_{\mathbf{O}}$ as an open subset of $\mathbb{D}_{\mathbf{B}}$.
By abuse of notation, we will denote $\Xi_1^* \operatorname{M}^\vee(\iota)^*(\mathcal{E})$ just by $\Xi_1^*(\mathcal{E})$.
Recall from (5.7) that there is section

$$l : \mathbb{D}_{\mathbf{B}} = \mathbb{D}_{\mathbf{H}_0} \times \mathbb{D}^\circ \to P_{\mathbf{B}}(\mathbb{C}) = U_{\mathbb{C}} \rtimes \mathbb{C}^*$$

(splitting, as usual, determined by I^*) given by

$$(\alpha, ZI') \mapsto (Z, \alpha(1)).$$

Now choose a *primitive* vector $z' \in I'$ and consider the section

$$\begin{aligned} &\mathbb{D}_{\mathbf{B}} \times P_{\mathbf{B}}(\mathbb{A}^{(\infty)})/K_1 \to \mathcal{E} \\ &\quad (\alpha, Z < I' >), [p] \mapsto l(\alpha, ZI') \circ z' = \alpha(1)^{-1}(\exp(Z)z') \qquad \text{for } p \in K_U K(1). \end{aligned}$$

Recall that for $Z = z \wedge k$, $\exp(Z)z' = z' + \langle z, z'\rangle k - (\langle k, z'\rangle + \langle z, z'\rangle Q_L(k))z$.
Extend it to a $P_{\mathbf{B}}(\mathbb{Q})$-equivariant section over all of $\mathbb{D}_{\mathbf{B}} \times P_{\mathbf{B}}(\mathbb{A}^{(\infty)})/K_1$. The quotient mod $P_{\mathbf{B}}(\mathbb{Q})$ constitutes an *integral* trivializing section $s_{z'}$ of $\Xi_1^*(\mathcal{E})$ over $\operatorname{M}({}^{K_1}\mathbf{B})$, extending to a trivializing section over $\operatorname{M}({}^{K_1}_{\Delta_1}\mathbf{B})$ (5.7), see also the analytic description of Ξ_1^* (9.1.2). Similarly, we get trivializing sections $s_{z'}^{\otimes l}$ for any $\Xi^*(\mathcal{E})^{\otimes l}$.
Starting with a (meromorphic) section f of $\Xi^*(\mathcal{E})^{\otimes l}$ on $\operatorname{M}({}^{K}_{\Delta}\mathbf{X})$ and trivializing the pullback via the boundary map by means of $s_{z'}^{\otimes l}$, we get Fourier coefficients

$$FC(f, \alpha, I, z', \rho, k, \sigma) \quad \text{“} := FC(\frac{f}{s_{z'}^{\otimes l}}, \alpha, I, I', \rho, k, \sigma)\text{”}.$$

Observe that they now depend on the special $z' \in I'$. However: $FC(f, \alpha, < z >, \lambda z', \rho, k, \sigma) = \lambda^l FC(f, \alpha, < z >, z', \rho, k, \sigma)$.
The corresponding formal function $\frac{f}{s_{z'}^{\otimes l}}$ on the canonical model $\operatorname{M}({}^{K_K K(N)}_{\Delta_1}\mathbf{B})$ for some $K_U \rtimes K(N) \subset K_1$ is then given by

$$\sum_{\lambda \in U_{\mathbb{Q}}^*} FC(f, \alpha, I, z', \rho\beta, \alpha(\lambda), \sigma)[\lambda]$$

in the fibre over some $\zeta_N = \exp(\alpha(\beta/N))$ for $\beta \in K(1)$.

Note that the Fourier expansions considered in [4] and [11] of modular forms (of weight l) differ by $(\pm 2\pi i)^l$, the sign being determined by ι.

(10.4.9) *Assume for the rest of this section that $m \geq 4$, and that, if $m = 4$, the Witt rank of $L_{\mathbb{Q}}$ (which can be 0, 1 or 2) is one.* This is because we want the Koecher principle to hold, i.e. the boundary in the Baily-Borel compactification should be of codimension ≥ 2. Furthermore, we want to prove and use a q-expansion principle, i.e. there should be at least a non-empty boundary.
Choose a common orientation of the negative definite subspaces of $L_{\mathbb{R}}$. Recall $\mathbb{D}_{\mathbf{H}_0} \cong \pi_0(\mathbb{D}_{\mathbf{O}})$ (isomorphism determined by this orientation — 10.2.1). Choose representatives $g_i \in \mathrm{SO}(\mathbb{A}^{(\infty)})$, $\alpha_i \in \mathbb{D}_{\mathbf{H}_0}$ of the classes

$$[\alpha_i, g_i] \in \mathrm{SO}(\mathbb{Q}) \backslash \mathbb{D}_{\mathbf{H}_0} \times \mathrm{SO}(\mathbb{A}^{(\infty)})/K'.$$

Denote by ι_i the embedding $\mathrm{Grass}^-(L_{\mathbb{R}})$ in $\mathbb{D}_{\mathbf{O}}$ with 'image α_i'. Let $L_{\mathbb{Z}}^{(i)} \subset L_{\mathbb{Q}}$ be the corresponding lattices, i.e. determined by $L_{\widehat{\mathbb{Z}}}^{(i)} = g_i L_{\widehat{\mathbb{Z}}}$.
Call a function: $\mathrm{Grass}^-(L_{\mathbb{R}}) \to \mathcal{E} \subset L_{\mathbb{C}}$ an ι-holomorphic orthogonal modular form with respect to $\Gamma^{(i)}$, if it is $\Gamma^{(i)}$-invariant, and if it is holomorphic when considered via the embedding $\iota : \mathrm{Grass}^-(L_{\mathbb{R}}) \hookrightarrow \mathbb{D}_{\mathbf{O}}$. If the function is ι-meromorphic, call it a ι-meromorphic orthogonal modular form.
We have an isomorphism

$$\bigcup_i \Gamma^{(i)} \backslash \mathrm{Grass}^-(L_{\mathbb{R}}) \to \mathrm{SO}(L_{\mathbb{Q}}) \backslash \mathbb{D}_{\mathbf{O}} \times \mathrm{SO}(\mathbb{A}^{(\infty)})/K = \mathrm{M}({}^K\mathbf{O}(L))(\mathbb{C}) \tag{4}$$

given by $\nu \mapsto [\iota_i(\nu), g_i]$ and where $\Gamma^{(i)} = \mathrm{SO}(L_{\mathbb{Q}}) \cap (\mathrm{SO}(L_{\mathbb{R}})^+ K)$.
We know (10.2.7) that these geometric connected components and their closures in any $\mathrm{M}({}^K_\Delta\mathbf{O}(L))$ are defined over $R = \mathbb{Z}_{(p)}[\zeta_{M,\mathbb{C}}]$ considered as a subring of $\mathbb{C}$ for some $M \in \mathbb{N}$. A section $f \in H^0(\mathrm{M}({}^K\mathbf{S}(L))_{\mathbb{C}}, \Xi^*\mathcal{E})$ pulled back via the isomorphism (4), yield orthogonal ι_i-modular forms $f^{(i)}$ with respect to $\Gamma^{(i)}$ on every connected component and vice versa, by the Koecher principle (we excluded the cases where it does not hold). The same holds true for meromorphic sections, resp. collections of ι_i-modular forms.
We also have

$$H^0(\mathrm{M}({}^K\mathbf{S}(L))_{\mathbb{C}}, \Xi^*\mathcal{E}) = H^0(\mathrm{M}({}^K_\Delta\mathbf{S}(L))_{\mathbb{C}}, \Xi^*\mathcal{E})$$

for any Δ, and again the same for meromorphic sections.

(10.4.10) Theorem (q-EXPANSION PRINCIPLE FOR ORTHOGONAL MODULAR FORMS). *Let I, z', K be as before. There is an M such that for $R := \mathbb{Z}_{(p)}[\zeta_{N,\mathbb{C}}] \subset \mathbb{C}, M|N$, we have (for any Δ) an isomorphism*

$$H^0(\mathrm{M}({}^K_\Delta\mathbf{O})_R, \Xi^*\mathcal{E}) =$$
$$\left\{ f \in H^0(\mathrm{M}({}^K\mathbf{O})_{\mathbb{C}}, \Xi^*\mathcal{E}) \middle| \forall i \ \forall \lambda \in U_{\mathbb{Q}}^*, \ FC(f, \alpha_i, I, z', g_i, \alpha(\lambda)) \in R \right\}$$

and similarly[6]

$$H^0(\mathrm{M}({}^K_\Delta\mathbf{O})_R, \Xi^*\mathcal{E}_{horz}) =$$
$$\left\{ f \in H^0(\mathrm{M}({}^K\mathbf{O})_\mathbb{C}, \Xi^*\mathcal{E}_{merom.}) \middle| \begin{smallmatrix} \forall i\ \exists \sigma \subset U_\mathbb{R}(-1)\ open\ r.p.\ cone, \\ \forall \lambda \in U_\mathbb{Q}^*,\ FC(f,\alpha_i,I,z',\alpha(\lambda),\sigma)\in R \end{smallmatrix} \right\}$$

Moreover, we have

$$FC(f^\tau, \alpha, I, z', \rho, \alpha(\lambda), \sigma) = FC(f, \alpha, I, z', \rho\mu_{I,I'}(k_\tau), \alpha(\lambda), \sigma)^\tau \qquad \forall \tau \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q}),$$

hence f is (in both cases) defined over $\mathbb{Z}_{(p)}$ instead of R, if it satisfies

$$FC(f, \alpha, I, z', \rho, \alpha(\lambda), \sigma) = FC(f, \alpha, I, z', \rho\mu_{I,I'}(k_\tau), \alpha(\lambda), \sigma)^\tau \qquad \forall \tau \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q}).$$

Here k_τ is the image of τ under the natural isomorphism $\mathrm{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q}) \simeq K(1)/K(N)$.

Proof. We may show the theorem for a particular Δ. A posteriori, it will then be true for any Δ (for the first statement, this involves the Koecher principle).
Assume, by choosing M appropriately, that R contains the ring of definition of every connected component of $\mathrm{M}({}^K\mathbf{O})$ (10.2.7).
Let f be a complex section of the bundle $\Xi^*(\mathcal{E})^{\otimes l}$ over $\mathrm{M}({}^K_\Delta\mathbf{O})$. By the abstract q-expansion principle (5.9.1) a section of a locally free sheaf is defined over $\mathrm{M}({}^K_\Delta\mathbf{O})_R$ if on any connected component (over R), there is a point p, defined over R, such that after passing to the completion at p, the corresponding formal series is defined over R. Now, by assumption on any connected component described by α_i, g_i, there is a boundary point corresponding to I and an open (= maximal dimensional) r.p.c. cone $\sigma \subset K_R(-1)$ such that there exists a Fourier expansion with respect to it, with values in R. Choosing Δ such that it contains these cones or is a refinement of one containing them, we see that there is a boundary point, around which f has only poles along the boundary. By the boundary isomorphism $\mathrm{M}(\iota, g_i)$ (3.3.5), it suffices to show that the pullback of f to $\Xi_1^*(\mathcal{E})$ over $\widehat{\mathrm{Sh}({}^{K_1}_{\Delta_1}\mathbf{B})}$ is defined over R. By the consideration in (10.4.8) above, the formal function $f/s_{z'}$ on some $\mathrm{M}(\widehat{{}^{K_K K(N')}_{\Delta_1}\mathbf{B}})$ for $K_K K(N') \subset K_1$ (splitting via I') is given by

$$\left(\frac{f}{s}\right)_{\zeta_{N'},\mathbb{C}} = \sum_{v \in K(-1)^*} FC(f, \alpha_i, I, z', g_i\mu_{I,I'}(k), \alpha(\lambda), \sigma)[\lambda].$$

Here, $[\alpha, k] \in \mathbb{Q}^* \backslash \mathbb{D}_{\mathbf{H}_0} \times \mathbb{A}^{(\infty)^*}/K(N')$ describes $\zeta_{N'}$. (It suffices of course to look at the fibre over the $\zeta_{M'}$ corresponding to $[\alpha_i, 1]$). We may assume that for all i, M divides the occurring N''s here.
Then, since by assumption these coefficients lie in R, $f/s_{z'}$ is defined over R and hence f itself. The first statements follows because we investigated at least 1 boundary point on each connected component.
Furthermore the Galois operation on a formal function (with possible poles along the

[6] $horz.$ means that we inverted all functions, which have no component of the fibre above p in its divisor.

boundary)

$$f \in \widehat{\mathcal{O}}_{\mathrm{M}({}^{K_U K(N')}_{\Delta}\mathbf{O})} \otimes R = \bigoplus_{\zeta_{N',\mathbb{C}}} R[\![U_{\mathbb{Z}}^* \cap \alpha(\sigma^\vee)]\!][U_{\mathbb{Z}}^*],$$

$$f = \sum_{\lambda \in U_{\mathbb{Z}}^*} f_{\zeta_{N'},\lambda}[\lambda]$$

is given by

$$(f^\tau)_{\zeta_{N',\mathbb{C}},v} = f^\tau_{\zeta_{N',\mathbb{C}}^{\tau^{-1}},v}.$$

And hence, a formal function is in $\widehat{\mathcal{O}}_{\mathrm{M}({}^{K_U K(M)}_{\Delta_1}\mathbf{B})}$, if and only if,

$$f_{\zeta_{N',\mathbb{C}},v} = f^\tau_{\zeta_{N',\mathbb{C}}^{\tau^{-1}},v}.$$

If we are given a section f of $\Xi^*(\mathcal{E})_R$ and we know this for all boundary components, and ρ, we may infer that f a section of $\Xi^*(\mathcal{E})$ defined over $\mathrm{spec}(\mathbb{Z}_{(p)})$. $\square$

(10.4.11) Recall from (7.8.1) the definition of the Borcherds lift $\Phi(\nu, h; F)$ associated with an automorphic form

$$F : \mathrm{Sp}(\mathfrak{M}_{\mathbb{Q}}) \backslash \mathrm{Sp}'(\mathfrak{M}_{\mathbb{A}^{(\infty)}}) \to S(L_{\mathbb{A}^{(\infty)}})$$

for the Weil representation, corresponding to a weakly holomorphic classical modular form.

Let $p \neq 2$ be a prime[7].

Assume that F has Fourier coefficients of the form $c(q) = \chi_{L_{\mathbb{Z}_p}} \otimes c^p(q)$, $c^p(q)$ as in (7.8.1) with values in $S(L_{\mathbb{A}^{(\infty,p)}})$. F then is invariant under an (w.l.o.g.) *admissible* compact open subgroup $K \subseteq \mathrm{SO}(\mathbb{A}^{(\infty)})$, with Fourier coefficients $c(q) \in S(L_{\mathbb{A}^{(\infty)}})$, $q \in \mathbb{Q}$.

For this to exist, it is necessary that p does not divide a minimal discriminant of $L_{\mathbb{Q}}$.

Let R be a huge[8] ring of roots of unity $\mathbb{Z}_{(p)}[\zeta_{N,\mathbb{C}}] \subset \mathbb{C}$, $p \nmid N$.

Using the q-expansion principle proven in this section (10.4.10), we will show the following strengthening of the main results [4, Theorem 13.3] and [57, Theorem 2.12] (resp. [13, Theorem 4.11]):

(10.4.12) Theorem. *Up to multiplication of F by a large constant $\in \mathbb{Z}$, we have*

i. For any Δ, there is a section

$$\Psi(F) \in H^0(\mathrm{M}({}^K_\Delta\mathbf{O})_R, (\Xi^*\mathcal{E})^{\otimes c_0(0)/2}_{horz.})$$

[7] $\neq 2$ in order to be able to apply results of part I

[8] containing the ring given by the last theorem (q-expansion principle) but maybe bigger. Its size will become clear from the proof.

such that on $\mathrm{M}({}^K\mathbf{O})(\mathbb{C})$*, we have*

$$\Phi(\nu, h; F) = -2\log h(\Psi(\nu, h; F)).$$

There is a locally constant, invertible function Λ *on* $\mathrm{M}({}^K_\Delta\mathbf{O})_R$ *such that*

$$\Lambda\Psi(F) \in H^0(\mathrm{M}({}^K_\Delta\mathbf{O}), (\Xi^*\mathcal{E})^{\otimes c_0(0)/2}_{horz.})$$

(i.e. $\Lambda\Psi(F)$ *is defined over* $\mathbb{Z}_{(p)}$*).*

ii.

$$\operatorname{div}(\Psi(\nu, h, F)^2) = \sum_{q<0} \mathrm{Z}(L_\mathbb{Z}, < -q >, c(q); K) + \Xi$$

(cf. 10.3.1), where Ξ *has support only within the support of the exceptional divisor.*

iii.

$$-\int_{\mathrm{M}({}^K\mathbf{O})_\mathbb{C}} \log h(\Lambda\Psi(\nu, h; F)) c_1(\Xi^*\mathcal{E}, \Xi^*h)^n$$
$$\equiv \sum_{q<0} \mathrm{vol}_\mathcal{E}(\mathrm{M}({}^K\mathbf{O})) \cdot \frac{\mathrm{d}}{\mathrm{d}\, s}\widetilde{\mu}(L, < -q >_\mathbb{Z}, c(q); s)\Big|_{s=0}$$

in $\mathbb{R}^{(p)}$*.*

iv. A Fourier expansion of $\Psi(F)$[9]*:*

$$\sum_{\lambda \in U^*_\mathbb{Q}} FC(\Psi(F), \alpha, I, z', 1, \alpha(\lambda), \sigma)[\lambda]$$

is up to a constant of absolute value

$$C^{-\frac{c(0,0)}{2}} \prod_{\substack{\delta \in I_{\mathbb{A}^{(\infty)}} \\ \chi(\langle \delta, z'\rangle) \neq 1}} (1 - \chi(\langle \delta, z'\rangle))^{\frac{c(\delta,0)}{2}\,\mathrm{d}\langle z'\delta,\rangle} \tag{5}$$

given by:

$$[\rho(I, \sigma, F)] \prod_{\lambda \in U^*_\mathbb{Q} \cap \sigma^\vee} \prod_{\delta \in I_{\mathbb{A}^{(\infty)}}} (1 - \chi(\langle z', \delta\rangle)[\lambda])^{c({}^t(C\lambda)_{z'} \circ z' + \delta, Q_{z'}(C\lambda))\,\mathrm{d}\langle z', \delta\rangle} \tag{6}$$

here $\mathrm{d}\cdots$ *is any invariant measure on* $\mathbb{A}^{(\infty)}$ *which has the property that all sets* $\{\langle z', \delta\rangle \mid \delta \in I_{\mathbb{A}^{(\infty)}}, F(x + \delta, \cdot) = f(\cdot)\}$ *for all* x, f *have volumes in* $\mathbb{Z}_{\geq 0}$*. The products (in fact multiplicative integrals) over* $\mathbb{A}^{(\infty)}$ *then have the obvious meaning.* χ *is the corresponding character (such that* $\mathrm{d}\,\delta$ *is self-dual with respect to it) and* C *is the conductor, i.e. the volume of* $\widehat{\mathbb{Z}}$ *with respect to it.* ${}^t(\lambda)_{z'}$ *is the isomorphism*

[9] This suffices to know every Fourier expansion of $\Psi(F)$ by property (12) above.

$U^* \to U$ induced by the quadratic form

$$Q_{z'}(z_1 \wedge k) = \langle z_1, z' \rangle^2 Q(k) = Q((z_1 \wedge k) \circ z').$$

The Weyl vector $\in U^*$ is characterized by

$$8\pi\rho(< z >, \sigma, F)\nu = \sqrt{Q_{z'}(\nu)}\Psi(< \nu >, F^{U,z'}) \qquad \text{for } \nu \in \sigma \subset U_{\mathbb{R}},$$

where $F^{U,z'}$ is defined as:

$$F^{U,z'}(w, g') := C^{-1} \int_{\delta \in I_{\mathbb{A}^{(\infty)}}} F(w \circ z' + \delta, g') \, \mathrm{d}\langle z', \delta \rangle$$

(any invariant measure).

Proof. First observe that F is a finite linear combination of functions valued in $\mathbb{C}[L_{\mathbb{Z}}/L_{\mathbb{Z}}^*]$ for various lattices $L_{\mathbb{Z}}$ of $L_{\mathbb{Z}_{(p)}}$. Hence, we may assume that F is of this kind, observing that all properties are stable under taking finite linear combinations, resp. change of K. Hence w.l.o.g. F corresponds (as above) to an f considered in [4].

Recall the normalization of the Hermitian metric h (10.4.1). Consider a boundary component associated with some I and z'. $\frac{\Psi}{s}$ may be seen as a function on $L_{0,\mathbb{C}}$. We have

$$h_{\mathcal{E}}(\Psi) = |\frac{\Psi}{s}|^2 h_E(s),$$

and

$$h_{\mathcal{E}}(s) = ((2\pi)^{-1} e^{-\gamma} \langle \Im(Z), \Im(Z) \rangle (2\pi)^2)^{\frac{c_0(0)}{2}}.$$

Hence we get

$$\log h_{\mathcal{E}}(\Psi) = \log |\frac{\Psi}{s}|^2 + \frac{c_0(0)}{2}(\log(\langle \Im(Z), \Im(Z) \rangle) + \log(2\pi) - \gamma)$$

and (i) follows from [4, Theorem 13.3], (iii) is essentially [57, Theorem 2.12 (ii)], using (7.10.1) and (7.6.12). See [13, Theorem 4.11] for an independent proof (use 6.4.5 for the comparison). Observe that the function Ψ differs from the one in [57] or [4] by $(2\pi i)^{\frac{c_0(0)}{2}}$. However, important here is only the overall constant relating an F with given *integral* Fourier coefficients to a Greens function which is to be integrated. This constant has been adjusted, such that (i) and (iii) hold. See also (11.2), where the same constant also appears (almost) naturally.

That some multiple of F exists, such that *all* coefficients are integral and even is due to the classical q-expansion principle. Further multiplication with the order of the character of $\mathrm{SO}(\mathbb{Q})$ occurring in [4], we get a section of the *complex* bundle.

Before proving (iv), we need some lemmas stating well-definition and invariance properties of the expressions (6) and (5). $\square$

(10.4.13) Lemma. *The expressions (6) and (5) do not depend on the choice of measure, provided it has the property stated above.*

Proof. Rewrite the first integral as

$$\left.\frac{\prod_{\delta\in I_{\mathbb{A}(\infty)}}(1-\chi(\langle z',\delta\rangle)[C^{-1}\lambda])^{\frac{c(\delta,0)}{2}\,\mathrm{d}\langle z',\delta\rangle}}{(1-[C^{-1}\lambda])^{\frac{c(0,0)}{2}}}\right|_{\lambda=0}.$$

Now consider the product

$$\prod_{\delta\in(z')^{-1}(C\widehat{\mathbb{Z}})+a}(1-\chi(\langle z',\delta\rangle)[\lambda C^{-1}])^{\mathrm{d}\langle z',\delta\rangle}=(1-\chi(\langle z',a\rangle)[C^{-1}\lambda]).$$

The above multiplicative integrals split into a product of these integrals by hypothesis on the measure (substitute $\lambda\mapsto C^{-1}\lambda$ in the second product). Now changing the measure to $N\,\mathrm{d}\,\delta$, we get the expression

$$\begin{aligned}&\prod_{\delta\in(z')^{-1}(C\widehat{\mathbb{Z}})}(1-\chi(N^{-1}\langle z',a\rangle)\chi(N^{-1}\langle z',\delta\rangle)[(NC)^{-1}\lambda])^{\mathrm{d}\,N^{-1}\langle z',\delta\rangle}\\ =&\prod_{i\in\mathbb{Z}/N\mathbb{Z}}(1-\chi(N^{-1}\langle z',a\rangle)\zeta^i[(NC)^{-1}\lambda])\\ =&(1-\chi(\langle z',a\rangle)[C^{-1}\lambda])\end{aligned}$$

here ζ is some primitive N-th root of unity in $\mathbb{C}$. This shows invariance of the products. We have to check what happens to

$$\frac{1}{(1-[\lambda C^{-1}])^{\frac{c(0,0)}{2}}}.$$

It changes to $\frac{1}{(1-[\lambda(NC)^{-1}])^{\frac{c(0,0)}{2}}}$ hence gets multiplied by

$$(1+[\lambda(CN)^{-1}]+[2\lambda(CN)^{-1}]+\cdots+[(N-1)\lambda(CN)^{-1}])^{\frac{c(0,0)}{2}},$$

which for $\lambda=0$ is equal to $N^{\frac{c(0,0)}{2}}$. $C^{-\frac{c(0,0)}{2}}$ gets $(NC)^{-\frac{c(0,0)}{2}}$. Therefore the whole expression is invariant as well. □

(10.4.14) Lemma. *This expression multiplied with the trivialization is invariant under $z'\mapsto\beta z'$ for $\beta\in\mathbb{Q}_{>0}$*

Proof. If we multiply z' by a factor $\beta\in\mathbb{Q}_{>0}$, the trivialization gets multiplied by β^l, where $l=c(0,0)/2$ is the weight of the modular form. On the other hand changing z' in the various $\langle z', \delta \rangle$ has the same effect as scaling the measure, hence (by the last lemma) as multiplying C by β. The expression $\lambda \circ z'$ gets divided by β (note that it depends on z' in several ways), $Q(\lambda)$ gets divided by β^2. Hence we get the same expression.

