# Effective divisors on moduli spaces of pointed stable curves

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**Abstract**

This thesis investigates various questions concerning the birational geometry of the moduli spaces $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$, with a focus on the computation of effective divisor classes.

In Chapter 2 we define, for any $n$-tuple $\underline{d}$ of integers summing up to $g-1$, a geometrically meaningful divisor on $\overline{\mathcal{M}}_{g,n}$ that is essentially the pullback of the theta divisor on a universal Jacobian variety under an Abel-Jacobi map. It is a generalization of various kinds of divisors used in the literature, for example by Logan to show that $\overline{\mathcal{M}}_{g,n}$ is of general type for all $g \geq 4$ as soon as $n$ is big enough. We compute the class of this divisor and show that for certain choices of $\underline{d}$ it is irreducible and extremal in the effective cone of $\overline{\mathcal{M}}_{g,n}$.

Chapter 3 deals with a birational model $X_6$ of $\overline{\mathcal{M}}_6$ that is obtained by taking quadric hyperplane sections of the degree 5 del Pezzo surface. We compute the class of the big divisor inducing the birational map $\overline{\mathcal{M}}_6 \dashrightarrow X_6$ and use it to derive an upper bound on the moving slope of $\overline{\mathcal{M}}_6$. Furthermore we show that $X_6$ is the final non-trivial space in the log minimal model program for $\overline{\mathcal{M}}_6$. We also give a few results on the unirationality of Weierstraß loci on $\overline{\mathcal{M}}_{g,1}$, which for $g = 6$ are related to the del Pezzo construction used to construct the model $X_6$.

Finally, Chapter 4 focuses on the case $g = 0$. Castravet and Tevelev introduced combinatorially defined *hypertree divisors* on $\overline{\mathcal{M}}_{0,n}$ that for $n = 6$ generate the effective cone together with boundary divisors. We compute the class of the hypertree divisor on $\overline{\mathcal{M}}_{0,7}$, which is unique up to permutation of the marked points. We also give a geometric characterization of it that is analogous to the one given by Keel and Vermeire in the $n = 6$ case.

## Zusammenfassung


Diese Arbeit untersucht verschiedene Fragen hinsichtlich der birationalen Geometrie der Modulräume $\overline{\mathcal{M}}_g$ und $\overline{\mathcal{M}}_{g,n}$, mit besonderem Augenmerk auf der Berechnung effektiver Divisorklassen.

In Kapitel 2 definieren wir für jedes $n$-Tupel ganzer Zahlen $\underline{d}$, die sich zu $g-1$ aufsummieren, einen geometrisch bedeutsamen Divisor auf $\overline{\mathcal{M}}_{g,n}$, der im Wesentlichen durch Zurückziehen des Thetadivisors einer universellen Jacobi-Varietät mittels einer Abel-Jacobi-Abbildung erhalten wird. Er ist eine Verallgemeinerung verschiedener in der Literatur verwendeten Arten von Divisoren, beispielsweise durch Logan im Beweis, dass $\overline{\mathcal{M}}_{g,n}$ für alle $g \geq 4$ von allgemeinem Typ ist, sobald $n$ groß genug ist. Wir berechnen die Klasse dieses Divisors und zeigen, dass er für bestimmte $\underline{d}$ irreduzibel und extremal im effektiven Kegel von $\overline{\mathcal{M}}_{g,n}$ ist.

Kapitel 3 beschäftigt sich mit einem birationalen Modell $X_6$ von $\overline{\mathcal{M}}_6$, das durch quadrische Hyperebenenschnitte auf der del-Pezzo-Fläche vom Grad 5 erhalten wird. Wir berechnen die Klasse des großen Divisors, der die birationale Abbildung $\overline{\mathcal{M}}_6 \dashrightarrow X_6$ induziert, und benutzen sie, um eine obere Schranke an die bewegliche Steigung von $\overline{\mathcal{M}}_6$ zu erhalten. Wir zeigen außerdem, dass $X_6$ der letzte nichttriviale Raum im log-minimalen Modellprogramm für $\overline{\mathcal{M}}_6$ ist. Weiterhin geben wir einige Resultate bezüglich der Unirationalität der Weierstraßorte auf $\overline{\mathcal{M}}_{g,1}$. Für $g = 6$ hängen diese mit der del-Pezzo-Konstruktion zusammen, die in Kapitel 3 benutzt wurde, um das Modell $X_6$ zu konstruieren.

Kapitel 4 konzentriert sich schließlich auf den Fall $g = 0$. Castravet and Tevelev führten auf $\overline{\mathcal{M}}_{0,n}$ kombinatorisch definierte *Hyperbaumdivisoren* ein, die für $n = 6$ zusammen mit den Randdivisoren den effektiven Kegel erzeugen. Wir berechnen die Klasse des Hyperbaumdivisors auf $\overline{\mathcal{M}}_{0,7}$, der bis auf Permutation der markierten Punkte eindeutig ist. Wir geben außerdem eine geometrische Charakterisierung für ihn an, die zu der von Keel und Vermeire für den Fall $n = 6$ gegebenen analog ist.

# Contents

# List of Figures

# Chapter 1

# Introduction

## 1.1 Moduli spaces of curves

One of the most basic problems in algebraic geometry is the classification of smooth algebraic curves up to isomorphism. It was long known that the topological structure of a curve is determined by its genus, which provides a *discrete* invariant that does not vary in families. In contrast, in 1857 Riemann published a calculation [75] that showed that the algebraic structure of a curve of given genus $g \geq 2$ depends on $3g - 3$ *continuous* parameters that he called moduli.

These parameters are naturally interpreted as coordinates on some kind of space, whose points correspond to isomorphism classes of smooth curves of genus $g$, and in fact algebraic geometers cheerfully proceeded proving lots of properties about this space, despite the fact that there was no proof its existence yet, much less an actual construction. This situation persisted for almost a century, before Teichmüller in 1940 [82] and Mumford in 1965 [74] gave constructions of the moduli space $\mathcal{M}_g$ as respectively an analytic and an algebraic variety.

In the latter setting, $\mathcal{M}_g$ is an irreducible quasi-projective algebraic variety defined over $\mathbb{Z}$ with finite quotient singularities. The presence of singularities stems from the fact that while the general curve of genus $g \geq 4$ has no non-trivial automorphisms, some specific ones do, and the automorphism group of such a curve acts on its space of first-order infinitesimal deformations. The quotient of the deformation space by this action is a local model for $\mathcal{M}_g$ around the chosen curve, giving rise to a finite quotient singularity. For $g = 2$ and 3, the principle remains the same, with the proviso that hyperelliptic involutions for these genera do not actually produce singularities.

The existence of non-trivial automorphisms is also the reason that the spaces $\mathcal{M}_g$ fail to qualify as fine moduli spaces, i. e. they do not represent the natural moduli functor that to a scheme $B$ associates the set of isomorphism classes of families of smooth genus $g$ curves over $B$. On the other hand, such a family over $B$ does indeed give rise to a moduli map $B \to \mathcal{M}_g$, and this correspondence is bijective at least if $S = \mathrm{Spec}(k)$ for a field $k$. Moreover, $\mathcal{M}_g$ is universal with respect to these two properties, making it what is called a *coarse moduli space*.

Both of the above mentioned difficulties can be solved simultaneously (at the expense of geometric vividness) by the introduction of *stacks* as initiated by Deligne and Mumford in 1969 [23]. This construction basically boils down to keeping track of automorphisms instead of factoring them out, and produces an object that is smooth (in an appropriate sense) and represents the moduli functor. The fact that the locus of curves with "bad" automorphisms (i. e. those giving rise to singularities) always has codimension greater than 1 in $\mathcal{M}_g$ will allow us for the most part to skim these issues, as we are almost exclusively concerned with divisorial calculations.

A natural extension of the classification problem for smooth curves is the inclusion of marked points on the curves forming part of the data to be classified. The construction runs through almost unchanged, and the resulting spaces (or stacks) that classify smooth genus $g$ curves with $n$ distinct ordered marked points are denoted by $\mathcal{M}_{g,n}$. They are irreducible of dimension $3g-3+n$. The forgetful functors between the various moduli problems induce forgetful maps $\pi_n : \mathcal{M}_{g,n} \to \mathcal{M}_{g,n-1}$ that drop one of the marked points. Considered as a morphism of stacks, the map $\pi_1 : \mathcal{M}_{g,1} \to \mathcal{M}_g$ is actually the universal family (the family that has the identity map of $\mathcal{M}_g$ as its moduli map), and for the associated coarse moduli spaces the same is true if one restricts to the locus of pointed curves without non-trivial automorphisms. The introduction of marked points has the additional advantage of allowing a unified treatment of the hitherto neglected cases $g = 0$ and 1, where the automorphism group of the generic curve is infinite: Requiring at least three marked points for genus 0 and one marked point in genus 1 rigidifies the problem enough to enable Mumford's construction to go through as before.

A further obvious defect of the space $\mathcal{M}_g$ is that it is not compact (that is to say, only quasi-projective). Embedding $\mathcal{M}_g$ via a very ample line bundle and taking the closure in the resulting projective space gives a compactification that turns out not to have good modular properties. However, in the same article [23], Deligne and Mumford showed that one obtains a natural modular compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ by slightly enlarging the class of the parameterized objects from smooth curves to so-called stable curves, where the only allowed singularities are ordinary nodes.

**Definition 1.1.1.** A curve $C$ is *stable* if it is nodal and $\omega_C$ is ample. A pointed curve $(C; p_1, \ldots, p_n)$ is stable if $C$ is nodal, the $p_j$ are smooth points of $C$, and $\omega_C(\sum_{i=1}^n p_i)$ is ample.

In both cases, the ampleness condition is equivalent to postulating that the curve has only finitely many automorphisms (where automorphisms of pointed curves are required to leave the marked points invariant). The key property of stable curves that gives rise to the compactification $\overline{\mathcal{M}}_g$ is expressed in the *stable reduction theorem* (see Figure 1.1):

**Theorem 1.1.2.** *Let $\mathscr{C} \to B^*$ be a flat family of stable curves over a punctured base scheme $B^* = B \setminus \{0\}$. Then there is a unique stable curve C such that there exists a finite base change $B' \to B$ fully ramified over $0$ and a flat family $\mathscr{C}' \to B'$ of stable curves, with the property that $\mathscr{C}$ and $\mathscr{C}'$ agree over $B' \times_B B^*$ and the fiber of $\mathscr{C}'$ over $0$ is isomorphic to C.*

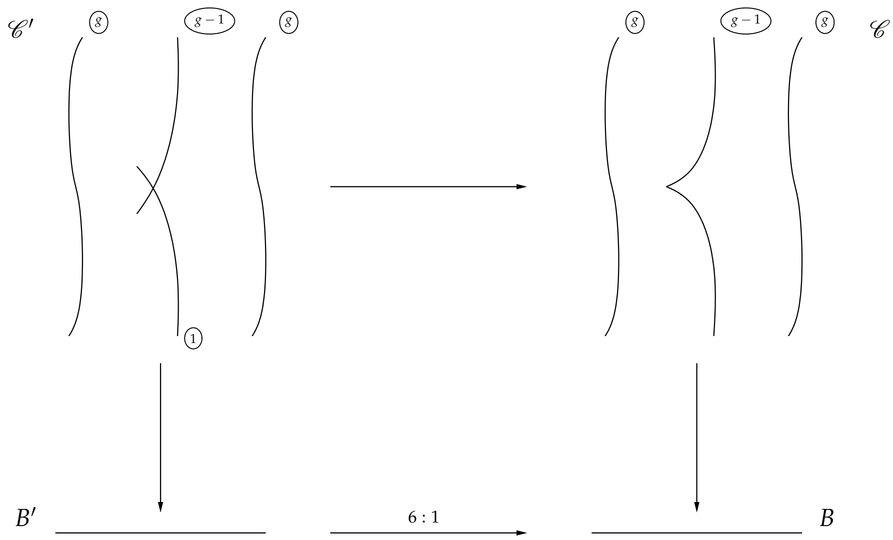


Figure 1.1: Stable reduction for smooth curves acquiring a cusp
(circled numbers denote geometric genus)

This is essentially saying that the moduli stack is separated and proper. The possible need for a finite base change reiterates the fact that not every morphism $B \to \overline{\mathcal{M}}_g$ comes from a family of curves.

As before, the same construction also works for pointed stable curves. The spaces $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$ are then again coarse moduli spaces and have associated stacks that represent the respective moduli functors. Their boundaries have codimension 1 and form a normal crossings divisor, making these spaces accessible to log geometric methods (see Section 1.6).

## 1.2 Birational classification of varieties

Much of the content of this thesis is concerned, directly or indirectly, with the birational geometry of moduli spaces of curves, or subsets of them. One of the coarsest possible birational invariants that still carries significant geometric meaning is the *Kodaira dimension*, which we proceed to define.

**Definition 1.2.1.** Let $X$ be an algebraic variety, $\mathscr{L}$ a line bundle on $X$ and

$$R(X, \mathscr{L}) := \bigoplus_{d \geq 0} H^0(X, \mathscr{L}^{\otimes d})$$

its ring of sections. Then the *Iitaka dimension* of $\mathscr{L}$ is defined as

$$\kappa(X, \mathscr{L}) := \begin{cases} -\infty & \text{if } R(X, \mathscr{L}) = 0, \\ \dim \operatorname{Proj} R(X, \mathscr{L}) & \text{else.} \end{cases}$$

The line bundle $\mathscr{L}$ is called *big* if $\kappa(X, \mathscr{L}) = \dim(X)$, i. e. if the map induced by $\mathscr{L}^{\otimes d}$ is birational onto its image for $d \gg 0$.

The Iitaka dimension of any line bundle obviously fulfills $-\infty \leq \kappa(X, \mathscr{L}) \leq \dim(X)$. It can be alternatively determined as the minimal $\kappa$ such that $h^0(X, \mathscr{L}^{\otimes d}) = \mathcal{O}(d^\kappa)$ as $d \to \infty$. As shown in [64], bigness of a line bundle depends only on its numerical equivalence class. Two equivalent chararacterizations of big line bundles are given in the following:

**Proposition 1.2.2.** *A line bundle $\mathscr{L}$ on $X$ is big if and only if*

- $\mathscr{L} \in \operatorname{int}(\operatorname{Eff}(X))$, *where* $\operatorname{Eff}(X)$ *is the cone of effective line bundles on $X$, or equivalently*
- $\mathscr{L} = \mathscr{A} \otimes \mathscr{E}$, *where $\mathscr{A}$ is ample and $\mathscr{E}$ is effective.*

The case $\mathscr{L} = K_X$ has a special designation:

**Definition 1.2.3.** The *Kodaira dimension* of a smooth algebraic variety $X$ is defined as

$$\kappa(X) := \kappa(X, K_X).$$

The Kodaira dimension of a singular variety is defined to be that of any desingularization of it.

The Kodaira dimension is a birational invariant (in particular, it does not depend on the desingularization chosen) and fulfills $-\infty \leq \kappa(X) \leq \dim X$. If $\kappa(X) = \dim(X)$, the variety $X$ is said to be *of general type*. While in a sense the Kodaira dimension is a very coarse invariant of the birational equivalence class of a variety, it nevertheless has a certain influence on how well one can parameterize subschemes of $X$.

**Definition 1.2.4.** A variety $X$ over an algebraically closed field is called

- *rational* if there is a birational map $\mathbb{P}^{\dim X} \dashrightarrow X$,
- *unirational* if there is a dominant map $\mathbb{P}^N \dashrightarrow X$ for some $N$,

- *rationally connected* if for two general points in $X$ there exists a rational curve that connects them,
- *uniruled* if through a general point of $X$ there passes a rational curve.

It is known that each of these properties implies the next, and that uniruled varieties have Kodaira dimension $-\infty$. The reverse implication of the last statement is known in dimension up to 3 [69], and it is known that $X$ is uniruled under the slightly stronger hypothesis that $K_X$ is not pseudo-effective [9].

In the next two sections, we will review what is known about the birational classification of the moduli spaces $\mathcal{M}_g$ and $\mathcal{M}_{g,n}$. As an illustration of the consequences that birational properties of moduli spaces have on the parameterized objects, we mention the following (Theorem 1.5.1 specifies in which cases the hypothesis is actually satisfied):

**Proposition 1.2.5.** *Suppose $\mathcal{M}_g$ is of general type. Then any surface $S$ containing a general curve $C$ of genus $g$ such that $h^0(S, \mathcal{O}_S(C)) \geq 2$ is birational to $C \times \mathbb{P}^1$.*

*Proof.* Choose a pencil $\ell \subseteq |\mathcal{O}_S(C)|$ containing $C$. Since the general element of $\ell$ is smooth, it induces a rational map $\mathbb{P}^1 \dashrightarrow \mathcal{M}_g$, whose image is a rational curve passing through $[C]$. As $C$ is general and $\mathcal{M}_g$ is not uniruled, the map has to be constant. Blowing up the base points of $\ell$ and removing singular fibers, we find that $S$ is birational to $C \times \mathbb{P}^1$. □

## 1.3 The Picard groups of $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$

As remarked in Section 1.1, the Deligne-Mumford compactifications $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$ are projective with divisorial boundary, giving us both a well-defined intersection theory as well as a plethora of naturally defined divisors. However, before we can start computing Picard groups, a few words are in order regarding the relation between the fine moduli stacks and the coarse moduli spaces, as well as the issue of singularities of the latter.

Since $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$ have only finite quotient singularities, any Weil divisor is actually $\mathbb{Q}$-Cartier. As there are both naturally defined line bundles on these spaces as well as plenty of geometrically interesting codimension 1 subloci on these spaces, we will unify their treatment by allowing $\mathbb{Q}$-coefficients from now on.

The relationship between $\mathbb{Q}$-divisor classes on the moduli space and those on the associated stack is beautifully explained in [47, Chapter 3.D]. A rational divisor class $\gamma$ on the moduli stack is a prescription which to any flat family $\varphi \colon \mathscr{C} \to B$ of stable curves functorially associates a rational divisor class $\gamma(\varphi)$ on the base $B$. Given a class $[D]$ on the moduli space, we get such a class on the stack by setting $\gamma(\varphi) := m_\varphi^*([D])$, where $m_\varphi \colon B \to \overline{\mathcal{M}}_g$ is the moduli map associated to the family $\varphi$. Conversely, given a class on the stack we can take its value on some finite covering of $\overline{\mathcal{M}}_g$ that has a universal family (such coverings were constructed in [66]) and push forward this class to $\overline{\mathcal{M}}_g$, dividing by the degree of the covering. These processes are inverse to each other.

As noted in [47], there is another way to associate a class $\sigma$ on the moduli stack to a codimension 1 sublocus $\Sigma \subseteq \overline{\mathcal{M}}_g$. Let $\varphi : \mathscr{C} \to B$ be a family with smooth 1-dimensional base $B$ such that the image of the associated moduli map $m_\varphi : B \to \overline{\mathcal{M}}_g$ does not lie completely inside $\Sigma$ (it is easy to see that specifying $\sigma(\varphi)$ for families of this type already suffices to define a class on the moduli stack). Then to the family $\varphi$ we associate the class $\sigma(\varphi)$ of the divisor on $B$ consisting of those points $b \in B$ that satisfy $[\varphi^{-1}(b)] \in \Sigma$, counted with the appropriate multiplicity. This multiplicity is computed as follows: Let $C_b := \varphi^{-1}(b)$, let $\mathrm{Def}(C_b)$ be the versal deformation space of $C_b$, and let $U \subseteq B$ be a small neighbourhood of $b$ over which $\varphi$ is a pullback of the versal family $\Phi\colon \mathscr{C} \to \mathrm{Def}(C_b)$, i. e. in the diagram

$$\begin{array}{ccccc} \varphi^{-1}(U) & \longrightarrow & \mathscr{C} & & \\ \varphi\big\downarrow & & \big\downarrow\Phi & & \\ U & \xrightarrow{\psi} & \mathrm{Def}(C_b) & \xrightarrow{m_\Phi} & \overline{\mathcal{M}}_g \end{array}$$

the left side is a fiber square. Then we define the multiplicity of $b$ in $\sigma(\varphi)$ to be the multiplicity of $m_\Phi^*(\Sigma)$ at the point $\psi(b)$. Unless otherwise noted, we will usually be referring to this construction when talking about the class on the moduli stack associated to a codimension 1 sublocus of $\overline{\mathcal{M}}_g$. Letting $[C] \in \Sigma$ be a generic curve, we then have

$$m_\varphi^*([\Sigma]) = |\mathrm{Aut}(C)| \cdot \sigma(\varphi),$$

as the versal deformation space of $C$ is an $|\mathrm{Aut}(C)|$-fold cover of a neighbourhood of $[C]$ in $\overline{\mathcal{M}}_g$.

We now have all the necessary equipment to describe the Picard groups of $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$. Since nodal curves are Gorenstein, any flat family $\varphi\colon X \to B$ of stable curves has a relative dualizing sheaf $\omega_\varphi$. Its pushforward $\varphi_*\omega_\varphi$ is a vector bundle of rank $g$ on $B$ called the *Hodge bundle* of $\varphi$. We denote its first Chern class by $\lambda(\varphi) := c_1(\varphi_*\omega_\varphi)$. Since all the operations involved in the definition of $\lambda$ are functorial, this defines a divisor class on the moduli stack. Harer [45] showed that $\mathrm{Pic}(\mathcal{M}_g)$ is in fact infinite cyclic, generated by the class $\lambda$.

Next, as already mentioned in Section 1.1, the boundary of $\overline{\mathcal{M}}_g$ is divisorial. By analyzing the deformation theory of nodal curves, one sees that it consists of $\lfloor g/2 \rfloor + 1$ irreducible components $\Delta_0, \ldots, \Delta_{\lfloor g/2 \rfloor}$. The general element of $\Delta_0$ is an irreducible 1-nodal curve of geometric genus $g-1$, while the general element of $\Delta_i$ for $i \geq 1$ consists of two irreducible components of genera $i$ and $g-i$ meeting at a node. The latter curves are said to be *of compact type* as their Jacobians are compact (in contrast to curves in $\Delta_0$, whose Jacobians always have a toric component). Following general usage, we denote by $\delta_0, \ldots, \delta_{\lfloor g/2 \rfloor}$ the divisor classes on the moduli stack associated to the $\Delta_i$ via evaluation on families, i. e. by the second of the two methods described above. We then have the following result:

**Theorem 1.3.1.** *For $g \geq 3$, the Picard group of $\overline{\mathcal{M}}_g$ is freely generated by $\lambda$ and the $\delta_i$, $0 \leq i \leq \lfloor g/2 \rfloor$.*

This theorem was proven by Arbarello and Cornalba [5] in characteristic 0, where it is true even with integer coefficients. It was extended to positive characteristic by Moriwaki [70].

For $g = 2$, the situation is a bit special, as $\mathcal{M}_2$ is in fact affine, so $\lambda$ is a $\mathbb{Q}$-linear combination of boundary divisors. Calculating degrees on two test families, one finds the single relation

$$\lambda = \frac{1}{10}\delta_0 + \frac{1}{5}\delta_1 \tag{1.1}$$

in genus 2 (see e. g. [71]).

If we additionally have marked points, there are further natural classes to consider. A family of pointed stable curves consists of a flat family $\varphi\colon \mathscr{C} \to B$ together with $n$ sections $\sigma_1, \ldots, \sigma_n\colon B \to \mathscr{C}$ such that for every $b \in B$, the fiber $\big(\varphi^{-1}(b); \sigma_1(b), \ldots, \sigma_n(b)\big)$ is a pointed stable curve. Given such a family, the pullback of the relative dualizing sheaf $\omega_\varphi$ via the sections $\sigma_i$ gives rise to a divisor class $\psi_i(\varphi) := c_1(\sigma_i^*\omega_\varphi)$. Again by functoriality, this defines divisor classes $\psi_1, \ldots, \psi_n$ on the moduli stack.

Turning now to geometrically defined subloci, in the presence of marked points the boundary divisors parameterizing reducible curves break up into irreducible components according to the distribution of the markings. More concretely, for any $i$ with $0 \leq i \leq \lfloor g/2 \rfloor$ and any $S \subseteq [n] := \{1, \ldots, n\}$, we have a divisor $\Delta_{i:S}$ whose general point corresponds to a reducible 1-nodal curve with components of genera $i$ and $g - i$, with the markings in $S$ lying on the former. If $i = 0$ we require that $|S| \geq 2$, as we do not get a stable curve otherwise. It will be convenient to also introduce the notation $\Delta_{i:S} := \Delta_{g-i:[n]\setminus S}$ for $\lceil g/2 \rceil \leq i \leq g$, where we require $|S| \leq n-2$ if $i = g$. Note there is an redundancy of notation if $g$ is even, as then $\Delta_{g/2:S} = \Delta_{g/2:[n]\setminus S}$. For these $i$ and $S$, we again denote by $\delta_{i:S}$ the classes on the moduli stack associated to $\Delta_{i:S}$.

The sum of all boundary classes $\delta_i$ on $\overline{\mathcal{M}}_g$, or $\delta_0$ and $\delta_{i:S}$ on $\overline{\mathcal{M}}_{g,n}$, is usually denoted by $\delta$. Similarly, we let $\psi$ be the sum of all the $\psi_j$ on $\overline{\mathcal{M}}_{g,n}$. As a matter of convention, when we sum over boundary divisor classes, we will take a summation range like $\sum_{i,S}$ to mean that only admissible combinations of $i$ and $S$ occur, and that every boundary divisor is used exactly once.

We then have the following description, also proved in [5] and again also true with $\mathbb{Z}$-coefficients:

**Theorem 1.3.2.** *For $g \geq 3$, the Picard group of $\overline{\mathcal{M}}_{g,n}$ is freely generated by $\lambda$, $\delta_0$, the $\delta_{i:S}$, and the $\psi_j$ with $1 \leq j \leq n$.*

For $g = 2$, we still have the single $\lambda$-relation (1.1), which pulls back under the forgetful maps. On $\overline{\mathcal{M}}_{1,1}$ there are additional relations

$$\lambda = \frac{1}{12}\delta_0 = \psi_1,$$

which can be seen by analyzing a pencil of plane cubics. The first of these is pulled back without change to the higher $\overline{\mathcal{M}}_{1,n}$, while the second one transforms to

$$\psi_j = \frac{1}{12}\delta_0 + \sum_{j\in S}\delta_{0:S}$$

for $1 \leq j \leq n$. These are all the relations in genus 1.

For $g = 0$, the situation can be made much more explicit, and a complete description of the Chow ring of $\overline{\mathcal{M}}_{0,n}$ has been given by Keel [61]. First of all, we have $\lambda = \delta_0 = 0$, as the Hodge bundle of a family of rational curves is 0 and an irreducible nodal curve always has positive genus (i. e. stable pointed rational curves are always marked trees of $\mathbb{P}^1$'s). Next, as $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, we have the relations $\delta_{0:12} = \delta_{0:13} = \delta_{0:14}$, which pull back to

$$\sum_{\substack{i,j\in S\\ k,l\notin S}}\delta_{0:S} = \sum_{\substack{i,k\in S\\ j,l\notin S}}\delta_{0:S} = \sum_{\substack{i,l\in S\\ j,k\notin S}}\delta_{0:S} \tag{1.2}$$

for any $\{i,j,k,l\} \subseteq [n]$. Finally, one can see that

$$\psi_j = \sum_{k=2}^{n-2}\frac{(n-k)(n-k-1)}{(n-1)(n-2)}\sum_{\substack{j\in S\\ |S|=k}}\delta_{0:S}$$

using e. g. the Kapranov construction of $\overline{\mathcal{M}}_{0,n}$ described in Section 1.9. Keel then gives the following characterization of the Chow ring of $\overline{\mathcal{M}}_{0,n}$:

**Theorem 1.3.3** ([61]). *One has*

$$A^*(\overline{\mathcal{M}}_{0,n}) \cong \mathbb{Z}\left\langle \delta_{0:S} \;\middle|\; S \subseteq [n] \text{ with } 2 \leq |S| \leq n-2 \right\rangle \Big/ I,$$

*where $I$ is the ideal generated by*

- *the relations* (1.2) *for any subset $\{i,j,k,l\} \subseteq [n]$,*
- *the notational artefacts $\delta_{0:S} = \delta_{0:[n]\setminus S}$ for any $S$, and*
- *the relation $\delta_{0:S} \cdot \delta_{0:T} = 0$ for any pair $S$, $T \subseteq [n]$ that does* not *satisfy one of the four inclusions $S \subseteq T$, $T \subseteq S$, $S \subseteq [n] \setminus T$ or $T \subseteq [n] \setminus S$.*

## 1.4 Divisor classes on $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$

Having now in hand a basis for the Picard groups of $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$ for all genera, we go on to describe some naturally defined divisors on these spaces. On the one hand, they are important for the problem of determining the Kodaira dimensions of these spaces, while on the other hand they give an idea of the kind of problems investigated in this thesis.

The first interesting divisor that comes to mind is the canonical divisor, by which we mean the unique extension of the canonical divisor of the smooth part to the whole space. As explained beatifully in [34], it turns out that it is in fact easier to compute the canonical class of the moduli stack, once one has defined what that is.

