HUMBOLDT-UNIVERSITÄT ZU BERLIN

Donaldson Hypersurfaces and Gromov-Witten Invariants

Dissertation
zur Erlangung des akademischen Grades

doctor rerum naturalium
(Dr. rer. nat.)

im Fach Mathematik

eingereicht an der

Mathematisch-Naturwissenschaftlichen Fakultät
der Humboldt-Universität zu Berlin

von

Dipl.-Math. Alexandre Krestiachine

Präsident der Humboldt-Universität zu Berlin:

Prof. Dr. Jan-Hendrik Olbertz

Dekan der Mathematisch-Naturwissenschaftlichen Fakultät:

Prof. Dr. Elmar Kulke

Gutachter:

1. Prof. Dr. Klaus Mohnke
2. Prof. Dr. Kai Cieliebak
3. Dr. Chris Wendl

eingereicht am: 25.11.2014
Tag der Verteidigung: 27.05.2015

Summary. The question of understanding the topology of symplectic manifolds (M, ω) has received great attention since the work of A. Weinstein and V. Arnold. One of the established tools is the theory of Gromov-Witten invariants. A Gromov-Witten invariant counts intersections of rational cycles in M with the moduli space of J-holomorphic curves representing a fixed class $A \in H_2(M, \mathbb{Z})$ for an ω-tame almost complex structure $J : TM \to TM$. However, without imposing additional assumptions on (M, ω) such counts are difficult to define in general due to the occurence of multiply covered J-holomorphic curves with negative Chern numbers.

This thesis deals with an alternative approach to Gromov-Witten invariants introduced by K. Cieliebak and K. Mohnke. Their approach delivers a pseudocycle for any fixed $A \in H_2(M, \mathbb{Z})$, provided M is closed and $[\omega] \in H^2(M, \mathbb{R})$ admits a lift to a rational class. The main advantage is that the analysis of (domain-dependent) J-holomorphic curves involves standard Fredholm theory. Transversality is achieved by adding additional marked points at the intersections of a curve with a symplectic hypersurface $V \subset M$, whose Poincaré dual is $D[\omega]$ for $D > 0$ an integer chosen sufficiently large. The existence of such hypersurfaces follows from a theorem of S. Donaldson, provided $[\omega]$ is a rational class.

Here this approach is extended to the case of an arbitrary symplectic form $\omega \in \Omega^2(M, \mathbb{Z})$. As in the original work we consider only the case of holomorphic spheres. We show that for any class $[\omega]$ there exists an open neighbourhood $[\omega] \in U \subset H^2(M, \mathbb{R})$, such that for any two rational symplectic forms ω_1, ω_2 with $[\omega_1], [\omega_2] \in U$ the corresponding pseudocycles are rationally cobordant. The proof is based on an adaptation of the arguments from the original Cieliebak-Mohnke approach to a more general situation - a presence of two transversely intersecting hypersurfaces V_1 and V_2 coming from different symplectic forms (ω_1 and ω_2). We pay additional attention to the construction of such hypersurfaces and their properties.

Zusammenfassung. Die Frage nach dem Verständnis der Topologie symplektischer Mannigfaltigkeiten (M, ω) erhielt immer größere Aufmerksamkeit, insbesondere seit den Arbeiten von A. Weinstein und V. Arnold. Ein bewährtes Mittel ist dabei die Theorie der Gromov-Witten-Invarianten. Eine Gromov-Witten-Invariante zählt Schnitte von rationalen Zyklen in M mit Modulräumen J-holomorpher Kurven, die eine fixierte Homologieklasse $A \in H_2(M, \mathbb{Z})$ repräsentieren, für eine ω-zahme fast komplexe Struktur $J : TM \to TM$. Allerdings ist es im Allgemeinen schwierig, solche Schnittzahlen zu definieren, ohne zusätzliche Annahmen an (M, ω) zu treffen, da mehrfach überlagerte J-holomorphe Kurven mit negativer Chernzahl vorkommen können.

Die vorliegende Dissertation folgt einem alternativen Ansatz zur Definition von Gromov-Witten-Invarianten, der von K. Cieliebak und K. Mohnke eingeführt wurde. Dieser Ansatz liefert für jede fixierte Homologieklasse einen Pseudozykel für jede geschlossene glatte Mannigfaltigkeit M mit einer rationalen symplektischen Form $[\omega] \in H^2(M, \mathbb{Z})$. Der Hauptvorteil einer solchen Vorgehensweise ist, dass die Analysis (domainabhängiger) J-holomorpher Kurven nur etablierte nichtlineare Fredholm-Theorie erfordert. Die Transversalität wird durch Hinzufügen zusätzlicher markierter Punkte erreicht, indem diese auf die Schnitte mit einer symplektischen Hyperfläche $V \subset M$ abgebildet werden. Dabei ist die Fundamentalklasse von V Poincaré-dual zu $D[\omega]$ für eine hinreichend große ganze Zahl $D > 0$. Die Existenz solcher Hyperflächen folgt aus einem Theorem von S. Donaldson.

Wir erweitern diesen Ansatz in der vorliegenden Arbeit für eine beliebige symplektische Form $\omega \in \Omega^2(M, \mathbb{Z})$. Wie bereits in der ursprünglichen Arbeit betrachten wir nur den Fall holomorpher Sphären. Wir zeigen, dass für die Koholomogieklasse $[\omega]$ eine offene Umgebung $[\omega] \in U \subset H^2(M, \mathbb{R})$ existiert, so dass für zwei beliebige rationale symplektische Formen ω_1, ω_2 mit $[\omega_1], [\omega_2] \in U$ die dazugehörigen Pseudozykel rational kobordant sind. Der Beweis basiert auf einer Modifikation der Argumente des Ansatzes von Cieliebak und Mohnke für den Fall von zwei sich transversal schneidenden Hyperflächen V_1 und V_2, die jeweils zu verschiedenen symplektischen Formen gehören (ω_1 und ω_2). Dabei schenken wir der Konstruktion und den Eigenschaften solcher Hyperflächen besondere Aufmerksamkeit.

Contents

Introduction

The present thesis deals with closed symplectic manifolds (M, ω), i.e. M is a compact smooth manifold without boundary and a differential form $\omega \in \Omega^2(M, \mathbb{R})$, which is closed ($d\omega = 0$) and non-degenerate ($\omega$ induces an isomorphism $TM \to T^*M$). The latter condition implies that M is orientable and even-dimensional.
The study of symplectic manifolds as such emerged from the theory of dynamical systems. Especially in the aftermath of the work of Alan D. Weinstein and Vladimir I. Arnold in the early 1970's questions concerning symplectic geometry attracted more and more attention.
Naturally, one would look for symplectic invariants. Besides the obvious one (like the class $[\omega] \in H^2(M, \mathbb{R})$) an ideal invariant would be one that takes into account symplectic cycles or symplectic submanifolds of M. Unfortunately, no general existence results are available. However, Michail L. Gromov introduced pseudo holomorphic curves in his seminal paper [Gro85] giving a starting point for invariants of a similar type.
Consider the space

$$\mathcal{J}_\tau(M, \omega) = \{J \in \mathrm{Aut}(TM) \mid J \circ J = -\mathbf{Id}_{TM},\, \omega(v, Jv) > 0 \text{ for all } v \neq 0\}$$

of ω-tame[1] almost complex structures on M. This space is contractible and hence $c_1(TM, \omega) := c_1(TM, J)$ for $J \in \mathcal{J}_\tau(M, \omega)$ is well-defined. Consider a (closed) Riemann surface (Σ, j) of genus g and take $J \in \mathcal{J}_\tau(M, \omega)$ a J-holomorphic (or pseudo holomorphic) curve in M is a smooth map

$$u : \Sigma \longrightarrow M, \text{ satisfying } \bar{\partial}_J u := du + J \circ df \circ j = 0.$$

Note that an embedded J-holomorphic curve is actually a symplectic submanifold.
A rich source for symplectic manifolds comes from complex geometry. Since

[1] We will denote the space of all ω-compatible almost complex structures by $\mathcal{J}_c(M, \omega)$. Any such $J \in \mathcal{J}_c(M, \omega)$ induces a Riemannian metric via $g_J := \omega(\cdot, J\cdot)$.

any Kähler manifold is symplectic and any complex submanifold of a Kähler manifold is also symplectic, it follows that any smooth projective variety carries a symplectic structure (induced by restricting the Fubini-Study form). However, the class of symplectic manifolds is different[1] from that of complex manifolds. Indeed the Nijenhuis tensor N_J might not vanish, so $J \in \mathcal{J}_\tau(M,\omega)$ might be not complex[2].

Moduli spaces of curves: idea vs. reality

We start with a simplified and idealistic version of what **one could expect** a Gromov-Witten invariant for a symplectic manifold to be.
Fix a class $A \in H_2(M,\mathbb{Z})$, $k \geq 0$ and $J \in \mathcal{J}_\tau(M,\omega)$ define the space

$$\begin{aligned}\mathcal{M}_{g,k}(A,J) := \big\{ & u : \Sigma \to M \mid u \text{ is J-holomorphic, } [u] = A, \\ & z_i \in S^2 \text{pairwise distinct for } i = 1,\dots,k \big\} / \mathrm{Aut}(S^2).\end{aligned}$$

If the space $\tilde{\mathcal{M}}_{g,k}(A,J)$ turns out to be a closed smooth finite-dimensional (say the dimension is d) manifold. Then the evaluation map $\mathrm{ev}^k : \tilde{\mathcal{M}}_{g,k}(A,J) \to M^k$ via $u \mapsto (u(z_1),\dots,u(z_k))$ would represent a d-cycle in M^k. Then, by taking cycles $\alpha_1,\dots,\alpha_k \in H_*(M,\mathbb{Z})$ with $\deg(\alpha_1)+\ldots+\deg(\alpha_k) = d$, an **idealistic** invariant would be given by intersection of cycles $[\mathrm{ev}^k] \cdot (\alpha_1,\dots,\alpha_k)$. However, such a situation almost **never** occurs due to the following problems.

(Transversality) One cannot expect the space $\tilde{\mathcal{M}}_{g,k}(A,J)$ to be a smooth manifold for all $J \in \mathcal{J}_\tau(M,\omega)$, even restricted to simple (non-multiply covered) curves[3]. One can expect this statement to hold only for a generic J (i.e. J is contained in a subset of second Baire category in $\mathcal{J}_\tau(M,\omega)$).

(Automorphisms) A reparametrization group $G = \mathrm{Aut}(\Sigma, j)$ acts on the space $\tilde{\mathcal{M}}_{g,k}(A,J)$. Even in the case $\Sigma = S^2$ it is $G = \mathrm{PSL}(2,\mathbb{Z})$ the group of Möbius transformations, i.e. a non-compact group.

(Compactness) It was observed by K. Uhlenbeck that the quotient $\tilde{\mathcal{M}}_{g,k}(A,J)/G$ might still be non-compact due to the bubbling phenomenon (see section 4.2 in [MS04]). This can be resolved by including Gromov limits of curves (or even better, Kontsevich's stable maps).

[1] See also section 1.4 for more facts on this topic.

[2] There exist symplectic manifolds with $N_J \neq 0$ for all $J \in \mathcal{J}_\tau(M,\omega)$. The first example was found in [Thu76], see also [McD84] for a simply-connected example.

[3] In general the linearization of Cauchy-Riemann operator $\bar{\partial}_J$ might be not surjective.

(Pseudocycle) Even after establishing compactness it is not clear that the evaluation map defines a cycle in M^k, since a compactification might contribute as a topological boundary, so it is not clear how to define the fundamental class $[\mathrm{ev}^k]$ in order to get a well-defined intersection theory.

(Independence) In order to achieve a symplectic invariant, the intersection product should not depend on the choice of $J \in \mathcal{J}_\tau(M,\omega)$ required in the definition of $\tilde{\mathcal{M}}_{g,k}(A,J)$.

The first step towards a solution of above problems was done in the seminal work of Y. Ruan and G. Tian [RT95]. They established Gromov-Witten invariants for all genera $g(\Sigma) \geq 0$ assuming semi-positivity[1] of (M,ω).

Independently, a treatment of the case $g(\Sigma) = 0$, i.e. that of J-holomorphic spheres, appeared in [MS94] and a very detailed exposition can be found in [MS04], again under the assumption of semi-positivity of (M,ω). The outline of the argument is as follows. Consider the space of stable maps[2] with $k \geq 3$ marked points $\bar{\mathcal{M}}_k$. Stability implies that there are no symmetries[3] on constant (ghost) components. Perturbing J (actually, the existence of such perturbations follows from Sard-Smale theorem) one can achieve a stratification $\mathcal{M}_T(A,J)$ of $\bar{\mathcal{M}}_k$, with each stratum $\mathcal{M}_T(A,J)$ being a smooth orientable manifold[4] of real dimension $2(n-3+c_1(A)+k-e(T))$, where T is a k-labelled tree and $e(T) = |T|-1$. Then the evaluation map of the top stratum (T with only one vertex) yields a pseudocycle (see Appedix A.2), since by dimension formula all other strata have codimension at least two. The existence of a pseudocycle is sufficient to establish intersection theory (see section 6.5 in [MS04].).

(Multiply covered curves with $c_1 < 0$) The semi-positivity assumption is essential in the above approach. Consider the following geometric situation (cf. p. 937 in [FO99]). Restrict to spheres without marked points and consider homology classes $A, A_1, A_2 \in H_2(M,\mathbb{Z})$ with $A = A_1 + lA_2$ for some integer $l > 0$. Assume that $n - 3 + c_1(A_2) \geq 0$ and $c_1(A_2) < 0$. Consider the moduli space $\tilde{\mathcal{M}}_0(A,J) = \{u : S^2 \to M \mid [u] = A, \bar{\partial}_J u = 0\}/\mathrm{Aut}(S^2)$. Assume that there exists sequence $u^\nu \in \tilde{\mathcal{M}}_0(A,J)$ that has a Gromov limit

---

[1] A symplectic manifold $/M,\omega)$ is called **semi-positive** if for any spherical class $A \in H_2(M,\mathbb{Z})$ with $\omega(A) > 0$ and $c_1(A) \geq 3-n$ it follows that $c_1(A) \geq 0$.

[2] Cf. section 3.4.

[3] See p. 110 in [MS04].

[4] One associates to J a Cauchy-Riemann operator $\bar{\partial}_J$, then its linearization is a Fredholm operator between Banach spaces; if it is surjective, for generic J the implicit function theorem implies that the kernel has finite dimension equal to the Fredholm index. The index is given the Riemann-Roch theorem for (real) linear Cauchy-Riemann operators. Smoothness follows by elliptic regularity. See also [Wen13] for a detailed exposition.

consisting of two J-holomorphic spheres $u_1, u_2 : S^2 \to M$ with $[u_1] = A_1$ and $[u_2] = lA_2$ and u_2 is multiply covered. So there exists a holomorphic map $\phi : S^2 \to S^2$ of degree l and a J-holomorphic sphere u_2', s.t $u_2 = u_2' \circ \phi$ so $[u_2'] = A_2$. Assuming transversality, Riemann-Roch theorem implies that the (expected) dimension $\dim \tilde{\mathcal{M}}_0(A_2, J) = 2n - 6 + 2c_1(A_2) \geq 0$, on the other hand $\dim \tilde{\mathcal{M}}_0(lA_2, J) = 2n - 6 + 2lc_1(A_2) < 0$ for l large. But any curve $u \in \tilde{\mathcal{M}}_0(A_2, J)$ induces a curve $u \circ \phi \in \tilde{\mathcal{M}}_0(lA_2, J)$ so $\tilde{\mathcal{M}}_0(A_2, J) \subset \tilde{\mathcal{M}}_0(lA_2, J)$. Hence such dimension count cannot be correct, i.e. the space $\tilde{\mathcal{M}}_0(lA_2, J)$ can not be made transversal for any $J \in \mathcal{J}_\tau(M, \omega)$. A similar issue occurs if one looks at the strata needed for a compactification of $\tilde{\mathcal{M}}_0(A, J)$. Note that above situation might occur only if $n > 3$. The presence of curves with negative Chern numbers causes transversality problems in other situations - see section 5.1 in [Sal97].

Symplectic manifolds which are not semi-positive exist in abundance - see section 6.4 in [MS04]. A simple example is provided by a symplectic blow up of $(\mathbb{C}P^4, \omega_{\mathrm{FS}})$ at one point. Then the exceptional divisor has a negative Chern number.

The definition of Gromov-Witten invariants for general symplectic manifold was etablished in

- [FO99] using Kuranishi structures and multi-valued perturbations.
- [LT98b] adapting arguments from the definition of Gromov-Witten invariants of an algebraic variety (given in [LT98a]).
- [Sie99b] using a similar approach.
- Moreover, it was shown in [Sie99a] and [LT99] that in the case of a projective algebraic variety the symplectic definitions coincide with a definition coming from algebraic geometry, given in [Beh97].
- It is also expected that Hofer's polyfold theory [Hof08] gives a solution.
- Recently, a more topological approach was presented in [Par14].

However, all above mentioned methods have one similarity - the introduction of more general perturbation tools in order to achieve transversality for moduli spaces of J-holomorphic curves.

Cieliebak-Mohnke approach and main result

In [CM07] a geometric approach to genus zero Gromov-Witten invariants was introduced. One of the main advantages is that the Fredholm analysis of J-holomorphic curves is kept standard, just as in [MS04]. The idea is that moduli spaces of J-holomorphic maps with domain-dependent J, whose underlying curves are already stable, give rise to pseudocycles. Such (domain) stability is achieved by putting additional marked points on the intersection points with a fixed symplectic hypersurface[1]. The drawback of this approach is that the

[1] A symplectic submanifold of real codimension two.

perturbation spaces (subsets of $\mathcal{J}_\tau(M,\omega)$) become quite complicated.
A sequence of symplectic hypersurfaces is provided by the celebrated result of S. Donaldson in [Don96]. It provides for any fixed $J \in \mathcal{J}_c(M,\omega)$ and a positive[1] $D \gg 0$ a symplectic hypersurface $V \subset M$ with $\mathrm{PD}([V]) = D[\omega]$, assuming that the symplectic form represents a rational class, i.e. $[\omega] \in H_2(M,\mathbb{Z})$. Given such a pair (V,J), denote by $\mathcal{J}(M,V,J,\theta) \subset \mathcal{J}_\tau(M,\omega)$ the space of tame almost complex structures leaving TV invariant and being θ-close[2] to J. For $l \geq 3$ let $\bar{\mathcal{M}}_{l+1}$ be the Deligne-Mumford space of stable curves with $l+1$ marked[3] points. The (perturbation) space of coherent almost complex structures is a subset[4]

$$\mathcal{J}_{l+1}(M,V,J,\theta) \subset \mathcal{C}^\infty(\bar{\mathcal{M}}_{l+1}, \mathcal{J}(M,V,J,\theta_1)).$$

For a $K \in \mathcal{J}_{l+1}(M,V,J,\theta)$ and $k \geq 1$ let $\pi_l : \bar{\mathcal{M}}_{k+l+1} \to \bar{\mathcal{M}}_{k+1}$ be the map that forgets first k marked points and stabilizes. Then any $K \in \mathcal{J}_{l+1}(M,V,J,\theta)$ induces $\pi_l^* K \in \mathcal{J}_{k+l+1}(M,V,J,\theta)$. Fix a $A \in H_2(M,\mathbb{Z})$ and denote the moduli space of $\pi_l^* K$-holomorphic spheres representing class A with $k+l$ marked points mapping last l points to hypersurface V by $\mathcal{M}_{k+l}(A,K,V)$.

Theorem A (Theorem 1.2 in [CM07]) *Assume that (V,J) is a Donaldson pair[5]. Let $l = D\omega(A)$, then there exists a nonempty set $K \in \mathcal{J}^{\mathrm{reg}}_{l+1}(M,V,J,\theta) \subset \mathcal{J}_{l+1}(M,V,J,\theta)$, such that for any $k \geq 1$ the evaluation map at the first k marked points*

$$\mathrm{ev}^k : \mathcal{M}_{k+l}(A,K,V) \to M^k$$

represents a pseudocycle $\mathrm{ev}^k(A,V,J,K)$ of dimension $2n - 6 + k + c_1(A)$.

Moreover, it was shown in [CM07] (Theorem 1.3) that the pseudocycle $\mathrm{ev}^k(A,V,J,K)$ does not depend on perturbation K, hypersurface V and a compatible almost complex structure J. In the sense that any two such pseudocycles are rationally cobordant (see Appendix A.2 for the definition). Hence, Theorem A actually yields (up to multiplication with a positive rational number) a pseudocycle $\mathrm{ev}^k(\omega,A)$. The proof requires Auroux's asymptotic uniqueness result for Donaldson hypersurfaces [Aur97].

Remark 1 *A generalization to the curves of higher genus was recently resolved in [Ger13] and independently also in [IP13]. The Cieliebak-Mohnke approach was used in [Wen14] in order to obtain results on hypersurfaces of contact type (avoiding the semi-positivity assumption).*

---

[1] We will often call D the degree of V.

[2] If not explicitly stated, we always use C^0 norms induced by (ω,J).

[3] The extra "+1" marked point plays the role of a variable for domain-dependence.

[4] The definition is located in section 3.2.

[5] See section 4.1 for the precise definition.

However, the assumption that ω represents an integer (or rational) homology class is essential for the approach. One cannot expect the existence of a symplectic hypersurface V (Poincaré dual to $D[\omega]$) for non-rational ω in order to control the intersection of V with holomorphic curves. Our main result is the following.

Theorem B *Given **any** symplectic form ω on M. Fix $A \in H_2(M,\mathbb{Z})$. There exists an open neighbourhood of ω, say $U \subset \Omega_2(M)$, such that for any pair of rational symplectic forms $\omega_1, \omega_2 \in U$ the corresponding (coming from Theorem A) pseudocycles $ev^k(\omega_1, A)$ and $ev^k(\omega_2, A)$ are rationally cobordant, up to multiplication with positive rational weights, for any $k \geq 3$.*

Outline of the proof

Consider a symplectic manifold (M,ω) with an integral class $[\omega] \in H^2(M,\mathbb{Z})$ and fix $J \in \mathcal{J}_c(M,\omega)$. We sketch the main steps:

(I) Given a fixed energy level $E > 0$ and a rational symplectic form ω' near ω. There exists an ω-symplectic hypersurface V Poincaré dual to $D[\omega']$, such that assuming regularity of spaces of simple holomorphic spheres of energy at most $D \gg 0$ implies that all holomorphic spheres of energy at most E in V are constant and all non-constant spheres intersect V in at least three points in the domain. Holomorphicity means here with respect to a tame almost complex structure K near J (cf. section 3.7).

(II) We adapt the definition of a Donaldson quadruple from [CM07] to the case where one of the hypersurfaces is Poincaré dual to $D'[\omega']$ with $\omega' \neq \omega$. Denote such a quadruple by (V, V', ω, J), see section 4.2 for a precise definition. Associate to such a quadruple the moduli space $\mathcal{M}_{k+l+l'}(A, K, V \cup V')$ of K-holomorphic spheres (here K is allowed to be domain-dependent) with $k + l + l'$ marked points in class A mapping middle l points to V and last l' points to V'.

(III) Establish perturbation spaces $\mathcal{J}^*_{l+1}(M, V \cup V', J, \theta, E)$ of coherent[1] ω-tame almost complex structures leaving V and V' invariant and being θ-close to J. Using similar arguments as in [CM07] we show compactness for domain-stable maps:

Theorem C.1 (see Theorem 4.10) *Fix an energy level $E > 0$ and a Donaldson quadruple (ω, J, V, V'). For $A \in H_2(M,\mathbb{Z})$ assume $\max\{\omega(A), \omega'(A)\} \leq E$ and set $l := D\omega(A)$, $l' := D'\omega'(A)$. For $k \geq 0$ take a subset $I \subset \{k+1, \dots, k+l+l'\}$ with $\{k+1, \dots, k+l\} \subset I$ and fix $K \in \mathcal{J}^*_{|I|+1}(M, V \cup V', J, \theta_1, E)$. Assume that a sequence of K-holomorphic spheres*

[1] See section 3.2.

$f^\nu \in \mathcal{M}_{k+l+l'}(A, K, V \cup V')$ has a Gromov limit - the stable map $(\mathbf{f}, \mathbf{z})$. Then the underlying nodal curve $\mathbf{z}$ is I-stable. Same statement holds if $\{k+l+1, \ldots, k+l+l'\} \subset I$.

I-stability means that a nodal curve is stable after removing marked points outside of I.

(IV) Providing the existence of regular perturbations $\mathcal{J}^{\text{reg}}_{l+1}(M, V \cup V', J, \theta, E) \subset \mathcal{J}^*_{l+1}(M, V \cup V, J, \theta, E)$ in order to achieve transversality of strata required for the compactification of $\mathcal{M}_l(A, K, V \cup V')$. This combined with Theorem C.1 implies

Theorem C.2 *Assumptions as in Theorem C.1 imply that for any $k \geq 1$ and any $K \in \mathcal{J}^{\text{reg}}_{|I|+1}(M, V \cup V', J, \Theta_1, E)$ the evaluation map that evaluates first k-marked points $ev^k : \mathcal{M}_{k+l+l'}(A, K, V_0 \cup V_1) \longrightarrow X^k$ defines the (rational) pseudocycle $\mathrm{ev}^k(A, V, V', J)$ of real dimension $d := 2(n - 3 + k + c_1(A))$.*

(V) In section 4.3 we show that, assuming the existence of a Donaldson quadruple (ω, J, V, V'), arguments from [CM07] together with Theorem C.2 yield rational cobordisms of pseudocycles (provided by Theorem A) $\mathrm{ev}^k(A, \omega)$ and $\mathrm{ev}^k(A, \omega')$ up to a multiplication with positive rational weights. Note that the existence of a Donaldson quadruple is not just a transversal intersection of two symplectic hypersurfaces. We require that perturbation spaces $\mathcal{J}^*_{l+1}(M, V \cup V, J, \theta, E)$ are nonempty.

(VI) Given **any** symplectic form ω_0 on M and fix $J_0 \in \mathcal{J}_c(\omega)$. Using the results from section 2.2 we can find rational symplectic forms ω, ω' ρ-nearby ω_0 and $J \in \mathcal{J}_c(\omega)$, $J' \in \mathcal{J}_c(\omega')$ ρ-nearby J for some $\rho > 0$. Then a modification of the Donaldson hypersurface theorem from chapter 2 yields a pair of ω-symplectic hypersurfaces V and V' that intersect transversely. In section 4.4 we show that such V and V' yield Donaldson quadruple (V, V', ω, J) provided ρ is chosen sufficiently small. Then Theorem C.2 implies our main result - Theorem B.

Note that our quadruples depend on a previously fixed energy level $E > 0$. Our geometric construction starts with rational (ω, ω'), however, we measure energy for each $A \in H_2(M, \mathbb{Z})$ with respect to $E(A) := \max\{N\omega(A), N\omega(A))$ with $N := \min\{n \in \mathbb{N} \mid [n\omega] \in H^2(M, \mathbb{Z})$ and $[n\omega'] \in H^2(M, \mathbb{Z})\}$. In a sense, we are using ω and ω' for the geometric construction and $(n\omega, n\omega')$ for transversality discussion, since assumptions on (Kähler) angles are invariant under scaling of the symplectic form ω.

It is important to understand that after considering (ω, ω') we get N, hence the energy $E(A)$. Only after that we construct hypersurfaces of high degree in order to obtain a quadruple.

Another essential point is that we use Opshtein's observation (about the transversality constant η in Donaldson's construction) described in section 2.5 in order to construct our quadruples. More precisely, we need to guarantee that the corresponding perturbation spaces are actually nonempty.

Discussion and remarks

Observe that we are not defining moduli spaces of holomorphic curves for an irrational symplectic form ω directly. However, Theorem B allows us to define (genus zero) Gromov-Witten invariants for such an ω. For any fixed $A \in H_2(M, \mathbb{Z})$ and $k \geq$ pick a rational ω' from the open neighbourhood U of ω, provided by Theorem B. Then, there exists a positive rational weight l, such that the cobordism class of pseudocycle $l\mathrm{ev}^k(A, \omega')$ from Theorem A does not depend on the particular choice of ω'.
Hence, the definition works exactly as in [CM07]. Let $\alpha_1, \ldots, \alpha_k$ be nontorsion cohomology classes in M of total degree $2n - 6 + 2k - 2c_1(A)$. Represent the Poincaré dual of the cup product of pullbacks of these classes to M^k by a cycle a in M. Assume that a is strongly transverse[1] to $\mathrm{ev}^k(A, \omega')$. Then the (genus zero) Gromov-Witten invariant is given by the intersection

$$\mathrm{GW}^{\omega}_{A,k}(\alpha_1, \ldots, \alpha_k) = \mathrm{GW}^{\omega'}_{A,k}(\alpha_1, \ldots, \alpha_k) = l\mathrm{ev}^k(A, \omega) \cdot a.$$

The statement of Theorem B is actually not that surprising. In the semi-positive case Gromov-Witten invariants are known to be deformation[2] invariant (see remark 7.1.11 in [MS04]) as long as the deformation (M, ω_t) is semi-positive for all t. Other approaches to Gromov-Witten invariants assert similar deformation invariance. It was observed in [CM07] that in the semi-positive case both invariants are equal (the regularity condition for simple curves holds without any hypersurface).

Structure of the thesis

We emphasize that the thesis is not self-contained. Our main focus lies on geometric aspects of the theory. Regarding the analysis of holomorphic curves we heavily rely on [CM07], which in turn is based on a very detailed exposition in [MS04]. Whenever possible our notation is kept identical to that used in [CM07].

[1] See Appendix A.2.
[2] See survey [Sal12] on deformation relations of symplectic structures.

The first part of Chapter 1 contains an overview of Donaldson's construction of symplectic hypersurfaces. We provide some geometric details. In the second part we review some topological properties of such submanifolds and discuss related open questions.

In the second chapter we prove technical statements needed to control deformations of symplectic and almost complex structures. Then we show a modification of Donaldson's argument which produces transversal intersections of symplectic hypersurfaces. We finish the chapter with Opshtein's observation.

Chapter 3 contains definitions and statements from [CM07] and [MS04] needed for our main result. The last section deals with intersections between holomorphic curves and symplectic hypersurfaces of high degree (constructed by starting with a different symplectic form).

The last chapter contains our definition of a Donaldson quadruple together with compactness and transversality results for corresponding moduli spaces. In the final part we combine results from Chapters 2 and 3 in order to show Theorem B.

Acknowledgement. I would like to express particular gratitude to my advisor Prof. Klaus Mohnke for the support over the past years and for spending many hours in our meetings.
I would like to thank Chris Wendl and Viktor Fromm for answering many questions about holomorphic curves and Jonathan David Evans for a discussion on Donaldson hypersurfaces. I benefitted a lot from discussions with Helga Baum, Sebastian Boldt, Frank Lapp and Dorothee Schüth.
I am truly grateful to Judith Berthold and Marko Berghoff for proofreading the manuscript.
Finally, I would like to thank Berlin Mathematical School, SFB 647 and Dirk Kreimer for providing financial support.

1

Approximate holomorphic geometry

Here we review the celebrated Donaldson hypersurface theorem from [Don96]. Consider a closed symplectic manifold (M,ω) with $[\omega]\in H^2(M,\mathbb{Z})$ and a fixed ω-compatible almost complex structure $J\in\mathcal{J}_c(M,\omega)$. Consider a complex line bundle[1] $L\to M$ equipped with a Hermitian connection whose curvature form is given by $-\frac{i}{2\pi}\omega$. We show that for any (sufficiently large) $k\gg 0$ there exist (approximate holomorphic) sections of $s_k : M\to L^k$. The sections s_k are uniformly transversal to the zero section, cutting out ω-symplectic hypersurfaces V_k. Also, see Section 2.2 of [AS08] for a short exposition.
We review the main steps of the argument proving some geometric details. At the end of the chapter we collect some properties and open questions regarding submanifolds V_k.

1.1 Localized sections

Consider an $\mathbb{R}$-linear map $A:\mathbb{C}^n\to\mathbb{C}$ and define

$$A'(z):=\frac{A(z)-iA(iz)}{2} \text{ and } A''(z):=\frac{A(z)+iA(iz)}{2}.$$

Then we have $A=A'+A''$, $A'(iz):=iA'(z)$ and $A''(iz):=-iA''(z)$, .i.e. A' and A'' decompose A in its complex linear and complex anti-linear part. The following lemma is fundamental to the Donaldson hypersurface theory.

Lemma 1.1. *If $\|A''\|<\|A'\|$, then the subspace* $\ker A\subset\mathbb{C}^n$ *is symplectic with respect to the standard symplectic form ω_0.*

Proof. The following argument is due to Patrick Massot. Define the adjoint map $A_*:\mathbb{C}\to\mathbb{C}^n$ via $\langle v, A_*z\rangle=\langle Av,z\rangle$ for $v\in\mathbb{C}^n$ and $z\in\mathbb{C}$.
The splitting of A enduces the corresponding $\mathbb{C}$-linear map $A'_*:\mathbb{C}\to\mathbb{C}^n$ and

[1] See Appendix A.1.