Furthermore the expression claimed for the Weyl vector does not depend on the choices: Scaling z' to $\beta z'$ changes the $w \circ z'$ to $\beta w \circ z'$. $F^U(w, z')$ changes to $\beta^{-1} F^U(\beta w, g')$. Recall the definition of the Weyl vector above. Here, since the argument of the quadratic form on $U_{\mathbb{Q}}$ and the argument in $K_{\mathbb{A}^{(\infty)}}$ of F^U are both scaled by β, $\Psi(\dots)$ is only scaled by β^{-1} coming from scaling F^U, hence invariance of the Weyl vector. σ is invariant under scaling. $\square$

(10.4.15) Lemma. *If the Fourier expansion of a section*

$$f \in H^0(\mathrm{M}({}^K\mathbf{O})_{\mathbb{C}}, \Xi^*(\mathcal{E})_{merom.})$$

for the trivialization defined by z' is given as in (6), it is for any $z'' \in I''$, given by the same expression involving z'', with respect to the trivialization defined z''.

Proof. The change of z' to $z'' = z' + \langle z_1, z' \rangle k - (\langle k, z' \rangle + \langle z_1, z' \rangle Q(k)) z_1 = \exp(w) z'$ has on the trivialization the effect of translation by $w = z_1 \wedge k$ in the argument. We have to see that this is the same effect that occurs in the expression (6) above. First of all $Q_{z'} = Q_{z''}$ because $\langle z, z' \rangle = \langle z, z'' \rangle$. $c({}^t(\lambda)_{z'} \circ z' + \delta z$ is changed to $c({}^t(\lambda)_{z'} \circ z' + (\delta + \frac{\lambda w}{\langle z, z' \rangle}) z$. Now substitute $\delta - \frac{\lambda w}{\langle z, z' \rangle})$ for δ and get the same expression, but $[\lambda C^{-1}]$ multiplied by $\exp(-2\pi i (\lambda w) C^{-1})$, the same happening under translation. The expression claimed for the Weyl vector, which implicitly depends on z', is invariant under this operation as well. In the end, every z' can be transformed in any vector with $\langle z, z' \rangle > 0$ by the two operations of translation by $U_{\mathbb{Q}}$ and scaling. $\square$

Proof of theorem, iv. By the previous lemmas it suffices to prove it for any choice of z', so take $z' := z'_{Borcherds} - Q(z'_{Borcherds}) z_{Borcherds}$, which is now isotropic, but satisfies still $\langle z_{Borcherds}, z' \rangle = 1$. As measure we may choose the standard invariant measure on $\mathbb{A}^{(\infty)}$, giving $\widehat{\mathbb{Z}}$ the volume 1. The trivialization now coincides (up to the factor $(2\pi i)^{\frac{c(0,0)}{2}}$ that we incorporated in the characterization of $\Psi(F)$) with the one given in [4], up to identification of $I^\perp / I$ with $I \otimes I^\perp / I$ with by $k \mapsto k \wedge z_{Borcherds}$. This interchanges Q and $Q_{z'}$.

Recall from [4, Theorem 13.3 (v)] that the Fourier expansion with respect to this particular choice is (up to a constant of absolute value 1) given by

$$\prod_{\delta \in \mathbb{Z}/N\mathbb{Z}} (1 - e(\delta/N))^{c_{\delta z/N}(0)/2} [\rho(U, W, F_U)] \prod_{\substack{\lambda \in (U_{\mathbb{Z}}^{(\rho)})^* \\ \langle \lambda, W \rangle > 0}} \prod_{\substack{\delta \in (L_{\mathbb{Z}})^*/L_{\mathbb{Z}} \\ \delta|_{z^\perp} = \lambda}} (1 - e(\langle \delta, z' \rangle)[\lambda])^{c_\delta(\lambda^2/2)}. \quad (7)$$

(5) now may be rewritten as

$$\prod_{\delta \in \frac{1}{N}\mathbb{Z}/\mathbb{Z}} (1 - e(\delta))^{\frac{c(\delta z, 0)}{2}}$$

and coincides with the product in Borcherds' expression. The Weyl vector is the same as well.
Claim: There is a bijection between

$$\{k_{Borcherds} \in (< z >_{\mathbb{Z}}^{\perp} \; / < z >_{\mathbb{Z}})^*, \delta_{Borcherds} \in L_{\mathbb{Z}}^*/L_{\mathbb{Z}}, \delta|_{z^{\perp}} = k\}$$

and

$$\{k \in U_{\mathbb{Q}}, \delta \in \mathbb{Q}/\mathbb{Z} = \mathbb{A}^{(\infty)}/\widehat{\mathbb{Z}} \;|\; k \circ z' + \delta z \in L_{\mathbb{Z}}^*\}.$$

Proof: The map is given sending k to $k_{Borcherds} \otimes z$ and δ to $k \circ z' + \delta z$. From the fact

$$k \circ z' + \delta z = k_{Borcherds} - < k_{Borcherds}, z' > z + \delta z \in L_{\mathbb{Z}}^*,$$

we get $k_{Borcherds} \in (< z >_{\mathbb{Z}}^{\perp} \; / < z >_{\mathbb{Z}})^*$. In the other direction, we map $\delta_{Borcherds}$ to $\delta = \langle k_{Borcherds}, z' \rangle$. These maps are inverse to each other. One direction is clear; for the other we have to show:

$$\delta_{Borcherds} \stackrel{!}{=} k_{Borcherds} + (\langle \delta_{Borcherds}, z' \rangle - \langle k_{Borcherds}, z' \rangle) z.$$

This expression, however, is true for $\delta_{Borcherds} \in L_{\mathbb{Q}}$ because $L_{\mathbb{Q}}$ is generated by $L_0 \cong L_0$ and z and z'. Furthermore changing it by an element in $L_{\mathbb{Z}}$ changes the expression by some integral multiple of $z \in L_{\mathbb{Z}}$. This shows that the 2nd product occurring in Borcherds' Fourier expansion is equal to our 2nd product above.
To prove the theorem, it remains to check the integrality properties stated. By multiplying F again by a large constant, we may also assume that the Weyl vector with respect to I and with respect to any i is integral.
We can just multiply $\Psi(F)$ one every connected component by an appropriate absolute value of 1, hence may assume that *one* Fourier expansion for each connected component is given *precisely* by the product of (5) and (6). We get i different expansions with coefficients in some ring $R = \mathbb{Z}_{(p)}[\zeta_{N,\mathbb{C}}]$, $p \nmid N$ which we assume to contain the ring described by (10.4.10).

Therefore by (10.4.10) $\Psi(F)$ extends to an (integral) section of $H^0(\mathrm{M}({}^K_\Delta\mathbf{O})_R, \Xi^*(\mathcal{E})_{horz.}^{\frac{c(0,0)}{2}})$ for any Δ.
Furthermore, the expansion is invertible integrally, hence we may infer that $\Psi(F)^{-1}$ is a section[10] of $H^0(\mathrm{M}({}^K_\Delta\mathbf{O})_R, \Xi^*(\mathcal{E})_{horz.}^{-\frac{c(0,0)}{2}})$. It follows that $\Psi(F)$ cannot have any connected (=irreducible) component of the fibre above p in its divisor.

To prove that $\Psi(F)$ is actually defined over $\mathbb{Z}_{(p)}$, i.e. lies in $H^0(\mathrm{M}({}^K_\Delta\mathbf{O}), \Xi^*(\mathcal{E})_{horz.}^{\frac{c(0,0)}{2}})$, let us check the requested invariance property of the Fourier coefficients.

[10] Up to the problem of constants, this is $\Psi(-F)$.

Let us compute

$$FC\left(\Psi(F)^{\tau}, \alpha, I, z', \rho, \lambda, \sigma\right) = FC\left(\Psi(F), \alpha, I, z', \rho\mu_{I,I'}(k_{\tau}), \lambda, \sigma\right)^{\tau}$$

for all $\tau \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q})$.

By the transformation property (12), we are reduced to compute

$$FC\left(\Psi(F')^{\tau}, \alpha, I, z', 1, \lambda, \sigma\right) = FC\left(\Psi(\omega(\mu_{I'}(k_{\tau}))^{-1}F'), \alpha, I, z', 1, \lambda, \sigma\right)^{\tau}$$

for $F' = \omega(\rho)^{-1}F$. We will, however, keep the letter F.

We have

$$c(\rho, m, \Psi(\omega(\mu_{I,I'}(k_{\tau}))^{-1}F) = c(\mu_{I'}(k_{\tau})\rho, m, \Psi(F))$$

by definition of the Weil representation of $\mathrm{SO}(L_{\mathbb{A}^{(\infty)}})$ and $\mu_{I'}(k_{\tau})$ operates trivial on ${}^t(C\lambda)_{z'} \circ z'$ because the latter is perpendicular to $I \oplus I'$ and it operates by multiplication by k_{τ} on δ. Substituting $k_{\tau}^{-1}\delta$ for δ and using invariance of the measure ($k_{\tau} \in \widehat{\mathbb{Z}}^*$!), we get the same expression, but all $\chi(\langle z', \delta\rangle)$ changed to $\chi(\langle z', k_{\tau}\delta\rangle) = \chi(k_{\tau}\langle z', \delta\rangle) = \chi(\langle z', \delta\rangle)^{\tau^{-1}}$.

However, note that the Fourier expansion in (iv) is only defined up to a complex constant of absolute value 1. Since for the computation above, we need maybe more than one Fourier expansion for the same connected component, we are not able to normalize anymore. However, our previous normalization shows that the occurring constants lie in R^*. They are invertible because otherwise there would be a connected component of the fibre above p in the divisor. We hence can infer from the above calculation that

$$f^{\tau} = \Lambda_{\tau} f$$

for all $\tau \in \mathrm{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q})$, where Λ_{τ} locally constant invertible on $\mathrm{M}({}^K\mathbf{O})_R$ with $|\Lambda_{\tau}|_{\infty} \equiv 1$ constant. From (an easily derived variant of) Hilbert 90 follows that there exists a Λ, locally constant invertible on $\mathrm{M}({}^K\mathbf{O})_R$ such that

$$(\Lambda f)^{\tau} = \tau f$$

(i.e. $\Lambda^{\tau}/\Lambda = \Lambda_{\tau}$).

The evaluation of

$$\int \log \parallel \Lambda f \parallel c_1(\Xi^*(\mathcal{E}))^d$$

differs from the same expression for f by

$$\sum_{\alpha \in \pi_0(\mathrm{M}({}^K\mathbf{O}))} \mathrm{vol}(\alpha) \log(|\Lambda(\alpha)|_{\infty}).$$

Since $|\Lambda(\tau^{-1}\alpha)^\tau|_\infty = |\Lambda(\alpha)|_\infty$ we may write this as

$$\begin{aligned}
&\frac{1}{\phi(M)} \sum_{\tau\in\mathrm{Gal}(\mathbb{Q}(\zeta_M)|\mathbb{Q}),\alpha\in\pi_0(\mathrm{M}(^K\mathbf{O}))} \mathrm{vol}(\alpha)\log(|\Lambda(\tau^{-1}\alpha)^\tau|_\infty)\\
=&\frac{1}{\phi(M)} \sum_{\tau\in\mathrm{Gal}(\mathbb{Q}(\zeta_M)|\mathbb{Q}),\alpha\in\pi_0(\mathrm{M}(^K\mathbf{O}))} \mathrm{vol}(\tau\alpha)\log(|\Lambda(\alpha)^\tau|_\infty)\\
=&\frac{1}{\phi(M)} \sum_{\alpha\in\pi_0(\mathrm{M}(^K\mathbf{O}))} \mathrm{vol}(\alpha)\log(|N_{\mathbb{Q}(\zeta_M)|\mathbb{Q}}(\Lambda(\alpha))|_\infty) \equiv 0 \qquad \text{in } \mathbb{R}^{(p)}
\end{aligned}$$

Here we used $\mathrm{vol}(\tau\alpha) = \mathrm{vol}(\alpha)$, which is true because $c_1(\Xi^*(\mathcal{E}))$ is Galois invariant.

□

(10.4.16) We will briefly investigate orthogonal modular forms, i.e. the bundle $\Xi^*(\mathcal{E})$ along higher dimensional boundary strata (only over $\mathbb{C}$). Let I be a 2-dimensional isotropic subspace. Denote $\mathbf{B}$ the corresponding boundary component (10.2.17). We have a morphism

$$\mathbf{B} \to \mathbf{H}(I)$$

and also a splitting of this morphism corresponding to a choice of an isotropic I^* in $L_{\mathbb{Z}_{(p)}}$. The 'dual' morphism maps $< z > \in \mathrm{M}^\vee(\mathbf{H}(I))$ to the corresponding $< z >^\perp \in \mathbb{P}(I^*) \subseteq \mathrm{M}^\vee(\mathbf{B}) \subseteq \mathrm{M}^\vee(\mathbf{O})$. Assume such a splitting is chosen. Denote by L_0 the orthogonal complement of $I \oplus I^*$.

The fibre over a point $[< z >, \rho] \in \mathrm{GL}(I_\mathbb{Q})\backslash\mathbb{D}_{\mathbf{H}(I)} \times \mathrm{GL}(I_{\mathbb{A}^{(\infty)}})/K$ is of the form (K appropriate)

$$W_\mathbf{B}(\mathbb{Q})\backslash W_\mathbf{B}(\mathbb{C}) \times W_\mathbf{B}(\mathbb{A}^{(\infty)})/F^0(W_\mathbf{B})K,$$

where $[w_\mathbb{C}, w_{\mathbb{A}^{(\infty)}}]$ is mapped to $[w_\mathbb{C} < z >^\perp, w_{\mathbb{A}^{(\infty)}}\rho]$ ($[< z >, \rho]$ embedded via the splitting). $F^0(W_\mathbf{B})$ is the stabilizer of $< z >^\perp$ in $W_\mathbf{B}(\mathbb{C})$; it maps down to $F^0(L) =< z > \otimes L_0$ in $V_\mathbf{B}(\mathbb{C}) = I \otimes L_0$.

The fibre itself is a $\mathbb{G}_m$-torsor over the Abelian variety $V_\mathbf{B}(\mathbb{Q})\backslash V_\mathbf{B}(\mathbb{C}) \times V_\mathbf{B}(\mathbb{A}^{(\infty)})/F^0(L) \times K$.

(10.4.17) Lemma. $\Xi^*(\mathcal{E})$ *is trivial* locally on the base.

Proof. This is because $F^0(V)$ has no nontrivial 1-dimensional representations. Hence we have a map

$$W(\mathbb{Q})\backslash(W(\mathbb{C})/F^0(W)) \times (W(\mathbb{A}^{(\infty)})/K) \to W(\mathbb{Q})\backslash\mathcal{E} \times (W(\mathbb{C})/F^0(W)) \times (W(\mathbb{A}^{(\infty)})/K)$$

given by mapping $[w, 1]$ to w applied to any chosen point (trivialization) in the fibre $\mathcal{E}_x$. We may do the same with a local section. □

We will give the trivialization more explicitly as follows: Chose a basis of I and a corresponding dual basis of I^*. This induces as well $U(\mathbb{C}) = \Lambda^2 I_\mathbb{C} \simeq \mathbb{C}$ and an isomorphism $I \otimes I^\perp / I = I \otimes L_0 \simeq L_0^2$. We will write vectors as $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_I$, resp. $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{I^*}$

Let $z' = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{I^*} \in I^*$, in the fibre of $\mathcal{E}$ over $< z' >$. It lies in $< z >^\perp$,for $\begin{pmatrix} \beta \\ -\alpha \end{pmatrix}_I$.

Writing an element in $w \in W_\mathbf{B}(\mathbb{C})$ as $w = (u, k_1, k_2)$, $u \in \mathbb{C} = U_\mathbf{B}(\mathbb{C})$, $k_1, k_2 \in L_{0,\mathbb{C}}$, we get

$$wz' = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}_{I^*} + u \begin{pmatrix} \beta \\ -\alpha \end{pmatrix}_I + \alpha k_1 + \beta k_2 - \begin{pmatrix} \alpha Q(k_1) \\ \beta Q(k_2) + \alpha \langle k_1, k_2 \rangle \end{pmatrix}_I.$$

Note that we have an isomorphism $I_\mathbb{R} \otimes_\mathbb{R} L_{0,\mathbb{R}} = (I_\mathbb{C} / < z >) \otimes_\mathbb{C} L_{0,\mathbb{C}}$ (because $< z > \in \mathbb{D}_\mathbf{O}$ by assumption). Hence we have an isomorphism (as usual)

$$U(\mathbb{C})W(\mathbb{R}) = W(\mathbb{C})/F^0(W).$$

This means that we may write an element in $W(\mathbb{C})/F^0(W)$ uniquely as (u, k_1, k_2) with $u \in U(\mathbb{C})$ and $k_1, k_2 \in L_{0,\mathbb{R}}$ The map

$$(u, k_1, k_2) \mapsto \Im(u)$$

is then the projection onto the imaginary part (2.4.6). The norm of the trivializing section of $\mathcal{E}$ determined by $M\alpha\beta_{I^*}$ as explained, is hence given by

$$|2\Im(\overline{\alpha}\beta)\Im(u)|.$$

If we choose a local lift $s : \mathrm{M}({}^K\mathbf{H}(I)) \to \mathbb{D}_{\mathbf{H}(I)} \to I^*$ mapping a point $< z > \subset I$ to some point in $< z >^\perp \subset I^*$, we may extend this trivializing section to a neighborhood of any point in $\mathrm{M}({}^K\mathbf{H}(I))$.

Now, one may find a local map $\rho : L_{0,\mathbb{C}} \to L_{0,\mathbb{C}}^2$ such that the image is a direct summand of $F^0(L)$ for all points on the base.

A local chart around any point of the exceptional divisor hence may be described as follows:

$$B(R, \mathbb{C}) \times B(R, \mathbb{C})^{m-4} \times B(R, \mathbb{C}).$$

The third type ball mapping to a neighborhood of $< z > \in \mathbb{D}_{\mathbf{H}(I)}$ to the corresponding α, β. the second type are just the parametrization of $L_{0,\mathbb{C}}$ by means of some basis followed by ρ. The first is a ball mapping via log to the u-coordinate. The trivializing section here has norm:

$$2\Im(u)\Im(\alpha\overline{\beta}) + 2|\alpha|^2(Q(y_1)) + 2|\beta|^2(Q(y_2)) + \Im(\alpha\overline{\beta}\langle k_1, y_2 \rangle),$$

(here we wrote $k_j = x_j + iy_j$). In the sequel we only need to remember that it has the form

$$\log(r_1)\psi_1(z_{m-2}) + \psi_0(z_2, \ldots, z_{m-3}),$$

where ψ_j are bounded C^∞-functions on the neighborhood chosen.

10.5. Main results: Geometric and arithmetic volume of Shimura varieties of orthogonal type and of their special cycles

The results of this section finally connect Arakelov geometry on the models constructed in part I with the analytic functions μ and λ defined in part II. *We emphasize again that the 'main theorems' of part I hold true conditionally on the conjecture (3.3.2), hence all 'theorems' in this section as well.*

(10.5.1) Recall (7.10.4) from the definition of the complex function

$$\lambda(L; s) = \prod_\nu \lambda_\nu(L; s)$$

and its modification $\widetilde{\lambda}$. There are explicit computations of them in (8.2) for the local factors and in (8.3) for the product. They are related to the Fourier coefficients of Eisenstein series by the orbit equation (7.10.4).

(10.5.2) Main theorem (GLOBAL FORMULATION). *Let $L_\mathbb{Z}$ be a lattice with quadratic form of discriminant $D \neq 0$ and signature $(m-2, 2)$. Let K be the* discriminant kernel *of $L_{\widehat{\mathbb{Z}}}$. It is an admissible compact open subgroup for all $p \nmid D$. Let Δ be a complete and smooth K-admissible rational polyhedral cone decomposition and let $\mathrm{M} = \mathrm{M}({}^K_\Delta\mathbf{O}(L))$. Let $\overline{\mathcal{E}}$ be as before. We have*

$$\begin{array}{lrcll} (i) & \mathrm{vol}_E(\mathrm{M}) & = & 4(-1)^m \lambda^{-1}(L_\mathbb{Z}; 0) & \\ (ii) & \widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}(\mathrm{M}) & \equiv & \frac{d}{ds} 4(-1)^m \lambda^{-1}(L_\mathbb{Z}; s)|_{s=0} & \text{in } \mathbb{R}_{2D} \end{array}$$

Proof of theorem (10.5.2, i). The theorem is true for Heegner points (i.e. all lattices $L_\mathbb{Z}$ with signature $(0,2)$) by the classical class number formula. A proof of this in our language is given in section (11.2). We are thus left with the case $m \geq 3$ and will give 2 different proofs in this case:
First proof:
Recall the definition of the function $\mu(L_\mathbb{Q}, M_\mathbb{Q}, \varphi; s)$ (in 7.10.1) related strongly to Fourier coefficients of an associated Eisenstein series. Let a lattice $L_\mathbb{Z} \subset L_\mathbb{Q}$ be given and $\kappa \in L^*_{\widehat{\mathbb{Z}}}/L_{\widehat{\mathbb{Z}}}$ and $M_\mathbb{Z} =< q >$. Let especially φ be the characteristic function of $\kappa \otimes M^*_\mathbb{Z}$. We denote the corresponding cycle by $\mathrm{Z}(L_\mathbb{Z}, < q >, \kappa; K)$.
Recall the orbit equation (7.10.4):

$$\mu(L_\mathbb{Z}, < q >, \kappa; s)\lambda(L_\mathbb{Z}; s)^{-1} = \sum_j \lambda((g_j^{-1}x)^\perp; s)^{-1}.$$

From formula (7.6.9), we get for the discriminant kernel K:

$$\mu(L_{\mathbb{Z}}, < q >, \kappa; 0) = \frac{\mathrm{vol}_E(\mathrm{Z}(L_{\mathbb{Z}}, < q >, \kappa; K))}{\mathrm{vol}_E(\mathrm{M}({}^K\mathbf{O}(L)))}. \tag{8}$$

Hence the assertion (10.5.2, i) is true for lattices of signature $(m-2,2)$ if and only if it is true for lattices of signature $(m-3,2)$, whence it is proven by induction on m. (Observe that $\sum_i 4(-1)^{m-1}\lambda((g_j^{-1}x)^{\perp}; 0)^{-1}$ is $\mathrm{vol}_E(\mathrm{Z}(L_{\mathbb{Z}}, < q >, \kappa; K))$ by the induction hypothesis).

Second proof: We use $\tau(\mathrm{SO}(L_{\mathbb{Q}})) = 2$, where τ is the Tamagawa number.

We have the following elementary relation (see also the proof of (7.6.8)):

$$\Lambda\tau = \mathrm{vol}(K)\,\mathrm{vol}_E(\mathrm{M}({}^K\mathbf{O}(L))),$$

where $\Lambda = 2\lambda_\infty^{-1}(L; 0)$ is the comparison factor from lemma (10.4.5) and $\mathrm{vol}(K)$ is computed with respect to the product of the canonical volumes. Recall that Λ involved the canonical volume on $\mathrm{SO}(L_{\mathbb{R}})$ and their product over all ν is a Tamagawa measure. We have $\mathrm{vol}(K) = \prod_p \lambda_p(L; 0)$ by definition. Everything put together yields $\mathrm{vol}_E(\mathrm{M}({}^K\mathbf{O}(L))) = 4(-1)^m\lambda^{-1}(L_{\mathbb{Z}}; 0)$ (which equals also $4(-1)^m\widetilde{\lambda}^{-1}(L_{\mathbb{Z}}; 0)$). □

(10.5.3) The proof of (10.5.2, ii) is the heart of this thesis and will occupy the rest of part III, see (10.5.7) for the starting point. In principle, we will use a similar induction proof, using a comparison of the *derivative* of the orbit equation with the formula for the *height* in Arakelov geometry. In a sense the 2 terms of the derivative of the product in the orbit equation correspond to the 2 terms defining (or characterizing) the $*$-product of Greens functions.

In more detail: If we take the **derivative at 0** of the orbit equation (7.10.4):

(10.5.4) Corollary. *Let $M_{\mathbb{Z}}$ be another lattice of dimension n with positive definite form Q_M, let K be the discriminant kernel of $L_{\mathbb{Z}}$, D be the discriminant of $L_{\mathbb{Z}}$ and D' be the discriminant of $M_{\mathbb{Z}}$ and $m-n \geq 2$.*

$$\frac{\mathrm{d}}{\mathrm{d}\,s}\left(4(-1)^m\lambda^{-1}(L_{\mathbb{Z}}; s)\mu(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s)\right)\Big|_{s=0} \equiv \mathrm{ht}_{\overline{\mathcal{E}}}(\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K))$$

in $\mathbb{R}_{2DD'}$.