The deformation theory of a stable curve $C$ canonically identifies its first-order deformation space as

$$\mathrm{Def}_1(C) \cong H^0(C, \Omega_C \otimes \omega_C)^\vee,$$

where $\omega_C$ is the dualizing sheaf and $\Omega_C$ is the sheaf of Kähler differentials on $C$. Thus the cotangent space to the moduli stack at $[C]$ can be identified with $H^0(C, \Omega_C \otimes \omega_C)$, at least when $C$ is automorphism-free. Accordingly, one defines the canonical class of the moduli stack by associating to a family $\varphi\colon \mathscr{C} \to B$ the class

$$K(\varphi) := c_1\big(\varphi_*(\Omega_\varphi \otimes \omega_\varphi)\big)$$

on $B$. By applying the Grothendieck-Riemann-Roch formula to the universal curve $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$, Mumford was able to compute the class of $K$ in terms of the basis given in Theorem 1.3.1:

**Theorem 1.4.1** ([48, §2]). *The canonical class of the moduli stack is given by*

$$K = 13\lambda - 2\delta.$$

Due to the fact that curves in $\Delta_1$ have an extra automorphism of order 2 (namely, the involution on the elliptic tail fixing the point of attachment), the natural map from the moduli stack to the moduli space is simply ramified along this locus. Using a stack version of the Riemann-Hurwitz formula, one can deduce from this the following result:

**Corollary 1.4.2.** *For $g \geq 4$, the canonical class of $\overline{\mathcal{M}}_g$ is given by*

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{\lfloor g/2 \rfloor}.$$

In genus 3 the map from the stack to the space is additionally ramified along the locus of hyperelliptic curves, which gives $K_{\overline{\mathcal{M}}_3} = 4\lambda - \delta_0$. For $g = 2$ one can directly compute that $K_{\overline{\mathcal{M}}_2} = -\frac{11}{10}\delta_0 - \frac{32}{5}\delta_1$.

By pulling back via forgetful maps, one can also compute the canonical class in the pointed case:

**Corollary 1.4.3** ([65, Theorem 2.6])**.** *For $g \geq 4$, the canonical class of $\overline{\mathcal{M}}_{g,n}$ is given by*

$$K_{\overline{\mathcal{M}}_{g,n}} = 13\lambda + \sum_{j=1}^{n} \psi_j - 2\delta_0 - 2\sum_{i,S} \delta_{i:S} - \sum_{S} \delta_{1:S}.$$

Two families of geometrically natural divisors were introduced by Harris and Mumford in [48] and Eisenbud and Harris in [29] to show the results about the Kodaira dimension of $\overline{\mathcal{M}}_g$ that we review in Section 1.5. The first are the so-called *Brill-Noether divisors* consisting of curves having an unexpected $g^r_d$. More precisely, we have the following theorem first formulated by Brill and Noether [85] and later rigorously proven by Griffiths and Harris [42]:

**Theorem 1.4.4.** *A general curve $[C] \in \mathcal{M}_g$ has a $g^r_d$ if and only if*

$$\rho(g, r, d) := g - (r+1)(g - d + r) \geq 0.$$

Thus if the parameters $g$, $r$ and $d$ are chosen such that $\rho(g, r, d) < 0$, the general genus $g$ curve does not admit a $g^r_d$. On the other hand, the locus $\mathcal{M}^r_{g,d}$ of curves in $\mathcal{M}_g$ that do possess a $g^r_d$ can locally be written as the degeneracy locus of a map between vector bundles, and from this description one can show that the codimension of $\mathcal{M}^r_{g,d}$ in $\mathcal{M}_g$ is at most $-\rho(g, r, d)$. In particular, if $\rho(g, r, d) = -1$, both results taken together imply that $\mathcal{M}^r_{g,d}$ is a divisor on $\mathcal{M}_g$. In [30] it is shown to be irreducible. The class of its closure in $\overline{\mathcal{M}}_g$ was computed in [29] up to a rational multiple $c$ as

$$\left[\overline{\mathcal{M}}^r_{g,d}\right] = c\left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} i(g-i)\delta_i\right). \tag{1.3}$$

Astonishingly, apart from $c$ the coefficients do not depend on $r$ and $d$, i. e. the classes of all Brill-Noether divisors of a fixed genus lie on a single ray in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$.

Naturally, the condition $\rho(g, r, d) = -1$ can only be fulfilled if $g + 1$ is composite. In particular, there are spaces $\mathcal{M}_g$ for even $g$ on which there are no Brill-Noether divisors. To compensate for that defect, Eisenbud and Harris introduced another family of divisors for parameters $r$ and $d$ such that $\rho(g, r, d) = 0$. The *Petri map* of a linear system $\ell = (\mathscr{L}, V)$ on a curve $C$ is the product map of sections

$$\mu_0(\ell)\colon V \otimes H^0(C, K_C \otimes \mathscr{L}^\vee) \to H^0(C, K_C).$$

The following theorem was first proven by Gieseker [41], with simpler proofs given later by Eisenbud and Harris [26] by means of limit linear series, and by Lazarsfeld [63] using the geometry of K3 surfaces:

**Theorem 1.4.5.** *For a general curve $[C] \in \mathcal{M}_g$ and any linear series $\ell$ of type $g^r_d$ on $C$, the Petri map $\mu_0(\ell)$ is injective.*

Thus for any $r$ and $d$ the locus of curves having a $g^r_d$ with non-injective Petri map is a proper sublocus of $\mathcal{M}_g$. In general, it has multiple components of varying dimension, but in the special case where $g$ is even, $r = 1$ and $d = g/2 + 1$, it can identified with the branch locus of the finite map $\mathcal{H}_{d,3g} \to \mathcal{M}_g$ from the Hurwitz scheme that forgets the covering and retains only the source curve (see Section 1.8 for details on the Hurwitz scheme). It is thus a divisor, whose closure on $\overline{\mathcal{M}}_g$ we denote by $\overline{\mathcal{GP}}_g$. Its class was also computed in [29] as

$$[\overline{\mathcal{GP}}_g] = 2\frac{(2d-4)!}{d!(d-2)!}\left((6d^2+d-6)\lambda - d(d-1)\delta_0 - \sum_{i=1}^{\lfloor g/2\rfloor} b_i\delta_i\right), \qquad (1.4)$$

where for $1 \le i \le \lfloor g/2 \rfloor$

$$\begin{aligned} b_i =& -i(i-2)(2d-3)(3d-2) + \frac{3}{2}i(i-1)(d-2)(4d-3) \\ &+ (i-1)(i-2)\frac{(g-2)!}{(d-1)!(d-2)!} \\ &- \sum_{k=1}^{\lfloor i/2-1\rfloor} 2(i-2k-1)\frac{(2k)!(g-2k-2)!}{(k+1)!k!(d-k)!(d-k-1)!}. \end{aligned}$$

The role of this divisor in computing the Kodaira dimension of $\mathcal{M}_g$ is explained in Section 1.5. For the case of $g = 6$, see Chapter 3.

Moving on to the world of pointed curves, the first interesting divisor that comes to mind is the *Weierstraß divisor* $\mathcal{W}_g$ consisting of 1-pointed curves $(C; p)$ such that $p$ is a Weierstraß point on $C$. The class of its closure was computed by Cukierman [22] to be

$$[\overline{\mathcal{W}}_g] = \binom{g+1}{2}\psi_1 - \lambda - \sum_{i=1}^{\lfloor g/2\rfloor}\binom{g-i+1}{2}\delta_i.$$

A *pointed Brill-Noether divisor* is any divisor on $\overline{\mathcal{M}}_{g,1}$ consisting of pointed curves $(C; p)$ such that $C$ has a $g^r_d$ with ramification sequence $(\alpha_0, \dots, \alpha_r)$ at $p$, where the *adjusted Brill-Noether number*

$$\rho(g, r, d; \alpha) := \rho(g, r, d) - \sum_{i=0}^{r}\alpha_i$$

is equal to $-1$ (see Definition 1.7.1 for the notion of a ramification sequence). In [30], Eisenbud and Harris showed that while the classes of pointed Brill-Noether divisors do not lie on a single ray of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ as in the unpointed case, they all lie in the cone spanned by $[\overline{\mathcal{W}}_g]$ and the pullback of the Brill-Noether divisor on $\overline{\mathcal{M}}_g$.

An obvious generalization of the Weierstraß divisor to the case of multiply pointed curves was considered by Logan [65]: For $n$ non-negative integers $a_1, \ldots, a_n$ summing up to $g$, let $D_{g;a_1,\ldots,a_n}$ denote the divisor of $n$-pointed curves $(C; p_1, \ldots, p_n)$ with the property that $h^0(C, \sum_{j=1}^n a_j p_j) \geq 2$. For $n = 1$ and $a_1 = g$, this is just the Weierstraß divisor. By partially computing the class of the closure of $D_{g;a_1,\ldots,a_n}$ in $\overline{\mathcal{M}}_{g,n}$, Logan was able to prove results about the Kodaira dimension of these spaces for $n$ large enough (see Section 1.5). In Chapter 2, we will compute the class of a divisor that generalizes Logan's result to the case where the sum of the $a_i$ can be larger than $g$, and we also fix some points in a second fiber of $\left|\sum a_i p_i\right|$.

## 1.5 The Kodaira dimension of $\mathcal{M}_{g,n}$

In this section, we will review what is known about the birational classification of the spaces $\mathcal{M}_g$ and $\mathcal{M}_{g,n}$. In general, their Kodaira dimensions increase with $g$ and $n$, but very diverse techniques are necessary to give upper and lower bounds.

Considering first the unpointed case, the state of the art is summarized in the following comprehensive list:

**Theorem 1.5.1.** *The moduli space $\mathcal{M}_g$ is*

- *rational for $2 \leq g \leq 6$,*
- *unirational for $7 \leq g \leq 14$,*
- *rationally connected for $g = 15$,*
- *uniruled for $g = 16$,*
- *of Kodaira dimension $\geq 2$ for $g = 23$, and*
- *of general type for $g = 22$ and $g \geq 24$.*

We will focus on a few of these results. The most classical one is due to Severi [79], who in 1915 proved the unirationality of $\mathcal{M}_g$ for $2 \leq g \leq 10$. His method was to represent the generic curve of genus $g$ by a planar model that has $\delta$ nodes and no other singularities, and whose degree $d$ is minimal with respect to the condition that $\rho(g, 2, d) \geq 0$. Severi then showed that that the nodes can be chosen in general position precisely for $g \leq 10$, i. e. for these genera the map that assigns to a planar model its set of nodes maps dominantly onto the configuration space of $\delta$ points in $\mathbb{P}^2$. The incidence correspondence that parameterizes plane curves with their nodes thus maps with rational fibers onto a rational space, hence is rational itself. By the choice of $d$, it also maps dominantly onto $\mathcal{M}_g$.

The first qualitative improvement of Severi's result occurred when Igusa showed the rationality of $\mathcal{M}_2$ in 1960 [56], followed in the late 80's by Shepherd-Barron and Katsylo, who in [80], [81], [59] and [60] proved rationality for $g = 4, 6, 5$ and 3, respectively (in historical order). Quantitavely, the bounds where unirationality is known were extended using modern methods by Sernesi ($g = 12$ [78]), Chang and Ran ($g = 11, 13$ [14]) and Verra ($g = 14$ [84]). Finally, the spaces $\mathcal{M}_{15}$ and $\mathcal{M}_{16}$ were shown to have Kodaira dimension $-\infty$ by Chang and Ran ([15], [16]). More recently, $\mathcal{M}_{15}$ was shown to be rationally connected by Bruno and Verra [10], while Farkas [34, Theorem 2.7] observed that in view of [9], the results in genus 16 actually imply uniruledness of $\mathcal{M}_{16}$.

On the other end of the spectrum, Harris and Mumford first showed in 1982 that $\mathcal{M}_g$ is of general type for odd $g \geq 25$ [48] and even $g \geq 40$ [46] using the theory of admissible covers. This essentially forced them to restrict their attention to pencils, so it was not surprising that the advent of the theory of limit linear series (discussed in Section 1.7), which works for linear systems of any dimension, enabled Eisenbud and Harris in 1987 to extend this result to all $g \geq 24$ [29]. The genus 22 case was established in 2010 by Farkas [34] using Koszul divisors. Finally, the lower bounds $\kappa(\mathcal{M}_{23}) \geq 0, 1, 2$ were shown in [48], [29] and [31], respectively.

The canonical method for showing that some $\mathcal{M}_g$ is of general type had been outlined by Harris and Mumford and has stayed the same ever since. First of all, they showed that, in classical language, the singularities of $\overline{\mathcal{M}}_g$ impose no *adjoint conditions*, that is to say if $\nu\colon \widetilde{\mathcal{M}}_g \to \overline{\mathcal{M}}_g$ is any desingularization, the pullback of pluricanonical forms induces an isomorphism

$$\nu^*\colon H^0\left(\overline{\mathcal{M}}_g, K_{\overline{\mathcal{M}}_g}^{\otimes k}\right) \xrightarrow{\cong} H^0\left(\widetilde{\mathcal{M}}_g, K_{\widetilde{\mathcal{M}}_g}^{\otimes k}\right)$$

for any $k \geq 0$ (recall that in Section 1.4 we defined $K_{\overline{\mathcal{M}}_g}$ as the unique extension of the canonical bundle of the smooth part to the whole space). Thus we can compute the Kodaira dimension of $\overline{\mathcal{M}}_g$, which by Definition 1.2.3 is the Kodaira dimension of any desingularization of it, as the Iitaka dimension of $K_{\overline{\mathcal{M}}_g}$.

Now suppose that $D$ is an effective divisor on $\overline{\mathcal{M}}_g$ whose class is given by

$$[D] = a\lambda - \sum_{i=0}^{\lfloor g/2 \rfloor} b_i \delta_i.$$

It is known that any $D$ that is the closure of an effective divisor on $\mathcal{M}_g$ has such an expression with $a, b_i \geq 0$. The fact to note then is that as long as the inequalities

$$\frac{a}{b_i} \leq \frac{13}{2} \text{ for } i \neq 1, \quad \text{and } \frac{a}{b_1} \leq \frac{13}{3}. \tag{1.5}$$

are fulfilled, we can write $K_{\overline{\mathcal{M}}_g} \equiv \alpha\lambda + \beta\,[D] + [E]$, with $E$ an effective divisor supported on the boundary, and positive numbers $\alpha$, $\beta$. Since $\lambda$ is big, the same then holds for $K_{\overline{\mathcal{M}}_g}$ by Proposition 1.2.2, hence $\overline{\mathcal{M}}_g$ is of general type in this situation.

Thus the problem is reduced to the quest for divisors whose classes satisfy (1.5). From the formulas (1.3) and (1.4) in Section 1.4, one sees that the Brill-Noether and Gieseker-Petri divisors fulfill this criterion for $g \geq 24$. The construction of *Koszul divisors* (whose definition in fact generalizes those of both the Brill-Noether and Gieseker-Petri divisors) enabled Farkas to conclude in the same way that $\mathcal{M}_{22}$ is also of general type.

Logan [65] extended these techniques to the pointed case, and by constructing suitable divisors on $\overline{\mathcal{M}}_{g,n}$ found for any $g$ with $4 \leq g \leq 22$ a number $n_g$ such that $\overline{\mathcal{M}}_{g,n}$ is of general type as soon as $n \geq n_g$. Almost all of these bounds were subsequently improved by Farkas using again Koszul divisors (see [33] or [32, Theorem 4.3] for a consolidated list).

## 1.6 The log minimal model program for $\overline{\mathcal{M}}_g$

One of the most rewarding yet also the most ambitious projects in current higher-dimensional algebraic geometry is the execution of the *minimal model program*, which aims to find, for any reasonably singular variety $X$, a rational map $\varphi\colon X \dashrightarrow X'$ such that $K_{X'}$ is nef and $\varphi$ is either birational (if $\kappa(X) \geq 0$) or a fibration with Fano type fibers (in case $\kappa(X) = -\infty$). It was shown in [8] that when $X$ is smooth and of general type, the canonical ring

$$R(X, K_X) := \bigoplus_{d \geq 0} H^0(X, K_X^{\otimes d})$$

is in fact finitely generated, so the *canonical model* $X' := \operatorname{Proj} R(X, K_X)$ has the required properties (and its canonical divisor is even ample). This takes care of the cases where $g = 22$ or $g \geq 24$ (see Theorem 1.5.1). However, other techniques are needed for smaller values of $g$.

In keeping with the general drift in higher-dimensional algebraic geometry, the log minimal model program (log MMP) for $\overline{\mathcal{M}}_g$ as initiated by Hassett and Keel [51] focuses instead on the *log canonical divisors* $K_{\overline{\mathcal{M}}_g} + \alpha\delta$, for rational $\alpha \in [0, 1]$, and the log canonical models

$$\overline{\mathcal{M}}_g(\alpha) := \operatorname{Proj} R(\overline{\mathcal{M}}_g, K_{\overline{\mathcal{M}}_g} + \alpha\delta).$$

Improving on a result of Mumford, Cornalba and Harris [20, Theorem (1.3)] showed that the divisor $a\lambda - b\delta$ is ample on $\overline{\mathcal{M}}_g$ if and only if $a/b > 11$. By this criterion, the log canonical divisor $K_{\overline{\mathcal{M}}_g} + \alpha\delta$ is ample for $9/11 < \alpha \leq 1$. Thus for these values one has $\overline{\mathcal{M}}_g(\alpha) \cong \overline{\mathcal{M}}_g$. On the other hand, $\overline{\mathcal{M}}_g(0)$ is the conjectural canonical model, so the sequence of models $\overline{\mathcal{M}}_g(\alpha)$ as $\alpha$ decreases from 1 to 0 is expected to yield information about the minimal model program for $\overline{\mathcal{M}}_g$. Conjecturally, all the in-between steps have modular interpretations as parameter spaces for curves with increasingly bad singularities, and satisfying certain stability conditions. A precise list of predictions for critical $\alpha$-values, together with the type of singularities which appear at each step, was obtained by heuristic methods in [1].

In the case of pointed stable curves, the additional presence of the $\psi$ divisor (which is also big) yields a further degree of freedom when choosing a big line bundle in order to construct a birational model. Given $n$ fixed 1-pointed curves of genera $g_1, \dots, g_n$, there is a natural map $i : \overline{\mathcal{M}}_{g,n} \hookrightarrow \overline{\mathcal{M}}_{g'}$ given by attaching the fixed curves at the marked points, where $g' = g + g_1 + \cdots + g_n$. Under this map, the class $K_{\overline{\mathcal{M}}_{g'}} + \alpha\delta$ pulls back to $K_{\overline{\mathcal{M}}_{g,n}} + \alpha\delta + (1-\alpha)\psi$, so constructing the models corresponding to these divisors with $\alpha$ going from 1 to 0 amounts to analyzing the effect that the log MMP for $\overline{\mathcal{M}}_{g'}$ has on the image of $i$.

Although the log minimal model programs for $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$ are far from completed in general, some low genus cases have been explicitly worked out, namely those of $\overline{\mathcal{M}}_2$, $\overline{\mathcal{M}}_3$, $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{1,n}$ (see [38] and references therein). Moreover, the first three steps of the log MMP for $\overline{\mathcal{M}}_g$ are the same in every genus and have been worked out by Hassett and Hyeon in [52] and [53], and very recently by Alper, Fedorchuk, Smyth and van der Wyck in [2]. The first is a divisorial contraction in which elliptic tails (i. e. genus 1 components that are attached to the rest of the curve at only one point) are replaced by cusps, while the second is a flip that contracts elliptic bridges (genus 1 components attached at two points) and introduces tacnodes instead. The third step is again a flip, replacing genus 2 tails attached at Weierstraß points by $A_4$-singularities (see Figure 1.2).

On the other end of the spectrum, since the spaces $\overline{\mathcal{M}}_g$ with $g \leq 16$ have Kodaira dimension $-\infty$, their log canonical models become empty for $\alpha \ll 1$. In genus 4, the last non-trivial step in the log MMP was worked out by Fedorchuk [37]. He considers the space $V$ of curves of bidegree (3, 3) on $\mathbb{P}^1 \times \mathbb{P}^1$, upon which acts the linearly reductive group $G = \mathrm{SL}(2) \times \mathrm{SL}(2) \rtimes \mathbb{Z}/2\mathbb{Z}$. This group contains $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$, and moreover $\mathcal{O}_V(1)$ has a natural linearization with respect to it. Since the generic curve of genus 4 has exactly two $g^1_3$'s and no non-trivial automorphisms, the GIT quotient $V^{ss} /\!\!/ G$ is a birational model of $\overline{\mathcal{M}}_4$. By an explicit analysis of the GIT stability of various types of (3,3)-curves on $\mathbb{P}^1 \times \mathbb{P}^1$, Fedorchuk shows the following:

**Theorem 1.6.1.** *The log canonical model $\overline{\mathcal{M}}_4(\alpha)$ is*

- *isomorphic to $V^{ss} /\!\!/ G$ for $8/17 < \alpha \leq 29/60$,*
- *a point for $\alpha = 8/17$, and*
- *empty for $\alpha \leq 8/17$.*

*The birational map $\overline{\mathcal{M}}_4 \dashrightarrow \overline{\mathcal{M}}_4(29/60)$ contracts $\Delta_1$ and $\overline{\mathcal{GP}}_4$ to points, while the hyperelliptic locus is flipped to the closure of the locus of curves with an $A_8$ singularity.*

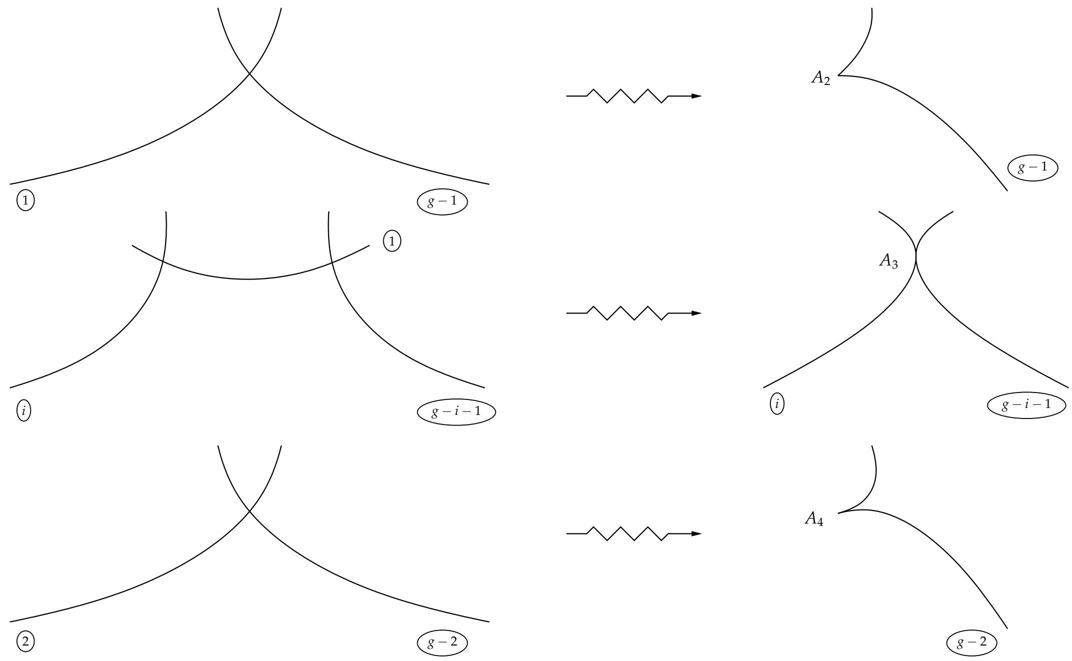


Figure 1.2: The first three steps in the log minimal model program for $\overline{\mathcal{M}}_g$ (circled numbers denote geometric genus)

A more general construction using variation of GIT linearizations on the space of (2, 3)-complete intersections in $\mathbb{P}^3$ was given in [11] and enabled the authors to describe all the spaces $\overline{\mathcal{M}}_4(\alpha)$ for $\alpha \leq 5/9$.

Likewise, the genus 5 case can be treated by considering nets of quadrics in $\mathbb{P}^4$, whose intersection is generically a canonically embedded genus 5 curve. Such nets are parameterized by the Grassmannian $\mathbb{G} = \mathbb{G}(3, 15)$ of 3-dimensional subspaces of $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$. The action of $\mathrm{SL}(5)$ on $\mathbb{P}^4$ induces an action on $\mathbb{G}$, and since any automorphism of a canonically embedded curve comes from an automorphism of the ambient space, the quotient $\mathbb{G}^{ss} /\!\!/ \mathrm{SL}(5)$ is a birational model of $\overline{\mathcal{M}}_5$. It was studied by Fedorchuk and Smyth [39], and they showed the following holds in analogy to the genus 4 case:

**Theorem 1.6.2.** *The log canonical model $\overline{\mathcal{M}}_5(\alpha)$ is*

- *isomorphic to $\mathbb{G}^{ss} /\!\!/ \mathrm{SL}(5)$ for $3/8 < \alpha \leq 14/33$,*
- *a point for $\alpha = 3/8$, and*
- *empty for $\alpha \leq 3/8$.*

*The birational map $\overline{\mathcal{M}}_5 \dashrightarrow \overline{\mathcal{M}}_5(14/33)$ is a divisorial contraction mapping $\Delta_1$ and $\Delta_2$ respectively to the locus of cuspidal curves and curves with a rational tail attached at an $A_5$ singularity, while the trigonal divisor is contracted to a point.*

In Chapter 3 we follow in Fedorchuk's footsteps and describe the final non-trivial log minimal model of $\overline{\mathcal{M}}_6$. Note that in one respect this is actually easier than the genus 4 and 5 cases, as the general canonical curve of genus 6 can be obtained as a hypersurface section of a surface whose automorphism group is finite (see Section 3.1). Thus one does not need to use GIT when constructing the model. However, this is the first time one needs to deal with genus 3 tails, which are much harder to obtain in families. For example, stable reduction of a generic $A_6$ or $A_7$ singularity does not yield a generic genus 3 curve (only hyperelliptic ones arise in this fashion, as shown in [49]). This constitutes one of the main obstructions to constructing suitable test curves in Section 3.3.

## 1.7 Limit linear series

As is known from classical algebraic geometry, almost any geometric property of an algebraic curve can be phrased in terms of linear systems on it. We quickly give the relevant definitions:

**Definition 1.7.1.** Let $C$ be a smooth curve. A *linear series* (or *linear system*) of degree $d$ and dimension $r$ (in short, a $g^r_d$) on $C$ is given by a pair $\ell = (\mathscr{L}, V)$, where $\mathscr{L}$ is a line bundle of degree $d$ on $C$ and $V \subseteq H^0(C, \mathscr{L})$ is a subspace of projective dimension $r$. The *vanishing sequence*

$$a^\ell(p) = \left(0 \leq a^\ell_0(p) < \cdots < a^\ell_r(p) \leq d\right)$$

of $\ell$ at a point $p \in C$ is the set $\left\{\mathrm{ord}_p(\sigma) \,\middle|\, \sigma \in V\right\}$ of vanishing orders of sections of $\ell$, arranged in ascending order. The *ramification sequence* of $\ell$ at $p$ is the sequence

$$\alpha^\ell(p) = \left(0 \leq \alpha^\ell_0(p) \leq \cdots \leq \alpha^\ell_r(p) \leq d - r\right),$$

where $\alpha^\ell_i(p) := a^\ell_i(p) - i$.

When the construction of the moduli space $\overline{\mathcal{M}}_g$ by Deligne and Mumford made it clear that stable curves are the appropriate modular limits for families of smooth curves, the question quickly became what should be similar limiting objects for linear systems on smooth curves. Definition 1.7.1 a priori also works for singular curves, but the resulting objects bear in general no direct relation to linear series on nearby smooth curves. If $\varphi\colon \mathscr{C} \to B$ is a family of generically smooth curves with a singular special fiber, the relative Picard scheme of $\varphi$ in general needs to be neither universally closed nor separated in a neighbourhood of the special fiber. That is to say, a line bundle on the generic fiber may not extend to a locally free sheaf on the singular curve, and if it does, it may do so in more than one way.

The first step towards a more adequate notion was taken by Beauville [7], whose notion of *admissible double coverings* was subsequently generalized to arbitrary degrees by Harris and Mumford [48].

**Definition 1.7.2.** Let $C$ be a connected nodal curve and $(B; p_1, \ldots, p_n)$ a stable $n$-pointed genus 0 curve. A map $\pi : C \to B$ is an *admissible covering* of degree $d$ if

- $\pi^{-1}(B_{\mathrm{reg}}) = C_{\mathrm{reg}}$ and $\pi^{-1}(B_{\mathrm{sing}}) = C_{\mathrm{sing}}$,
- when restricted to $C_{\mathrm{reg}}$, $\pi$ becomes a covering of degree $d$, simply branched at the points $p_j$ and unbranched elsewhere, and
- at every node of $C$, the map $\pi$ has the same ramification index when restricted to either one of the two branches meeting there.

Harris and Mumford showed [48] that for $r = 1$, admissible covers are indeed the right limits of $g^1_d$'s on smooth curves:

**Theorem 1.7.3.** *Let $C$ be a stable curve. Then $[C] \in \overline{\mathcal{M}}^1_{g,d}$ if and only if there is a nodal curve $C'$ that admits an admissible covering of degree $d$ and is stably equivalent to $C$.*

Here *stably equivalent* means that $C$ is obtained from $C'$ by contracting all rational components on which $\omega_{C'}$ is not ample. For the notation $\overline{\mathcal{M}}^r_{g,d}$ we refer to Section 1.4. The advantage of using admissible coverings is provided by the fact that this criterion actually works for all stable curves. On the other hand, one has to look at all possible curves that are stably equivalent to $C$, and of course the theory only works for 1-dimensional linear systems.

The theory of *limit linear series* was developed by Eisenbud and Harris [27] as a generalization of the theory of admissible coverings to the case of higher-dimensional linear systems. It gives a clearer picture also of the 1-dimensional case and dispenses with the need for looking at stably equivalent curves. On the other hand, the theory only works for stable curves of *compact type*, i. e. those whose Jacobian is compact, or equivalently whose dual graph is a tree. This makes it necessary to deal with curves in $\Delta_0$ separately.