$\mathbb{C}$-antilinear map $A''_* : \mathbb{C} \to \mathbb{C}^n$.
Consider two vectors $v := A'_* 1$ and $w := A''_* i$. Observe that $\|A'_*\| = \|v\|$ and $\|A''_*\| = \|w\|$, so the inequality $\|A''\| < \|A'\|$ implies $\|v\| < \|w\|$. We compute

$$A_* 1 = A'_* 1 + A''_* 1 = v - iA''_* i = v - iw$$

$$A_* i = A'_* i + A''_* i = iv + w$$

$$\omega_0(A_* 1, A_* i) = \omega_0(v - iw, iv + w) = \|v\|^2 - \|w\|^2 - \underbrace{\omega_0(v, w) - \omega_0(iw, iv)}_{=0}.$$

Hence, $\omega_0(A_* 1, A_* i) \neq 0$, i.e. $\operatorname{span}_{\mathbb{R}} \{A_* 1, A_* i\} \subset \mathbb{C}^n$ is symplectic. Finally,

$$\ker A = (\operatorname{im} A_*)^{\perp} = i(\operatorname{im} A_*)^{\omega_0},$$

and the claim follows. ∎

Now, consider the trivial line bundle $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$, equip the base $\mathbb{C}^n$ with the standard symplectic and complex structure (ω_0, J_0) and define the 1-form

$$A := \frac{1}{4} \sum_{j=1}^{n} z_j d\bar{z}_j - \bar{z}_j dz_j, \;\; A \in \Omega^1(\mathbb{C}^n).$$

Then $dA = -i\omega_0$. Denote the complex-antilinear part of A by $A^{0,1} := \sum_j z_j d\bar{z}_j$, so we can define the modified Cauchy-Riemann operator on sections

$$\bar{\partial}_A f := \bar{\partial} f + f \cdot A^{0,1} \text{ for a smooth } f : \mathbb{C}^n \to \mathbb{C}.$$

Note that here we write $\bar{\partial} := d^{0,1}$. Now, consider a specific real valued section $\tilde{\sigma}(z) := e^{-\frac{|z|^2}{4}}$ for $z \in \mathbb{C}^n$. Since $|z|^2 = z \cdot \bar{z}$, we see that

$$\bar{\partial}\tilde{\sigma} = \frac{1}{4} \left(\sum_{j=1}^{n} z_j d\bar{z}_j \right) e^{-\frac{|z|^2}{4}}.$$

We arrive at the next fundamental observation, namely $\bar{\partial}_A \tilde{\sigma} = 0$, since

$$\bar{\partial}_A \tilde{\sigma} = \bar{\partial}\tilde{\sigma} + \tilde{\sigma} \cdot A^{0,1} = \frac{1}{4} \left(\sum_{j=1}^{n} -z_j d\bar{z}_j + z_j d\bar{z}_j \right) e^{-\frac{|z|^2}{4}} = 0.$$

Remark 1.2. In the literature this is sometimes referred to as the effect of positive curvature and might be interpreted as follows. One might think of A as a connection form on the trivial bundle. So this bundle possesses holomorphic sections (with respect to modified Cauchy-Riemann operator $\bar{\partial}_A$) which are rapidly decreasing at infinity, in contrast to the flat case.

On the other hand the complex linear part of A defines the operator

$$\partial_A f := \partial f + A^{1,0} f, \text{ for a smooth } f : \mathbb{C}^n \to \mathbb{C},$$

and again, here we denote $\partial f := d^{1,0}$. Together both operators form a connection on the trivial line bundle, we denote it by $\nabla := \bar{\partial}_A + \partial_A$. Moreover, observe that for $\tilde{\sigma}$ we have

$$\nabla\tilde{\sigma} = \underbrace{\bar{\partial}_A\tilde{\sigma}}_{=0} + \partial_A\tilde{\sigma} = -\frac{1}{2}\left(\sum_{j=1}^{n} z_j d\bar{z}_j\right) e^{-\frac{|z|^2}{4}}.$$

Recall from Appendix A.1 that we can consider the complex line bundle $L \to M$ together with a Hermitian connection with a curvature form $-i/(2\pi)\omega$, if $[\omega] \in H^2(M, \mathbb{Z})$. For a given integer $k > 0$ denote the tensor bundle by $L^k := \underbrace{L \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} L}_{k \text{ - times}}$. L^k is again a complex line bundle equipped with an induced Hermitian connection with a curvature form $-ik/(2\pi)\omega$.
The main point of this section is to transport section $\tilde{\sigma}$ to a section of the line bundle L^k. We begin with preliminary definitions. Denote by g the metric induced by ω and J, then g induces the distance function $d : M \times M \to \mathbb{R}$. Then the rescaled metric $g_k := kg$ induces the rescaled distance function set $d_k = k^{\frac{1}{2}} d$ and we define $e_k : M \times M \to \mathbb{R}$ via

$$e_k(p,q) := \begin{cases} \exp\left(-\frac{d_k(p,q)^2}{5}\right) & \text{if } d_k(p,q) \leq k^{\frac{1}{4}} \\ 0 & \text{else.} \end{cases}$$

Proposition 1.3 (cf. Proposition 9 in [Don96]). *For any $p \in M$ and $k \gg 0$ there exists a smooth section σ_p of the line bundle L^k and constant $C > 0$ (independent of k), such that at any $q \in M$ we have*

1. *there exists an $R > 0$, indepedent of q, such that $d_k(p,q) \leq R$ implies $|\sigma_p(q)| \geq \frac{1}{C}$*
2. *$|\sigma_p(q)| \leq e_k(p,q)$*
3. *$|\nabla\sigma_p(q)| \leq C(1 + d_k(p,q))e_k(p,q)$*
4. *$|\bar{\partial}\sigma_p(q)| \leq Ck^{-\frac{1}{2}} d_k(p,q)^2 e_k(p,q)$*
5. *$|\nabla\bar{\partial}\sigma_p(q)| \leq Ck^{-\frac{1}{2}}(d_k(p,q) + d_k(p,q)^3)e_k(p,q)$*

The operators $\bar{\partial}$ and ∇ on L^k are induced by the corresponding operators defined on L.

Remark 1.4. The lower bound from (1) together with the upper bound (2) imply that section σ_p is actually concentrated around the point p, which justifies the name of this section.
Sections with $|\bar{\partial} s| \ll |\partial s|$ are often called **approximate holomorphic**. Note that sections σ_p are approximate holomorphic, since a lower bound for $|\partial \sigma_p|$ follows from statement (1) in the above proposition.

Proof. The proof is basically Section 2 in [Don96]. The idea is simple, one cuts off section $\tilde{\sigma}$ and it transports to L^k via a suitable Dabroux chart. Here, we give the main steps of the argument.

(I) Take a standard cut-off function $\beta : [0, \infty) \to \mathbb{R}$ with

$$\beta(x) := \begin{cases} 1 & \text{if } x \leq \frac{1}{2} \\ 0 & \text{if } x \geq 1 \\ \text{smooth monotone} & \text{else} \end{cases}$$

define a k-dependent cut-off function $\beta_k : \mathbb{C}^n \to \mathbb{R}$ via $\beta_k(z) := \beta(k^{-1/6}|z|)$. Note that $\operatorname{supp}(\beta_k) \subset \left\{|z| \leq k^{1/6}\right\}$.

(II) Let $B_R := B(0, R) \subset \mathbb{C}^n$ be the Euclidean ball of radius R centered at the origin. Choose a Darboux chart $\phi : B_R \to V$, $\phi^*\omega = \omega_0$, such that $\phi(0) = p$. Note that ϕ can be chosen in a way that all its derivatives with respect to metric g do not depend on point p. Moreover, we assume that $\phi^* J(x)_{|x=0} = J_0$. Define the rescaled chart $\phi_k : B_{\sqrt{k}R} \to M$ via $\phi_k(x) = \phi(k^{-1/2}x)$, and we have then $\phi_k^* k\omega = \omega_0$.

(III) Lift ϕ_k to a bundle map. More precisely, consider the trivial line bundle $B_{\sqrt{k}R} \times \mathbb{C} \to B_{\sqrt{k}R}$ equipped with the connection $d + A$ (see above discussion). Using parallel transport one can lift the chart ϕ_k to $\tilde{\phi_k}$, i.e. the following diagram commutes

$$\begin{tikzcd} B_{\sqrt{k}R} \times \mathbb{C} \arrow[r, "\tilde{\phi}_k"] \arrow[d, "\pi"'] & L^k \arrow[d, "\pi"] \\ B_{\sqrt{k}R} \arrow[r, "\phi_k"] & M \end{tikzcd}$$

with the property that $\tilde{\phi}_k^* \nabla = d + A$, where ∇ is the Hermitian connection on the complex line bundle L^k.

(IV) Define section $\sigma_p : M \to L^k$ by setting

$$\sigma_p(x) := \begin{cases} \tilde{\phi}_k \circ (\beta_k \cdot \tilde{\sigma}) \circ \phi_k^{-1} & \text{if } x \in \operatorname{Im}(\phi_k) \\ 0 & \text{if } x \notin \operatorname{Im}(\phi_k). \end{cases}$$

Such σ_p satisfies above inequalities (cf. pp. 675-677 in [Don96]). The main point is that the chart ϕ_k is very close to being an isometry (it is one at the origin, by assumption). ■

1.2 Controlled transversality

Fix points $p_i \in M$ with $i = 1 \ldots m$ for some integer $m > 0$ and consider sections $\sigma_i := \sigma_{p_i}$ from Proposition 1.3. Moreover, fix a collection of complex numbers $w := \{w_1, \ldots, w_m\}$ with all $|w_i| \leq 1$. Then a linear combination yields a new section of L^k

$$s_w := \sum_{i=1}^{m} w_i \sigma_i.$$

The aim of this passage is twofold - first, it is to show that there exists an appropriate choice of points p_i, such that section s_w satisfies similar upper bounds as in Proposition 1.3. This is the easier part. Second, it is to show that there is an appropriate choice of the coefficients w_i, such that ∂s_w is bounded from below near the zero section. This part is considerably harder.

Definition 1.5. *We call a cover $\{B(p_i)\}_{i \in \{1..m\}}$ of M with g_k-unit balls centered at $p_i \in M$* ***admissible*** *if for any $q \in M$*

$$\sum_{i=1}^{m} d_k(q, p_i)^r e_k(q, p_i) \leq C, \text{ for } r = 0, .., 3.$$

Lemma 1.6 (cf. Lemma 12 in [Don96]). *For any $k > 0$ there exists an admissible covering of M with a constant C which does not depend on k.*

Proof. The main point of the proof is that in the Euclidean case taking lattice

$$\Lambda := \frac{1}{2}\sqrt{\frac{n}{2k}}\,(\mathbb{Z}^n \oplus i\mathbb{Z}^n) \subset \mathbb{C}^n$$

and Euclidean balls of g_k-radius $\frac{1}{2}$ centered at the points of Λ cover $\mathbb{C}^n$. Choose a k-independent Darboux atlas consisting of charts $\phi_j : U_j \to M$ with bounded domains U_j and transport the lattice to M. See proof of Lemma 2.30[1] for a detailed argument. ■

Once the existence of an admissible covering is clarified, we have the following

Proposition 1.7 (cf. Lemma 14 in [Don96]). *For any k and any collection $w_1, \ldots, w_m \in \mathbb{C}$ with $|w_i| \leq 1$ section σ_w associated to an admissible cover satisfies at any point of M the following inequalities*

[1] It deals with the case of a submanifold $V \subset M$, here we can just take $V = \emptyset$.

- $|s_w| \leq C$
- $|\bar{\partial} s_w| \leq C\frac{1}{\sqrt{k}}$
- $|\nabla\bar{\partial} s_w| \leq C\frac{1}{\sqrt{k}}$

where C is independent of k, ∇ and $\bar{\partial}$ are the corresponding operators on L^k.

Proof. Fix $q \in M$, then the first inequality follows from Proposition 1.3 (2):

$$|s_w(q)| \leq \sum_{i=1}^{m} |w_i||\sigma_i(q)| \leq \sum_{i=1}^{m} e_k(p_i, q) \leq m.$$

For the second and third statement we use Proposition 1.3 (4) resp. (5)

$$|\bar{\partial} s_w| \leq \sum_{i=1}^{m} |w_i||\bar{\partial}\sigma_i| \leq C' \frac{1}{\sqrt{k}} \sum_{i=1}^{m} d_k(p_i, q)^2 e_k(p_i, q),$$

$$|\nabla\bar{\partial} s_w| \leq \sum_{i=1}^{m} |w_i||\nabla\bar{\partial}\sigma_i(q)| \leq C' \frac{1}{\sqrt{k}} \sum_{i=1}^{m} (d_k(p_i, q) + d_k(p_i, q)^3) e_k(p, q).$$

Now, observe that $e^{(k^{1/2}x)^2/5}(k^{1/2}x)^r \leq 5$ for any $k \geq 1$, $x \geq 0$ and $r = 1, 2, 3$ implies

$$|\bar{\partial} s_w| \leq 5C'm\frac{1}{\sqrt{k}} \text{ and } |\nabla\bar{\partial} s_w| \leq 10C'm\frac{1}{\sqrt{k}}.$$

Claim follows by taking the maximum of all occuring constants. ■

Recall from [Don96] the following

Definition 1.8. *A smooth map $f : U \subset \mathbb{C}^n \to \mathbb{C}$ is η-**transverse** to w for $\eta > 0$ and $w \in \mathbb{C}^n$, if for any $z \in U$: $|f(z) - w| < \eta$ implies $|(D_x f)_z| > \eta$.*
*A smooth section $s : M \to L$ of a complex line bundle is η-**transverse** to 0, if $|s(x)| < \eta$ implies $|\nabla_x s| > \eta$.*

Corollary 1.9. *Given smooth maps $f, g : U \to \mathbb{C}$ with $\|f - g\|_{\mathcal{C}^1} \leq \delta$. If f is η-transverse to w then g is $(\eta - \delta)$-transverse to w.*

Remark 1.10. Given a trivial line bundle $\mathbb{C} \to \mathbb{C}^n$ together with a smooth section $s : \mathbb{C}^n \to \mathbb{C}$. Assume that $s(0) = 0$, then for a fixed $\eta > 0$ η-transversality to 0 of s implies that s is transversal to the zero section over some neighbourhood of 0 in $\mathbb{C}^n$. However, the effect this definition becomes evident if one cosiders a sequence of sections s_k. Then η-transversality for all k implies uniform (independent of k) transversality near the zero section.

The main statement of this section is the following

Proposition 1.11 (cf. Proposition 15 in [Don96]). *There exists an $\epsilon > 0$, such that for any sufficiently large k one can choose coefficients w_i with $|w_i| \leq 1$, such that the associated section satisfies $|\partial s_w| > \epsilon$ on the zero-set.*

Proof. We indicate the main steps of the proof.
(I) Observe (see Lemma 16 in [Don96]) that for any $D > 0$ there exists a number N independent of k, such that there exists a partition of the index set $\{1, \ldots, m\}$ into N disjoint subsets, i.e. $I = I_1 \cup \ldots \cup I_N$ with the property that

$$d_k(p_i, p_j) \geq D \text{ for } i, j \in I_\alpha \text{ and all } \alpha = 1, \ldots, N.$$

One might think of this step as coloring balls centered at p_i in N different colors, such that this number is independent of the stage (current k), once D is fixed.

(II) Fix any $D > 0$, hence the previous step gives us a partition $\{I_\alpha\}$ of I. Given this, define sets (denoting by B_k the g_k-unit balls)

$$M_\alpha := \bigcup_{i \in I_\beta,\ \beta \leq \alpha} B_k(p_i).$$

One gets a sequence of nested sets, exhausting whole M

$$\emptyset = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_N = M.$$

The idea is to achieve transversality stepwise, i.e. to find a (finite) sequence of sections s^α satisfying a lower bound $|\partial s^\alpha| > \epsilon$ on $M_\alpha \cap (s^\alpha)^{-1}(0)$.
Take charts $\phi_j : U_j \to M$ from Proposition 1.3 and assume that

$$\phi_j^{-1}(B_k(p_j)) \subset \varDelta \subset \varDelta^+$$

with $\varDelta = \frac{11}{10} B_k(0)$ and $\varDelta^+ := \frac{22}{10} B_k(0)$. Then over $\phi_j(\varDelta)$ we have a standard trivialization of L^k together with section σ_i constructed in Proposition 1.3, hence locally section s_w is represented by $s_w = f_i \sigma_i$ for a function $f_i : \varDelta^+ \to \mathbb{C}$. And we say that section s_w is η-transverse over B_i if the function f_i is η-transverse to 0 over $\varDelta$.

(III) Given a section s_w with $|w_i| \leq 1$ then local representation functions f_i defined over $\varDelta^+$ satisfy (see Lemma 18 in [Don96])

- $\|f_i\|_{C^1(\varDelta^+)} \leq C$
- $\left\|\bar{\partial} f_i\right\|_{C^1(\varDelta^+)} \leq C k^{1/2}$
- For $k \gg 0$ and any $\epsilon > 0$, the lower bound $|\partial f_i| > \epsilon$ on $f^{-1}(0) \cap \varDelta$ implies $|\partial_L s_w| > C^{-1}\epsilon$ on $s_w^{-1}(0) \cap B_i$.

Hence, approximate holomorphicity of functions f_i imply approximate holomorphicity of section s_w near the the zero section.

(IV) Now, consider local representation functions of a nearby section. More precisely, let $w' := (w'_1, \ldots, w'_m)$ be another coefficient vector with $|w'_i| \leq 1$, such that for some fixed $\alpha \in \{1, \ldots, N\}$ and $\delta > 0$ we have

$$w'_j = \begin{cases} w_j & \text{if } j \notin I_\alpha \\ w'_j \text{ with } |w_j - w'_j| \leq \delta & \text{if } j \in I_\alpha. \end{cases}$$

Denote the corresponding section by $s_{w'}$. Then all local representation functions f'_j satisfy (cf. Lemma 19 in [Don96]):

- $\|f'_i - f_i\|_{C^1(\Delta^+)} \leq C\delta$, for all $i \in \{1, \ldots, m\}$
- $\|f'_i - f_i - (w'_i - w_i)\|_{C^1(\Delta^+)} \leq C \exp(-D^2/5)\delta$, if $i \in I_\alpha$.

(V) Here, the existence of local perturbations is justified. Fix a $\sigma > 0$ and denote

$$\mathcal{H}_\sigma := \left\{ f : \Delta^+ \to \mathbb{C} \;\middle|\; \|f\|_{C^0(\Delta^+)} \leq 1 \text{ and } \left\|\bar{\partial} f\right\|_{C^1(\Delta^+)} \leq \sigma \right\}.$$

Moreover, for an integer p define $Q_p : (0, \infty) \to \mathbb{R}$ via $Q_p(\delta) := \ln(\delta^{-1})^{-p}$.

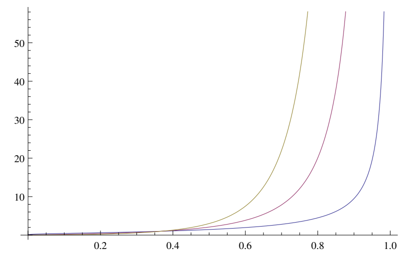

Fig. 1.1. Graph of Q_p for $p = 3$, $p = 2$ and $p = 1$.

Then we have the following quantitative result for elements of $\mathcal{H}_\sigma$ (cf. Theorem 20 in [Don96]):
There exists an integer $p = p(n)$, such that for any real $0 < \delta < 1/2$ the inequality $Q_p(\delta)\delta \geq \sigma$ (with some σ fixed above) implies that for any $f \in \mathcal{H}_\sigma$ there is a complex number w with $|w| \leq \delta$ with the property that $(f - w)$ is $Q_p(\delta)\delta$-transverse to 0.

(VI) Fix an $0 < \alpha \leq N$ and some section $s_{w^{\alpha-1}}$ of L^k which is $\eta_{\alpha-1}$-transverse over $V_{\alpha-1}$ for some $0 \leq \eta_{\alpha-1} < 1$. Then an application of (V) together with (III) and (IV) yield the following statement (cf. Proposition 23 in [Don96]):
There exist constants $\rho < 1$ and $p \in \mathbb{N}$, such that assuming the inequalities

- $\eta_{\alpha-1} \leq \rho$
- $k^{1/2} \leq Q_p(\eta_{\alpha-1})\eta_{\alpha-1}$
- $e^{-D^2/5} \leq Q_p(\eta_{\alpha-1})$

one can find a perturbation w_α of the vector $w_{\alpha-1}$, such that the associated section s_α of L^k is $\eta_\alpha := \eta_{\alpha-1} Q_p(\eta_{\alpha-1})$-transverse over V_α.

Note that $\eta_\alpha \leq \eta_{\alpha-1}$. The setup for the inductive process is now complete, since starting with any s_{w^0} (eg. $s_{w^0} = 0$), the above statement produces a section s_{w^1} which is η_1-transverse over V_1 for an $0 < \eta < 1$.

(VII) Finally, one has to see that assumptions from (VI) are satisfied at each step α and hence the induction produces the desired section transverse over the whole $V_N = M$. Observe (for an $\eta_0 \leq \rho$)

$$-\ln \eta_\alpha = -\ln(\underbrace{\eta_{\alpha-1} Q_p(\eta_{\alpha-1})}_{=\eta_\alpha}) = \ln \frac{1}{\eta_{\alpha-1}} - \ln Q_p(\eta_{\alpha-1}).$$

Then Lemma 24 in [Don96] implies that for any $q > p$ there exists an $\alpha_1 = \alpha_1(q,\rho)$, such that $-\ln \eta_\alpha \leq q(\alpha+\alpha_1)\ln(\alpha+\alpha_1)$, hence we have (assuming ρ sufficiently small)

$$Q_p(\eta_{\alpha-1})^{-\frac{1}{p}} \leq \ln \frac{1}{\eta_{\alpha-1}} - \ln Q_p(\eta_{\alpha-1}) \leq q(\alpha+\alpha_1)\ln(\alpha+\alpha_1).$$

So we conclude that for some constant $C = C(p, \alpha_1)$

$$Q_p(\eta_{\alpha-1}) \geq (q(\alpha+\alpha_1)\ln(\alpha+\alpha_1))^{-p} \geq C\,(\alpha \ln \alpha)^{-p} \geq C\,(N \ln N)^{-p}.$$

Moreover, since by construction $N \leq \bar{C} D^{2n}$ and choosing $D \gg 0$ implies

$$Q_p(\eta_{\alpha-1}) \geq \bar{C}\left(D^{2np+1}\right) \geq e^{-D^2/5}.$$

Hence, the conditions in (VI) depend now only on the value k. So choosing k sufficiently large the inductive process yields (after a finite number of steps) a section s_{w^N} which is η_N-transverse over M. ■

Remark 1.12. Clearly, the central point of the proof sketched above is part (VI). The original proof of this statement occupies sections 4 and 5 in [Don96] and uses Y. Yomdin's work about complexity of real algebraic sets. However, later on D. Auroux presented a significantly easier proof of a slightly weaker statement in [Aur02].

Combining the statements of Proposition 1.7 and Proposition 1.11 one gets the celebrated result of Donaldson

Theorem 1.13 (cf. Theorem 5 in [Don96]). *There exists a constant $C > 0$, such that for all $k \gg 0$ there exist sections s_k of $L^k \to M$ which restricted to its zero-set satisfies*

$$|\bar{\partial} s_k| < \frac{C}{\sqrt{k_k}} |\partial s|.$$

Given previous result together with considerations about sections of complex line bundles from Appendix A.1 yield

Corollary 1.14 (Donaldson hypersurface theorem). *For any $k \gg 0$ there exists symplectic $2n-2$-dimensinal submanifolds $V_k \subset M$ with the property* $\mathrm{PD}[V_k] = k[\omega]$.

Combining the statement of the above theorem with the definition of the Kähler angle (see Section 2.1 for the definition and properties) we get

Corollary 1.15. *For any $k \gg 0$ the Kähler angles of Donaldson hypersurfaces satisfy $\theta(V_k) = O(k^{-1/2})$.*

Proof. At any point $p \in M$ the Kähler angle satisfies

$$\theta(T_pV_k) = \arctan\left[2\frac{\left(|\partial_p s|^2|\bar{\partial}_p s|^2 - |\langle \partial_p s, \bar{\partial}_p s\rangle|\right)^{-1/2}}{|\partial_p s|^2 - |\bar{\partial}_p s|^2}\right] \leq 2\frac{|\bar{\partial}_p s|}{|\partial_p s|}.$$

Together with $\sqrt{k}|\bar{\partial}_p s| < C|\partial_p s|$ this yields

$$\theta(V_k) = \sup_{p\in M} \theta(T_pV_k) \leq 2Ck^{-1/2}.$$

■

1.3 Lefschetz hyperplane theorem

We begin with several historical remarks. In the early 1920s Solomon Lefschetz showed a remarkable theorem. Given a nonsingular projective algebraic variety (over $\mathbb{C}$) V_n of dimension n and a generic hyperplane section V_{n-1} of it. Then the inclusion map $V_{n-1} \hookrightarrow V_n$ induces a bijection on homology groups with integer coefficients of dimension less than $n-1$ and a surjection in dimension $n-1$.
Andreotti and Frankel have shown a cohomological version of the Lefschetz theorem in [AF59]. Their proof relied on an idea of R. Thom to use Morse theory of Stein manifolds. The key to their proof is that for an n-dimensional Stein manifold Y, they have shown that $H_i(Y,\mathbb{Z}) = 0$ for $i > n$ and $H_n(Y,\mathbb{Z})$ is torsion free.
A modern version of the Lefschetz theorem was obtained by R. Bott in [Bot59]. Again using Morse theory, he showed the following:

Theorem 1.16 (R. Bott). *Let X be a compact complex n-dimensional manifold and E a positive line bundle over X. Then for any nonsingular holomorphic section $s : X \to E$, X can be obtained by attaching cells of real dimension $\geq n$ to the zero-set $s^{-1}(0)$. So there exists an $r > 0$, such that*

$$X \cong s^{-1}(0) \cup e_1 \cup \ldots \cup e_r \ \textit{with}\ \dim e_i \geq n.$$

Corollary 1.17. *In the setting of above theorem denote* $S := s^{-1}(0)$. *Then the inclusion map* $j : S \hookrightarrow X$ *induces*

- *isomorphisms for* $0 \leq p \leq n-2$*:* $\pi_p(S) \to \pi_p(X)$*,* $H_p(S,\mathbb{Z}) \to H_p(X,\mathbb{Z})$ *and* $H^p(X,\mathbb{Z}) \to H^p(S,\mathbb{Z})$
- *surjections:* $\pi_{n-1}(S) \to \pi_{n-1}(X)$*,* $H_{n-1}(S,\mathbb{Z}) \to H_{n-1}(X,\mathbb{Z})$
- *injection:* $H^{n-1}(X,\mathbb{Z}) \to H^{n-1}(S,\mathbb{Z})$.

Proof. The statement follows by the standard tool from algebraic topology applied to the cellular decomposition from Theorem 1.16. For holomogy see p. 137-146 in [Hat09], coholomolgy p. 202-203 in [Hat09] and for homotopy Section 4.1 in [Hat09]. ■

Note that since projective algebraic varieties always admit a positive line bundle, Bott's theorem contains Lefschetz' original statement. The main advantage of Bott's proof is the statement for homotopy groups. Note that the homology statement would not automatically imply the homotopy version, since $\pi_1(S)$ may act nontrivially on higher relative groups $\pi_k(X,S)$.

Remark 1.18 (affine Lefschetz theorem). A similar statement is still true if one allows a projective variety V to contain a finite number of singular points at infinity, i.e. in $V_\infty = \{[z_0 : \ldots : z_{n+1}] \in V : z_0 = 0\}$, such that $V - V_\infty$ is smooth and is nowhere tangent to the hyperplane $\{z_0 = 0\}$. Then it was shown in [How66] that a generic hyperplane section of $V - V_\infty$ has the same properties as the set S in Corollary 1.17.

Considering the differences between Kähler and the symplectic category, it seems surprising that Donaldson hypersurfaces satisfy a Lefschetz-type theorem. Indeed, Donaldson has shown that a slight modification of Bott's argument yields

Proposition 1.19 (cf. Proposition 39 in [Don96]). *Let* V_k *be a sequence of Donaldson hypersurfaces in* (M^{2n},ω,J). *Then for* $k \gg 0$ *the inclusion maps* $j : V_k \hookrightarrow M$ *have the same properties as in Corollary 1.17.*

Proof. The argument is a slight modification of the morse-theoretic proof in the complex case as it can be found in Chapter 3 in [Nic11]. Consider corresponding sections $s_k : M \to L^k$ and set $\psi_k(x) := \ln s_k(x)$ for $x \in M - V_k$. It is sufficient to show that any critical point of ϕ_k has Morse index at least n. It is equivalent to show that $\bar\partial_J \partial_J \psi_k$ is negative definite at any critical point of ψ_l. The main difference to the complex case is that $\bar\partial_J \partial_J \psi_k$ depends on $\bar\partial_{L^k} s_k$ which might not vanish, since s_k is not a holomorphic section. However, the fact that $|\bar\partial_{L^k} s_k| \ll |\partial_{L^k} s_k|$ is still sufficient to show the claim. ■

Remark 1.20. Observe that the Lefschetz hyperplane theorem implies that the restriction of ω to the complement $M - V_k$ is an exact symplectic form.

1.4 Remarks and questions

Practically, nothing from this section is used later in the present thesis. However, we collect here several facts and questions concerning Donaldson's construction. Some of them might be useful within a further development of the Cieliebak-Mohnke approach to Gromov-Witten theory.

(A) Consider $\mathbb{C}P^2$ equipped with a symplectic form ω. The homology group $H_2(\mathbb{C}P^2,\mathbb{Z})=\mathbb{Z}$ has a spherical generator, since $\pi_2(\mathbb{C}P^2)=H_2(\mathbb{C}P^2,\mathbb{Z})$, we call it A. It was shown in [Tau95] that A can be represented as a fundamental class of an embedded pseudo-holomorphic (so ω-symplectic) sphere. On the other hand Donaldson's result shows that for any symplectic 4-manifold (M,ω) there is an integer $D>0$ and a closed connected[1] symplectic (embedded) surface $V\subset M$ such that $PD[V]=D[\omega]$. The adjunction formula then yields

$$[V]\cdot[V]-\langle c_1(TM),[V]\rangle+\chi(V)=0$$

combined with $\chi(V)=2-2g(V)$ and Lemma 3.27 (Auroux's lemma, applied for some $K\mathcal{J}_\tau(\omega)$ with $KTV\subset TV$) it follows that

$$g(V)\geq\frac{1}{2}\left[D^2\mathrm{PD}(\omega\wedge\omega)-DD_*\mathrm{PD}(\omega\wedge\omega)+1\right].$$

Note that the constant D_* depends on V at rate[2] $D^{-1/2}$, so for $D\gg 0$ it can be made D-independent. So, for large degree D the genus of V becomes very large. So Donaldson's construction is an existence proof for embedded symplectic curves in four dimensions, however there exists no symplectic surgery operation that would kill the degree in order to obtain Taubes' result[3].

(B) One might ask if Donaldson's construction would imply a version of a Kodaira embedding theorem for almost complex manifolds. This is indeed the case, as in [MPS02] the existence of a sequence of asymptotically holomorphic embeddings $\phi_k:(M^{2n},\omega)\to(\mathbb{C}P^{2n+1},\omega_{FS})$ was shown, such that $\phi_k^*[\omega_{FS}]=[k\omega]$, provided ω is an integral class. However, a symplectic embedding of a symplectic manifold into $\mathbb{C}P^N$ for large N is a classical result due to Gromov (cf. Section 3.4.2 in [Gro86] and [Tis77]). Although Gromov used h-principle to obtain the result, the assumption on integrality of ω is still required. Since any symplectic form can be perturbed into a rational one and then multiplied by a positive integer in order to obtain an integral form (cf. last chapter), "symplectic projectivity" seems not that restrictive. This is a huge contrast to the complex case, where a celebrated result in [Voi04] and [Voi02] yields examples of Kähler manifolds of complex dimension $n\geq 4$,

[1] Follows from the Lefschetz property of V.

[2] Distance of K to some previously fixed $J\in\mathcal{J}_c(\omega)$.

[3] Taubes' result is actually true for any closed symplectic 4-manifold with $b_2^+>1$.

whose homotopy type is not of a complex projective one. So, one cannot deform them in order to be projective. A natural question in this context is if asymptotically holomorphic embeddings of such Kähler manifolds are actually approximating singular complex subvarieties of $\mathbb{C}P^{2n+1}$.

(C) It is a folklore fact that on an almost complex manifold (M^{2n}, J) one cannot expect to find any closed complex[1] submanifolds of complex dimension greater than 1. Intuitively, for generic J the Cauchy-Riemann equation yields an overdetermined system of PDEs, which is non-integrable. A precise treatment of this question was given in [Kru03]. B. Kruglikov showed that there exists an open and dense[2] subset $\mathcal{J}' \subset \mathcal{J}(M)$ of all almost complex structures on M. Such that for any $J \in \mathcal{J}'$ one has no local complex submanifolds of dimension $2m$ with $2 \geq m \geq n-1$. Hence, the Cieliebak-Mohnke approach starts constructing perturbation data with non-generic data, which is quite remarkable, because the outcome is a symplectic invariant.

(D) Hypersurfaces from Donaldson's construction seem to inherit certain properties from the ambient manifold. A smooth simply connected manifold M is called formal if its real homotopy groups $\pi_*(M) \otimes \mathbb{R}$ can be computed from the real cohomology ring $H^*(M, \mathbb{R})$. A fundamenal result in [DGMS75] states that any Kähler manifold is formal. However, there exist simply connected symplectic manifolds that are not formal - the first example was given in [BT00]. It was shown in [FM05] that Donaldson hypersurfaces might inherit formality from the ambient manifold. See also [Kut12] on essential manifolds. Proofs of above facts use Lefschetz hyperplane theorem for Donaldson hypersurfaces.