Plugging in formulas (10.5.2) and (8), we get

$$\begin{aligned} &\mu'(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0)\,\mathrm{vol}_E(\mathrm{M}({}^K_\Delta\mathbf{O}(L_{\mathbb{Q}}))) \\ &+ \deg_E(\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K))\widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}(\mathrm{M}({}^K_\Delta\mathbf{O}(L))) \\ \equiv &\sum_i \widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}(\mathrm{M}({}^{K_i}_{\Delta_i}\mathbf{O}((g_i^{-1}x)^{\perp}))) \qquad (= \mathrm{ht}_{\overline{\mathcal{E}}}(\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K)) \end{aligned}$$

in $\mathbb{R}_{2DD'}$. *Here*

$$\deg_E(\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K)) = \frac{\mathrm{vol}_E(\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K))}{\mathrm{vol}_E(\mathrm{M}({}^K\mathbf{O}(L)))}$$

is the (relative) geometric degree.

The idea is, however, to prove a certain average of (10.5.4) directly (using Borcherds' products and the calculation of its integral [13, 54]) and then to deduce theorem (10.5.2, ii) by induction. This average version is (11.6.2) below. The main difficulties are

- The multiple of $\widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}(\mathrm{M}({}^K\mathbf{O}(L)))$ occurring in the average version of the formula should not be 0. This corresponds to the task of constructing Borcherds products of non-zero weights (11.3).
- All quantities $\widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}(\mathrm{M}({}^{K_i}_{\Delta_i}\mathbf{O}((g_i^{-1}x)^{\perp})))$ have to be known already. This is not so easy as in the geometric volume case because the method of using Borcherds products works only if the Witt rank of L is not zero (i.e. if $\mathrm{M}({}^K\mathbf{O}(L))$ has cusps). Therefore, we first calculate the arithmetic volume of the surrounding variety, avoiding cycles without boundary in the divisor of the constructed Borcherds product. Then we allow certain cycles without boundary (in a controlled way) and reverse the argument to calculate the arithmetic volume of those (11.3).
- Certain boundary terms in the integrals over star products of the occurring Greens functions (log of the Hermitian norm of sections) have to be shown to vanish (11.6.3).

Corollary (10.5.4) is very weak because it contains no information from bad primes whatsoever. We will also prove the following strengthening of (10.5.4). In our opinion this is the most general statement one can obtain by this method, i.e. by using Borcherds products without knowledge of the bad fibers of the varieties involved. It partially answers conjectures of Bruinier, Kudla, Kühn, Rapoport, Yang, and others.

(10.5.5) Main theorem. *Let* $M_{\mathbb{Z}}$ *be a lattice of dimension* n *with positive definite* Q_M. *Let* K *be the discriminant kernel of* $L_{\mathbb{Z}}$, D *be the discriminant of* $L_{\mathbb{Z}}$ *and* D'' *be the product of primes* p *such that* $M_{\mathbb{Z}_p}/M^*_{\mathbb{Z}_p}$ *is not cyclic. Assume* $m-n \geq 4$, *or* $m=4, n=1$ *and* $L_{\mathbb{Q}}$ *is has Witt rank 1. We have*

$$\frac{\mathrm{d}}{\mathrm{d}\,s}\, 4(-1)^m \left(\lambda^{-1}(L_{\mathbb{Z}}; s)\mu(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; s)\right)\Big|_{s=0} \equiv \mathrm{ht}_{\overline{\mathcal{E}}}(\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K))$$

in $\mathbb{R}_{2DD''}$.

Note that for $n=1$ always $D''=1$. This theorem will follow from its local version (10.5.8).

(10.5.6) Remark. The last theorem has been proven in full generality, also including information at $2D$ and ∞ in [68–70] for Shimura curves and in [97] for the modular curve ($n = 1$). For the equality at ∞, with μ_∞ replaced by the full ∞-factor of the Fourier coefficient of the Eisenstein series, the special cycles have to be complemented by the Kudla-Millson Greens functions (7.6.8) made dependent on the imaginary part as well, cf. also the introduction.

There are stronger results also for Hilbert modular surfaces [15, 65, 67] and for Siegel threefolds [67]. It was proven also for the product of modular curves (The Z's for $n = 1$ are the Hecke correspondences in this case) in [19, section 7.8].

The global formulation of (10.5.2, ii) immediately follows from the following local one:

(10.5.7) Main theorem (Local formulation). *Let $p \neq 2$. Let $L_{\mathbb{Z}_{(p)}}$ be a unimodular (at p) lattice and signature $(m-2, 2)$. Let K be any admissible compact open subgroup and Δ be complete and smooth and let $\mathrm{M} = \mathrm{M}({}^K_\Delta\mathbf{O}(L))$. Let $\overline{\mathcal{E}}$ be as before. We have*

$$\widehat{\mathrm{vol}}_{\overline{\mathcal{E}},p}(\mathrm{M}) \equiv \mathrm{vol}_E(\mathrm{M}_\mathbb{C}) \left.\frac{\frac{d}{ds}\lambda^{-1}(L_\mathbb{Z}; s)}{\lambda^{-1}(L_\mathbb{Z}; s)}\right|_{s=0}$$

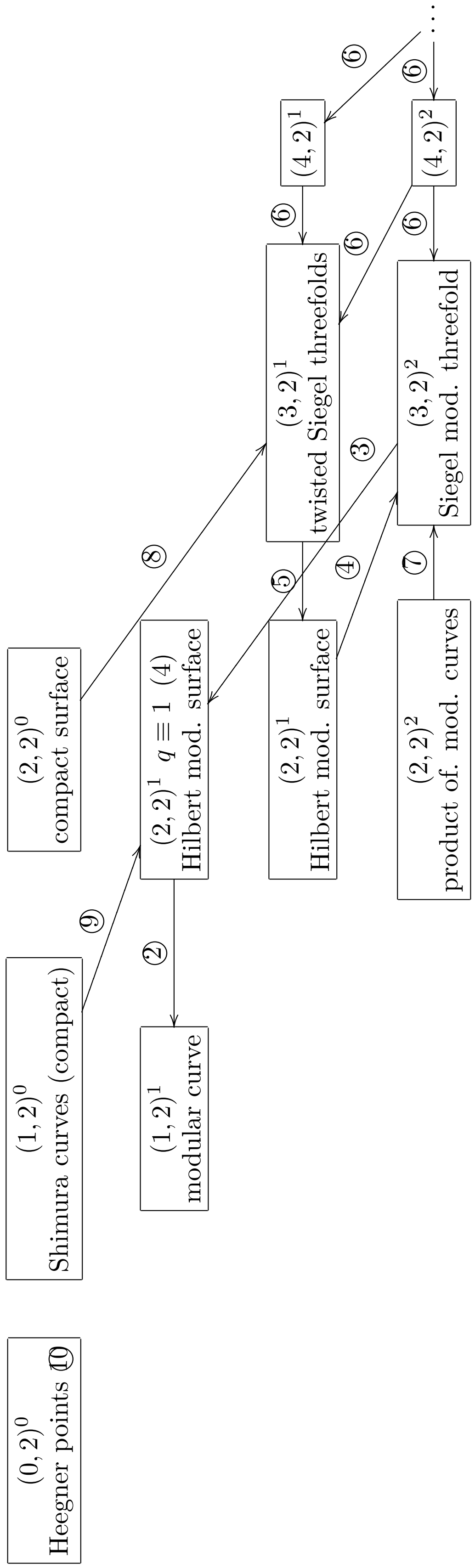
in $\mathbb{R}^{(p)}$ and any $\mathbb{Z}$-model $L_\mathbb{Z} \subset L_{\mathbb{Z}_{(p)}}$.

Proof. First of all the statement is independent of multiplication of Q_L by a scalar $\in \mathbb{Z}_{(p)}^*$ because the Shimura datum and $\mathcal{E}$ as a SO-equivariant bundle are not affected by this and the Hermitian metric changes by a factor in $\mathbb{Z}_{(p)}^*$ with does not change the arithmetic volume (considered in $\mathbb{R}^{(p)}$).

The strategy of induction, similar to the proof of (10.5.2, i), is to walk through the set of (unimodular at p) lattices in a special way, starting from lattices $L_{\mathbb{Z}_{(p)}}$ with known

$$\widehat{\mathrm{vol}}_{\overline{\mathcal{E}},p}(\mathrm{M}({}^K_\Delta\mathbf{O}(L_{\mathbb{Z}_{(p)}})))$$

(for admissible K) and then to construct special Borcherds products by the results in section (11.3) in a way such that all quantities in (11.6.2) - except the next unknown one - are already known. This path through the set of lattices will be (mostly) according to the dimension and Witt rank of the lattice. It is illustrated by the following scheme, wherein an arrow indicates logical dependence (i.e. the inverse walking direction):


$(0,2)^0$
Heegner points ⑩
$(1,2)^0$
Shimura curves (compact)
$(1,2)^1$
modular curve
$(2,2)^0$
compact surface
$(2,2)^1$ $q \equiv 1$ (4)
Hilbert mod. surface
$(2,2)^1$
Hilbert mod. surface
$(2,2)^2$
product of. mod. curves
$(3,2)^1$
twisted Siegel threefolds
$(3,2)^2$
Siegel mod. threefold
$(4,2)^1$
$(4,2)^2$
...
②
③
④
⑤
⑥
⑦
⑧
⑨

STEP 1: We start with the case of a lattice $L_{\mathbb{Z}_{(p)}}$ of signature $(1,2)$ and Witt rank 1. These are up to multiplication of Q_L by scalars in $\mathbb{Z}_{(p)}^*$ of the form $Q_L(z) = z_1z_2 - z_3^2$ which is equivalently the space $\{X \in M_2(\mathbb{Z}_{(p)}) \mid {}^tX = X\}$ with $Q_L(X) = \det(X)$, considered in (11.1.1). The volume $\widehat{\mathrm{vol}}_{\overline{\mathcal{E}},p}(\mathrm{M}({}^K_\Delta\mathbf{O}(L_{\mathbb{Z}_{(p)}})))$ has been calculated in Kühn's thesis, see (11.1.1) for the comparison.
STEP 2: **Hilbert modular surfaces of prime discriminant** $q \equiv 1 \ (4)$. This is the case considered in [15]. We reproduce their argument here as follows: (11.3.15, i) shows that a Borcherds lift of non-zero weight can be found, such that all occurring Shimura varieties in $\mathrm{Z}(L_\mathbb{Z}, < -q >, c_q; K)$ for the occurring c_q are of the form already treated in STEP 1. Hence comparison of the formula in (11.6.2) with the derivative of the orbit equation (7.10.4) shows the truth of theorem (10.5.7).
STEP 3: **Siegel modular threefold**. In (11.3.16) it is shown that a Borcherds lift of non-zero weight can be found, such that every Shimura variety occurring in $\mathrm{Z}(L_\mathbb{Z}, < -q >, c_q; K)$ for the non-zero c_q's is of the form considered in STEP 2 (even copies of the same). Then proceed as in STEP 2.
STEP 4: **general Hilbert modular surfaces**. In (11.3.16) it is shown that a Borcherds lift of non-zero weight can be found, such that every Shimura variety occurring in $\mathrm{Z}(L_\mathbb{Z}, < -q >, c_q; K)$ for the non-zero c_q's is a $\mathrm{M}({}^K_\Delta\mathbf{O}(L_{\mathbb{Z}_{(p)}}))$ for a given lattice $L_{\mathbb{Z}_{(p)}}$ of signature $(2,2)$, Witt rank 1. Now $\widehat{\mathrm{vol}}_{\overline{\mathcal{E}},p}(\mathrm{M}({}^K_\Delta\mathbf{O}(L)))$ is known by STEP 3, and $\widehat{\mathrm{vol}}_{\overline{\mathcal{E}},p}(\mathrm{Z}(L_\mathbb{Z}, < -q >, c_q; K))$ may be deduced from the comparison of the formula in (11.6.2) with the derivative of the orbit equation.
STEP 5: **twisted Siegel threefolds**. This is the case L is of signature $(2,3)$ and Witt rank 1. We have only to avoid $\mathrm{Z}(L_\mathbb{Z}, < -q >, c_q; K)$'s with occurring Shimura varieties for lattices with are not unimodular at p *or* compact ones. This is achieved by theorem (11.3.14, i).
STEP 6: **dimension 4 and higher**. We know all arithmetic volumes for orthogonal Shimura varieties of dimension 3. Hence we may proceed by induction on the dimension and have only to avoid $\mathrm{Z}(L_\mathbb{Z}, < -q >, c_q; K)$'s containing Shimura varieties for lattices with are not unimodular at p. This is achieved by (11.3.13).
We are left with a couple of cases that have been omitted in the above process. They can be treated analogously to STEP 4 above, by reversing the usual argument.
STEP 7: **product of modular curves**. This is the case of signature $(2,2)$, Witt rank 2. Use (11.3.16) again. This can also be treated directly, using STEP 1 (Kühn's thesis), see [19, section 7.8] for a proof.
STEP 8: **compact surfaces**. This is the case of signature $(2,2)$, Witt rank 0. Use (11.3.14, ii).
STEP 9: **Shimura curves**. This is the case of signature $(1,2)$, Witt rank 0. Use (11.3.15, ii)
STEP 10: **Heegner points**. These are quadratic positive definite lattices. They may up to multiplication of Q_L by scalars in $\mathbb{Z}_{(p)}^*$ be embedded in a space considered in STEP 1. (11.6.2) and also the construction of Borcherds products are not available in this case, but this is the easiest case and we prove the truth of (10.5.7) directly in (11.2) using the Δ modular form (which is, in a sense, a Borcherds product, too). $\square$

The global formulation of (10.5.5) follows immediately from the following local version:

(10.5.8) Main theorem (Local formulation). *Let $p \neq 2$. Let $L_{\mathbb{Z}_{(p)}}$ be a unimodular (at p) lattice and of signature $(m-2,2)$, Let $M_{\mathbb{Z}_{(p)}}$ be a positive definite lattice of dimension n with cyclic $M^*_{\mathbb{Z}_p}/M_{\mathbb{Z}_p}$. Assume $m-n \geq 4$, or $m=4, n=1$ and $L_{\mathbb{Q}}$ is isotropic, or $m=5, n=2$ and $L_{\mathbb{Q}}$ has Witt rank 2. Let $L_{\mathbb{Z}}$, $M_{\mathbb{Z}}$ any $\mathbb{Z}$-models of $L_{\mathbb{Z}_{(p)}}$ and $M_{\mathbb{Z}_{(p)}}$ respectively and let K be any admissible compact open subgroup in the discriminant kernel of $L_{\mathbb{Z}}$. We have:*

$$\begin{aligned}
&\mu'(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; 0)\,\mathrm{vol}_E(\mathrm{M}({}^K_\Delta\mathbf{O}(L))) \\
&\quad + \deg_E(\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K))\widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}(\mathrm{M}({}^K_\Delta\mathbf{O}(L))) \\
&\equiv \mathrm{ht}_{\overline{\mathcal{E}}}(\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K))
\end{aligned}$$

in $\mathbb{R}^{(p)}$. Here $\deg_E(\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K))$ is the (relative) geometric degree $\frac{\mathrm{vol}_E(\mathrm{Z}(L_{\mathbb{Z}}, M_{\mathbb{Z}}, \kappa; K))}{\mathrm{vol}(\mathrm{M}({}^K\mathbf{O}(L)))}$.

Proof. We first prove it for $n=1$: Here it follows immediately from (11.6.2), using a Borcherds product such that all Z's in the divisor consist of Shimura varieties corresponding to lattices, which are unimodular at p, *except* for $\mathrm{Z}(L_{\mathbb{Z}}, < q >, \kappa; K)$ which shall occur, too, with non-zero multiplicity. Theorem (11.3.13, ii) enables us to construct such a product. Note that the assumptions imply that $\mathrm{M}({}^K\mathbf{O}(L))$ has cusps.
Proof for $n \geq 2$: Because $M^*_{\mathbb{Z}_{(p)}}/M_{\mathbb{Z}_{(p)}}$ is cyclic, we may find lattices $M'_{\mathbb{Z}}$, unimodular at p, and $< q >_{\mathbb{Z}}$ such that

$$M_{\mathbb{Z}} =< q > \perp M'_{\mathbb{Z}}$$

is a model of $M_{\mathbb{Z}_p}$. Let $\kappa = \kappa_q \oplus \kappa'$ be a corresponding decomposition. Let $\mathrm{SO}'(L_{\widehat{\mathbb{Z}}})\alpha_i \subset \mathrm{I}(L_{\widehat{\mathbb{Z}}}, M'_{\widehat{\mathbb{Z}}}) \cap \kappa'$ be a decomposition into orbits. We may form the cycles $\mathrm{Z}(\alpha_i^\perp, < q >, \kappa_q)$ on $\mathrm{M}({}^{K_i}_{\Delta_i}\mathbf{O}(\alpha_i^\perp))$, where the K_i are the respective discriminant kernels (all admissible by construction of $M'_{\mathbb{Z}}$). The latter Shimura varieties are all equipped with morphisms into $\mathrm{M}({}^K_\Delta\mathbf{O}(L_{\mathbb{Z}_{(p)}}))$ (the union of their images is the cycle $\mathrm{Z}(L, M', \kappa')$). (These Shimura varieties all have the same Shimura datum, and one could see their images as conjugated images of a single Shimura variety with varying K as in the definition of special cycle (10.3)). In each of these Shimura varieties, we have cycles $\mathrm{Z}(\alpha_i^\perp, < q >, \kappa_q)$. Identifying them with their image in $\mathrm{M}({}^K_\Delta\mathbf{O}(L_{\mathbb{Z}_{(p)}}))$ we get

$$\mathrm{Z}(L, M, \kappa) = \bigcup_{\mathrm{SO}'(L_{\widehat{\mathbb{Z}}})\alpha \subset \mathrm{I}(L_{\widehat{\mathbb{Z}}}, M'_{\widehat{\mathbb{Z}}}) \cap \kappa'} \mathrm{Z}(\alpha_i^\perp, < q >, \kappa_q).$$

By the proof for $n=1$ we know the height of the right hand side. For this note that the assumptions imply that $\mathrm{M}({}^{K_i}\mathbf{O}(\alpha_i^\perp))$ has cusps[11]. Comparing this with global Kitaoka

[11] One could certainly weaken those assumption slightly, to allow more cases with $m-n=3$, and even with $m-n=2$, using the formula claimed in (11.2.12) for the modular curve, and [70] for Shimura curves.

(7.10.5), we get the general result and its truth does, of course, not depend on the models $L_{\mathbb{Z}}$, $M_{\mathbb{Z}}$ chosen and not on K (cf. also 9.4.6). $\square$

Since the local orbit equation for $\widetilde{\lambda}_p$ and $\widetilde{\mu}_p$ remains true, if $L_{\mathbb{Z}_p}$ is unimodular, $M_{\mathbb{Z}_p} =< q >$, where $p^2 \nmid M$, see also (7.10.4), we get as a direct corollary of (10.5.8):

(10.5.9) Corollary (LOCAL FORMULATION). *Let $p \neq 2$. Let $L_{\mathbb{Z}_{(p)}}$ be a lattice with quadratic form of discriminant p and signature $(m-2,2)$, $m \geq 3$. For any bigger unimodular lattice $L'_{\mathbb{Z}_{(p)}}$ and signature $(m'-2,2)$ such that $L_{\mathbb{Z}_{(p)}}$ is a saturated sublattice (and these exist), the following holds true:*
Let $\mathrm{M} = \mathrm{M}({}^K_\Delta\mathbf{O}(L))$ and $\mathrm{M}' = \mathrm{M}({}^{K'}_{\Delta'}\mathbf{O}(L'))$, where K' is an admissible compact open and such that we have an embedding (of rational ECMSD): ${}^K_\Delta\mathbf{O}(L_{\mathbb{Q}}) \to {}^{K'}_{\Delta'}\mathbf{O}(L'_{\mathbb{Q}})$. ($K$ will then be 'admissible' with respect to the discriminant kernel $K_p \subset \mathrm{SO}(L_{\mathbb{Q}_p})$ which does not come from a reductive group scheme anymore (i.e. is not hyperspecial) - compare 6.4.2).
Let $\overline{\mathcal{E}}$ be as before. We have

$$\mathrm{ht}_{\overline{\mathcal{E}},p}(\mathrm{M}) \equiv \mathrm{vol}_E(\mathrm{M}_{\mathbb{C}}) \left. \frac{\frac{d}{ds}\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};s)}{\widetilde{\lambda}^{-1}(L_{\mathbb{Z}};s)} \right|_{s=0}$$

in $\mathbb{R}^{(p)}$ and any $\mathbb{Z}$-model $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$, where the height is computed in M'.

(10.5.10) Question. *(10.5.5) in conjunction with determination of orbits in (6.4.18) allows to calculate the height at p for embeddings of Shimura varieties of lattices with cyclic $L^*_{\mathbb{Z}_p}/L_{\mathbb{Z}_p}$ and $m \geq 3$ into such associated lattices unimodular at p. It would be the first derivative of a weighted sum over some $4(-1)^m\lambda^{-1}(L'_{\mathbb{Z}};s)\widetilde{\mu}(L',<q>,\kappa;s)$'s for the unimodular L' and different q's. But we do not know how to characterize this value independently of $L'_{\mathbb{Z}}$ and even whether it is independent at all from the embedding. Note that the naive orbit equation is not true for $\widetilde{\lambda}$ anymore (cf. also 11.2.12).*
How is the situation for arbitrary $L_{\mathbb{Z}}$? Is the height independent of the embedding? Is there a canonical model, or at least a canonical *arithmetic volume? If yes, is it the first derivative of a natural function in s, depending only on $L_{\mathbb{Z}}$? How is the situation for different K, which do not occur as discriminant kernels of lattices?*

11. Calculation of arithmetic volumes

11.1. Kühn's thesis

(11.1.1) Consider the lattice

$$L_{\mathbb{Z}_{(p)}} = \{X \in M_2(\mathbb{Z}_{(p)}) \mid {}^tX = X\}$$

$Q_L = \det$. It has signature $(1, 2)$.
The group GL_2 acts by $g \circ X = gX^tg$ on $L_{\mathbb{Z}_{(p)}}$, preserving Q_L up to scalar. This defines a group isomorphism $\mathrm{GSpin}(L) = \mathrm{GL}_2$ and yields an isomorphism of Shimura data

$$\mathbf{S}(L) = \mathbf{H}_1.$$

Here, the underlying map $\mathbb{D}_{\mathbf{H}_1} \to \mathbb{D}_{\mathbf{S}(L)}$ is the restriction of the following map on compact duals $\mathrm{M}^\vee(\mathbf{H}_1) \to \mathrm{M}^\vee(\mathbf{S}(L))$:

$$< \begin{pmatrix} a \\ b \end{pmatrix} > \mapsto < \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} >, \tag{1}$$

where $< \begin{pmatrix} a \\ b \end{pmatrix} >$ is a Lagrangian subspace (in this case an arbitrary one dimensional subspace) in $\mathbb{Z}_{(p)}^2$ and $< \begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix} >$ is the corresponding isotropic subspace of $L_{\mathbb{Z}_{(p)}}$.
We have a boundary component $\mathbf{B}$, associated with the parabolic group $Q = \{\begin{pmatrix} * & * \\ & * \end{pmatrix}\}$, resp. its quotient modulo the center, fixing $< \begin{pmatrix} 1 \\ 0 \end{pmatrix} >$, resp. $< \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} >$. $P_{\mathbf{B}}$ is the subgroup scheme $\begin{pmatrix} 1 & * \\ & * \end{pmatrix}$ of Q, resp. its image in the quotient.
We have $\mathrm{M}^\vee(\mathbf{B}) = \mathbb{A}^1$, where $x \in \mathbb{A}^1$ is mapped to $< \begin{pmatrix} x \\ 1 \end{pmatrix} >$, resp. $< \begin{pmatrix} x^2 & x \\ x & 1 \end{pmatrix} >$ under the boundary map. In addition $\mathbb{D}(\mathbf{B}) = \mathbb{C} \times \mathbb{D}_{\mathbf{H}_0}$, where $\mathbb{D}_{\mathbf{H}_0} = \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}(1))$ canonically. The image of $\mathbb{D}_{\mathbf{H}_1}$ (or equivalently $\mathbb{D}_{\mathbf{S}(L)}$) is the union of $\mathbb{H}^+ \times (2\pi i)$, $\mathbb{H}^- \times (-2\pi i)$. This is the usual identification with the upper and lower half plane.
Recall that the bundle $\mathcal{E}$ was defined as the restriction to $\mathrm{M}^\vee(\mathbf{S}(L))$ of the tautological bundle on $\mathbb{P}(L)$ with the natural group action of SO, hence by the action $g \circ X = \frac{1}{\det(g)} gX^tg$. Similarly on $\mathrm{M}^\vee(\mathbf{H}_1)$ we have an $\mathcal{L} = \mathcal{O}(1)$ (anti-tautological bundle) coming

from identification with $\mathbb{P}^1$. We see from (1) that the bundles $\mathcal{E}$ and $\mathcal{L}^{\otimes -2}$ are isomorphic under the above identification, if we forget the group action.