**Definition 1.7.4.** Let $C$ be a nodal curve of compact type with irreducible components $C_1, \ldots, C_s$, and $r$, $d$ two natural numbers. A *limit* $g^r_d$ on $C$ is a collection $\ell$ of linear series $\ell_i = (\mathscr{L}_i, V_i)$ of degree $d$ and dimension $r$ on each component $C_i$, satisfying the compatibility conditions

$$a^{\ell_i}_m(\nu) + a^{\ell_j}_{r-m}(\nu) \geq d, \qquad m = 0, \ldots, r,$$

for each node $\nu$ at which the components $C_i$ and $C_j$ meet. The $\ell_i$ are called the *aspects* of $\ell$. A *section* of $\ell$ is a collection $\sigma = (\sigma_1, \ldots, \sigma_s)$ of sections $\sigma_i \in V_i$ satisfying the compatibility conditions

$$\mathrm{ord}_\nu(\sigma_i) + \mathrm{ord}_\nu(\sigma_j) \geq d, \qquad m = 0, \ldots, r,$$

for each node $\nu$ at which $C_i$ and $C_j$ meet. If $p \in C$ is a smooth point, the *vanishing sequence* and *ramification sequence* of $\ell$ at $p$ and the *vanishing order* of a section $\sigma$ of $\ell$ at $p$ are respectively defined to be $a^{\ell}(p) := a^{\ell_i}(p)$, $\alpha^{\ell}(p) := \alpha^{\ell_i}(p)$ and $\operatorname{ord}_p(\sigma) := \operatorname{ord}_p(\sigma_i)$, where $C_i$ is the component of $C$ on which $p$ lies.

It is shown in [27, Section 2 and Proposition 3.1] that limit linear series are indeed adequate limiting objects for ordinary linear series on smooth curves:

**Theorem 1.7.5.** *If a nodal curve of compact type lies in the closure of the locus of curves admitting a $g^r_d$, then it admits a limit $g^r_d$. For $r = 1$, the converse is also true.*

Both statements remain true even if one prescribes fixed vanishing sequences at points specializing to smooth points on the nodal curve. We formulate the latter explicitly for later use in Chapter 2:

**Theorem 1.7.6.** *Let $(C; p_1, \ldots, p_n)$ be an $n$-pointed stable curve of compact type, and for $i = 1, \ldots, n$ let*

$$a^{(i)} = (0 \leq a^{(i)}_0 < a^{(i)}_1 \leq d)$$

*be vanishing sequences. Then $C$ admits a limit $g^1_d$ with vanishing sequences $a^{(i)}$ at $p_i$ if and only if there is a family*

$$(\pi\colon \mathscr{C} \to B; \sigma_1, \ldots, \sigma_n\colon B \to \mathscr{C})$$

*of pointed stable curves with central fiber $(C; p_1, \ldots, p_n)$, whose generic fiber $C_b$ is smooth and admits a $g^1_d$ with vanishing sequences $a^{(i)}$ at $\sigma_i(b)$.*

## 1.8 Hurwitz spaces

We introduce here the basic techniques in the theory of Hurwitz spaces, which we will need in Section 2.6. These are parameter spaces of degree $d$ coverings from a genus $g$ curve to $\mathbb{P}^1$ with prescribed branching behaviour. They were first introduced by Hurwitz [55]; our treatment follows Fulton [40].

**Definition 1.8.1.** Fix $g \geq 0$ and $d \geq 1$ and set $w := 2g + 2d - 2$. The *Hurwitz space* $\mathcal{H}_{d,w}$ is a scheme whose geometric points parameterize equivalence classes of degree $d$ coverings $\varphi\colon C \to \mathbb{P}^1$ that are simply ramified at $w$ points, with $C$ a smooth genus $g$ curve. A covering $\varphi' : C' \to \mathbb{P}^1$ is considered equivalent to $\varphi$ if there exists an isomorphism $\rho\colon C \to C'$ with $\varphi = \varphi' \circ \rho$.

For any $g$ and $d$ the Hurwitz scheme exists and is smooth (this was shown by Hurwitz over $\mathbb{C}$ and by Fulton over $\operatorname{Spec} \mathbb{Z}\,[1/d!]$). It comes equipped with a finite unramified morphism $\delta : \mathcal{H}_{d,w} \to \mathbb{P}^w \setminus \Delta$ that associates to $\varphi$ its set of branch points. Here $\mathbb{P}^w$ is considered as the parameter space of unordered $w$-tuples of points in $\mathbb{P}^1$, and $\Delta$ is the discriminant hypersurface.

**Theorem 1.8.2** (Hurwitz [55] and Clebsch [18]). *The space $\mathcal{H}_{d,w}$ is connected.*

*Sketch of proof.* Let $B \in \mathbb{P}^w \setminus \Delta$ be a fixed configuration of unordered points on $\mathbb{P}^1$, and choose a base point $b \in \mathbb{P}^1 \setminus B$. Then by the Riemann existence theorem, the elements of the fiber $\delta^{-1}(B)$ are in bijective correspondence with equivalence classes of homomorphisms $\xi\colon \pi_1(\mathbb{P}^1 \setminus B, b) \to S_d$ having transitive image. Two such homomorphisms $\xi$ and $\xi'$ are considered equivalent if they differ by an inner automorphism of $S_d$, i. e. if there exists a $\sigma \in S_d$ such that $\xi(\gamma) = \sigma^{-1}\xi'(\gamma)\sigma$ for all $\gamma \in \pi_1(\mathbb{P}^1 \setminus B, b)$. Choosing generators for $\pi_1(\mathbb{P}^1 \setminus B, b)$ (and thereby also fixing an ordering of $B$), this set is in turn in bijection with $A_{d,w}/S_d$, where

$$A_{d,w} := \Big\{(\tau_1, \dots, \tau_w) \in (S_d)^w \,\Big|\, \text{the } \tau_i \text{ are transpositions generating a transitive subgroup of } S_d \text{ and fulfill } \tau_1 \cdot \ldots \cdot \tau_w = 1\Big\},$$

and $S_d$ acts on $A_{d,w}$ by simultaneous conjugation, i. e.

$$\sigma \cdot (\tau_1, \dots, \tau_w) := (\sigma^{-1}\tau_1\sigma, \dots, \sigma^{-1}\tau_w\sigma).$$

The idea is now to analyze the monodromy action of $\pi_1(\mathbb{P}^w \setminus \Delta, B)$ on $\delta^{-1}(B)$ and show that it is transitive. Hurwitz [55] computed the action of certain generators (nowadays called *braid moves*) of $\pi_1(\mathbb{P}^w \setminus \Delta, B)$ on $A_{d,w}$, and Clebsch [18] showed that any $w$-tuple of transpositions can be brought into a normal form independent of the $\tau_i$ by moves of this sort. Since $\mathbb{P}^b \setminus \Delta$ is connected, so is then $\mathcal{H}_{d,w}$. □

As it pertains to what follows, we should remark that by the same topological construction as above, the set $A_{d,w}$ itself (without quotienting out by $S_d$) also has a modular meaning: For fixed $B \in \mathbb{P}^w \setminus \Delta$ and $b \in \mathbb{P}^1 \setminus B$ it parameterizes equivalence classes of tuples $(\varphi, \psi)$, where $\varphi\colon C \to \mathbb{P}^1$ is a simply branched degree $d$ covering with branch locus $B$ as before, and $\psi\colon \varphi^{-1}(b) \xrightarrow{\sim} [d]$ is a *marking* of the points in the special fiber. Two such tuples are considered equivalent if there is an equivalence between the coverings that is compatible with the markings.

Hurwitz spaces parameterizing simply branched coverings can be generalized by allowing one *special fiber* where higher order branching may occur. Fix $B$ and $b$ as before, as well as one further distinct marked point $c \in \mathbb{P}^1$, and choose a generating set of $\pi_1(\mathbb{P}^1 \setminus (B \cup \{c\}), b)$ consisting of simple loops $\gamma_1, \dots, \gamma_w$ around the points of $B$. Then the set

$$A^{\sigma}_{d,w} := \Big\{(\tau_1, \dots, \tau_w) \in (S_d)^w \,\Big|\, \text{the } \tau_i \text{ are transpositions generating a transitive subgroup of } S_d \text{ and fulfill } \tau_1 \cdot \ldots \cdot \tau_w = \sigma\Big\}$$

parameterizes equivalence classes of tuples $(\varphi, \psi)$, where $\varphi$ is a degree $d$ covering that is simply branched over $B$ and has a special fiber over $c$, and $\psi$ is an identification of $\varphi^{-1}(b)$ with $[d]$ such that the loop $\gamma_1 \dots \gamma_w$ around $c$ induces the permutation $\sigma$ on $\varphi^{-1}(b)$ under this identification. In particular, $\varphi$ ramifies over $c$ according to the partition of $d$ corresponding to the conjugacy class of $\sigma$.

We let $\mathcal{H}^{\sigma}_{d,w}$ be the generalized Hurwitz space parameterizing such tuples. It is classically known that this space exists and is smooth, see for example [68, §1.4]. Moreover, we have the following result:

**Proposition 1.8.3** ([68, §2.2])**.** *The free group generated by braid moves acts transitively on $A^{\sigma}_{d,w}$. Thus $\mathcal{H}^{\sigma}_{d,w}$ is connected, and hence irreducible.*

We note that given any equivalence class of a marked covering $\big[(\varphi, \psi)\big] \in \mathcal{H}^{\sigma}_{d,w}$ and a permutation $\sigma' \in S_d$ having the same cycle type as $\sigma$, we can always find a basis $(\gamma'_1, \dots, \gamma'_w)$ of $\pi_1(\mathbb{P}^1 \setminus (\Delta \cup \{c\}, b)$ such that the loop $\gamma'_1 \dots \gamma'_w$ corresponds via $\psi$ to the permutation $\sigma'$ on the marked fiber. Thus the generalized Hurwitz space depends only on the partition $\mu \vdash d$ determining the conjugacy class of $\sigma$, and we will therefore denote it by $\mathcal{H}^{\mu}_{d,w}$.

Finally, we want to introduce Hurwitz spaces with *marked special fiber* parameterizing equivalence classes of triples $(\varphi, \psi, \chi)$, where $\varphi$ and $\psi$ are as before, and

$$\chi \colon \varphi^{-1}(c) \xrightarrow{\sim} [\ell(\mu)]$$

is a marking of the special fiber. Here $\ell(\mu)$ is the length of the partition. We denote these spaces by $\widetilde{\mathcal{H}}^{\mu}_{d,w}$. The obvious forgetful map

$$\varepsilon \colon \widetilde{\mathcal{H}}^{\mu}_{d,w} \to \mathcal{H}^{\mu}_{d,w}, \quad (\varphi, \psi, \chi) \mapsto (\varphi, \psi)$$

is étale of degree $|\mathrm{Aut}(\mu)|$, so $\widetilde{\mathcal{H}}^{\mu}_{d,w}$ is also smooth. In Section 2.6 we will show that in some cases it is connected as well, and use this to derive the irreducibility of some of the divisors $\overline{D}_{\underline{d}}$ considered there.

## 1.9 Hypertree divisors on $\overline{\mathcal{M}}_{0,n}$

Contrary to the case of the higher genus space $\overline{\mathcal{M}}_{g,n}$, which are of general type as soon as $g$ or $n$ is large enough, the spaces $\overline{\mathcal{M}}_{0,n}$ parameterizing stable pointed rational curves are rational for all $n$. This enables one to give a much more detailed account of their geometry, resulting in open problems that rapidly gain more of a combinatorial flavor, rather than geometric. We start by giving an explicit realization of $\overline{\mathcal{M}}_{0,n}$ as a successive blow-up of projective space. The first such construction was given by Kapranov [58], the one we describe here is due to Hassett [50].

**Theorem 1.9.1.** *Fix $n \geq 3$ and points $p_1, \ldots, p_{n-1} \in \mathbb{P}^{n-3}$ in general linear position.*

- *Let $X_0[n] := \mathbb{P}^{n-3}$.*
- *For $k = 1, \ldots, n-4$, let $X_k[n]$ be the blow-up of $X_{k-1}[n]$ at the proper transforms of all the $(k-1)$-dimensional subspaces spanned by $k$-tuples of the points $p_i$.*

*Then $X_{n-4}[n] \cong \overline{\mathcal{M}}_{0,n}$.*

Under this isomorphism, the boundary $\partial\overline{\mathcal{M}}_{0,n}$ corresponds to the union $\Delta$ of all the exceptional divisors in $X_{n-4}[n]$ and the proper transforms of all the $(n-4)$-dimensional subspaces spanned by $(n-3)$-tuples of the points $p_i$. Any point $p \in X_{n-4}[n] \setminus \Delta$ corresponds to a smooth pointed rational curve, which can be obtained as the proper transform of the unique rational normal curve of degree $n-3$ passing through $p_1, \ldots, p_{n-1}$ and $p$, with these points as markings.

**Example 1.9.2.** Thus the first few spaces $\overline{\mathcal{M}}_{0,n}$ can be realized as follows:

- $\overline{\mathcal{M}}_{0,3}$ is a point.
- $\overline{\mathcal{M}}_{0,4}$ is just the projective line $\mathbb{P}^1$. The boundary consists of three points, corresponding to the three possible non-smooth stable rational 4-pointed curves (see Figure 1.3).

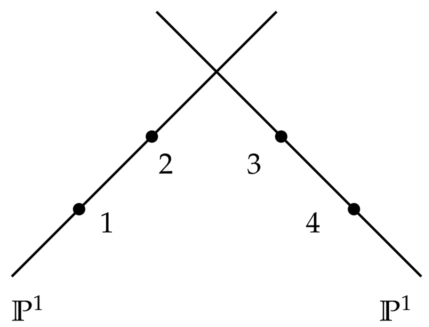


Figure 1.3: One of the three singular stable rational 4-pointed curves

- $\overline{\mathcal{M}}_{0,5}$ is $\mathbb{P}^2$ blown up in four general points, hence the unique smooth del Pezzo surface $S$ of degree 5. The boundary consists of the four exceptional divisors together with the proper transforms of the six lines through pairs of the points, i. e. the ten $(-1)$-curves on $S$.

While the generating system for $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$ given in Theorem 1.3.3 is symmetric and geometrically very natural, it does not come without relations. The construction 1.9.1 on the other hand gives a basis for $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$, which is relation free but unsymmetric, as it prefers one point over the others.

**Definition 1.9.3.** Let $\psi : \overline{\mathcal{M}}_{0,n} \xrightarrow{\cong} X_{n-4}[n]$ be the isomorphism from Theorem 1.9.1, let $\beta : X_{n-4}[n] \to X_0[n] = \mathbb{P}^{n-3}$ be the blow-down map, and let $\varphi := \beta \circ \psi$. Set $H := \varphi^*\mathcal{O}_{\mathbb{P}^{n-3}}(1)$, and for $S \subseteq [n-1]$ with $1 \leq |S| \leq n-4$ set $E_S := \psi^*\mathcal{O}_{X_{n-4}[n]}(\widetilde{E_S})$, where $\widetilde{E_S}$ is the exceptional divisor introduced in blowing up the proper transform of the subspace $\langle p_i \mid i \in S\rangle$. Then $H$ together with the $E_S$ form a basis for $\mathrm{Pic}(\overline{\mathcal{M}}_{0,n})$, which we call the *Kapranov basis* with respect to the marking $n$.

Note that $E_S$ correponds to the boundary divisor class $\delta_{0:S\cup\{n\}}$. Using the action of $S_n$ on the marked points, one can also construct Kapranov bases with respect to the other markings.

In [62], Keel and McKernan raised a question due to Fulton, which "in the interest of drama" they refer to as *Fulton's conjecture*.

**Definition 1.9.4.** A *vital cycle* of codimension $k$ on $\overline{\mathcal{M}}_{0,n}$ is any irreducible cycle whose general element is a curve having exactly $k$ nodes.

In codimension 1, the vital cycles are thus just the boundary divisors. In codimension $(n-4)$, i. e. dimension 1, the vital curves are all gotten by attaching to $\mathbb{P}^1$ four stable pointed rational curves without moduli, and varying the cross-ratio of the four points of attachment (see Figure 1.4).

**Conjecture 1.9.5** (Fulton). *Every effective cycle of codimension $k$ on $\overline{\mathcal{M}}_{0,n}$ is linearly equivalent to an effective sum of vital cycles of codimension $k$.*

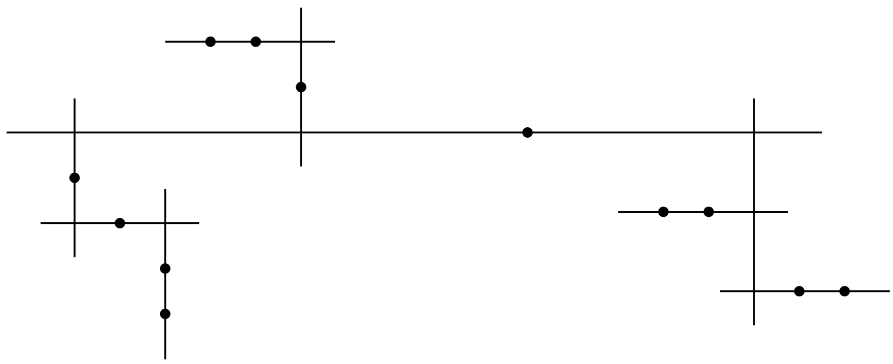

Figure 1.4: A vital curve on $\overline{\mathcal{M}}_{0,12}$ (the cross ratio on the central component varies)

Keel and McKernan proved the conjecture in codimension $(n-4)$ for $n \leq 7$. For divisors the conjecture is true up to $n = 5$ (this can be shown directly, but it also follows from the much stronger result by Batyrev and Popov [6] describing Cox rings of del Pezzo surfaces). However, Keel and Vermeire [83] found a counterexample in codimension 1 for $n = 6$, which lifts to all higher $n$. It is described in detail in Section 4.1.

Hassett and Tschinkel [54] showed that for $n = 6$ the boundary divisors together with the Keel-Vermeire divisors generate the cone of effective divisors. Moreover, Castravet and Tevelev [13] for all $n \geq 6$ found a series of further divisors violating Fulton's conjecture, which they called *hypertree divisors*, as they are based on the combinatorial concept of hypertrees.

**Definition 1.9.6** ([13, Definition 1.2])**.** A *hypertree* on $n$ vertices is given by a collection $\Gamma = \{\Gamma_1, \dots, \Gamma_d\}$ of subsets of $[n]$ satisfying the following conditions:

- $|\Gamma_j| \geq 3$ for $j = 1, \dots, d$,
- any $i \in [n]$ is contained in at least two $\Gamma_j$, and
- for any $S \subseteq [d]$,
$$\sum_{j\in S}(|\Gamma_j| - 2) \leq \Big|\bigcup_{j\in S}\Gamma_j\Big| - 2, \tag{1.6}$$
  with equality holding if $|S| = d$.

The hypertree $\Gamma$ is called *irreducible* if (1.6) is a strict inequality for $1 < |S| < d$.

There are no hypertrees on fewer than 5 points, while for $n = 6$ and 7 there is (up to renumbering) a unique irreducible hypertree in each case (see Figure 4.1). After that, the number of irreducible hypertrees grows rapidly with $n$. The name hypertree is motivated by the fact that if one requires $|\Gamma_j| = 2$ for all $j$, and in (1.6) replaces both instances of the number 2 by 1, one obtains the condition on $\Gamma$ to be the set of edges of a tree.

**Definition 1.9.7.** Let $\Gamma = \{\Gamma_1, \dots, \Gamma_d\}$ be a hypertree on $n$ vertices. A *planar realization* of $\Gamma$ is a collection of points $p_1, \dots, p_n \in \mathbb{P}^2$ such that for any $S \subseteq [n]$ with $|S| \geq 3$, the points $\{p_i \mid i \in S\}$ are collinear if and only if $S \subseteq \Gamma_j$ for some $j$.

It is shown in [13, Theorem 6.1] that any hypertree has a planar realization. One can thus define the following locus:

**Definition 1.9.8.** Let $\Gamma$ be an irreducible hypertree on $n$ vertices. Then we define $D_\Gamma$ to be the closure in $\overline{\mathcal{M}}_{0,n}$ of the locus of pointed curves $(\mathbb{P}^1; q_1, \dots, q_n)$ for which there exists a planar realization $p_1, \dots, p_n$ of $\Gamma$, and a projection $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ defined at all the $p_i$, such that $\varphi(p_i) = q_i$ for all $i = 1, \dots, n$.

Castravet and Tevelev then proceed to show that $D_\Gamma$ is an irreducible effective divisor on $\overline{\mathcal{M}}_{0,n}$ [13, Theorem 1.5], and moreover that it generates an extremal ray of the effective cone. If $\Gamma$ is the unique irreducible hypertree on 6 vertices, they show that $D_\Gamma$ is just the Keel-Vermeire divisor on $\overline{\mathcal{M}}_{0,6}$. In Chapter 4 we will derive an analogous characterization for the hypertree divisor on $\overline{\mathcal{M}}_{0,7}$ and use it to compute its class in a Kapranov basis with respect to the last marking.

## 1.10 Outline of results

In Chapter 2 we will define the divisor $\overline{D}_{\underline{d}}$ on $\overline{\mathcal{M}}_{g,n}$, which is the closure of the pullback of the theta divisor from the universal Jacobian of degree $g-1$ via a global weighted Abel-Jacobi map that sends $(C; x_1, \ldots, x_n)$ to the line bundle $\mathcal{O}_C(\sum d_j x_j)$ on $C$. Here $\underline{d} = (d_1, \ldots, d_n)$ is an $n$-tuple of integers, not all positive, that sum up to $g-1$. We compute the class of $\overline{D}_{\underline{d}}$ using a vector bundle computation, a pushdown argument reducing the number of marked points, and the method of test curves. We then show that for parameter choices of the special form $\underline{d} = (d_1, \ldots, d_{n-2}; -1, -1)$ with $d_1, \ldots, d_{n-2}$ positive, the divisor $\overline{D}_{\underline{d}}$ is extremal in the effective cone of $\overline{\mathcal{M}}_{g,n}$. This is accomplished by constructing covering families of curves that intersect $\overline{D}_{\underline{d}}$ negatively and showing that $\overline{D}_{\underline{d}}$ is irreducible by establishing irreducibility of an appropriate Hurwitz space which maps dominantly onto it.

Chapter 3 discusses the birational model $X_6$ of $\overline{\mathcal{M}}_6$ that is given by quadric hyperplane sections of the degree 5 del Pezzo surface. We epxlicitly describe the natural birational map from $\overline{\mathcal{M}}_6$ to $X_6$ and determine its exceptional locus. We then compute the class of the divisor inducing this map, again by computing its intersection with enough test curves. We use this to get a new upper bound on the moving slope of $\overline{\mathcal{M}}_6$ and show that $X_6$ is the final non-trivial space in the log minimal model program for $\overline{\mathcal{M}}_6$. We conclude by some notes on the unirationality of the Weierstraß loci $W_{g,k}$ on $\overline{\mathcal{M}}_{g,1}$. It turns out that the case where $g = k = 6$ can be treated via the same construction used to establish the model $X_6$.

Finally, Chapter 4 deals with the hypertree divisor on $\overline{\mathcal{M}}_{0,7}$, which is an effective divisor that is known to lie outside of the effective cone generated by the boundary divisors. We derive a geometric characterization of this divisor and compute its class in a suitable basis. The effective cone of $\overline{\mathcal{M}}_{0,7}$ is conjecturally generated by boundary divisors together with the $S_7$-orbits of this divisor and the pullback of the hypertree divisor on $\overline{\mathcal{M}}_{0,6}$ under the action that permutes the marked points.

Chapter 2 is based on the article [73], with the exception of the newly added Section 2.6 on the extremality of the divisors $\overline{D}_{\underline{d}}$. Chapter 3 is essentially the preprint [72].

Chapter 2

# 2 The pullback of a theta divisor to $\overline{\mathcal{M}}_{g,n}$

## 2.1 Introduction

It has long been known classically that if $C$ is a smooth curve of genus $g \geq 2$ and $C_{g-1}$ denotes its $(g-1)$-fold symmetric product, the Abelian sum map $C_{g-1} \to \mathrm{Pic}^{g-1}(C)$, which associates to an unordered $(g-1)$-tuple $x_1 + \cdots + x_{g-1} \in C_{g-1}$ the line bundle $\mathcal{O}_C(x_1 + \cdots + x_{g-1})$, has as image a divisor, which becomes a theta divisor under an identification of $\mathrm{Pic}^{g-1}(C)$ with the Jacobian of $C$. This result can be globalized to a map $\mathcal{C}_{g,g-1} \to \mathrm{Pic}_g^{g-1}$, where

$$\mathcal{C}_{g,g-1} = \left(\mathcal{M}_{g,1} \times_{\mathcal{M}_g} \cdots \times_{\mathcal{M}_g} \mathcal{M}_{g,1}\right) / S_{g-1}$$

is the $(g-1)$-fold symmetric product of the universal curve, and $\mathrm{Pic}_g^{g-1}$ is the universal Picard variety of degree $g-1$. The image of this map is again a divisor, which we denote by $\Theta_g$. Given an integer vector $\underline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ satisfying $\sum_{j=1}^n d_j = g-1$, we can define a map $\varphi_{\underline{d}} \colon \mathcal{M}_{g,n} \to \mathrm{Pic}_g^{g-1}$ associating to a pointed curve $[C; x_1, \ldots, x_n]$ the line bundle $\mathcal{O}_C(d_1x_1 + \cdots + d_nx_n)$ on $C$. If at least one of the $d_j$ is negative the image of $\varphi_{\underline{d}}$ is not contained in $\Theta_g$, and we can ask what is the class of the pullback $D_{\underline{d}} := \varphi_{\underline{d}}^*\Theta_g$ and its closure on $\overline{\mathcal{M}}_{g,n}$. Unraveling the concepts involved, we arrive at the following equivalent definition:

**Definition 2.1.1.** Let $\underline{d} = (d_1, \ldots, d_n)$ be an $n$-tuple of integers summing up to $g-1$, with at least one $d_j$ negative. Denote by

$$D_{\underline{d}} := \left\{[C; x_1, \ldots, x_n] \in \mathcal{M}_{g,n} \,\middle|\, h^0\left(C, d_1x_1 + \cdots + d_nx_n\right) \geq 1\right\},$$

which is a divisor on $\mathcal{M}_{g,n}$, and let $\overline{D}_{\underline{d}}$ be its closure in $\overline{\mathcal{M}}_{g,n}$.

Note that since the $x_j$ are distinct, the condition $h^0\left(C, d_1x_1 + \cdots + d_nx_n\right) \geq 1$ is equivalent to postulating that there is a pencil of degree $d_{S_+} := \sum_{j:d_j>0} d_j$ on $C$ that contains the divisor $\sum_{j:d_j>0} d_j x_j$ and has a section that vanishes to order $-d_j$ at $x_j$ for all $j \in S_- := \{j \mid d_j < 0\}$. As it ties in nicely with the limit linear series characterization on reducible curves, we will always use this reformulation from now on.

The main result of this chapter, which is proven in Theorem 2.5.6, is the computation of the class of this divisor in $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$. It is given by

$$\begin{aligned}[\overline{D}_{\underline{d}}] = & -\lambda + \sum_{j=1}^{n} \binom{d_j+1}{2} \psi_j - 0 \cdot \delta_0 \\ & - \sum_{\substack{i,S \\ S \subseteq S_+}} \binom{|d_S - i| + 1}{2} \delta_{i:S} - \sum_{\substack{i,S \\ S \not\subseteq S_+}} \binom{d_S - i + 1}{2} \delta_{i:S},\end{aligned} \tag{2.1}$$

where $S_+ := \{j \mid d_j > 0\}$ and $d_S := \sum_{j \in S} d_j$. Thus the next to last summand corresponds to boundary classes that parameterize reducible curves where the points indexed by $S_-$ lie on a single component, while the last one corresponds to classes parameterizing curves which have points from $S_-$ on both components.

In the special case $\underline{d} = (d_1, \ldots, d_{n-1}; -1)$ with $d_1, \ldots, d_{n-1} > 0$, the divisor $\overline{D}_{\underline{d}}$ is just the pullback to $\overline{\mathcal{M}}_{g,n}$ of the divisor of pointed curves $[C; x_1, \ldots, x_{n-1}] \in \overline{\mathcal{M}}_{g,n-1}$ having a $g^1_g$ containing $d_1 x_1 + \cdots + d_{n-1} x_{n-1}$, which was considered by Logan [65]. For $n = 2$, it is the pullback of the Weierstraß divisor on $\overline{\mathcal{M}}_{g,1}$, whose class has been computed by Cukierman [22] to be

$$[\overline{\mathcal{W}}_g] = -\lambda + \binom{g+1}{2} \psi_1 - \sum_{i=1}^{g-1} \binom{g-i+1}{2} \delta_{i:1}. \tag{2.2}$$

For more details on this, see Remarks 2.5.2 and 2.5.5.

A divisor similar to $\overline{D}_{\underline{d}}$ was studied by Hain [44]: On an open subset $U$ of $\overline{\mathcal{M}}_{g,n}$ (or a covering of such) where there is a globally defined theta characteristic $\alpha$, one can define a morphism $\varphi'_{\underline{d}} \colon U \to \mathrm{Pic}^0_g$ mapping a pointed curve $[C; x_1, \ldots, x_n]$ to the line bundle $\mathcal{O}_C(d_1 x_1 + \cdots + d_n x_n - \alpha) \in \mathrm{Pic}^0(C)$. The class of the closure in $\overline{\mathcal{M}}_{g,n}$ of the pullback $D'_{\underline{d}} := (\varphi'_{\underline{d}})^* \Theta_\alpha$ is computed in [44, Theorem 11.7]; expressed in our notation it is

$$\left[\overline{D}'_{\underline{d}}\right] = -\lambda + \sum_{j=1}^{n} \binom{d_j+1}{2} \psi_j + \delta_0/8 - \sum_{i,S} \binom{d_S - i + 1}{2} \delta_{i:S} \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n}) \otimes \mathbb{Q}.$$

Both this result and our Theorem 2.5.6 are reproven in a recent preprint by Grushevsky and Zakharov [43, Theorem 6], where it is also shown that the divisor considered by Hain is reducible and decomposes as $\overline{D}_{\underline{d}}$ together with some boundary components, with multiplicities according to the generic vanishing order of the theta function.