(E) A pretty unexpected application of the Donaldson construction was found in [Eva12]. Given a symplectic manifold (M^{2n}, ω), for a $J \in \mathcal{J}_c(\omega)$ the Nijenhuis energy is given by $E_J = \int_M \|N_J\|_J^2 \, \omega^n$, where N_J is the Nijenhuis tensor of J. It was shown that rationality of $[\omega]$ implies that the infimum of N_J taken over all $J \in \mathcal{J}_c(M)$ is zero. The statement follows by stretching the neck with respect to a tubular neighbourhood of a fixed Donaldson hypersurface (made J-complex), giving sequence of compatible almost complex structures J_ν whose N_{J_ν} converges to zero. It is an open question, whether N_J is zero for an irrational class $[\omega]$.
Note that a simple deformation argument does not work here. By taking a $J \in \mathcal{J}_c(\omega)$ we might approximate it by a rational ω' together with a $J' \in \mathcal{J}_c(\omega')$, which would leave the corresponding Donaldson hypersurface invariant (for large degree). However, the neckstretching process would produce a sequence J'_ν which is not close to J' and hence to J.

[1] In the sense that their tangent bundle is J-invariant. Such submanifolds are also called pseudo-holomorphic.

[2] With respect to $\mathcal{C}^r$-topology for $r = \max\{2, 6-n\}$.

(F) Consider a closed symplectic manifold (M^{2n},ω), then any $J \in \mathcal{J}_c(M,\omega)$, induces the Riemannian metric g_J, which in turn induces Laplace operator Δ_J acting on functions on M. Denote by $\lambda_1(M,g_J)$ the first eigenvalue of Δ_J. It was conjectured in [Pol98] that $\sup_{J\in\mathcal{J}_c(M,J)} \lambda_1(M,g_J) = \infty$ and proved in the case of the existence of an isotropic Hörmander distribution on M. This stays in contrast to the Kähler case where an upper bound exists if one considers only integrable J in $\mathcal{J}_c(M,\omega)$ - see [BLY94]. This conjecture was recently proven by L. Buhovky in full generality [Buh13] with a method that seems to be a real-analytic version of Donaldson's construction. He constructs a sequence of vector fields whose integral curves tend to fill out whole M, then, by associating complex-subspaces to them he constructs a sequence of almost complex structures and then, after rescaling (just as in [Pol98]) the sequence, produces metrics with arbitrary large λ_1.
Somehow, Buhovky's argument approximates an isotropic distribution that might not exist at all on M for topological reasons. Observe that it was shown in Proposition 40 from [Don96] that hypersurfaces V_k converge as currents to $\frac{k}{2\pi}\omega$ for $k \to \infty$, i.e. there exists a constant $C > 0$, such that for any form $\psi \in \Omega^{2n-2}(M)$ we have

$$\left| \int_{V_k} \psi - \frac{k}{2\pi} \int_M \psi \wedge \omega \right| \leq C k^{1/2} \|d\psi\|_{L^\infty(M)} .$$

Hence, one could conjecture that there exists a sequence J_k of compatible almost complex structures associated to V_k having the same properties as in [Buh13]. One access point could be, considering S^1-bundles over V_k, obtaining embedded real hypersurfaces in M and then appealing to Cheeger's isoperimetric inequality [Che70].

(F) In Chapter 2 we construct Donaldson hypersurfaces V_k that intersect transversely any fixed symplectic hypersurface $W \subset M$. We consider a situation in which we can find a ω-tamed almost complex structure K that leaves both TW and TV_k (for a fixed k) invariant. Such, K cannot be ω-compatible in general. It is an open question whether there exists another symplectic form ω', such that $K \in \mathcal{J}_c(M,\omega')$. If yes, is there any relation between the classes $\mathrm{PD}([W])$, $[\omega]$ and $[\omega']$? This question is related to Donaldson's "tame vs. compatible" problem - see [TW11].

(G) One might wonder if one could obtain an analytic (non-constructive) proof of Donaldson hypersurface theorem. This question was attacked in [BU00]. Using Fourier integral operators and a spectral gap for high tensor powers of the line bundle $L \to M$ they showed the existence of approximate holomorphic sections and obtained a sharper version of Kodaira embedding as in (B), see also [MM08] for an approach using Bergam kernels. However, it is still an open problem to deduce Donaldson's theorem by above approaches. One might speculate that a solution to this problem would give a better understanding in what sense (log) Kodaira dimension is a symplectic invariant, see also the recent work [McL14].

2

Preliminaries and modifications of Donaldson's construction

Consider a symplectic manifold (M, ω) with $[\omega] \in H^2(M, \mathbb{Z})$ and closed ω-symplectic submanifold $W \subset M^{2n}$ with $k < n$. In this section we prove the following

Proposition 2.1. *(M, ω) admits Donaldson hypersurfaces V_k which intersect W transversely, provided $k \gg 0$.*

Remark 2.2. The statement of the above proposition is not new. It is stated for the case $\mathrm{PD}([W]) = D[\omega]$ in [CM07] (see Theorem 8.1). The general statement can be found in [Pao01] (see Proposition 1.1), see also [Moh03]. The case of more than two hypersurfaces is considered in [Ops13]. The idea in all three cases is roughly the same. However, here we carry out some technical details.

The actual statement proven in this section is the following

Proposition 2.3. *Fix an almost complex structure $J \in \mathcal{J}_c(M, \omega)$. For a given compact complex submanifold $W \subset M$, i.e. $JTW \subset TW$, of real codimension 2, there exists an $\eta > 0$, such that for all $D \gg 0$ there is a Donaldson hypersuface of degree D that intersects W η-transversely.*

Results presented in the next section show that Proposition 2.3 implies Proposition 2.1. Moreover, it contains several definitions and technical tools used later on. At the end of the chapter we discuss Opshtein's observation from [Ops13] that in a special case one can find a lower bound for the transversality constant η appearing in the statement of Proposition 2.3.

Remark 2.4. Complementing the statement of Proposition 2.1, it was shown in [Pao01] that in case of $\dim_{\mathbb{R}} W < 2n - 2$ one can construct a Donaldson hypersurface containing the whole submanifold W.

2.1 Hermitian linear algebra and deformations

Consider a symplectic manifold (M, ω_1) with compatible almost complex structure J_1. Denote by g_1 and $\|\cdot\|_1$ the induced Riemannian metric on M

and norm on $\Gamma(TM)$, respectively. Fix another symplectic form ω_2, with $\|\omega_1 - \omega_2\|_1 < \epsilon$, for some $\epsilon > 0$.

The following lemma is very basic, but a key to the main result.

Lemma 2.5. *Fix $J_0 \in \mathcal{J}_c(\omega_0)$ and $J \in \mathcal{J}_\tau(\omega_0)$. Let ω_1 be another symplectic structure with $\|\omega_0 - \omega_1\|_0 < \epsilon < 1$ and assume $\|J - J_0\|_0 \leq \frac{1-\epsilon}{1+\epsilon}$ then $J \in \mathcal{J}_\tau(\omega_1)$.*

Proof. Assume that J does not tame ω_1, so there exists an $v \neq 0$, such that $\omega_1(v, Jv) \leq 0$. Since $\omega_0(v, Jv) > 0$, the inequality $\|\omega_0 - \omega_1\|_0 < \epsilon$ implies

$$\epsilon > \frac{|\omega_0(v, Jv) - \omega_1(v, Jv)|}{\|v\|_0 \|Jv\|_0} = \frac{\omega_0(v, Jv) - \omega_1(v, Jv)}{\|v\|_0 \|Jv\|_0}.$$

Now, by the triangle inequality we get

$$\|Jv\|_0 \leq \|Jv - J_0 v\|_0 + \|J_0 v\|_0 \leq \frac{2}{1+\epsilon} \|v\|_0 .$$

Similarly, we get $\|Jv\|_0 \geq \frac{2\epsilon}{1+\epsilon} \|v\|_0$. Moreover, we observe

$$\|(J - J_0)v\|_0^2 = \omega_0\left((J - J_0)v, J_0(J - J_0)v\right) = \|Jv\|_0^2 + \|v\|_0^2 - 2\omega_0(v, Jv).$$

Hence we get $\omega_0(v, Jv) \geq \frac{1}{2}\left(\|Jv\|_0^2 + \|v\|_0^2 - \left(\frac{1-\epsilon}{1+\epsilon}\right)^2 \|v\|_0^2\right)$.
Summarizing above facts

$$\begin{aligned}
\epsilon >& \frac{1+\epsilon}{2\|v\|_0^2}\left[\frac{1}{2}\left(\|Jv\|_0^2 + \|v\|_0^2 - \left(\frac{1-\epsilon}{1+\epsilon}\right)^2 \|v\|_0^2\right) - \omega_1(v, Jv)\right] \\
\geq& \frac{1+\epsilon}{2\|v\|_0^2}\left[\frac{1}{2}\underbrace{\left(\left(1 - \frac{1-\epsilon}{1+\epsilon}\right)^2 \|v\|_0^2 + \|v\|_0^2 - \left(\frac{1-\epsilon}{1+\epsilon}\right)^2 \|v\|_0^2\right)}_{=2\frac{2\epsilon}{1+\epsilon}\|v\|_0^2} - \omega_1(v, Jv)\right] \\
=& \epsilon - \frac{1+\epsilon}{2\|v\|_0^2}\omega_1(v, Jv) \geq \epsilon,
\end{aligned}$$

which is a contradiction. Hence, J tames ω_1. The non-linear version of the statement follows by taking supremum over all points of M. ∎

Observe that once $\epsilon < \sqrt{2} - 1$, the above lemma implies a simpler bound $\|J - J_0\| < \sqrt{2} - 1$.

Remark 2.6. Note that once ω_0, ω_1 and J, J_0 satisfy the assumptions of the previous lemma, i.e. J tames ω_1, then J tames any positive multiple of ω_1.

Lemma 2.7. *Given two symplectic forms ω_0 and ω_1. Fix $J_0 \in \mathcal{J}_c(\omega_0)$ and denote by $\|\cdot\|_0$ the norm induced by (ω_0, J_0). Assume that $\|\omega_0 - \omega_1\|_0 \le \epsilon < 1$. Then $c_1(M, \omega_0) = c_1(M, \omega_1) \in H^2(M, \mathbb{Z})$.*

Proof. For $J \in \mathcal{J}_\tau(\omega_0)$ one defines $c_1(M, \omega_0) := c_1(TM, J)$. This definition is independent of the choice of J, since the space $\mathcal{J}_\tau$ is contractible and hence all complex bundles (TM, J) with $J \in \mathcal{J}_\tau(\omega_0)$ are homotopy equivalent.
It follows from $|\underbrace{\omega_0(x, Jx)}_{=\|x\|_0^2} - \omega_1(x, Jx)| \le \epsilon \|x\|_0^2$, that $J \in \mathcal{J}_\tau(\omega_1)$. Hence, by definition we have $c_1(M, \omega_0) = c_1(M, \omega_1)$. ∎

Given a (linear) Hermitian space (V, ω, J) with the Hermitian metric $h(\cdot, \cdot) := \omega(\cdot, J\cdot) + i\omega(\cdot, \cdot)$.

Definition 2.8 (cf. p. 79 in [CM07]). *Consider a subspace $X \subset V$ with $\dim_{\mathbb{R}}(X) = 2k$. The **Kähler angle** of X is given by*

$$\theta(X) = \theta(X, \omega, J) = cos^{-1}\left(\frac{\omega^k_{|X}}{k! \Omega_X}\right),$$

where Ω_X is the volume form on X.
*The **Kähler angle** of a closed even dimensional submanifold $V \subset M$ of a Hermitian manifold (M, ω, J) is given by*

$$\theta(V) = \theta(V, \omega, J) = \sup_{x \in V} \theta(T_x V, \omega_x, J_x).$$

Lemma 2.9 (cf. Lemma 8.3 in [CM07]).

1. *An even-dimensional submanifold $V \subset (M, \omega)$ is ω-symplectic iff $\theta(V) < \pi/2$.*
2. *For a smooth oriented real hypersurface W (i.e. $\dim W = 2n - 2$) the Kähler angle satisfies*

$$\theta(W) = \theta(JW) = \theta(W^\omega) = \theta(W^\perp).$$

Proof. All statements are simply the non-linear analogs of the linear counterparts from Lemma 8.3 in [CM07]. ∎

Lemma 2.10. *Given a symplectic hypersurface $V \subset (M, \omega)$, let $J \in \mathcal{J}_c(\omega)$ then the Kähler angle of V is given by*

$$\theta(V) = \angle_M(V, JV) := \sup_{z \in V} \sup_{y \neq 0 \in J_z T_z V} \inf_{x \neq 0 \in T_z V} cos^{-1}\left(\frac{|\langle x, y\rangle|}{\|x\| \, \|y\|}\right).$$

Proof. Follows from Lemma 8.3(d) in [CM07] assuming that V is symplectic. ■

Note that $\frac{|\langle x,y\rangle|}{\|x\|\|y\|} \le 1$ by Cauchy-Schwarz is equal to 1 iff x and y are linear dependent, hence $\angle_M(V, JV) = 0$ iff V is J-complex.

Moreover, for a fixed $z \in V$ we see that

$$\begin{aligned}\angle_M(T_zV, J_zT_zV) &= \sup_{y\neq 0\in J_zT_zV} \inf_{x\neq 0\in T_zV} \cos^{-1}\left(\frac{|\langle x,y\rangle|}{\|x\|\,\|y\|}\right)\\ &= \sup_{y'\neq 0\in T_zV} \inf_{x\neq 0\in T_zV} \cos^{-1}\left(\frac{|\langle x,J_zy'\rangle|}{\|x\|\,\|J_zy'\|}\right)\\ &= \sup_{y'\neq 0\in T_zV} \inf_{x\neq 0\in T_zV} \cos^{-1}\left(\frac{|\omega_z(x,y')|}{\|x\|\,\|y'\|}\right).\end{aligned}$$

We recall some facts from [CM07] (cf. p. 84).

Definition 2.11. *For an Euclidean space V, consider two subspaces $X, Y \subset V$. The **minimal angle** between X and Y is given by*

$$\angle_m(X,Y) := \begin{cases} 0 & \text{if } X \text{ and } Y \text{ are not transverse,}\\ \displaystyle\inf_{\substack{x\neq 0\in X'\\ y\neq 0\in Y'}} \angle(x,y) & \text{where } X' = (X\cap Y)^\perp \cap X, Y' = (X\cap Y)^\perp\cap Y.\end{cases}$$

We now consider two pairs on M: $(\omega_0, J_0 \in \mathcal{J}_c(\omega_0))$ and $(\omega_1, J_1 \in \mathcal{J}_c(\omega_1))$ and denote the induced norms by $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively. Then we see that

$$\|J_0\|_0^2 = \sup_{\|v\|_0=1} \|J_0v\|_0^2 = \sup_{\|v\|_0=1} \omega_0(J_0v, J_0J_0v)^2 = \sup_{\|v\|_0=1} \omega_0(v, J_0v)^2 = 1,$$

$$\begin{aligned}\|\omega_0\|_0^2 &= \sup_{\|v\|_0=\|w\|_0=1} |\omega_0(v,w)|^2 = \sup_{\|v\|_0=\|J_0w'\|_0=1} |\omega_0(v, J_ow')|^2\\ &\le \sup_{\|v\|_0=\|J_0w'\|_0=1} \|v\|_0\,\|w'\|_0 \le 1,\end{aligned}$$

hence, for $v = w'$ we get $\|\omega_0\|_0^2 = 1$.

Now, we show that the norms $\|\cdot\|_{0,1}$ are equivalent.

Lemma 2.12. *Assume that $\|\omega_0 - \omega_1\|_0 < \epsilon$ and $\|J_0 - J_1\|_0 < \eta$ for $\epsilon, \eta > 0$. Then for any $v \in V$ we have*

$$(1-\eta-\epsilon-\eta\epsilon)\,\|v\|_0^2 \le \|v\|_1^2 \le (1+\epsilon)(1+\eta)\,\|v\|_0^2.$$

Proof. Observe first that $\|\omega_1\|_0 \leq \|\omega_1 - \omega_0\|_0 + \|\omega_0\|_0 < \epsilon + 1$. The same argument yields $\|J_1\|_0 < \eta + 1$. Using this we get

$$\begin{aligned}
\|v\|_1^2 &= \omega_1(v, J_1 v) = \omega_1\left(\tfrac{v}{\|v\|_0}, \tfrac{J_1 v}{\|J_1 v\|_0}\right) \|v\|_0 \|J_1 v\|_0 \\
&\leq \sup_{\|x\|_0 = 1} \omega_1\left(x, \tfrac{J_1 x}{\|J_1 x\|_0}\right) \|v\|_0^2 \|J_1\|_0 = \|\omega_1\|_0 \|J_1\|_0 \|v\|_0^2 \\
&< (1+\epsilon)(1+\eta) \|v\|_0^2 .
\end{aligned}$$

Now, consider the following:

$$\begin{aligned}
|\, \|v\|_0^2 - \|v\|_1^2 \,| &= |\omega_0(v, J_0 v) - \omega_0(v, J_1 v) + \omega_0(v, J_1 v) - \omega_1(v, J_1 v)| \\
&\leq |\omega_0(v, (J_0 - J_1)v)| + |(\omega_0 - \omega_1)(v, J_1 v)| \\
&\leq \|v\|_0 \|(J_0 - J_1)v\|_0 + \epsilon \|v\|_0 \|J_1 v\|_0 \\
&\leq \eta \|v\|_0^2 + \epsilon(\eta + 1) \|v\|_0^2 = (\eta + \epsilon\eta + \epsilon) \|v\|_0^2 .
\end{aligned}$$

Using the preceding inequality we finally get

$$\begin{aligned}
\|v\|_0^2 &\leq |\, \|v\|_0^2 - \|v\|_1^2 \,| + \|v\|_1^2 \\
&\leq (\eta + \epsilon\eta + \epsilon) \|v\|_0^2 + \|v\|_1^2 .
\end{aligned}$$

■

Next, we discuss that there is always a good choice for an almost complex structure.

Lemma 2.13. *Consider $\mathbb{R}^{2n}$ equipped with the standard structure $(\omega_0, J_0, \langle , \rangle)$. Let ω be another (linear) symplectic structure. Assume that $\|\omega_0 - \omega\|_0 < \epsilon$, then there exists an ω-compatible complex structure J with $\|J_0 - J\|_0 < 3\epsilon$.*

Proof. We start with a standard approach (cf. Appendix in [IP03]). Given ω defines $A \in GL_{\mathbb{R}}(2n)$ via $\langle A\cdot, \cdot \rangle = \omega(\cdot, \cdot)$. For any $x, y \in \mathbb{R}^{2n}$ we have

$$x^T A^T y = \omega(x, y) = -\omega(y, x) = -y^T A^T x = -x^T A y$$

and hence $A^T = -A$. Since $-A^2 = A^T A$ and for any $x \neq 0 \in \mathbb{R}^{2n}$ we have $x^T A^T A x = \|Ax\|^2 > 0$ (note that $A^T A$ has no kernel). Hence $\sqrt{-A^2}$ is well defined and we set $J := A\sqrt{-A^2}$.
Observe that iA is Hermitian[1], so we can write it as $iA = U \Lambda U^{-1}$ for a unitary matrix U and a real diagonal matrix Λ. With $(iA)^2 = -A^2$ we get $\sqrt{-A^2} = U|\Lambda|U^{-1}$. Hence, $\sqrt{-A^2}$ and A commute, so it follows that $J^2 =$

[1] $(iA)^* = -iA^T = iA$

$-\mathbb{1}$ and $J^T = -J$, so that we get $\omega(Jx, Jy) = \langle AJx, Jy\rangle = x^T J^T A^T Jy = x^T A^T y = \omega(x, y)$. Moreover, $\omega(x, Jx) = x^T A^T Jx = x^T(-A^2)\sqrt{-A^2}x > 0$ for any $x \neq 0$, since the product $(-A^2)\sqrt{-A^2}$ is positive definite. Hence, J is ω-compatible.
Now, observe that for any $x, y \in \mathbb{R}^{2n}$

$$|\langle (A - J_0)x, y\rangle| = |\langle Ax, y\rangle - \langle J_0 x, y\rangle| = |\omega(x, y) - \omega_0(x, y)| < \epsilon \left\|x\right\|_0 \left\|y\right\|_0,$$

hence, $\left\|(A - J_0)x\right\|_0^2 \leq \epsilon \left\|(A - J_0)x\right\|_0 \left\|x\right\|_0$ and so $\left\|J - A\right\|_0 \leq \epsilon$. Same arguments as in Lemma 2.15 yield $\left\|\sqrt{-A^2} - I_{2n}\right\|_0 \leq \epsilon$. Moreover,

$$\left\|A\right\|_0 \leq \left\|A - J_0\right\|_0 + \left\|J_0\right\|_0 \leq 1 + \epsilon.$$

So that we finally get the bound

$$\begin{aligned}\left\|J_0 - J\right\|_0 &= \left\|J_0 - AJ_0 + AJ_0 - A\sqrt{-A^2}\right\|_0 \\ &\leq \left\|J_0 - A\right\|_0 \left\|J_0\right\|_0 + \left\|A\right\|_0 \left\|J_0 - \sqrt{-A^2}\right\|_0 \leq \epsilon^2 + 2\epsilon.\end{aligned}$$

Hence, for $\epsilon < 1$ we have $\left\|J_0 - J\right\| \leq 3\epsilon$. ∎

The next lemma gives some control over Kähler and minimal angles under a small deformation.

Lemma 2.14. *For any two pairs (ω_i, J_i) for $i = 1, 2$ such that $\left\|\omega_0 - \omega_1\right\|_0 < \epsilon$ and $\left\|J_0 - J_1\right\|_0 < \epsilon$. Denote by $\theta_{0,1}$ the induced Kähler angles. Given a $2k$-dimensional submanifold $V \subset M$ with $\theta_0(V) < \eta < \frac{\pi}{2}$ then $\epsilon < \frac{1}{20}$ implies*

$$\theta_1(V) \leq \theta_0(V) + 2\epsilon^{\frac{1}{4}}.$$

Given two submanifolds $V, W \subset M$ with $V \cap W \neq \emptyset$. Denote by $\angle_m^i(V, W)$ the corresponding minimal angles induced by the pairs (ω_i, J_i) for $i = 1, 2$, then $\epsilon < \frac{1}{50}$ implies

$$\angle_m^1(V, W) \geq \angle_m^0(V, W) - \epsilon^{\frac{1}{4}}.$$

Proof. First, consider the linear case. For any $x, y \in \mathbb{R}^{2n}$

$$\begin{aligned}1 &\geq \frac{|\omega_1(x, y)|}{\left\|x\right\|_1 \left\|y\right\|_1} \geq \frac{1}{(1+\epsilon)^2} \frac{|\omega_1(x, y)|}{\left\|x\right\|_0 \left\|y\right\|_0} \geq \frac{1}{(1+\epsilon)^2}\left[\frac{|\omega_0(x, y)| - |\omega_0(x, y) - \omega_1(x, y)|}{\left\|x\right\|_0 \left\|y\right\|_0}\right] \\ &\geq \frac{1}{(1+\epsilon)^2}\left[\frac{|\omega_0(x, y)|}{\left\|x\right\|_0 \left\|y\right\|_0} - \left\|\omega_0 - \omega_1\right\|_0\right] \geq \frac{1}{(1+\epsilon)^2}\left[\frac{|\omega_0(x, y)|}{\left\|x\right\|_0 \left\|y\right\|_0} - \epsilon\right] \\ &\geq -\frac{\epsilon}{(1+\epsilon)^2} \geq -\frac{1}{4}.\end{aligned}$$

Hence, we have $\cos^{-1}\left(\frac{|\omega_1(x,y)|}{\left\|x\right\|_1 \left\|y\right\|_1}\right) \leq \cos^{-1}\left(\frac{1}{(1+\epsilon)^2}\left[\frac{|\omega_0(x,y)|}{\left\|x\right\|_0 \left\|y\right\|_0} - \epsilon\right]\right)$.
Now, applying the following inequalities:

$$\cos^{-1}(\alpha x) \leq \cos^{-1}(x) + \cos^{-1}(\alpha), \text{ for } -1 \leq x \leq 1 \text{ and } 0 \leq \alpha \leq 1$$
$$\cos^{-1}\left(\frac{1}{(1+\epsilon)^2}\right) \leq \epsilon^{\frac{1}{4}}, \text{ for } 0 \leq \epsilon \leq \frac{1}{20}$$
$$\cos^{-1}(x-\epsilon) \leq \cos^{-1}(x) + \epsilon^{\frac{1}{4}}, \text{ for } 0 \leq x \leq 1 \text{ and } 0 \leq \epsilon \leq \frac{1}{5}$$

we get the estimate

$$\cos^{-1}\left(\frac{|\omega_1(x,y)|}{\|x\|_1 \|y\|_1}\right) \leq \cos^{-1}\left(\frac{|\omega_0(x,y)|}{\|x\|_0 \|y\|_0}\right) + 2\epsilon^{\frac{1}{4}}.$$

By taking supremum over $x \neq 0$ and infimum over $y \neq 0$ the statement for the linear case follows. The general case follows by taking the supremum over every tangent space.
The proof for minimal angles is similar. Denote by $g_i := \omega_i(\cdot, J_i\cdot)$ the induced metric. Then Lemma 2.12 implies for any non-zero vectors $v, w \in T_xM$ (and a fixed $x \in M$):

$$\begin{aligned}
\frac{|g_0(v,w) - g_1(v,w)|}{\|v\|_0 \|w\|_0} &\leq \frac{|\omega_0(v,(J_0w - J_1)w)|}{\|v\|_0 \|w\|_0} + \frac{|\omega_0(v,J_1w) - \omega_1(v,J_1w)|}{\|v\|_0 \|w\|_0} \\
&\leq \underbrace{\|\omega_0\|_0}_{=1} \|J_0 - J_1\|_0 + \frac{|\omega_0(v,J_1w) - \omega_1(v,J_1w)|}{(1-\epsilon)^2 \|v\|_1 \|w\|_1} \\
&\leq \epsilon + \frac{|\omega_0(v,w) - \omega_1(v,w)|}{(1-\epsilon)^2 \|v\|_1 \underbrace{\|J_1w\|_1}_{=\|w\|_1}} \leq \epsilon + \frac{\epsilon}{(1-\epsilon)^2}.
\end{aligned}$$

Since $\epsilon(1 + 1/(1-\epsilon)^2) \leq 10\epsilon$ for $\epsilon < \frac{1}{2}$, we get $\|g_0 - g_1\|_0 \leq 10\epsilon$. Now let $x \in V \cap W$, take any two vectors $v \in T_xV$, $w \in T_xW$ with $v \notin T_xW$ and $w \notin T_xV$. Then

$$\cos(\angle^1(v,w)) = \frac{|g_1(v,w)|}{\|v\|_1 \|w\|_1} \leq \frac{10\epsilon}{(1-\epsilon)^2} + \frac{1}{(1-\epsilon)^2}\frac{|g_0(v,w)|}{\|v\|_0 \|w\|_0}, \text{ so}$$

$$\cos(\angle^1(v,w)) \leq \frac{1}{(1-\epsilon)^2}\left[10\epsilon + \cos(\angle^0(v,w))\right].$$

Observe that for $\epsilon < \frac{1}{50}$ and $\angle^0(v,w) \leq \pi/4$ we have

$$\cos^{-1}\left(\frac{1}{(1-\epsilon)^2}\left[10\epsilon + \cos(\angle^0(v,w))\right]\right) \geq \angle^0(v,w) - \epsilon^{1/4}.$$

Note, the case $\pi/2 \geq \angle^0(v,w) \geq \pi/4$ is not relevant, since it would already imply that the minimal angle will not become small after an ϵ-perturbation, so the claim follows. ■

Given a $V \subset M$, such that $\theta_0(V) = \eta < \frac{\pi}{2}$ (i.e. V is ω_0-symplectic), then for any (ω_1, J_1) in the ϵ-ball around (ω_0, J_0) it follows that V is ω_1-symplectic, as long as $\epsilon < \left(\frac{\pi-4\eta}{8}\right)^4$.

Next, we show a slight improvement of Lemma 8.9 from [CM07].

Lemma 2.15. *Given two complex structures J_0, J_1 compatible wrt. the standard linear symplectic structure on $\mathbb{R}^{2n}$ with $\|J_0 - J_1\| < \theta$, then there exists a path J_t of compatible complex structures, such that $\|J_t - J_1\| < \theta^2 + 2\theta$ for all $0 \leq t \leq 1$.*
In the case of tame complex structures we get the same statement with $\|J_t - J_1\| < \frac{\theta}{1-\theta}$ for all $0 \leq t \leq 1$.

Proof. Recall the standard construction of the connecting path (cf. proof of Proposition 2.50 in [MS98]): we may assume that $J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. Then the linear homotopy $g_t(\cdot,\cdot) = (1-t)\omega_0(\cdot, J_0\cdot) + t\omega_0(\cdot, J_1\cdot)$ with $0 \leq t \leq 1$ defines via $\omega_0(\cdot,\cdot) = g_t(A_t\cdot,\cdot)$ a family of skew-symmetric non-degenerate matrices A_t. Then $J_t := \sqrt{-A_t^2}A_t$ defines a compatible complex structure connecting J_0 and J_1. And we compute (with norm induced by $\omega_0(\cdot, J_0\cdot)$)

$$\begin{aligned}\|J_0 - J_t\| &\leq \left\|J_0 - \sqrt{-A_t^2}J_0\right\| + \left\|\sqrt{-A_t^2}J_0 - \sqrt{-A_t^2}A_t\right\| \\ &\leq \left\|I_{2n} - \sqrt{-A_t^2}\right\| + \left\|\sqrt{-A_t^2}\right\| \|J_0 - A_t\|.\end{aligned}$$

Moreover, we have

$$\|J_0 - A_t\| = \|J_0 - (1-t)J_0 - tJ_1\| \leq \|J_0 - J_1\| \leq \theta.$$

For estimating $\left\|I_{2n} - \sqrt{-A_t^2}\right\|$ note that, since A_t is anti-symmetric and non-degenerate it has purely imaginary eigenvalues[1], say $\pm\sqrt{-1}\lambda_i$ for $i = 1, \ldots, n$ and $\lambda_i \in \mathbb{R}$. Then for an eigenvector $v_i = a_i + \sqrt{-1}b_i$ we have $A_t a_i = -\lambda_i b_i$ and $A_t b_i = \lambda_i a_i$. The bound $\|J_0 - A_t\| \leq \theta$ leads to

$$\|a_i\|^2 \theta^2 \geq \|a_i - \lambda_i b_i\| = \|a_i\|^2 + \lambda_i^2 \|b_i\|^2 + 2\lambda_i\langle J_0 a_i, b_i\rangle,$$

and a similar bound with b_i, then by adding both inequalities and dividing by $\|v_i\|^2$ gives

$$1 + \lambda_i^2 + 2\lambda_i \frac{2\langle J_0 a_i, b_i\rangle}{\|v_i\|^2} \leq \theta^2.$$

Moreover, since $0 \leq \|J_0 a_i - b_i\|^2 = \|J_0 a_i\|^2 + \|b_i\|^2 + 2\langle J_0 a_i, b_i\rangle = \|v_i\|^2 + 2\langle J_0 a_i, b_i\rangle$, the previous inequality turns into $(1-\lambda_i)^2 \leq \theta^2$.

[1] After complexifying the real vector space $\mathbb{R}^{2n}$.