We recall the *integral* trivializing section (5.7.5 - determined by the point $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ above $< \begin{pmatrix} 0 \\ 1 \end{pmatrix} >$) of $\Xi^* \mathcal{L}^{-1}$ on $\mathrm{M}({}^K\mathbf{B})$ which is (in this case) given by

$$(x \times \alpha, 1_f) \mapsto \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \alpha(1) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha(1) \begin{pmatrix} x \\ 1 \end{pmatrix},$$

resp. of $\Xi^* \mathcal{E}$:

$$(x \times \alpha, 1_f) \mapsto \alpha(1) \begin{pmatrix} x^2 & x \\ x & 1 \end{pmatrix}.$$

(11.1.2) To a classical modular form for $\mathrm{SL}_2(\mathbb{Z})$ of weight $2k$ with Fourier series

$$f = \sum_k a_k q^k,$$

we may therefore associate the section f' of $\Xi^* \mathcal{L}^{\otimes 2k}$, whose restriction to $\mathbb{H}^+ \times 1_f$ is given by

$$\tau \mapsto f(\tau)(2\pi i)^{-2k} \begin{pmatrix} \tau \\ 1 \end{pmatrix}^{\otimes -2k},$$

resp. the section f' of $\Xi^* \mathcal{E}^{\otimes k}$ given by

$$\tau \mapsto f(\tau)(2\pi i)^k \begin{pmatrix} 1 & \tau \\ \tau & \tau^2 \end{pmatrix}^{\otimes k}$$

of $\Xi^* \mathcal{E}$.

The latter has norm (10.4.1):

$$h_{\mathcal{E}}(f') : \tau \mapsto |f(\tau)|^2 (2\pi)^{2k} 2^k (2\pi)^{-k} e^{-k\gamma} \Im(\tau)^{2k}.$$

We have

$$\begin{aligned} \log(h_{\mathcal{E}}(f')) &= \log(\|f\|^2_{Pet}) - 2k \log(4\pi) + 2k \log(2\pi) + k \log(2) - k \log(\gamma) - k \log(2\pi) \\ &= \log(\|f\|^2_{Pet}) - k \log(2\pi) - k \log(\gamma) - k \log(2). \end{aligned}$$

(11.1.3) We have the well-known formula:

$$\mathrm{vol}_{\mathcal{L}}(\mathrm{Sh}({}^{K(1)}\mathbf{S}(L)) = -\frac{1}{2}\zeta(-1) = \frac{1}{24},$$

and therefore

$$\mathrm{vol}_{\mathcal{E}}(\mathrm{Sh}({}^{K(1)}\mathbf{S}(L)) = \zeta(-1) = -\frac{1}{12},$$

and

$$\mathrm{vol}_{\mathcal{E}}(\mathrm{Sh}({}^{K(1)}\mathbf{O}(L)) = 2\zeta(-1) = -\frac{1}{6}.$$

Recall that the volume is computed as an orbifold volume and $\mathcal{E} = \mathcal{L}^{\otimes -2}$ forgetting the group action.

Since multiplying the norm (square root of the metric) by a scalar ρ changes the arithmetic volume of an arithmetic variety X of *arithmetic* dimension $\dim(X)$ by

$$-\dim(X)\log(\rho)\,\mathrm{vol},$$

comparing to [71, Theorem 6.1], we see that

$$\widehat{\mathrm{vol}}_{\overline{\mathcal{E}}}(\mathrm{Sh}({}^{K(1)}\mathbf{O}(L)) = \mathrm{vol}_{\mathcal{E}}(\mathrm{Sh}({}^{K(1)}\mathbf{O}(L))(-2\frac{\zeta'(-1)}{\zeta(-1)} - 1 + \log(2) + \log(2\pi) + \gamma).$$

On the other hand, we have (8.3.4):

$$4(-1)^3\widetilde{\lambda}^{-1}(L,s) = 2\zeta(-1) + 2\zeta(-1)\left(-2\frac{\zeta'(-1)}{\zeta(-1)} - 1 + \frac{1}{2}\log(2) + \log(2\pi) + \gamma\right)s + O(s^2).$$

Hence formulas (10.5.2, i, ii) are true. (ii) in this case up to a $\frac{1}{2}\log(2)$, hence in $\mathbb{R}_2$.

11.2. Heegner points

(11.2.1) For $L'_{\mathbb{Z}}$ a quadratic negative definite lattice of discriminant D, we may find an embedding $L'_{\mathbb{Z}} \hookrightarrow L$ where $L_{\mathbb{Z}}$ is the lattice occurring in the last section, but maybe only with Q_L multiplied by a scalar. This does not affect the arithmetic volume $\widehat{\mathrm{vol}} \in \mathbb{R}_D$ (see the comment in the proof of 10.5.7).

We want to compute the height (cf. 9.4.2) of $Y = \mathrm{M}({}^{K'}\overline{\mathbf{O}(L')})$ corresponding to the embedding above. It is equal to (the closure of) $Y(\mathbb{C}) + Y(\mathbb{C})^\sigma$, where σ is complex conjugation.

There exitst a modular form Δ of weight 12 with integral Fourier coefficients, such that $a_1 = 1$ and such that the divisor of Δ' on $\mathrm{M}({}^{K(1)}_{\Delta}\mathbf{S}(L))$ (with the canonical Δ) does not intersect Y. This is because points in Y correspond to elliptic curves with complex multiplication which have good reduction everywhere.

Hence the height of Y with respect to $(\Xi^*\mathcal{E}, \Xi^*h)$ is given by (cf. 11.1.2 above):

$$-\frac{1}{6}\sum_{z\in Y(\mathbb{C})} \log(\sqrt{h_{\mathcal{E}}(\Delta')(z)}) = -\frac{1}{12}(\log(|\Delta|^2) - \frac{1}{2}\log(2\pi) + \frac{1}{2}\log(\gamma) - \frac{1}{2}\log(2)).$$

Comparison with (8.1) yields

$$\sum_{z\in Y(\mathbb{C})} 2\frac{Z(2s)}{Z(s)}E(z,s) = \mathrm{vol}_{\mathcal{E}}(\mathrm{Sh}({}^{K(1)}\mathbf{O}(L'))+ \quad \mathrm{ht}_{\overline{\mathcal{E}}}(\mathrm{Sh}({}^{K(1)}\mathbf{O}(L'))\cdot(s-1)+ \quad O((s-1)^2),$$

where ht is the global height (9.4.2), equal to the (canonical) global arithmetic volume $\widehat{\mathrm{vol}}$ in $\mathbb{R}_{2D}$.

(11.2.2) Theorem. *Let $L'_{\mathbb{Z}}$ be a primitive negative definite lattice of square-free discriminant D and Y as above. We claim:*

$$\sum_{z\in Y(\mathbb{C})} \frac{Z(2s+2)}{Z(s+1)}E(z;s+1) = 2\widetilde{\lambda}^{-1}(L'_{\mathbb{Z}};s).$$

If D is not square-free, we have at least

$$\sum_{z\in Y(\mathbb{C})} \frac{Z(2s+2)}{Z(s+1)}E(z;s+1)\bigg|_{s=0} = 2\widetilde{\lambda}^{-1}(L'_{\mathbb{Z}};0)$$

and

$$\sum_{z\in Y(\mathbb{C})} \frac{\mathrm{d}}{\mathrm{d}\,s}\frac{Z(2s+2)}{Z(s+1)}E(z;s+1)\bigg|_{s=0} \equiv \frac{\mathrm{d}}{\mathrm{d}\,s}\, 2\widetilde{\lambda}^{-1}(L'_{\mathbb{Z}};s)\bigg|_{s=0}$$

in all $\mathbb{R}^{(p)}$, where $p^2 \nmid D$.

The proof will occupy the rest of this section.

(11.2.3) Corollary. *Formulas (10.5.2, i, ii) are true for two dimensional primitive lattices, in the case of (ii) even as an identity in $\mathbb{R}_{2N}$, where N is the product of primes p with $p^2|D$, if instead of the arithmetic volume the height of the Zariski closure in* $\mathrm{M}({}^{K(1)}\mathbf{O}(L))$ *is considered.*

(11.2.4) Remark. (11.2.2) shows in particular that a better definition of a function in s giving the correct volumes for *all* two-dimensional lattices (with primitive form) would be $\sum_{z\in Y(\mathbb{C})}\frac{Z(2s+2)}{Z(s+1)}E(z;s+1)$. This value may be decomposed into a product over all primes as well, which satisfy the local orbit equation, too (for $\widetilde{\lambda}_p$ and $\widetilde{\mu}_p$ instead of λ_p and μ_p)! This fact will be discussed in (11.2.12) in the end of this section.

(11.2.5) Let us now take the task of proving (11.2.2). Recall the notation from (11.1.1). Via $\perp$, maximal negative definite sublattices $N\cap L_{\mathbb{Z}}$ correspond bijectively to vectors $S=\begin{pmatrix} a & b\\ b & c\end{pmatrix}$ of positive length $ac-b^2>0$ with $a<0, c<0$, $2|a, 2|c$ and $(\frac{a}{2}, b, \frac{c}{2})=1$.

They correspond as usual to certain complex vectors of length 0

$$Z_\tau = \begin{pmatrix} 1 & \tau \\ \tau & \tau^2 \end{pmatrix},$$

too, with $\tau \in \mathbb{H}$, by virtue of

$$a\tau^2 - 2b\tau + c = 0, \text{i. e. } \tau = \frac{b + i|S|}{a} \tag{2}$$

$$N = Z_\tau \oplus Z_{\overline{\tau}} = \mathbb{R}\begin{pmatrix} \frac{a}{2} & 0 \\ 0 & -\frac{c}{2} \end{pmatrix} + \mathbb{R}\begin{pmatrix} b & \frac{c}{2} \\ \frac{c}{2} & 0 \end{pmatrix} \tag{3}$$

(11.2.6) Remark. Be aware that the lattice $N_\tau \cap \mathbb{Z}^3$ is not equivalent to $\mathbb{Z}^2, S$. This is however true locally, if $|\det(S)|_p = 1$, but not necessarily, if $|\det(S)|_p \neq 1$. The discriminant of $N \cap L_\mathbb{Z}$ is equal to $\det(S)$ if S satisfies the conditions above (the generators of N given above span a lattice of index $\frac{c}{2}$ in a primitive one).

(11.2.7) Let K_∞ be the stabilizer of N_i, where i is a square root of -1 in $\mathbb{C}$. Its image in $\mathrm{PGL}(\mathbb{R})$ is a maximal compact subgroup. Explicitly: $K_\infty = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \middle| a, b \in \mathbb{R} \right\}$. We identify the symmetric space associated with this quadratic form with $\mathrm{GL}_2(\mathbb{R})/K_\infty$ via $g \mapsto {}^t g N_i g = N_{gi}$. Each $g \in \mathrm{GL}(L_\mathbb{R})$ decomposes as

$$g = z g_\tau k_\infty$$

with

$$\tau = x + iy = g \circ i, \quad g_\tau = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & \\ & \sqrt{y}^{-1} \end{pmatrix} = u(x) g_l(\sqrt{y}).$$

Suppose given an *oriented* negative definite subspace N_τ, defined over $\mathbb{Q}$. Its stabilizer in $\mathrm{GL}_2(\mathbb{Q})$ is given by

$$T = \{g \in \mathrm{GL}_2(\mathbb{Q}) \mid \det(g)^{-1t} g S g = S\},$$

hence $PT := T/\mathbb{G}_m$ is a quotient mod ± 1 of the orthogonal group associated with the bilinear form given by S, or equivalently it is the special orthogonal group SO of the quadratic form given by N_τ.
Consider $K = \prod_p \mathrm{PGL}_2(\mathbb{Z}_p)$. It is equally the discriminant kernel associated with the lattice $L_\mathbb{Z}$ (because the discriminant is 2). $K \cap PT(\mathbb{A}^{(\infty)})$ hence is the discriminant kernel of the quadratic from $< S >^\perp = N_\tau \cap L_\mathbb{Z}$ (6.4.2).
Consider the respective complex Shimura varieties associated with $\mathbf{O}(L)$ and $\mathbf{O}(< S >^\perp$):

$$X = [\mathrm{PGL}_2(\mathbb{Q}) \backslash (\mathrm{PGL}_2(\mathbb{A})/K_\infty K)] \cong [\mathrm{PGL}_2(\mathbb{Z}) \backslash \mathbb{H}^\pm]$$

and

$$Y_S = \left[PT(\mathbb{Q})\backslash(PT(\mathbb{A})/PT(\mathbb{R})(K \cap PT(\mathbb{A}^{(\infty)})))\right],$$

which we consider as analytic stacks (orbifolds) with a natural embedding:

$$\iota : Y_S \hookrightarrow X \tag{4}$$

$$[h_f, h_\infty] \mapsto [h_f, h_\infty g_\tau] \tag{5}$$

We will now consider the standard Eisenstein series (of weight 0) associated with $\Psi(s) = \prod_\nu \Psi_\nu(s)$, with $\Psi_\nu(s, \begin{pmatrix} \alpha & \\ & \delta \end{pmatrix} u(\beta)k) = |\alpha\delta^{-1}|^s$ and compute its 'trace' over the set Y_S. Observe that we have a different normalization than in (7.4). Assume throughout that the real part of s is big enough. All infinite series, resp. integrals considered in this section will then converge absolutely.

First we have

$$\sum_{h \in Y_S} \frac{1}{\#\operatorname{Aut}(h)} E(\Psi(s), h) = \sum_{h \in Y_S} \frac{1}{\#\operatorname{Aut}(h)} \sum_{g \in P(\mathbb{Q})\backslash \operatorname{GL}_2(\mathbb{Q})} \Psi(s; ghg_\tau) \tag{6}$$

$$= \sum_{h \in PT(\mathbb{A}^{(\infty)})/K \cap PT(\mathbb{A}^{(\infty)})} \Psi(s; hg_\tau) =: f(s), \tag{7}$$

where we used $\operatorname{GL}_2(\mathbb{Q}) = P(\mathbb{Q})T(\mathbb{Q})$ in the last step. *We will denote this quantity by $f(s)$ in this section.*

Considering the right invariance of $\Psi(s)$ under K, this may be rewritten as:

$$f(s) = \Psi_\infty(s; g_\tau) \prod_p \frac{\int_{\mathbb{G}_m(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)} \Psi_p(s; x)\mu}{\operatorname{vol}_\mu(K \cap T(\mathbb{A}^{(\infty)}))}, \tag{8}$$

where μ is any translation invariant measure on $\mathbb{G}_m(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)$.

(11.2.8) Lemma. *We have* $\Psi_\infty(s; g_\tau) = \left|\frac{\sqrt{\det(S)}}{a}\right|_\infty^s$ *and* $\Psi_p(s; h) = \left|\frac{\det(h)}{\gcd(h_{12}, h_{22})^2}\right|_p^s$.

Proof. The first assertion follows from the evaluation of the relation of g_τ and S, given above. For the second assertion consider the case $\nu_p(h_{12}) > \nu_p(h_{22})$,
hence $\gcd(h_{12}, h_{22}) = h_{22}$. We get:

$$\begin{pmatrix} 1 & -h_{21}det(h)^{-1}h_{22} \\ & 1 \end{pmatrix} \begin{pmatrix} \det(h)^{-1}h_{22} & \\ & h_{22}^{-1} \end{pmatrix} \begin{pmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{pmatrix} = \begin{pmatrix} 1 & \\ \frac{h_{12}}{h_{22}} & 1 \end{pmatrix}$$

where the matrix on the right hand side is integral. Analogously for the case $\nu_p(h_{12}) < \nu_p(h_{22})$. $\square$

(11.2.9) The lemma shows that it is convenient to parameterize the torus T as follows (recall $a \neq 0$):

$$\iota : \mathbb{Q}_p^2 - \{0\} \to T(\mathbb{Q}_p)$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + 2\frac{b}{a}y & \frac{c}{a}y \\ -y & x \end{pmatrix}$$

We have $\det(\iota(v)) = \frac{Q_L(v)}{\frac{a}{2}}$, where $Q_L(v) = \frac{a}{2}x^2 + bxz + \frac{c}{2}y^2$. A translation invariant measure on $T(\mathbb{A}^{(\infty)})$ is given by $\det(\iota(v))^{-1}\,\mathrm{d}\,v$, where $\mathrm{d}\,v$ is the standard measure on ${\mathbb{Q}_p}^2$. We choose the measure μ, quotient of this by a measure giving ${\mathbb{Z}_p}^*$ volume 1.
This yields:

$$\int_{\mathbb{Z}_p^2} |\frac{Q_L(v)}{\frac{a}{2}}|_p^{s-1}\,\mathrm{d}\,v = \sum_{i=0..\infty} p^{-2is} \int_{\mathbb{G}_m(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)} \Psi_p(s;x)\mu$$

and therefore:

$$f(s) = \frac{|\det(S)|_\infty^{\frac{1}{2}s}}{2^s\zeta(2s)} \prod_p \frac{\int_{\mathbb{Z}_p^2} |Q_L(v)|_p^{s-1}\,\mathrm{d}\,v}{|\frac{a}{2}|_p^{-1}\,\mathrm{vol}\{x,y \mid \frac{c}{a}y, x, x+2\frac{b}{a}y, y \in \mathbb{Z}_p, x^2 + 2\frac{b}{a}xy + \frac{c}{a}y^2 \in {\mathbb{Z}_p}^*\}}.$$

Substituting $\frac{a}{2}y$ for y and then $x - \frac{b}{2}y$ for x in the volume computation in the denominator, we get:

$$|\frac{a}{2}|_p^{-1}\,\mathrm{vol}\{\cdots\} = \mathrm{vol}\{x,y \in {\mathbb{Z}_p}^2 \mid x^2 + \frac{ac-b^2}{4}y^2 \in {\mathbb{Z}_p}^*\}.$$

if b is even or $p \neq 2$, otherwise substitute in addition $x + \frac{1}{2}$ for x and get

$$|\frac{a}{2}|_p^{-1}\,\mathrm{vol}\{\cdots\} = \mathrm{vol}\{x,y \in {\mathbb{Z}_p}^2 \mid x^2 + xy + \frac{ac-b^2+1}{4}y^2 \in {\mathbb{Z}_p}^*\}.$$

Hence in any case:

$$f(s) = 2^{-s}\frac{|\det(S)|_\infty^{\frac{1}{2}s}}{\zeta(2s)} \prod_p \zeta_p(L_{\mathbb{Z}_p}, s),$$

where $\zeta_p(L_{\mathbb{Z}_p}, s)$ is the normalized local zeta function of the lattice, by definition (see 6.5). Note that it depends only on the discriminant for 2 dimensional lattices.
Write

$$f(s) = 2^{-s}\frac{|\det(S)|_\infty^{\frac{1}{2}s}}{\zeta(2s)}\zeta_K(s) \prod_{p^2|D} R_p(p^{-s})$$
$$R(p^{-s}) = \frac{\zeta_p(L_{\mathbb{Z}_p}, s)}{\zeta_{K,p}(s)}$$

where $K = \mathbb{Q}(\sqrt{-D_0})$ and $D = D_0 f^2$, D_0 is fundamental.

We have $R_p(1) = \left|\frac{D_0}{D}\right|_p^{\frac{1}{2}} = |f|_p^{-1}$, or $R_p(1) = \left|\frac{D_0}{D}\right|_p^{\frac{1}{2}} (1 - (\frac{-D_0}{p})p^{-1})$, in case $\nu_p(D) \geq 2$ is even.

Now calculate

$$\frac{Z(2s+2)}{Z(s+1)} f(s+1) = 2^{-s} \frac{\Gamma(s+1)\pi^{\frac{s}{2}+\frac{1}{2}}}{\pi^{s+1}\Gamma(\frac{s}{2}+\frac{1}{2})} |\det(S)|^{\frac{1}{2}s+1} L(\chi, s+1).$$

Applying the doubling formula for the Γ-function, we get:

$$\frac{Z(2s+2)}{Z(s+1)} f(s+1) = \frac{\Gamma(\frac{s}{2}+1)}{\pi^{\frac{s}{2}+1}} |\det(S)|^{\frac{1}{2}(s+1)} L(\chi, s+1).$$

On the other hand, we have by (8.2.1) and the definition of $\widetilde{\lambda}$ (7.10.4) for square-free discriminant:

$$\widetilde{\lambda}^{-1}(L'; s) = \frac{1}{2} \frac{\Gamma(\frac{s}{2}+1)}{\pi^{\frac{s}{2}+1}} |D|^{\frac{1}{2}(s+1)} L(\chi, s+1).$$

Hence the first part of (11.2.2) is proven. Note that multiplying the discriminant by some p^j $j \geq 2$ multiplies both quantities by a rational function in $p^{-\frac{s}{2}}$ with the same value at $s = 0$. Hence the second statement of (11.2.2) is true as well.

(11.2.10) More explicitly, we get for bad discriminants the following correction factors: If $\nu_p(D)$ is odd, this yields (6.5):

$$\frac{R_p'(1)}{R_p(1)} = l - 1 + \frac{1 - p^{\frac{l-1}{2}}}{p^{\frac{l+1}{2}} - p^{\frac{l-1}{2}}}$$

and if $\nu_p(D) \geq 2$ is even (6.5):

$$\frac{R_p'(1)}{R_p(1)} = l - \frac{1}{p-1} + \frac{((\frac{-D_0}{p}) - 1)}{((\frac{-D_0}{p}) - p)(p^{\frac{l}{2}} - p^{\frac{l}{2}-1})} + \frac{(\frac{-D_0}{p})}{(p - (\frac{-D_0}{p}))}$$

(cf. 11.2.12).

(11.2.11) Remark. The functional equation of the Eisenstein series forces a functional equation for the R_p's, namely

$$R_p(p^{-s}) = R_p(p^{s-1}).$$

(11.2.12) Remark. Let $L_{\mathbb{Z}}$ be the lattice of (11.1.1) again and q a positive integer. Recall that the quantities $\mu(L, < q >; s)$, $\lambda(L; s)$, $\lambda(\alpha^{\perp}; s)$ are products over all ν of μ_ν's and λ_ν's, respectively. The λ_ν's satisfy the local orbit equation:

$$\lambda_p^{-1}(L_{\mathbb{Z}_p}; s)\mu_p(L_{\mathbb{Z}_p}, < q >; s) = \sum_{\mathrm{SO}'(L_{\mathbb{Z}_p})\alpha \subseteq \mathrm{I}(<q>,L)(\mathbb{Z}_p)} \lambda_p^{-1}(\alpha^\perp; s).$$

If we substitute $\widetilde{\lambda}$ and $\widetilde{\mu}$ (correcting factors, e.g. $|D|_\infty^{\frac{1}{2}s}$, distributed to the respective primes) for the respective λ and μ's, this equation is destroyed, unless there is only one orbit (q square-free at p). If we let $(\lambda')_p^{-1}(\alpha^\perp; s)$ be the quantity $|D|_p^{-\frac{1}{2}(s+1)} \frac{\zeta_p(\alpha^\perp; s+1)}{\zeta_p(s+1)}$ (local factor of $\frac{Z(2s+2)}{Z(s+1)} E(Y, s+1)$, for Y corresponding to $\alpha^\perp$ as above), which agrees with $\widetilde{\lambda}_p^{-1}(\alpha^\perp; s)$ if $p^2 \nmid D$, then the following equation *is* true for all q:

$$\widetilde{\lambda}_p^{-1}(L_{\mathbb{Z}_p}; s)\widetilde{\mu}_p(L_{\mathbb{Z}_p}, < q >; s) = \sum_{\mathrm{SO}'(L_{\mathbb{Z}_p})\alpha \subseteq \mathrm{I}(<q>,L)(\mathbb{Z}_p)} (\lambda')_p^{-1}(\alpha^\perp; s).$$

This is particularly striking from our point of view because classical resp. Arakelov geometry only tell us that the first two coefficients in the expansion at $s = 0$ in its product over all ν should be equal. From the point of view of the Shimura correspondence, this is maybe not so amazing because the two Eisenstein series determining $\widetilde{\mu}$, resp. λ' are (roughly) Shimura lifts of each other. I am indebted to J. Bruinier and T. Yang for this last comment.

11.3. Preparation of Borcherds forms

(11.3.1) We will work first in the classical setting, with modular forms on $\mathbb{H}$ for any subgroup $\Gamma \subset \mathrm{Sp}_2'(\mathbb{R})$ commensurable with $\mathrm{Sp}_2'(\mathbb{Z})$ with values in any representation V of $\mathrm{Sp}_2'(\mathbb{Z})$. Here Sp_2' is either $\mathrm{Sp}_2 = \mathrm{SL}_2$ or Mp_2. Let k be a weight, half integral, if $\mathrm{Sp}' = \mathrm{Mp}$ and an integer, if $\mathrm{Sp}' = \mathrm{Sp}$.
Let

$$\mathrm{ModForm}(\Gamma, V, k)$$

be the space of modular forms of weight k for Γ of representation V, meromorphic at the cusps. Let

$$\mathrm{HolModForm}(\Gamma, V, k)$$

be the space of modular forms of weight k for Γ of representation ρ, which are holomorphic at the cusps.