This chapter is organized as follows: In Section 2.2 we will collect some results on pullbacks and pushforwards of divisors on $\overline{\mathcal{M}}_{g,n}$ that we will need during the course of the chapter. In Section 2.3 the coefficients of the $\lambda$ and $\psi_j$ classes in the expression for $[\overline{D}_{\underline{d}}]$ are computed by a vector bundle technique. The rest of the coefficients are computed via test curves. The actual test curve computations are done in Section 2.4, and the results are applied in Section 2.5 together with a pushdown technique to finish the proof of the main result.

### Notation

By a *nodal curve*, we shall mean a reduced connected 1-dimensional scheme of finite type over a field $k$ whose only singularities are ordinary nodes. A nodal curve is said to be of *compact type* if its dual graph is a tree, or equivalently if its Jacobian is compact.

As always, we use the shorthand $[n] := \{1, \dots, n\}$. If $a$ is any expression, we write $(a)_+ := \max(a, 0)$. Occasionally we will write down a binomial coefficient $\binom{a}{2}$ with $a < 0$, by which we just mean $a(a-1)/2$.

If $\underline{d} = (d_1, \dots, d_n)$ is an $n$-tuple of integers, we write $S_+$ (resp. $S_-$) for the set of indices $j \in [n]$ with $d_j > 0$ (resp. $d_j < 0$). Moreover, if $S \subseteq [n]$ is an arbitrary set of indices, we write $d_S := \sum_{j \in S} d_j$. When convenient, we will assume that the positive $d_j$ come first and in the notation $\overline{D}_{\underline{d}}$ separate them with a semicolon from the negative ones.

As mentioned before, when summing over boundary classes $\delta_{i:S}$ in $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$, the summation range $\sum_{i,S}$ (and obvious analogues) will be implicitly taken to involve only admissible combinations (e. g. $|S| \geq 2$ for $i = 0$) and to contain every divisor only once (e. g. by postulating $1 \in S$ or $i \leq g/2$). By $\pi_n\colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ we will denote the forgetful map which forgets the $n$-th point, while by $\pi_{(jk \mapsto \bullet)}$ we mean the map which identifies the divisor $\Delta_{0:jk} \subseteq \overline{\mathcal{M}}_{g,n}$ with $\overline{\mathcal{M}}_{g,n-1}$ by removing the rational component and introducing the new marking $\bullet$ for the former point of attachment. By $\pi\colon \mathcal{M}_{g,1} \times_{\mathcal{M}_g} \mathcal{M}_{g,n} =: \mathcal{U} \to \mathcal{M}_{g,n}$ we denote the universal family over $\mathcal{M}_{g,n}$ having sections $\sigma_1, \dots, \sigma_n\colon \mathcal{M}_{g,n} \to \mathcal{U}$, and by $\omega_\pi \in \mathrm{Pic}(\mathcal{U}/\mathcal{M}_{g,n})$ the relative dualizing sheaf of the map $\pi$. Picard groups are always understood in the functorial sense, i. e. as groups of divisor classes on the moduli stack.

## 2.2 Pushforward and pullback formulas

For computing pullbacks of divisor classes, we need the following formulas, which can be found in [5, p. 161]:

**Lemma 2.2.1.** *If $\pi_n : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ is the forgetful map forgetting the last point, then we have the following formulas for pullbacks of divisor classes:*

*(i)* $\pi_n^*\lambda = \lambda$,

*(ii)* $\pi_n^*\psi_j = \psi_j - \delta_{0:jn}$,

*(iii)* $\pi_n^*\delta_0 = \delta_0$,

*(iv)* $\pi_n^*\delta_{i:S} = \delta_{i:S} + \delta_{i:S\cup\{n\}}$, *except that* $\pi_1^*\delta_{g/2:\emptyset} = \delta_{g/2:\emptyset}$.

To apply the Grothendieck-Riemann-Roch formula in Section 2.3, we need certain formulas for pushforwards of intersections of cycles on the universal family, which can be found for example in [36, Lemma 3.13]. We reproduce the ones that concern us here:

**Lemma 2.2.2.** *With notation as given in Section 2.1,*

*(i)* $\pi_*\big(c_1(\omega_\pi)^2\big) = 12\lambda$,

*(ii)* $\pi_*\big(c_1(\omega_\pi)c_1(\sigma_j)\big) = \psi_j$, *and*

*(iii)* $\pi_*\big(c_1(\sigma_j)^2\big) = -\psi_j$.

In order to be able to apply a pushdown technique in Section 2.5, we also need various formulas for pushforwards of intersections of basis divisor classes via the map $\pi_{(jk\mapsto\bullet)}$ which identifies the divisor $\Delta_{0:jk}$ with $\overline{\mathcal{M}}_{g,n-1}$. They can be found in a table in [65, Theorem 2.8]; we list the relevant ones here:

**Lemma 2.2.3.** *The following formulas for pushforwards of intersection cycles hold:*

*(i)* $\pi_{(1n\mapsto\bullet)*}(\lambda\cdot\delta_{0:1n}) = \lambda$,

*(ii)* $\pi_{(1n\mapsto\bullet)*}(\psi_j\cdot\delta_{0:1n}) = \begin{cases} 0 & \text{for } j = 1, n, \\ \psi_j & \text{for } j = 2, \ldots, n-1, \end{cases}$

*(iii)* $\pi_{(1n\mapsto\bullet)*}(\delta_0\cdot\delta_{0:1n}) = \delta_0$,

*(iv)* $\pi_{(1n\mapsto\bullet)*}(\delta_{0:1n}^2) = -\psi_\bullet$,

*(v)* $\pi_{(1n\mapsto\bullet)*}(\delta_{i:S}\cdot\delta_{0:1n}) = \begin{cases} \delta_{i:S} & \text{if } 1, n \notin S, \\ \delta_{i:S'} & \text{if } 1, n \in S, \\ 0 & \text{if } 1 \in S, n \notin S \text{ or } 1 \notin S, n \in S, \end{cases}$

*where* $S' := \big(S \setminus \{1, n\}\big) \cup \{\bullet\}$.

The corresponding formulas for the pushforwards of intersections of divisors with other boundary divisor classes of the form $\delta_{0:jk}$ can easily be obtained from Lemma 2.2.3 by applying the $S_n$-action permuting the points on $\overline{\mathcal{M}}_{g,n}$. Note that when we take out the basis elements of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ that get mapped to 0 in the above formulas, the map $\alpha \mapsto \pi_{(1n\mapsto\bullet)*}(\alpha\cdot\delta_{0:1n})$ is injective on the span of the remaining basis elements, a fact we will make use of in Section 2.5 (see Remark 2.5.3).

Finally, for applying the pushdown technique we also need to know how the divisor $\overline{D}_{\underline{d}}$ behaves under intersection and pushforward:

**Lemma 2.2.4.** *If $j$, $k \in [n]$ are two indices such that $d_j$ and $d_k$ have the same sign, then*

$$\pi_{(jk \mapsto \bullet)*}(\overline{D}_{\underline{d}} \cdot \delta_{0:jk}) = \overline{D}_{\underline{d}'},$$

*where $\underline{d}' = (d_1, \ldots, \widehat{d_j}, \ldots, \widehat{d_k}, \ldots, d_n, d_\bullet = d_j + d_k)$.*

*Proof.* This is an easy generalization of the proof of [65, Proposition 5.3]. □

## 2.3 Computation of the main coefficients

We write the class of the divisor $\overline{D}_{\underline{d}}$ as

$$[\overline{D}_{\underline{d}}] = a\lambda + \sum_{j=1}^{n} c_j \psi_j + b_0 \delta_0 + \sum_{i,S} b_{i:S} \delta_{i:S}. \tag{2.3}$$

In this section we determine the coefficients $a$ and $c_j$ by expressing $D_{\underline{d}}$ as the degeneracy locus of a map of vector bundles of the same rank and applying Porteous' formula. These calculations will also be instrumental in computing some of the boundary coefficients $b_0$ and $b_{i:S}$ in Section 2.5, while the remaining ones will be obtained by intersecting the closure $\overline{D}_{\underline{d}}$ with suitably chosen test curves.

The top Chern class $\lambda_g := c_g(\mathbb{E})$ of the Hodge bundle is known to have class 0 in $A^g(\mathcal{M}_{g,n})$ (see [67]). Therefore we can find a nowhere vanishing section of $\mathbb{E}$, or equivalently, a relative section of $\omega_\pi$ over $\mathcal{M}_{g,n}$, whose zero locus cuts out a canonical divisor on every fiber of $\pi$. We denote that zero locus by $\mathscr{K}$. Furthermore, we denote by $\mathscr{D} := \sum_{j=1}^{n} d_j \sigma_j \in \operatorname{Pic}(\mathcal{U}/\mathcal{M}_{g,n})$ the relative divisor which on every fiber cuts out the divisor given by the linear combination of the marked points.

We now consider the restriction map $\rho\colon \omega_\pi(\mathscr{D}) \to \omega_\pi(\mathscr{D})\big|_{\mathscr{K}}$ and its direct image

$$\varphi := R^0\pi_*\rho\colon R^0\pi_*\big(\omega_\pi(\mathscr{D})\big) \to R^0\pi_*\big(\omega_\pi(\mathscr{D})\big|_{\mathscr{K}}\big). \tag{2.4}$$

Since $\mathscr{D}$ has relative degree $g-1$, we find that $R^1\pi_*(\omega_\pi(\mathscr{D})) = 0$. Similarly, $\omega_\pi(\mathscr{D})|_{\mathscr{K}}$ is torsion on fibers, so we also have $R^1\pi_*(\omega_\pi(\mathscr{D})|_{\mathscr{K}}) = 0$. Thus by Grauert's theorem, both sheaves in (2.4) are in fact locally free, and by Riemann-Roch they are easily seen to both have rank $2g-2$.

We are now in a position to compute the main coefficients of $\overline{D}_{\underline{d}}$.

**Proposition 2.3.1.** *In the expression* (2.3) *for $[\overline{D}_{\underline{d}}]$, we have $a = -1$ and $c_j = \binom{d_j+1}{2}$.*

*Proof.* The short exact sequence

$$0 \to \mathcal{O}_{\mathcal{U}}(\mathscr{D}) \to \omega_\pi(\mathscr{D}) \xrightarrow{\rho} \omega_\pi(\mathscr{D})\big|_{\mathscr{K}} \to 0, \tag{2.5}$$

yields after pushing down the long exact sequence

$$\begin{aligned} 0 &\to R^0\pi_*\big(\mathcal{O}_{\mathcal{U}}(\mathscr{D})\big) \to R^0\pi_*\big(\omega_\pi(\mathscr{D})\big) \xrightarrow{\varphi} R^0\pi_*\big(\omega_\pi(\mathscr{D})\big|_{\mathscr{K}}\big) \\ &\to R^1\pi_*\big(\mathcal{O}_{\mathcal{U}}(\mathscr{D})\big) \to 0. \end{aligned} \tag{2.6}$$

Since $\sum_{j=1}^n d_j = g-1$ implies that $h^0(C, \sum_{j=1}^n d_j x_j) = h^1(C, \sum_{j=1}^n d_j x_j)$ for every point $[C; x_1, \ldots, x_n] \in \mathcal{M}_{g,n}$, the sequence (2.6) stays exact after passing to a fiber. Thus, the divisor $D_{\underline{d}}$ is exactly the degeneracy locus of the map $\varphi$, and by Porteous' formula it follows that

$$\big[D_{\underline{d}}\big] = c_1\Big(R^0\pi_*\big(\omega_\pi(\mathscr{D})\big|_{\mathscr{K}}\big)\Big) - c_1\Big(R^0\pi_*\big(\omega_\pi(\mathscr{D})\big)\Big). \tag{2.7}$$

We can calculate the two terms in (2.7) by a Grothendieck-Riemann-Roch computation. For the first one, we obtain

$$\begin{aligned} \operatorname{ch}\big(\pi_!\big(\omega_\pi(\mathscr{D})\big|_{\mathscr{K}}\big)\big) &= \operatorname{ch}\big(\pi_*\big(\omega_\pi(\mathscr{D})\big|_{\mathscr{K}}\big)\big) \\ &= \pi_*\Big[\operatorname{ch}\big(\omega_\pi(\mathscr{D})\big|_{\mathscr{K}}\big)\cdot \operatorname{td}\big(\omega_\pi^\vee\big)\Big] \\ &= \pi_*\Big[\big(\operatorname{ch}\big(\omega_\pi(\mathscr{D})\big) - \operatorname{ch}\big(\mathcal{O}_{\mathcal{U}}(\mathscr{D})\big)\big)\cdot \operatorname{td}\big(\omega_\pi^\vee\big)\Big] \qquad \text{(by (2.5))} \\ &= \pi_*\Big[\big(\operatorname{ch}(\omega_\pi) - 1\big)\cdot \operatorname{ch}(\mathscr{D})\cdot \operatorname{td}\big(\omega_\pi^\vee\big)\Big] \\ &= \pi_*\Big[\big(c_1(\omega_\pi) + \frac{1}{2}c_1^2(\omega_\pi) + \ldots\big)\cdot\big(1 + c_1(\mathscr{D}) + \frac{1}{2}c_1^2(\mathscr{D}) + \ldots\big)\cdot \\ &\qquad \cdot\big(1 - \frac{1}{2}c_1(\omega_\pi) + \frac{1}{12}c_1^2(\omega_\pi) + \ldots\big)\Big] \\ &= (2g-2) + \pi_*\Big[c_1(\omega_\pi)c_1(\mathscr{D}) + \ldots\Big] \\ &= (2g-2) + \sum_{j=1}^n d_j\psi_j + \ldots \qquad \text{(by Lemma 2.2.2)}, \end{aligned}$$

while for the second one we compute

$$\begin{aligned} \operatorname{ch}\big(\pi_!\big(\omega_\pi(\mathscr{D})\big)\big) &= \operatorname{ch}\big(\pi_*\big(\omega_\pi(\mathscr{D})\big)\big) \\ &= \pi_*\Big[\operatorname{ch}(\omega_\pi)\cdot \operatorname{ch}(\mathscr{D})\cdot \operatorname{td}\big(\omega_\pi^\vee\big)\Big] \\ &= \pi_*\Big[\big(1 + c_1(\omega_\pi) + \frac{1}{2}c_1^2(\omega_\pi) + \ldots\big)\cdot\big(1 + c_1(\mathscr{D}) + \frac{1}{2}c_1^2(\mathscr{D}) + \ldots\big)\cdot \\ &\qquad \cdot\big(1 - \frac{1}{2}c_1(\omega_\pi) + \frac{1}{12}c_1^2(\omega_\pi) + \ldots\big)\Big] \\ &= (2g-2) + \lambda + \frac{1}{2}\sum_{j=1}^n (d_j - d_j^2)\psi_j + \ldots \qquad \text{(by Lemma 2.2.2)}. \end{aligned}$$

Putting these together into (2.7) yields the result. □

## 2.4 Intersections with test curves

For later use in Section 2.5, we will gather here several computations of intersections of $\overline{D}_{\underline{d}}$ with families of pointed curves which are wholly contained in the boundary of $\overline{\mathcal{M}}_{g,n}$. This constitutes the main work in computing the class of $\overline{D}_{\underline{d}}$, the remaining part being mainly a properly engineered application of the results presented here.

We recall from Theorem 1.4.4 that a generic curve $C$ of genus $g$ has a $g^r_d$ if and only if the *Brill-Noether-number*

$$\rho(g, r, d) = g - (r+1)(g - d + r)$$

is non-negative. Moreover, postulating a vanishing sequence $a = (a_0, \dots, a_r)$ at a generic point of $C$ imposes $\sum_{i=0}^r (a_i - i)$ conditions on the space of $g^r_d$'s on $C$. We will also constantly use the result stated in Theorem 1.7.6, saying that a pointed nodal curve of compact type lies in $\overline{D}_{\underline{d}}$ if and only if it carries a limit $g^1_d$ with the required vanishing.

**Remark 2.4.1.** In proving the results of this section, we will often come across questions of the following form: Given a curve $C$ of genus $g$ and a positive integer $d$, how many $g^1_d$'s $\ell$ are there on $C$ satisfying some ramification conditions whose codimensions add up to $\rho(g, 1, d)$?

In our cases, among the conditions there will always be one of *full ramification*, where we require $\ell$ to contain some fixed effective divisor $D$ of degree $d$. This reduces the problem to a Schubert calculus computation in the Grassmannian $\mathbb{G}(1, r)$, where we set $r := r(D) = h^0(C, D) - 1$. Postulating the vanishing sequence $(a, b)$ at a generic point of $C$ corresponds to the Schubert cycle $\sigma_{a,b-1}$, and requiring $\ell$ to contain $D$ amounts to intersecting with $\sigma_{r-1} := \sigma_{0,r-1}$. Since

$$\sigma_{\alpha_1,\beta_1} \cdot \ldots \cdot \sigma_{\alpha_k,\beta_k} \cdot \sigma_{r-1} = 1 \qquad \text{for } \sum_{i=1}^{k} (\alpha_i + \beta_i) = r - 1,$$

in such cases $\ell$ is always unique.

We first consider the case $n = 2$, where we write $\underline{d} = (g + b - 1; -b)$ with $b > 0$. Here and in the following, the intersection numbers of the families in question with generators of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ that are not explicitly mentioned in the Lemmas are implied (and easily seen) to be 0.

**Lemma 2.4.2.** *Let $(C; x_1, x_2, y)$ be a generic 3-pointed curve of genus $g - 1$, and let $F$ be the family in $\overline{\mathcal{M}}_{g,2}$ obtained by gluing the marked point $y$ to a base point of a generic plane cubic pencil. Then we have*

$$F \cdot \overline{D}_{\underline{d}} = 0,$$
$$F \cdot \lambda = 1, \qquad F \cdot \delta_0 = 12, \qquad F \cdot \delta_{g-1:12} = -1.$$

*Proof.* A member of $F$ lying in $\overline{D}_{\underline{d}}$ has a limit $g^1_{d_1}$ whose $C$-aspect $\ell_C$ is spanned by $d_1 x_1$ and $bx_2 + \sigma$ for some $\sigma \in |d_1 x_1 - bx_2|$. By Riemann-Roch,

$$h^0(C, (g+b-1)x_1 - bx_2) = 1$$

for $x_1$, $x_2$ generic, so $\ell_C$ is unique, and since $y$ is also generic, it has vanishing sequence $a^{\ell_C}(y) = (0, 1)$. Thus the aspect on the elliptic tail would have to have vanishing sequence $(d_1 - 1, d_1)$ at the base point, which is impossible.

The remaining intersection numbers are well known and can be found for example in [47, p. 173f.]. □

**Lemma 2.4.3.** *Let $(C; x_2)$ be a generic 1-pointed curve of genus $g$, and let $F$ be the family in $\overline{\mathcal{M}}_{g,2}$ obtained by letting a point $x_1$ move along $C$. Then we have*

$$F \cdot \overline{D}_{\underline{d}} = g(d_1^2 - 1),$$
$$F \cdot \psi_1 = 2g - 1, \qquad F \cdot \psi_2 = 1, \qquad F \cdot \delta_{0:12} = 1.$$

*Proof.* We compute the intersection number $F \cdot \overline{D}_{\underline{d}}$ by degenerating $C$ to a *comb curve* $R \cup_{y_1} E_1 \cup \cdots \cup_{y_g} E_g$ consisting of a rational spine $R$ to which are attached $g$ elliptic tails at generic points $y_1, \ldots, y_g$, with the point $x_2$ lying on $R$ (see Figure 2.1). As shown in [25, Section 9], the variety of limit $g^r_d$'s is reduced on a generic such curve, so all we have to do is count the number of limit linear series $\ell = (\ell_R, \ell_{E_1}, \ldots, \ell_{E_g})$ of type $g^1_{d_1}$ satisfying the given vanishing conditions at $x_1$ and $x_2$.

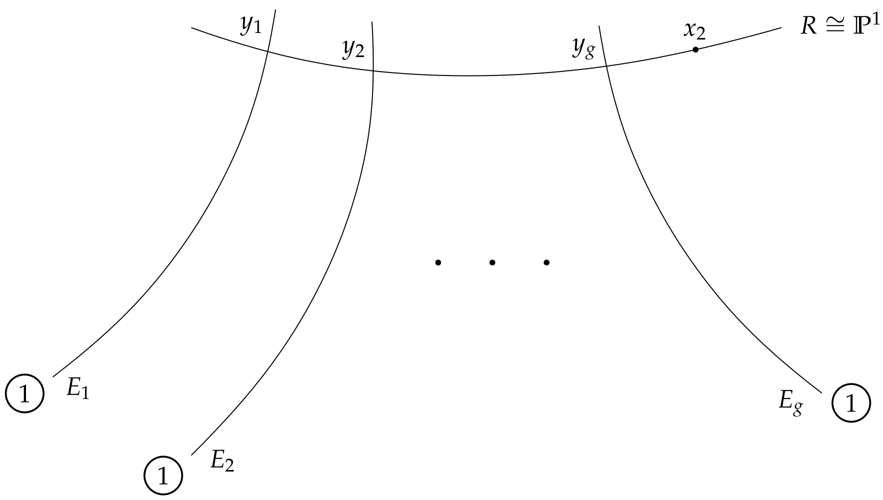


Figure 2.1: A comb curve (circled numbers denote geometric genus)

By [27, Proposition 1.1], we must have $x_1 \in E_i$ for some $i$. The $E_j$-aspect of each elliptic tail $E_j$ with $j \neq i$ must satisfy $a^{\ell_{E_j}}(y_j) \leq (d_1 - 2, d_1)$, giving $a^{\ell_R}(y_j) \geq (0, 2)$ for these $j$. Thus the $R$-aspect of $\ell$ is a $g^1_{d_1}$ that contains the divisor $d_1 y_i$, vanishes to order $b$ at $x_2$ and is simply ramified at $(g-1)$ further points, corresponding to the Schubert cycle

$$\sigma_{a_0^{\ell_R}(y_i), d_1-1} \cdot \sigma_{b-1} \cdot \sigma_1^{g-1} \qquad \text{in } \mathbb{G}(1, d_1).$$

Counting dimensions, this is non-empty only if $a_0^{\ell_R}(y_i) = 0$, and then $\ell_R$ is unique by Remark 2.4.1. We thus get the upper bound $a^{\ell_R}(y_i) \leq (0, d_1)$, which by the compatibility conditions is equivalent to $a^{\ell_{E_i}}(y_i) \geq (0, d_1)$. Since also $a^{\ell_{E_i}}(x_1) \geq (0, d_1)$, this is possible only if equality holds everywhere and $x_1 - y_i$ is a non-trivial $d_1$-torsion point in $\mathrm{Pic}^0(E_i)$. Thus each of the $g$ elliptic tails gives exactly $(d_1^2 - 1)$ possibilities for $x_1$.

The remaining intersection numbers can be found by standard techniques. □

**Lemma 2.4.4.** *Let $(C; x_1)$ be a generic 1-pointed curve of genus $g$, and let $F$ be the family in $\overline{\mathcal{M}}_{g,2}$ obtained by letting a point $x_2$ move along $C$. Then we have*

$$F \cdot \overline{D}_{\underline{d}} = g(b^2 - 1),$$
$$F \cdot \psi_1 = 1, \qquad F \cdot \psi_2 = 2g - 1, \qquad F \cdot \delta_{0:12} = 1.$$

*Proof.* We proceed as in the proof of Lemma 2.4.3, degenerating $C$ to a comb curve where now $x_1 \in R$. Reasoning as before, we find that we must have $x_2 \in E_j$ for some $j$ and $a^{\ell_R}(y_j) \leq (0, b)$ for dimension reasons, so $a^{\ell_{E_j}}(y_j) \geq (g-1, g+b-1)$. Together with $a^{\ell_{E_j}}(y_j) \geq (0, b)$ this implies that $x_2 - y_j$ is a non-trivial $b$-torsion point in $\mathrm{Pic}^0(E_j)$, so each of the $g$ elliptic tails contributes $(b^2 - 1)$ possibilities for $x_2$. □

**Lemma 2.4.5.** *Let $(C_1; x_1, x_2, y)$ be a generic 3-pointed curve of genus $g - i$, $C_2$ a generic curve of genus $i \geq 2$, and let $F$ denote the family in $\overline{\mathcal{M}}_{g,2}$ obtained by gluing $y$ to a moving point of $C_2$. Then we have*

$$F \cdot \overline{D}_{\underline{d}} = i(i^2 - 1),$$
$$F \cdot \delta_{g-i:12} = 2 - 2i.$$

*Proof.* Let $\ell = (\ell_{C_1}, \ell_{C_2})$ be a limit $g^1_{d_1}$ on $C$. By genericity, the family of $g^1_{d_1}$'s on $C_1$ with the required vanishing at $x_1$ and $x_2$ has dimension

$$\rho(g-i, 1, d_1) - (d_1 - 1) - (b - 1) = i - 1,$$

so for $y \in C_1$ also generic we must have $a_1^{\ell_{C_1}}(y) \leq i$. The compatibility relations then force $a_0^{\ell_{C_2}}(y) \geq d_1 - i$. Since $\ell_{C_2}$ contains the divisor $d_1 y$, this means that $|iy|$ is a $g^1_i$ on $C_2$, i. e. $y$ is one of the $i(i^2 - 1)$ Weierstraß point of $C_2$. Since $C_2$ is generic, it has only ordinary Weierstraß points, so we must have equality, and $\ell$ is unique by Remark 2.4.1. □

We now turn to cases where $n = 3$. We will first suppose that $d_1, d_2 > 0$, while $d_3 < 0$, and we write $b := -d_3$ and $d := d_1 + d_2 = g + b - 1$.

For $b = 1$, the following result has been proven already in [65, Proposition 3.3] and [24, Lemma 6.2].

**Lemma 2.4.6.** *Let $(C; x_2, x_3)$ be a generic 2-pointed curve of genus $g$, and let $F$ be the family in $\overline{\mathcal{M}}_{g,3}$ obtained by letting a point $x_1$ vary on $C$. Then we have*

$$\begin{aligned} &F \cdot \overline{D}_{\underline{d}} = gd_1^2 - (g - d_2)_+, \\ &F \cdot \psi_1 = 2g, \qquad F \cdot \psi_2 = 1, \qquad F \cdot \psi_3 = 1, \\ &F \cdot \delta_{0:12} = 1, \qquad F \cdot \delta_{0:13} = 1. \end{aligned}$$

*Proof.* Suppose first that $g = 1$, i. e. $b = d$. Then a $g^1_d$ containing the divisors $d_1x_1 + d_2x_2$ and $dx_3$ exists if and only if these are linearly equivalent, and since $d_2 > 0$ this gives $d_1^2$ possibilities for $x_1$ as claimed.

If $g > 1$, we degenerate $C$ to a transverse union $C = E \cup_y C'$ such that $(E; x_2, y)$ is a generic 2-pointed elliptic curve and $(C'; y, x_3)$ is a generic 2-pointed curve of genus $g - 1$. Then there is a decomposition $F = F_E + F_{C'}$ of 1-cycles on $\overline{\mathcal{M}}_{g,3}$, where $F_E$ and $F_{C'}$ correspond to the cases $x_1 \in E$ and $x_1 \in C'$. These are in a natural way pushforwards via gluing morphisms of 1-cycles $F'_E$ and $F'_{C'}$ on $\overline{\mathcal{M}}_{1,3}$ and $\overline{\mathcal{M}}_{g-1,3}$, respectively. We will show that

$$F_E \cdot \overline{D}_{\underline{d}} = F'_E \cdot \overline{D}_{(d_1,d_2;-d)} \quad (= d_1^2 \text{ by the above}), \tag{2.8}$$

$$F_{C'} \cdot \overline{D}_{\underline{d}} = \begin{cases} F'_{C'} \cdot \overline{D}_{(d_1,d_2-1;d_3)} & \text{if } d_2 > 1, \\ (g-1)(d_1^2 - 1) & \text{if } d_2 = 1, \end{cases} \tag{2.9}$$

and by induction we conclude that

$$F \cdot \overline{D}_{\underline{d}} = \sum_{i=1}^{d_2} d_1^2 + \sum_{i=d_2+1}^{g} (d_1^2 - 1) = gd_1^2 - (g - d_2)_+.$$

For showing (2.8), let $\ell = (\ell_E, \ell_{C'})$ be a $g^1_d$ having the required vanishing. Then $\ell_E$ has a section not vanishing at $y$, so by the compatibility conditions $\ell_{C'}$ must be totally ramified there. Counting dimensions as in the proof of Lemma 2.4.5, we find that the latter cannot have a base point at $y$, so again by compatibility $\ell_E$ needs to have a section vanishing to order $d$ at $y$. This is equivalent to requiring that $(E; x_1, x_2, y)$ lies in $\overline{D}_{(d_1,d_2;-d)}$.