Now, we see that $-A_t^2 a_i = A_t \lambda_i b_i = \lambda_i^2 a_i$ and $-A_t^2 b_i = \lambda_i^2 b_i$, i.e. λ_i^2 are eigenvalues of $-A_t^2$ and hence $|\lambda_i|$ are eigenvalues of $\sqrt{-A_t^2}$. Choosing an orthonormal basis of eigenvectors w_i for $\sqrt{-A_t^2}$ we get for any vector $w = \sum \alpha_i w_i$

$$\left\|(I_{2n} - \sqrt{-A_t^2})w\right\|^2 = \sum_i (1 - |\lambda_i|)^2 \alpha_i^2 \|w_i\|^2 \leq \theta^2 \|w\|^2,$$

since $(1 - |\lambda_i|)^2 \leq (1 - \lambda_i)^2$, so $\left\|I_{2n} - \sqrt{-A_t^2}\right\| \leq \theta$.
Finally combining previous inequalities with

$$\left\|\sqrt{-A_t^2}\right\| \leq \left\|\sqrt{-A_t^2} - I_{2n}\right\| + \|I_{2n}\| \leq \theta + 1$$

yields

$$\|J_0 - J_t\| \leq \theta + (\theta + 1)\theta = \theta^2 + 2\theta.$$

Proof for the tame case
The map $\Phi(J) := (J + J_0)^{-1} \circ (J - J_0)$ defines a diffeomorphism from $\mathcal{J}_\tau(\omega)$ to the space of matrices $\{S \in M(2n, \mathbb{R}) | SJ_0 + J_0 S = 0,\ \|S\| < 1\}$ (cf. Propositon 1.1.6 in [Aud94]).
Hence, we can define the path J_t via $J_t := \Phi^{-1}((1 - t)\Phi(J_0) + t\Phi(J_1))$, but $\Phi(J_0) = 0$ and $\Phi^{-1}(S) = J_0 \circ (\mathbf{Id} + S) \circ (\mathbf{Id} - S)^{-1}$, so we can bound $\|J_t\|$ for all $0 \leq t \leq 1$. Let $S_t := t\Phi(J_1)$, hence $\|S_t\| \leq t \left\|(J_1 + J_0)^{-1}\right\| \|J_1 - J_0\| \leq \theta \left\|(J_1 + J_0)^{-1}\right\|$.
Next, we use the Neumann series in the following way. Assume that A and B are square matrices, B is invertible and $\|A - B\| \leq p \left\|B^{-1}\right\|^{-1}$ for $0 < p < 1$, where the norm is the operator norm, then $\left\|A^{-1}\right\| \leq \frac{1}{1-p} \left\|B^{-1}\right\|$. Indeed, by writing $A = B(\mathbf{Id} - (\mathbf{Id} - B^{-1}A))$ and observing that

$$\left\|\mathbf{Id} - B^{-1}A\right\| \leq \|B\| \|B - A\| \leq p < 1,$$

the Neumann series yields $\left\|T^{-1}\right\| \leq \left\|(\mathbf{Id} - (\mathbf{Id} - B^{-1}A))^{-1}\right\| \left\|B^{-1}\right\| \leq (1 - p)^{-1} \left\|B^{-1}\right\|$. Now we can bound $\left\|(J_1 + J_0)^{-1}\right\|$. Since $\|J_1 + J_0 - 2J_0\| \leq \theta = \frac{\theta}{2} \left\|(2J_0)^{-1}\right\|^{-1}$, it follows that $\left\|(J_1 + J_0)^{-1}\right\| \leq \frac{2}{2-\theta} \left\|(2J_0)^{-1}\right\| = \frac{1}{2-\theta}$, moreover $\|S_t\| \leq \frac{\theta}{2-\theta}$. Finally, we compute

$$\begin{aligned}
\|J_0 - J_t\| &= \left\|J_0 - \Phi^{-1}(S_t)\right\| = \left\|J_0 - J_0(\mathbf{Id} + S_t)(\mathbf{Id} - S_t)^{-1}\right\| \\
&\leq \left\|\mathbf{Id} - (\mathbf{Id} + S_t)(\mathbf{Id} - S_t)^{-1}\right\| \\
&= \left\|\mathbf{Id} - (\mathbf{Id} + S_t) \sum_{k=0}^{\infty} S_t^k\right\| \\
&\leq 2 \|S_t\| \left\|(\mathbf{Id} - S_t)^{-1}\right\| \leq 2 \frac{\theta}{2 - \theta} \frac{1}{1 - \frac{\theta}{2-\theta}} = \frac{\theta}{1 - \theta}.
\end{aligned}$$

■

Corollary 2.16. *Assuming* $0 \leq \theta \leq \frac{1}{2}$ *in the previous lemma yields a simpler bound* $\|J_0 - J_t\| \leq \frac{5}{2}\theta$*, which is valid in both cases.*

Corollary 2.17. *Given a symplectic manifold* (M, ω) *together with a symplectic submanifold* $V \subset M$*. Fix a* $J \in \mathcal{J}_c(\omega)$ *and assume that* $\theta(V) \leq \theta_1$ *for some* $\theta_1 > 0$*. Then there exists another* ω*-compatible* $K \in \mathcal{J}_c(\omega)$*, such that*

$$K(TV) \subset TV \text{ and } \|K - J\| \leq \frac{5}{2}\theta_1,$$

where the norm is induced by the pair (ω, J)*.*

Proof. Lemma 2.19 implies that on V there exists an almost complex structure $K \in \mathcal{J}_c(V, \omega_{|V})$, such that on V we have $\|J - K\| \leq \theta_1$. Hence, we need to extend K to an almost complex structure on M.
Denote by $d : M \times M \to \mathbb{R}$ the distance function induced by the metric $\omega(\cdot, J\cdot)$. Then the set $U_\epsilon := \{x \in M \mid d(x, V) \leq \epsilon\}$ forms a tubular neighbourhood of V, provided $\epsilon > 0$ is sufficiently small.
Consider a standard cut-off function $f : [0, \epsilon] \to \mathbb{R}_+$, i.e. f is monotone decreasing, $f(0) = 1$, $f(\epsilon) = 0$ and all derivatives of f vanish near 0 and ϵ.
Now, Lemma 2.15 implies that there is a family of compatible almost complex structures K_t with $K_0 = K$ and $K_1 = J$ with $\|K - K_t\| \leq \frac{5}{2}\theta_1$. Hence, we extend K over $U\epsilon$ by setting $K_x := K_{f(x,V)}$ at any base point $x \in M$. Outside of U we just extend K by setting it equal to J. ∎

2.2 Preliminaries

Before continuing with the proof of Proposition 2.3 we consider the following

Example 2.18. Equip $\mathbb{R}^6$ with the standard symplectic structure $\omega = \sum_{i=1}^{3} dx_i \wedge dy_i$. Consider the following subspaces

$$V = \mathrm{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2} \right\}$$

$$W = \mathrm{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1} + a\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} + a\frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_2} \right\} \text{ for some } a > 0.$$

Both V and W are symplectic, but their intersection

$$V \cap W = span_{\mathbb{R}} \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_2} \right\}$$

is not, since $\omega_{|V \cap W} = 0$.
Now assume that there is a (linear) complex structure $K \in \mathcal{J}_\tau(\omega)$ which leaves both V and W invariant, i.e. $KV \subseteq V$ and $KW \subseteq W$. This implies that $K(V \cap W) \subseteq V \cap W$. Since K tames ω, it follows that $V \cap W$ is symplectic, giving a contradiction. Hence, no such K exists in this case[1].

[1] One might say that the intersection $V \cap W$ is not positive.

First, recall the following

Lemma 2.19 (cf. Lemma 8.5 (c) in [CM07]). *Consider an ω-symplectic subspace of codimension two $W \subset V$. Then there exists an ω-compatible almost complex structure $K : V \to V$ that leaves W and W^ω invariant. And for a fixed $J \in \mathcal{J}_c(\omega)$ we have*

$$\|K - J\| \leq 2 \sin\left(\frac{\theta(W)}{2}\right) \leq \theta(W).$$

Because of the central role played by the above lemma we present the proof here.

Proof. Denote the intersection by $W_0 := W \cap JW$. For $\dim W_0 = 2n-2$ it follows that W is J-invariant and one simply sets $K := J$. The other possible[1] case is $\dim W_0 = 2n-4$. Then we set $K_{|W_0} := J$. Now by replacing V by $W_0^\perp = W_0^\omega$ we reduce the proof to the four-dimensional case.
For $\dim V = 4$ let x, y be an oriented orthonormal basis of W. Denote by $\pi_W : V \to W$ the orthogonal projection. On W we define $K : W \to W$ via a positive rotation, i.e. $Kx := y$ and $Ky := -x$, hence we have $K^2 = -\mathbf{1}$. Now observe that

$$\|Jy - Ky\| = \|Jy + x\| = \|Jx - y\| = \|Jx - Kx\|.$$

From $\langle Jx, x\rangle = 0$ it follows that for some $\theta \in [0, \pi]$ we have $\langle Jx, y\rangle = \cos\theta \cdot y$. Moreover $\langle Jx, y\rangle = \omega(x, y) > 0$ implies $\theta < \frac{\pi}{2}$ and (cf. Lemma 2.9 statement 1) yields $\theta \leq \theta(W)$. Combining this with

$$\|Jx - y\|^2 = \|Jx\|^2 + \|y\|^2 - 2\omega(x, y) = 2 + 2\cos\theta = 4\sin^2\frac{\theta}{2},$$

gives $\|J - K\| \leq 2\sin\frac{\theta(W)}{2}$ on W. The constructed $K : W \to W$ is compatible with ω, since $\omega(x, Kx) = \omega(x, y) > 0$ and W is two-dimensional.
The only thing left is to define K on W^ω. Since W is a symplectic hyperplane, we have $\dim W^\omega = 2$ and for a fixed oriented orthogonal basis $\{x', y'\}$ of W^ω we define $Kx' := y'$ and $Ky' := -x'$. Now, the same arguments apply for W^ω as for W and we get $\|J - K\| \leq 2\sin\frac{\theta(W^\omega)}{2}$ on W^ω. Since $\theta(W) = \theta(W^\omega)$ (cf. Lemma 2.9 statement 2), we get the estimate on the whole space $V = W \oplus W^\omega$. ∎

Now, we continue with the case of two symplectic hypersurfaces and start with the following

Example 2.20. Equip $\mathbb{R}^6$ with the standard symplectic structure $\omega = \sum_{i=1}^3 dx_i \wedge dy_i$. Consider the following subspaces, given via inclusions (for a fixed real a):

[1] Note that other dimensions for W_0 are not possible by the assumption of codimension two of W and the fact that W_0 is by definition a complex subspace.

$$\phi_a : \mathbb{R}^4 \longrightarrow \mathbb{R}^6, \text{ via } (x_1, y_1, x_2, y_2) \mapsto (x_1, y_1, x_2, y_2, a \cdot y_1, a \cdot y_2).$$

Denote the correpsonding linear subspaces by $V_a := \mathrm{im}(\phi_a)$. By computing the pullback

$$\omega_a := \phi_a^* \omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + a^2 \cdot dy_1 \wedge dy_2$$

we see that $\omega_a \wedge \omega_a = 2 \cdot dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$. Hence, for any $a \in \mathbb{R}$ the spaces V_a are symplectic hyperplanes of $\mathbb{R}^6$.
Let J be the standard complex structure. Then V_0 is a complex subspace. Moreover, for any $a \in \mathbb{R}$ we compute the corresponding Kähler angle. The pullback of the standard metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ is given by

$$g_a := \phi_a^* g = dx_1 \circ dx_1 + (1 + a^2) dy_1 \circ dy_1 + dx_2 \circ dx_2 + (1 + a^2) dy_2 \circ dy_2,$$

hence, for the volume form on V_a we get

$$\Omega_{V_a} = \sqrt{|g_a|} dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 = (1 + a^2) dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2.$$

Combining the previous statements we get the Kähler angle for V_a:

$$\theta(V_a) = \cos^{-1}\left(\frac{\omega^2_{|V_a}}{2! \cdot \Omega_{V_a}}\right) = \cos^{-1}\left(\frac{\omega_a \wedge \omega_a}{2! \cdot \Omega_{V_a}}\right) = \cos^{-1}\left(\frac{1}{1 + a^2}\right).$$

It follows that V_a is a complex subspace if and only if $a = 0$.
Now fix some $a \neq 0$, then for the intersection of V_0 and V_a we have

$$V_0 \cap V_a = \{(x_1, 0, x_2, 0, 0, 0) \mid x_1, x_2 \in \mathbb{R}\}.$$

The subspace $V_0 \cap V_a$ is not symplectic, since $\omega_{|V_0 \cap V_a} = 0$, i.e. this intersection is not positive (for any $a \neq 0$). Now we compute the minimal angle $\angle_m(V_0, V_a)$. First, consider the orthogonal complement

$$(V_0 \cap V_a)^\perp = \{(0, y_1, 0, y_2, x_3, y_3) \mid y_1, y_2, x_3, y_3 \in \mathbb{R}\}$$

and the following intersections

$$A := (V_0 \cap V_a)^\perp \cap V_0 = \{(0, y_1, 0, y_2, 0, 0) \mid y_1, y_2 \in \mathbb{R}\},$$

$$B := (V_0 \cap V_a)^\perp \cap V_a = \{(0, y_1', 0, y_2', a y_1', a y_2') \mid y_1', y_2' \in \mathbb{R}\}.$$

Now take $v \in A$ and $w \in B$ and consider

$$\frac{|\langle x, y \rangle|}{\|v\| \, \|w\|} = \frac{|y_1 y_1' + y_2 y_2'|}{\sqrt{1 + a^2}\sqrt{y_1^2 + y_2^2}\sqrt{y_1'^2 + y_2'^2}} \leq \frac{1}{\sqrt{1 + a^2}},$$

hence, it follows for the minimal angle

$$\angle_m(V_0, V_a) := \underset{0 \neq x \in A, 0 \neq y \in B}{inf} cos^{-1}\left(\frac{|\langle x, y \rangle|}{\|v\| \, \|w\|}\right) \geq cos^{-1}\left(\frac{1}{\sqrt{1 + a^2}}\right).$$

Assuming $a \neq 0$, above statement implies that hyperplanes V_0 and V_a intersect transversely. However, there is no ω-tame almost complex structure which would leave both V_0 and V_a invariant, since their intersection is not symplectic, despite the fact that the maximum of the Kähler angle $\max\{\theta(V_0), \theta(V_a)\} = \theta(V_a)$ can be made arbitrary small.

Remark 2.21. The previous example seems to contradict the statement of Lemma 8.7(b) from [CM07]. However, we show a slight modification of that lemma below by making an additional assumption about the intersection of hyperplanes (being symplectic). The main point is that this assumption will be always satisfied during the later construction.

Lemma 2.22 (modification of Lemma 8.7(b) **from [CM07]).** *For a given pair of codimension two ω-symplectic subspaces W and W' of (V, ω, J), such that their minimal angle satisfies $\angle_m(W, W') \geq \epsilon$ for an $\epsilon > 0$. Moreover, assume that the maximum of their Kähler angles is bounded by $\theta(W) \leq \theta$ and $\theta(W') \leq \theta'$ (with $0 \leq \theta, \theta' < \frac{\pi}{2}$), moreover, the intersection $W \cap W'$ is symplectic.*
Then there exists an w-tame complex structure $K \in \mathcal{J}_\tau(V)$ which leaves both W and W' invariant, such that

$$\|K - J\| < \frac{4}{\epsilon}\max(\theta, \theta'), \text{ provided that } \max(\theta, \theta') < 1.$$

Proof. First we construct a complex structure on the intersection $W \cap W'$, in the case where $dim(W \cap W') > 0$. The case $dim(W \cap W') = 0$ appears if $dimV = 4$. We construct K by multiple application of Lemma 2.19.
By applying Lemma 2.19 to (M, W, J, ω) we get a compatible complex structure J', such that $\|J - J'\| \leq 2\sin(\theta/2)$ and W is J'-complex.
Restrict (ω, J') to W. Since $\angle_m(W, W') \geq \epsilon$, the intersection $W \cap W'$ is a symplectic (by assumption) hypersurface. Hence, applying Lemma 2.19 to $(W, W \cap W', J')$ we get a complex structure $K \in \mathcal{J}_c(W, \omega_{|W})$, such that $\|J' - K\| \leq 2\sin(\theta(W \cap W')/2)$ and $W \cap W'$ is K-invariant.
Let $\{x, y\}$ be an oriented J'-orthonormal basis of $A := (W \cap W')^\omega \cap W'$, extend K to V via $Kx := y$ and $Ky := -x$. Hence, K leaves W and W' invariant. And from the proof of Lemma 2.19 follows that, restricted to A, we get $\|J' - K\| \leq 2\sin(\theta(A)/2)$.
Now, for the Kähler angles we have: $\theta(W \cap W') \leq \min(\theta, \theta')$, hence $\theta((W \cap W')^\omega) \leq \min(\theta, \theta')$ and so $\theta((W \cap W')^\omega \cap W') \leq \min(\theta, \theta')$. This implies $\|K - J'\| \leq \max(\theta, \theta')$ on $W \cup W'$ and hence

$$\|K - J\| \leq \|K - J'\| + \|J - J'\| \leq \max(\theta, \theta') + 2\sin(\frac{\theta(A)}{2}) \leq 2\max(\theta, \theta').$$

To get the inequality on the whole space V, we look at $(W \cap W')^\omega$ which is by assumption 4-dimensional. Let $\dim V = 4$, $W = \mathrm{span}\{x_1, y_1\}$, $W' = \mathrm{span}\{x_2, y_2\}$ and $W \cap W' = 0$. Take $v = ax_1 + by_1 + cx_2 + dy_2 \in V$ and compute

$$(K-J)v = a(y_1 - Jx_1) - bJ(y_1 - Jx_1) + c(y_2 - Jx_2) - dJ(y_2 - Jx_2).$$

The assumption on the minimal angle implies $\langle w, w' \rangle \leq \cos\epsilon \, \|w\| \, \|w'\|$ for $w \in W$ and $w' \in W'$, which together with $\|y_i - Jx_i\| \leq \sin\max(\theta, \theta')$ yields

$$\begin{aligned}
\|(K-J)v\|^2 &\leq (a^2+b^2)\, \|y_1 - Jx_1\|^2 + (c^2+d^2)\, \|y_2 - Jx_2\|^2 \\
&\qquad + 2\cos\epsilon \sqrt{(a^2+b^2)\, \|y_1 - Jx_1\|^2 \, (c^2+d^2)\, \|y_2 - Jx_2\|^2} \\
&\leq \sin^2\max(\theta,\theta') \left(a^2+b^2+c^2+d^2 + 2\cos\epsilon\sqrt{(a^2+b^2)(c^2+d^2)} \right) \\
&\leq \sin^2\max(\theta,\theta')(1+\cos\epsilon)(a^2+b^2+c^2+d^2).
\end{aligned}$$

Same reasoning together with $\|y_i\| \geq \cos\max(\theta, \theta')$ yields

$$\begin{aligned}
\|v\|^2 &\geq a^2 + b^2\, \|y_1\|^2 + c^2 + d^2\, \|y_2\|^2 - 2\cos\epsilon \sqrt{(a^2 + b^2\, \|y_1\|^2)(c^2 + d^2\, \|y_2\|^2)} \\
&\geq \cos^2\max(\theta,\theta')(1-\cos\epsilon)(a^2+b^2+c^2+d^2).
\end{aligned}$$

Hence, we get $\|K - J\| \leq \frac{\sin\max(\theta,\theta')}{\cos\max(\theta,\theta')}\sqrt{\frac{1+\cos\epsilon}{1-\cos\epsilon}}$.
Note that $\sqrt{\frac{1+\cos\epsilon}{1-\cos\epsilon}} \leq 2/\epsilon$ as long as $0 < \epsilon < 1$ and $\frac{\sin\max(\theta,\theta')}{\cos\max(\theta,\theta')} \leq 2\max(\theta,\theta')$ as long as $0 \leq \max(\theta, \theta') < 1$. ■

As already mentioned in [CM07] the complex structure K from the previous lemma does not lie in $\mathcal{J}_c(V)$ in general. This happens exactly when the subspaces $(W \cap W')^\omega \cap W'$ and W^ω do not coincide. We recall the following

Example 2.23. (cf. Remark 8.8 in [CM07]) Consider $\mathbb{R}^4$ equipped with the standard structure (ω_0, J_0). Let

$$W := \{(x_1, y_1, 0, 0) \mid x_1, y_1 \in \mathbb{R}\} \text{ and } W' := \{(\epsilon x_2, 0, x_2, y_2) \mid x_2, y_2 \in \mathbb{R}\}.$$

Since $W \cap W' = (0,0,0,0)$ and W, W' are both 2-dimensional subspaces (for any ϵ) their intersection is transverse. The Kähler angles are given by $\theta(W) = 0$ (since W is J_0-invariant) and

$$\theta(W') = \cos^{-1}\left(\frac{\omega_{0|W'}}{\Omega_{W'}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{\epsilon^2+1}}\right) \leq |\epsilon|,$$

i.e. W' is approximately J_0-holomorphic for ϵ small.
Now assume we have an ω_0-compatible complex structure K that leaves both W and W' invariant. Observe that $e_1 := (1,0,0,0) \in W$ and $\omega(v, e_1) = \langle v, J_0 e_1 \rangle = 0$ for any $v \in W'$. Now, since K should leave W' invariant, we get

$$\omega(v, Ke_1) = \omega(Kv, e_1) = \langle Kv, J_0 e_1 \rangle = 0,$$

and using that $J_0 e_1 \in W$ and that K leaves W invariant, we get

$$0 \neq \langle v, e_1 \rangle = \omega(v, J_0 e_1) = \omega(\underbrace{Kv}_{\in W'}, \underbrace{KJ_0 e_1}_{\in W}), \text{ as long as } \epsilon \neq 0. \tag{2.1}$$

On the other hand $\{e_1, Ke_1\}$ is a basis for W, which together with (2.1) would imply that $\langle v, e_1 \rangle = 0$ - a contradiction, hence no such ω-compatible K can exist if $\epsilon \neq 0$.
Note that for $\epsilon \neq 0$ we have $(\underbrace{W \cap W'}_{0})^\omega \cap W' = \mathbb{R}^4 \cap W' = W' \neq W^\omega$.

Remark 2.24. Arguments from the previous example are not limited to dimension four, since any higher dimensional case can be reduced to four dimensions just by dividing out the (nonempty) intersection of hyperplanes.

2.3 Ball cover relative to a hypersurface

Consider a compact Riemannian manifold (M, g). For any $k > 0$ define the rescaled metric $g_k := k \cdot g$. Since any Riemannian metric (on a complete manifold) gives a distance function $d : M \times M \to \mathbb{R}$, we denote by d_k the distance function induced by g_k.

Lemma 2.25. $d_k(x, y) = k^{\frac{1}{2}} d(x, y)$ *for any* $x, y \in M$

Proof. The distance is given by

$$d_k(x, y) := \underset{\gamma}{inf} \left\{ L_k(\gamma) | \gamma \in \mathcal{C}^0([0, 1], M), \gamma(0) = x, \gamma(1) = y \right\}, \text{ where}$$

$$L_k(\gamma) := \int_0^1 \|\gamma'(t)\|_k \, dt = \int_0^1 \sqrt{g_k(\gamma'(t), \gamma'(t))} dt = \sqrt{k} \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))} dt.$$

■

We recall that (following [Don96]) we have defined for $x, y \in M$

$$e_k(x, y) := \begin{cases} e^{-d_k(x,y)^2/5} & \text{if } d_k(x, y) \leq k^{1/4} \\ 0 & \text{if } d_k(x, y) > k^{1/4} \end{cases}$$

and that a cover $\{B(p_i)\}_{i \in \{1..s\}}$ of (M, g) with g_k-radius r balls centered at $p_i \in M$ is called **admissible** if there exists a constant (independent of k) $C > 0$, such that

$$\sum_{i=1}^{s} d_k(q, p_i)^r e_k(q, p_i) \leq C, \text{ for } r = 0, \ldots, 3.$$

Definition 2.26. *Given a submanifold* $W \subset M$*, we call a covering* $\{B(p_i)\}_{i \in \{1..s\}}$ *of* (M, g) *with* g_k*-radius* r *balls centered at* $p_i \in M$ ***admissible wrt.*** W *if it is admissible and the restriction of the covering to* W *is also an admissible covering of* W *wrt. the restricted metric* $g_{|W}$*.*

Given an equidistant lattice Λ on $\mathbb{C}^n$, i.e. $\Lambda = a(\mathbb{Z}^n \oplus i\mathbb{Z}^n) \subset \mathbb{C}^n$. Then there is a condition on a, such that we could cover $\mathbb{C}^n$ by balls of radius r centered at the lattice points of Λ.

Lemma 2.27. *The set $\{B_r(p) \subset \mathbb{R}^n \mid p \in \Lambda\}$ covers $\mathbb{R}^n$, if $a < \frac{2r}{\sqrt{n}}$.*

Proof. A sufficient condition for such set to be a covering is that a main hypercube diagonal (of an n-dimensional hypercube) is covered by two balls (of radius r) centered at its ends, i.e. the length of a main hypercube diagonal should be smaller than twice the radius of the balls, hence $a\sqrt{n} < 2r$. ∎

First, we look at the local situation, for which we need the following technical result

Lemma 2.28. *Fix a real number $\alpha > 0$ and consider the lattice $\Lambda := \alpha(\mathbb{Z}^n \oplus i\mathbb{Z}^n)$. Then for any real numbers $a, r > 0$ and $w \in \mathbb{C}^n$ the series*

$$\sum_{z\in\Lambda} \|z-w\|^r \, e^{-a\|z-w\|^2}$$

is uniformly bounded in w.

Proof. By symmetry of the lattice we can assume that $w \in [0,\alpha]^{2n}$, hence we can bound $\|z-w\| < \alpha\sqrt{2n}$ if $z \in (\{0,1\}^n \oplus i\{0,1\}^n)$. So, we can bound the series

$$\sum_{z\in\Lambda} \|z-w\|^r \, e^{-a\|z-w\|^2} \leq \sum_{l\in\mathbb{N}} l^{2n}(\alpha\sqrt{2n})^r e^{-a\alpha^2 2n}.$$

Since the latter series is of the form $\sum_{l\in\mathbb{N}} l^c e^{-c'l}$ with $c, c' > 0$ is convergent, the claim follows. ∎

Lemma 2.29. *Fix the subspace $\mathbb{C}^{n-1} \subset \mathbb{C}^n$ by taking the first $n-1$ components. Then the collection $\{B(p_i)\}$ of g_k-unit balls centered at the lattice points $p_i \in \Lambda = \frac{1}{\sqrt{2kn}}(\mathbb{Z}^n \oplus i\mathbb{Z}^n)$ is an admissible covering of $\mathbb{C}^n$ wrt. $\mathbb{C}^{n-1}$ for any $k > 0$.*

Proof. Since $\frac{1}{\sqrt{2kn}} < \frac{2}{\sqrt{2nk}}$ (radius of g_k-unit ball is $k^{-1/2}$) Lemma 2.27 implies, that $\{B(p_i)\}$ covers $\mathbb{C}^n$. Moreover, for the balls centered at the sublattice $\Lambda' := \frac{1}{\sqrt{2kn}}(\mathbb{Z}^{n-1} \oplus i\mathbb{Z}^{n-1})$ cover $\mathbb{C}^{n-1}$. A direct application of Lemma 2.28 implies admissibility of both coverings ($\mathbb{C}^n$ and $\mathbb{C}^{n-1}$). ∎

Such sequence (wrt. k) of coverings will be used to apply Donaldson's argument twice in order to get the main result of this section. We show that such coverings exist on a smooth manifold.

Lemma 2.30. *For a given symplectic hypersurface $W \subset M$, there exists a constant $C > 0$, such that for all $k \gg 0$ there exists an admissible covering wrt. W by g_k-unit balls centered at $p_i \in M$ with $i = 1 \ldots N$.*

Proof. The proof is a modification of the arguments on pp. 678-679 in [Don96]. Since M is compact, we can cover it by charts $\phi_s : \tilde{U}_s \to M$ (with $s = 1 \dots S$), such that $\tilde{U}_s \subset \mathbb{C}^n$ are bounded and there exists a (small) $\gamma > 0$, such that for any $x, y \in \tilde{U}_s$

$$(1-\gamma)\left\|x-y\right\| \leq d(\phi_s(x), \phi_s(y)) \leq (1+\gamma)\left\|x-y\right\|.$$

Note that the same inequalities are valid for rescaled distance. Moreover, we require that $\phi_s^{-1}(W)$ is either empty or is contained in $\mathbb{C}^{n-1}$ (first $n-1$ components in $\mathbb{C}^n$).
Denote by $d_{W,k} : W \times W \to \mathbb{R}$ the distance function on W, induced by restricting ω and J to W and rescaling by $k^{1/2}$. Recall that the rescaled distance function on M is denoted by d_k. Since W is also compact, we can assume, after suitable refinement of $\{\tilde{U}_s\}$ and denoting the new covering by $\{U_s\}$, that $x, y \in U_s$ implies

$$(1-\delta)d_{W,k}(x,y) \leq d_k(x,y) \leq (1+\delta)d_{W,k}(x,y)$$

for some small $\delta > 0$, provided we have chosen k sufficiently large.
Let Λ be the lattice from Lemma 2.29 and denote the corresponding sublattice by $\Lambda' := \Lambda \cap \mathbb{C}^{n-1}$. Set $\Lambda_s := \phi_s(\Lambda \cap U_s)$ and $\Lambda'_s := \phi_s(\Lambda' \cap U_s)$. Then, by construction, g_k-unit balls centered at the points of Λ_s cover $\phi_s(U_s)$. Similarly, g_k-unit balls centered at the points of Λ'_s cover $\phi_s(U_s \cap \mathbb{C}^{n-1})$.
In analogy to e_k we define its restricted version (for $x, y \in W$)

$$e_{W,k}(x,y) := \begin{cases} e^{-d_{W,k}(x,y)^2/5} & \text{if } d_{W,k}(x,y) \leq k^{1/4} \\ 0 & \text{if } d_{W,k}(x,y) > k^{1/4}. \end{cases}$$

To show admissibility observe that, since the finite collection of charts ϕ_s does not depend on k, we have to bound for any fixed s the sums for any $x \in M$ and $y \in W$

$$R_s(x) := \sum_{p \in \Lambda_s} d_k(p,x)e_k(p,x) \text{ and } \sum_{p \in \Lambda'_s} R'_s(y) := d_{W,k}(p,y)e_{W,k}(p,y).$$

By definition, $x \notin U_s$ and $y \notin U_s \cap \mathbb{C}^{n-1}$ imply $R_s(x) = 0$ and $R'_s(y) = 0$ respectively. Now, let $x = \phi_s(z)$ for some $z \in \mathbb{C}^n$, so we can bound

$$R_s(x) \leq \sum_{p \in \Lambda} (1+\gamma)^r \sqrt{k^r} \left\|z-p\right\|^r e^{-\frac{k\|z-p\|^2}{5(1-\gamma)^2}}$$
$$= \sum_{p \in \Lambda_0} (1+\gamma)^r \left\|z-p\right\|^r e^{-\frac{\|z-p\|^2}{5(1-\gamma)^2}} \leq C,$$

with a constant $C > 0$ coming from Lemma 2.28 and the (unrescaled) lattice $\Lambda_0 := \frac{1}{\sqrt{2n}}(\mathbb{Z}^n \oplus i\mathbb{Z}^n)$. For $y \in W$ let $y = \phi_s(w)$ for some $w \in \mathbb{C}^{n-1} \subset \mathbb{C}^n$ and consider

$$R'_s(y) \le \sum_{p\in\Lambda'_s} (1+\gamma)^r (1+\delta)^r \sqrt{k^r}\, \|w-p\|^r \, e^{-\frac{k\|w-p\|^2}{5(1-\gamma)^2(1-\delta)^2}}$$
$$= \sum_{p\in\Lambda'_0} (1+\gamma)^r (1+\delta)^r \|w-p\|^r \, e^{-\frac{\|w-p\|^2}{5(1-\gamma)^2(1-\delta)^2}} \le C',$$

again, with a constant $C' > 0$ coming from Lemma 2.28 and the (unrescaled) sublattice $\Lambda'_0 := \frac{1}{\sqrt{2n}}(\mathbb{Z}^{n-1} \oplus i\mathbb{Z}^{n-1})$. Since both constants C and C' do not depend on k, the claim follows. ■

Remark 2.31. One of the central aspects why Donaldson's construction works at all is the independence of k in the above proof. It can only be achieved, by considering infinite sums in Lemma 2.28.
Moreover, we do not require that the balls from the covering constructed above, which are not centered a the submanifold W, do not intersect W.

2.4 Proof of Propositon 2.3

Again, we divide the argument into several steps.

(A) Take a covering of M with g_k-unit balls from Lemma 2.30, which is both admissible for M and the submanifold W. Again, denote centers of the balls by p_i with $i \in I$ and let I' be the subset indexing the centers of the balls lying on W.

(B) As in step (I) in Proposition 1.11 for any fixed $D > 0$ we get a partition $\{I_\alpha\}$ of the index set I with $\alpha = 1, \ldots, N$ (note that $N > 0$ is independent of k). This defines a partition of I' via $I'_\alpha := I_\alpha \cap I$. Hence, we have

$$d_{W,k}(p_i, p_j) \ge \frac{1}{1+\gamma} D \text{ for } i, j \in I'_\alpha \text{ and all } \alpha = 1 \ldots N.$$

Moreover, by setting
$$W_\alpha := \bigcup_{i\in I'_\beta,\ \beta\le\alpha} B_k(p_i)$$
we obtain an increasing sequence of sets covering W.

(C) For $p \in W$ let σ_p be a compactly supported section of the line bundle L^k from Proposition 1.3. Provided k is chosen sufficiently large[1], the pullback section $\sigma'_p : W \to L^k_{|W}$ satisfies the following inequalities for any $q \in W$:

1. $d_{W,k}(p,q) \le R$ implies $|\sigma'_p(q)| \ge 1/C$, for a fixed $R > 0$ independent of q

[1] Note that chosing k large implies that local sections σ_p become supported over small balls of radius $k^{-1/4}$.

2. $|\sigma_p'(Q)| \leq e_{W,k}(p,q)$
3. $|\nabla'\sigma_p(q)| \leq C(1 + d_{W,k}(p,q))e_{W,k}(p,q)$
4. $|\bar{\partial}'\sigma_p(q)| \leq Ck^{-1/2}d_{W,k}(p,q)^2 e_{W,k}(p,q)$
5. $|\nabla'\bar{\partial}'\sigma_p(q)| \leq Ck^{-1/2}(d_{W,k}(p,q) + d_{W,k}(p,q)^3)e_{W,k}(p,q)$

with a k-independent constant $C > 0$ and ∇', $\bar{\partial}'$ denoting the restrictions of the corresponding operators to the bundle $L^k \to W$. The proof follows by combining Proposition 1.3 with the arguments from the proof of Lemma 2.29.

(D) Consider the section $s_1 : M \to L^k$ concentrated around W:

$$s_1 := \sum_{i \in I'} w_i \sigma_{p_i} \text{ with } w_i \in \mathbb{C} \text{ and } |w_i| \leq 1.$$

Observe that the restriction $s_1' := (s_1)_{|W}$ satisfies $|s_1'| \leq C$, $|\bar{\partial}' s_1'| \leq C(1/\sqrt{k})$ and $|\nabla'\bar{\partial} s_1'| \leq C(1/\sqrt{k})$ for some k-independent $C \geq 0$. This follows analogously to Proposition 1.7 using inequalities from step (C). Now applying Proposition 1.11 to s_1' we get coefficients w_i, such that the restricted section s_1' is η_1-transverse to zero over W for some $\eta_1 > 0$.

(E) Consider another section $s_2 : M \to L^k$, which is zero in the neighbourhood of W:

$$s_2 := \sum_{i \in (I - I')} w_i \sigma_{p_i} \text{ with } w_i \in \mathbb{C} \text{ and } |w_i| \leq 1.$$

Any such section satisfies inequalities from Proposition 1.7 by construction. Hence, applying Proposition 1.11 to s_2 we get new coefficients w_i, such that the new section (which we still denote by the same name) s_2 is η_2-transverse to 0. Finally, the section $s := s_1 + s_2$ is $\min(\eta_1, \eta_2)$-transverse to 0 over whole M and its restriction to W is also $\min(\eta_1, \eta_2)$-transverse to 0. Therefore s is our desired section.