(11.3.2) We consider now a more abstract setting. Consider the modular curve

$$\mathrm{M}({}^{K(N)}\mathbf{H}(\mathfrak{M})),$$

$\mathfrak{M} = M \oplus M^*$ with one dimensional M, and standard symplectic form, where $K(N)$ is formed with respect to a latice $M_{\mathbb{Z}} \oplus M_Z^*$. We are interested merely in the rational canonical model in this section. There is a boundary component $\mathbf{B}$ associated with the line $< M >$ with group $P_{\mathbf{B}} = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ and $W_{\mathbf{B}} = U_{\mathbf{B}} = \mathbb{W}(M \otimes M) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. Let $U := (M \otimes M)_{\mathbb{Z}}$. The boundary component is of the type considered in (5.7). There we saw that there is a morphism $\mathbf{B} \to \mathbf{H}_0$ and for each complementary line, e.g. $< M^* >$, there is a splitting of this morphism. The morphism determines a diagram

$$\begin{array}{ccc} \mathrm{M}({}^{K_{NU} \rtimes K(N,\mathbb{G}_m)}\mathbf{B})_{\mathbb{Q}} & \xrightarrow{\sim} & \mathrm{spec}(\mathbb{Q}[\zeta_N][\frac{1}{N}U^*]) \\ \downarrow & & \downarrow \\ \mathrm{M}({}^{K(N,\mathbb{G}_m)}\mathbf{H}_0)_{\mathbb{Q}} & \xrightarrow{\sim} & \mathrm{spec}(\mathbb{Q}[\zeta_N]) \end{array}$$

where the obvious splitting on the right hand side corresponds to the splitting determined by $< M^* >$ on the left. There is a trivializing section s, defined over $\mathbb{Q}$, of the bundle of (nearly holomorphic) modular forms of weight k as described in (5.7).

The expansion of a classical modular form, pulled back via the boundary map and trivialized by means of this trivializing section looks as follows. For each $\zeta_{N,\mathbb{C}} \in \mathbb{C}$, primitive N-th root of unity, we have a series

$$\sum_{\lambda \in \frac{1}{N}U^*} a_{\lambda,\zeta_{N,\mathbb{C}}}[\lambda]$$

(function, defined over $\mathbb{C}$, of a suitable completion of the top-right element of the diagram). In (5.7.2) and (5.7.3) we saw that, up to a power of $(2\pi i)$, the $a_{i,\zeta_{N,\mathbb{C}}}$ are the classical Fourier coefficients of f pulled back via the map

$$\begin{aligned} \Gamma\backslash\mathbb{D}^{\alpha}_{\mathbf{H}} &\to \mathrm{GL}(\mathfrak{M}_{\mathbb{Q}})\backslash\mathbb{D}_{\mathbf{H}} \times \mathrm{GL}(\mathfrak{M}_{\mathbb{A}^{(\infty)}})/K(N) \\ \tau &\mapsto [\tau, k] \end{aligned}$$

where $(\alpha, k) \in \mathbb{D}_{\mathbf{H}_0} \times K(1)$ describes $\zeta_{N,\mathbb{C}} = \exp(\alpha(\frac{k}{n}))$.

Hence f^τ for $\tau \in \mathrm{Gal}(\mathbb{C}|\mathbb{Q})$ has the series

$$\sum_{\lambda \in \frac{1}{N}U^*} a^{\tau}_{\lambda,\zeta^{\tau^{-1}}_{N,\mathbb{C}}}[\lambda],$$

in the fibre over $\zeta_{N,\mathbb{C}}$.

Modular forms of level $K(N)$ are the same as modular forms for $K(1)$ with values the induced representation

$$\mathrm{ind}^{K(1)}_{K(N)}(\mathbb{C}).$$

Such a form f has an expansion

$$\sum_{\lambda \in \frac{1}{N}U^*} a_{\lambda,\zeta_{N,\mathbb{C}}}[\lambda],$$

too, with $a_{i,\zeta_{N,\mathbb{C}}} \in \mathrm{ind}_{K(N)}^{K(1)}(\mathbb{C})$. This is now, however, completely determined by its coefficients $a_i := a_{i,\exp(2\pi i/n)}$.
The coefficients of f^τ are hence given by

$$a_i' = \begin{pmatrix} k_\tau^{-1} & 0 \\ 0 & 1 \end{pmatrix} (a_i^\tau) \tag{9}$$

where $k_\tau \in K(1, \mathbb{G}_m)/K(N, \mathbb{G}_m)$ is such that $\exp(2\pi i/n)^\tau = \exp(2\pi i k_\tau/n)$.
We are interested in the subspace $\mathrm{ind}_{K_0(N)}^{K(1)}(\chi, 1) \subseteq \mathrm{ind}_{K(N)}^{K(1)}(\mathbb{C})$ transforming under $K_0(N)$ by the character $(\chi, 1)$, where χ is some character of $\mathbb{A}^{(\infty)*}$ with values in ± 1. The Galois operation on modular forms restricts to those with values in this subspace, as it should be.

(11.3.3) Let Z be the kernel of $\mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z})$ and define $\Gamma_0(N)'$ as $\Gamma_0(N)$ in the integral case and as its pre-image in $\mathrm{Mp}_2(\mathbb{Z})$ in the half-integral case. In the half-integral case, we always assume $4|N$ and that we are given a character

$$\chi : \Gamma_0(N)' \to \mu_4,$$

such that Z acts nontrivially.
We have the space

$$\mathrm{ModForm}(\mathrm{Sp}_2'(\mathbb{Z}), k, \mathrm{ind}_{\Gamma_0(N)'}^{\mathrm{Mp}_2(\mathbb{Z})}(\chi)) = \mathrm{ModForm}(\Gamma_0(N)', k, \chi).$$

Consider now the integral weight case again. There are isomorphisms

$$\begin{array}{c} \Gamma_0(N)\backslash\mathbb{H} \\ \Big\downarrow \wr \\ \mathrm{SL}_2(\mathbb{Q})\backslash\mathbb{H} \times \mathrm{SL}(\mathbb{A}^{(\infty)})/K_0(N, \mathrm{SL}) \\ \Big\downarrow \wr \\ \mathrm{GL}_2(\mathbb{Q})\backslash\mathbb{H}^\pm \times \mathrm{GL}(\mathbb{A}^{(\infty)})/K_0(N, \mathrm{GL}). \end{array}$$

and a modular form on the bottom with character $(\chi, 1)$ pulls back to a modular form of character χ for $\Gamma_0(N)$. We have an isomorphism

$$\mathrm{ind}_{\Gamma_0(N)}^{\mathrm{Sp}_2(\mathbb{Z})}(\chi) = \mathrm{ind}_{K_0(N)}^{K(1)}(\chi) \subset \mathrm{ind}_{K(N)}^{K(1)}(\mathbb{C})$$

and can read off the Galois action on Fourier coefficients.

(11.3.4) Let N be an integer such that the Weil representation $\mathrm{Weil}(L_\mathbb{Z}^*/L_\mathbb{Z})$ factors through $\mathrm{Sp}'(\mathbb{Z}/N\mathbb{Z})$, and consider the line bundle of modular forms $\Xi^*\mathcal{L}^{\otimes k}$ of integral weight k over $\mathrm{M}({}_{\Delta}^{K(N)}\mathbf{H}(\mathfrak{M}))_{\mathbb{Q}}$. We have a morphism

$$M({}_{\Delta}^{K(N)}\mathbf{H}(\mathfrak{M}))_{\mathbb{Q}} \to M({}^{K(N,\mathbb{G}_m)}\mathbf{H}_0)_{\mathbb{Q}}, \tag{10}$$

induced by the determinant $\mathbf{H}(\mathfrak{M})) \to \mathbf{H}_0$, and the fibers of this map are geometrically connected (Δ is the canonical complete one). The automorphism group of the bundle $(\Xi^*\mathcal{L}^{\otimes k})^M$ hence is $\mathrm{GL}_M(\mathcal{O}_{M({}^{K(N,\mathbb{G}_m)}\mathbf{H}_0)})$ We have $\mathcal{O}_{M({}^{K(N,\mathbb{G}_m)}\mathbf{H}_0)} \cong \mathbb{Q}(\zeta_N)$ (for any $x \in \mathrm{GL}_N(\mathbb{Q}(\zeta_N))$ and we denote the corresponding automorphism by $a(x)$), such that the inclusion $\mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$, $\zeta_N \mapsto \zeta_{N,\mathbb{C}}$ is identified with (the class of) $(\alpha, k) \in \mathbb{D}_{\mathbf{H}_0} \times K(1)$, where α, k are such that $\zeta_{N,\mathbb{C}} = \exp(\alpha(\frac{1}{N}k))$. The Weil representation defines a morphism

$$w : \mathrm{SL}(\mathbb{Z}/N\mathbb{Z}) \mapsto \mathrm{GL}(\mathbb{Q}(\zeta_N)[L^*/L]),$$

using the identification $\zeta_N \mapsto \exp(2\pi i \frac{1}{N})$, where i is the root of -1 used in the definition (via the underlying characters on the various $\mathbb{Q}_\nu$).

On $\Xi^*\mathcal{L}^{\otimes k} \to M({}_{\Delta}^{K(N)}\mathbf{H}(\mathfrak{M}))_{\mathbb{Q}}$ we have the (right) action of the Hecke operators in $\mathrm{GL}(\mathbb{Z}/N\mathbb{Z})$, defined over $\mathbb{Q}$. We denote the corresponding action by multiplication on the right.

From the morphism (10) and the explicit description of its target, we may infer the formula

$$(a(x^{\tau_k})Z)k = a(x)(Zk) \tag{11}$$

for $Z \in (\Xi^*\mathcal{L}^{\otimes k})^M$, where τ_k is the image of k under

$$\mathrm{GL}(\mathbb{Z}/N\mathbb{Z}) \xrightarrow{\det} (\mathbb{Z}/N\mathbb{Z})^* \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}(\zeta_N)|\mathbb{Q}).$$

From the explicit formulæ for the Weil representation (7.2.1) follows[1]:

(11.3.5) Lemma.

$$w(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} k \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}) = w(k)^{\tau_\alpha}. \tag{12}$$

Proof. The operators $x^* \mapsto \varphi({}^t\alpha x^*)$ are Galois invariant. If we act on the operator $x^* \mapsto \exp(2\pi i Q_L(x^*)\beta)\varphi(x^*)$ by τ_α, we get the same as by substituting β by $\alpha\beta$. If we act on the operator $x^* \mapsto \int_{L_{\mathbb{A}^{(\infty)}}} \varphi(\gamma x) \exp(-2\pi i x^* x)\, \mathrm{d}\, x$ by τ_α, we get the same as by substituting γ by $\alpha^{-1}\gamma$, provided $\mathrm{d}\, x$ is chosen Galois invariant (e.g. such that $L_{\widehat{\mathbb{Z}}}$ has a volume in $\mathbb{Q}$). Now an actual operator $w(g_l(\alpha))$ differ by those considered by a $\frac{\Upsilon_f(\alpha Q_L)}{\Upsilon_f(Q_L)}$ in the first case, which is a sign because m is even, hence Galois invariant. In the third case $w(d(\gamma))$ differs by $c(\gamma) := \frac{\Upsilon_f(\gamma Q_L)}{|\gamma|^{\frac{1}{2}}}$, where $|\gamma|$ is calculated with respect to

[1]In other words, the Weil representation extends semi-linearly to GL_2 if we let $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ act by τ_α

the chosen measure and its dual. Now consider the equation $\left(u(-\gamma^{-1})d(\gamma)\right)^3 = 1$. Since $c(\gamma)^2 \in \mathbb{Q}$, it shows $c(\alpha^{-1}\gamma) = c(\gamma)^{\tau_\alpha}$. □

Therefore we may define a twisted right action of $\mathrm{GL}(\mathbb{Z}/N\mathbb{Z})$ on $\Xi^*\mathcal{L}^{\otimes k} \otimes_{\mathbb{Q}} \mathbb{Q}[L_{\mathbb{Z}}^*/L_{\mathbb{Z}}]$ by $Z \cdot k := a(w(k^{-1}))(Zk)$ for $k \in \mathrm{SL}(\mathbb{Z}/N\mathbb{Z})$ and by $Z \cdot \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} := Z \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$. Formula (11) and (12) show that this is a well defined action of $\mathrm{GL}(\mathbb{Z}/N\mathbb{Z})$.
Let us denote the quotient bundle by this action by $\mathcal{WEIL}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})$. It is a bundle over the stack $\mathrm{M}({}^{K(1)}_{\Delta}\mathbf{H}(\mathfrak{M}))_{\mathbb{Q}}$. A section $f \in H^0(\mathrm{M}_{\mathbb{C}}, \mathcal{WEIL}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}))$ is a modular form for the Weil representation in the classical sense. It has a 'q-expansion'

$$\sum_{\lambda \in \frac{1}{N} M_{\mathbb{Z}}^*} a_\lambda [\lambda],$$

where $a_\lambda \in \mathbb{C}[L_{\mathbb{Z}}^*/L_{\mathbb{Z}}]_\lambda$ and they coincide with the usual Fourier coefficients (up to a power of $\pm 2\pi i$, which we may and will ignore). In this case the discussion shows that f^τ for $\tau \in \mathrm{Gal}(\mathbb{C}|\mathbb{Q})$ has the coefficients a_λ^τ (usual Galois action). Hence f is defined over $\mathbb{Q}$, iff $a_\lambda \in \mathbb{Q}[L_{\mathbb{Z}}^*/L_{\mathbb{Z}}]$ for all λ, and, more importantly for us:

(11.3.6) Lemma. *Let k be a (half-)integral weight and Sp_2' either Sp_2 or Mp_2 according to whether $k \in \mathbb{Z}$ or not. For $f \in \mathrm{ModForm}(\mathrm{Sp}_2', \mathrm{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}), k)$ with Fourier expansion*

$$f = \sum_i a_i q^i,$$

$a_i \in \mathrm{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})_i$, we have for $\tau \in \mathrm{Gal}(\mathbb{C}|\mathbb{Q})$ and

$$f^\tau := \sum_i a_i^\tau q^i,$$

that $f^\tau \in \mathrm{ModForm}(\mathrm{Sp}_2', \mathrm{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}), k)$, too.

Proof. We have proven the statement for $k \in \mathbb{Z}$ above. For half integral weight, consider the space $L_{\mathbb{Z}}' = L_{\mathbb{Z}} \oplus < 1 >$. We have

$$\mathrm{Weil}((L_{\mathbb{Z}}')^*/L_{\mathbb{Z}}') = \mathrm{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}) \otimes \mathrm{Weil}(< 1 >^* / < 1 >).$$

We have the classical theta function

$$\theta \in \mathrm{HolModForm}(\mathrm{Mp}_2, \mathrm{Weil}(< 1 >^* / < 1 >), \frac{1}{2})$$

(with integral Fourier coefficients!). Multiplication with it yields a map

$$\mathrm{ModForm}(\mathrm{Mp}_2, \mathrm{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}), k) \to \mathrm{ModForm}(\mathrm{Mp}_2, \mathrm{Weil}((L_{\mathbb{Z}}')^*/L_{\mathbb{Z}}'), k + \frac{1}{2}),$$

which commutes with the Galois action on Fourier coefficients. By an elementary calculation, one sees that any holomorphic function on $\mathbb{H}$ (or even formal Fourier series) with values in $\mathrm{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})(\mathbb{C})$ which lies in $\mathrm{ModForm}(\mathrm{Mp}_2, \mathrm{Weil}((L'_{\mathbb{Z}})^*/L'_{\mathbb{Z}}), k+\frac{1}{2})$ after multiplication with θ has to lie in $\mathrm{ModForm}(\mathrm{Mp}_2, \mathrm{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}), k)$. Hence the statement of the lemma. ☐

(11.3.7) Recall the following construction from [5].
Denote

$$\begin{aligned}\mathrm{PowSer}(\Gamma) &= \bigoplus_{\kappa} \mathbb{C}[\![q_\kappa]\!],\\ \mathrm{Laur}(\Gamma) &= \bigoplus_{\kappa} \mathbb{C}[\![q_\kappa]\!][q_\kappa^{-1}],\\ \mathrm{Sing}(\Gamma) &= \bigoplus_{\kappa} \mathbb{C}[\![q_\kappa]\!][q_\kappa^{-1}]/q_\kappa\mathbb{C}[\![q_\kappa]\!],\end{aligned}$$

where κ runs through all cusps for Γ. $\mathrm{Mp}_2(\mathbb{Z})$ operates on these sets.
Let $\Gamma' \subset \Gamma$ be a subgroup of finite index, which acts trivial on V. We define

$$\mathrm{PowSer}(\Gamma, V) = \left(\mathrm{PowSer}(\Gamma') \otimes V\right)^{\Gamma},$$

where Γ acts on both factors.
Similarly we define $\mathrm{Laur}(\Gamma, V)$ and $\mathrm{Sing}(\Gamma, V)$. We have still (abstractly) a 1:1 correspondence between $f \in \mathrm{PowSer}(\Gamma, V)$ (say) and a collection f_κ for each cusp of Γ (not Γ'!) where

$$f_\kappa = \sum_{m \in \mathbb{Q}_{\geq 0}} a_m q_\kappa^m, \tag{13}$$

and a_m lies in a subvector space V_m of V, depending on m.
For $\Gamma = \mathrm{Sp}'_2(\mathbb{Z})$, we have explicitly $V_m = \{v \in V \mid \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} v = e^{2\pi i m} v\}$ and in particular V_m is zero if $m \notin \frac{1}{N}\mathbb{Z}$ for some N, depending on Γ'.
There are maps

$$\lambda : \mathrm{HolModForm}(\Gamma, k, V^*) \to \mathrm{PowSer}(\Gamma, V^*), \tag{14}$$

$$\lambda : \mathrm{ModForm}(\Gamma, 2-k, V) \to \mathrm{Sing}(\Gamma, V), \tag{15}$$

which in the representation (13) for the right hand side, this corresponds to taking Fourier series.
There is a non-degenerate pairing between $\mathrm{PowSer}(\Gamma, V)$ and $\mathrm{Sing}(\Gamma, V^*)$ given by

$$\langle f, \phi \rangle = \sum_{\kappa} \mathrm{res}(f_\kappa \phi_\kappa q_\kappa^{-1} \,\mathrm{d}\, q_\kappa).$$

Here, the product $f_\kappa \phi_\kappa$ involves duality between V and V^*. This pairing is invariant under the operation of $\mathrm{Mp}_2(\mathbb{Z})$ and hence gives a well defined pairing between the spaces in question.

For example if $\Gamma = \mathrm{Mp}_2(\mathbb{Z})$ and standard parameter q, up to a scalar this pairing is the same as

$$\langle f, \phi \rangle = \sum_m \langle a_{-m}, b_m \rangle,$$

for $f = \sum_m a_m q^m$ and $\phi = \sum_m b_m q^m$.

In [loc. cit., Theorem 3.1] it is sketched that one can deduce from Serre duality that

$$\lambda(\mathrm{ModForm}(\mathrm{Mp}_2(\mathbb{Z}), 2-k, V)) = (\lambda(\mathrm{HolModForm}(\mathrm{Mp}_2(\mathbb{Z}), k, V^*))^{\perp},$$

and in [loc. cit., Lemma 4.3] this is refined to

$$\lambda(\mathrm{ModForm}(\mathrm{Mp}_2(\mathbb{Z}), 2-k, V)_{\mathbb{Z}}) \otimes \mathbb{C} = \left(\mathrm{Gal}(\overline{\mathbb{Q}}|\mathbb{Q})\lambda(\mathrm{HolModForm}(\mathrm{Mp}_2(\mathbb{Z}), k, V^*)\right)^{\perp}.$$

It is also shown that the last space has finite index in $\mathrm{Sing}(\mathrm{Mp}_2(\mathbb{Z}), V)$.

(11.3.8) Let $L_{\mathbb{Z}_{(p)}}$ a quadratic lattice with non-degenerate quadratic form Q_L of signature $(m-2, 2)$ and $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$ a $\mathbb{Z}$-lattice s.t. $Q \in \mathrm{Sym}^2(L_{\mathbb{Z}}^*)$.

Consider the Weil representation Weil $= \mathrm{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})$ associated with a lattice $L_{\mathbb{Z}}$. We know from (7.7) that on the characteristic function $\chi_{L_{\widehat{\mathbb{Z}}}}$ the Weil representation of $\Gamma_0(N)'$ is given by a character, which on $\Gamma_0(N)$ may be described by $a \mapsto \chi(a) := \pm\Upsilon(aQ_L)\Upsilon(Q_L)^{-1}$ extended to $\Gamma_0(N)$ by $\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \mapsto \chi(a)$. It has values in ± 1 if m is even and in μ_4 if m is odd. The sign of $\chi(a)$ is determined by the lift to $\Gamma_0(N)'$. It is always $+1$, if m is even.

Hence, by the adjunctions

$$\mathrm{Hom}_{\Gamma_0(N)'}(\chi, \mathrm{Weil}) = \mathrm{Hom}_{\mathrm{Sp}_2'(\mathbb{Z})}(\mathrm{ind}_{\Gamma_0(N)'}^{\mathrm{Sp}_2'(\mathbb{Z})} \chi, \mathrm{Weil})$$

and

$$\mathrm{Hom}_{\Gamma_0(N)'}(\mathrm{Weil}, \chi) = \mathrm{Hom}_{\mathrm{Sp}_2'(\mathbb{Z})}(\mathrm{Weil}, \mathrm{ind}_{\Gamma_0(N)'}^{\mathrm{Sp}_2'(\mathbb{Z})} \chi)$$

and considering injection of $\chi_{L_{\widehat{\mathbb{Z}}}}$, resp. evaluation at 0, on the left hand sides, we get maps

$$\alpha : \mathrm{ind}_{\Gamma_0(N)'}^{\mathrm{Sp}_2'(\mathbb{Z})}(\chi) \to \mathrm{Weil}$$

and

$$\beta : \mathrm{Weil} \to \mathrm{ind}_{\Gamma_0(N)'}^{\mathrm{Sp}_2'(\mathbb{Z})}(\chi)$$

on the right hand sides.

Accordingly we get maps

$$\alpha' : \mathrm{HolModForm}(\mathrm{Sp}_2'(\mathbb{Z}), k, \mathrm{ind}_{\Gamma_0(N)'}^{\mathrm{Sp}_2'(\mathbb{Z})}(\chi)) \to \mathrm{HolModForm}(\mathrm{Sp}_2'(\mathbb{Z}), k, \mathrm{Weil})$$

and

$$\beta' : \mathrm{HolModForm}(\mathrm{Sp}_2'(\mathbb{Z}), k, \mathrm{Weil}) \to \mathrm{HolModForm}(\mathrm{Sp}_2'(\mathbb{Z}), k, \mathrm{ind}_{\Gamma_0(N)'}^{\mathrm{Sp}_2'(\mathbb{Z})}(\chi))$$

as well.
This yields a corresponding decomposition:

$$\mathrm{HolModForm}(\Gamma_0(N)', k, \mathbb{C}) = \ker(\alpha') \oplus \mathrm{im}(\beta').$$

(11.3.9) Lemma. *α and β are Galois invariant with the coordinate-wise Galois action on* Weil *and the action (9) on* ind*. The above decomposition is defined over $\mathbb{Q}(i)$ (or $\mathbb{Q}$ if m is even).*

Proof. This second statement follows immediately from the first because the condition of being in $\ker(\alpha')$ or $\mathrm{im}(\beta')$ is Galois invariant.
The first statement is proven by an explicit examination (cf. 12) of Weil for invariance under $\mathrm{Gal}(\mathbb{C}|\mathbb{Q}(i))$ in any case and for $\mathrm{Gal}(\mathbb{C}|\mathbb{Q})$, if m is even. ☐

(11.3.10) Remark. For prime-cyclic $L_{\mathbb{Z}}^*/L_{\mathbb{Z}} \cong \mathbb{F}_p$ (in particular m even), we have $\chi(a) = \left(\frac{a}{p}\right)$, and the decomposition above is well-known as decomposition into + and − space: We have $\mathrm{im}(\alpha)_m = \mathbb{C}\chi_{\pm\kappa+L_{\widehat{\mathbb{Z}}}}$ where $Q_L(\kappa) \equiv \frac{m}{p}$ (1), hence for $(m,p) = 1$ we get

$$\mathrm{im}(\beta)_{\frac{m}{p}} \cong \begin{cases} 0 & \frac{m}{p} \text{ is not represented by } L_{\mathbb{Z}}^*/L_{\mathbb{Z}}, \\ \mathbb{C} & \frac{m}{p} \text{ is represented by } L_{\mathbb{Z}}^*/L_{\mathbb{Z}}, \end{cases}$$

and with interchanged conditions for $\ker(\alpha)$. (For the comparison with other work, note: the $a_{\frac{m}{p}}$ via the above identification with $\mathbb{C}$ are the coefficients a_m of the image of f under the Fricke involution).