Now consider (2.9). Since $\ell_E$ contains $d_1y + d_2x_2$, and by genericity $d_2x_2 \not\equiv d_2y$, it cannot also contain the divisor $dy$, so we must have $a_1^{\ell_E}(y) \leq d - 1$. By the compatibility condition then $a_0^{\ell_{C'}}(y) \geq 1$, and after removing the base point we obtain a $g^1_{d-1}$ on $C'$

containing the divisor $d_1x_1 + (d_2 - 1)y$ and having a section vanishing to order $b$ at $x_3$. For $d_2 > 1$ this is equivalent to $(C'; x_1, y, x_3) \in \overline{D}_{(d_1, d_2-1; d_3)}$, while for $d_2 = 1$ the answer is given in Lemma 2.4.3. □

**Lemma 2.4.7.** *Let $(C_1; y)$ be a generic 1-pointed curve of genus $i \geq 1$, $(C_2; x_2, x_3, y)$ a generic 3-pointed curve of genus $g - i$, $(C = C_1 \cup_y C_2; x_2, x_3)$ the 2-pointed curve obtained by gluing $C_1$ and $C_2$ at $y$, and $F$ the family in $\overline{\mathcal{M}}_{g,3}$ obtained by letting a point $x_1$ move along $C_1$. Then we have*

$$F \cdot \overline{D}_{\underline{d}} = i(d_1^2 - 1) + (i - d_1)_+,$$
$$F \cdot \psi_1 = 2i - 1, \qquad F \cdot \delta_{i:1} = -1, \qquad F \cdot \delta_{i:\emptyset} = 1.$$

*These formulas also hold for $d_2 = 0$.*

*Proof.* Let $\ell = (\ell_{C_1}, \ell_{C_2})$ be a limit $g^1_d$ on $C$ satisfying the given vanishing conditions and write $\ell_{C_2} = a_0 y + \ell'_{C_2}$, where $a_0 := a_0^{\ell_{C_2}}(y)$. Then $\ell'_{C_2}$ must contain the divisor $(d_1 - a_0)y + d_2x_2$ and have a section vanishing to order $b$ at $x_3$, so it corresponds to the Schubert cycle $\sigma_{r-1} \cdot \sigma_{b-1}$ in $\mathbb{G}(1, r)$, where

$$r := h^0(C_2, (d_1 - a_0)y + d_2x_2) - 1 = d - a_0 - g + i$$

by Riemann-Roch. This is non-empty only if $b \leq r$, or equivalently if $a_0 \leq i - 1$.

In case $a_0 < d_1$, we have $a_1^{\ell_{C_2}}(y) = d_1$, so $a_0^{\ell_{C_1}}(y) \geq d_2$. Then $\ell'_{C_1} := \ell_{C_1} - d_2y$ is a $g^1_{d_1}$ fully ramified at $x_1$. Since $C_1$ is generic and therefore has only ordinary Weierstraß points, this is possible only if $d_1 \geq i$. Since

$$a_1^{\ell_{C_1}}(y) \geq d - a_0 \geq d - i + 1,$$

$\ell'_{C_1}$ vanishes to order $d_1 - i + 1$ at $y$, so by Lemma 2.4.3 the number of possibilities is given by

$$F \cdot \overline{D}_{\underline{d}} = i(d_1^2 - 1).$$

If on the other hand $a_0 = d_1$, we have $d_1 \leq i - 1$ by the above. By another Schubert cycle computation for $\ell'_{C_2}$ we find that we need to have $a_1^{\ell_{C_2}}(y) \leq i$, so $a_0^{\ell_{C_1}}(y) \geq d - i$. Thus $\ell_{C_1} - (d - i)y$ is now a $g^1_i$ having $d_1x_1 + (i - d_1)y$ as a section. Applying Lemma 2.4.6 with $\underline{d} = (d_1, i - d_1; -1)$, we find that

$$F \cdot \overline{D}_{\underline{d}} = id_1^2 - d_1.$$

Both arguments also go through when $d_2 = 0$. □

Still considering cases where $n = 3$, we now suppose that $d_1 > 0$, while $d_2, d_3 < 0$, and we write $b_j := -d_j$ for $j = 2, 3$ and $b := -d_2 - d_3$, so that $d_1 = g + b - 1$.

**Lemma 2.4.8.** *Let $(C; x_1, x_3)$ be a generic 2-pointed curve of genus $g$, and let $F$ be the family in $\overline{\mathcal{M}}_{g,3}$ obtained by letting a point $x_2$ vary on $C$. Then we have*

$$
\begin{aligned}
&F \cdot \overline{D}_{\underline{d}} = g d_2^2, && && \\
&F \cdot \psi_1 = 1, && F \cdot \psi_2 = 2g, && F \cdot \psi_3 = 1, \\
&F \cdot \delta_{0:12} = 1, && F \cdot \delta_{0:23} = 1. &&
\end{aligned}
$$

*Proof.* This is similar to the proof of Lemma 2.4.6: Let $C = E \cup_y C'$ again with now $x_1 \in E$ and $x_3 \in C'$. Then $F = F_E + F_{C'}$ with

$$
\begin{aligned}
F_E \cdot \overline{D}_{\underline{d}} &= F'_E \cdot \overline{D}_{(d_1; d_2, d_3 - g + 1)} = d_2^2 \quad \text{and} \\
F_{C'} \cdot \overline{D}_{\underline{d}} &= F'_{C'} \cdot \overline{D}_{(d_1 - 1; d_2, d_3)}.
\end{aligned}
$$

The only difference to before is that now $\ell_{C'}$ has a $b_2$-fold base point at $y$ in case $x_2 \in E$. The result follows by induction. □

**Lemma 2.4.9.** *Let $(C_1; y)$ be a generic 1-pointed curve of genus $i \geq 1$, $(C_2; x_1, x_3, y)$ a generic 3-pointed curve of genus $g - i$, $\big(C = C_1 \cup_y C_2; x_1, x_3\big)$ the 2-pointed curve obtained by gluing $C_1$ and $C_2$ at $y$, and $F$ the family in $\overline{\mathcal{M}}_{g,3}$ obtained by letting a point $x_2$ move along $C_1$. Then we have*

$$
\begin{aligned}
&F \cdot \overline{D}_{\underline{d}} = i(d_2^2 - 1), && && \\
&F \cdot \psi_2 = 2i - 1, && F \cdot \delta_{i:2} = -1, && F \cdot \delta_{i:\emptyset} = 1.
\end{aligned}
$$

*These formulas also hold for $d_3 = 0$.*

*Proof.* Let $\ell = (\ell_{C_1}, \ell_{C_2})$ be a limit $g^1_{d_1}$ on $C$ satisfying the given vanishing conditions. Then $\ell_{C_2}$ must include the divisor $d_1 x_1$, so $a_0^{\ell_{C_2}}(y) = 0$. The section of $\ell_{C_2}$ vanishing to order $b_3$ at $x_3$ must also vanish to order $a_1 := a_1^{\ell_{C_2}}(y)$ at $x_1$: otherwise the corresponding section of $\ell_{C_1}$ would have to be fully ramified at $y$ while at the same time vanishing to order $b_2$ at $x_2$, which is absurd. We thus need

$$h^0(C_2, d_1 x_1 - a_1 y - b_3 x_3) = b_2 + i - a_1 \geq 1 \iff a_1 \leq b_2 + i - 1,$$

where we used Riemann-Roch and the genericity of the points on $C_2$. By compatibility, $a_0^{\ell_{C_1}}(y) \geq g - i + b_3$, and thus $\ell_{C_1} - (g - i + b_3)y$ is a $g^1_{i+b_2-1}$ which is fully ramified at $y$ and vanishes to order $b_2$ at $x_2$. By Lemma 2.4.4 there are $i(d_2^2 - 1)$ possibilities for $x_2$. □

We now finally consider the situation $n = 4$ with $d_1, d_2 > 0$ and $d_3, d_4 < 0$. We write $b_j := -d_j$ for $j = 3, 4$ and $b := b_3 + b_4$, so that $d := d_1 + d_2 = g + b - 1$.

**Lemma 2.4.10.** *Let $(C_1; x_1, x_3)$ be a generic 2-pointed curve of genus $i$ with $1 \leq i \leq g$, $(C_2; x_2, x_4, y)$ a generic 3-pointed curve of genus $g - i$, and let $F$ be the family in $\overline{\mathcal{M}}_{g,4}$ obtained by gluing $y$ to a moving point of $C_1$. Then we have*

$$F \cdot \overline{D}_{\underline{d}} = i(d_1 + d_3 - i + 1)^2 - (i - d_1)_+,$$
$$F \cdot \psi_1 = 1, \qquad F \cdot \psi_3 = 1,$$
$$F \cdot \delta_{i:13} = -2i, \qquad F \cdot \delta_{i:1} = 1, \qquad F \cdot \delta_{i:3} = 1.$$

*Proof.* Let $\ell = (\ell_{C_1}, \ell_{C_2})$ be a limit $g^1_d$ on $C$ satisfying the given vanishing conditions. Then $\ell$ contains the divisor $(d_1x_1 + d_2y, d_1y + d_2x_2)$.

Suppose first that $d_1 + d_3 \geq i$, or $d_2 + d_4 \leq g - i - 1$. Then the base locus of $\ell_{C_2}$ cannot contain $d_1y$, since $h^0(C_2, d_2x_2 - b_4x_4) = 0$ by Riemann-Roch and genericity. Hence $a_1^{\ell_{C_2}}(y) = d_1$, and by a dimension count $a_0^{\ell_{C_2}}(y) \leq i - d_3 - 1$, with equality attained for a unique $g^1_d$. Thus $a^{\ell_{C_1}}(y) \geq (d_2, g - i - d_4)$, and we can apply Lemma 2.4.8 with $\underline{d} = (d_1; d_2 + d_4 - g + i, d_3)$ to find

$$F \cdot \overline{D}_{\underline{d}} = i(d_2 + d_4 - g + i)^2.$$

If $d_1 + d_3 < i - 1$, then $h^0(C_1, d_1x_1 - b_3x_3) = 0$, so $d_2y$ cannot be in the base locus of $\ell_{C_1}$, forcing $a_1^{\ell_{C_1}}(y) = d_2$ and thus $a_0^{\ell_{C_2}}(y) = d_1$. As in the proof of Lemma 2.4.9, we find that $a^{\ell_{C_2}}(y) \leq (d_1, i - d_3 - 1)$, so $a^{\ell_{C_1}}(y) \geq (g - i - d_4, d_2)$. Applying Lemma 2.4.6 with $\underline{d} = (d_2 + d_4 - g + i, d_1; d_3)$ then gives

$$F \cdot \overline{D}_{\underline{d}} = i(d_2 + d_4 - g + i)^2 - (i - d_1)_+.$$

Finally, if $d_1 + d_3 = i - 1$ we obtain $a^{\ell_{C_2}}(y) = (d_1, d_1 + 1)$ and $\ell_{C_2} - d_1y$ must have a section vanishing to order 1 at $y$ and $b_4$ at $x_4$. Since $h^0(C_2, d_2x_2 - y - b_4x_4) = 0$, this is impossible, so in this case

$$F \cdot \overline{D}_{\underline{d}} = 0,$$

which is consistent with the other two formulas. □

## 2.5 Computation of the boundary coefficients

For computing the boundary coefficients of $\overline{D}_{\underline{d}}$ we will use a bootstrapping approach, considering first the easiest non-trivial case $n = 2$, then generalizing to the case $n > 2$ with exactly one $d_j < 0$, and finally tackling the most general situation.

### 2.5.1 The case $n = 2$

For ease of notation, we will write $\underline{d} = (d_1, d_2) = (g+b-1, -b)$ with $b \geq 1$ and denote the corresponding divisor by $\overline{D}_{(g+b-1,b)} =: \overline{D}_b$.

**Proposition 2.5.1.** *The class of $\overline{D}_b$ is given by*

$$\begin{aligned}\left[\overline{D}_b\right] = & -\lambda + \binom{g+b}{2}\psi_1 + \binom{b}{2}\psi_2 - 0 \cdot \delta_0 - \binom{g+1}{2}\delta_{0:12} \\ & - \sum_{i=1}^{g-1}\left[\binom{g-i+b}{2}\delta_{i:1} + \binom{g-i+1}{2}\delta_{i:12}\right].\end{aligned}$$

*Proof.* From Section 2.3 we know that $a = -1$, $c_1 = \binom{g+b}{2}$ and $c_2 = \binom{-b+1}{2} = \binom{b}{2}$. Intersecting $\overline{D}_b$ with the family from Lemma 2.4.5, we find that $b_{g-i:12} = -\binom{i+1}{2}$, or dually $b_{i:12} = -\binom{g-i+1}{2}$ for $i = 0, \dots, g-2$. From the family in Lemma 2.4.7 (taking $d_2 = 0$), we get

$$b_{i:1} = (2i-1)c_1 + b_{g-i:12} - i((g+b-1)^2 - 1) = -\binom{g-i+b}{2} \quad \text{for } i = 2, \dots, g-1,$$

and Lemma 2.4.9 with $d_3 = 0$ gives $b_{g-1:12} = -1$. Using Lemma 2.4.7 once more, we get the value for $b_{1:1}$, while finally Lemma 2.4.2 leads to $b_0 = (b_{g-1:12} - a)/12 = 0$. □

**Remark 2.5.2.** Note that when we pull back from $\overline{\mathcal{M}}_{g,1}$ the Weierstraß divisor $\overline{\mathcal{W}}_g$, whose class is given in (2.2), we get by Lemma 2.2.1 that

$$\begin{aligned}\left[\pi_2^*\overline{\mathcal{W}}_g\right] &= -\lambda + \binom{g+1}{2}\psi_1 - \binom{g+1}{2}\delta_{0:12} - \sum_{i=1}^{g-1}\binom{g-i+1}{2}\left(\delta_{i:1} + \delta_{i:12}\right) \\ &= \left[\overline{D}_1\right]\end{aligned}$$

as expected. Furthermore it is easy to see that a 2-pointed curve $(C = C' \cup_y \mathbb{P}^1; x_1, x_2)$ with $x_1, x_2 \in \mathbb{P}^1$ is in $\overline{D}_b$ exactly when it has a limit $g^1_{g+b-1}$ whose $C'$-aspect satisfies $a^{\ell_{C'}}(y) = (b-1, g+b-1)$, which is the case if and only if $y$ is a Weierstraß point of $C'$. From Lemma 2.2.3 we obtain accordingly

$$\pi_{(12\mapsto\bullet)*}\left(\left[\overline{D}_b\right] \cdot \delta_{0:12}\right) = -\lambda + \binom{g+1}{2}\psi_\bullet - \sum_{i=1}^{g-1}\binom{g-i+1}{2}\delta_{i:\bullet} = \left[\overline{\mathcal{W}}_g\right].$$

### 2.5.2 The case of exactly one negative $d_j$

We now consider the next simplest case where exactly one of the $d_j$ is negative (for definiteness, and without loss of generality, we take $d_n < 0$).

**Remark 2.5.3.** Here and in the next section we will several times apply a "pushdown" argument which runs as follows: Let $j, k \in [n]$ be two indices such that $d_j$ and $d_k$ have the same sign, and suppose that $\alpha \in \mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ is one of the basic divisor classes described in Section 1.3 satisfying $\beta := \pi_{(jk\mapsto\bullet)*}(\alpha \cdot \delta_{0:jk}) \neq 0$. Since then no other basis element is mapped to $\beta$ and $\pi_{(jk\mapsto\bullet)*}([\overline{D}_{\underline{d}}] \cdot \delta_{0:jk}) = \left[\overline{D}_{\underline{d}'}\right]$ with $\underline{d}'$ as in Lemma 2.2.4, the coefficient of $\alpha$ in the expression for $\overline{D}_{\underline{d}}$ is the same as the coefficient of $\beta$ in the class of $\overline{D}_{\underline{d}'}$.

**Proposition 2.5.4.** *If $d_j > 0$ for $j = 1, \dots, n-1$, then the class of $\overline{D}_{\underline{d}}$ is given by*

$$[\overline{D}_{\underline{d}}] = -\lambda + \sum_{j=1}^{n} \binom{d_j + 1}{2} \psi_j - 0 \cdot \delta_0 - \sum_{i,S \subseteq [n-1]} \binom{|d_S - i| + 1}{2} \delta_{i:S}.$$

*Proof.* We already know from Section 2.3 that $a = -1$ and $c_j = \binom{d_j+1}{2}$.

For the $b_{0:jk}$ with $j, k \in [n-1]$, we can apply the pushdown argument explained in Remark 2.5.3 to the divisor class $\delta_{0:jk}$ itself, which gets mapped to $-\psi_\bullet$. Thus we have $b_{0:jk} = -c_\bullet$, where $c_\bullet$ is the coefficient of $\psi_\bullet$ in the expression for $\pi_{(jk\mapsto\bullet)*}(\overline{D}_{\underline{d}} \cdot \delta_{0:jk})$. Since $j, k \leq n-1$, we have $d_j, d_k > 0$, so we can apply Lemma 2.2.4 to find that $b_{0:jk} = -\binom{d_j+d_k+1}{2}$. Similarly, in order to compute $b_{0:S}$ for $S \subseteq [n-1]$, we can intersect with one divisor $\delta_{0:jk}$ with $j, k \in S$ at a time and push down via the appropriate forgetful maps; by inductively reasoning as before we find

$$b_{0:S} = -\binom{d_S + 1}{2} \qquad \text{for } S \subseteq [n-1].$$

Looking at Lemma 2.2.3 and using a simple induction again, we see that when we successively let all of the points $x_1, \dots, x_{n-1}$ come together and push down via the appropriate forgetful maps, the divisor $\delta_{i:\emptyset}$ is mapped to $\delta_{i:\emptyset} = \delta_{g-i:12}$ on $\overline{\mathcal{M}}_{g,2}$, so by Lemma 2.2.4 and Proposition 2.5.1 again we see that

$$b_{i:\emptyset} = -\binom{i+1}{2} \qquad \text{for } i \geq 1.$$

Next, using the test family from Lemma 2.4.7 we get that

$$b_{i:j} = (2i-1)c_j + b_{i:\emptyset} - i(d_j^2 - 1) - (i - d_j)_+ = -\binom{|d_j - i| + 1}{2},$$

for $j \in [n-1]$, and using a pushdown argument once again we arrive at

$$b_{i:S} = -\binom{|d_S - i| + 1}{2} \qquad \text{for } S \subseteq [n-1] \text{ and } i \geq 1.$$

Finally, the fact $b_0 = 0$ follows again from letting all of the points $x_1, \ldots, x_{n-1}$ coalesce, pushing down to $\overline{\mathcal{M}}_{g,2}$ and recurring to Proposition 2.5.1. $\square$

**Remark 2.5.5.** If $b = 1$, we expect $\overline{D}_{\underline{d}}$ to be the pullback to $\overline{\mathcal{M}}_{g,n}$ of the divisor

$$D = \left\{ [C; x_1, \ldots, x_{n-1}] \,\middle|\, h^0\left(C, d_1x_1 + \cdots + d_{n-1}x_{n-1}\right) \geq 2 \right\}$$

which was considered by Logan [65]. Indeed, we have for $S \subseteq [n-1]$ that

$$b_{i:S\cup\{n\}} = b_{g-i:[n-1]\setminus S} = -\binom{|g - d_S - g + i| + 1}{2} = b_{i:S},$$

and moreover $c_n = 0$ and $b_{0:jn} = -c_j$ for $j \in [n-1]$. Lemma 2.2.1 thus shows that

$$[D] = -\lambda + \sum_{j=1}^{n-1} \psi_j - 0 \cdot \delta_0 - \binom{|d_S - i| + 1}{2} \delta_{i:S},$$

which is consistent with the computations in [65].

### 2.5.3 The general case

We will now finally deal with the most general case where there are at least two $d_j$ of either sign, thereby proving formula (2.1). We exclude the degenerate case where some $d_j$ equals 0, since in this case the divisor $\overline{D}_{\underline{d}}$ is just a pullback of some $\overline{D}_{\underline{d}'}$ from some moduli space with fewer marked points, so its class can easily be computed from Theorem 2.5.6 with the help of the formulas in Lemma 2.2.1.

**Theorem 2.5.6.** *The class of $\overline{D}_{\underline{d}}$ in $\operatorname{Pic}(\overline{\mathcal{M}}_{g,n})$ is given by*

$$\begin{aligned}[\overline{D}_{\underline{d}}] = &- \lambda + \sum_{j=1}^{n} \binom{d_j + 1}{2} \psi_j - 0 \cdot \delta_0 \\ &- \sum_{\substack{i,S \\ S \subseteq S_+}} \binom{|d_S - i| + 1}{2} \delta_{i:S} - \sum_{\substack{i,S \\ S \not\subseteq S_+}} \binom{d_S - i + 1}{2} \delta_{i:S}.\end{aligned}$$

*Proof.* From Section 2.3 we know that $a = -1$ and $c_j = \binom{d_j+1}{2}$. Using the by now familiar pushdown technique, we get from Proposition 2.5.4 that $b_0 = 0$ and

$$b_{i:S} = -\binom{|d_S - i| + 1}{2} \qquad \text{for } S \subseteq S_+.$$

Thus we are left with computing the $b_{i:S}$ where the points indexed by $S_-$ do not all lie on the same component.

Suppose first that $\emptyset \neq S \subsetneq S_-$. By letting the points from $S_+$, $S$ and $S_- \setminus S$ respectively come together, we can reduce to the case $n = 3$ with $d_1 = d_{S_+} > 0$, $d_2 = d_S < 0$ and $d_3 = d_{S_- \setminus S} < 0$. The divisor $\delta_{i:S}$ is mapped to $-\psi_2$ for $i = 0$ and to $\delta_{i:2}$ for $i > 0$. We know that $c_2 = \binom{d_2+1}{2}$, while for $i > 0$ we get from Lemma 2.4.9 that

$$b_{i:2} = (2i-1)\binom{d_2+1}{2} + b_{i:\emptyset} - i(d_2^2 - 1) = -\binom{d_2 - i + 1}{2}.$$

Thus in total we deduce by Lemma 2.2.4 that

$$b_{i:S} = -\binom{d_S - i + 1}{2} \qquad \text{for } \emptyset \neq S \subsetneq S_-.$$

Finally, let $S = S_1 \cup S_2$ with $\emptyset \neq S_1 \subsetneq S_+$ and $\emptyset \neq S_2 \subsetneq S_-$. Letting the points from $S_1$, $S_+ \setminus S_1$, $S_2$ and $S_- \setminus S_2$ respectively come together, we reduce to the computation of $b_{i:13}$ in the case $n = 4$. Taking the family from Lemma 2.4.10 we find

$$\begin{aligned} b_{i:13} &= \frac{1}{2i}\left(c_1 + c_3 + b_{i:1} + b_{i:3} - i(d_1 + d_3 - i + 1)^2 + (i - d_1)_+\right) \\ &= -\binom{d_1 + d_3 - i + 1}{2}. \end{aligned}$$

Note that although in Lemma 2.4.10 we require $i \geq 1$, the above formula is invariant under the substitution $(i, d_1, d_3) \mapsto (g - i, d_2, d_4)$, so it holds also for $i = 0$. Thus in total we get

$$b_{i:S} = -\binom{d_S - i + 1}{2} \qquad \text{for } S = S_+ \cup S_- \text{ with } \emptyset \neq S_1 \subsetneq S_+ \text{ and } \emptyset \neq S_2 \subsetneq S_-,$$

which finishes the computation of $\left[\overline{D}_{\underline{d}}\right]$. □

## 2.6 Extremality of the divisors $\overline{D}_{\underline{d}}$

In [17], Chen and Coskun consider a divisor on $\overline{\mathcal{M}}_{1,n}$ that is defined analogously to our $\overline{D}_{\underline{d}}$, namely as the locus of pointed curves $(E; x_1, \ldots, x_n)$ for which $\sum d_j x_j$ is linearly equivalent to 0. They exhibit an infinite series of non-proportional divisors on $\overline{\mathcal{M}}_{1,3}$ that they show to be extremal in the effective cone. This proves in particular that $\overline{\mathcal{M}}_{1,3}$ is not a Mori Dream Space. Pulling back these divisors to $\overline{\mathcal{M}}_{1,n}$ with $n \geq 4$ gives the same result for these spaces.

The geometry however differs somewhat from our situation where $g \geq 2$, as the translation automorphism essentially fixes one of the marked points on the elliptic curve. Some intersection numbers change slightly due to this phenomenon, and the proof does not run through in the same way. However, we can show at least in some special cases that our divisors $\overline{D}_{\underline{d}}$ are still extremal. As the proof involves topological arguments, we assume $k = \mathbb{C}$ in this section.

**Theorem 2.6.1.** *Let $\underline{d} = (d_1, \ldots, d_{n-2}; -1, -1)$ with $d_i > 0$ for $i = 1, \ldots, n-2$. Then $\overline{D}_{\underline{d}}$ is extremal in the effective cone of $\overline{\mathcal{M}}_{g,n}$.*

In order to prove Theorem 2.6.1, we will use the well-known criterion, a proof for which is given in [17, Lemma 4.1]:

**Lemma 2.6.2.** *Let $D$ be an irreducible effective divisor on a projective variety $X$, and let $C$ be a curve on $X$ with $C \cdot D < 0$ and such that through a general point of $D$ there passes a curve that is algebraically equivalent to $C$. Then $D$ is extremal and rigid.*

We call $C$ a *covering curve* for $D$. Chen and Coskun construct such a curve in $\overline{\mathcal{M}}_{1,3}$ by fixing a 1-pointed elliptic curve $(E; x_1)$ and letting two points $x_2, x_3$ vary on it under the condition that $\sum d_j x_j \equiv 0$. We will do the same, but due to the slightly different intersection multiplicities we only get a finite number of extremal divisors for each $n$.

**Proposition 2.6.3.** *Let $\underline{d}$ be as in Theorem 2.6.1, let $[C'; x_1', \ldots, x_{n-2}'] \in \overline{\mathcal{M}}_{g,n-2}$ be a generic $(n-2)$-pointed curve, let $\pi_{n-1,n} : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-2}$ be the forgetful map that drops the last two markings, and define*

$$X := \Big\{ [C; x_1, \ldots, x_n] \,\Big|\, \pi_{n-1,n}([C; x_1, \ldots, x_n]) = [C'; x_1', \ldots, x_{n-2}'] \text{ and } h^0(C, d_1x_1 + \cdots + d_nx_n) \geq 1 \Big\}.$$

*Then $X$ is a covering curve for $\overline{D}_{\underline{d}}$ with $X \cdot \overline{D}_{\underline{d}} < 0$.*

*Proof.* The covering property is immediate, since the curves $X$ resulting from different choices of $[C'; x_1', \ldots, x_{n-2}']$ are all algebraically equivalent. It is also clear from the construction that $X \cdot \delta_0 = 0$, $X \cdot \delta_{i:S} = 0$ for $1 \leq i \leq g-1$ and any $S$, and $X \cdot \delta_{0:S} = 0$ if $\left|S \cap [n-2]\right| \geq 2$, as the first $(n-2)$ points do not move. Moreover, $X \cdot \lambda = 0$ since the underlying unpointed family is trivial.

For computing the intersection number of $X$ with $\delta_{0:n-1,n}$, we apply the pushdown argument from Remark 2.5.3 and let all the positive as well as the two negative points come together. We are then in the situation where $\underline{d}' = (g+1, -2)$ and the second point is moving. Using Lemma 2.4.4 we find that $X \cdot \delta_{0:n-1,n} = 3g$.

Suppose that there were a point in $X \cap \Delta_{0:j,n-1}$ for some $j$ with $1 \leq j \leq n-2$. Analyzing the possible limit linear series, one finds that the aspect on the genus $g$ component would have to have a base point at the node. Subtracting it would result in a $g^1_g$ containing $d_1x_1 + \cdots + (d_j - 1)x_j + \cdots + d_{n-2}x_{n-2}$. But for a generic choice of $x_1, \ldots, x_{n-2}$, this divisor does not move, so $X \cdot \delta_{0:j,n-1} = 0$, and by symmetry also $X \cdot \delta_{0:j,n} = 0$.

A curve in $X \cap \Delta_{0:j,n-1,n}$ would similarly have to have at least a simple base point in the genus $g$ aspect, and subtracting it would result in a degree $g$ divisor, which generically does not move. Thus $X \cdot \delta_{0:j,n-1,n} = 0$ as well.

Finally, we note that $X \cdot \psi_j = 0$ for $1 \leq j \leq n-2$, as these marked points correspond to constant sections in the family that are not blown up, so their self-intersection is 0.

Now since the coefficients of $\psi_{n-1}$ and $\psi_n$ in the class of $\overline{D}_{\underline{d}}$ are 0, we find that

$$X \cdot \overline{D}_{\underline{d}} = -\binom{-2+0+1}{2} X \cdot \delta_{0:n-1,n} = -3g < 0. \qquad \square$$

It thus remains to show that $\overline{D}_{\underline{d}}$ is irreducible. For this we use the generalized Hurwitz spaces $\widetilde{\mathcal{H}}^{\mu}_{d,w}$ with marked special fiber that were introduced in Section 1.8. Note that if $\underline{d}$ is as in Theorem 2.6.1, the space $\widetilde{\mathcal{H}}^{\mu}_{d,w}$ for $\mu = (d_1, \ldots, d_{n-2})$, $d = g+1$ and $w = 2g+2d-2 = 4g$ maps dominantly onto $\overline{D}_{\underline{d}}$. It thus suffices to show that this space is connected. The technique is essentially the same as for the simple Hurwitz spaces, except that we do not use the monodromy action on the base to get from one point of a fiber to another, but rather keep $C$ constant and construct a path directly on $\widetilde{\mathcal{H}}^{\mu}_{d,w}$ using the geometry of the curve.

**Proposition 2.6.4.** *The spaces $\widetilde{\mathcal{H}}^{\mu}_{g+1,4g}$ are connected.*

*Proof.* Let $\varepsilon\colon \widetilde{\mathcal{H}}^{\mu}_{g+1,4g} \to \mathcal{H}^{\mu}_{g+1,4g}$ be the map forgetting the marking of the special fiber. Since the latter space is irreducible, it suffices to show that we can go from one element of a fiber of $\varepsilon$ to another. If all $\mu_i$ are distinct, then $|\mathrm{Aut}(\mu)| = 1$, so $\varepsilon$ is an isomorphism and the claim is proven. Thus without loss of generality suppose that $\mu_1 = \mu_2$.