(F) The final ingredient is that η-transversality of section s restricted to W implies a lower bound for the minimal angle between W and $s^{-1}(0)$. But this is exactly the content of Lemma 8.7(a) in [CM07]. More precisely, denoting $V := s^{-1}(0)$ we have for any $x \in V \cap W$

$$\angle_m(T_x W, T_x V) \geq \frac{\nu(\ker(\nabla_x s)_{|W})}{\|\nabla_x s\|},$$

where $\nu(\cdot)$ denotes the minimal norm of the right inverse (which exists, since $\nabla_x s$ is surjective). Now, **(E)** implies that $\|\nabla_x s\| \geq \eta$ with $\eta := \min(\eta_1, \eta_2)$ and $\nu(\ker(\nabla_x s)_{|W}) \geq 1/\left\|\nabla_x s_{|W}\right\| \geq 1/\eta'$ for some $0 < \eta' < \eta$, hence

$$\angle_m(T_x W, T_x V) \geq \frac{\eta}{\eta'} > 0.$$

Remark 2.32. Step **(E)** from the proof involves the application of Proposition 1.11 starting not with arbitrary coefficients w_i, but with some of them already chosen to achieve controlled transversality. The subsequent choice of all remaining coefficients corresponds to a small perturbation which does not destroy controlled transversality over W.

2.5 Singular polarizations and η-transversality

The arguments from the sections above show the existence of two transversely intersecting closed symplectic hypersurfaces, say W_1 and W_2, with the property that the Poincaré dual of W_2 is a multiple of the symplectic form, while the fundamental class of W_1 might be arbitrary. However, nothing essential prevents the above method from being applied more than twice. The main assumption is still rationality of the symplectic form.
Now, let ω be any (possibly non-rational) symplectic form, then using perturbations of ω (just like in the final part of the present thesis) one can find positive real $a_i \in \mathbb{R}$ and rational symplectic forms ω_i, such that we get a decomposition of ω on the cohomology level

$$[\omega] := a_1[\omega_1] + \ldots + a_N[\omega_N], \text{ for some } N > 0.$$

Clearly, each of these rational symplectic forms can be represented by a Donaldson hypersurface, but even more is true.

Theorem 2.33 (cf. Theorem 2 in [Ops13]). *For $dim_{\mathbb{R}} M = 4$ there exist symplectic hypersurfaces $W_1, \ldots, W_N$, which intersect pairwise transversely and positively, i.e. there is a decomposition (or singular polarization)*

$$[\omega] := \sum_{i=1}^{N} a_i PD[\Sigma_i],\ a_i > 0.$$

In higher dimensions a similar statement is available.

The proof can be deduced by an iterative application of Propositon 2.3, however, there is an alternative to that - an observation about η-transversality made by E. Opshtein. We will use this observation in the final part of the thesis.

Theorem 2.34 (cf. Theorem 5 in [Ops13]). *Given a symplectic manifold (M,ω), fix $J \in \mathcal{J}_c(\omega)$ and denote by g the induced metric. For any sufficiently small $\epsilon > 0$ and rational symplectic forms ω_1, ω_2. Fix $J_1 \in \mathcal{J}_c(\omega_1)$, $J_2 \in \mathcal{J}_c(\omega_2)$ and assume that*

$$\|\omega_j - \omega\|_g \leq \epsilon \text{ and } \|J_j - J\|_g \leq \epsilon \text{ for } j \in \{1,2\}.$$

Then there exists an $\eta = \eta(\epsilon) > 0$, such that the following holds. Let $L_j \to M$ be Hermitian line bundles with a connection of curvature $-iq/(2\pi)\omega_j$ (with q

chosen, such that $[q\omega_j]$ is an integer class). Then for any $k \gg 0$ there exist sequences of sections $s_j = (s_j^k) : M \to L_j^k$ with the following properties

1. *s_j are approximately J_j-holomorphic (wrt. $g_k := kqg$), i.e.*

$$\left\| s_j^k \right\|_{\mathcal{C}^1} \leq C, \ \left\| \bar{\partial}_{J_j} s_j^k \right\|_{\mathcal{C}^1} \leq \frac{C}{\sqrt{k}}$$

for a k-independent constant $C = C(\epsilon) > 0$.

2. *Each s_j is η-transverse to 0, i.e. $\left\| s_j^k \right\|_{g_k} \leq \eta$ implies $\left\| \partial_{J_j} s_j^k \right\|_{g_k} \geq \eta$.*

3. *The pair of (sequences of) sections $(s_1, s_2) : M \to L_1^k \oplus L_2^k$ is η-transverse to 0, i.e. for any $x \in M$, $\left\| (s_1^k(x), s_2^k(x)) \right\|_{g_k} \leq \eta$ implies the linear map $(\partial_{J_1} s_1^k, \partial_{J_2} s_2^k) : T_x M \to \mathbb{C}^2$ has a right inverse of g_k-norms less than $1/\eta$.*

Remark 2.35. Although above statement seems to be a direct consequence of Donaldson's construction combined with Auroux's extensions, the main point is that η does not depend on the choice of (ω_j, J_j) but on the ϵ-neighbourhood around (ω, J). This is not obvious, since the bundles L_j are topologically different. However, the main point is that the construction of localized sections varies continuously wrt. the choice of ω_j.

Remark 2.36. Observe that Theorem 2.34 in the case $\omega = \omega_1 = \omega_2$ and $J = J_1 = J_2$ implies the existence of a Lefschetz pencil on M, that was shown for any symplectic manifold with rational symplectic form in [Don99]. Recall that the proof of it [Don99] is a straightforward generalization of the hypersurface statement from [Don96] combined with a more refined transversality result.
So in some sense Opshtein's result is an approximate version of a Lefschetz pencil, even if such might not exist if ω represents an irrational class itself.
Note that we state Theorem 2.34 only for a pair of sections. Opshtein's original result is stated for any finite number of sections. The difference is that for more than two sections it is not quite clear how to show that the mutual transversality can be achieved. This problem disappears in dimension 4, since then transversal intersection of three sections means that a pair of them is disjoint from the third. However, we are interested in the case of only two sections.

Corollary 2.37. *Consider sections $(s_1, s_2) : M \to L_1^k \oplus L_2^k$ from the preceding theorem. Then $V_1 := s_1^{-1}(0)$ and $V_2 := s_2^{-1}(0)$ are closed symplectic hypersurfaces of M, Poincaré dual to $kq[\omega_1]$ resp. $kq[\omega_2]$. The minimal angle wrt. g_k satisfies*

$$\angle_m(V_1, V_2) \geq \eta.$$

Proof. The first part of the statement follows from η-transversality, i.e. statement (2) of the Theorem 2.34. Hence, the only issue is the lower bound for the minimal angle.

Fix a point $p \in V_1 \cap V_2$, for $j \in \{1,2\}$ set $u_j := \partial_{J_j} s_j(p)$ and $H_j := \ker u_j$. Denote by π_2 the g_k-orthogonal projection on H_2. For a $y \in H_1$ we then have

$$\|y\|^2 = \|\pi_2(y)\|^2 + \|y - \pi_2(y)\|^2 .$$

Moreover, since π_2 projects onto the kernel of u_2 we get

$$\|u_2(y)\| = \|u_2(y - \pi_2(y))\| \leq \|u_2\| \, \|y - \pi_2(y)\| .$$

Hence, we have

$$\frac{\|\pi_2(y)\|^2}{\|y\|^2} \leq 1 - \frac{\|u_2(y)\|^2}{\|u_2\|^2 \|y\|^2}.$$

Now assume $\|y\| = 1$, then $u_1(y) = 0$ together with statement (3) of Theorem 2.34 implies that $\|u_2(y)\| \geq \eta \|y\|$. Moreover, the global bound on the sections $\|s_j\| \leq C$ (with $C > 0$ independent of k, ω_1 and ω_2) implies $\|u_2\| \leq 2C$, provided $k \gg 0$. So, the ratio $\|\pi_2(y)\|^2 / \|y\|^2$ is bounded away from 1 by $\eta^2/(4C)^2$, i.e. a constant that depends on η. Hence, the minimal angle between H_1 and H_2 is bounded below by a constant that depends on η, but is independent of k, ω_1 and ω_2, which we denote again by η.
Finally, since $k \gg 0$, hyperplanes $H_j := \ker u_j$ and $T_pV_j := \ker d_p s_j$ become very close for $j = 1, 2$ respectively. So the minimal angle between T_pV_1 and T_pV_2 is again bounded below by η for any $p \in V_1 \cap V_2$. ■

Finally, we indicated the main steps used for the proof of Theorem 2.34 (cf. Section 5.2 from [Ops13]).
(I) Observe that although the line bundles involving L_1 and L_2 are different, one can achieve a version of local sections $\sigma_{p,j}$ for $j = 1, 2$ as in Proposition 1.3 with constants independent of a sufficiently small perturbation of the metric (i.e. independent of (ω_j, J_j)) and with a higher decay rate of the section away from p (see Lemma 5.4 in [Ops13]).
(II) Then the local to global construction as in [Don99] yields for any k approximate holomorphic sections $s_j : M \to L_j^k$.
(III) Transversality can be now deduced using Auroux's simplification as stated in [Aur02], yielding a transversality constant η which is independent of the perturbation (ω_j, J_j), since all constants in (II) can be chosen wrt. to (ω, J) and η does not depend on k (just as in the original case of a Lefschetz pencil).

3

Trees, stable curves and domain-stable nodal maps

In this chapter (excluding the last section) we recollect the theory of J-holomorphic maps with a domain-dependent J developed in [CM07]. Basically, we recall the main definitions and cite compactness and transversality results. This exposition is kept as dense as possible. The main advantage of the Cieliebak-Mohnke approach is that the analysis of holomorphic curves is mainly based on the exposition from [MS04], which is very detailed. Note also that the idea of using domain-dependent J is not new, as it was already applied in Gromov's original work[1] [Gro85].
The last section contains modifications of the arguments from section 8 in [CM07] adapted to our slightly general situation - symplectic hypersurfaces whose Poincaré dual is a multiple of a possibly different symplectic form.

3.1 Trees and nodal curves

Definition 3.1. *Given $k \geq 0$, a triple $T = (T, E, \Lambda)$ is called a* ***k-labelled tree*** *if (T, E) is a connected cycle-free graph with vertices T and edges $E \subset T \times T$, and $\Lambda = \{\Lambda_\alpha\}_{\alpha \in T}$ is a decomposition of the index set $\{1, \ldots, k\} = \coprod_{\alpha \in T} \Lambda_\alpha$.*

The labelling set Λ defines the map $\{1, \ldots, k\} \to T$ via $i \mapsto \alpha_i$, such that $i \in \Lambda_{\alpha_i}$.

Denote the number of edges of T by $e(T) := \#(T) - 1$. Moreover, we write $\alpha E \beta$ if $(\alpha, \beta) \in E$. A map $\tau : T \to \tilde{T}$ is called a **tree homomorphism** if for any $\alpha' \in \tilde{T}$ the preimage $\tau^{-1}(\alpha')$ is a tree and $\alpha E \beta$ implies either $\tau(\alpha) = \tau(\beta)$ or $\tau(\alpha) \tilde{E} \tau(\beta)$ for any $\alpha, \beta \in T$. If such a map τ is bijective and the inverse τ^{-1} is also a tree homomorphism, then τ is called a **tree isomorphism**.
Intuitively, a tree homomorphism might collapse subtrees to vertices while an isomorphism is just a reordering of edges and vertices.

[1] There it occurs in the form of perturbing the right side of $\bar{\partial}_J f = 0$.

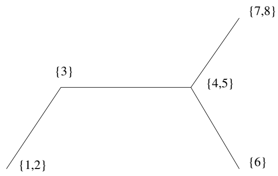

Fig. 3.1. An 8-labelled tree.

Definition 3.2. *A k-labelled tree T is called **stable** if for any vertex $\alpha \in T$*

$$n_\alpha := \#\Lambda_\alpha + \#\{\beta \mid \alpha E \beta\} \geq 3.$$

Given any k-labelled T it can be **stabilized** by collapsing vertices with $n_\alpha < 3$ and modifying the edge relation. The resulting tree is then a stable k-labelled tree and we denote it by $st(T)$.

Definition 3.3. *A **weighted k-labelled tree** $(T, \{A_\alpha\})$ is a k-labelled tree T together with $A_\alpha \in H_2(M, \mathbb{Z})$ for $\alpha \in T$. Such a tree is called **stable** if each vertex α with A_α carries at least 3 special points. Such a vertex is called a **ghost component** and a maximal subtree consisting of ghost components is called a **ghost tree**. A subset $R \subset \{1, \ldots, k\}$ is called the **reduced index set** if it contains all marked points on non-ghost components and the unique marked point z_i with maximal index i on **each** ghost tree.*

Definition 3.4. *A **nodal curve** of genus zero with k marked points modelled over a k-labelled tree T is a tuple*

$$\mathbf{z} = (\{z_{\alpha\beta}\}, \{z_i\}) \text{ such that } \alpha E \beta \text{ and } 1 \leq i \leq k$$

*with $z_{\alpha\beta}, z_i \in S^2$, moreover we require that the **special points***

$$SP_\alpha := \{z_{\alpha\beta} \mid \alpha E \beta\} \cup \{z_i \mid \alpha_i = \alpha\}$$

*are pairwise distinct. A nodal curve is called **stable** if the underlying tree T is stable. We will denote the stabilization of $\mathbf{z}$ by $st(\mathbf{z})$.*
*Given two nodal curves $\mathbf{z}$, $\tilde{\mathbf{z}}$ modelled over T and $\tilde{T}$, a **morphism** $\phi : \mathbf{z} \to \tilde{\mathbf{z}}$ between them is a tuple $\phi = (\tau, \{\phi_\alpha\}_{\alpha \in T})$ consisting of*

$$\tau : T \longrightarrow \tilde{T} \text{ - a tree homomorphism,}$$
$$\phi_\alpha : S_\alpha \longrightarrow S_{\tau(\alpha)} \text{ - holomorphic maps,}$$

such that for $1 \leq i \leq k$ and any $\alpha, \beta \in T$ with $\alpha E \beta$ we have

$$\tilde{z}_{\tau(\alpha)\tau(\beta)} = \phi_\alpha(z_{\alpha\beta}) \text{ if } \tau(\alpha) \neq \tau(\beta)$$
$$\phi_\alpha(z_{\alpha\beta}) = \phi_\beta(z_{\beta\alpha}) \text{ if } \tau(\alpha) = \tau(\beta).$$

In addition we require that marked points are mapped onto marked points on the corresponding component, i.e. $\tilde{z}_i = \phi_{\alpha_i}(z_i)$ and $\tilde{\alpha}_i = \tau(\alpha_i)$.
*A morphism of nodal curves is an **isomorphism** if τ is a tree isomorphism and each ϕ_α is biholomorphic.*

Consider the set $\{S_\alpha\}$ with $\alpha \in T$ and each S_α a standard Riemann sphere. To a given nodal curve $\mathbf{z}$ we associate a **nodal Riemann surface**

$$\Sigma_{\mathbf{z}} := \coprod_{\alpha \in T} S_\alpha / \sim$$

with $z \sim w$ for $z \in S_\alpha$, $w \in S_\beta$ and $z = z_{\alpha\beta}$, $z = z_{\beta\alpha}$ and keep the marked points z_i on each component.

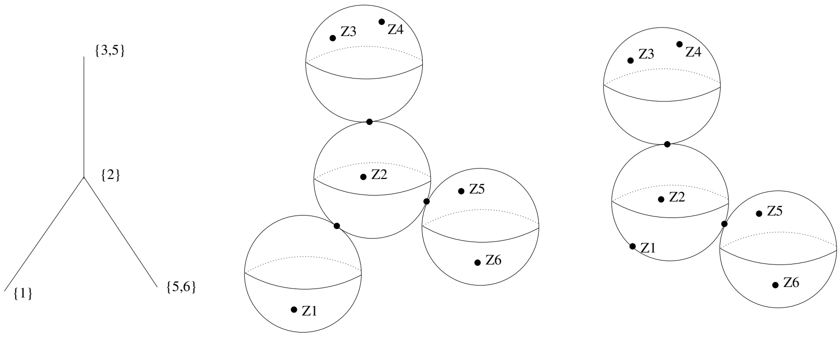

Fig. 3.2. A 6-labelled tree, a nodal Riemann surface (with 6 marked points) modelled over it and the stabilization of this curve.

Remark 3.5. Note that a labelled tree alone does not encode a nodal surface, since the latter contains marked and attaching points. However, information contained in a nodal curve is the same as in a nodal surface. A morphism of nodal curves $\tau : \mathbf{z} \to \tilde{\mathbf{z}}$ induces a continuous map $\Sigma_{\mathbf{z}} \to \Sigma_{\tilde{\mathbf{z}}}$, which is holomorphic if restricted to any spherical component.

Denote the space of all nodal curves (modelled over a fixed tree T with k marked points) by $\tilde{\mathcal{M}}_T \subset (S^2)^E \times (S^2)^k$.

Proposition 3.6. *We summarize important statements about $\tilde{\mathcal{M}}_T$.*

1. *If $\mathbf{z} \in \tilde{\mathcal{M}}_T$ is stable, then the only isomorphism $\mathbf{z} \to \mathbf{z}$ is the identity.*
2. *Denote by G_T the group of isomorphisms of nodal curves, fixing T. For a stable tree T the action of G_T on $\tilde{\mathcal{M}}_T$ is free and proper.*
3. *For a stable tree T the quotient $\mathcal{M}_T := \tilde{\mathcal{M}}_T / G_T$ is a smooth (complex) manifold of dimension $\dim_{\mathbb{R}} \mathcal{M}_T = 2k - 6 - 2e(T)$.*

Proof. For (1) see Remark D.3.3 and p. 580 in [MS04]. (2) and (3) are contained in the statement of Remark 2.1 in [CM07]. ■

Let $k \geq 3$ denote the space of all nodal curves with k marked points modelled over a tree T with only one vertex by $\tilde{\mathcal{M}}_k$ and the corresponding quotient by $\mathcal{M}_k = \tilde{\mathcal{M}}_k / G_T$.

Definition 3.7. *Let $k \geq 3$. The **Deligne-Mumford space** of genus zero with k marked points is defined as*

$$\bar{\mathcal{M}}_k := \coprod_{T \text{ stable } k\text{-labelled tree}} \mathcal{M}_T.$$

Proposition 3.8. *We summarize facts about the topology of $\bar{\mathcal{M}}_k$ for $k \geq 3$.*

1. *The set $\bar{\mathcal{M}}_k$ equipped with Gromov topology is a compact connected metrizable space.*
2. *The space $\bar{\mathcal{M}}_k$ carries a structure of a smooth (complex) compact manifold of dimension $dim_{\mathbb{R}} \bar{\mathcal{M}}_k = 2k - 6$.*
3. *For a given stable k-labelled tree T the closure (wrt. Gromov topology) of $\mathcal{M}_T$ in $\bar{\mathcal{M}}_T$ is given by $\bar{\mathcal{M}}_T = \coprod_{\tilde{T}} \mathcal{M}_{\tilde{T}}$, with $\tilde{T}$ a k-labelled stable tree, such that there exists a surjective tree homomorphism $\tilde{T} \to T$.*
4. *The subspace $\bar{\mathcal{M}}_T \subset \bar{\mathcal{M}}_k$ is a (complex) submanifold of real codimension $2e(T)$ for any stable k-labelled tree T.*

Proof. The statements follow from Theorem 2.7 in [Knu83]. However, for a less algebro-geometric argument see sections D.5 and D.6 in [MS04]. ■

Proposition 3.9. *We consider the projection*

$$\pi : \bar{\mathcal{M}}_{k+1} \longrightarrow \bar{\mathcal{M}}_k,$$

given by forgetting the last marked point and then stabilizing the resulting nodal curve.

1. *π is a holomorphic map.*
2. *The fiber $\pi^{-1}([\mathbf{z}])$ is biholomorphic to $\Sigma_{\mathbf{z}}$.*
3. *For any $[\mathbf{z}] \in \bar{\mathcal{M}}_k$ each component of the preimage $\pi^{-1}([\mathbf{z}])$ is an embedded holomorphic sphere in $\bar{\mathcal{M}}_{k+1}$.*
4. *Denote the l-time composition of projections by $\pi_l := \pi \circ \ldots \circ \pi$. The map π_l induces a morphism between corresponding nodal Riemann surfaces $\Sigma_{\mathbf{z}} \to \Sigma_{\pi_l[\mathbf{z}]}$ for any $[\mathbf{z}] \in \bar{\mathcal{M}}_{k+l}$, then there exists a collection of subtrees $T' \subset T$, such that the morphism is constant on all components $\alpha \in T'$ and biholomorphic otherwise.*

Proof. For (1) see p. 581 in [MS04], (2) and (3) follow from Section D.4 in [MS04]. Point (4) is Lemma 2.6 in [CM07] and it follows from the definition of stabilization. ■

We finish this section by supplementing examples for $\bar{\mathcal{M}}_k$ (usually denoted by $\bar{\mathcal{M}}_{0,k}$). All examples are taken from Section D.7 in [CM07].

k	$\bar{\mathcal{M}}_k$	comment
3	$\{*\}$	$\exists!\{\text{marked points}\} \mapsto \{0,1,\infty\} \subset \mathbb{C}P^1$
4	$\mathbb{C}P^1$	consists of stable trees with one edge and with two marked points on each component
5	$\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$	consider singular fibration $\bar{\mathcal{M}}_5 \to \bar{\mathcal{M}}_4$ with generic fiber $\mathbb{C}P^1$ and singular fiber homeomorphic to an intersection of two copies of $\mathbb{C}P^1$

3.2 Coherent almost complex structures

In this section we recall results and definitions from Section 3 in [CM07]. It is not essential for the understanding of our main result. However, the approach in [CM07] uses the existence of almost complex structures parametrized by $\bar{\mathcal{M}}_{k+1}$ in a coherent way, i.e. they are independent of the domain near the double points. This should simplify gluing arguments in the future work.
Consider $\bar{\mathcal{M}}_{k+1}$. For a $(k+1)$-labelled stable tree T we define an equivalence relation on the marked points via $i \sim j$ if $z_{\alpha_0 i} = z_{\alpha_0 j}$ for $i,j \in \{0,\ldots,k\}$. Equivalence classes yield a decomposition $\{0,\ldots,k\} = I_0 \cup \ldots \cup I_l$. Stability condition implies $l+1 = n_{\alpha_0} \geq$. We call a decomposition $\mathbf{I} = (I_0,\ldots,I_l)$ **stable** if $I_0 = 0$ and $|\mathbf{I}| := l+1 \geq 3$. We assume that I_j is ordered, such that the integers $i_j := \min\{i|i \in I_j\}$ satisfy $0 = i_0 < \ldots < i_l$.
Fix a stable decomposition $\mathbf{I} = (I_0,\ldots,I_l)$, denote the union of all stable trees that induce $\mathbf{I}$ by $\mathcal{M}_{\mathbf{I}} \subset \bar{\mathcal{M}}_{k+1}$. Such $\mathcal{M}_{\mathbf{I}}$ yields a stratification of $\bar{\mathcal{M}}_{k+1}$. The ordering convention defines the map $p_{\mathbf{I}} : \mathcal{M}_{\mathbf{I}} \to \mathcal{M}_{|\mathbf{I}|}$ by sending a stable curve $\mathbf{z}$ to special points on $S_{\alpha 0}$.
Given a Banach space Z, a map $F : \bar{\mathcal{M}}_{k+1} \to Z$ is called **coherent** if

1. $F = 0$ in a neighbourhood of $\mathcal{M}_{\mathbf{I}} \subset \bar{\mathcal{M}}_{k+1}$ with $|\mathbf{I}| = 3$
2. for any stable decomposition $\mathbf{I}$ with $|\mathbf{I}| \geq 4$ there exists a smooth map $F_{\mathbf{I}} : \mathcal{M}_{|\mathbf{I}|} \to Z$, such that $F_{|\mathcal{M}_{\mathbf{I}}} = F_{\mathbf{I}} \circ p_{\mathbf{I}} \to Z$.

The space of such maps is denoted by $\mathrm{Coh}(\bar{\mathcal{M}}_{k+1}, Z)$.
Let P be a smooth manifold. Following [CM07] we call a **tamed almost complex structure on** (M,ω) **parametrized by** P a smooth section in the (pullback) bundle $\mathbf{J}(TM,\omega) \to P \times M$, where $\mathbf{J}(TM,\omega)$ is the space of all ω-tame almost complex structures. For a fixed section J_0 we set[1]

$$T_{J_0}\mathcal{J}_P := C^\epsilon(P\times, T_{J_0}\mathbf{J}(TM,\omega))$$

[1] The spaces $C^\epsilon(M,E)$ were introduced by A. Floer in [Flo88], see also Remark 3.7 in [CM07]. $B(0,\rho)$ is an open ball of radius ρ - injectivity radius of the exponential map $\exp_J : T_J\mathbf{J}(M,\omega) \to \mathbf{J}(M,\omega)$.

$$\mathcal{J}_P(M,\omega) := \exp_{J_0}(\{Y \in T_{J_0}\mathcal{J}_P \mid Y(p,x) \in B(0,\rho(g(x),J_0(x)))\}).$$

The main point is that one can think of any $J \in \mathcal{J}_P$ as a map $J : P \to \mathcal{J}$. For practical use we define:[1]

$$\mathcal{J}_{S^2} := \mathcal{J}_{S^2}(M,\omega) \text{ and } \mathcal{J}_{\bar{\mathcal{M}}_{k+1}} := \mathcal{J}_{\bar{\mathcal{M}}_{k+1}}(M,\omega).$$

Then the space $\mathcal{J}_{k+1}$of **coherent almost complex structures** is given by[2]

$$T_{J_0}\mathcal{J}_{k+1} := \mathrm{Coh}(\bar{\mathcal{M}}_{k+1}, T_{J_0}\mathcal{J}) \subset T_{J_0}\mathcal{J}_{\bar{\mathcal{M}}_{k+1}}$$

$$\mathcal{J}_{k+1} := \exp_{J_0}(T_{J_0}\mathcal{J}_{k+1}) \subset \mathcal{J}_{\bar{\mathcal{M}}_{k+1}}.$$

Lemma 3.10 (cf. Lemma 3.6 in [CM07]). *For $I \subset \{1,\dots,k\}$ with $|I| \geq 3$ let $\pi_I : \bar{\mathcal{M}}_{k+1} \to \bar{\mathcal{M}}_{|I|+1}$ be the projection forgetting marked points outside the set $I \cup \{0\}$ and stabilizing. Then we have an induced pullback map*

$$\pi_I^* : \mathcal{J}_{|I|+1} \longrightarrow \mathcal{J}_{k+1}.$$

3.3 Symplectic energy

We consider a closed Riemann surface $(\Sigma, j, dvol_\Sigma)$ and fix $J \in \mathcal{J}_\tau(M,\omega)$. Any such J induces a Riemannian metric on M via

$$g_J(\cdot,\cdot) := \frac{1}{2}\left(\omega(\cdot, J\cdot) - \omega(J\cdot,\cdot)\right).$$

Recall from [MS04] the following

Definition 3.11. *For any smooth map $u : \Sigma \to M$ the* ***energy*** *is given by*

$$E(u) := \frac{1}{2}\int_\Sigma \|du\|_J^2 \, dvol_\Sigma,$$

where the norm of $du \in \Omega^1(\Sigma, u^*TM)$, viewed as a linear map, is induced by g_J.

Lemma 3.12 (cf. Lemma 2.2.1 in [MS04]). *For any J-holomorphic curve $u : \Sigma \to M$ we have the following* ***energy identity***

$$E(u) := \int_\Sigma u^*\omega.$$

[1] For the second space recall that for any $\mathbf{z} \in \bar{\mathcal{M}}_k$, $\pi^{-1}(\mathbf{z})$ is biholomorphic to a nodal curve $\Sigma_{\mathbf{z}}$ and the restriction of $J \in \mathcal{J}_{\bar{\mathcal{M}}_{k+1}}$ to it yields a continuous map, which is smooth on any component of $\mathbf{z}$.

[2] $\mathcal{J}$ is the space of almost complex structures of class C^ϵ, wrt. J_0.

Now, consider two symplectic forms ω_0 and ω_1, both tamed by the same almost complex structure $J \in \mathcal{J}_\tau(M,\omega_0) \cap \mathcal{J}_\tau(M,\omega_1)$. Assume there is a J-holomorphic curve $u : \Sigma \to M$ representing some homology class $[u] \in H_2(M,\mathbb{Z})$, then

Lemma 3.13. *both pairings are positive:* $\omega_0([u]) > 0$ *and* $\omega_1([u]) > 0$.

However, the above statement is only valid if a given homology class $A \in H_2(M,\mathbb{Z})$ is represented by a J-holomorphic curve. Consider the following example

Example 3.14. For $\epsilon \geq 0$ equip $S^2 \times S^2$ with the family of symplectic forms

$$\omega_\epsilon = \pi_1^* dS^2 + (1+\epsilon)\pi_2^* dS^2,$$

where $\pi_i : S^2 \times S^2 \to S^2$ are the canonical projections and dS^2 is the volume form of S^2. Let J be the standard complex structure on $S^2 \times S^2$. It is compatible with ω_0. Assuming that ϵ is taken sufficiently small, J tames ω_ϵ.
For some $x \in S^2$ let $A := [S^2 \times \{x\}] - [\{x\} \times S^2]$, we get

$$\int_A \omega_\epsilon = \mathrm{vol}(S^2) - (1+\epsilon)\mathrm{vol}(S^2),$$

hence $\omega_0(A) = 0$, but $\omega_\epsilon(A) > 0$ for $\epsilon > 0$.

3.4 Domain-dependent nodal and holomorphic maps

In this section we give a short exposition of Sections 4 and 5 from [CM07]. We adapt standard pseudo-holomorphic curve theory to domain-dependent almost complex structures.

Definition 3.15. *Given (a family of ω-tame almost complex structures parametrized by S^2) $J \in \mathcal{J}_{S^2}$, we define the* ***Cauchy-Riemann operator associated to J***

$$\bar{\partial}_J f := \frac{1}{2}\left(df + J(z, f(z)) \circ df \circ j\right),$$

for any smooth map $f : S^2 \to M$, $z \in S^2$ and a fixed complex structure j on S^2. We call f ***J-holomorphic*** *if $\bar{\partial}_J f = 0$.*

The common definition of energy of a smooth map $f : S^2 \to M$ is given by $E(f) := \frac{1}{2}\int_{S^2} |df|^2$. Note that norm $|df| := \|df\|_J$ is given by the induced metric $g_J(\cdot,\cdot) := \frac{1}{2}(\omega(\cdot, J\cdot) + \omega(J\cdot,\cdot))$. Here the metric is domain-dependent, i.e. $|df|(z) := \|df_z\|_{J(z)}$ for $z \in S^2$. However, for a J-holomorphic map f we have still the usual energy identity (cf. p. 20 in [MS04]):

$$E(f) = \int_{S^2} f^*\omega.$$

Next, we set up the standard nonlinear setting. Fix integers m, p, such that $m \geq 1$, $p > 1$ and $mp > 2$. Fix distinct points $z_1, \dots z_k \in S^2$ and consider the following diagram (denote by $W^{m,p}$ the (m,p)-Sobolev space).

$$\begin{array}{c} \mathcal{E} \\ \Big\downarrow {\scriptstyle \mathcal{E}_f := W^{m-1,p}(S^2, \Omega^{0,1}(f^*TM))} \\ \mathcal{B} := W^{m,p}(S^2, M) \\ \Big\downarrow {\scriptstyle ev^k : f \mapsto (f(z_1), \dots, f(z_k))} \\ M^k := \underbrace{M \times \ldots \times M}_{k \text{ times}} \end{array}$$

For a fixed $J \in \mathcal{J}_{S^2}$ the Cauchy-Riemann operator defines a section $\bar{\partial}_J : \mathcal{B} \to \mathcal{E}$, it induces the section $\bar{\partial} : \mathcal{B} \times \mathcal{J}_{S^2} \to \mathcal{E}$ via $(f, J) \mapsto \bar{\partial}_J f$ ($\bar{\partial}$ is called the universal Cauchy-Riemann operator). Moreover, $\mathcal{B}$ is a Banach manifold, $\mathcal{E} \to \mathcal{B}$ is a Banach bundle, and both $\bar{\partial}_J$ and $\bar{\partial}$ are smooth sections of Banach bundles. A first transversality observation is the following

Proposition 3.16 (see Lemma 4.1 and Lemma 4.2 in [CM07]). *For $k \in \mathbb{N}$ fix pairwise distinct points $\{z_1, \dots, z_k\} \subset S^2$ and $J \in \mathcal{J}_{S^2}$. Then for any nonconstant $f \in \mathcal{B}$ with $\bar{\partial}_J f = 0$ the linearization of*

$$(\bar{\partial}, \mathrm{ev}^k) : \mathcal{B} \times \mathcal{J}_{S^2} \to \mathcal{E} \times M^k$$

is surjective at (f, J). Moreover, if $f \in \mathcal{B}$, then the linearization of

$$(\bar{\partial}, \mathrm{ev}^1) : \mathcal{B} \times \mathcal{J}_{S^2} \to \mathcal{E} \times M$$

is also surjective at (f, J).