(11.3.11) Assume now that $n \geq 3$ or $n = 2$ and Witt rank of $L_{\mathbb{Q}} = 1$.
With the result above, we are able to construct a Borcherds product $\Psi(F)$ whose divisor

$$\mathrm{div}(\Psi(F)^2) = \sum_{m \in \mathbb{Q}_{<0}} \mathrm{Z}(L_{\mathbb{Z}}, < -m >, a_m; K) + \Xi$$

satisfies certain special properties (see theorems 11.3.13-11.3.16 below).
First we need a lemma:

(11.3.12) Lemma. *Let $L_{\mathbb{Z}}$ be a lattice of signature $(m-2, 2)$, $m \geq 4$. Let* Weil_0 *be a subrepresentation of the Weil representation* $\mathrm{Weil}(L_{\widehat{\mathbb{Z}}}^*/L_{\widehat{\mathbb{Z}}})$*, which is, as a subspace defined over $\mathbb{Q}$. Let $\{M(m)\}_{m \in \mathbb{Q}_{\geq 0}}$ be a collection of subvector spaces $M(m) \subseteq \mathrm{Weil}_{0,m}^*$*

(for the notation, see 11.3.8), such that $\sum_m \dim(M(m)) \to \infty$.
Assume that any modular form in $\mathrm{HolModForm}(\mathrm{Sp}_2'(\mathbb{Z}), \mathrm{Weil}_0^*, \frac{m}{2})$ *with Fourier coefficients supported only on* M *vanishes.*

i. *There is an* $F \in \mathrm{ModForm}(\mathrm{Sp}_2'(\mathbb{Z}), \mathrm{Weil}_0, 2 - \frac{m}{2})$ *with Fourier expansion:*

$$\lambda(F) = \sum_{m \in \mathbb{Q}} c_m q^m$$

with $c_0(0) \neq 0$ *and all* $c_m \perp M(-m), m < 0$,

ii. *Let* $0 > l \in \mathbb{Q}$ *and* $c_l \in \mathrm{Weil}_{0,l}$ *with* $c_l \not\perp M(-l)$ *be given.*
There is an $F \in \mathrm{ModForm}(\mathrm{Sp}_2'(\mathbb{Z}), \mathrm{Weil}_0, 2 - \frac{m}{2})$ *with Fourier expansion:*

$$\lambda(F) = \sum_{m \in \mathbb{Q}} c_m q^m$$

with $c_0(0) \neq 0$, *all* $c_m \perp M(-m), m < 0, m \neq l$ *and* c_l *is the given one.*

Proof. (Compare [15, Lemma 4.11])
(i) Let $\mathrm{Sing}_M(\mathrm{Sp}_2'(\mathbb{Z}), \mathrm{Weil}_0) \subset \mathrm{Sing}(\mathrm{Sp}_2'(\mathbb{Z}), \mathrm{Weil}_0)$ be the subspace, where the coefficients satisfy $c_m \perp M(m), m > 0$. Obviously $\mathrm{Sing}_M(\mathrm{Sp}_2'(\mathbb{Z}), \mathrm{Weil}_0)^\perp$ is the subspace $\mathrm{PowSer}^M(\mathrm{Sp}_2'(\mathbb{Z}), \mathrm{Weil}_0^*)$, where $c_m^* \in M(m)$ for all $m > 0$ and $c_0^* = 0$. Since $\lambda(\mathrm{ModForm}(\mathrm{Sp}_2'(\mathbb{Z}), 2 - \frac{m}{2}, \mathrm{Weil}_0)_\mathbb{Q}) \otimes \mathbb{C}$ has finite index in $\mathrm{Sing}(\mathrm{Sp}_2'(\mathbb{Z}), \mathrm{Weil}_0)$ [5] and $\sum_m \dim(M(m)^\perp) \to \infty$, we have

$$\lambda(\mathrm{ModForm}(\mathrm{Sp}_2'(\mathbb{Z}), 2 - \frac{m}{2}, \mathrm{Weil}_0)_\mathbb{Q}) \cap \mathrm{Sing}_M(\mathrm{Sp}_2'(\mathbb{Z}), \mathrm{Weil}_0)_\mathbb{Q} \neq 0.$$

We want to show that the application $[c_0(0)] : \mathrm{Sing}(\mathrm{Sp}_2'(\mathbb{Z}), \mathrm{Weil}_0)_\mathbb{Q} \to \mathbb{Q}$ does not vanish on this intersection. Since

$$\begin{aligned}&\left(\lambda(\mathrm{ModForm}(\mathrm{Sp}_2'(\mathbb{Z}), 2 - \frac{m}{2}, \mathrm{Weil}_0)_\mathbb{Q}) \cap \mathrm{Sing}_M(\mathrm{Sp}_2'(\mathbb{Z}), \mathrm{Weil}_0)_\mathbb{Q}\right)^\perp \\ =&\lambda(\mathrm{HolModForm}(\mathrm{Sp}_2'(\mathbb{Z}), \frac{m}{2}, \mathrm{Weil}_0^*)) + \mathrm{PowSer}^M(\mathrm{Sp}_2'(\mathbb{Z}), \mathrm{Weil}_0^*)_\mathbb{Q},\end{aligned}$$

Note that $\lambda(\mathrm{HolModForm}(\mathrm{Sp}_2'(\mathbb{Z}), \frac{m}{2}, \mathrm{Weil}_{0,\mathbb{C}}^*))$ is a Galois invariant subspace because of (11.3.6), and because Weil_0 is, as a subspace, defined over $\mathbb{Q}$.
Let us assume that there is a relation

$$[c_0(0)] = \lambda(f) + \phi,$$

where $f \in \mathrm{HolModForm}(\mathrm{Sp}_2'(\mathbb{Z}), 2 - \frac{m}{2}, \mathrm{Weil}_0^*)$ and ϕ is in $\mathrm{PowSer}^M(\mathrm{Sp}_2'(\mathbb{Z}), \mathrm{Weil}_0^*)$.
From this follows that f is a modular form, whose coefficients satisfy $c_m^* \in M(m)$.
Therefore $f = 0$, a contradiction.

(ii) Since $c_l \not\perp M(l)$, we find a $c_l^* \in M(l)$ such that $c_l^* c_l \neq 0$. We have to show that also the element $[c_l^* q^l] : \mathrm{Sing}(\mathrm{Sp}_2'(\mathbb{Z}), \rho)_{\mathbb{Q}} \to \mathbb{Q}$ does not vanish on the intersection above, where now $M(l)$ has been set to 0.
For if this is not the case, we can add an appropriate element not being in its kernel to our original F to get the result. Suppose that $[c_l^* q^l]$ vanishes. Then we get a relation:

$$[c_l^* q^l] = \lambda(f) + \phi,$$

where $f \in \mathrm{HolModForm}(\mathrm{Sp}_2'(\mathbb{Z}), 2 - \frac{m}{2}, \mathrm{Weil}_0^*)$ and ϕ is in $\mathrm{PowSer}^M(\mathrm{Sp}_2'(\mathbb{Z}), \mathrm{Weil}_0^*)$. Therefore $f = 0$ as above, a contradiction. □

We will now show several theorems, stating that there exist Borcherds lifts, whose divisor has special properties. This is an essential ingredient in the calculation of arithmetic volumes later. To not interrupt the discussion, we refer to the next section (11.4) for several elementary lemmas on quadratic forms which are needed, and to section (11.5) for several facts about vanishing of modular forms with sparse Fourier coefficients.

(11.3.13) Theorem. *i. Let $L_{\mathbb{Z}_{(p)}}$ be an isotropic lattice **of signature** $(m-2, 2)$**,** $m \geq 4$, there is a lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$, such that up to multiplication of Q_L with a scalar $\in \mathbb{Z}_{(p)}^*$ there is an $F \in \mathrm{ModForm}(\mathrm{Sp}_2'(\mathbb{Z}), 2 - \frac{m}{2}, \mathrm{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}))$ with integral Fourier coefficients, such that $\Psi(F)$ has nonzero weight,*

all occurring $\mathrm{Z}(L_{\mathbb{Z}}, < -m >, a_m; K)$ in $\mathrm{div}(\Psi(F))$ are p-integral, i.e. consist of canonical models of Shimura varieties $\mathrm{M}({}^K\mathbf{O}(L'_{\mathbb{Z}_{(p)}}))$, for various lattices $L'_{\mathbb{Z}_{(p)}}$ with non-degenerate quadratic form.

ii. We find an F above, such that $\mathrm{div}(\Psi(F))$ contains, in addition, precisely one $\mathrm{Z}(L_{\mathbb{Z}}, < m >, \kappa; K)$, $p|m$ with non-zero multiplicity.

Proof. (i) Choose any lattice of the form $L_{\mathbb{Z}} = H \oplus L'_{\mathbb{Z}}$ in $L_{\mathbb{Z}_{(p)}}$. Let

$$M(m) = \begin{cases} 0 & |m|_p = 1, \\ \mathrm{Weil}_m^* & |m|_p \neq 1. \end{cases}$$

Then all $\mathrm{Z}(L_Z, < -m >, c_m; K)$ with $c_m \perp M(m)$ consist of p-integral canonical models of Shimura varieties $\mathrm{M}({}^K\mathbf{O}(L'_{\mathbb{Z}_{(p)}}))$ because $v^\perp$, for any $v \in L_{\mathbb{Z}_p}$ with $Q_L(v) = m$, $|m|_p = 1$, is unimodular.
To construct the required Borcherds form, by lemma (11.3.12, i), we have to show that any modular form $f \in \mathrm{HolModForm}(\mathrm{Sp}_2'(\mathbb{Z}), \frac{m}{2}, \mathrm{Weil}^*)$ whose Fourier coefficients are supported only on M, vanishes. But every component of f is (in particular) a modular form for some $\Gamma(N)$. It vanishes by lemma (11.5.1).
For (ii), use lemma (11.3.12, ii). □

(11.3.14) Theorem. *i. Let $L_{\mathbb{Z}_{(p)}}$ be a lattice* ***of signature*** $(3,2)$ ***and Witt rank 1****, there is a lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$, such that up to multiplication of Q_L with a scalar $\in \mathbb{Z}_{(p)}^*$. there is an $F \in \mathrm{ModForm}(\mathrm{Mp}_2(\mathbb{Z}), 2-\frac{m}{2}, \mathrm{Weil})$ with integral Fourier coefficients, such that $\Psi(F)$ has nonzero weight, all occurring $\mathrm{Z}(L_Z, < -m >, a_m; K)$ in $\mathrm{div}(\Psi(F))$ are p-integral, i.e. consist of canonical models of Shimura varieties $\mathrm{M}({}^K\mathbf{O}(L'_{\mathbb{Z}_{(p)}}))$, for various lattices $L'_{\mathbb{Z}_{(p)}}$ with non-degenerate quadratic form, such that L' has signature $(2,2)$ and Witt rank 1.*

ii. Up to multiplication of Q_L with a scalar $\in \mathbb{Z}_{(p)}^*$, for every $L'_{\mathbb{Z}_{(p)}}$ ***of signature*** $(2,2)$***, Witt rank 0****, we find a lattice L of signature $(3,2)$, Witt rank 1, such that in $\mathrm{div}(\Psi(F))$ above there occurs, in addition to the subvarieties above, precisely one $\mathrm{Z}(L, < l >, \varphi; K)$ with non-zero coefficient, consisting of canonical models of Shimura varieties $\mathrm{M}({}^{K_i}\mathbf{O}(L'_{\mathbb{Z}_{(p)}}))$ for various different admissible K_i's.*

Proof. (i) By lemma (11.4.3), we may multiply Q_L by a scalar such that there is a lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$, with cyclic $L^*_{\mathbb{Z}}/L_{\mathbb{Z}}$ of order $2D'$ where D' is square-free of the form

$$L_{\mathbb{Z}} = H \oplus L'_{\mathbb{Z}}.$$

By the very construction (7.1) of the Weil representation, we know that $\mathrm{Weil}(L^*_{\mathbb{Z}}/L_{\mathbb{Z}})$ decomposes

$$\mathrm{Weil}(L^*_{\mathbb{Z}}/L_{\mathbb{Z}}) = \bigotimes_{l|D} \mathrm{Weil}(L^*_{\mathbb{Z}_l}/L_{\mathbb{Z}_l}),$$

and for each l we have a decomposition

$$\mathrm{Weil}(L^*_{\mathbb{Z}_l}/L_{\mathbb{Z}_l}) = \mathrm{Weil}(L^*_{\mathbb{Z}_l}/L_{\mathbb{Z}_l})^+ \oplus \mathrm{Weil}(L^*_{\mathbb{Z}_l}/L_{\mathbb{Z}_l})^-,$$

into irreducible representations, here $L^*_{\mathbb{Z}_l}/L_{\mathbb{Z}_l}$ cyclic of prime order (resp. order 2 or 4) is used. (If the order is 2, $\mathrm{Weil}(L^*_{\mathbb{Z}_2}/L_{\mathbb{Z}_2})^-$ is zero).
We will work with the *irreducible* representation

$$\mathrm{Weil}(L^*_{\mathbb{Z}}/L_{\mathbb{Z}})^+ := \bigotimes_{l} \mathrm{Weil}(L^*_{\mathbb{Z}_l}/L_{\mathbb{Z}_l})^+,$$

with product basis build from the basis $\chi_{\pm\kappa+L_{\mathbb{Z}_l}} \in \mathrm{Weil}(L^*_{\mathbb{Z}_l}/L_{\mathbb{Z}_l})^+$. There will be a basis vector of the form

$$\gamma := \sum \chi_{\kappa+L_{\widehat{\mathbb{Z}}}},$$

where the sum runs only over *primitive* $\kappa \in L^*_{\mathbb{Z}}/L_{\mathbb{Z}}$.
Let

$$M(m) = \begin{cases} (\mathbb{C}\gamma)^{\perp} & |m|_p = 1, \\ (\mathrm{Weil}(L^*_{\mathbb{Z}}/L_{\mathbb{Z}})^+)^*_m & |m|_p \neq 1. \end{cases}$$

Note: Every $v^{\perp}$ such that $\gamma(v) \neq 0$ with $Q_L(v) = m$, $|m|_p = 1$ is unimodular. It is also isotropic because by lemma (11.4.3) m is already represented by $L'_{\widehat{\mathbb{Z}}}$.

To construct the required Borcherds form, by lemma (11.3.12, i), we have to show that any modular form $f \in \mathrm{HolModForm}(\mathrm{Mp}_2(\mathbb{Z}), \frac{m}{2}, (\mathrm{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^+)^*)$ with Fourier coefficients supported only on M vanishes. Note that $\mathrm{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^+$ is defined over $\mathbb{Q}$ as a subvectorspace.
Now $\gamma \circ f$ vanishes by (11.5.1). Since $\mathrm{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}})^+$ is irreducible, lemma (11.5.3) tells us $f = 0$.
(ii) Use (11.4.5) to construct the lattice $L_{\mathbb{Z}}$ and $\mathrm{Z}(L_{\mathbb{Z}}, < x >, L_{\mathbb{Z}}; K)$ consists obviously of the required models. Since $\chi_{L_{\widehat{\mathbb{Z}}}} \not\perp \gamma^{\perp}$, apply lemma (11.3.12, ii). ☐

(11.3.15) Theorem. *i. Let $L_{\mathbb{Z}_{(p)}}$ be an* ***isotropic lattice of signature*** $(2,2)$ ***and fundamental discriminant*** $-q$***,*** q ***a prime*** $\equiv -1 \ 4$*, there is a lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$, such that up to multiplication of Q_L with a scalar $\in \mathbb{Z}_{(p)}^*$, there is an $F \in \mathrm{ModForm}(\mathrm{Sp}_2(\mathbb{Z}), 2-\frac{m}{2}, \mathrm{Weil}(L_{\mathbb{Z}}^*/L_{\mathbb{Z}}))$ with integral Fourier coefficients, such that $\Psi(F)$ has nonzero weight,*

all occurring $\mathrm{Z}(L, < -m >, a_m; K)$ in $\mathrm{div}(\Psi(F))$ are p-integral, i.e. consist of canonical models of Shimura varieties $\mathrm{M}({}^K\mathbf{O}(L'_{\mathbb{Z}_{(p)}}))$, for various lattices $L'_{\mathbb{Z}_{(p)}}$ with non-degenerate quadratic form, such that L' has signature $(1,2)$ and Witt rank 1.

ii. Up to multiplication of Q_L with a scalar $\in \mathbb{Z}_{(p)}^*$, for every $L'_{\mathbb{Z}_{(p)}}$ ***of signature*** $(1,2)$***, Witt rank 0****, we find a lattice L of signature $(2,2)$, Witt rank 1 and fundamental discriminant $-q$, q a prime $\equiv -1 \ 4$ (as in i), such that in $\mathrm{div}(\Psi(F))$ above there occurs, in addition to the subvarieties above, precisely one $\mathrm{Z}(L, < l >, \varphi; K)$ with non-zero coefficient, consisting of canonical models of Shimura varieties $\mathrm{M}({}^{K_i}\mathbf{O}(L'_{\mathbb{Z}_{(p)}}))$ for various different admissible K_i's.*

Proof. (i) Let Weil_0 be $< \mathrm{Sp}_2(\mathbb{Z})\chi_{L_{\widehat{\mathbb{Z}}}} >$, which is irreducible in this case, hence isomorphic to $\mathrm{im}(\beta)$ (11.3.8), and as a subspace defined over $\mathbb{Q}$.
By lemma (11.4.3) we may multiply Q_L by a scalar such that there is a lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$, with cyclic $L_{\mathbb{Z}}^*/L_{\mathbb{Z}}$ of order q, of the form

$$L_{\mathbb{Z}} = H \oplus < x^2 + xy + \frac{1-q}{4} y^2 > .$$

A $\mathrm{Z}(L, < m >, \kappa; K)$ (for admissible K) consists of Shimura varieties of the required form, if $v^{\perp}$ for $v \in \pm\kappa + L_{\widehat{\mathbb{Z}}}$, $Q_L(v) = m$ is isotropic. This is the case, if and only if m is represented by $< x^2 + xy + \frac{1-q}{4} y^2 >$. Hence define

$$M(m) := \begin{cases} \mathrm{Weil}_{0,m}^* & |m|_p \neq 1, \\ \mathrm{Weil}_{0,m}^* \cap R(m)^{\perp} & |m|_p = 1, \end{cases}$$

where $R(m) = \{f \in \mathrm{Weil}_{0,m} \mid \exists v \in < x^2 + xy + \frac{1-q}{4} y^2 >_{\mathbb{A}^{(\infty)}} : f(v) \neq 0, Q_L(v) = m\}$.

We know (11.3.10) that $\mathrm{Weil}^*_{0,\frac{j}{q}}$ is zero, if $\chi_q(j) = -1$. The definition of $M(m)$ and (11.4.4) imply that for *primes* $p' \neq p$ with $\chi_q(p') = 1$, $M(\frac{p'}{q})$ is also zero.
To construct the required meromorphic modular form, by lemma (11.3.12), we have to show that any modular form $f \in \mathrm{HolModForm}(\mathrm{Sp}_2(\mathbb{Z}), \frac{m}{2}, \mathrm{Weil}^*_0)$ with Fourier coefficients supported only on M vanishes. Let $f' := \rho(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})f$. We have $f'_0 = f_0 + \sum_{\pm i} f_{\pm i}$. We know that $f'_0(q\tau)$ is a modular form for $\Gamma_0(q)$ again. From the above follows that all Fourier coefficients $a_{p'}$ of $f'_0(q\tau)$ for $p' \neq p$ vanish. The vanishing of f'_0 now follows from theorem (11.5.2) and that of f by lemma (11.5.3).
(ii) Use (11.4.2) to construct the lattice $L_\mathbb{Z}$ and $\mathrm{Z}(L_\mathbb{Z}, < x >, L_\mathbb{Z}; K)$ consists obviously of the required models. Since $\chi_{L_{\widehat{\mathbb{Z}}}} \not\perp \gamma^\perp$, apply lemma (11.3.12, ii). ☐

(11.3.16) Theorem. *Let $L_{\mathbb{Z}_{(p)}}$ be a* ***lattice of signature*** $(3,2)$ ***of Witt rank 2****. There is a lattice $L_\mathbb{Z} \subset L_{\mathbb{Z}_{(p)}}$, such that up to multiplication of Q_L with a scalar $\in \mathbb{Z}^*_{(p)}$, there is an $F \in \mathrm{ModForm}(\mathrm{Mp}_2(\mathbb{Z}), 2 - \frac{m}{2}, \mathrm{Weil}(L^*_\mathbb{Z}/L_\mathbb{Z}))$ with integral Fourier coefficients, such that $\Psi(F)$ has nonzero weight,* $\mathrm{div}(\Psi(F))$ *consist of exactly one $\mathrm{Z}(L_\mathbb{Z}, < -l >, a_l)$ with non-zero coefficient, which itself consists of canonical models of a Shimura varieties* $\mathrm{M}({}^K\mathbf{O}(L'_{\mathbb{Z}_{(p)}}))$, *for* any *(a priori) given lattice $L'_{\mathbb{Z}_{(p)}}$ of signature $(2,2)$, Witt rank ≥ 1.*

Proof. Define $L_{\mathbb{Z}_{(p)}} = L'_{\mathbb{Z}_{(p)}} \perp < x >$, where x is represented by $L'_{\mathbb{Z}_{(p)}}$ and $|x|_p = 1$. Up to multiplication of Q_L by a scalar, we find a lattice $L_\mathbb{Z} = H^2 \perp < 1 > \subset L_{\mathbb{Z}_{(p)}}$. $\mathrm{Weil} = \mathrm{Weil}(L^*_\mathbb{Z}/L_\mathbb{Z})$ is irreducible in this case. Define

$$M(m) = \begin{cases} \mathrm{Weil}^*_m & m \neq x, \\ 0 & m = x. \end{cases}$$

To construct the required meromorphic modular form, by lemma (11.3.12), we have to show that any modular form $f \in \mathrm{HolModForm}(\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \mathrm{Weil}^*)$ with Fourier coefficients supported only on M vanishes. This follows because

$$\mathrm{HolModForm}(\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \mathrm{Weil}^*) =$$
$$\mathrm{HolModForm}(\mathrm{Mp}_2(\mathbb{Z}), \frac{5}{2}, \mathrm{im}(\alpha^*)) \subset \mathrm{HolModForm}(\Gamma_0(4)', \frac{5}{2}, \chi'_4).$$

The last space is 1-dimensional in this case, hence the first contains only the Eisenstein series, which has a non-zero coefficient a_x. ☐

11.4. Lemmata on quadratic forms

(11.4.1) Lemma. *Let $L_{\mathbb{Z}_{(p)}}$ be an unimodular lattice of signature $(2,2)$ and Witt rank 1, with (fundamental) discriminant D. Up to multiplication of Q_L by a unit, there exists* *a lattice* $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$ *of the form*

$$L_{\mathbb{Z}} = H \oplus < x^2 + xy + \frac{1-D}{4} y^2 >$$

if $D \equiv 1$ *modulo 4 or*

$$L_{\mathbb{Z}} = H \oplus < x^2 - \frac{D}{4} y^2 >$$

if $4|D$.

Proof. We may write $L_{\mathbb{Z}_{(p)}} = H \oplus L'_{\mathbb{Z}_{(p)}}$ and assume that $L'_{\mathbb{Z}_{(p)}}$ represents 1. $L'_{\mathbb{Z}_{(p)}}$ is anisotropic by assumption. Then it is well-known and elementary that $L_{\mathbb{Z}_{(p)}}$ contains a $\mathbb{Z}$-lattice of the required form. □

(11.4.2) Lemma. *Let* $L_{\mathbb{Z}_{(p)}}$ *be a unimodular anisotopic lattice of signature* $(1,2)$. *Up to multiplication of* Q_L *with a scalar, there exists a unimodular lattice* $L'_{\mathbb{Z}_{(p)}} = L_{\mathbb{Z}_{(p)}} \oplus < x >$ *which is isotropic and of* prime *fundamental discriminant* $-q$, $q \neq p$ *and a lattice* $L'_{\mathbb{Z}} \subset L'_{\mathbb{Z}_{(p)}}$ *of the form*

$$L'_{\mathbb{Z}} = H \oplus < x^2 + xy + \frac{1-q}{4} y^2 > .$$

Proof. Write $L_{\mathbb{Z}_{(p)}} =< \alpha_1, \ldots, \alpha_3 >$, with α_i square-free. Let D be the square-free part of $\prod \alpha_i$. We may find a prime q, different from p, such that $q \equiv -1(4)$ and for any $l \neq 2$ with $\nu_l(\prod \alpha_i) = 2$ (hence $l \nmid D$), we have $Dq \equiv -\alpha_j$ modulo $\mathbb{Q}_l^2$, where α_j is the one not divisible by l.
We may also prescribe its residue mod 8, such that $L_{\mathbb{Q}_2} \oplus < Dq >$ is isotropic. For if they were anisotropic for q, q' congruent to $-1, -5$ (8) respectively, we would get $< Dq > \simeq < Dq' >$ from Witt's theorem and the uniqueness of the 4-dimensional anisotropic space, which is absurd.
Hence

$$L_{\mathbb{Z}_{(p)}} + < Dq >$$

is unimodular, of signature $(2,2)$ with square-free discriminant q. It is isotropic at all $l \neq 2$ either because of the congruence condition on Dq. Hence the Witt rank is 1 and we may apply the previous lemma to it (this changes also the form on $L_{\mathbb{Z}_{(p)}}$ by a scalar) to get the result. □

(11.4.3) Lemma. *Let* $L_{\mathbb{Z}_{(p)}}$ *be a unimodular anisotropic lattice of signature* $(1,2)$. *Up to multiplication of* Q_L *by a scalar, there is a lattice* $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$ *of discriminant* $D = 2D'$, *where* D' *is square-free and with* $L_{\mathbb{Z}}^*/L_{\mathbb{Z}}$ *cyclic.*
It has the property that a primitive $\kappa \in L_{\hat{\mathbb{Z}}}^*/L_{\hat{\mathbb{Z}}}$ *represents an* $m \in \mathbb{Q}$ *if and only if* $q(\kappa) \equiv m \ (1)$.