Let $(\varphi\colon C \to \mathbb{P}^1, \psi, \chi)$ be a marked covering. Let $p_i := \chi^{-1}(i)$ for $1 \leq i \leq \ell := \ell(\mu)$, and $q_j := \psi^{-1}(j)$ for $1 \leq j \leq d$. Keeping the points $p_3, \ldots, p_\ell$ and $q_1$ fixed, we pick inside $C$ non-intersecting paths $\gamma_1, \gamma_2$ from $p_1$ to $p_2$ and vice versa, such that for all $t \in [0, 1]$ the linear system $|\mu_1\gamma_1(t) + \mu_2\gamma_2(t) + \mu_3 p_3 + \cdots + \mu_\ell p_\ell - q_1|$ contains a reduced divisor. This is possible since these linear systems are always non-empty by Riemann-Roch, and the locus of divisors classes having a non-reduced effective representative is of complex codimension 1 inside $\mathrm{Pic}^g(C)$. We then choose paths for the $q_j$ corresponding to these reduced divisors, i. e. such that

$$\mu_1 p_1(t) + \mu_2 p_2(t) + \mu_3 p_3 + \cdots + \mu_\ell p_\ell \equiv q_1 + q_2(t) + \cdots + q_d(t)$$

and the $q_j(t)$ are distinct from each other and from $q_1$ for all $t \in [0, 1]$. In the case where $h^0(C, \sum_{i=1}^{\ell} \mu_i p_i - q_1) > 1$, it may happen that the points $q_2(1), \ldots, q_d(1)$ lie in a different fiber than before, but in that case we can just pick a path inside that linear system which takes them back to the original fiber, keeping all the $p_i$ as well as $q_1$ fixed. At this point, the position of the points $q_2, \ldots, q_d$ is a permutation of their original configuration.

We now pick a path inside the linear system $\left|\sum_{i=1}^{\ell} \mu_i p_i\right|$ that takes the $q_j$ back to their original position. Since $C$ is connected, the monodromy group of $\pi_1(\mathbb{P}^1 \setminus (B \cup \{c\}), b)$ acts transitively on the fiber over $b$ (see Section 1.8 for the notation). As any transitive subgroup of $S_d$ that is generated by transpositions must be the full symmetric group, there is a path $\gamma \in \pi_1(\mathbb{P}^1 \setminus (B \cup \{c\}), b)$ whose action on $\varphi^{-1}(b)$ keeps $q_1$ fixed and induces that permutation on $q_2, \ldots, q_d$ which returns them to their original positions. Moving the points $q_j$ simultaneously along the respective homotopy liftings of the path $\gamma$, we arrive back at the original configuration with the points $p_1$ and $p_2$ interchanged, thus proving the claim. $\square$

**Corollary 2.6.5.** *Let $\underline{d} = (d_1, \ldots, d_{n-2}; -1, -1)$ with $d_i > 0$ for $i = 1, \ldots, n-2$. Then $\overline{D}_{\underline{d}}$ is irreducible.*

*Proof.* The space $\widetilde{\mathcal{H}}^{\mu}_{g+1,4g}$ is smooth and connected by Proposition 2.6.4, hence irreducible. Since it maps dominantly onto $\overline{D}_{\underline{d}}$, the claim follows. $\square$

The Theorem is now an easy consequence:

*Proof of Theorem 2.6.1.* By Corollary 2.6.5, $\overline{D}_{\underline{d}}$ is irreducible, and by Proposition 2.6.3 there is a covering curve that has negative intersection with it. Thus it is extremal by the criterion from Lemma 2.6.2. $\square$

# Chapter 3

# The final log canonical model of $\overline{\mathcal{M}}_6$

## 3.1 Introduction

A general smooth curve $C$ of genus 6 has five planar sextic models with four nodes in general linear position. Blowing up these four points, and embedding the resulting surface in $\mathbb{P}^5$ via its complete anticanonical linear series, one finds that the canonical model of $C$ is a quadric hyperplane section of a degree 5 del Pezzo surface $S$. As any four general points in $\mathbb{P}^2$ are projectively equivalent, this surface is unique up to isomorphism. Its automorphism group is finite and isomorphic to the symmetric group $S_5$ (see e. g. [81]). The surface $S$ contains ten $(-1)$-curves, which are the four exceptional divisors of the blowup, together with the proper transforms of the six lines through pairs of the points. There are five ways of choosing four non-intersecting $(-1)$-curves on $S$, inducing five blowdown maps to $S \to \mathbb{P}^2$, and restricting to the five $g^2_6$'s on $C$. Residual to the latter are five $g^1_4$'s, which can be seen in each planar model as the projection maps from the four nodes, together with the map that is induced on $C$ by the linear system of conics passing through the nodes.

This description gives rise to a birational map

$$\varphi : \overline{\mathcal{M}}_6 \dashrightarrow X_6 := \big| -2K_S \big| / \operatorname{Aut}(S),$$

which is well-defined and injective on the sublocus $(\mathcal{M}_6 \cup \Delta_0^{\mathrm{irr}}) \setminus \overline{\mathcal{GP}}_6$. Here $\Delta_0^{\mathrm{irr}}$ denotes the locus of irreducible singular stable curves, and $\overline{\mathcal{GP}}_6$ is the closure of the Gieseker-Petri divisor of curves having fewer than five $g^1_4$'s (or residually, $g^2_6$'s). Its class is computed in [29] as

$$\left[\overline{\mathcal{GP}}_6\right] = 94\lambda - 12\delta_0 - 50\delta_1 - 78\delta_2 - 88\delta_3,$$

cf. equation (1.4). It is an extremal effective divisor of minimal slope on $\overline{\mathcal{M}}_6$ (see [16]). Curves in $\overline{\mathcal{GP}}_6$ have planar sextic models in which the nodes fail to be in general linear position, which forces the anticanonical image of the blown-up $\mathbb{P}^2$ to become singular. In the generic case, three of the nodes become collinear, and the line through them is a $(-2)$-curve that gets contracted to an $A_1$ singularity.

The aim of this chapter is to study the birational model $X_6$, determine its place in the log minimal model program of $\overline{\mathcal{M}}_6$, and use it to derive an upper bound on the moving slope of this space. In order to do so, we will start in Section 3.2 by determining explicitly the way in which $\varphi$ extends to the generic points of the divisors $\Delta_i$, $i = 1, 2, 3$, and $\overline{\mathcal{GP}}_6$. The divisors $\Delta_1$ and $\Delta_2$ are shown to be contracted by 1 and 4 dimensions, as the low genus components are replaced by a cusp and an $A_5$ singularity, respectively. The divisors $\Delta_3$ and $\overline{\mathcal{GP}}_6$ turn out to be contracted to points, and the curves parameterized by them are shown to be mapped to the classes of certain non-reduced degree 10 curves on $S$.

In Section 3.3, we will then construct test families along which $\varphi$ is defined and determine their intersection numbers with the standard generators of $\mathrm{Pic}(\overline{\mathcal{M}}_6)$ as well as with $\varphi^*\mathcal{O}_{X_6}(1)$. Having enough of those enables us in Section 3.4 to finally compute the class of the latter. This computation is then used that to establish the upper bound $s'(\overline{\mathcal{M}}_6) \leq 102/13$ for the moving slope of $\overline{\mathcal{M}}_6$, as well as to show that log canonical model $\overline{\mathcal{M}}_6(\alpha)$ is isomorphic to $X_6$ for $16/47 < \alpha \leq 35/102$ and becomes trivial below this range.

## 3.2 Defining $\varphi$ in codimension 1

In this section we will see how $\varphi$ is defined on the generic points of the codimension 1 subloci of $\overline{\mathcal{M}}_6$ parameterizing curves whose canonical image does not lie on $S$. As mentioned in the introduction, these are the divisors $\Delta_i$, $i = 1, 2, 3$, as well as $\overline{\mathcal{GP}}_6$, and they will turn out to constitute exactly the exceptional locus of $\varphi$.

**Proposition 3.2.1.** *A curve $C = C_1 \cup_p C_2 \in \Delta_1$ with $p$ not a Weierstraß point on $C_2 \in \mathcal{M}_5$ is mapped to the class of a cuspidal curve whose pointed normalization is $(C_2, p)$. In particular, the map $\varphi$ contracts $\Delta_1$ by one dimension.*

*Proof.* This follows readily from the existence of a moduli space for pseudostable curves (see [76]). More concretely, let $\pi : \mathscr{C} \to B$ be a flat family of genus 6 curves whose general fiber is smooth and Gieseker-Petri general, and with special fiber $C$. Then the twisted linear system $|\omega_\pi(C_1)|$ maps $\mathscr{C}$ to a flat family of curves in $|-2K_S|$. It restricts to $\mathcal{O}_{C_1}$ on $C_1$ and to $\omega_{C_2}(2p)$ on $C_2$, so it contracts $C_1$ and maps $C_2$ to a cuspidal curve of arithmetic genus 6, which lies on a smooth del Pezzo surface. $\square$

**Proposition 3.2.2.** *Let $C = C_1 \cup_p C_2 \in \Delta_2$ be a curve such that*

- *the component $C_2 \in \mathcal{M}_4$ is Gieseker-Petri general, and*
- *$p$ is not a Weierstraß point on either component.*

*Then $C$ is mapped to the class of a curve consisting of $C_2$ together with a line that is 3-tangent to it at $p$. In particular, the map $\varphi$ restricted to $\Delta_2$ has 4-dimensional fibers.*

*Proof.* Let $\mathscr{C} \to B$ be a flat family of genus 6 curves whose general fiber is smooth and Gieseker-Petri general, and with special fiber $C$. Blow up the hyperelliptic conjugate $\widetilde{p} \in C_1$ of $p$ and let $\pi\colon \mathscr{C}' \to B$ be the resulting family with central fiber $C'$ and exceptional divisor $R$. Then the twisted line bundle $\mathscr{L} := \omega_\pi(2C_2)$ restricts to $\omega_{C_2}(3p)$, $\mathcal{O}_{C_1}$ and $\mathcal{O}_R(1)$ on the respective components of $C'$. By a detailed analysis of the family of linear systems $(\mathscr{L}, \pi_*\omega_\pi)$, one can see that it restricts to $|\omega_{C_2}(3p)|$ on $C_2$ and maps $R$ to the 3-tangent line at $p$, while contracting $C_1$. A similar but harder analysis of this type is carried out in Lemma 3.2.5 for the case of $\Delta_3$, to which we refer.

In order to see that the central fiber lies on $S$ as a section of $-2K_S$, it suffices to observe that a generic pointed curve $(C_2, p) \in \mathcal{M}_{4,1}$ has three quintic planar models with a flex at $p$. Each such model has two nodes, projecting from which gives the two $g^1_3$'s. The 3-tangent line $R$ at $p$ meets $C_2$ at two other points, so $C_2 \cup R$ is a plane curve of degree 6 with four nodes (and an $A_5$ singularity). Blowing up the four nodes, which for generic $(C_2, p)$ will be in general linear position, gives the claim.

For showing that the flat limit is unique, it suffices by [37, Lemma 3.10] to show that if $C'$ is any small deformation of $R \cup_p C_2$, then $C_1 \cup_p C_2$ is not the stable reduction of $C'$ in any family in which it occurs as the central fiber. If $C'$ is smooth, this is obviously satisfied. If $p$ stays an $A_5$ singularity in $C'$, then $(C_4, p)$ must move in $\mathcal{M}_{4,1}$, which is also fine. On the other hand, if $(C_4, p)$ stays the same, then the singularity must get better, since there is only a finite number of $g^2_5$'s on $C_4$ having a flex at $p$. For $A_k$ singularities with $k \leq 3$, any irreducible component arising in the stable reduction has genus at most 1, while for $A_4$ singularities the stable tail is a hyperelliptic curve attached at a Weierstraß point. $\square$

**Proposition 3.2.3.** *Let $C = C_1 \cup_p C_2 \in \Delta_3$ be a curve such that on both components,*

- *$p$ is not a Weierstraß point, and*
- *$p$ is not in the support of any odd theta characteristic (in particular, neither component is hyperelliptic).*

*Then $C$ is mapped to the class of a non-reduced degree 10 curve on $S$ consisting of two pairs of intersecting $(-1)$-curves, together with two times a twisted cubic joining the nodes. In particular, $\varphi$ contracts $\Delta_3$ to a point.*

*Proof.* Let $\mathscr{C} \to B$ be a flat family of genus 6 curves whose general fiber is smooth and Gieseker-Petri general, and with special fiber $C$. By assumption, the two base points of $|\omega_{C_i}(-2p)|$ are distinct from each other and from $p$ for $i = 1, 2$. Blow up the total space $\mathscr{C}$ at $p$ and at these four base points. Let $\pi\colon \mathscr{C}' \to B$ denote the resulting family with central fiber $C' = C_1 + C_2 + R + \sum R_{ij}$, where $C_i$ are the proper transforms of the genus 3 components, and $R$ and $R_{ij}$ are the exceptional divisors over $p$ and the base points, respectively. For $i, j = 1, 2$, denote by $p_{ij}$ the point of intersection of $C_i$ with $R_{ij}$, and by $p_i$ the point of intersection of $C_i$ with $R$ (see Figure 3.1).

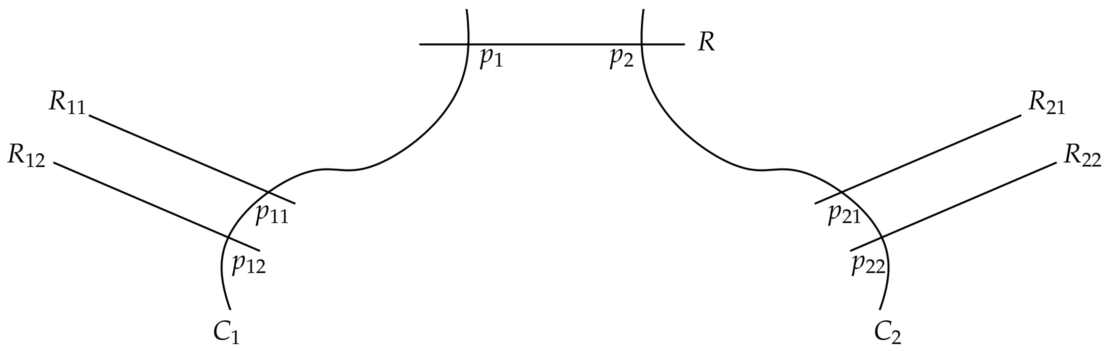


Figure 3.1: The central curve $C'$

Consider the twisted sheaf $\mathscr{L} := \omega_\pi\big(3(C_1 + C_2) + \sum R_{ij}\big)$ on $\mathscr{C}'$. On the various components of $C'$, it restricts to $\mathcal{O}_{C_i}$, $\mathcal{O}_R(6)$ and $\mathcal{O}_{R_{ij}}(1)$, respectively. The pushforward $\pi_*\mathscr{L}$ is not locally free (the central fiber has dimension 7 instead of 6), but it contains $\pi_*\omega_\pi$ as a locally free rank 6 subsheaf. The central fiber $V$ of the image of this sheaf in $\pi_*\mathscr{L}$ is described in Lemma 3.2.5. The induced linear system $(\mathscr{L}\big|_{C'}, V)$ maps $C'$ to the curve $C'' = R + 2R_1 + 2R_2 \subseteq \mathbb{P}^5$, which consists of the middle rational component $R$ embedded as a degree 6 curve, together with twice the tangent lines $R_1$ and $R_2$ at $p_1$ and $p_2$. The genus 3 components $C_i$ are contracted to the points $p_i$. If one introduces coordinates $[x_0 : \cdots : x_5]$ in $\mathbb{P}^5$ corresponding to the basis of $V$ given in Lemma 3.2.5, the image curve lies on the variety

$$\begin{aligned}
\widetilde{S_{2,3}} &= \bigcup_{[\lambda:\mu]\in\mathbb{P}^1} \overline{\varphi_1([\lambda:\mu])\varphi_2([\lambda:\mu])}, \text{ where} \\
\varphi_1([\lambda:\mu]) &:= [\lambda^3 : 0 : \lambda^2\mu : \lambda\mu^2 : 0 : \mu^3] \text{ and} \\
\varphi_2([\lambda:\mu]) &:= [0 : \lambda^2 : 0 : 0 : \mu^2 : 0],
\end{aligned}$$

which is a projection of the rational normal scroll $S_{2,3} \subseteq \mathbb{P}^6$ from a point in the plane of the directrix. This surface is among the possible degenerations of the degree 5 del Pezzo surface investigated in [21, Proposition 3.2], and has the same Betti diagram. In equations, it is given by

$$\widetilde{S_{2,3}} = \left\{ \operatorname{rk}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{pmatrix} \le 1 \right\} \cap \left\{ \operatorname{rk}\begin{pmatrix} x_0 & x_2 & x_3 \\ x_2 & x_3 & x_5 \end{pmatrix} \le 1 \right\},$$

and $C''$ is a quadric section cut out for example by $x_1x_4 - x_0x_5$. When restricted to the directrix, the image of the projection is the line $\widetilde{L} = \{x_0 = x_2 = x_3 = x_5 = 0\}$, which is the singular locus of $\widetilde{S_{2,3}}$. The two branch points $q_i$ of this restriction are the intersection points of the double lines $R_i$ with $\widetilde{L}$.

The image of $\mathscr{C}'$ under the family of linear systems $(\mathscr{L}, \pi_*\omega_\pi)$ lies on a flat family of surfaces $\mathscr{S} \subseteq \mathbb{P}^5 \times B$ with general fiber $S$ and special fiber $\widetilde{S_{2,3}}$. We will construct a birational modification of $\mathscr{S}$ whose central fiber is isomorphic to $S$. Let $\pi' \colon \mathscr{S}' \to B$ be the family obtained by blowing up $\widetilde{L}$, and $S' \subseteq \mathscr{S}'$ the exceptional divisor. The proper transform of $\widetilde{S_{2,3}}$ in $\mathscr{S}'$ is $S_{2,3}$, and the intersection curve $L = S_{2,3} \cap S'$ is its directrix.

We want to show that $S' \cong S$. The ten $(-1)$-curves of the generic fiber cannot all specialize to points in the central limit, since then the whole surface $S$ would be contracted, contradicting flatness. Any exceptional curve that is not contracted must go to $\widetilde{L}$ in the limit, since it is the only curve on $\widetilde{S_{2,3}}$ having a normal sheaf of negative degree. By a chase around the intersection graph of the $(-1)$-curves on $S$, one can see that if one of them is mapped dominantly to $\widetilde{L}$, then at least four of them are. Since the graph is connected, the rest of them get mapped to points that lie on $\widetilde{L}$. Using a base change ramified over 0 if necessary, we may assume that limits of non-contracted curves get separated in $\mathscr{S}'$, while the contracted ones are blown up to lines. Thus there are ten distinct $(-1)$-curves on $S'$, which by the list of possible limits in [21] forces it to be isomorphic to $S$ (note that there are at most seven $(-1)$-curves on a singular degree 5 del Pezzo surface, see [19, Proposition 8.5]).

It remains to see what happens to the central curve $C''$ in the process. Denote by $\psi \colon \mathscr{S}' \to \mathbb{P}^5 \times B$ the map induced by the family of linear systems $(\omega_{\pi'}^\vee(S_{2,3}),\ \pi'_*\omega_{\pi'}^\vee)$. This restricts to $-K_{S'}$ on $S'$, and to a subsystem of $|3F|$ on $S_{2,3}$. Thus the map $\psi$ contracts the latter and has degree 3 on $L$. This implies that $\mathcal{O}_{S'}(L) = \rho^*\mathcal{O}_{\mathbb{P}^2}(1)$ for one of the five maps $\rho \colon S' \to \mathbb{P}^2$, and there are exactly four exceptional curves $E_1, \dots, E_4 \subseteq S'$ that do not meet $L$. The blowdown fibration on $S'$ is given by $|2L - \sum E_i|$, and it contains exactly 3 reducible conics. The flat pullback of $C''$ to $\mathscr{S}'$ contains the two conics in the fibration that meet $L$ at the ramification points of the map $L \to \widetilde{L}$, and the map $\psi$ restricted to $C''$ contracts the two double lines $R_i$ to the points $q_i$ and maps $R$ doubly onto $L$. Thus the flat limit of $C''$ consists of twice the line $L$ together with the two conics in the fibration which are tangent to $L$ at the points $q_i$. Since the non-reduced singularity that is locally given by $y^2(y - x^2)$ has no smooth genus 3 curves in its variety of stable tails, the two conics must actually be reducible and meet $L$ at their nodes. This configuration is unique up to the $\mathrm{Aut}(S)$-action, so the map is well-defined. □

**Remark 3.2.4.** Under the five blowdown maps $S \to \mathbb{P}^2$, the image curve $\varphi(C)$ has two different planar models: One is a double line meeting two of the three reducible conics through the blowup points at their nodes, while the other is a double conic through three blowup points, with the tangent lines at two of them meeting at the fourth (see Figure 3.2). Using an appropriate family, one can see directly that the non-reduced planar curve singularity $y^2(y^2 - x^2)$ has the generic smooth genus 3 curve in its variety of stable tails.

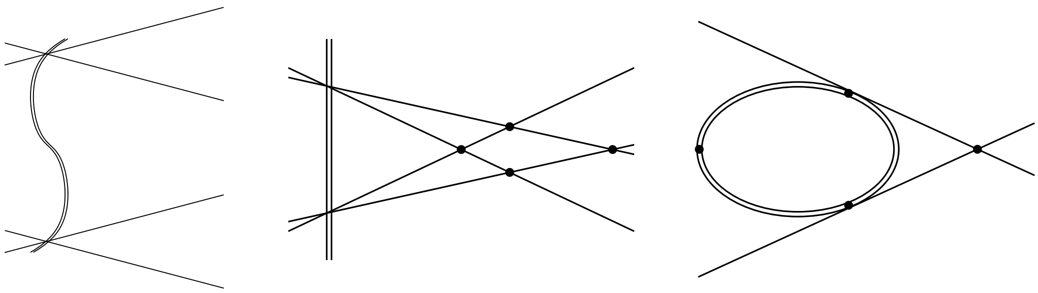

Figure 3.2: The image of $C$ under $\varphi$ and its two planar models

**Lemma 3.2.5.** *Let $\mathscr{C}'$ and $\mathscr{L}$ be constructed as in the proof of Proposition 3.2.3, and let $V$ be the central fiber of the image of $\pi_*\omega_\pi \hookrightarrow \pi_*\mathscr{L}$. Choose coordinates $[s : t]$ on each rational component such that on $R_{1j}$ the coordinate $t$ is centered at $p_{1j}$, on $R_{2j}$ the coordinate $s$ is centered at $p_{2j}$ ($j = 1, 2$), and on $R$ the coordinate $s$ is centered at $p_1$ and $t$ at $p_2$. Then $V$ is spanned by the following sections (on $C_i$ the sections are constants and not listed in the table):*

| $R_{11}$ | $R_{12}$ | $R$ | $R_{21}$ | $R_{22}$ |
|---|---|---|---|---|
| $0$ | $0$ | $s^6$ | $t$ | $t$ |
| $0$ | $0$ | $s^5t$ | $s$ | $s$ |
| $0$ | $0$ | $s^4t^2$ | $0$ | $0$ |
| $0$ | $0$ | $s^2t^4$ | $0$ | $0$ |
| $t$ | $t$ | $st^5$ | $0$ | $0$ |
| $s$ | $s$ | $t^6$ | $0$ | $0$ |

*Proof.* Let $\ell_R = (\mathscr{L}_R, V_R)$ be the $R$-aspect of the unique limit canonical series on the central fiber of $\mathscr{C}'$. By [28, Theorem 2.2], we have that

$$\mathscr{L}_R = \omega_\pi\big(5(C_1 + C_2) + 4\textstyle\sum R_{ij}\big)\Big|_R = \mathcal{O}_R(10)$$

and $\ell_R$ has vanishing sequence $a_R^\ell(p_i) = (2, 3, 4, 6, 7, 8)$ at both $p_i$, so

$$V_R = s^2t^2\langle s^6, s^5t, s^4t^2, s^2t^4, st^5, t^6\rangle.$$

Since on $R$ the inclusion $\mathscr{L}\big|_R \hookrightarrow \mathscr{L}_R$ restricts to $\mathcal{O}_R(6) \hookrightarrow \mathcal{O}_R(10)$, $\sigma \mapsto s^2t^2\sigma$, we have that $s^2t^2V\big|_R \subseteq V_R$. Since the dimensions match, the claim for the central column follows. By dimension considerations, it is clear that $\mathscr{L}$ must restrict to the complete linear series $\big|\mathcal{O}_{R_{ij}}(1)\big|$ on $R_{ij}$.

It remains to show that if a section $\sigma \in V$ fulfills $\mathrm{ord}_{p_i}(\sigma\big|_R) \geq 2$, then $\sigma\big|_{R_{ij}} = 0$ for $j = 1, 2$. For this, let $\sigma_{C_i} \in H^0\big(C, \mathcal{O}_{\mathscr{C}'}(C_i)\big|_C\big)$ be the restriction of a generating section, and let $\varphi_i : H^0\big(C, \mathscr{L}(-C_i)\big|_C\big) \to H^0\big(C, \mathscr{L}\big|_C\big)$ be the map given by $\sigma \mapsto \sigma_{C_i} \cdot \sigma$. For a divisor $D$ on $\mathscr{C}'$ and $k \in \mathbb{N}$ introduce the subspaces

$$V_{i,k}(D) := \left\{\sigma \in H^0\big(C, \mathscr{L} \otimes \mathcal{O}_{\mathscr{C}'}(D)\big|_C\big) \;\middle|\; \operatorname{ord}_{p_i}(\sigma|_R) \geq k\right\},$$
$$V_{i,k} := V_{i,k}(0).$$

Since $\mathscr{L}\big|_{C_i} = \mathcal{O}_{C_i}$, we have that $\operatorname{im}(\varphi_i) = V_{i,1}$. Moreover, we certainly have that $\varphi_i(V_{i,1}(-C_i)) \subseteq V_{i,2}$ and

$$\begin{aligned}\operatorname{codim}\big(\varphi_i(V_{i,1}(-C_i)), V_{i,1}\big) &\leq \operatorname{codim}\big(V_{i,1}(-C_i), H^0\big(C, \mathscr{L}(-C_i)\big|_C\big)\big)\\ &\leq 1.\end{aligned}$$

But from the description of the sections on $R$ it is apparent that $V_{i,2} \subsetneq V_{i,1}$, so we have in fact $\varphi_i(V_{i,1}(-C_i)) = V_{i,2}$. Thus we get

$$\begin{aligned}V_{i,2} &= \varphi_i(V_{i,1}(-C_i))\\ &= \varphi_i\big(\left\{\sigma \in H^0(C, \mathscr{L}(-C_i)\big|_C) \;\middle|\; \sigma\big|_{R_{ij}} = 0 \text{ for } j = 1,2\right\}\big)\\ &\subseteq \left\{\sigma \in H^0(C, \mathscr{L}\big|_C) \;\middle|\; \sigma\big|_{R_{ij}} = 0 \text{ for } j = 1,2\right\}.\end{aligned}$$

□

**Proposition 3.2.6.** *Let $C$ be a smooth Gieseker-Petri special curve whose canonical image lies on a singular del Pezzo surface with a unique $A_1$ singularity, but not passing through that singularity. Then $\varphi$ maps $C$ to a non-reduced degree 10 curve on $S$ consisting of four times a line together with two times each of the three lines meeting it. In particular, $\varphi$ contracts $\overline{\mathcal{GP}}_6$ to a point.*

*Proof.* This can be done by a geometric construction similar to [37, Theorem 3.13]. Here we follow a simpler approach from [57]: A curve $C$ as above has a planar sextic model with three collinear nodes, so the map $\mathcal{G}^1_4 \to \mathcal{M}_6$ is simply ramified over $C$. Thus a neighbourhood of the ramification point will map a (double cover of a) neighbourhood of $C$ to a family of curves of bidegree $(4,4)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. The image of the general fiber will be an irreducible curve with three nodes, while the special fiber goes to four times the diagonal. Blowing up the nodes gives a flat family on $S$ with central fiber as described. □

**Remark 3.2.7.** A pencil of anti-bicanonical curves on a singular del Pezzo surface as above has slope $47/6$ like in the smooth case (for which see Lemma 3.3.1). This would seem to contradict the fact that $\varphi$ contracts the Gieseker-Petri divisor, which has the same slope, to a point. However, any such pencil will contain a curve $C$ having a node at the singular point. The normalization of such a curve is a trigonal curve of genus 5, since blowing up the node and blowing down four disjoint $(-1)$-curves gives a planar quintic model of $C$ together with a line. Using this model, one can show that $\varphi$ maps $C$

to a configuration consisting of three times a line on $S$ together with three lines and two conics meeting it. This arrangement obviously has moduli, so we deduce that $\varphi$ is not defined on $\Delta_0^{\mathrm{trig}} := \{C \in \Delta_0 | C \text{ has a trigonal normalization}\}$, which is a component of $\Delta_0 \cap \overline{\mathcal{GP}}_6$.

## 3.3 Test families

In order to compute the class of $\varphi^*\mathcal{O}_{X_6}(1)$ we now construct some test families and record their intersection numbers with the standard generators of $\mathrm{Pic}(\overline{\mathcal{M}}_6)$ and with $\varphi^*\mathcal{O}_{X_6}(1)$. Those numbers not mentioned in the statements of the lemmas are implied to be 0.