Fix $A \in H_2(M, \mathbb{Z})$, $J \in \mathcal{J}_{S^2}$, $k \geq 1$, a smooth submanifold $Z \subset M^k := M \times \ldots \times M$ and pairwise distinct points $z_1, \dots, z_k \in S^2$. Define the space

$$\tilde{\mathcal{M}}(A, J, Z) := \left\{ f : S^2 \to M \mid \bar{\partial}_J f = 0, [f] = A, (f(z_1), \dots, f(z_k)) \in Z \right\}.$$

Proposition 3.17 (see Corollary 4.4 in [CM07]). *For any submanifold $Z \subset M$ there exists a Baire set $\mathcal{J}^{\mathrm{reg}}_{S^2}(Z) \subset \mathcal{J}_{S^2}$, such that for any class $A \in H_2(M, \mathbb{Z})$, a fixed point $z_1 \in S^2$ and a $J \in \mathcal{J}^{\mathrm{reg}}_{S^2}(Z)$ the space $\tilde{\mathcal{M}}(A, J, Z)$ is a smooth manifold of dimension*

$$dim_{\mathbb{R}} \tilde{\mathcal{M}}(A, J, Z) = 2n + c_1(A) - codim_{\mathbb{R}} Z.$$

Moreover, for any $Z \subset M^k$ (with $k \geq 1$) exists a Baire set $\mathcal{J}^{\mathrm{reg}}_{S^2}(Z) \subset \mathcal{J}_{S^2}$, such that for any nontrivial class $A \in H_2(M, \mathbb{Z})$, fixed distinct points $z_1, \dots, z_k \in S^2$ and $J \in \mathcal{J}^{\mathrm{reg}}_{S^2}(Z)$, the space $\tilde{\mathcal{M}}(A, J, Z)$ is a smooth manifold of dimension

$$dim_{\mathbb{R}} \tilde{\mathcal{M}}(A, J, Z) = 2n + c_1(A) - codim_{\mathbb{R}} Z.$$

Above-mentioned concept naturally generalizes to maps modelled over a tree. Fix a k-labelled tree T and define $\mathcal{J}_T := \prod_{\alpha\in T}\mathcal{J}_{S_\alpha}$. Let $\mathbf{z}$ be a nodal curve modelled over T and $\Sigma_{\mathbf{z}}$ the corresponding nodal Riemann surface.

Definition 3.18 (stable map). *A* ***continuous map*** $\mathbf{f} : \Sigma_{\mathbf{z}} \to M$ *is a collection of continuous maps* $\{f_\alpha\}$ $f_\alpha : S_\alpha \to M$ *that match at the nodal points, i.e.* $f_\alpha(z_{\alpha\beta}) = f_\beta(z_{\beta\alpha})$ *if* $\alpha E\beta$.
Given a $\mathbf{J} \in \mathcal{J}_T$ *define the Cauchy-Riemann operator* $\bar\partial_{\mathbf{J}}\mathbf{f}$ *to be equal* $\bar\partial_{J_\alpha} f_\alpha(z)$ *at a point* $z \in S_\alpha$. *A continuous* $\mathbf{f}$ *is called* $\mathbf{J}$***-holomorphic***[1] *if* $\bar\partial_{\mathbf{J}}\mathbf{f} = 0$. *We call a pair* $(\mathbf{z}, \mathbf{f})$ *a* ***nodal*** $\mathbf{J}$***-holomorphic map with k marked points****. Define the homology class and energy of* $(\mathbf{z}, \mathbf{f})$ *via*

$$[\mathbf{f}] := \sum_{\alpha\in T}[f_\alpha] \in H_2(M,\mathbb{Z}) \text{ and } E(\mathbf{f}) := \sum_{\alpha\in T} E(f\alpha).$$

Hence, $(\mathbf{z}, \mathbf{f})$ *is* ***modelled over the weighted tree*** $(T, \{A_\alpha\})$ *if* $\mathbf{z}$ *is modelled over* T *and* $A_\alpha = [f_\alpha]$ *for all* $\alpha \in T$. *And* $(\mathbf{z}, \mathbf{f})$ *is called* ***stable*** *if* $(T, \{A_\alpha\})$ *is a weighted stable tree. The space of all nodal maps modelled over* $(T, \{A_\alpha\})$ *is denoted by* $\tilde{\mathcal{M}}_T(\{A_\alpha\}, \mathbf{J})$. *For a fixed class* $A \in H_2(M,\mathbb{Z})$ *the space of stable nodal maps is given by*

$$\tilde{\mathcal{M}}_T(A, \mathbf{J}) := \coprod_{\sum A_\alpha = A} \tilde{\mathcal{M}}_T(\{A_\alpha\}, \mathbf{J}), \text{ such that } (T, \{A_\alpha\}) \text{ is weighted stable.}$$

Let $k \geq 3$, fix a stable curve $\mathbf{z}$ modelled over a k-labelled tree T. Recall that the restriction of $J \in \mathcal{J}_{k+1}$ to $\pi^{-1}(\mathbf{z}) \cong \Sigma_{\mathbf{z}}$ produces an element $J_{\mathbf{z}} \in \mathcal{J}_T$, since $\Sigma_{\mathbf{z}} = \cup_{\alpha\in T} S_\alpha$ and the restriction of $J_{\mathbf{z}}$ to each component is smooth. Hence, above $\bar\partial$-operator can also be used here, namely

Definition 3.19 (domain-stable map). *Given* $J \in \mathcal{J}_{k+1}$. *A continuous map* $\mathbf{f} : \Sigma_z \to M$ *is called* $J_{\mathbf{z}}$*-holomorphic if* $\bar\partial_{J_{\mathbf{z}}}\mathbf{f} = 0$. *If* $\mathbf{z}$ *is a stable curve, then the pair* $(\mathbf{z}, \mathbf{f})$ *is called a* ***domain-stable map***[2].

Definition 3.20 (nodal J-holomorphic map for $J \in \mathcal{J}_{k+1}$). *Given a nodal curve* $\mathbf{z}$ *with* k*-marked points, its stabilization induces a holomorphic map* $st : \Sigma_{\mathbf{z}} \to \Sigma_{st(\mathbf{z})}$. *Hence, as in the previous definition,* J *yields an element* $J^{st}_{\mathbf{z}} \in \mathcal{J}_T$, *so* $\bar\partial_{J^{st}_{\mathbf{z}}}\mathbf{f}$ *is well-defined, and we call such* $(\mathbf{z}, \mathbf{f})$ *a* J***-holomorphic nodal map*** *if* $\bar\partial_{J^{st}_{\mathbf{z}}}\mathbf{f} = 0$. *Again, for a fixed class* $A \in H_2(M,\mathbb{Z})$ *the space of stable nodal* J*-holomorphic maps modelled over a* k*-labelled tree* T *is denoted by*

$$\tilde{M}_T(A, J) := \coprod_{\sum A_\alpha = A} \tilde{M}_T(\{A_\alpha\}, J).$$

[1] Note that $\bar\partial_{\mathbf{J}}\mathbf{f} = 0$ implies here that the map over sphere S_α is J_α-holomorphic, so it is actually smooth by elliptic regularity for each α.

[2] Clearly, domain-stable maps are stable, but the converse is false in general.

Any two nodal J-holomorphic maps $(\mathbf{z}, \mathbf{f})$ $(\mathbf{z}', \mathbf{f}')$ are called ***isomorphic***[1] *if the nodal curves $\mathbf{z}$, $\mathbf{z}'$ are isomorphic via $(\tau, \{\phi_\alpha\})$ and $f'_{\tau(\alpha)} \circ \phi_\alpha = f_\alpha$ for all $\alpha \in T$.*

Remark 3.21. The outcome of the above construction is that $J^{st}_{\mathbf{z}}$ is constant on the components which are killed by the stabilization operation.

Denote the group of all isomorphisms of the space $\tilde{\mathcal{M}}(A, J)$ by G_T. The action of G_T is proper and stability (of holomorphic maps) implies that all isotropy groups are finite (cf. p. 55 in [CM07]). Hence, we have

Definition 3.22 (moduli spaces with $J \in \mathcal{J}_{k+1}$). *For a fixed $k \geq 0$ the* ***moduli space of stable maps*** *is given by*

$$\bar{\mathcal{M}}_k(A, J) := \bigcup_{T \text{ } k\text{-labelled tree}} \mathcal{M}_T(A, J) := \bigcup_{T \text{ } k\text{-labelled tree}} \tilde{\mathcal{M}}_T(A, J)/G_T.$$

The spaces $\mathcal{M}_T(A, J)$ are called ***strata*** *of $\bar{\mathcal{M}}_k(A, J)$ and $\mathcal{M}_k(A, J) := \mathcal{M}_{T_k}(A, J)$ is the* ***top stratum*** *for T_k, a k-labelled tree with one vertex. The* ***moduli space of domain-stable maps*** *is denoted by*

$$\bar{\mathcal{M}}^{ds}_k(A, J) := \bigcup_{T \text{stable } k\text{-labelled tree}} \mathcal{M}_T(A, J) \subset \bar{\mathcal{M}}_k(A, J).$$

Note that just as in [MS04] the space $\bar{\mathcal{M}}_k(A, J)$ can be equipped with the Gromov topology, becoming a metrizable space. Moreover, since the underlying tree of a domain-stable map is stable, the group G_T acts freely on $\tilde{\mathcal{M}}_T(A, J)$, hence, one has a decomposition

$$\mathcal{M}_T(A, J) = \coprod_{\Sigma A_\alpha = A} \mathcal{M}_T(\{A_\alpha\}, J).$$

3.5 Transversality results and compactness

A central feature of stable holomorphic maps with uniformly bounded energy is that one establishes (Gromov) compactness, i.e. any such sequence has a (Gromov) convergent subsequence. First, recall[2] the definition of Gromov convergence.
Let $J \in \mathcal{J}_{k+1}$. A sequence of stable maps $(\mathbf{z}^\nu, \mathbf{f}^\nu) \in \bar{\mathcal{M}}_k(A, J)$ **converges in the sense of Gromov** to a stable map $(\mathbf{z}, \mathbf{f}) \in \bar{\mathcal{M}}_k(A, J)$, if for any $\nu \gg 0$ there exists a surjective tree homomorphism $H\nu : T \to T^\nu$ and a collection of automorphisms $\{\phi^\nu_\alpha\}$ with $\alpha \in T$, such that the following holds

[1] Note that Lemma 5.1 in [CM07] states that J-holomorphicity is preserved under such isomorphisms.

[2] See Section 5.5. in [MS04] for the definition in the domain-independent case. However, as asserted in [CM07], compactness issues carry over to the case of coherent almost complex structures.

- At any vertex $\alpha \in T$ the sequence $f_{H^\nu(\alpha)} \circ \phi_\alpha^\nu : S^2 \to M$ converges uniformly to f_α on any compact subset of $S^2 \backslash \{z_{\alpha\beta} \mid \beta \in T, \alpha E \beta\}$.
- At any node $\alpha E \beta$ the engergy[1] equality holds:

$$E_{\alpha\beta}(\mathbf{f}) = \lim_{\epsilon \to 0} \lim_{\nu \to \infty} \left[E(f^\nu(\alpha), \phi_\alpha^\nu(B_\epsilon(z_{\alpha\beta}))) + \sum_{\substack{\gamma \in T \\ \alpha E \gamma, z_{\alpha\gamma} \in \phi_\alpha^\nu(B_\epsilon(z_{\alpha\beta}))}} E_{\alpha\gamma}(\mathbf{f}) \right].$$

- If $\alpha E \beta$ and (after passing to a subsequence of ν) $f^\nu(\alpha) = f^\nu(\beta)$, then $(\phi_\alpha^\nu)^{-1} \circ \phi_\beta^\nu$ converges uniformly on any compact subset of $S^2 \backslash \{z_{\alpha\beta}\}$.
- If $\alpha E \beta$, then (after passing to a subsequence of ν) $f^\nu(\alpha) \neq f^\nu(\beta)$ implies $z_{\alpha\beta} = \lim_{j \to \infty} (\phi_\alpha^\nu)^{-1} \left(z^\nu_{f^\nu(\alpha) f^\nu(\beta)} \right)$.
- For all $i = 1, \ldots, n$ we have $\alpha_i^\nu = f^\nu(\alpha_i)$ and $z_i = \lim_{\nu \to \infty} (\phi_{\alpha_i}^\nu)^{-1}(z_i^\nu)$.

Then, just as in the domain-independent case, a uniform energy bound implies compactness:

Theorem 3.23 (compactness, cf. Theorem 5.2 in [CM07]). *Given $J \in \mathcal{J}_{k+1}$ and consider a sequence of stable J-holomorphic maps $(\mathbf{z}^\nu, \mathbf{f}^\nu) \in \bar{\mathcal{M}}_k(A, J)$, such that $E(\mathbf{f}^\nu) \leq C$ for some $C > 0$ (and all ν). Then there exists a subsequence ν_j, such that $(\mathbf{z}^{\nu_j}, \mathbf{f}^{\nu_j})$ converges in the sense of Gromov to a stable J-holomorphic map $(\mathbf{z}, \mathbf{f}) \in \bar{\mathcal{M}}_k(A, J)$.*
Moreover, after passing to this subsequence the following holds:

- *There exists a stable weighted tree $(T', \{A_{\alpha'}\})$, such that $(\mathbf{z}^\nu, \mathbf{g}^\nu) \in \mathcal{M}_{T'}(\{A_{\alpha'}\}, J)$.*
- *There exists a stable weighted tree $(T, \{A_\alpha\})$ and a surjective tree homomorphism $\tau : T \to T'$ with $\tau(\alpha_i) = \alpha_i'$, $\sum_{\alpha \in \tau^{-1}(\alpha')} A_\alpha = A_{\alpha'}$ and $(\mathbf{z}, \mathbf{f}) \in \mathcal{M}_T(\{A_\alpha\}, J)$.*
- *Stabilizations $\mathrm{st}(\mathbf{z}^\nu)$ converge to stabilization $\mathrm{st}(\mathbf{z})$ and $\mathbf{f}$ is $J_{\mathrm{st}(\mathbf{z})}$-holomorphic.*

Note that here compactness holds for stable maps. In general, the subset of domain-stable maps would not be compact. However, it is an essential aspect in the Cieliebak-Mohnke approach that in a certain geometric situation the space of domain-stable maps with constraints[2] becomes actually compact.

We continue by recapitulating transversality results from sections 5, 6 and 9 from [CM07]. We will use them without any modification, since they are valid for all closed symplectic submanifolds without specifying their fundamental classes. Transversality is basically achieved by perturbing almost complex structure on the complements of these submanifolds. We consider the following geometric setting:

[1] Here $E_{\alpha\beta}$ is the sum energies of all components belonging to a maximal subtree which is attached to α and contains β.

[2] Intersection condition with a symplectic hypersurface of high degree - see Section 9 in [CM07] and the last chapter.

- Fix a tame almost complex structure $J \in \mathcal{J}_\tau(M, \omega)$.
- Let $\mathcal{V} = \{V_0, V_1, \ldots\}$ be a countable set of submanifolds of M of positive codimension.
- Assume that all V_i are J-invariant (i.e. $JTV_i \subset TV_i$).
- Denote the complement of $\mathcal{V}$ by $V^c := M - \cup_i V_i$.
- For any $l \geq 3$ consider $\mathcal{J}_{l+1}(V^c)$ the set of coherent almost complex structures that agree along all V_i.

In this context we have

Proposition 3.24 (cf. Proposition 9.1 in [CM07]). *There exists a Baire set*

$$\mathcal{J}^{reg}_{l+1}(V^c, \mathcal{V}, J) \subset \mathcal{J}_{l+1}(V^c),$$

such that for any I-stable[1] *k-labelled tree T, homology classes $A_\alpha \in H_2(M, \mathbb{Z})$ with $\alpha \in T$ and $K \in \mathcal{J}^{reg}_{|I|+1}(V^c, \mathcal{V}, J_0)$ the following holds.*

- *The moduli space $\mathcal{M}^*_T(\{A_\alpha\}, K, \mathcal{V})$ of stable K-holomorphic maps modelled over $(T, \{A_\alpha\})$, intersecting V_i at the points z_j with $j \in R$ (and R is the reduced index set) and without a non-constant component entirely contained in V_i, is a smooth manifold of dimension*

$$\dim \mathcal{M}^*_T(\{A_\alpha\}, K, \mathcal{V}) = 2n - 6 + 2k - 2e(T) + \sum_{\alpha \in T} 2c_1(A_\alpha) - \sum_{j \in R} \operatorname{codim}_{\mathbb{R}} V_j.$$

- *The evaluation map ev^k factors as*

$$\mathrm{ev}^k : \mathcal{M}^*_T(\{A_\alpha\}, K, \mathcal{V}) \longrightarrow \mathcal{M}^*_{\pi_R(T)}(\{A_\alpha\}, K, \mathcal{V}) \longrightarrow M^k$$

through a smooth manifold of dimension

$$\dim \mathcal{M}^*_{\pi_R(T)}(\{A_\alpha\}, K, \mathcal{V}) = 2n - 6 + 2k + 2|R| + \sum_{\alpha \in T} 2c_1(A_\alpha) - \sum_{j \in R} \operatorname{codim}_{\mathbb{R}} V_j - 2e(\pi_R(T)).$$

- *Suppose that only one of $\{A_\alpha\}$ is non-trivial, say $A_{\alpha_0} \neq 0$, fix integers $l_j \geq -1$ for $j \in R$. Then the moduli space $\mathcal{M}^*_{\pi_R(T)}(A_{\alpha_0}, K, \mathcal{V}, \{l_j\})$ of stable K-holomorphic maps modelled over $(T, \{A_\alpha\})$ tangent to V_i of order l_j at special points $z_{\alpha_0 j}$ and not entirely contained in $\mathcal{V}$, is a smooth manifold of dimension*

[1] We take $I \subset \{1, \ldots, k\}$ with $|I| \geq 3$.

$$\dim \mathcal{M}^*_T(A_{\alpha_0}, K, \mathcal{V}, \{l_j\}) = 2n - 6 + 2k - 2e(T) + 2c_1(A_{\alpha_0}) - \sum_{j \in R} (l_j + 1)\mathrm{codim}_{\mathbb{R}} V_j.$$

- *The corresponding evaluation map* ev^k *factors as*

$$\mathrm{ev}^k : \mathcal{M}^*_T(A_{\alpha_0}, K, \mathcal{V}, \{l_j\}) \longrightarrow \mathcal{M}^*_{|R|}(A_{\alpha_0}, K, \mathcal{V}, \{l_j\}) \longrightarrow M^k$$

through a smooth manifold of dimension

$$\dim \mathcal{M}^*_{|R|}(A_{\alpha_0}, K, \mathcal{V}, \{l_j\}) = 2n - 6 + 2k + 2|R| + 2c_1(A_{\alpha_0}) - \sum_{j \in R} \mathrm{codim}_{\mathbb{R}} V_j.$$

3.6 Tangencies and intersections

In this section we consider intersections between a complex hypersurface and a holomorphic curve. Everything here is taken from Section 7 in [CM07]. We fix an almost complex manifold (M, J) and consider Riemann sphere S^2 with a standard complex structure.

Definition 3.25. *Given any smooth submanifold $V \subset M$ and a smooth map $f : S^2 \to M$. Assume that $f(z) \in V$ for some $z \in S^2$, then we call $f(z)$ an* ***isolated intersection of*** f ***and*** V*, if there exists a closed disc $D \subset S^2$ containing z and a closed ball (of the same dimension as V) $B \subset V$ which contains $f(z)$, such that $f^{-1}(B) \cap D = \{z\}$.*
Fix $z \in S^2$, then we define the ***local intersection number*** *via*

$$\iota(f, V, z) := (f_{|D}) \cdot B,$$

after applying a small perturbation to f and counting with signs.
Assuming that $\partial V = \emptyset$, we define the ***intersection number*** *of f and V by setting*

$$\iota(f, V) := f \cdot V.$$

Fix a tree T and $k \geq 1$, let $\mathbf{f}$ be a non-constant (genus zero) nodal J-holomorphic map with k modelled over T. Denote by $\mathbf{z}$ the underlying curve and by f_α with $\alpha \in T$ the corresponding components. Assume that V is a closed J-complex hypersurface and that no non-constant components f_α are contained in V. Given a marked point z_i we define the ***intersection number*** *via*

$$\iota(\mathbf{f}, V, z_i) := \begin{cases} \iota(f_{\alpha_i}, V, z_i) & \text{if } f_{\alpha_i} \text{ is non-constant} \\ \sum_{\beta \in T_2} \iota(f_\beta, V, z_{\beta i}) & \text{if } f_{\alpha_i} \text{ is constant} \end{cases}$$

where T_2 is defined as follows. Let T_1 be the ghost tree containing α_i, then $T_2 \subset T - T_1$ is given by vertices adjacent to T_1. Here, $z_{\beta i}$ denotes the nodal point connecting f_β and T_1.

As the following statements show, J-holomorphicity has strong implications[1] on the intersection of such objects, even in a non-integrable setting.

Proposition 3.26 (cf. Proposition 7.1 and Lemma 7.2 in [CM07]). ***(A)*** *Given a J-holomorphic map $f : S^2 \to M$ and a J-complex closed hypersurface $V \subset M$. Assume that the image of f is not entirely contained in V then the set $f^{-1}(V)$ is finite and*

$$\iota(F, V) = \sum_{z \in f^{-1}(V)} \iota(f, V, z).$$

Moreover, at any intersection point $z \in f^{-1}(V)$ denote the tangency order l of f to V by l_z. Then l_z is finite, $l_z \geq 0$ and we have $\iota(f, V, z) = l_z + 1$. Hence, the numbers $\iota(f, V, z)$ are positive for any $z \in f^{-1}(V)$.
(B) *Fix a $k \geq 1$ and consider a sequence $(\mathbf{z}^\nu, \mathbf{f}^\nu)$ of non-constant nodal J-holomorphic maps with k marked points. Assume that $(\mathbf{z}^\nu, \mathbf{f}^\nu) \to (\mathbf{z}, \mathbf{f})$ in the Gromov topology (cf. Section 3.5) and that $\mathbf{f}^\nu$ and $\mathbf{f}$ have no non-constant components entirely contained in V. Let z_i be a marked point of $\mathbf{z}$ with $i \geq 1$. If z_1 is contained in a non-constant component of $\mathbf{f}$, then*

$$\iota(\mathbf{f}, V, z_i) \geq \limsup_{\nu \to \infty} \iota(\mathbf{f}^\nu, V, z_i^\nu).$$

If z_i lies on a constant component of $\mathbf{f}$, then

$$\iota(\mathbf{f}, V, z_i) \geq \limsup_{\nu \to \infty} \sum_{\alpha_j T_1} \iota(\mathbf{f}^\nu, V, z_j^\nu),$$

where $T_1 \subset T$ is a ghost tree containing the corresponding vertex and the above sum counts for each ghost tree $T' \subset T^\nu$ at most one of the z_j^ν with $\alpha_j \in T'$.

Proof. Part (**A**) follows from the Carleman similarity principle from [MS04], see pp. 74-75 in [CM07]. The second part follows from (**A**) and the definition of Gromov convergence, see pp. 75-76 in [CM07]. ∎

3.7 Holomorphic curves and symplectic hypersurfaces

First, we recall the following fact from [CM07] which was suggested by D. Auroux. Because of its central role we give also a proof of it here. For a given pair (M, ω) we fix $J \in \mathcal{J}_c(M)$ and $\alpha \in \Omega^2(M)$ with $[\alpha] = c_1(M, \omega)$.

[1] Such a phenomenon is often called positivity of intersections.

Lemma 3.27 (cf. Lemma 8.11 in [CM07]). *Let $K \in \mathcal{J}_\tau(M,\omega)$ with $\|J-K\| < \theta_0$. Then for any class $A \in H_2(M,\mathbb{Z})$ containing a non-constant closed K-holomorphic curve we have*

$$\langle c_1(TX), A\rangle \leq \frac{1+\theta_0}{1-\theta_0}\,\|\alpha\|\,\omega(A) =: D_*\omega(A).$$

Proof. Take any $v \in T_xM$. By definition we have $\|v\| = \omega_0(v, Jv)$ and continuity yields following estimates

$$\alpha(v,Kv) \leq |\alpha(v,Kv)| \leq \|\alpha\|\,\|v\|\,\|Kv\| \leq \|\alpha\|\,\|v\|^2\,(1+\|J-K\|),$$

$$\omega(v,Kv) \geq (1-\|J-K\|)\,\|v\|^2 \geq 0.$$

Combining both statements with $\|J-K\| < \theta_0$ gives

$$\alpha(v,Kv) \leq \frac{1+\theta_0}{1-\theta_0}\,\|\alpha\|\,\omega(A).$$

Now, for a closed K-holomorphic curve $f : \Sigma \to M$ representing A we have

$$\langle c_1(TX), A\rangle = \int_\Sigma f^*\alpha \leq D_* \int_\Sigma f^*\omega = D_*\omega(A).$$

■

Note that the constant D_* depends on the pair (ω, J) and on θ_0. However, it does not depend on the scaling of ω. We also remark that there is an apriori estimate in case of a deformation of ω. Consider, two pairs (ω_1, J_1) and (ω_2, J_2) denote the corresponding norms by $\|\cdot\|_i$ for $i = 1, 2$. Assume that $\|\omega_1 - \omega_2\|_1 < \epsilon$ and $\|J_1 - J_2\|_1 < \epsilon$ for some $0 < \epsilon < 1$, i.e. the intersection $\mathcal{J}_\tau(\omega_0) \cap \mathcal{J}_\tau(\omega_1)$ is not empty by Lemma 2.5. Then the combination of Lemma 2.9 and Lemma 2.12 yields

$$D_*(\omega_1, J_1, \theta_0) < 2D_*(\omega_0, J_0, \theta_0 + \epsilon^{1/4}),$$

provided that $\epsilon^{1/4} < \max\{1-\theta_0, \sqrt{2}-1\}$. Note that we have used the fact $c_1(M,\omega_0) = c_1(M,\omega_1)$.

Definition 3.28. *Given a symplectic manifold (M,ω), a tame almost complex structure $K \in \mathcal{J}_\tau(M,\omega)$, a K-complex submanifold $V \subset M$, positive integer $l > 0$ and an energy level $E > 0$. Then, the* ***regularity condition*** *$\mathcal{R}(M,\omega,V,K,E,l)$ is satisfied, if*

- *all moduli spaces of simple K-holomorphic spheres in M of energy at most E are smooth manifolds of the expected dimension and*
- *all moduli spaces of non-constant simple K-holomorphic spheres in M of energy at most E with prescribed tangency order to V of at most l are smooth manifolds of the expected dimension.*

In case $V = \emptyset$ *the second assumption becomes empty and we just write* $\mathcal{R}(M, \omega, K, E)$.

Now, we observe that for a Donaldson hypersurface coming from a different symplectic structure but with additional assumption we have a similar statement as Proposition 8.13 from [CM07].

Proposition 3.29. *Consider a symplectic hypersurface* $V \subset M$ *with* $\mathrm{PD}(V) = D[\omega_1]$ *for some integer* $D > 0$ *and an integer class* $[\omega_1] \in H^2(M, \mathbb{Z})$. *Assume that* $\theta(V, \omega_0, J_0) < \theta_2$. *Fix a* $K \in \mathcal{J}(M, \omega_0, V, J_0, \theta_0)$[1], $E > 0$ *and assume* $K \in \mathcal{J}_\tau(\omega_1)$. *Then the regularity assumption* $\mathcal{R}(M, \omega_0, V, K, E, D_*E + n)$[2] *implies that*

1. *if* $D > (D_*E + n - 4)$, *then all* K*-holomorphic spheres in* V *of energy at most* E *are constant.*
2. *If* $D > 2(D_*E + n - 2)$, *then every non-constant* K*-holomorphic sphere in* M *of energy at most* E *intersects* V *in at least* 3 *distinct points in the domain.*

Proof. The proof is very similar to the proof of Proposition 8.13 from [CM07]. For the first statement the only difference is the index calculation:

$$\begin{aligned} \mathrm{ind}(A) &= 2(n-1) - 6 + 2\langle c_1(TV), A\rangle \\ &= 2n - 8 + 2\langle c_1(M), A\rangle - 2D\omega_1(A) \\ &\leq 2n - 8 + 2D_*\omega_0(A) - 2D\omega_1(A) \\ &\leq 2(n - 4 + D_*E - D) \end{aligned}$$

where the inequalities follow from Lemma 3.27, $\omega_0(A) \leq E$ and $\omega_1(A) \geq 1$. Hence, the index is negative if $D > (D_*E + n - 4)$. Note that the constant $D_* = D_*(\omega_0, J_0, \theta_0)$ is chosen wrt. (ω_0, J_0).
For the second statement, arguing in the same manner, we consider $f : S^2 \to M$ a non-constant K-holomorphic curve representing class A of energy at most E intersecting V in at most 2 distinct points in the domain. We assume that f is simple, otherwise replace it by the underlying simple curve.
We consider the moduli space of simple K-holomorphic spheres representing A with the local intersection number at least $L \leq \lfloor D_*E \rfloor + n + 1$ with V at one point, say $\tilde{\mathcal{M}}^s(M, V, L, A, K)$. By Proposition 3.24 this space is a smooth manifold of dimension

$$\dim_{\mathbb{R}} \tilde{\mathcal{M}}^*(M, V, L, A, K) = 2n - 4 + 2c_1(A) - 2L \geq 0.$$

Hence, we have again by Lemma 3.27:

[1] Recall, that $\mathcal{J}(M, \omega_0, V, J_0, \theta_0)$ is the space of all ω_0-tame almost-complex structures J, leaving TV invariant, such that $\|J - J_0\|_0 < \theta_0$.

[2] We actually consider an upper bound for the tangency order given by $l \leq \lfloor D_*E \rfloor + n$.

$$L \leq c_1(A) + n - 2 \leq D_* \omega_0(A) + n - 2 \leq D_* E + n - 2.$$

Since K tames ω_1, we get $\omega_1(A) \geq 1$, so

$$[f] \cdot [V] = D\omega_1(A) > 2(D_* E + n - 2) \geq 2L \geq 2.$$

Hence, f intersects V in at least 3 distinct points in the domain. ∎

We get an analogous statement for families of almost complex structures. We recall from p. 89 of [CM07] the following:

Let $\mathcal{K} \subset \mathcal{J}(M, \omega_0, V, J_0, \theta_0)$ be a family of almost complex structures smoothly depending on a parameter $\tau \in P$ with P a smooth k-dimensional manifold. Then the moduli spaces of $\mathcal{K}$-holomorphic spheres are moduli spaces of pairs (u, τ) with $\tau \in P$ and u a K_τ-holomorphic sphere. The corresponding transformation groups should act on u only. So the expected dimension of such moduli spaces is increased exactly by k and we get the following

Proposition 3.30 (analog of Proposition 8.14 from [CM07], cf. Propositon 3.29). *Consider a symplectic hypersurface $V \subset M$ with $\mathrm{PD}(V) = D[\omega_1]$ for some integer $D > 0$, an integer class $[\omega_1] \in H^2(M, \mathbb{Z})$ and $\theta(V, \omega_0, J_0) < \theta_2$. Fix a $\mathcal{K} \subset \mathcal{J}(M, \omega_0, V, J_0, \theta_0)$, $E > 0$ and assume $\mathcal{K} \subset \mathcal{J}_\tau(\omega_1)$. Then the regularity condition $\mathcal{R}(M, \omega_0, V, \mathcal{K}, E, D_* E + n)$[1] implies that*

1. *if $D > (D_* E + n - 4 + k)$, then all $\mathcal{K}$-holomorphic spheres in V of energy at most E are constant,*
2. *if $D > 2(D_* E + n - 2) + k$, then every non-constant $\mathcal{K}$-holomorphic sphere in M of energy at most E intersects V in at least 3 distinct points in the domain.*

The next statement is an adaptation of Lemma 8.18 from [CM07].

Lemma 3.31. *Given two transversely intersecting symplectic hypersurfaces $V_0, V_1 \subset M$ with $PD(V_0) = D_0[\omega_0]$ and $PD(V_1) = D_1[\omega_1]$ for integers $D_0, D_1 > 0$ and an integer class $[\omega_1] \in H^2(M, \mathbb{Z})$. Fix a $K \in \mathcal{J}(M, V_0 \cup V_1, J, \theta_0)$[2] and an energy level $E > 0$. Assume the regularity condition $\mathcal{R}(V_0 \cap V_1, \omega_0, K, E)$ and that $K \in \mathcal{J}_\tau(\omega_1)$.*
Then $D_0 > max\{D_*, D_* + n - 5\}$ implies that any K-holomorphic sphere contained in $V_0 \cap V_1$ with energy at most E is constant.

Proof. First, observe the decomposition of the pullback bundle

$$TX_{|V_0 \cap V_1} = T(V_0 \cap V_1) \oplus N(V_0 \cap V_1) = T(V_0 \cap V_1) \oplus N(V_0) \oplus N(V_1),$$

but since $c_1(NV_i) = \mathrm{PD}[V_i]$ and $\mathrm{PD}[V_i] = D_i[\omega_i]$, it implies

[1] Here we consider $\mathcal{K}$-holomorphic spheres.