Proof. First we prove the existence of the lattice. Write $L_{\mathbb{Z}_{(p)}} =< \alpha_1, \dots, \alpha_3 >$, with $\alpha_i \in \mathbb{Z}$ square-free. It suffices to construct them locally. The lattice $L_{\mathbb{Z}} =< \alpha_1, \dots, \alpha_3 >_{\mathbb{Z}}$ already satisfies the assumption for all $l \neq 2$. For $l = 2$ there are 2 cases:
1. The space is isotropic, whence there is, up to multiplication of the lattice by 2, a lattice of the form

$$L_{\mathbb{Z}_2} = H \perp < 1 >,$$

which has discriminant 2.
2. The space is anisotropic, hence, up to multiplication of the lattice by 2, of the form

$$L_{\mathbb{Q}_2} =< 1, 3, 2\epsilon >$$

with $\epsilon \in \{1, 3, -3, -1\}$. In it there exists the lattice

$$L_{\mathbb{Z}_2} =< x^2 + xy + y^2 > \perp < 2\epsilon >$$

of discriminant 4, with $L^*_{\mathbb{Z}_2}/L_{\mathbb{Z}_2}$ cyclic of order 4.
The only-if part of the claimed property is clear. For the if part, it suffices to show that m is represented by $L_{\mathbb{Q}_l}$ for all l. Consider the space $L_{\mathbb{Q}_l} \oplus < -m >$. It is automatically isotropic at all $l \nmid D$. But at all $l | D$ it is isotropic as well because of the condition $q(\kappa) \equiv m \ (1)$. For $l = 2$, we need only to consider the case, where $L_{\mathbb{Z}_2} =< x^2 + xy + y^2 > \perp < 2\epsilon >$. We get the equation

$$\epsilon z^2 \equiv 8m \ (8),$$

with $z \in (\mathbb{Z}/8\mathbb{Z})^2$. Hence $m(\mathbb{Q}_2^*)^2 = 2\epsilon(\mathbb{Q}_2^*)^2$. This forces the form $L_{\mathbb{Q}_l} \oplus < -m >$ to be isotropic. □

The strong form of the lemma is wrong for lattices of dimension 2. However, we have the following weaker form:

(11.4.4) Lemma. *Let $L_{\mathbb{Z}_{(p)}}$ be a unimodular anisotropic lattice of signature $(1, 1)$ and of prime discriminant $q \equiv -1 \ (4)$. Up to multiplication of Q_L by a unit $\in \mathbb{Z}^*_{(p)}$, there is a lattice $L_{\mathbb{Z}} \subset L_{\mathbb{Z}_{(p)}}$ of discriminant q of the form*

$$L_{\mathbb{Z}} =< x^2 + xy + \frac{1-q}{4} y^2 > .$$

It has the property that a primitive *$\kappa \in L^*_{\widehat{\mathbb{Z}}}/L_{\widehat{\mathbb{Z}}}$ represents an $m = \frac{q}{p} \in \mathbb{Q}$, where q* is prime, *if and only if $q(\kappa) \equiv m \ (1)$.*

Proof. The existence of the lattice is well-known (compare also 11.4.1). The property follows directly from the law of quadratic reciprocity. □

(11.4.5) Lemma. *Let $L_{\mathbb{Z}_{(p)}}$ be a unimodular anisotopic lattice of signature $(2,2)$. Up to multiplication of Q_L with a scalar, there exists a unimodular lattice $L'_{\mathbb{Z}_{(p)}} = L_{\mathbb{Z}_{(p)}} \oplus < x >$ of signature $(3,2)$ and a $\mathbb{Z}$-lattice $L'_{\mathbb{Z}} \subset L'_{\mathbb{Z}_{(p)}}$ of discriminant $D = 2D'$, where D' is square-free of the form*

$$L'_{\mathbb{Z}} = H \oplus L''_{\mathbb{Z}}.$$

such that $(L'_{\mathbb{Z}})^*/L'_{\mathbb{Z}}$ is cyclic.

Proof. Take $L'_{\mathbb{Z}_{(p)}} = L_{\mathbb{Z}_{(p)}} \oplus < x >$ for an arbitrary negative $x \in \mathbb{Z}^*_{(p)}$. It is automatically isotropic, hence of the form

$$L'_{\mathbb{Z}_{(p)}} = H \oplus L''_{\mathbb{Z}_{(p)}}.$$

Applying lemma (11.4.3) to $L''_{\mathbb{Z}_{(p)}}$, we get the result (this multiplies also Q_L on $L_{\mathbb{Z}_{(p)}}$ by a scalar — note that multiplying the quadratic form on H by a scalar does not affect its class). □

11.5. Lacunarity of modular forms

(11.5.1) Lemma. *Let $N > 0$ be an integer and p a prime with $p \nmid N$.*
Let f be a holomorphic modular form of (half-)integral weight $k \neq 0$ for the group $\Gamma(N)$. If f has a Fourier expansion of the form

$$f = \sum_{n \in \mathbb{Q}_{\geq 0}} a_n q^n,$$

where a_n is zero, unless n is an integral multiple of $\frac{p}{N}$, then $f = 0$.

Proof. The assumption implies that f is periodic with period $\frac{N}{p}$. It is hence a modular form for the group Γ, generated by $\Gamma(N)$ and the matrix $\begin{pmatrix} 1 & \frac{N}{p} \\ 0 & 1 \end{pmatrix}$. This group contains the product

$$C = \begin{pmatrix} 1 & \frac{N}{p} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{N^2}{p} & \frac{N}{p} \\ N & 1 \end{pmatrix}.$$

The trace of C is equal to $2 + \frac{N^2}{p}$. Since $p \nmid N$, its p-adic valuation is > 1. Hence at least one of the eigenvalues of C has p-adic valuation > 1. (We choose some fixed extension of the p-adic valuation to $\overline{\mathbb{Q}}$). It follows that $\|C^i\| \to \infty$ for any chosen p-adic matrix norm $\| \cdot \|$. From this, it follows that $[\Gamma : \Gamma(N)] = \infty$. For assume that there are finitely many representatives α_i. Let ν be the maximum of their p-adic matrix norms. Every element $\gamma \in \Gamma$ is of the form

$$\alpha_i \gamma`$$

for $\gamma` \in \Gamma(N)$. The matrix norm of γ is hence $\leq \nu$. A contradiction. Hence Γ cannot be a discrete subgroup, and since $k \neq 0$, we have $f = 0$. □

(11.5.2) Theorem. *Let $q \equiv 1 \ (4)$ be a prime and S a finite set of primes. Let $\chi_q(x) = \left(\frac{x}{q}\right)$ and $k \geq 2$ be an integer.*
If $f \in \mathrm{HolModForm}(\Gamma_0(q), k, \chi_q)$ has a Fourier expansion of the form

$$f = \sum_{n \in \mathbb{Z}_{\geq 0}} a_n q^n,$$

with algebraic a_n, where (i) $a_n = 0$, whenever $\chi_p(n) = -1$, and (ii) $a_p = 0$, whenever $p \notin S$.
Then $f = 0$.

Proof. This follows from an idea of [82], see also [15, Lemma 4.14]: Condition (i) forces f to be in the subspace $\mathrm{HolModForm}(\mathrm{SL}_2(\mathbb{Z}), k, \mathrm{im}(\alpha))$ (see 11.3.10). Since $q \equiv 1 \ (4)$, there are no forms of type 'CM' in this space.
Since the a_n are algebraic, we can write f as a linear combination

$$f = c_0 E_0 + c_\infty E_\infty + \sum_{i=1}^{n} c_i f_i,$$

where the c_i are algebraic, the f_i are cuspidal Hecke eigenforms and E_0, E_∞ are the Eisenstein series. Assume n minimal.
In [82, Lemma 1] it is shown that for all sufficiently large primes l the mod l representation $\rho := \rho_{1,l} \times \cdots \times \rho_{n,l}$ contains a subgroup conjugated to

$$G = \mathrm{SL}_2(\mathbb{F}_1) \times \cdots \times \mathrm{SL}_2(\mathbb{F}_n)$$

and $\mathrm{im}(\rho)/G$ is Abelian, where the $\mathbb{F}_i$'s are defined in [loc. cit.]. Choose l such that all $|c_i|_l = 1$, whenever $c_i \neq 0$.
If c_0 or c_∞ is $\neq 0$, we proceed as follows: By Chebotarev, there is a positive density of primes p, such that the image of Frob_p is conjugated to

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Choose $\epsilon = \pm 1$ according to whether $c_0 \equiv c_\infty \ (l)$ or not. Since G has no nontrivial Abelian quotient ($l > 4$) and because $\mathrm{im}(\rho)/G$ is Abelian, by possibly multiplying A by a scalar matrix, we may assume in addition that $\chi_q(p) = \epsilon$ and $p^{k-1} \equiv \epsilon \ (l)$. Hence $|a_p(c_0 E_0 + c_\infty E_\infty)|_l = 1$ and $|a_p(f_i)|_l < 1$, We get $a_p(f) \neq 0$ for infinitely many primes, a contradiction. If $c_0 = c_\infty = 0$, choose

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and argue as before. $\square$

(11.5.3) Lemma. *Let $\Gamma \subset \mathrm{Mp}_2(\mathbb{R})$ be an arithmetic subgroup and V_ρ, ρ an* irreducible *representation of* Γ. *Let $f \in \mathrm{HolModForm}(\Gamma, \rho, k)$ and $\beta : V_\rho \to \mathbb{C}$ be a non-zero linear form. If $\beta \circ f = 0$ then $f = 0$.*

Proof. This follows immediately from the irreducibility: Choose a basis $\{e_i\}$ of V_ρ such that $\beta = e_0^*$. Since V_ρ is irreducible, for any i there is an operator of the form

$$O_i = \sum_j \alpha_j \rho(\gamma_j),$$

which interchanges e_i and e_0. Consider the form $O_i \circ f$. $e_0^* \circ O_i \circ f$ is equal to $e_i^* \circ f$ on the one hand. On the other hand it is equal to $\sum_i \alpha_i (e_0^* \circ f)|_k \gamma_i$, which is zero by assumption. Hence $f = 0$. □

11.6. Borcherds products and Arakelov geometry

(11.6.1) Let the signature of $L_\mathbb{Q}$ be $(m-2, 2)$ with $m \geq 4$ and assume the Witt rank to be 1 if $m = 4$. Let $L_\mathbb{Z}$ be a lattice of the form $L_\mathbb{Z} = H \perp L'_\mathbb{Z}$.
Take a modular form F as in (11.3) with Fourier expansion

$$F(\tau) = \sum_{m \in \mathbb{Q}} c_m q^m,$$

where $c_m \in \mathrm{Weil}(L_\mathbb{Z}^*/L_\mathbb{Z}) \subset S(\mathbb{A}^{(\infty)})$, with $c_0(0) \neq 0$.
We will now prove an average and local version of (10.5.4). It is used in an essential way in the proofs of (10.5.7) and (10.5.8).

(11.6.2) Theorem. *Under the conditions above, we have:*

$$\begin{aligned} &\sum_q \mu'(L_\mathbb{Z}, < -q >, c_q; 0)\, \mathrm{vol}_E(\mathrm{M}({}^K_\Delta\mathbf{O}(L))) \\ &\quad + \sum_q \deg_E(\mathrm{Z}(L_\mathbb{Z}, < -q >, c_q; K)) \widehat{\mathrm{vol}}_{\overline{\mathcal{E}},p}(\mathrm{M}({}^K_\Delta\mathbf{O}(L))) \\ = &\sum_q \widehat{\mathrm{vol}}_{\overline{\mathcal{E}},p}(\mathrm{Z}(L_\mathbb{Z}, < -q >, c_q; K)) \end{aligned}$$

in $\mathbb{R}^{(p)}$. Here deg *denotes the relative geometric degree.*

Proof. Let $f_0 := \Psi(F)$ be the Borcherds lift of F — cf. (7.8) and (10.4.12).
We may calculate $\widehat{\mathrm{vol}}_{\overline{\mathcal{E}},p}(\mathrm{M}({}^K_\Delta\mathbf{O}(L)))$ In the following way:
First assume w.l.o.g. (e.g. by taking a lattice with large discriminant in the construction of F or by just pulling back $\Psi(F)$ afterwards) that K is neat.
We know by (10.4.4) that $\Xi^*(\mathcal{E})^{-1}$ is ample on $\mathrm{M}({}^K\mathbf{O})$ and that some power of it has no

base points on $\mathrm{M}({}^K_\Delta\mathbf{O})$. Choose some sufficiently fine, smooth, complete, and projective Δ (by lemma 9.4.5 the arithmetic volume does not depend on this choice) such that all models exist and all special cycles involved embed nicely as projective schemes (cf. 10.3). $\Xi^*(\mathcal{E})^{\otimes k}$, for some negative k, defines a morphism

$$\alpha : \mathrm{M}({}^K_\Delta\mathbf{O}) \to \mathbb{P}^r(\mathbb{Z}_{(p)}),$$

whose restriction to $\mathrm{M}({}^K\mathbf{O})$ is an embedding. Let D be the boundary divisor. Recall: M and D are defined over $\mathbb{Z}_{(p)}$. We have $\dim(\mathrm{im}(D)_\mathbb{C}) \le 1$ because $\Xi^*(\mathcal{E})_\mathbb{C}^{\otimes k}$ induces the Baily Borel compactification. In addition, we may also choose k and F such that $f_0 \in H^0(\mathrm{M}({}^K_\Delta\mathbf{O}), \Xi^*(\mathcal{E})^{\otimes k})$ (10.4.12).

Up to increasing k again, we may find hyperplanes $H_1, \dots, H_n$ intersecting properly with $\mathrm{im}(\mathrm{div}(f_0))$, not intersecting (simultaneously) in $\mathrm{im}(D)$ at all, and not intersecting (simultaneously) in the locus, where $\alpha_{\mathbb{F}_p}$ is not finite. Furthermore, we may assume that already H_1 and H_2 do not intersect in $\mathrm{im}(D)_\mathbb{C}$. Let $f_1, \dots, f_n$ be the corresponding sections. Furthermore by (10.4.12, ii)

$$\mathrm{div}(f_0) \cdot \mathrm{div}(f_1) \cdot \dots \cdot \mathrm{div}(f_n) = \frac{1}{2} \sum_{m<0} \mathrm{Z}(L_\mathbb{Z}, < -m >, c(m); K) \cdot \mathrm{div}(f_1) \cdot \dots \cdot \mathrm{div}(f_n)$$

because the $f_i, i \ge 1$ do not intersect in D simultaneously.

Note that the arithmetic volume is the sum of this expression and

$$(\frac{1}{2\pi i})^n \int \mathfrak{g}_0 * \mathfrak{g}_1 * \dots * \mathfrak{g}_n,$$

where

$$\mathfrak{g}_i = (k\Omega, \log \Xi^* h(f_i))$$

is the corresponding Green object. Here

$$\Omega := \mathrm{c}_1(\Xi^*(\mathcal{E}), \Xi^*(h_\mathcal{E}))$$

is the first Chern form of the bundle $\Xi^*(\mathcal{E})$ with respect to the (log-singular) Hermitian metric $\Xi^* h_\mathcal{E}$. On any parametrization defined by a point-like boundary component as in (10.2.17) it is given by

$$\mathrm{d}\,\mathrm{d}^c \log(Y^2)$$

where $Z = X + iY \in L_0(\mathbb{C})$.

By (11.6.3) below, we may write this integral as

$$\begin{aligned} &= \frac{1}{(2\pi i)^n} \int_{\mathrm{M}({}^K_\Delta\mathbf{O})_\mathbb{C}} G_0 k\Omega \wedge \dots \wedge k\Omega \\ &+ \frac{1}{(2\pi i)^{n-1}} \int_{\mathrm{supp}(\mathrm{div}(f_0)) \cap \mathrm{M}({}^K_\Delta\mathbf{O})_\mathbb{C}} \mathfrak{g}_1 * \dots * \mathfrak{g}_n \end{aligned}$$

hence we get the equation ($\in \mathbb{R}^{(p)}$)

$$\begin{aligned}&\frac{1}{(2\pi i)^n}\int_{\mathrm{M}({}_\Delta^K\mathbf{O})_\mathbb{C}} G_0 k\Omega\wedge\cdots\wedge k\Omega\\&+k^{n+1}\widehat{\mathrm{vol}}_{\overline{\mathcal{E}},p}(\mathrm{M}({}_\Delta^K\mathbf{O}(L)))\\=&k^n\sum_q\widehat{\mathrm{vol}}_{\overline{\mathcal{E}},p}(\mathrm{Z}(L_\mathbb{Z},<-q>,c_q;K))\end{aligned}$$

(cf. also 9.3.7) and therefore the required one, taking into account that

$$\frac{1}{2}\sum_q \deg_E(\mathrm{Z}(L_\mathbb{Z},<-q>,c_q;K))=k=\frac{c(0,0)}{2}$$

(relative degree) and by (10.4.12, iii):

$$\begin{aligned}&\frac{1}{(2\pi i)^n}\int_{\mathrm{M}({}_\Delta^K\mathbf{O})_\mathbb{C}} G_0\Omega\wedge\cdots\wedge\Omega\equiv\\-&\mathrm{vol}_\mathcal{E}(\mathrm{M}({}_\Delta^K\mathbf{O})_\mathbb{C})\sum_q\mu'(L_\mathbb{Z},<-q>,c_q;0),\end{aligned}$$

in $\mathbb{R}^{(p)}$. □

(11.6.3) Theorem. *Let $L_\mathbb{Q}$ be as before, i.e. of signature $(m-2,2)$ with $m\geq 5$, or $m=4$ and the Witt rank is 1.*
Let $f_0,\ldots,f_n$ be sections of $\Xi^*(\mathcal{E})^{\otimes k}$ on $\mathrm{M}({}_\Delta^K\mathbf{O})$, intersecting properly:

$$\bigcap_i \mathrm{supp}(\mathrm{div}(f_i))=\emptyset,$$

such that

$$\mathrm{supp}(\mathrm{div}(f_1))\cap\mathrm{supp}(\mathrm{div}(f_2))\cap\mathrm{supp}(D)=\emptyset$$

(Witt rank 2), resp.

$$\mathrm{supp}(\mathrm{div}(f_1))\cap\mathrm{supp}(D)=\emptyset$$

(Witt rank 1).
We have

$$\begin{aligned}&\frac{1}{(2\pi i)^n}\int_{\mathrm{M}({}_\Delta^K\mathbf{O})_\mathbb{C}}\mathfrak{g}_0*\cdots*\mathfrak{g}_n\\=&\frac{1}{(2\pi i)^n}\int_{\mathrm{M}({}^K\mathbf{O})_\mathbb{C}} G_0k\Omega\wedge\cdots\wedge k\Omega\\+&\frac{1}{(2\pi i)^{n-1}}\int_{\mathrm{supp}(\mathrm{div}(f_0))\cap\mathrm{M}({}^K\mathbf{O})_\mathbb{C}}\mathfrak{g}_1*\cdots*\mathfrak{g}_n.\end{aligned}$$

All occurring integrals exist.

Proof. According to [15, Theorem 1.14], we have

$$\begin{aligned}
&\frac{1}{(2\pi i)^n}\int_{\mathrm{M}({}^K_\Delta\mathbf{O})_\mathbb{C}} \mathfrak{g}_0 * \cdots * \mathfrak{g}_n \\
=&\lim_{\varepsilon\to 0}\left(\frac{1}{(2\pi i)^n}\int_{\mathrm{M}({}^K_\Delta\mathbf{O})_\mathbb{C}-B_\varepsilon(D)} G_0\omega_1\wedge\cdots\wedge\omega_n\right.\\
&\left.-\frac{2}{(2\pi i)^{n-1}}\int_{\partial B_\varepsilon(D)} G\wedge \mathrm{d}^c G_0 - G_0\wedge \mathrm{d}^c G\right)\\
&+\frac{1}{(2\pi i)^{n-1}}\int_{\mathrm{supp}(\mathrm{div}(f_0))\cap \mathrm{M}({}^K\mathbf{O})_\mathbb{C}} \mathfrak{g}_1 * \cdots * \mathfrak{g}_n
\end{aligned}$$

where we take any representation

$$\mathfrak{g}_1 * \cdots * \mathfrak{g}_n = (\omega_1\wedge\cdots\wedge\omega_{g_n}, G).$$

The integral

$$\frac{1}{(2\pi i)^n}\int_{\mathrm{M}({}^K\mathbf{O})} G_0\omega_1\wedge\cdots\wedge\omega_n$$

exists by [12, Theorem 2] because

$$\omega_1\wedge\cdots\wedge\omega_n \sim \Omega^n,$$

and we excluded the cases $n=1$, Witt rank 1 and $n=2$, Witt rank 2.
Any point on the boundary maps either to a 0 or 1-dimensional boundary stratum in the Baily-Borel compactification. We will prepare special neighborhoods of these points and call them of **first (resp. second) type** for the rest of this section.

(11.6.4) A point of the first type lies (identification via boundary map) on the exceptional divisor of a torus embedding constructed by means of the torus $\mathbb{G}_m\otimes K_\mathbb{Z}$, where $K_\mathbb{Z}$ is a lattice in $<z>\otimes<z>^\perp/<z>$, where $<z>$ is a corresponding isotropic line. The bundle $\Xi^*\mathcal{E}$ is trivial over the whole torus and the trivializing section s has norm

$$h_\mathcal{E}(s)=\sum_{i,j}\langle\lambda_i,\lambda_j\rangle\log(r_i)\log(r_j),$$

where λ_i constitutes a basis on $K_\mathbb{Z}$ and r_i is the corresponding absolute value in $\mathbb{A}^n\cong \mathrm{M}({}^{K_1}_{\Delta_1}\mathbf{B})$ (here Δ' is just 1 top-dimensional cone together with all its faces, generated by $\lambda_1,\cdots,\lambda_n$. At a point of any other stratum of the compactification we may write this as

$$\sum_{i<m,j<m}\langle\lambda_i,\lambda_j\rangle\log(r_i)\log(r_j)+\sum_{i<m}\log(r_i)\psi_i(z)+\psi_0(z),$$

where the ψ_i are smooth functions in a neighborhood of the point, satisfying $\mathrm{d}\,\mathrm{d}^c\,\psi_i=0$. (just incorporate all terms with $\log r_i$, $i\geq m$ which are non-singular at the point in question.).

(11.6.5) A point p of the second type lies (again, identification via boundary map) at the zero section of the line bundle over $\mathrm{M}({}^{K_1}_{\Delta_1}\mathbf{B})$ over $\mathrm{M}({}^{K_1}_{\Delta_1}\mathbf{B}/U_{\mathbf{B}})$ which a family of Abelian varieties over $\mathrm{M}({}^{K}\mathbf{H}(I))$. Choose a trivializing section s of $\Xi^*\mathcal{E}$ on a small neighborhood of the projection of p to $\mathrm{M}({}^{K}\mathbf{H}(I))$. Such a section was calculated in (10.4.16), with norm of the form

$$h_{\mathcal{E}}(s) = \log(r_1)\psi_1(z_n) + \psi_0(z),$$

where ψ_1 is harmonic and does only depend on z_n (coordinate on $\mathrm{M}({}^{K}\mathbf{H}(I))$) and ψ_0 is smooth.

(11.6.6) We will prepare an ε-tube neighborhood of D as follows. We prepared neighborhoods around every point of the boundary above. Take a finite cover U_i consisting of these. Let σ_i be a partition of unity *defined on* D, corresponding to the chosen cover. Consider a neighborhood U_i with coordinates $z_1, \cdots z_i$. Assume that D has the equation $(z_1)^{a_1} \cdots (z_n)^{a_n} = 0$. In any other cover U_j overlapping with U_i. D has the equation $(z_1^j)^{a_1^j} \cdots (z_n^j)^{a_n^j} = 0$, where after a renumbering of the coordinates a_k is either equal to a_k^j or one of them is 0. If they are equal (i.e. the components of the divisor correspond), we have

$$z_k^j = z_k(f_{k,0}^j(z_1, \ldots \widehat{k}, z_n) + z_k f_k^j(z_1, \ldots, z_n)),$$

where $f_{k,0}^j(z_1, \ldots \widehat{k}, z_n)$ does not depend on z_k and is everywhere non zero on the overlap in question. Define $r_k' = \sum \sigma_j |z_k^j|$. A global ε-tube neighborhood around D may now be described as $r_k' \leq \varepsilon$ for all k, where $a_k \neq 0$ if ε is chosen small enough.