**Lemma 3.3.1.** *A generic pencil $T_1$ of quadric hyperplane sections of $S$ has the following intersection numbers:*

$$T_1 \cdot \lambda = 6, \quad T_1 \cdot \delta_0 = 47, \quad T_1 \cdot \varphi^*\mathcal{O}_{X_6}(1) = 1.$$

*Proof.* Since all members of $T_1$ are irreducible it suffices to show that $\varphi_*\lambda = \mathcal{O}_V(6)$ and $\varphi_*\delta = \mathcal{O}_V(47)$ on $V := \big| -2K_S \big| \cong \mathbb{P}^{15}$. This is completely parallel to the computation in [37, Proposition 3.2]: If $\mathscr{C} \subseteq S \times V =: Y$ denotes the universal curve, we have $\mathcal{O}_Y(\mathscr{C}) = \mathcal{O}_Y(-2K_S, 1)$, so by adjunction $\omega_{\mathscr{C}/V} = \mathcal{O}_{\mathscr{C}}(-K_S, 1)$. Applying $\pi_{2*}$ to the exact sequence

$$0 \to \mathcal{O}_Y(K_S, 0) \to \mathcal{O}_Y(-K_S, 1) \to \omega_{\mathscr{C}/V} \to 0,$$

we find that

$$\pi_{2*}\omega_{\mathscr{C}/V} \cong \pi_{2*}\mathcal{O}_Y(-K_S, 1) \cong H^0(S, -K_S) \otimes \mathcal{O}_V(1),$$

since $\pi_{2_*}\mathcal{O}_Y(K_S, 0) = R^1\pi_{2*}\mathcal{O}_Y(K_S, 0) = 0$ by Kodaira vanishing. Therefore we get $\varphi_*\lambda = \det \pi_{2*}\omega_{\mathscr{C}/V} = \mathcal{O}_V(6)$.

We also find that

$$\varphi_*\kappa = \pi_{2*}(\omega^2_{\mathscr{C}/V}) = \pi_{2*}\big((-2K_S, 1) \cdot (-K_S, 1)^2\big) = \mathcal{O}_V(25).$$

From Mumford's relation $\kappa = 12\lambda - \delta$ we deduce that $\varphi_*\delta = \mathcal{O}_V(47)$. □

**Lemma 3.3.2.** *The family $T_2$ of varying elliptic tails has the following intersection numbers:*

$$T_2 \cdot \lambda = 1, \quad T_2 \cdot \delta_0 = 12, \quad T_2 \cdot \delta_1 = -1, \quad T_2 \cdot \varphi^*\mathcal{O}(1) = 0.$$

*Proof.* The first three intersection numbers are standard. By Proposition 3.2.1, $\varphi$ is defined on $T_2$ and contracts it to a point. □

**Lemma 3.3.3.** *The family $T_3$ of genus 2 tails attached at non-Weierstraß points has the following intersection numbers:*

$$T_3 \cdot \lambda = 3, \quad T_3 \cdot \delta_0 = 30, \quad T_3 \cdot \delta_2 = -1, \quad T_3 \cdot \varphi^*\mathcal{O}(1) = 0.$$

*Proof.* This family and its intersection numbers are described in [37, Section 3.2.2]. By Proposition 3.2.2, $\varphi$ is defined on $T_3$ and contracts it to a point. $\square$

The following computation is used in the proof of Lemma 3.3.5.

**Lemma 3.3.4.** *Let $X$ be a smooth threefold, $\mathscr{C} \subseteq X$ a surface with an ordinary $k$-fold point, $\pi\colon \widetilde{X} \to X$ the blowup at that point, and $\widetilde{\mathscr{C}}$ the proper transform of $\mathscr{C}$. Then*

$$\chi(\mathcal{O}_{\widetilde{\mathscr{C}}}) = \chi(\mathcal{O}_{\mathscr{C}}) - \binom{k}{3}.$$

*Proof.* Let $E \subseteq \widetilde{X}$ be the exceptional divisor and $C = E \cap \widetilde{\mathscr{C}}$. By adjunction,

$$K_{\widetilde{\mathscr{C}}} = (K_{\widetilde{X}} + \widetilde{\mathscr{C}})\big|_{\widetilde{\mathscr{C}}} = (\pi^*K_X + 2E + \pi^*\mathscr{C} - kE)\big|_{\widetilde{\mathscr{C}}} = \pi^*K_{\mathscr{C}} - (k-2)C,$$

so Riemann-Roch for surfaces gives

$$\chi(\mathcal{O}_{\widetilde{\mathscr{C}}}) = \chi(\mathcal{O}_{\widetilde{\mathscr{C}}}(-kC)) - kC^2 = \chi(\mathcal{O}_{\widetilde{\mathscr{C}}}(-kC)) + k^2.$$

From the exact sequence

$$0 \to \mathcal{O}_X(-\mathscr{C}) \to \mathcal{O}_{\widetilde{X}}(-kE) \to \mathcal{O}_{\widetilde{\mathscr{C}}}(-kC) \to 0,$$

we get that

$$\chi(\mathcal{O}_{\widetilde{\mathscr{C}}}(-kC)) = \chi(\mathcal{O}_{\widetilde{X}}(-kE)) - \chi(\mathcal{O}_X) + \chi(\mathcal{O}_{\mathscr{C}}).$$

Finally, using induction on the exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(-(i+1)E) \to \mathcal{O}_{\widetilde{X}}(-iE) \to \mathcal{O}_{\mathbb{P}^2}(i) \to 0,$$

for $i = 0, \ldots, k-1$, we conclude that

$$\chi(\mathcal{O}_{\widetilde{X}}(-kE)) = \chi(\mathcal{O}_X) - \sum_{i=0}^{k-1} \frac{i^2 + 3i + 2}{2} = \chi(\mathcal{O}_X) - \frac{k^3 + 3k^2 + 2k}{6}.$$

Putting these three equations together gives the result. $\square$

**Lemma 3.3.5.** *There is a family $T_4$ of stable genus 6 curves having the following intersection numbers:*

$$T_4 \cdot \lambda = 16, \quad T_4 \cdot \delta_0 = 118, \quad T_4 \cdot \delta_3 = 1, \quad T_4 \cdot \varphi^*\mathcal{O}(1) = 4.$$

*Proof.* Let $X$ be the blowup of $\mathbb{P}^2 \times \mathbb{P}^1$ at four constant sections of the second projection, and let $\mathscr{C}, \mathscr{C}' \subseteq X$ denote the proper transforms of degree 4 families of plane sextic curves, with assigned nodes at the blown-up points. Suppose $\mathscr{C}$ is chosen in such a way that it contains the curve pictured in Figure 3.2 as a member, and that the fourfold points of this fiber are also ordinary fourfold points of the total space, while away from this special fiber the family is smooth and all singular fibers are irreducible nodal. Furthermore, suppose $\mathscr{C}'$ is chosen generically, so that all its members are irreducible stable curves.

Let $\pi : \widetilde{X} \to X$ be the blowup of $X$ at the two fourfold points of $\mathscr{C}$, denote by $\widetilde{\mathscr{C}}$ the proper transform of $\mathscr{C}$, and by $E_1, E_2 \subseteq \widetilde{X}$ the exceptional divisors of $\pi$. Then $\widetilde{\mathscr{C}} = \pi^*\mathscr{C} - 4E_1 - 4E_2$ and $K_{\widetilde{X}} = \pi^* K_X + 2E_1 + 2E_2$, so

$$\begin{aligned} K^2_{\widetilde{\mathscr{C}}} &= (K_{\widetilde{X}} + \widetilde{\mathscr{C}})^2 \widetilde{\mathscr{C}} \\ &= (\pi^*(K_X + \mathscr{C}) - 2(E_1 + E_2))^2 (\pi^*\mathscr{C} - 4(E_1 + E_2)) \\ &= (K_X + \mathscr{C}')^2 \mathscr{C}' - 16(E_1^3 + E_2^3) = K^2_{\mathscr{C}'} - 32. \end{aligned}$$

By Lemma 3.3.4, we find that

$$\chi(\mathcal{O}_{\widetilde{\mathscr{C}}}) = \chi(\mathcal{O}_{\mathscr{C}}) - 2\binom{4}{3} = \chi(\mathcal{O}_{\mathscr{C}'}) - 8,$$

so $c_2(\widetilde{\mathscr{C}}) = c_2(\mathscr{C}') - 64$ by Noether's formula. If $T_4$ and $T_4'$ denote the families in $\overline{\mathcal{M}}_6$ induced by $\widetilde{\mathscr{C}}$ and $\mathscr{C}'$, respectively, we thus find that $T_4 \cdot \lambda = T_4' \cdot \lambda - 8 = 4 \cdot 6 - 8 = 16$ (note that $T_4'$ is numerically equivalent to $4T_1$, where $T_1$ is the pencil described in Lemma 3.3.1). Moreover, since the difference in topological Euler characteristics between a general (smooth) fiber and the special (blown-up) fiber of $\widetilde{\mathscr{C}}$ is 6, we compute that $T_4 \cdot \delta_0 = T_4' \cdot \delta_0 - 64 - 6 = 4 \cdot 47 - 70 = 118$. Finally, $T_4$ is constructed in such a way that $T_4 \cdot \delta_3 = 1$ and $T_4 \cdot \varphi^*\mathcal{O}(1) = 4$. $\square$

**Lemma 3.3.6.** *There is a family $T_5$ of stable genus 6 curves having the following intersection numbers:*

$$T_5 \cdot \lambda = 21, \quad T_5 \cdot \delta_0 = 164, \quad T_5 \cdot \varphi^*\mathcal{O}(1) = 10.$$

*Proof.* In order to construct $T_5$, we take a family of quadric hyperplane sections of a family of generically smooth anticanonically embedded del Pezzo surfaces, with special fibers having $A_1$ singularities. More concretely, let $\widetilde{\mathscr{S}}$ be the blowup of $\mathbb{P}^2 \times \mathbb{P}^1$ along the four sections

$$\begin{aligned} \Sigma_1 &= \big([1:0:0], [\lambda:\mu]\big), \\ \Sigma_2 &= \big([0:1:0], [\lambda:\mu]\big), \\ \Sigma_3 &= \big([0:0:1], [\lambda:\mu]\big), \\ \Sigma_4 &= \big([\lambda+\mu:\lambda:\mu], [\lambda:\mu]\big), \end{aligned}$$

where $[\lambda : \mu] \in \mathbb{P}^1$ is the base parameter. We map $\widetilde{\mathscr{S}}$ into $\mathbb{P}^7 \times \mathbb{P}^1$ by taking a system of eight (3, 1)-forms that span the space of anticanonical forms in every fiber, as given for example by the following:

$$\begin{aligned} f\big([x_0 : x_1 : x_2]\big) = \big[\; & x_0x_1(\lambda x_0 - (\lambda+\mu)x_1) : x_0^2(\mu x_1 - \lambda x_2) \\ & : x_0x_2(\mu x_0 - (\lambda+\mu)x_2) : x_0x_2(\mu x_1 - \lambda x_2) \\ & : x_0x_1(\mu x_1 - \lambda x_2) \qquad : x_1^2(\mu x_0 - (\lambda+\mu)x_2) \\ & : x_1x_2(\mu x_1 - \lambda x_2) \qquad : x_2^2(\lambda x_0 - (\lambda+\mu)x_1)\;\big]. \end{aligned}$$

This maps every fiber anticanonically into a 5-dimensional subspace of $\mathbb{P}^7$ that depends on $[\lambda : \mu] \in \mathbb{P}^1$. The image of the blown-up $\mathbb{P}^2$ is isomorphic to $S$ except for the parameter values $[\lambda : \mu] = [1 : 0]$, $[0 : 1]$ and $[1 : -1]$. In each of these cases, three of the four base points lie on a line, which gets contracted to an $A_1$ singularity under the anticanonical embedding.

Denote the image of $f$ by $\mathscr{S}$, let $H_1$, $H_2$ be the generators of $\operatorname{Pic}(\mathbb{P}^7 \times \mathbb{P}^1)$ and $\widetilde{H_1}$, $\widetilde{H_2}$, $E_1, \ldots, E_4$ those of $\operatorname{Pic}(\widetilde{\mathscr{S}})$. Note that

$$f^*H_1 = 3\widetilde{H_1} - \sum E_i + \widetilde{H_2} \text{ and } f^*H_2 = \widetilde{H_2}.$$

We claim that $\mathscr{S} \equiv 5H_1^5 + 9H_1^4H_2 \in A^*(\mathbb{P}^7 \times \mathbb{P}^1)$. Indeed, the first coefficient is just the degree in a fiber, while the second one is computed as

$$\begin{aligned} \mathscr{S} \cdot H_1^3 &= (3\widetilde{H_1} - \sum_{i=1}^4 E_i + \widetilde{H_2})^3 = 27\widetilde{H_1}^2\widetilde{H_2} + 3\sum_{i=1}^4 \widetilde{H_2}E_i^2 - E_4^3 + 9\widetilde{H_1}E_4^2 \\ &= 27 - 12 + 3 - 9 = 9. \end{aligned}$$

Here we have used that $\widetilde{H_2}E_i^2 = -1$ for $i = 1, \ldots, 4$, as it is just the self-intersection of the exceptional $\mathbb{P}^1$ in a fiber. Moreover, by the normal bundle exact sequence,

$$E_i^3 = K_{\mathbb{P}^2\times\mathbb{P}^1} \cdot \Sigma_i - \deg K_{\Sigma_i} = (-3\widetilde{H_1} - 2\widetilde{H_2})\widetilde{H_1}^2 + 2 = 0$$

for $i = 1, 2, 3$, and similarly

$$E_4^3 = (-3\widetilde{H_1} - 2\widetilde{H_2})(\widetilde{H_1}^2 + \widetilde{H_1}\widetilde{H_2}) + 2 = -3.$$

Finally, $\widetilde{H_1}$ and $\widetilde{H_2}$ both restrict to the same thing on $E_4$ (namely, the class of a fiber of the fibration $E_4 \to \Sigma_4$), so $\widetilde{H_1}E_4^2 = \widetilde{H_2}E_4^2 = -1$.

Let $\mathscr{C}$ be the family of curves that is cut out on $\mathscr{S}$ by a generic hypersurface of bidegree (2, 2), so that $\mathscr{C} \equiv 10H_1^6 + 28H_1^5H_2$. Since $K_{\widetilde{\mathscr{S}}} = \mathcal{O}_{\widetilde{\mathscr{S}}}(-3\widetilde{H_1} + \sum E_i - 2\widetilde{H_2})$, we find that $K_{\mathscr{S}} = \mathcal{O}_{\mathscr{S}}(-H_1 - H_2)$. Thus $\omega_{\mathscr{S}/\mathbb{P}^1} = \mathcal{O}_{\mathscr{S}}(-H_1 + H_2)$, and by adjunction $\omega_{\mathscr{C}/\mathbb{P}^1} = \mathcal{O}_{\mathscr{C}}(H_1 + 3H_2)$. If $T_5$ denotes the family induced in $\overline{\mathcal{M}}_6$ by $\mathscr{C}$, we then find

that
$$T_5 \cdot \kappa = \omega^2_{\mathscr{C}/\mathbb{P}^1} = (H_1 + 3H_2)^2 \cdot (10H_1^6 + 28H_1^5 H_2) = 88.$$

Next we note that $\mathcal{O}_{\mathscr{S}}(-\mathscr{C}) = 2K_{\mathscr{S}}$, so applying Riemann-Roch for threefolds to the short exact sequence $0 \to 2K_{\mathscr{S}} \to \mathcal{O}_{\mathscr{S}} \to \mathcal{O}_{\mathscr{C}} \to 0$, we get

$$\begin{aligned}\chi(\mathcal{O}_{\mathscr{C}}) &= \chi(\mathcal{O}_{\mathscr{S}}) - \chi(2K_{\mathscr{S}}) \\ &= -\frac{1}{2}K_{\mathscr{S}}^3 + 4\chi(\mathcal{O}_{\mathscr{S}}) \\ &= -\frac{1}{2}(-H_1 - H_2)^3(5H_1^5 + 9H_1^4 H_2) + 4 \\ &= 16,\end{aligned}$$

where we used that $\chi(\mathcal{O}_{\mathscr{S}}) = 1$ because $\mathscr{S}$ is rational. Hence

$$T_5 \cdot \lambda = \chi(\mathcal{O}_{\mathscr{C}}) - (g(\mathbb{P}^1) - 1)(g(C) - 1) = 21,$$

where $C$ is a generic fiber of $\mathscr{C}$. Finally, by Mumford's relation we get

$$T_5 \cdot \delta_0 = 12 \cdot 21 - 88 = 164.$$

For computing $T_5 \cdot \varphi^* \mathcal{O}(1)$, we note that we can also construct $\mathscr{S}$ as follows: Blow up $\mathbb{P}^2 \times \mathbb{P}^1$ at $[1:0:0]$, $[0:1:0]$, $[0:0:1]$ and $[1:1:1]$ and embed it into $\mathbb{P}^7 \times \mathbb{P}^1$ via the map

$$\begin{aligned} &f'([x_0 : x_1 : x_2]) = \\ &\quad = [\ x_0x_1(x_0 - x_1) : x_0^2(x_1 - x_2) : x_0x_2(x_0 - x_2) : x_0x_2(x_1 - x_2) \\ &\qquad : x_0x_1(x_1 - x_2) : x_1^2(x_0 - x_2) : x_1x_2(x_1 - x_2) : x_2^2(x_0 - x_1)\ ]\end{aligned}$$

on the first component (and the identity on $\mathbb{P}^1$). Now take the proper transform of this constant family under the birational map $\psi \colon \mathbb{P}^7 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^7 \times \mathbb{P}^1$ given by

$$\begin{aligned}\psi([y_0 : \cdots : y_7]) = [\ &\lambda^2(\lambda+\mu)^2 y_0 : \lambda\mu(\lambda+\mu)^2 y_1 : \\ &\mu^2(\lambda+\mu)^2 y_2 : \lambda\mu^2(\lambda+\mu) y_3 : \\ &\lambda^2\mu(\lambda+\mu) y_4 : \lambda^2\mu(\lambda+\mu) y_5 : \\ &\lambda^2\mu^2 y_6 : \lambda\mu^2(\lambda+\mu) y_7\ ].\end{aligned}$$

Denoting by $\mathscr{S}' \cong S \times \mathbb{P}^1$ the image of $f'$, the intersection number $T_5 \cdot \varphi^* \mathcal{O}(1)$ is given by the number of curves in $T_5$ passing through a general fixed point of $S$. Since two general hyperplane sections cut out five general points on $S$, we compute that

$$T_5 \cdot \varphi^* \mathcal{O}(1) = \frac{1}{5} \mathcal{O}_{\mathscr{S}'}(H_1)^2 \cdot \psi^* \mathcal{O}_{\mathscr{S}}(\mathscr{C}) = \frac{1}{5} H_1^5 \cdot H_1^2 \cdot (2H_1 + 10H_2) = 10. \qquad \square$$

## 3.4 The moving slope of $\overline{\mathcal{M}}_6$

**Proposition 3.4.1.** *The moving slope of $\overline{\mathcal{M}}_6$ fulfills $47/6 \le s'(\overline{\mathcal{M}}_6) \le 102/13$.*

*Proof.* The lower bound is the slope of the effective cone of $\overline{\mathcal{M}}_6$ and was known before (see [35]). Using the test families $T_1$ through $T_5$ described in Section 3.3, we get that

$$\varphi^*\mathcal{O}(1) = 102\lambda - 13\delta_0 - 54\delta_1 - 84\delta_2 - 94\delta_3.$$

Since $\mathcal{O}(1)$ is ample on $X_6$ and $\varphi$ is a rational contraction, this is a moving divisor on $\overline{\mathcal{M}}_6$, which gives the upper bound on the moving slope. $\square$

**Remark 3.4.2.** Note that $102/13 \approx 7.846$ is strictly smaller than $65/8 = 8.125$, which was the upper bound previously obtained in [35]. However, since our families $T_4$ and $T_5$ are not covering families for divisors contracted by $\varphi$, we cannot argue as in [37, Corollary 3.7]. In particular, the actual moving slope may be lower than the upper bound given here.

**Proposition 3.4.3.** *The log canonical model $\overline{\mathcal{M}}_6(\alpha)$ is*

- *isomorphic to $X_6$ for $16/47 < \alpha \le 35/102$,*
- *a point for $\alpha = 16/47$, and*
- *empty for $\alpha < 16/47$.*

*Proof.* This is completely analogous to [37, Corollary 3.6]. Since

$$\begin{aligned}(K_{\overline{\mathcal{M}}_6} + \alpha\delta) - \varphi^*\varphi_*(K_{\overline{\mathcal{M}}_6} + \alpha\delta) &= \\ &= (13\lambda - (2-\alpha)\delta) - \varphi^*\varphi_*(13\lambda - (2-\alpha)\delta) \\ &= (\frac{35}{2} - 51\alpha)\left[\overline{\mathcal{GP}}_6\right] + (9-11\alpha)\delta_1 + (19-29\alpha)\delta_2 + (34-96\alpha)\delta_3\end{aligned}$$

is an effective exceptional divisor for $\varphi$ as long as $\alpha \le 35/102$, the upper bound follows. Moreover, $\varphi_*(13\lambda - (2-\alpha)\delta) = \mathcal{O}_{X_6}(47\alpha - 16)$, which gives the lower bound. $\square$

## 3.5 Unirationality of Weierstraß loci

The constructions discussed in this chapter can be also used to prove the unirationality of the Weierstraß locus in $\overline{\mathcal{M}}_{6,1}$, consisting of 1-pointed curves where the marked point is a Weierstraß point. This is a divisor whose class was computed by Cukierman, as mentioned in the introduction to Chapter 2. More generally, for $g \ge 2$ and $2 \le k \le g$ we can define the loci

$$\mathcal{W}_{g,k} := \left\{[(C,p)] \;\middle|\; h^0(C,kp) \ge 2\right\} \subseteq \mathcal{M}_{g,1}$$

and their closures $\overline{\mathcal{W}}_{g,k} \subseteq \overline{\mathcal{M}}_{g,1}$. For $k = g$ this is just the Weierstraß divisor $\mathcal{W}_g$, which is isomorphic to $\overline{D}_{g;-1}$ (see Remark 2.5.2 above). The loci $\mathcal{W}_{g,k}$ were first considered by Arbarello in [3] and shown to be irreducible of dimension $2g + k - 3$. The author mentions in [3, Remark 3.25] that these spaces can be shown to be unirational for $2 \leq k \leq 5$. The proof, which is a variant of the methods used in [4], does not seem to be readily accessible in the literature, so we give details here for the convenience of the reader.

**Theorem 3.5.1.** *The loci $\mathcal{W}_{g,k}$ are unirational for $2 \leq k \leq 5$.*

*Proof.* If $(C, p) \in \mathcal{M}_{g,1}$ is a pointed curve such that $|kp|$ is a $g^1_k$, then by B. Segre's theorem from [77, p. 539] the curve $C$ has a planar model $\Gamma$ of degree

$$n = (g + k + \varepsilon + 1)/2, \quad \varepsilon = 0 \text{ or } 1,$$

having one ordinary $(n-k)$-fold point $q$ and otherwise only nodes as singularities, and such that projection from $q$ gives back the $g^1_k$. In particular, the line $\overline{pq}$ is $k$-fold tangent to $\Gamma$ at $p$. The number of nodes is then

$$\delta = \binom{n-1}{2} - \binom{n-k}{2} - g. \tag{3.1}$$

Mimicking the proof of Theorem 5.3 in [4], we first show that the nodes are in general position, i. e. given two fixed points $p, q \in \mathbb{P}^2$ and $\delta$ generic points $q_1, \ldots, q_\delta \in \mathbb{P}^2$, there exists an irreducible curve in $\mathbb{P}^2$ that has an $(n-k)$-fold point at $q$, nodes at the $q_i$, and is $k$-tangent to $\overline{pq}$ at $p$.

The proof is a straightforward generalization of [4, Corollary 4.7]: Let $X_0$ be the blow-up of $\mathbb{P}^2$ at $q$, let $E$ be the exceptional divisor, and let $p_1 \in X_0$ be the preimage of $p$. For $1 \leq j \leq k$, construct $X_j$ from $X_{j-1}$ by blowing up $p_j$, thereby introducing the exceptional divisor $E_j$, and letting $p_{j+1}$ be the intersection of $E_j$ with the proper transform of the line $\overline{pq}$. On the final space $X := X_k$, we let $H$ be the pullback of the hyperplane class, and we keep denoting by $E_j$ the proper transforms of the exceptional divisor of the $j$-th step. The linear system of curves satisfying the above conditions is then isomorphic to the complete linear system $|D| = |nH - (n-k)E - \sum_{j=1}^k jE_j|$ on $X$.

As shown in [4, Corollary 4.6], the following conditions are sufficient to ensure that $|D|$ contains an irreducible curve with nodes at $\delta$ general points:

(i) a general element of $|D|$ is connected,

(ii) $p_a(D) \geq \delta$,

(iii) $\dim |D| \geq 3\delta$, and

(iv) given $\delta$ general points $p_1, \ldots, p_\delta$ on $X$, there is an element $C \in |D|$ which is singular at $p_1, \ldots, p_\delta$, and such that $K_X \cdot C' < 0$ for every irreducible component $C'$ of $C$.

The verification of conditions (i) and (ii) is straightforward, while for (iii) we note that

$$\begin{aligned}\dim |D| - 3\delta &= \binom{n+2}{2} - 1 - \binom{n-k+1}{2} - k - 3\delta \\ &= \begin{cases} (5-k)g + k + 1 & \text{if } \varepsilon = 0 \\ (5-k)g + 3 & \text{if } \varepsilon = 1, \end{cases}\end{aligned}$$

whence the restriction $k \leq 5$.

Checking condition (iv) is a bit more involved, since $-K_X$ is not ample on $X$. However, the only curve on $X$ on which $K_X$ has non-negative degree is the proper transform $L$ of the line $\overline{pq}$, which can be seen as follows: First, $K_X = -3H + E + \sum_{j=1}^{k} jE_j$, so

$$K_X \cdot (aH - bE - \sum_{j=1}^{k} b_j E_j) = -3a + b + b_k.$$

Since $L \equiv H - E - \sum_{j=1}^{k} jE_j$, we find that $K_X \cdot L = k - 2 \geq 0$. On the other hand, any curve $Z$ of degree $a$ that does not contain the line $L$ must fulfill $b + b_k \leq a$ by Bezout, and thus $K_X \cdot Z \leq -2a < 0$.

Now comparing dimensions, one finds that the space $|D - L|$ has codimension 1 in $|D|$, so the general element of the latter does not contain $L$. Since the points $q_i$ are generic (and in particular do not lie on $L$), the same holds if one postulates nodes at the points $q_i$. Thus condition (iv) is satisfied and the general element of $|D|$ with nodes at the $q_i$ is irreducible.

It remains to show that a map $f : C \to \mathbb{P}^1$ that corresponds to a generic $g^1_k$ with total ramification at $p$ comes from such a planar model. In other words, we have proven above that the incidence correspondence

$$V' := \left\{ (Y, q_1 + \cdots + q_\delta) \,\middle|\, Y \in |D| \text{ has nodes at the } q_i \right\}$$

has a component $V$ that is a projective bundle over $\operatorname{Sym}^\delta \mathbb{P}^2$, and we want to show that $V$ maps dominantly onto $\mathcal{W}_{g,k}$. The tangent space to $V$ at a point $(Y, q_1 + \cdots + q_\delta)$ is given by

$$T_{(Y, q_1 + \cdots + q_\delta)} V \cong H^0(C, N_\varphi),$$

where $\varphi \colon C \to X$ is the normalization map of $Y$ and $N_\varphi$ is its normal bundle. Following [4], we want to show that this space maps surjectively onto $T_{(C,p)} \mathcal{W}_{g,k}$, so the map from $V$ to $W_{g,k}$ is open and hence dominant.

Note that $H^0(C, N_\varphi) \cong H^0(C, N_{\varphi_0}(-kp))$, where $\varphi_0 : C \to X_0$ is the induced map after blowing down all the $E_j$, $j = 1, \ldots, k$. The normal sheaf $N_f$ has a subsheaf $\eta'$ whose global sections correspond to deformations of the map $f$ that retain the point of total ramification. It is given by the short exact sequence

$$0 \to T_C \to f^*T_{\mathbb{P}^1}(-(k-1)p) \to \eta' \to 0,$$

see [48, §5]. Let $\eta := \eta' \otimes \mathcal{O}_C(-p) \cong \eta'$. We will show that $H^0(C, N_{\varphi_0}(-kp))$ maps surjectively onto $H^0(C, \eta)$. Consider the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[d] & \mathcal{O}_C(H+E-kp) \arrow[d] & \mathcal{O}_C(H+E-p) \arrow[d] & \\
0 \arrow[r] & T_C(-kp) \arrow[r] \arrow[d] & \varphi_0^* T_{X_0}(-kp) \arrow[r] \arrow[d] & N_{\varphi_0}(-kp) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_C(-p) \arrow[r] \arrow[d] & f^*T_{\mathbb{P}^1}(-kp) \arrow[r] \arrow[d] & \eta \arrow[r] \arrow[d] & 0 \\
 & \mathcal{O}_{(k-1)p} \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}$$

where we denote by $H$ and $E$ also the pullbacks of these divisor classes to $C$. Here the middle column is as given in [4, p. 361] and comes from the relative tangent sequence of the map $\pi : X_0 \to \mathbb{P}^1$, while exactness of the rightmost column follows from the snake lemma.

It therefore suffices to show that $H^1(C, \mathcal{O}_C(H+E-p)) = 0$. But

$$h^0(C, \mathcal{O}_C(H+E-p)) = h^0(C, \mathcal{O}_C(H+E)) - 1,$$

as not every element of $|H+E|$ intersects $C$ in $p$, and thus

$$h^1(C, \mathcal{O}_C(H+E-p)) = h^1(C, \mathcal{O}_C(H+E)).$$

The latter was shown to be zero by Arbarello and Cornalba, and the theorem now follows by observing that the Hurwitz scheme of maps $f: C \to \mathbb{P}^1$ with total ramification at $p$, whose tangent space at $(C, f)$ is $H^0(C, \eta')$, maps dominantly onto $\mathcal{W}_{g,k}$. □

The loci $\mathcal{W}_{g,k}$ have been shown by various authors to be even rational when $k \leq 4$, and for $k = 5$ and $g = 20n - 4$ with $n \geq 1$ (see [12] and references therein). Complementing these results is the following proposition about the unirationality of the divisor

$\overline{\mathcal{W}}_{6,6} \subseteq \overline{\mathcal{M}}_{6,1}$, which can be derived from the construction of the general genus 6 curve as a quadric hyperplane section of the degree 5 del Pezzo surface. I am indebted to Frank-Olaf Schreyer for the following proof.