[2] We actually assume that the perturbation space $\mathcal{J}(M, V_0 \cup V_1, J, \theta_0)$ is not empty. In the last chapter we show that such V_0, V_1 and $\theta_0 < 1$ exist.

$$c_1(T(V_0 \cap V_1)) = c_1(TX_{|V_0 \cap V_1}) - D_0[\omega_0] - D_1[\omega_1].$$

By regularity condition, the expected dimension of simple K-holomorphic spheres representing class A is given by

$$\begin{aligned}
\operatorname{ind}(A) &= 2(n-2) - 6 + 2c_1(T(V_0 \cap V_1))(A) \\
&= 2n - 10 + 2c_1(TX_{|V_0 \cap V_1})(A) - 2D_0\omega_0(A) - \underbrace{\mathrm{PD}[V_1](A)}_{=D_1\omega_1(A)>0} \\
&\leq 2n - 10 + 2(c_1(TX_{|V_0 \cap V_1})(A) - D_0\omega_0(A)) \\
&\leq 2n - 10 + 2(D_* - D_0)\omega_0(A).
\end{aligned}$$

The last inequality follows from Lemma 3.27 (with $D_* = D_*(\omega_0, J_0, \theta_0)$) and $c_1(TX)(A) = c_1(TX_{|V_0 \cap V_1})(A)$ for any A represented by a K-holomorphic curve lying in $V_0 \cap V_1$.
Now, observe that $\omega_0(A) \geq 1$ for $A \neq 0$. Then $D_0 > \max\{D_*, D_* + n - 5\}$ would imply $\operatorname{ind}(A) < 0$ and hence together with the regularity assumption the claim follows. ■

Remark 3.32. Observe that the lower bound for the degree of V_0 does not depend on V_1, as long as $\mathcal{J}(M, V_0 \cup V_1, J, \theta_0) \neq \emptyset$.

We fix constants $0 < \theta_2 < \theta_1 < \theta_0 < 1$, $\eta > 0$, such that (everything measured wrt. (ω_0, J_0))

$$\angle_m(V_0, V_1) \geq \eta \text{ and } \theta(V_i, \omega_0, J_0) < \theta_2 \text{ for } i = 1, 2$$

and the space (of ω_0-tame almost complex structures) $\mathcal{J}(M, V_0 \cap V_1)$ contains a nonempty open subspace $\mathcal{J}(M, V_0 \cup V_1, J_0, \theta_1)$, whose any two elements can be connected in the space $\mathcal{J}(M, V_0 \cup V_1, J_0, \theta_0)$. Recall that $\mathrm{PD}[V_0] = D_0[\omega_0]$ and $\mathrm{PD}[V_1] = D_1[\omega_1]$ for integer classes $[\omega_0], [\omega_1] \in H^2(M, \mathbb{Z})$. We will see in the last chapter that such a choice of constants exists. Next, we give slight generalizations of the Definitions 8.15 and 8.19 from [CM07].

Definition 3.33. *For the hypersurface V_1 and fixed $E > 0$ we define*

$$\mathcal{J}^*(M, V_1, J_0, \theta_1, E) \subset \mathcal{J}(M, V_1, J_0, \theta_1)$$

to be the space of ω_0-tame almost complex structures K, such that

1. *all K-holomorphic spheres of energy at most E contained in V_1 are constant,*
2. *every non-constant K-holomorphic sphere of energy at most E in M intersects V_1 in at least 3 distinct points in the domain.*

For the pair of hypersurfaces (V_0, V_1) and again a fixed $E > 0$ we set

$$\mathcal{J}^*(M, V_0 \cup V_1, J_0, \theta_1, E) \subset \mathcal{J}(M, V_0 \cup V_1, J_0, \theta_1)$$

as the subset of those K, such that the following holds

1. *all K-holomorphic spheres of energy at most E contained in $V_0 \cup V_1$ are constant,*
2. *every non-constant K-holomorphic sphere of energy at most E in M intersects each V_i in at least 3 distinct points in the domain for $i = 1, 2$.*

Finally, we define the constant

$$D^* = D^*(\omega_0, J_0, E, \theta_0) := 2D_*(\omega_0, J_0, \theta_0)E + 2n.$$

The following statement is a direct analog of Corollaries 8.16 and 8.20. Proofs easily carry over to our situation.

Lemma 3.34. *Fix an energy level $E > 0$. Consider a hypersurface V_1 as above and assume $D_1 > D^* = D^*(\omega_0, J_0, E)$. Then the spaces $\mathcal{J}^*(M, V_1, J_0, \theta_1, E')$ are open and dense in $\mathcal{J}(M, V_1, J_0, \theta_1)$ for all $0 < E' \leq E$. Moreover, any two elements in $\mathcal{J}^*(M, V_1, J_0, \theta_1, E')$ can be connected by a path in $\mathcal{J}^*(M, V_1, J_0, \theta_2, E')$.*
For a pair of hypersurfaces V_0, V_1 the assumption $D_i > D^* = D^*(\omega_0, J_0, E)$ implies that the spaces $\mathcal{J}^*(M, V_0 \cap V_1, J_0, \theta_1, E')$ are open and dense in $\mathcal{J}(M, V_0 \cap V_1, J_0, \theta_1)$ for all $0 < E' \leq E$. Furthermore, any two elements from $\mathcal{J}^*(M, V_0 \cap V_1, J_0, \theta_2, E')$ can be connected by a path in $\mathcal{J}^*(M, V_0 \cap V_1, J_0, \theta_2, E')$.

Proof. We start with a single hypersurface V_1.
Openness. Assume there exists for some $K \in \mathcal{J}^*(M, V_1, J_0, \theta_1, E')$ a sequence $K^\nu \in \mathcal{J}(M, V_1, J_0, \theta_1)$ of non-constant K^ν-holomorphic spheres of energy at most E' in V_1 with $K^\nu \to K$. Hence, Gromov compactness would imply existence of a non-constant K-holomorphic sphere of energy at most $E' < E$, contradicting condition (1) in the Definition 3.33. So this condition is open.
Now, assume the existence of a sequence $K^\nu \in \mathcal{J}(M, V_1, J_0, \theta_1)$ converging to a $K \in \mathcal{J}^*(M, V_1, J_0, \theta_1, E')$ not satisfying the condition (2) from Definition 3.33. So, there exists a sequence of K^ν-holomorphic spheres of energy at most E' intersecting V_1 in at most 2 points. Again, Gromov compactness yields a non-constant K-holomorphic curve intersecting V_1 in at most 2 points, hence a contradiction. This shows that condition (2) is also open.
Density. The argument is literally the same as in [CM07]. The main point is that the set of all $K \in \mathcal{J}(M, V_1, J_0, \theta_1)$ satisfying the regularity condition $\mathcal{R}(M, \omega_0, V_0, K, E, D_*E + n)$ is actually dense. Indeed, the first point follows from Theorem 3.1.5 from [MS04] and the second from Proposition 6.9 in [CM07]. Observe that, once the regularity condition is satisfied, the fact that V_1 is not Poincaré dual to a multiple of ω_0, plays no role.
Connectedness. Given $K_0, K_1 \in \mathcal{J}^*(M, V_1, J_0, \theta_2, E')$, Lemma 2.15 implies that they can be connected by a path $K_t \in \mathcal{J}(M, V_1, J_0, \theta_1)$ for $t \in [0, 1]$. Again as in [CM07] we can achieve the regularity condition by arbitrary small perturbation of the path, say K_t', such that K_i' is arbitrary close to K_i for $i = 1, 2$ (so $K_t' \in \mathcal{J}^*(M, V_1, J_0, \theta_1, E')$). So, openness of $\mathcal{J}^*(M, V_1, J_0, \theta_1, E')$

implies that K_0 and K_1 can be connected by a path in $\mathcal{J}^*(M, V_1, J_0, \theta_1, E')$. The situation of the pair V_0, V_1 is analogous to that in Corollary 8.20 in [CM07]. The only difference is that we, according to Lemma 3.31, have to rule out non-constant spheres in $V_0 \cap V_1$. ∎

Hence, we arrive at the following definitions for perturbation spaces consisting of coherent tame almost complex structures required in the last chapter.

Definition 3.35 (cf. Definition 9.9 in [CM07]). *Fix $l \geq 3$, consider $\mathcal{J}_{l+1}$ the set of coherent almost complex structures from Section 3.2 and define*

$$\begin{aligned}\mathcal{J}_{l+1}(M, V_0 \cup V_1, J_0, \theta_1) := \{\, K \in \mathcal{J}_{l+1} \big| K(\zeta) \in \mathcal{J}(M, V_0 \cup V_1, J_0, \theta_1) \ \forall \zeta \in \bar{\mathcal{M}}_{l+1}, \\ K(\zeta)_{|V_0 \cup V_1} \text{ is independent of } \zeta \,\}\,.\end{aligned}$$

*For $\theta_2 < \theta_1$ and $E > 0$ a subset $B \subset \mathcal{J}^*(M, V_0 \cup V_1, J_0, \theta_1, E)$ is called **θ_2-contractible** if it is contractible to a point lying in $\mathcal{J}^*(M, V_0 \cup V_1, J_0, \theta_2, E)$. Using this define*

$$\begin{aligned}\mathcal{J}^*_{l+1}(M, V_0 \cup V_1, J_0, \theta_1, \theta_2, E) := \{\, K \in \mathcal{J}_{l+1} \big| K(\zeta) \in B \ \forall \zeta \in \bar{\mathcal{M}}_{l+1}, \\ B \subset \mathcal{J}^*(M, V_0 \cup V_1, J_0, \theta_1, E) \ \theta_2\text{-contractible}, \\ K(\zeta)_{|V_0 \cup V_1} \text{ is independent of } \zeta \,\}\,.\end{aligned}$$

Note that by Corollary 2.16 we can define for $0 \leq \theta_1 < \frac{1}{2}$

$$\mathcal{J}^*_{l+1}(M, V_0 \cup V_1, J_0, \theta_1, E) := \mathcal{J}^*_{l+1}(M, V_0 \cup V_1, J_0, \theta_1, \frac{2}{5}\theta_1, E).$$

This combined with Lemma 3.34 implies

Lemma 3.36. *For fixed $l \geq 3$ and $E > 0$ the subset*

$$\mathcal{J}^*_{l+1}(M, V_0 \cup V_1, J_0, \theta_1, E) \subset \mathcal{J}_{l+1}(M, V_0 \cup V_1, J_0, \theta_1)$$

is nonempty and open.

4

Moduli spaces and Donaldson hypersurfaces

4.1 Moduli spaces and Donaldson pairs

In this section we recollect definitions and theorems from [CM07] needed later on. We consider a symplectic manifold (M, ω) with $[\omega] \in H^2(M, \mathbb{Z})$ and $\dim_{\mathbb{R}} M = 2n \geq 4$.

Denote by $\bar{\mathcal{M}}_l$ the Deligne-Mumford space of stable genus zero curves with l marked points. Let $\pi_l : \bar{\mathcal{M}}_{l+k} \to \bar{\mathcal{M}}_l$ be the projection that forgets first k marked points. For a symplectic hypersurface $V \subset M$ we define the space

$$\begin{aligned}\mathcal{M}_{k+l}(A, K, V) := \big\{&(z_1, .., z_{k+l}, f) \mid f \in \mathcal{C}^\infty(S^2, M), [f] = A,\\ &f \text{ is } \pi_l^* K\text{-holomorphic, } z_i \in S^2 \text{ pairwise distinct and}\\ &\phi(z_{k+1}), \dots, \phi(z_l) \in V\big\}/\mathrm{Aut}(S^2),\end{aligned}$$

for a fixed $A \in H_2(M, \mathbb{Z})$, $V \subset M$ symplectic hypersurface and $K \in \mathcal{J}_{l+1}(M, V, J, \theta)$.
Let T be a $(k+l)$-labelled l-stable tree (cf. Chapter 3), and $A_\alpha \in H_2(M, \mathbb{Z})$ for vertices $\alpha \in T$. Define the space

$$\begin{aligned}\mathcal{M}_T(\{A_\alpha\}, K, V) := \big\{&(\mathbf{z}, \mathbf{f}) \mid (\mathbf{z}, \mathbf{f}) \text{ nodal map modelled over } l\text{-stable}\\ &\text{weighted tree } (T, \{A_\alpha\}), \mathbf{f} \text{ is } \pi_l^* K\text{-holomorphic, i.e.}\\ &(\mathbf{z}, \mathbf{f}) \in \mathcal{M}_T(\{A_\alpha\}, K) \text{ and the last } l \text{ marked points:}\\ &f_{\alpha_{k+1}}(z_{\alpha_{k+1}}), \dots, f_{\alpha_{k+l}}(z_{\alpha_{k+l}}) \in V\big\}.\end{aligned}$$

Definition 4.1. *Fix an energy level $E > 0$ and constants $0 < \Theta_2 < \Theta_1 < \Theta_0 < 1$. A **Donaldson pair** of degree $D > 0$ is a tuple (V, J) with $J \in \mathcal{J}_c(\omega)$ and an ω-symplectic hypersurface $V \subset M$, such that the following holds*

- *$PD([V]) = D[\omega]$ and $D > D^*(\omega, J, E, \Theta_0)$ (degree assumption)*

- *$\theta(V) = \theta(V, J, \omega) < \Theta_2$ (smallness of the Kähler angle)*
- *The space $\mathcal{J}(M, V, J, \Theta_1)$ is nonempty and any two elements from it can be connected by a path lying in $\mathcal{J}(M, V, J, \Theta_0)$.*

Remark 4.2. Existence of such pairs follows from the Donaldson hypersurface theorem combined with results from sections 3.7 and 2.1 (see end of this chapter). Although energy E was not in the original definition from [CM07], we included it here in order to match our general construction. In the case of only one hypersurface (Poincaré dual to $D[\omega]$) one can indeed take $E = \omega(A)$, obtaining identical statements.

Observe that for $l := D\omega(A)$ for any $A \in H_2(M, \mathbb{Z})$ with $\omega(A) \leq E$ assumption $D \geq D^*$ implies that the space $\mathcal{J}^*_{l+1}(M, V, J, \Theta_1, E)$ is nonempty. Note that l is the intersection number of V and A, since $D\omega(A) = \int_A D\omega = \int_A PD[V]$.

Theorem 4.3 (cf. Theorem 1.1 **in [CM07]).** *Fix an energy level $E > 0$. Then for any Donaldson pair (V, J) of degree D and integer multiples $l \geq 3$ of D there exist nonempty sets $\mathcal{J}^{\text{reg}}_{l+1}(M, V, J, \Theta_1) \subset \mathcal{J}_{l+1}(M, V, J, \Theta_1)$, such that for any $K \in \mathcal{J}^{\text{reg}}_{l+1}(M, V, J, \Theta_1)$ the following holds.*
Consider $A \in H_2(M, \mathbb{Z})$ with $l = D\omega(A)$ and $\omega(A) \leq E$. For $k \geq 3$ let T be an $(k+l)$ labelled l-stable tree with $A_\alpha \in H_2(M, \mathbb{Z})$ for $\alpha \in T$, such that $\sum A_\alpha = A$ and every ghost tree contains at most one of the last l marked points. Then the moduli space $\mathcal{M}_T(\{A_\alpha\}, K, V)$ is a smooth manifold of dimension

$$dim_{\mathbb{R}} \mathcal{M}_T(\{A_\alpha\}, K, V) = 2\,(n - 3 + k + c_1(A) - e(T))\,.$$

Proof. For the proof we refer to pp. 96-97 (Section 9) of [CM07]. Note that the difference is just of formal nature. The original proof uses the spaces $\mathcal{J}^*(M, V, J, \Theta_1, E_l)$ with $E_l := l/D = \omega(A)$. We are using the spaces $\mathcal{J}^*(M, V, J, \Theta_1, E)$ with $E_l < E$ instead, which are still open and dense by Lemma 3.34 and our (stronger) assumption $D \geq D^*(\omega, J, \Theta_0, E)$. ■

The key step in establishing Gromov-Witten invariants is to show that the evaluation map

$$ev^k : \mathcal{M}_{k+l}(A, K, V) \longrightarrow M^k$$

that evaluates first k marked points forms a pseudocycle. See Appendix A.2 for details on pseudocycles.

Theorem 4.4 (cf. Theorem 1.2 **in [CM07]).** *Fix an energy level $E > 0$. Given a Donaldson pair (V, J). Then for any $k \geq 1$ the evaluation map $ev^k : \mathcal{M}_{k+l}(A, K, V) \longrightarrow M^k$ forms a pseudocycle $ev^k(A, J, V, K)$ of real dimension $2(n - 3 + k + c_1(A)) =: 2d$.*

Proof. We only sketch the proof here, see pp. 97-98 (Section 9) of [CM07] for details. The first issue is to show that l-stability is preserved by Gromov convergence.

Assume that the sequence $(z^j, f^j) \in \mathcal{M}_{k+l}(A, K, V)$ converges to a stable map (z, f), such that z is not l-stable. By Gromov compactness (see Section 3.5) f is $K_{\pi_l(z)}$-holomorphic. The assumption implies that there is a non-constant component $f_\alpha : S_\alpha \to M$ with the domain S_α containing an intersection point with V, which is neither a node nor one of the last l points. Gromov convergence implies the same statement for f^j for large j. Hence, one gets $[V] \cdot [f^j] > [V] \cdot A =: l$, which is a contradiction, since $[f^j] = A$ (homology class is preserved by Gromov convergence).
It follows from Theorem 4.3 that for any stable tree T (with $e(T) > 0$) the associated space $\mathcal{M}_T(\{A_\alpha\}, K, V)$ has at least codimension 2 in the space $\mathcal{M}_{k+l}(A, K, V)$, if any ghost tree $T' \subset T$ contains at most one of the last l marked points.
If the ghost tree T' has more than one vertex, the statement follows immediately from Proposition 3.24. The case $|T'| = 1$ occurs if $A_\alpha \neq 0$ for exactly one vertex α, but by assumption that two of the last marked points belong to T', imply the intersection number of f_α with V is at least 2, leading again to a strata of codimension at least 2 (see p. 98 of [CM07]). ■

It was shown (see Theorem 1.3 in [CM07]) that up to a rational cobordism the pseudocycle $\frac{1}{l!}\mathrm{ev}^k(A, J, V, K)$ does not depend on a particular choice of the perturbation K and the Donaldson pair (V, J).

The following is actually not needed for the proof of the main result. However, considerations from Section 3.7 allow us to show analogous results as above for a more general version of a Donaldson pair.

Definition 4.5. *Fix an energy level $E > 0$ and constants $0 < \Theta_2 < \Theta_1 < \Theta_0 < 1$. A* ***(generalized) Donaldson pair*** *of degree $D > 0$ is a tuple (V, J) with $J \in \mathcal{J}_c(\omega)$ and an ω-symplectic hypersurface $V \subset M$, such that the following holds*

- *$PD([V]) = D[\omega']$ with ω' a symplectic form and $[\omega'] \in H_2(M, \mathbb{Z})$*
- *$D > D^*(\omega, J, E, \Theta_0)$*
- *$\theta(V) = \theta(V, J, \omega) < \Theta_2$*
- *The space $\mathcal{J}(M, V, J, \Theta_1)$ is nonempty and any two elements from it can be connected by a path lying in $\mathcal{J}(M, V, J, \Theta_0)$*
- *$\mathcal{J}(M, V, J, \Theta_0) =: \mathcal{J}(M, V, \omega, J, \Theta_0) \subset \mathcal{J}_\tau(M, \omega')$.*

The generalization of Theorems 4.3 and 4.4 to such a pair is straightforward. Once one assumes for any $A \in H_2(M, \mathbb{Z})$ that $\max\{\omega(A), \omega'(A)\} < E$ and sets $l = D\omega'(A)$, the proofs become identical, since the condition $D \geq D^*$ insures that the corresponding perturbation spaces are open and dense in $\mathcal{J}(M, V, J, \Theta_0)$. One also could take arguments from the next section for the case $V_0 = \emptyset$.

4.2 Moduli spaces and Donaldson quadruples

In this section we sligthly extend the notion of Donaldson quadruple introduced in [CM07] (cf. Definition 9.7, p. 99). In order to keep track of different symplectic structures, we fix a symplectic manifold (M, ω_0) with $[\omega_0] \in H^2(M, \mathbb{Z})$.

Definition 4.6. *Fix an energy level $E > 0$ and constants $0 < \Theta_3 < \Theta_2 < \Theta_1 < \Theta_0 < 1$, $0 < \eta$. A* ***Donaldson quadruple*** *of bi-degree $D_0, D_1 > 0$ consists of $J_0 \in \mathcal{J}_c(\omega)$ and ω_0-symplectic hypersurfaces $V_0, V_1 \subset M$, such that following conditions hold*

- $\angle_m(V_0, V_1) \geq \eta$
- $PD([V_0]) = D_0[\omega_0]$
- $PD([V_1]) = D_1[\omega_1]$ *with ω_1 a symplectic structure and $[\omega_1] \in H_2(M, \mathbb{Z})$*
- $\min(D_0, D_1) \geq \max\{D^*(\omega_0, J_0, E, \Theta_0), D^*(\omega_1, J_1, E, \Theta_0)\}$ *for $J_1 \in \mathcal{J}_c(\omega_1)$ with*[1] $\|J_0 - J_1\|_0 \leq \Theta_3$
- $\theta(V_i) = \theta(V_i, J_0, \omega_0) < \Theta_3$ *for $i = 0, 1$*
- *The space $\mathcal{J}(M, V_0 \cup V_1, J_0, \Theta_2)$ is nonempty and any two elements from it can be connected by a path lying in $\mathcal{J}(M, V_0 \cup V_1, J_0, \Theta_1)$*
- $\mathcal{J}(M, V_0 \cup V_1, J_0, \Theta_0) =: \mathcal{J}(M, V_0 \cup V_1, \omega_0, J_0, \Theta_0) \subset \mathcal{J}_\tau(M, \omega_1)$.

Remark 4.7. Taking $\omega_1 = \omega_0$ and $J_0 = J_1$ yields (up to E-dependency) the original definition. In that case the last condition is empty.
Existence of quadruples as defined above is shown at the end of the chapter. The definition might look asymmetric concerning V_0, V_1, since we measure anything wrt. (ω_0, J_0). A symmetric version (with $\mathrm{PD}([V_i]) = D_i[\omega_i]$ for $i = 1, 2$) is possible, but is not needed, since by assumption on degrees D_1 it follows that for a $J_1 \in \mathcal{J}_c(\omega_1)$ with $\|J_0 - J_1\|_0 \leq \Theta_3$ the tuple (V_1, J_1) is a Donaldson pair wrt. symplectic form ω_1 and a slightly smaller Θ_1.

In analogy to the previous section we define moduli space associated to a pair of symplectic hypersurfaces $V_0, V_1 \subset M$.

Definition 4.8. *Fix $k, l_0, l_1 > 0$, $A \in H_2(M, \mathbb{Z})$, and denote $z \in \bar{\mathcal{M}}_{k+l_0+l_1+1}$ via and $K \in \mathcal{J}_\tau(M, \omega)$ let*

$$\begin{aligned}\mathcal{M}_{k+l_0+l_1}(A, K, V_0 \cup V_1) := \big\{(f, z_0, \ldots, z_{k+l_0+l_1})\big| & f : S^2 \to M, \bar{\partial}_K f = 0, \\ & [f] = A,\ z_i \in S^2 \text{ pairwise distinct}; \\ & f(z_{k+1}), \ldots, f(z_{k+l_0}) \in V_0; \\ & f(z_{k+l_0+1}), \ldots, f(z_{k+l_0+l_1}) \in V_1\big\}/Aut(S^2).\end{aligned}$$

Denote by $\bar{\mathcal{M}}_{k+l_0+l_1+1}$ the Deligne-Mumford space. For $I \subset \{k+1, \ldots, k+l_0+l_1\}$ let $\pi_I : \bar{\mathcal{M}}_{k+l_0+l_1+1} \longrightarrow \bar{\mathcal{M}}_{|I|+1}$ be the standard projection that forgets

[1] We denote by $\|\cdot\|_i$ the norm induced by (ω_i, J_i) for $i = 0, 1$.

marked points outside I and stabilizes. Given an I-stable $(k+l_0+l_1)$-labelled tree T, $A_\alpha \in H^2(M,\mathbb{Z})$ with $\alpha \in T$ and $\sum A_\alpha = A$. We define[1]

$$\begin{aligned}\mathcal{M}_T(\{A_\alpha\},K,V_0\cup V_1) := \{\,(\mathbf{z},\mathbf{f}) \in \mathcal{M}_T(\{A_\alpha\},K) \;\big|\; & \\ & f_{\alpha_{k+1}}(z_{k+1}),\dots,f_{\alpha_{k+l_0}}(z_{k+l_0}) \in V_0, \\ & f_{\alpha_{k+l_0+1}}(z_{k+l_0+1}),\dots,f_{\alpha_{k+l_0+l_1}}(z_{k+l_0+l_1}) \in V_1\ \}\,.\end{aligned}$$

The next theorem is a transversality result for J-holomorphic spheres to a Donaldson quadruple. It extends Theorem 9.8 from [CM07] to our definition of the Donaldson quadruple.

Theorem 4.9. *Fix an energy level $E > 0$. Given a Donaldson quadruple $(\omega_0, J_0, V_0, V_1)$. For any fixed $A \in H_2(M,\mathbb{Z})$ with $\omega_0(A) > 0$ and*

$$\max\{\omega_0(A), \omega_1(A)\} \le E,$$

let $l_0 := deg(V_0)\omega_0(A)$ and let $l_1 := deg(V_1)\omega_1(A)$. For any $\bar{l} \ge 3$ there exist nonempty sets

$$\mathcal{J}^{\mathrm{reg}}_{\bar{l}+1}(M, V_0\cup V_1, J_0, \Theta_1, E) \subset \mathcal{J}_{\bar{l}+1}(M, V_0 \cup V_1, J_0, \Theta_1)$$

with the following property. Fix $k \ge 0$, a subset $I \subset \{k+1,\dots,k+l_0+l_1\}$ of length $|I| \ge max\ (3, min\ (l_0, l_1))$, and an I-stable $(k+l_0+l_1)$-labelled tree T. Take classes $A_\alpha \in H_2(M,\mathbb{Z})$ for $\alpha \in T$ with $\sum A_\alpha = A$ and assume that any ghost tree in $(T,\{A_\alpha\})$ contains at most one of the last l_0+l_1 marked points. Then for any $K \in \mathcal{J}^{reg}_{|I|+1}(M, V_0\cup V_1, J_0, \Theta_1, E)$ the moduli space $\mathcal{M}_T(\{A_\alpha\}, K, V_0 \cup V_1)$ is a smooth manifold of real dimension

$$dim_{\mathbb{R}}\mathcal{M}_T(\{A_\alpha\},K,V_0\cup V_1) = 2\,(n-3+k+c_1(A)-e(T))\,.$$

Proof. First, we set $E' := \max(\omega_0(A), \omega_1(A))$. Then it follows from Lemma 3.34 that the subset of ω-tame almost complex structures

$$\mathcal{J}^*(M, V_0\cup V_1, J, \Theta_1, E) \subset \mathcal{J}(M, V_0\cup V_1, J, \Theta_1)$$

is open and dense, moreover by Lemma 3.36 it follows that the subset of coherent ω-tame almost complex structures

$$\mathcal{J}^*_{|I|+1}(M, V_0\cup V_1, J, \Theta_1, E) \subset \mathcal{J}_{|I|+1}(M, V_0\cup V_1, J, \Theta_1)$$

is nonempty and open. Now, fix I as above and recall that

$$\pi_I : \bar{\mathcal{M}}_{k+l_0+l_1+1} \longrightarrow \bar{\mathcal{M}}_{|I|+1}$$

[1] See Section 3.4 for the definition of the space $\mathcal{M}_T(\{A_\alpha\}, K)$.

is the projection given by forgetting marked points outside of I and stabilizing. Note that for a given $K \in \mathcal{J}^*_{|I|+1}(M, V_0 \cup V_1, J, \Theta_1, E)$ a curve $f : S^2 \to M$ is called K-holomorphic, if $\bar{\partial}_{\pi_I^* K} f = 0$. So, for a fixed I-stable tree T and classes $\{A_\alpha\}$ as above take $K \in \mathcal{J}^*_{|I|+1}(M, V_0 \cup V_1, J, \Theta_1, E)$ and consider the moduli space of stable maps modelled over $(T, \{A_\alpha\})$, i.e. the space $\mathcal{M}_T(\{A_\alpha\}, K, V_0 \cup V_1)$. It follows by Lemma 3.34 that any non-constant component $(\mathbf{z}, \mathbf{f}) \in \mathcal{M}_T(\{A_\alpha\}, K, V_0 \cup V_1)$ intersects the complement $M - (V_0 \cup V_1)$, since $\omega_0(A_\alpha) \leq E'$ (and $\omega_1(A_\alpha) \leq E'$) for any α. Hence, for any $K \in \mathcal{J}^*(M, V_0 \cup V_1, J_0, \Theta_1, E)$ Proposition 3.24 yields the Baire set $\mathcal{J}^{\mathrm{reg}}_{|I|+1}(M - V_0 \cup V_1, \{V_0, V_1\}, K) \subset \mathcal{J}_{|I|+1}(M - V_0 \cup V_1)$ with k replaced by $k + l_0 + l_1$. Then define

$$\mathcal{J}^{\mathrm{reg}}_{|I|+1}(M, V_0 \cup V_1, J_0, \Theta_1, E') := \bigcup_{K \in \mathcal{J}^*(M, V_0 \cup V_1, J_0, \Theta_1, E')} \left[\mathcal{J}^{\mathrm{reg}}_{|I|+1}(M - V_0 \cup V_1, \{V_0, V_1\}, J_0) \cap \mathcal{J}_{|I|+1}(M, V_0 \cup V_1, J_0, \Theta_1, E')\right].$$

Then, for any $K \in \mathcal{J}^{\mathrm{reg}}_{|I|+1}(M, V_0 \cup V_1, J_0, \Theta_1, E)$ and any I-stable tree T as above, Proposition 3.24 implies that the space $\mathcal{M}_T(\{A_\alpha\}, K, V_0 \cup V_1)$ is a smooth manifold of dimension

$$\begin{aligned} \dim_{\mathbb{R}} \mathcal{M}_T(\{A_\alpha\}, K, V_0 \cup V_1) &= 2n - 6 + 2c_1(A) + 2(k + l_0 + l_1) - 2e(T) \\ &\quad - l_0(2n - \dim_{\mathbb{R}}(V_0)) - l_1(2n - \dim_{\mathbb{R}}(V_1)) \\ &= 2\,(n - 3 + c_1(A) + k - e(T))\,. \end{aligned}$$

■

The next statement is a compactness result. It shows that in our special situation the space of domain stable maps is actually compact. It is basically the statement of Proposition 9.10 in [CM07] for the case $\omega_0 \neq \omega_1$.

Theorem 4.10. *Fix an energy level $E > 0$ and a Donaldson quadruple $(\omega_0, J_0, V_0, V_1)$. For $A \in H_2(M, \mathbb{Z})$ assume $\max\{\omega_0(A), \omega_1(A)\} \leq E$ and set $l_0 := \deg(V_0)\omega_0(A)$, $l_0 := \deg(D_0)\omega_0(A)$. For $k \geq 0$ take a subset*

$$I \subset \{k+1, \ldots, k + l_0 + l_1\} \text{ with } \{k+1, \ldots, k + l_0\} \subset I$$

and fix $K \in \mathcal{J}^*_{|I|+1}(M, V_0 \cup V_1, J_0, \theta_1)$.
Assume that a sequence of K-holomorphic spheres $f^\nu \in \mathcal{M}_{k+l_0+l_1}(A, K, V_0 \cup V_1)$ has a Gromov-limit - the stable map $(\mathbf{f}, \mathbf{z})$. Then the underlying nodal curve $\mathbf{z}$ is I-stable.
Moreover, the same statement holds if $\{k + l_0 + 1, \ldots, k + l_0 + l_1\} \subset I$.

Proof. The proof is very similar to the proof of Proposition 9.10 in [CM07]. Assume that $\mathbf{z}$ is not I-stable, i.e. there is a non-constant component of $(\mathbf{f}, \mathbf{z})$, say (f_α, S_α), such that S_α contains at most two special points, ignoring points

from I.
By compactness (see Section 3.5) it follows that $\mathbf{f}$ is $K_{\pi_I(\mathbf{z})}$-holomorphic. Recall that π_I removes marked points not contained in I and stabilizes, hence the image of $\pi_I(S_\alpha)$ is a point. So there exists a $K_\alpha \in \mathcal{J}^*(M, V_0 \cup V_1, J_0, \theta_1, E)$, such that $\bar{\partial}_{K_\alpha} f_\alpha = 0$ and K_α does not depend on the points of S_α.
Since f_α is non-constant, Lemma 3.34 implies that $f_\alpha(S_\alpha) \not\subset V_0 \cup V_1$ and $f_\alpha(S_\alpha)$ intersects each V_0 and V_1 in at least three distinct points in the domain. Hence, there exist extra intersection points, say $x_0 \in V_0$ and (in the second case) $x_1 \in V_1$, which are neither nodes nor marked points contained in $\{k+1, \ldots, k+l_0\}$ resp. $\{k+l_0+1, \ldots, k+l_0+l_1\}$.
Since the intersection number does not change under small perturbations (cf. Section 3.6), for sufficiently large ν such intersection points occur for f^ν, say x_0^ν and x_1^ν. Observe that f^ν is by definition a K-holomorphic curve with distinct marked points, and that the Proposition 3.26 implies that each marked point from $\{k+1, \ldots, k+l_0\}$ resp. $\{k+l_0+1, \ldots, k+l_0+l_1\}$ contributes to the intersection number by at least 1.
Hence, existence of extra intersection points (after choosing ν sufficiently large) x_0^ν and x_1^ν would imply

$$[V_0] \cdot [f^\nu] > l_0 = [V_0] \cdot A \text{ and } [V_1] \cdot [f^\nu] > l_1 = [V_1] \cdot A,$$

which is a contradiction to the assumption $[f^\nu] = A$. ∎

The next theorem is an analog of Proposition 9.11 and Theorem 9.12 from [CM07], adapted to our definition of a Donaldson quadruple. Again, most arguments carry over, however, we give a detailed proof for the sake of completeness.