Write

$$r_k' = r_k \left(\sum \sigma_j |f_{0,k}^j(z_1, \ldots \widehat{k}, z_k)| + r_k g \right),$$

where g is a bounded C^∞ function. If ε is small enough, $0 < r_i' \leq \varepsilon$ will ensure $r_k = |z_k| > 0$. For each point $z_1, \ldots \widehat{k}, z_n$, $\{\varphi_k \in [0, 2\pi], r_k' = \varepsilon\}$ will parameterize a loop around the corresponding component of D, independent of the current chart.

We may cover $\partial B_\varepsilon(D)$ by sets (in our local neighborhood $U_i \cong B(R)$)

$$St_k'(R) = \{z \in B(R) \mid r_k' = \varepsilon, r_j' \geq \varepsilon \text{ if } a_j \neq 0\},$$

for all k, where $a_k \neq 0$.

We have for small r_k' and some bounded C^∞-function h:

$$r_k = r_k' \left(\left(\sum \sigma_j |f_{0,k}^j(z_1, \cdots \widehat{k}, z_k)| \right)^{-1} + r_k' h \right),$$

$$\begin{aligned}\mathrm{d}r_k =& \mathrm{d}{r_k}' \left(\sum \sigma_j |f_{0,k}^j(z_1, \cdots \overset{\widehat{k}}{} , z_n)|\right)^{-1} \\ &+ r_k' \mathrm{d}{r_k}' h \\ &+ r_k' \mathrm{d}\left(\left(\sum \sigma_j |f_{0,k}^j(z_1, \cdots \overset{\widehat{k}}{} , z_n)|\right)^{-1} + r_k' h\right).\end{aligned} \tag{16}$$

(11.6.7) It suffices to show that on each of the sets

$$St_k'(R) = \{z \in B(R) \mid r_k' = \varepsilon, r_j \geq \delta\varepsilon \text{ if } a_j \neq 0\},$$

for some small δ, the limit of the absolute integral

$$\lim_{\varepsilon \to 0} \int_{St_k'(R)} |G \wedge \mathrm{d}^c G_0 - G_0 \wedge \mathrm{d}^c G|,$$

for some (global!) representation

$$\mathfrak{g}_1 * \cdots * \mathfrak{g}_n \sim (\Omega^n, G)$$

is zero. First of all $\operatorname{supp}(\operatorname{div}(f_1)) \cap \operatorname{supp}(\operatorname{div}(f_2)) \cap \operatorname{supp}(D)$ by construction, hence we may represent

$$G = \sigma G_{1,2} \Omega^{n-2} + \mathrm{d}\,\mathrm{d}^c((1-\sigma) G_{1,2}) G',$$

where, however, σ is equal to 1 in a neighborhood of D. We may furthermore represent $\mathfrak{g}_1 * \mathfrak{g}_2 = (\Omega^2, G_{1,2})$ with

$$G_{1,2} = \sigma_1 G_1 \Omega + \mathrm{d}\,\mathrm{d}^c(\sigma_2 G_1) G_2,$$

where $\sigma_1 + \sigma_2 = 1$ is a partition of unity of the following form: The intersection of $\operatorname{div}(f_1)$ with D occurs precisely on the pre-image under the projection on the Baily-Borel compactification of a set of isolated points in the 1-dimensional boundary stratum isomorphic to some $\mathrm{M}({}^K\mathbf{H}(I))$. Choose a C^∞-function $p\sigma$ on $\mathrm{M}({}^K\mathbf{H}(I))$ with support on some disc $B(R)$ around one of these points, which is 1 on a smaller neighborhood of the point.

We may assume that in $B_\varepsilon(D)$ for very small ε, σ_1 is just the pre-image of $p\sigma$ under the projection $\mathrm{M}({}^K_{\Delta_1^\circ}\mathbf{B}) \to \mathrm{M}({}^K\mathbf{H}(I))$.

We have hence in a neighborhood of D:

$$\begin{aligned} G = \sigma_1 G_1 \Omega^{n-1} + \sigma_2 G_2 \Omega^{n-1} + (\mathrm{d}\,\sigma_2 \wedge \mathrm{d}^c G_1) G_2 \wedge \Omega^{n-2} \\ -(\mathrm{d}^c \sigma_2 \wedge \mathrm{d}\, G_1) G_2 \wedge \Omega^{n-2} + (\mathrm{d}\,\mathrm{d}^c \sigma_2) G_1 G_2 \wedge \Omega^{n-2}. \end{aligned}$$

Since in any of the neighborhoods constructed above

$$G_1 = \sum a_j \log(r_j) + \log(h_{\mathcal{E}}(s)) + \psi(z) \tag{17}$$

for some harmonic function ψ, where s is the corresponding trivializing section of $\Xi(\mathcal{E})$. Since in any case (by 11.6.10) the limit of integrals of log-log growth-forms is 0, we are reduced to show one the one hand that

$$\lim_{\varepsilon\to 0}\int_{St_k'(\varepsilon)} |\mathrm{d}\varphi_j \sigma_? G_? \Omega^{n-1}| = 0$$

and

$$\lim_{\varepsilon\to 0}\int_{St_k'(\varepsilon)} |\log(r_j)\, \mathrm{d}^c(\sigma_? G_?)\Omega^{n-1}| = 0$$

in neighborhoods of both types. Every other term in (17) yields a limit over an integral of a form of log-log type, which is zero by (11.6.10).

If $j = k$ we may rewrite r_j by means of r_k' and apply (20) of lemma (11.6.11). In the other case we apply (21) of lemma (11.6.11). (Note that $\mathrm{d}\varphi_j$ is trivially of the form $\log(r_j)$ times a $\log\log$-form.) After this, the vanishing of the limit $\varepsilon \to 0$ follows from (11.6.9).

On the other hand, we have to show

$$\lim_{\varepsilon\to 0}\int_{St_1'(\varepsilon)} |\mathrm{d}\varphi_1\, (\mathrm{d}\,\sigma_2 \wedge \mathrm{d}^c\, G_1) - \mathrm{d}^c\,\sigma_2 \wedge \mathrm{d}\, G_1 + (\mathrm{d}\,\mathrm{d}^c\,\sigma_2) G_1) \wedge G_2 \wedge \Omega^{n-2}| = 0 \tag{18}$$

$$\lim_{\varepsilon\to 0}\int_{St_1'(\varepsilon)} |\log(r_1)\, (\mathrm{d}\,\sigma_2 \wedge \mathrm{d}^c\, G_1) - \mathrm{d}^c\,\sigma_2 \wedge \mathrm{d}\, G_1 + (\mathrm{d}\,\mathrm{d}^c\,\sigma_2) G_1) \wedge \mathrm{d}^c\, G_2 \wedge \Omega^{n-2}| = 0 \tag{19}$$

(here $j = k = 1$ because there cannot be an intersection of components above $\mathrm{M}({}^K\mathbf{H}(I))$)

Claim: For (19), we may w.l.o.g. assume that G_2 is of the form $\log(h_{\mathcal{E}}(s))$, where s is the trivializing section valid around the whole neighborhood of a point on $\mathrm{M}({}^K\mathbf{H}(I))$, if we in addition show

$$\lim_{\varepsilon\to 0}\int_{St_1'(\varepsilon)} |\frac{\mathrm{d}r_1}{r_1}\, (\mathrm{d}\,\sigma_2 \wedge \mathrm{d}^c\, G_1 - \mathrm{d}^c\,\sigma_2 \wedge \mathrm{d}\, G_1 + (\mathrm{d}\,\mathrm{d}^c\,\sigma_2) G_1) \wedge \mathrm{d}^c\, G_2 \wedge \Omega^{n-2}| = 0.$$

For this, note that we constructed σ_2 (in $B_\varepsilon(D)$!) as the pre-image (by the projection $\mathrm{M}({}^{K_1}_{\Delta_1^\circ}\mathbf{B}) \to \mathrm{M}({}^K\mathbf{H}(I))$ of a function supported on a small disc around the zeros or poles of f_1 in the boundary of the Baily-Borel compactification. Let X be the pre-image of the disc in some neighborhood of the boundary in $\mathrm{M}({}^K_{\Delta_1^\circ}\mathbf{B})$. Applying the theorem of

Stokes to $X \cap \partial B_\varepsilon(D)$, we get

$$\begin{aligned}
0 = &- \int_{X\cap B_\varepsilon(D)} G_0\,(\mathrm{d}\,\sigma_2 \wedge \mathrm{d}^c\,G_1 - \mathrm{d}^c\,\sigma_2 \wedge \mathrm{d}\,G_1 + (\mathrm{d}\,\mathrm{d}^c\,\sigma_2)G_1) \\
&\quad \wedge \mathrm{d}^c\,G_2 \wedge \boldsymbol{\Omega} \wedge \Omega^{n-3} \\
&+ \int_{X\cap B_\varepsilon(D)} G_0\,(\mathrm{d}\,\sigma_2 \wedge \mathrm{d}^c\,G_1 - \mathrm{d}^c\,\sigma_2 \wedge \mathrm{d}\,G_1 + (\mathrm{d}\,\mathrm{d}^c\,\sigma_2)G_1) \\
&\quad \wedge \mathbf{d}\,\mathrm{d}^c\,G_2 \wedge \mathrm{d}^c\log(h_{\mathcal{E}}(s)) \wedge \Omega^{n-3} \\
&+ \int_{X\cap B_\varepsilon(D)} G_0 \left(-\,\mathrm{d}\,\sigma_2 \wedge \mathbf{d}\,\mathrm{d}^c\,G_1 \underbrace{-\mathbf{d}\,\mathrm{d}^c\,\sigma_2 \wedge \mathrm{d}\,G_1 + \mathrm{d}\,\mathrm{d}^c\,\sigma_2 \wedge \mathbf{d}G_1}_{=0} \right) \\
&\quad \wedge \mathrm{d}^c\,G_2 \wedge \mathrm{d}^c\log(h_{\mathcal{E}}(s)) \wedge \Omega^{n-3} \\
&+ \int_{X\cap B_\varepsilon(D)} \mathbf{d}G_0 \wedge (\mathrm{d}\,\sigma_2 \wedge \mathrm{d}^c\,G_1 - \mathrm{d}^c\,\sigma_2 \wedge \mathrm{d}\,G_1 + (\mathrm{d}\,\mathrm{d}^c\,\sigma_2)G_1) \\
&\quad \wedge \mathrm{d}^c\,G_2 \wedge \mathrm{d}^c\log(h_{\mathcal{E}}(s)) \wedge \Omega^{n-3}
\end{aligned}$$

The first line is the one we have to estimate. We may hence instead consider the remaining three lines. Here all problematic integrals are of the form

$$\int_{X\cap B_\varepsilon(D)} \log(r_1)\xi' \wedge \xi \wedge \mathrm{d}^c\log(h_{\mathcal{E}}(s)) \wedge \Omega^{n-2}$$

or

$$\int_{X\cap B_\varepsilon(D)} \frac{\mathrm{d}r_1}{r_1} \wedge \xi' \wedge \xi \wedge \mathrm{d}^c\,G_2 \wedge \mathrm{d}^c\log(h_{\mathcal{E}}(s)) \wedge \Omega^{n-3},$$

where ξ' is a smooth form generated by $r_n\mathrm{d}\varphi_n$ and $\mathrm{d}r_n$ and ξ is log-log. The latter poses no problem because $\frac{\mathrm{d}r_1}{r_1}$ is, up to terms involving $\mathrm{d}r_1'$, a smooth form (16).
Now, after the estimates (22, 23) of lemma (11.6.11) we get vanishing of the limit $\varepsilon \to 0$ by (11.6.9) again. □

(11.6.8) We will write the quantities in question in the basis

$$\mathrm{d}\varphi_1, \cdots, \mathrm{d}\varphi_n, \mathrm{d}r_1, \cdots\!\widehat{k}, \mathrm{d}r_n, \mathrm{d}r_k',$$

Only the term

$$\mathrm{d}\varphi_1 \wedge \cdots \wedge \mathrm{d}\varphi_n \wedge \mathrm{d}r_1 \wedge \cdots\!\widehat{k} \wedge \mathrm{d}r_n$$

gives a nonzero contribution.
Call a form a **(*)-form**, if it is generated by $\log(\log(r_i))^M$,

$$\mathrm{d}\varphi_1, \cdots, \mathrm{d}\varphi_n, \frac{\mathrm{d}r_1}{r_1\log(r_1)}, \cdots\!\widehat{k}, \frac{\mathrm{d}r_n}{r_n\log(r_n)}$$

and $f\mathrm{d}r_k'$ for any f, which is smooth outside D.

(11.6.9) Lemma. *For every (*)-form σ:*

$$\int_{St'_k(\varepsilon)} \sigma = O(\log(|\log \varepsilon|)^M)$$

for some M.

Proof. Easy, see e.g. [12]. The (maybe highly singular) terms $f\mathrm{d}r'_k$ play no role because terms involving $\mathrm{d}r'_k$ do not give any contribution to the integral. □

(11.6.10) Corollary. *For every log-log-growth-form ξ:*

$$\lim_{\varepsilon\to 0} \int_{St'_k(\varepsilon)} \xi = 0.$$

Proof. We have by (16)

$$\frac{1}{\log(r_k)}\mathrm{d}\varphi_i \prec \frac{1}{\log(r'_k)}\sigma,$$

and

$$\frac{\mathrm{d}r_k}{r_k\log(r_k)} \prec \frac{1}{\log(r'_k)}\sigma,$$

where the σ are of (*)-form.

If ξ has correct degree, by definition, it involves at least either $\frac{1}{\log(r_k)}\mathrm{d}\varphi_k$ or $\frac{\mathrm{d}r_k}{r_k\log(r_k)}$. Hence

$$\xi \prec \frac{1}{\log(r'_k)}\sigma,$$

for some (*)-form σ. The statement follows by (11.6.9) because, of course, log grows faster than any power of $\log - \log$. □

(11.6.11) Lemma. *If $U = B(R)$ is a neighborhood of any type and $i = 2$, or $i = 1$ if the Witt rank of $L_{\mathbb{R}}$ is 1. For any log-log-growth form ξ of rank $2(n-i)-1$, we have*

$$\xi \wedge \Omega^i \prec \frac{1}{\log(r'_k)^2}\sigma, \tag{20}$$

where σ is a (*)-form.

Also we have for any $j \neq k$

$$\xi \wedge \Omega^i \prec \frac{1}{\log(r'_k)\log(r_j)}\sigma, \tag{21}$$

where σ is another (*)-form.

If $U = B(R)$ is a neighborhood of the second type (11.6.5), we have for any log-log-growth *form* $\xi' \wedge \xi$ *of rank* $2n-5$ *where* ξ' *is a smooth form, generated by* $r_n d\varphi_n$ *and* dr_n. *where* r_n, φ_n *are the polar coordinates of the projection to a* $\mathrm{M}({}^K\mathbf{H}(I))$ *in a neighborhood of the point.*

$$d\varphi_k \wedge \xi' \wedge \xi \wedge \Omega \prec \frac{1}{\log(r'_k)}\sigma \tag{22}$$

and

$$\xi' \wedge \xi \wedge \mathrm{d}^c \log(h_{\mathcal{E}}(s)) \wedge \Omega \prec \frac{1}{\log(r'_k)^2}\sigma. \tag{23}$$

Proof. We have in any case:

$$\mathrm{d}^c \log(r_i) = \mathrm{d}\varphi_i, \tag{24}$$

$$\mathrm{d}\log(r_i) = \frac{\mathrm{d}r_i}{r_i}, \tag{25}$$

and

$$\mathrm{d}^c \log h_{\mathcal{E}}(s) = \frac{\mathrm{d}^c\, h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}, \tag{26}$$

$$\mathrm{d}\log h_{\mathcal{E}}(s) = \frac{\mathrm{d}\, h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}, \tag{27}$$

$$\Omega \sim \mathrm{d}\,\mathrm{d}^c \log h_{\mathcal{E}}(s) = \frac{\mathrm{d}\, h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} \frac{\mathrm{d}^c\, h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} + \frac{\mathrm{d}\,\mathrm{d}^c\, h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}. \tag{28}$$

In a neighborhood *of the first type* (11.6.4), we may write

$$h_{\mathcal{E}}(s) = \sum_{i,j<m} \langle \lambda_i, \lambda_j \rangle \log(r_i)\log(r_j) + \sum_{i<m} \log(r_i)\psi_i(z) + \psi_0(z),$$

where ψ_i are harmonic functions and ψ_0 is smooth. Here the λ_i are linearly independent and $\langle \lambda_i, \lambda_j \rangle > 0$ if $i \neq j$. Hence for $i \neq j$ always

$$h_{\mathcal{E}}(s) \gg \log(r_i)\log(r_j) \tag{29}$$

and if $Q_L(\lambda_i) > 0$

$$h_{\mathcal{E}}(s) \gg \log(r_i)^2. \tag{30}$$

Hence

$$\begin{aligned}\mathrm{d}^c\, h_{\mathcal{E}}(s) = & \log(r_k)(\sum_{j<m} \langle \lambda_k, \lambda_j \rangle \mathrm{d}\varphi_j + \mathrm{d}^c\, \psi_k) + \sum_{i<m} \mathrm{d}\varphi_i \psi_i \\ & + \sum_{i<m, j<m, i\neq k} \langle \lambda_i, \lambda_j \rangle \log(r_i)\mathrm{d}\varphi_j + \sum_{i<m, i\neq k} \log(r_i)\, \mathrm{d}^c\, \psi_k + \mathrm{d}^c\, \psi_0\end{aligned} \tag{31}$$

$$\begin{aligned}\mathrm{d}\, h_{\mathcal{E}}(s) =&\log(r_k)(\sum_{j<m}\langle\lambda_k,\lambda_j\rangle\frac{\mathrm{d}r_j}{r_j}+\mathrm{d}\,\psi_k)+\sum_{i<m}\frac{\mathrm{d}r_j}{r_j}\psi_i\\ &+\sum_{i<m,j<m,i\neq k}\langle\lambda_i,\lambda_j\rangle\log(r_i)\frac{\mathrm{d}r_j}{r_j}+\sum_{i<m,i\neq k}\log(r_i)\,\mathrm{d}\,\psi_k+\mathrm{d}\,\psi_0 \qquad (32)\\ \mathrm{d}\,\mathrm{d}^c\, h_{\mathcal{E}}(s) =&\mathrm{d}\varphi_k(\sum_{j<m}\langle\lambda_k,\lambda_j\rangle\frac{\mathrm{d}r_j}{r_j}+\mathrm{d}\,\psi_k)+\sum_{i<m}\frac{\mathrm{d}r_j}{r_j}\,\mathrm{d}^c\,\psi_i\\ &+\sum_{i<m,j<m,i\neq k}\langle\lambda_i,\lambda_j\rangle\mathrm{d}\varphi_i\frac{\mathrm{d}r_j}{r_j}+\sum_{i<m,i\neq k}\mathrm{d}\varphi_i\,\mathrm{d}\,\psi_k+\mathrm{d}\,\mathrm{d}^c\,\psi_0. \qquad (33)\end{aligned}$$

We now substitute $\frac{\mathrm{d}r_k}{r_k}$ by a form of shape

$$f\frac{\mathrm{d}r'_k}{r'_k}+\xi,$$

where ξ is smooth. This is possible by means of formula (16).
First assume $i=2$. Ω^2 is proportional to

$$\frac{\mathrm{d}\,\mathrm{d}^c\,h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}\wedge\frac{\mathrm{d}\,\mathrm{d}^c\,h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} \qquad (34)$$

$$+\frac{\mathrm{d}\,h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}\frac{\mathrm{d}^c\,h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}\wedge\frac{\mathrm{d}\,\mathrm{d}^c\,h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}. \qquad (35)$$

In (34), multiplying out the expression (33) squared, we have for every occurring summand S

$$S\prec\frac{1}{\log(r_k)^2}\sigma\prec\frac{1}{\log(r'_k)^2}\sigma,$$

where σ is a (*)-form, using the estimate (29).
In (35), multiplying out the product of (31-33), we have

$$S\prec\frac{1}{\log(r_k)^2}\sigma\prec\frac{1}{\log(r'_k)^2}\sigma,$$

using the estimate (29) again, except for the summands of the form

$$\log(r_k)\log(r_k)(\sum_{j<m}\langle\lambda_k,\lambda_j\rangle\mathrm{d}\varphi_j+\mathrm{d}^c\,\psi_k)\wedge(\sum_{j<m}\langle\lambda_k,\lambda_j\rangle\frac{\mathrm{d}r_j}{r_j}+\mathrm{d}\,\psi_k)\wedge S, \qquad (36)$$

where S is any summand of (33).
But now in the expression $\xi\wedge\Omega^2$ either occurs a $\frac{\mathrm{d}\varphi_k}{\log(r_k)}$ from ξ, hence the estimate (20) is true, or a $r_k\mathrm{d}\varphi_k$ occurs in S which satisfies a much stronger estimate, or a $\mathrm{d}\varphi_k$ occurs in S. It occurs, however, multiplied with $(\sum_{j<m}\langle\lambda_k,\lambda_j\rangle\frac{\mathrm{d}r_j}{r_j}+\mathrm{d}\,\psi_k)$ so (36) is zero in that case.
The estimate (21) is more easy and left to the reader.

Now assume $i = 1$ and Witt rank not 2. Then, by the estimate (30), already every summand S in (31-33) divided by $h_{\mathcal{E}}(s)$ satisfies

$$S \prec \frac{1}{\log(r_i)}\sigma \prec \frac{1}{\log(r_i')}\sigma.$$

The estimates (22, 23) do not involve neighborhoods of the first type.
In a neighborhood *of the second type* (11.6.5), we may write

$$h_{\mathcal{E}}(s) = \log(r_1)\psi_1(z_n) + \psi_0(z),$$

where ψ_1 is harmonic and depends only on z_n and ψ_0 is smooth.
Clearly in this case:

$$h_{\mathcal{E}}(s) \gg \log(r_1). \tag{37}$$

Also

$$\begin{aligned} \mathrm{d}^c\, h_{\mathcal{E}}(s) &= \log(r_1)\, \mathrm{d}^c\, \psi_1(z_n) + \mathrm{d}\varphi_1 \psi_1(z_n) + \mathrm{d}^c\, \psi_0(z), \\ \mathrm{d}\, h_{\mathcal{E}}(s) &= \log(r_1)\, \mathrm{d}\, \psi_1(z_n) + \frac{\mathrm{d}r_1}{r_1}\psi_1(z_n) + \mathrm{d}\, \psi_0(z), \\ \mathrm{d}\, \mathrm{d}^c\, h_{\mathcal{E}}(s) &= \mathrm{d}\varphi_1\, \mathrm{d}\, \psi_1(z_n) + \frac{\mathrm{d}r_1}{r_1}\, \mathrm{d}^c\, \psi_1(z_n) + \mathrm{d}\, \mathrm{d}^c\, \psi_0(z). \end{aligned}$$

We now substitute again $\frac{\mathrm{d}r_1}{r_1}$ by a form of shape

$$f\frac{\mathrm{d}r_1'}{r_1'} + \xi,$$

where ξ is smooth. This is possible by means of formula (16).
Here, for (20), we may argue exactly as before. (21) is vacuous in this type of neighborhood. For (22): Write once again Ω as

$$\frac{\mathrm{d}\, \mathrm{d}^c\, h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} \tag{38}$$

$$+\frac{\mathrm{d}^c\, h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} \wedge \frac{\mathrm{d}\, \mathrm{d}^c\, h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}. \tag{39}$$

In (38), using (37), we have for any summand

$$S \prec \frac{1}{\log(r_i')}\sigma.$$

In (39), using (37) again, we get the same for any summand except possibly for

$$\log(r_1)^2\, \mathrm{d}^c\, \psi_1(z_n) \wedge \mathrm{d}\, \psi_1(z_n),$$

which cancels, however, with ξ' because $\mathrm{d}^c\, \psi_1(z_n) \wedge \mathrm{d}\, \psi_1(z_n)$ is proportional to $r_n \mathrm{d}\varphi_n \wedge$

$\mathrm{d}r_n$.
For (22): Consider:

$$\mathrm{d}^c(h_{\mathcal{E}}(s))\Omega = \frac{\mathrm{d}^c\, h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)} \wedge \frac{\mathrm{d}\,\mathrm{d}^c\, h_{\mathcal{E}}(s)}{h_{\mathcal{E}}(s)}. \tag{40}$$

Using (37) again, we now have for any summand

$$S \prec \frac{1}{\log(r_i')^2}\sigma,$$

except possibly for

$$\mathrm{d}\varphi_i\,\mathrm{d}\,\psi_1(z_n)\log(r_1)\,\mathrm{d}^c\,\psi_1(z_n),$$

which cancels with ξ' because $\mathrm{d}^c\,\psi_1(z_n) \wedge \mathrm{d}\,\psi_1(z_n)$ is proportional to $r_n\mathrm{d}\varphi_n \wedge \mathrm{d}r_n$. □

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 31. März 2010 Fritz Hörmann