**Proposition 3.5.2.** *The locus $\mathcal{W}_{6,6}$ is unirational.*

*Proof.* Start with the del Pezzo surface $S \subseteq \mathbb{P}^5$. On $S$ we have a natural $\mathbb{P}^5$-bundle which is the push-forward of the relative $\mathcal{O}(1)$-bundle of the projection $\pi: S \times \mathbb{P}^5 \to S$. Its total space is $S \times \left|\mathcal{O}_{\mathbb{P}^5}(1)\right|$. Inside there, we can construct an incidence correspondence as a subbundle

$$\mathcal{F} := \left\{(p, H) \,\middle|\, p \in H\right\} \subseteq S \times \left|\mathcal{O}_{\mathbb{P}^5}(1)\right|,$$

which is then a $\mathbb{P}^4$-bundle over $S$ (though no longer trivial). In the same way as before we can construct a trivial $\mathbb{P}^{15}$-bundle on $S \times \left|\mathcal{O}_{\mathbb{P}^5}(1)\right|$, whose total space is $\left(S \times \left|\mathcal{O}_{\mathbb{P}^5}(1)\right|\right) \times \left|\mathcal{O}_{\mathbb{P}^5}(2)\right|$. Inside its restriction to $\mathcal{F}$ we define a second incidence correspondence by

$$\mathcal{E} := \left\{(p, H, Q) \,\middle|\, i(H, Q \cap S; p) \geq 6\right\} \subseteq \mathcal{F} \times \left|\mathcal{O}_{\mathbb{P}^5}(2)\right|.$$

As described in Chapter 3, $C := Q \cap S$ is generically a smooth canonically embedded genus 6 curve. As the requirement on the local intersection multiplicity poses 6 independent conditions on each fiber, $\mathcal{E}$ is now a $\mathbb{P}^9$-bundle on $\mathcal{F}$ (again, no longer trivial). Since $S$ is rational, so are $\mathcal{E}$ and $\mathcal{F}$.

Note that $Q$ and $C$ determine each other uniquely, as $\left|\mathcal{O}_{\mathbb{P}^5}(2)\right| \cong \left|\mathcal{O}_S(2)\right|$. By the geometric form of Riemann-Roch, $p$ is a Weierstraß point on $C$ if and only if the 5-osculating 4-plane to $C$ at $p$ is in fact 6-osculating, which shows that the image of the map $\varphi\colon \mathcal{E} \to \overline{\mathcal{M}}_{6,1}$, $\varphi(p, H, Q) := \left[(Q \cap S, p)\right]$ lies inside the Weierstraß divisor. On the other hand, given a general $\left[(C, p)\right] \in W_{6,6}$, embed $C$ canonically into $\mathbb{P}^5$, let $S_C$ be the del Pezzo surface on which it lies, let $Q$ be a quadric such that $C = S_C \cap Q$, and let $H$ be 5-osculating 4-plane to $C$ at $p$. As remarked above, we have in fact $i(H, C; p) \geq 6$ since $p$ is a Weierstraß point. Choosing an automorphism $\psi$ of $\mathbb{P}^5$ that takes $S_C$ to $S$, we find that $\left[(C, p)\right] = \varphi\big(\psi(p), \psi(H), \psi(Q)\big)$, so $\varphi$ is dominant and $\mathcal{W}_{6,6}$ is unirational. $\square$

Chapter 4

# 4 The hypertree divisor on $\overline{\mathcal{M}}_{0,7}$

## 4.1 Introduction

The cone of effective divisors on $\overline{\mathcal{M}}_{0,n}$ has been conjectured by Fulton to be generated by boundary divisors [62]. For $n \leq 5$ this is true, however on $\overline{\mathcal{M}}_{0,6}$, Keel and Vermeire [83] found 15 examples of irreducible effective divisors $D$, one for each partition of $\{1, \ldots, 6\}$ into three pairs, which are not linearly equivalent to any effective combination of boundary divisors. The divisors $D$ can be described in (at least) three essentially different ways:

(i) $D$ is the non-boundary component of the fixed locus of a permutation $\sigma \in S_6$ consisting of three disjoint transpositions, where $S_6$ acts on $\overline{\mathcal{M}}_{0,6}$ by permuting the markings.

(ii) If $\varphi : \overline{\mathcal{M}}_{0,6} \to \overline{\mathcal{M}}_3$ is the map identifying three pairs of marked points to give a nodal genus 3 curve, then

$$D = \overline{\varphi^*(\overline{\mathcal{M}}^1_{3,2}) \cap \mathcal{M}_{0,6}},$$

i. e. $D$ is the proper transform of the hyperelliptic locus in $\overline{\mathcal{M}}_3$.

(iii) $D$ is the closure of the locus of smooth 6-pointed curves having the property that when embedded as a conic in $\mathbb{P}^2$, the three chords joining pairs of points intersect in a common point.

Each characterization follows readily from the one before, but while the first one does not yield easily to generalization, the latter ones do, and moreover a "geometric" characterization like (iii) above can favorably be used to express $D$ in terms of a Kapranov basis of $\overline{\mathcal{M}}_{0,6}$.

As recounted in the introduction, Hassett and Tschinkel [54] proved that for $n = 6$ the Keel-Vermeire divisors together with the boundary divisors do indeed generate

$\mathrm{Eff}(\overline{\mathcal{M}}_{0,6})$. However, Castravet and Tevelev [13] produced a series of new effective divisors on $\overline{\mathcal{M}}_{0,n}$ for $n \geq 6$ via combinatorial objects called *hypertrees* (see Section 1.9), which they proved to be extremal in $\mathrm{Eff}(\overline{\mathcal{M}}_{0,n})$. For $n = 6$ their method just reproduces the Keel-Vermeire divisors, thereby giving yet another characterization for them. For a certain subclass of these divisors, so-called *bipyramid divisors*, they provided a "Brill-Noether characterization" like (ii) above, and via the method of admissible covers they also obtained a geometric characterization as in (iii).

In keeping with this theme, we derive an analogous geometric characterization for the hypertree divisor on $\overline{\mathcal{M}}_{0,7}$, enabling us to compute its class in terms of a Kapranov basis. In principle, one could use this result to analyze $\mathrm{Eff}(\overline{\mathcal{M}}_{0,7})$ by the methods used in [54] and investigate the question whether the boundary divisors, hypertree divisors and pullbacks of hypertree divisors from $\overline{\mathcal{M}}_{0,6}$ together generate this cone. However, the computations soon become too resource intensive to remain practical.

## 4.2 Geometric characterization

Let $\Gamma = \big\{\{1,3,5\}, \{2,4,6\}, \{1,2,7\}, \{3,4,7\}, \{5,6,7\}\big\}$ denote the up to permutations unique irreducible hypertree on seven points as shown in Figure 4.1, and denote by $D_\Gamma$ the corresponding divisor on $\overline{\mathcal{M}}_{0,7}$ as defined in Section 1.9. When working with hypertrees, we will denote our markings by $(p_1, \ldots, p_7)$, corresponding to the similarly numbered vertices of $\Gamma$. When working in a more geometric setting however, it will be convenient to denote them by $(p_1, p_2, p_3, q_1, q_2, q_3, r)$ instead, thus adopting the structure of the hypertree $\Gamma$ into the notation.

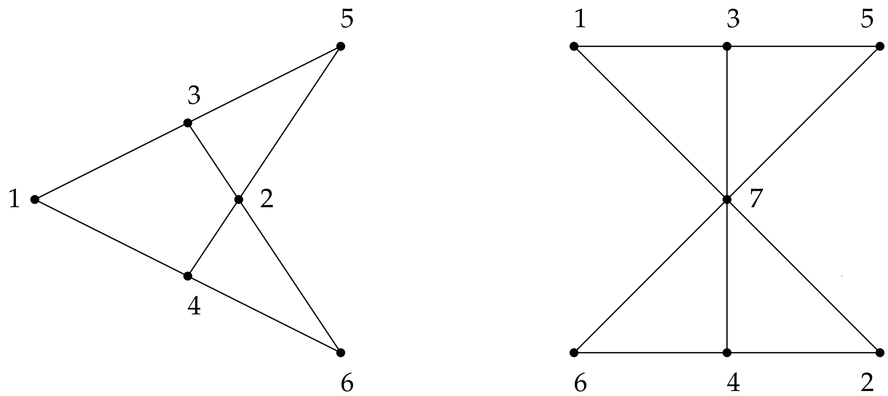


Figure 4.1: The unique irreducible hypertrees for $n = 6$ and $n = 7$

Our main result then is the following:

**Theorem 4.2.1.** *Let $D \subseteq \overline{\mathcal{M}}_{0,7}$ denote the closure of the locus of equivalence classes of smooth stable 7-pointed curves $(\mathbb{P}^1; p_1, p_2, p_3, q_1, q_2, q_3, r)$ satisfying the following geometric condition:*

*If $\iota : \mathbb{P}^1 \to \mathbb{P}^3$ is an embedding of $\mathbb{P}^1$ as a quartic in $\mathbb{P}^3$ such that the triples $\{p_1, p_2, p_3\}$ and $\{q_1, q_2, q_3\}$ both become collinear under $\iota$, then there is a line through $\iota(r)$ meeting all three chords $\overline{\iota(p_i)\iota(q_i)}$, $i = 1, 2, 3$.*

*Then $D = D_\Gamma$.*

We will prove Theorem 4.2.1 in several steps, the first of which consists in showing that $D$ is indeed a well-defined divisor. To this end, we show the following:

**Lemma 4.2.2.** *Given six distinct point $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{P}^1$, there is an embedding of $\mathbb{P}^1$ as a quartic in $\mathbb{P}^3$, unique up to automorphisms of $\mathbb{P}^3$, such that the triples $p_1, p_2, p_3$ and $q_1, q_2, q_3$ both become collinear.*

*Proof.* A $g^3_4$ on $\mathbb{P}^1$ is given by a 4-dimensional subspace $V \subseteq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$. The map $\iota$ associated to $|V|$ maps the $p_i$ to collinear points if and only if there is a pencil of hyperplanes in $\mathbb{P}^3$ containing their images. Pulling back via $\iota$, this means that $|V|$ must contain the projective line

$$l_1 = \left\{ p_1 + p_2 + p_3 + p \;\middle|\; p \in \mathbb{P}^1 \right\} \subseteq \left|\mathcal{O}_{\mathbb{P}^1}(4)\right|.$$

In the same way, it must contain the line $l_2 = \{q_1 + q_2 + q_3 + p \mid p \in \mathbb{P}^1\}$. Since the $p_i$ and $q_i$ are distinct, these lines are skew, so their span has projective dimension 3, and moreover their generic elements are disjoint. Thus, $|V|$ is uniquely determined by these conditions and base point free. $\square$

From now on, we denote by $\iota\colon \mathbb{P}^1 \to \mathbb{P}^3$ the embedding constructed in Lemma 4.2.2 with respect to the given markings, and by $C_i := \overline{\iota(p_i)\iota(q_i)}$, $i = 1, 2, 3$, the three chords.

Note that passing through a given line is a codimension 1 condition on lines in $\mathbb{P}^3$ (i. e. a cycle of codimension 1 in $\mathbb{G}(1, 3)$), while containing a given point is a condition of codimension 2. Since $\dim \mathbb{G}(1, 3) = 4$, the locus of points through which there is a line meeting all the chords $C_i$ will therefore generically have dimension 2, and may be even larger if the configuration of chords is too special. Thus, it could easily happen that the image curve $C := \iota(\mathbb{P}^1)$ lies wholly inside this locus, in which case $D$ would not be a divisor. In order to show that this does not happen, we need the following lemma:

**Lemma 4.2.3.** *There is a unique non-singular quadric surface $Q \subseteq \mathbb{P}^3$ containing the curve $C$. Moreover, $C$ is a divisor of type $(3, 1)$ on $Q$.*

*Proof.* As in the proof of Lemma 4.2.2, we see that there is a unique $g^1_3$ on $\mathbb{P}^1$ mapping each of the triples $\{p_i\}$ and $\{q_i\}$ to a single point: It is simply given as the line spanned by the elements $p_1 + p_2 + p_3$ and $q_1 + q_2 + q_3 \in |\mathcal{O}_{\mathbb{P}^1}(3)|$, and it is base point free, since these two divisors are disjoint. Letting $|W|$ denote this $g^1_3$, we have an embedding $\rho\colon \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ given by the pair of linear systems $\left(|W|, \left|\mathcal{O}_{\mathbb{P}^1}(1)\right|\right)$, with the triples

$\{p_i\}$ and $\{q_i\}$ each landing in a fibre of the first projection. Now the Segre embedding $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ has as image a non-singular quadric $Q \subset \mathbb{P}^3$, and the fibres of the two projections get mapped to the rulings of the quadric, so the triples $\{p_i\}$ and $\{q_i\}$ become collinear in $\mathbb{P}^3$. By the uniqueness statement of Lemma 4.2.2, we thus have $\iota = \sigma \circ \rho$, i. e. $C \subseteq Q$. By construction, $C$ is a divisor of type $(3, 1)$ on $Q$. Finally, if there were another quadric $Q'$ containing $C$, we would have to have $C = Q \cap Q'$ by degree reasons. But $Q' \sim 2H$ on $\mathbb{P}^3$, so $Q \cap Q'$ has type $(2, 2)$ on $Q$. □

We can now complete the first step in the proof of Theorem 4.2.1.

**Proposition 4.2.4.** *$D$ is a divisor.*

*Proof.* Since the space $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$ of quadrics in $\mathbb{P}^3$ is 10-dimensional, and containing a line imposes 3 conditions on a quadric surface, there is at least one quadric $Q' \subseteq \mathbb{P}^3$ containing the three chords $C_i$. If there were another quadric $Q''$ containing the chords, then by Bézout both $Q'$ and $Q''$ would also contain the lines $\overline{\iota(p_1)\iota(p_2)\iota(p_3)}$ and $\overline{\iota(q_1)\iota(q_2)\iota(q_3)}$, so their intersection would have total degree $5 > 2 \cdot 2$. Hence, the quadric $Q'$ containing the chords is unique.

Now through any point $p \in Q'$ there is a line intersecting the three chords $C_i$ (just take the member of the other ruling through $p$), and if $p \in \mathbb{P}^3$ is a general point possessing such a line, then $p$ will have to be contained in $Q'$ by degree reasons. Thus, $Q'$ is exactly the locus of points having the desired property, and it will suffice to show that $Q' \neq Q$, so $C \nsubseteq Q'$ by Lemma 4.2.3.

For this, we show that $C_i \nsubseteq Q$: Otherwise $C_i$ would have to be a member of the second ruling of $Q$, since it intersects the line $\overline{\iota(p_1)\iota(p_2)\iota(p_3)}$, which belongs to the first ruling. But then by Lemma 4.2.3, $C_i$ could intersect $C$ only once, contradiction. □

From the proof of Proposition 4.2.4 we can already derive a description of the geometry of the divisor $D$, at least away from the boundary:

**Corollary 4.2.5.** *$D$ is expressible as a 2-sheeted branched covering of $\overline{\mathcal{M}}_{0,6}$.*

*Proof.* It follows from Proposition 4.2.4 that given six distinct points $p_i, q_i \in \mathbb{P}^1$, there are exactly two points $r \in \mathbb{P}^1$ such that $[(\mathbb{P}^1; p_i, q_i, r)] \in D$, namely the preimages via $\iota$ of the two remaining points of intersection of $C$ with $Q'$ besides the $\iota(p_i)$ and $\iota(q_i)$. The morphism forgetting the seventh marking thus gives a 2-sheeted covering map $D \cap \mathcal{M}_{0,7} \to \mathcal{M}_{0,6}$, branched along the locus where these two points of intersection coincide. Since forgetful morphisms are flat and proper, the covering map carries over to the closure. □

This behavior is in marked contrast to the case of boundary divisors, which are always products of lower-dimensional moduli spaces. Note also that $D$ finds a natural realization as a cycle of codimension 2 on $\overline{\mathcal{M}}_{0,8}$, namely as $D \cong \pi_7^*(D) \cap \pi_8^*(D)$, where

$\pi_k : \overline{\mathcal{M}}_{0,8} \to \overline{\mathcal{M}}_{0,7}$ is the morphism forgetting the $k$-th marking. The covering map to $\overline{\mathcal{M}}_{0,6}$ is given by forgetting both additional markings.

We are now ready to tackle the next step in the proof of Theorem 4.2.1. The following proof is heavily influenced by the methods used in [13, §9].

**Proposition 4.2.6.** *We have $D_\Gamma \subseteq D$.*

*Proof.* Let $(\mathbb{P}^1; p_1, \ldots, p_7)$ be a curve whose equivalence class lies in $D_\Gamma$. In the dual projective plane, we get the picture shown in Figure 4.2. Here $L_1, \ldots, L_7$ are the lines dual to the seven vertices of $\Gamma$, numbered as in Figure 4.1. The points $P_i$ and $P_{lmn}$ correspond dually to the five lines of $\Gamma$, while the $P_{jk}$ are just the remaining points of intersection $L_j \cap L_k$. The line $L_\infty$ is the dual of the center of projection.

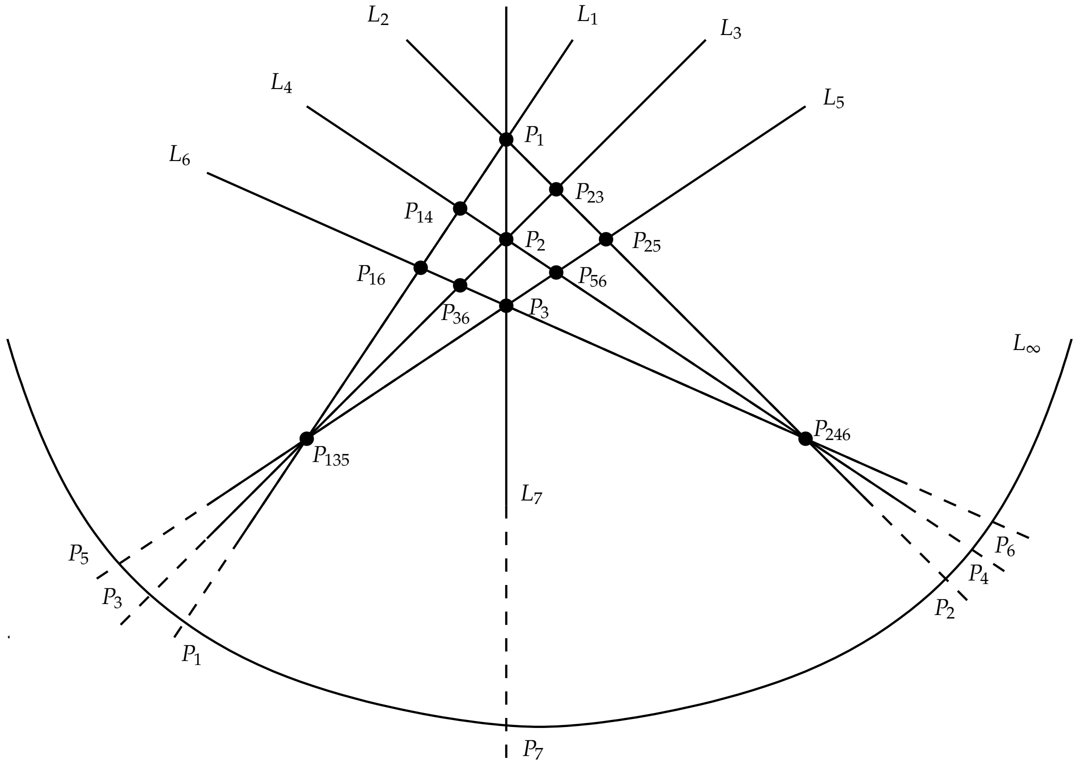


Figure 4.2: Projection of $D_\Gamma$ in the dual picture

Let $S$ be the blow-up of this $\mathbb{P}^2$ at all the points $P_i$, $P_{jk}$ and $P_{lmn}$, denote by $E_i$, $E_{jk}$ and $E_{lmn}$ the corresponding exceptional divisors, and consider the linear system

$$|V| = |4H - \sum E_i - \sum E_{jk} - \sum E_{lmn}|$$

on $S$, which is easily seen to be base point free. Since $\dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)) = 15$ and in total we are imposing 11 conditions, we have $\dim |V| \geq 3$. A closer examination shows that given any subcollection of the assigned base points, one can always find a quartic not passing through any of them, so in fact one has $\dim |V| = 3$.

We show next that $\dim |V|\big|_{L_\infty} = 3$, i. e. $L_\infty$ gets embedded as a quartic in $\mathbb{P}^3$. If not, there would be a divisor $Z \in |V|$ having $L_\infty$ as a component. The residual divisor $Z - L_\infty$ would then come from a cubic in $\mathbb{P}^2$ meeting the lines $L_1$, $L_3$ and $L_5$ in 4 points each, hence containing them. But then $Z = L_1 + L_3 + L_5 + L_\infty$, which does not contain $P_{246}$. Thus, $|V|$ has full rank when restricted to $L_\infty$.

Next, if $Z \in |V|$ is any divisor containing the points $p_1$ and $p_3$, then again by degree reasons $Z$ has to contain the lines $L_1$ and $L_3$, and the residual divisor $Z - L_1 - L_3$ likewise has to contain $L_5$. Thus every hyperplane in $\mathbb{P}^3$ containing the images of $p_1$ and $p_3$ also contains the image of $p_5$, so the three image points must lie on a line. By symmetry, the same holds for the images of $p_2$, $p_4$ and $p_6$.

Consider now the restriction of $|V|$ to the line $L_7$. As the remaining base points $P_{jk}$ and $P_{lmn}$ impose at most 8 conditions on the 10-dimensional space $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$, the dimension of $|V|$ drops by at least 2 when restricted to $L_7$. As above, a closer examination shows that in fact $\dim |V|\big|_{L_7} = 1$, so $L_7$ gets mapped to a line in $\mathbb{P}^3$. We want to show that its image meets the chords $C_i$.

When restricted to one of the lines $L_i$, $i = 1, \dots, 6$, the dimension of $|V|$ drops by at least $\dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) - 7 = 3$, so these lines get contracted to points. Now consider finally a member of $Z \in |V|$ containing the exceptional divisor $E_1$. Then $Z - E_1$ is a member of $|4H - 2E_1 - E_2 - E_3 - \sum E_{jk} - \sum E_{lm}|$, which has a 2-dimensional space of sections (passing twice through a point imposes 3 conditions on curves in $\mathbb{P}^2$), so we again find that $E_1$ gets mapped to a line. Since $E_1$ meets $L_1$ and $L_2$, which get contracted to points, $E_1$ must get mapped to the chord $C_1$ joining the images of $p_1$ and $p_2$. By symmetry, $E_2$ and $E_3$ get mapped to $C_2$ and $C_3$, respectively. Since $L_7$ meets all the $E_i$, its image meets all the chords, which was to be shown. □

## 4.3 The class of $D_\Gamma$

The only thing that is left to do in order to prove Theorem 4.2.1 is showing that $D$ is irreducible, which will emerge as a side result from the computation of its class. The proof of the next theorem employs the method used in [13, §10].

**Theorem 4.3.1.** *In terms of the Kapranov basis of* $\mathrm{Pic}(\overline{\mathcal{M}}_{0,7})$ *with respect to the seventh marking, the class of D is given by*

$$\begin{aligned} D \sim 2H - \textstyle\sum_{i=1}^{6} E_i \quad & -E_{12} - E_{34} - E_{56} \\ & -E_{35} - E_{15} - E_{13} \\ & -E_{46} - E_{26} - E_{24} \\ & -E_{135} - E_{246}. \end{aligned} \tag{4.1}$$

*Proof.* If $l_1, l_2, l_3 \in \mathbb{G}(1,3) \subseteq \mathbb{P}(\wedge^2 k^4)$ are mutually skew lines in $\mathbb{P}^3$, and $c \in \mathbb{P}^3$ is a point not lying on any $l_i$, then there is a line through $c$ meeting all of the $l_i$ if and only if the three planes $\langle l_i, c\rangle$ meet in a line, which is the case if and only if the corresponding points $l_i \wedge c \in \mathbb{P}(\wedge^3 k^4) \cong (\mathbb{P}^3)^\vee$ are collinear. Let $l_i = \overline{a_i b_i}$ for $i = 1, 2, 3$. If we choose coordinates on $\mathbb{P}^3$, then in the dual coordinates on $(\mathbb{P}^3)^\vee$ we have

$$l_i \wedge c = \left[\begin{vmatrix} a_{i1} & b_{i1} & c_1 \\ a_{i2} & b_{i2} & c_2 \\ a_{i3} & b_{i3} & c_3 \end{vmatrix} : \begin{vmatrix} a_{i0} & b_{i0} & c_0 \\ a_{i2} & b_{i2} & c_2 \\ a_{i3} & b_{i3} & c_3 \end{vmatrix} : \begin{vmatrix} a_{i0} & b_{i0} & c_0 \\ a_{i1} & b_{i1} & c_1 \\ a_{i3} & b_{i3} & c_3 \end{vmatrix} : \begin{vmatrix} a_{i0} & b_{i0} & c_0 \\ a_{i1} & b_{i1} & c_1 \\ a_{i2} & b_{i2} & c_2 \end{vmatrix}\right]. \tag{4.2}$$

We now construct our embedding $\iota : \mathbb{P}^1 \to \mathbb{P}^3$ by the method used in Lemma 4.2.3. Let $x_i$, $y_i$ and $z$ be the coordinates of the points $p_i, q_i, r \in \mathbb{P}^1$ in an affine chart. Then

$$t \mapsto \left[\prod(z - x_i) \cdot \prod(t - y_i) : \prod(z - y_i) \cdot \prod(t - x_i)\right]$$

is a $g^1_3$ as in Lemma 4.2.3, i. e. mapping each of the triples $x_i$ and $y_i$ to single points. Using $t \mapsto [t : 1]$ for the second factor and embedding $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$ via the Segre embedding $\big([\lambda_1 : \mu_1], [\lambda_2 : \mu_2]\big) \mapsto [\lambda_1\lambda_2 : \lambda_1\mu_2 : \mu_1\lambda_2 : \mu_1\mu_2]$, the coordinates of our lines and the point become

$$a_i = [x_i : 1 : 0 : 0], \quad b_i = [0 : 0 : y_i : 1], \quad c = [z : 1 : z : 1].$$

Plugging this into (4.2), we find that the coordinates of our hyperplanes become

$$\begin{aligned} C_i \wedge \iota(r) &= \left[\begin{vmatrix} 1 & 0 & 1 \\ 0 & y_i & z \\ 0 & 1 & 1 \end{vmatrix} : \begin{vmatrix} x_i & 0 & z \\ 0 & y_i & z \\ 0 & 1 & 1 \end{vmatrix} : \begin{vmatrix} x_i & 0 & z \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} : \begin{vmatrix} x_i & 0 & z \\ 1 & 0 & 1 \\ 0 & y_i & z \end{vmatrix}\right] \\ &= \big[\, y_i - z \,:\, x_i(y_i - z) \,:\, z - x_i \,:\, y_i(z - x_i) \,\big], \end{aligned}$$

and the divisor $D$ is in $\mathbb{A}^7$ given by the vanishing of the maximal minors of the coefficient matrix. Using for example Macaulay2, one sees that this ideal is actually determinantal, generated by the single irreducible quartic polynomial

$$f(x_i, y_i, z) = \begin{vmatrix} y_1 - z & y_2 - z & y_3 - z & 0 \\ x_1(y_1 - z) & x_2(y_2 - z) & x_3(y_3 - z) & 1 \\ z - x_1 & z - x_2 & z - x_3 & 0 \\ y_1(z - x_1) & y_2(z - x_2) & y_3(z - x_3) & 1 \end{vmatrix},$$

which finally finishes the proof of Theorem 4.2.1. Using Macaulay2 again, one can compute the multiplicities of $f$ along the various diagonals $\Delta_I \subseteq \mathbb{A}^7$, and using [13, Lemma 10.4] one finally arrives at the expression (4.1). □

**Remark 4.3.2.**

(i) A naive count would at first suggest that $D$ has no sections, since we are imposing 15 conditions on quadrics. However, the conditions are not independent: If one lets $p_1$ up to $p_5$ be the coordinate points and $p_6 = [1 : 1 : 1 : 1 : 1]$, then a section of $D$ is given by the proper transform of $x_0x_3 - x_1x_2 + x_1x_4 - x_3x_4$. It is not hard to see that in fact $h^0(D) = 1$, which shows directly that $D$ is not an effective linear combination of boundary divisors.

(ii) The polynomial $f$ is quadratic when considered as a polynomial in $z$ with coefficients in $k[x_i, y_i]$. Its discriminant gives exactly the branch locus of the covering map $D \to \overline{\mathcal{M}}_{0,6}$ constructed in Corollary 4.2.5. It would be interesting to give an accessible geometric interpretation to this locus.

(iii) Another obvious question is whether $D$ can also be expressed as a pullback of a Brill-Noether divisor on some $\overline{\mathcal{M}}_g$. Such a description should be related to our geometric characterization by the framework of admissible covers, and the fact that $D$ is a 2-sheeted covering of $\overline{\mathcal{M}}_{0,6}$ should be reflected in the properties of the linear systems arising in this description.

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# Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den 4. September 2013 Fabian Müller