Theorem 4.11. *Fix an energy level $E > 0$, consider a Donaldson quadruple $(\omega_0, J_0, \mathcal{V}_0, \mathcal{V}_1)$. Fix $A \in H_2(M, \mathbb{Z})$ with $\omega_0(A) > 0$ and $\max\{\omega_0(A), \omega_1(A)\} \leq E$. Set $l_0 := \deg(V_0)\omega_0(A)$ and $l_1 := \deg(V_1)\omega_1(A)$, then for any $k \geq 1$ and any $K \in \mathcal{J}^{\mathrm{reg}}_{|I|+1}(M, V_0 \cup V_1, J_0, \Theta_1, E)$, the evaluation map that evaluates first k-marked points*

$$ev^k : \mathcal{M}_{k+l_0+l_1}(A, K, V_0 \cup V_1) \longrightarrow X^k$$

defines a pseudocycle of real dimension $d := 2(n - 3 + k + c_1(A))$.

Proof. We start with computing the dimension of the strata.

(I) Consider an I-stable $(k + l_0 + l_1)$-labelled tree T with $e(T) > 0$ and fix a decomposition $\sum_\alpha A_\alpha = A$ with $\alpha \in T$.

(I.A) If any ghost tree in T contains at most one of the middle l_0 or last l_1 points, then we are exactly in the situation of Theorem 4.9, i.e. the corresponding moduli space is a smooth manifold of dimension

$$\dim_{\mathbb{R}} \mathcal{M}_T(\{A_\alpha\}, K, V_0 \cup V_1) = 2(n - 3 + k + c_1(A)) - 2e(T) \leq d - 2.$$

Recall that the reduced index set R is a subset of $k + l_0 + l_1$ marked points, which contains only one marked point (of maximal index) per ghost tree and all other marked points on non-constant components. Denote by $T_R := \pi_R(T)$ the stable tree corresponding to the reduced index set R.
Since $K \in \mathcal{J}^{reg}_{|I|+1}(M, V_0 \cup V_1, J_0, \theta_1, E)$, the second statement in Proposition 3.24 implies that the evaluation map ev^k factors through the smooth manifold

$$\mathcal{M}^*_{T_R}(\{A_\alpha\}, K, \{(M,k), (V_0, l_0), (V_1, l_1)\}).$$

Now, assume the contrary of $(I.A)$ - that there is a ghost tree $T' \subset T$ that contains at least two of the last $l_0 + l_1$ marked points.

(I.B) If $e(T_R) > 0$, then the dimension formula in Proposition 3.24 implies that the corresponding moduli space has the (real) dimension $d - e(T_R) \leq d-2$.

(I.C) Consider the case $e(T_R) = 0$, i.e. all other components of $(T, \{A_\alpha\})$ are ghost components, except one, say $A_{\alpha'} \neq 0$. So this component contains, by assumption, at least two of the last $l_0 + l_1$ marked points. Consider the following three subcases:

(I.C.1) Assume that the ghost tree T' contains two of the middle l_0 marked points, say z_l and $z_{l'}$. Let $z_{\alpha_0 i}$ be the special point at the node where the ghost tree T' is attached to the (only) non-constant component α_0. Using Proposition 3.26 we get a lower bound for the local intersection number at $z_{\alpha i}$:

$$\iota(\mathbf{f}, V_0, z_{\alpha i}) \geq \iota(\mathbf{f}, V_0, z_l) + \iota(\mathbf{f}, V_0, z_l') \geq 2,$$

i.e. the tangency order of $\mathbf{f}$ to V_0 is at least 1 in at least one intersection point. Consider the following collection

$$C := \{(V_0, v_0), \ldots, (V_0, v_{l_0-1}), (V_1, v_{l_0}), \ldots, (V_1, v_{l_0+l_1})\},$$

with $v_i \geq -1$ the corresponding orders of tangency at the last $l_0 + l_1$ marked points. Above discussion implies that at least for one $0 \leq j \leq l_0 - 1$ we have $v_j \geq 1$. Now, we are in the situation of the second case of the Proposition 3.24, i.e. the moduli space $\mathcal{M}^*_T(A_{\alpha_0}, K, C)$ is a smooth manifold and the evaluation map factors through a smooth manifold of dimension (note, that $|R| \leq k$)

$$\begin{aligned}\dim_{\mathbb{R}} \mathcal{M}^*_{|R|}(A_{\alpha_0}, K, C) &= 2n - 6 + 2c_1(A_{\alpha_0}) + 2|R| - 2\sum_{i \in R} v_i \\ &\leq 2n - 6 + 2c_1(A) + 2k - 2 = d - 2.\end{aligned}$$

(I.C.2) The case where T' contains two of the last l_1 points is similar to (I.C.1), since Proposition 3.26 applies also for intersections with V_1.

(I.C.3) The last case occurs if the ghost tree T' contains one of the middle l_0 points and one from the last l_1 points. Geometrically this implies that the ghost tree T' is attached to the point $z_{\alpha j}$ which is mapped to the intersection $V_0 \cap V_1$. Consider the collection $C := \{Z_i\}$ given by

$$Z_i := \begin{cases} V_0 & k+1 \leq i \leq k+l_0 \text{ and } i \neq j \\ V_1 & k+l_0+1 \leq i \leq k+l_0+l_1 \text{ and } i \neq j \\ V_0 \cap V_1 & i = j, \end{cases}$$

then the first case of Proposition 3.24 implies that the corresponding evaluation map ev^k factors through a smooth manifold of dimension

$$\begin{aligned} \dim_{\mathbb{R}} \mathcal{M}^*_{|R|}(A_{\alpha_0}, K, C) &\leq 2n - 6 + 2c_1(A_{\alpha_0}) + 2|R| - \sum_{i \in R} 2 \\ &\leq 2n - 6 + 2c_1(A) + 2k - 2 = d - 2. \end{aligned}$$

Hence, we have shown that for a tree T with $e(T) > 0$ the corresponding moduli space has codimension 2 with respect to the dimension of the top stratum $\mathcal{M}_{k+l_0+l_1}(A, K, V_0 \cup V_1)$.

(II) Now observe that Theorem 4.10 implies that the closure of the moduli space $\mathcal{M}_{k+l_0+l_1}(A, K, V_0 \cup V_1)$ consists of stable maps $(\mathbf{f}, \mathbf{z})$, such that the underlying curve $\mathbf{z}$ is I-stable. Hence, strata considered in (I) form a compactification, and since all of them, after evaluating at the first k points, factor through smooth manifolds of codimension at least 2, it follows by definition that the evaluation map $ev^k : \mathcal{M}_{k+l_0+l_1}(A, K, V_0 \cup V_1) \to M^k$ defines a pseudocycle.

■

4.3 Rational cobordisms for Donaldson quadruples

In this section we fix an energy level $E > 0$, constants $0 < \Theta_3 < \Theta_2 < \Theta_1 < \Theta_1 < \Theta_0 < 1$ and $\eta > 0$ as in the previous section. We consider Donaldson quadruple $(\omega_0, J_0, V_0, V_1)$ bi-degree (D_0, D_1).

For a given $I \subset \{1, .., l_0 + l_1\}$ with $|I| \geq 3$ recall that the map $\pi_I : \mathcal{M}_{l_0+l_1+1} \to \mathcal{M}_{|I|+1}$ is given by forgetting marked points outside the set $I \cup \{0\}$ and stabilizing. For $I = \{1, \ldots, l_0\}$ we set $\pi_{l_0} := \pi_I$.

Lemma 4.12. *Given a Donaldson pair as above and $I = \{1, \ldots, l_0\}$. Then for any $l_0, l_1 \geq 0$ with $E \geq \max\{l_0/D_0, l_1/D_1\}$ and any $K \in \mathcal{J}^{reg}_{l_0+1}(M, V_0 \cup V_1, J_0, \Theta_1, E)$ (the space from Theorem 4.9) we have*

1. *$K \in \mathcal{J}^{reg}_{l_0+1}(M, V_0, J_0, \Theta_1, E)$.*

2. $\pi_{l_0}^* K \in \mathcal{J}^{reg}_{l_0+l_1+1}(M, V_0 \cup V_1, J_0, \Theta_1, E)$.

Analogous statement holds in the case $I = \{l_0 + 1, \ldots, l_0 + l_1\}$*.*

Proof. First we observe the following inclusions for energy $E > 0$:

$$\begin{array}{ccc} \mathcal{J}(M, V_0 \cup V_1, J_0, \Theta_1) & \subset & \mathcal{J}(M, V_0, J_0, \Theta_1) \\ \text{open} \cup \text{dense} & & \text{open} \cup \text{dense} \\ \mathcal{J}^*(M, V_0 \cup V_1, J_0, \Theta_1, E) & \subset & \mathcal{J}^*(M, V_0, J_0, \Theta_1, E). \end{array}$$

The upper relation follows directly from the definition. The inclusions on the left and right side are open and dense by the degree condition ($D_1, D_0 \geq D^*$) of the Donaldson quadruple. The lower inclusion follows, since by definition for any $K' \in \mathcal{J}^*(M, V_0 \cup V_1, J_0, \theta_1, E)$ all K'-holomorphic spheres in $V_0 \cup V_1$ of energy below E are constant and all non-constant K'-holomorphic spheres of energy below E intersect **each** V_0 and V_1 in at least 3 distinct points in the domain. Hence, omitting V_1 yields the lower inclusion.
Since $K \in \mathcal{J}^{\mathrm{reg}}_{l_0+1}(M, V_0 \cup V_1, J_0, \theta_1)$, by definition there exists a

$$\bar{J}_0 \in \mathcal{J}^*(M, V_0 \cup V_1, J_0, \Theta_0, E), \text{ such that}$$

$$K \in \mathcal{J}^{\mathrm{reg}}_{l_0+1}(M - V_0 \cup V_1, \{V_0, V_1, V_0 \cap V_1\}, \bar{J}_0, \Theta_1) \cap \mathcal{J}^*_{l_0+1}(M, V_0 \cup V_1, \bar{J}_0, \Theta_0, E).$$

Lemma 3.10 implies that the map π_{l_0} induces a map on coherent almost complex structures $\pi_{l_0}^* : \mathcal{J}_{l_0+1} \to \mathcal{J}_{l_0+l_1+1}$, hence $\pi_{l_0}^* K \in \mathcal{J}^*_{l_0+l_1+1}(M, V_0 \cup V_1, \bar{J}_0, \Theta_0, E)$. Finally, it follows from Proposition 3.24 that $\pi_{l_0}^* K \in \mathcal{J}^{\mathrm{reg}}_{l_0+l_1+1}(M - V_0 \cup V_1, \{V_0, V_1, V_0 \cap V_1\}, \bar{J}_0, \Theta_1)$. This implies the second statement. Proof for the statement in the case $I = \{l_0 + 1, ..., l_0 + l_1\}$ follows, if one starts with V_1 instead of V_0. ∎

Same arguments (i.e. a choice of a smooth path $K_t \in \mathcal{J}^*_{l_0+l_1+1}(M, V_0 \cup V_1, J_0, \Theta_1, E)$) as in the proof of Proposition 10.2 from [CM07] yields independence of a perturbation, hence we have

Lemma 4.13. *For a Donaldson quadruple as above,* $A \in H_2(M, \mathbb{Z})$ *with* $\max\{\omega_0(A), \omega_1(A)\} \leq E$*. Set* $l_0 = D_0\omega_0(A)$ *and* $l_1 = D_1\omega_1(A)$*. Then for any* $K_0, K_1 \in \mathcal{J}^{\mathrm{reg}}_{l_0+l_1+1}(M, V_0 \cup V_1, J_0, \Theta_1, E)$ *the pseudocycles*

$$\mathrm{ev}^k : \mathcal{M}_{k+l_0+l_1}(A, K_i, V_0 \cup V_1) \longrightarrow M^k \textit{ for } i = 0, 1$$

are cobordant for any $k \geq 0$*.*

Hence, we will denote the pseudocycle given by above mentioned map ev^k as $\mathrm{ev}^k(A, V_0, V_1, J_0)$. Since an analogous statement for the Donaldson pair (V_0, J_0) was shown in [CM07], we denote a pseudocycle associated to it by $\mathrm{ev}^k(A, V_0, J_0)$. Recall that Proposition 10.4 in [CM07] implies the existence of a rational cobordism $\mathrm{ev}^k(A, V_0, J_0) \sim \mathrm{ev}^k(A, V_0, J_0')$ for any $J_0, J_0' \in \mathcal{J}_c(\omega_0)$.

Finally, we get an analog of Proposition 10.3 from [CM07]. The main difference is again that our definition of the Donaldson quadruple is slightly more general and that we actually fix an energy level.

Proposition 4.14. *Donaldson quadruple as above,* $A \in H_2(M, \mathbb{Z})$ *with* $\max\{\omega_0(A), \omega_1(A)\} \leq E$*. Set* $l_0 = D_0\omega_0(A)$ *and* $l_1 = D_1\omega_1(A)$*. Assume that there exists*[1] *a* $J_1 \in \mathcal{J}_c(\omega_1)$*, such that we have a pseudocycle* $\mathrm{ev}^k(A, V_1, J_1)$ *associated to the Donalson pair* (V_1, J_1)*. Moreover, assume that there exists a* $\Theta_1' > 0$*, such that the intersection* $\mathcal{J}^{\mathrm{reg}}_{l_1+1}(A, V_0 \cup V_1, J_1, \Theta_1', E) \cap \mathcal{J}^{\mathrm{reg}}_{l_1+1}(A, V_0 \cup V_1, J_0, \Theta_1, E)$ *is nonempty*[2]*.*
Then for any $k \geq 0$ *we have rational cobordisms of pseudocycles*

$$\frac{1}{l_0!}\mathrm{ev}^k(A, V_0, J_0) \sim \frac{1}{(l_0 l_1)!}\mathrm{ev}^k(A, V_0, V_1, J_0) \sim \frac{1}{l_1!}\mathrm{ev}^k(A, V_1, J_1).$$

Proof. Fix a perturbation $K \in \mathcal{J}^{\mathrm{reg}}_{l_0+1}(M, V_0 \cup V_1, J_0, \Theta_1, E)$ provided by Theorem 4.9. Then Lemma 4.12 implies that $\pi^*_{l_0}K \in \mathcal{J}^{\mathrm{reg}}_{l_0+l_1+1}(M, V_0 \cup V_1, J_0, \Theta_1, E)$ and $K \in \mathcal{J}^{\mathrm{reg}}_{l_0+1}(M, V_0, J_0, \Theta_1, E)$. Hence, we get two pseudocycles

$$\mathrm{ev}^k : \mathcal{M}_{l_0+l_1+1}(A, V_0 \cup V_1, \pi^*_{l_0}K) \to M^k \text{ and } \mathrm{ev}^k : \mathcal{M}_{l_0+1}(A, V_0, K) \to M^k.$$

Forgetting the last l_1 marked points, i.e. intersection points with V_1 (such points are pairwise distinct, since coincidence leads to tangency order and a stratum of positive codimension - see Proposition 3.26) induces a covering map of degree $l_1!$ and we have a commutative diagram of pseudocycles

$$\begin{tikzcd}
\mathcal{M}_{l_0+l_1+1}(A, V_0 \cup V_1, \pi^*_{l_0}K) \arrow[r, "\pi_{l_0}"] \arrow[d, "\mathrm{ev}^k"'] & \mathcal{M}_{l_0+1}(A, V_0, K) \arrow[d, "\mathrm{ev}^k"] \\
M^k \arrow[r, "\mathbf{Id}"] & M^k.
\end{tikzcd}$$

Hence, above statements imply equality as currents (i.e. a rational cobordism):

$$\mathrm{ev}^k(V_0, K) \sim \frac{1}{l_1!}\mathrm{ev}^k(V_0, V_1, \pi^*_{l_0}K).$$

For the second cobordism take a perturbation

$$K \in \mathcal{J}^{\mathrm{reg}}_{l_1+1}(A, V_0 \cup V_1, J_1, \Theta_1', E) \cap \mathcal{J}^{\mathrm{reg}}_{l_1+1}(A, V_0 \cup V_1, J_0, \Theta_1, E)$$

and observe that Lemma 4.12 implies again that

$$\pi^*_{l_1}K \in \mathcal{J}^{\mathrm{reg}}_{l_0+l_1+1}(A, V_0 \cup V_1, J_1, \Theta_1', E) \text{ and } K \in \mathcal{J}^{\mathrm{reg}}_{l_1+1}(A, V_1, J_1, \Theta_1', E),$$

so same reasoning as for V_0 yields the full statement. ■

[1] Existence of such J_1 is guaranteed in the proof of Theorem 4.21.

[2] In the proof of Theorem 4.21 such Θ_1' is an ϵ-small perturbation of Θ_1 with $\epsilon \ll \Theta_3$.

4.4 The irrational case

First, we recall some standard Hodge theory. Here we mainly follow the exposition in Chapter 6 from [War83]. Given two p-forms $\alpha, \beta \in \Omega^p(M)$ their inner-product is given by $\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta$. With $*$-Hodge star associated to the metric $\omega_0(\cdot, J_0 \cdot)$. We denote the induced norm by $\|\cdot\|_{L^2}$. The Hodge-Laplacian is given by $\Delta = \delta d + d\delta$ and we have

Theorem 4.15 (Hodge decomposition, cf. 6.8 **in [War83]).** *For each $0 \leq p \leq 2n$ the space of harmonic p-forms $H^p(M) = \{\alpha \in \Omega^p(M) | \Delta\alpha = 0\}$ is finite dimensional and the space of all p-forms $\Omega^p(M)$ decomposes into three orthogonal direct summands:*

$$\Omega^p(M) = H^p(M) \oplus d(\Omega^{p-1}(M)) \oplus \delta(\Omega^{p+1}(M)).$$

An immediate consequence of the Hodge decomposition is

Corollary 4.16 (cf. 6.11 **in [War83]).** *For any given de Rham cohomology class $A \in H^p_{DR}(M, \mathbb{R})$ there is a unique harmonic representative $\alpha \in \Omega^p(M)$, i.e. $[\alpha] = A$ and $\Delta\alpha = 0$. Moreover, α minimizes the norm within the class A.*

Proof. For the proof of the first statement see pp. 225−226 in [War83]. For the second statement take any $\beta \in \Omega^p(M)$ with $[\beta] = A$. Hodge decomposition yields $\beta = \beta_H + \beta_d + \beta_\delta$ with $\Delta\beta_H = 0$, β_d exact and β_δ co-exact. Since $[\beta_H] = A$, uniqueness implies $\beta_H = \alpha$. So, $\|\beta\|_{L^2} = \|\beta_H\|_{L^2} + \|\beta_d\|_{L^2} + \|\beta_\delta\|_{L^2} \geq \|\alpha\|_{L^2}$. ■

Remark 4.17. Note that for any ω-compatible J and an associated Laplacian Δ to the metric $\omega(\cdot, J\cdot)$ we have $\Delta\omega = 0$. Since the Riemannian volume form is $\frac{1}{n!}\omega^n$, we have $*\omega = \frac{1}{n-1}\omega^{n-1}$, hence $d * \omega = 0$ so ω and is harmonic.

Remark 4.18. At this point, one should point out that we rely on the choice of a compatible almost complex structure J, since we are using standard Hodge theory. Note that there exists a natural Hodge theory for symplectic manifolds. One can define a purely symplectic analog of Hodge-$*$ and $\delta := *d*$ and call a form symplectic harmonic if it is d-closed and δ-closed. J.-L. Brylinski proved in [Bry88] an analog of the uniqueness statement from Corollary 4.16 for Kähler manifolds. However, O. Mathieu proved in [Mat95] that uniqueness holds if and only if the manifold has the strong Lefschetz property.

The outcome of Corollary 4.16 is that the Hodge map $h : H^p(M, \mathbb{R}) \to \Omega_p(M)$ mapping a given class to the unique harmonic representative is well-defined and it induces a norm on $H^p_{DR}(M, \mathbb{R})$ via $\|A\| := \|h(A)\|_{L^2}$ for any $A \in H^p_{DR}(M, \mathbb{R})$.

Then, the next statement might be considered pretty pedantic, since it is a sort of a technical folklore, but since it seems to be hard to find an exact reference for it in literature, we give a proof of it here.

Lemma 4.19. *For any harmonic k-form α there exists a constant $C = C(n,k) > 0$ such that $\|\alpha\| \leq C\,\|\alpha\|_{L^2}$.*

Proof. First, recall from Proposition 4.7 in [Mor01] that $\alpha \wedge * \beta = \langle \alpha, \beta \rangle \, dvol_M$ with $\langle \cdot, \cdot \rangle$ the induced by $\omega(\cdot, J\cdot)$ scalar product on the space of k-forms. Next, recall from Corollary 7.11 from [GT83]: There exists a constant $C = C(n,p)$, such that $\sup_{x\in M} |f(x)| =: \|f\| \leq C\,\|f\|_{p,2}$ for any $p \geq n$. Here $\|\cdot\|_{p,2}$ is the $(p,2)$-Sobolev norm for functions

$$\|f\|_{p,2} := \left(\int_M \sum_{|\lambda|\leq p} |D^\lambda f|^2 dvol_M \right)^{\frac{1}{2}}.$$

This norm induces the $(p,2)$-Sobolev norm on forms. Now, since Hodge-Laplacian is an elliptic operator of order 2 we have (cf. 6.29 in [War83]) for some constant $C' = C'(n,p) > 0$ and $p \geq 0$

$$\|\alpha\|_{p+2,2} \leq C' \left(\|\Delta\alpha\|_{p,2} + \|\alpha\|_{p,2} \right).$$

Since α is harmonic, multiple application of previous inequality implies that there exists a constant $C'' = C''(p,n) > 0$ for $p \geq 2$, s.t $\|\alpha\|_{p,2} \leq C''\,\|\alpha\|_{L^2}$. Hence, we get for any $p \geq n$ and setting $\tilde{C} := CC''$

$$\sup_M \langle \alpha, \alpha \rangle^{\frac{1}{2}} \leq C\,\|\alpha\|_{p,2} \leq \tilde{C}\,\|\alpha\|_{L^2} = \tilde{C} \left(\int_M \langle \alpha, \alpha \rangle \, dvol_M \right)^{\frac{1}{2}}.$$

Now fix a base point $x \in M$ and choose orthonormal basis $(e^1, ..., e^{2n})$ of $T_x^* M$ hence $\alpha_x = \sum_I \alpha_I e^I$ with I an ordered index set of length $|I| = k$. Then $\langle \alpha, \alpha \rangle_x = \sum_I \alpha_I^2$. On the other hand

$$\begin{aligned} \sup_{|v_1|=1,\ldots,|v_k|=1} |\alpha_x(v_1, \ldots, v_k)| &\leq \sum_I |\alpha_I| |e^I(v_1, \ldots, v_k)| \leq \sum_I |\alpha_I| \\ &\leq \sqrt{\binom{2n}{k}} \sqrt{\sum_I |\alpha_I|^2}. \end{aligned}$$

We conclude that, $\|\alpha\| \leq \sqrt{\binom{2n}{k}}\tilde{C}\,\|\alpha\|_{L^2}$. ■

Lemma 4.20. *There is an $\epsilon = \epsilon(J,\omega) > 0$, such that any form in the image of the restriction $h : B(\epsilon, \omega) \subset H^2(M, \mathbb{R}) \to \Omega_2(M)$ is symplectic.*

Proof. Since the non-degeneracy condition is open and M is closed, we get an $\delta > 0$ such that for any form $\omega' \in \Omega_2(M)$, $\|\omega - \omega'\|_{g_J} < \delta$ implies $(\omega')^n \neq 0$. Hence, the statement follows by continuity of the Hodge map h, since its image contains only closed forms and $h([\omega]) = \omega$. ■

Finally, we can combine results from the current and previous chapters in order to prove the main result of the thesis.

Theorem 4.21. *Given any symplectic form ω on M. There exists an open neighbourhood of ω, say $U \subset \Omega_2(M)$, such that for any pair of rational symplectic forms $\omega_1, \omega_2 \in U$ the corresponding pseudocycles $ev^k(\omega_1)$ and $ev^k(\omega_2)$ are rationally cobordant, up to multiplication with a positive rational weight, for any $k \geq 3$.*

Proof. Fix a homology class $A \in H_2(M, \mathbb{Z})$ with $\omega(A) > 0$. Fix $\Theta_0 < 1$. Fix an ω-compatible almost complex structure $J \in \mathcal{J}_c(\omega)$.

(A) Take $\epsilon > 0$ from Lemma 4.20. Fix two classes $A_1, A_2 \in H^2(M, \mathbb{R}) \cap H^2(M, \mathbb{Q})$, such that $\|A_1 - [\omega]\|_J < \epsilon$ and $\|A_2 - [\omega]\|_J < \epsilon$. Then it follows that $\tilde{\omega}_1 := h(A_1)$ and $\tilde{\omega}_2 := h(A_2)$ are both (rational) symplectic forms, with h denoting the Hodge map. Moreover, Lemma 4.19 implies that $\|\tilde{\omega}_1 - \omega\|_J < C\epsilon$ and $\|\tilde{\omega}_2 - \omega\|_J < C\epsilon$, for some constant $C > 0$, depending on ω and J.
Now, using Lemma 2.13, we can find an $\tilde{\omega}_1$-compatible almost complex structure J_1 with $\|J - J_1\|_J < 3C\epsilon$. Now, $\tilde{\omega}_1$ and J_1 induce another metric and hence a norm which we denote by $\|\cdot\|_1$. Moreover, Lemma 2.12 implies that

$$\|\tilde{\omega}_1 - \tilde{\omega}_2\|_1 < 2\epsilon \left[(1 + C\epsilon)(1 + 3C\epsilon)\right]^{\frac{1}{2}} < C_1\epsilon^2,$$

for a suitably chosen $C_1 > 0$. Again, by Lemma 2.13 we can find a $J_2 \in \mathcal{J}_c(\omega_2)$ with the property

$$\|J_1 - J_2\|_1 \leq 3\, \|\tilde{\omega}_1 - \tilde{\omega}_2\|_1 \leq 3C_1\epsilon^2.$$

For simplicity we can say that the pairs $(\tilde{\omega}_1, J_1)$ and $(\tilde{\omega}_2, J_2)$ lie in an ϵ-neighbourhood of (ω, J). Hence, Theorem 2.34 (Opshtein's Theorem) and Corollary 2.37 delivers a transversality parameter $\eta = \eta(\epsilon, \omega, J)$, which does not depend on the choice of the pairs $(\tilde{\omega}_i, J_i)$. Note that we have $J_1 \in \mathcal{J}_\tau(\omega_2)$, as long as $3C_1\epsilon^2 < 1$ (cf. Lemma 2.5).

(B) For any given $0 < \rho < \epsilon$ step **(A)** delivers pairs $(\tilde{\omega}_i, J_i)$, lying in a ρ-neighbourhood of (ω, J). Now, let N be the smallest positive integer, such that

$$[N\tilde{\omega}_1] \in H^2(M, \mathbb{Z}) \text{ and } [N\tilde{\omega}_2] \in H^2(M, \mathbb{Z}).$$

Set $\omega_1 := N\tilde{\omega}_1$, $\omega_2 := N\tilde{\omega}_2$. Note that we still have $J_1 \in \mathcal{J}_c(\omega_1)$ and $J_1 \in \mathcal{J}_\tau(\omega_2)$. Moreover, $\|\omega_1 - \omega_2\|_{\omega_i, J_i} \leq \rho$. We set the energy level via

$$E := \max\{\omega_1(A), \omega_2(A)\},$$

and recall the constant D^* from Section 3.7 and set

$$D^* := \max\{D^*(\omega_1, J_1, E, \theta_0), D^*(\omega_2, J_2, E, \theta_0)\}.$$

(C) By Theorem 2.34 and Corollary 2.37 applied to the pairs $(\tilde{\omega}_1, J_1)$ and $(\tilde{\omega}_2, J_2)$ there exist ω-symplectic hypersurfaces V_0 and V_1 satisfying

- $\mathrm{PD}(V_1) = D[\omega_1]$ and $\mathrm{PD}(V_2) = D[\omega_2]$
- Kähler angles are bounded by $\theta(V_j) < CD^{-1/2}$ for $j = 1, 2$
- V_1 intersects V_2 transversely and their minimal angle is bounded from below by $\angle_m(V_1, V_2) \geq \eta$

again, with constants $\eta = \eta(\epsilon)$ from part **(A)** and $C > 0$ independent of D, provided D is chosen[1] sufficiently large. Above angles are measured wrt. a rescaled metric induced by (J, ω), but since conformal change of the metric does not affect angles, we can view them as measured by (ω, J). By increasing D if necessary, we assume that

$$D \geq D^* \text{ and } \theta(V_j) \leq \Theta_4 \text{ for } j = 1, 2.$$

(D) Now, since $\|\omega - \tilde{\omega}_1\|_J \leq \rho$ and $\|J - J_1\|_J \leq \rho$ Lemma 2.14 implies for the Kähler angle $\theta_1(\cdot)$ measured wrt. $(\tilde{\omega}_1, J_1)$:

$$\theta_1(V_j) \leq \theta(V_j) + 2\rho^{1/4} \leq \Theta_4 + 2\rho^{1/4} \text{ for } j = 1, 2$$

and for the minimal angle $\angle_m^1$ measured wrt. $(\tilde{\omega}_1, J_1)$:

$$\angle_m^1(V_1, V_2) \geq \angle_m(V_1, V_2) - 2\rho^{1/4} \geq \eta - 2\rho^{1/4}.$$

Again, since conformal change of the metric does not affect angles, we regard above angles as measured wrt. (ω_1, J_1).

(E) Observe that the constants ρ, Θ_4 and η are mutually independent in our construction. We proceed with a selection of constants (with $\Theta_0 < 1$ already fixed)

$$0 < \Theta_3 < \Theta_2 < \Theta_1 < \Theta_0 < 1$$

- Let $\Theta_2 < \frac{2}{5}\Theta_1 < \frac{2}{5}\Theta_0$, where the constant $2/5$ comes from Corollaries 2.17 and 2.16.
- Let $\Theta_3 < \frac{4}{\eta - 2\rho^{1/4}}(\Theta_4 + 2\rho^{1/4})$, cf. assumption in Lemma 2.22.

Since η is fixed, we can choose ρ and Θ_4, such that $\Theta_3 < \Theta_2$. Hence, we have shown that $(\omega_1, J_1, V_1, V_2)$ defines a Donaldson quadruple. Note that (ω_1, J_1, V_1) and (ω_2, J_2, V_2) are Donaldson Pairs, provided ρ was chosen sufficiently small. Recall that such choice of constants implies that the space

$$\mathcal{J}(M, V_1 \cup V_2, J_1, \Theta_1) = \{K \in \mathcal{J}(M, V_1 \cap V_2) \mid \|K - J_1\|_1 < \Theta_1\}$$

is nonempty and that any two elements in it can be connected by a path lying in the space $\mathcal{J}(M, V_1 \cup V_2, J_1, \Theta_0)$.

(F) Proposition 4.14 applied to $(\omega_1, J_1, V_1, V_2)$ yields a rational cobordism of pseudocycles

[1] Here we use D instead of k, since we are talking about degrees of hypersurfaces instead of twisting parameters of line bundles.

$$\frac{1}{l_1!}\mathrm{ev}^k(A, V_1, \omega_1) \sim \frac{1}{l_2!}\mathrm{ev}^k(A, V_2, \omega_2)$$

for any $k \geq 3$ with $l_1 := \deg(V_1)\omega_1(A)$ and $l_2 := \deg(V_2)\omega_2(A)$. Note that in our case we have $D = \deg(V_1) = \deg(V_2)$. Finally, since ω_i is $\tilde{\omega}_i$ multiplied by N for $j = 1, 2$, it follows that pseudocylcles associated to $(J_1, \tilde{\omega}_1)$ and $(J_2, \tilde{\omega}_2)$ are also rationally cobordant. ■

Remark 4.22. Clearly, one could try to substitute Opshtein's theorem in steps (**A**) and (**C**) by a combination of Theorem 1.13 and Proposition 2.3. Namely, take (ω_1, J_1) and (ω_2, J_2) from the first part of step (**A**). Now, applying Theorem 1.13 (Donaldson's theorem) for the pair (ω_2, J_2) we get an ω_2-symplectic hypersurface V_2 with $\mathrm{PD}(V_2) = D_2[\omega_2]$ for $D_2 \gg 0$ with the Kähler angle (measured wrt. ω_2 and J_2) $\theta_2(V_2) < 2C'D_2^{-1/2} =: \theta_2'$. We can assume that $D_2 > D_*$ and θ_2' is sufficiently small. Lemma 2.14 implies that the Kähler angle of V_2 measured wrt. the pair (ω_1, J_1) satisfies $\theta_1(V_2) < \theta_2' + 3C_1\epsilon^2 := \theta_2$. So, V_2 is ω_1-symplectic, provided $D_2 \gg 0$ and $\epsilon \ll 1$. Note that the constant C_1 depends only on (ω, J).
Lemma 2.19 provides an $\bar{J}_1 \in \mathcal{J}_c(\omega_1)$, st. $\left\|J_1 - \bar{J}_1\right\|_1 < \theta_2$ and $\bar{J}_1 TV_2 \subset TV_2$. Hence, applying Proposition 2.3 to $(M, V_2, \omega_1, \bar{J}_1)$ yields for any fixed $\eta > 0$ an ω_1-symplectic hypersurface V_1 satisfying

- $\mathrm{PD}(V_1) = D_1[\omega_1]$ for a $D_1 \gg 0$
- Kähler angle of V_1 is given by $\theta(V_1) < 2C''D_1^{-1/2} =: \theta_1$
- V_1 intersects V_2 transversely and the minimal angle is $\angle_m(V_1, V_2) \geq \eta$.

Note that latter angles are measured wrt. the pair $(\omega_1, \bar{J}_1)$. The problem now is that η seems to depend on V_2, which again depends on $\tilde{\omega}_2$. So, by choosing perturbation parameter ϵ small (which is necessary in order to make $\theta(V_2)$ small wrt. (ω_1, J_1)) there is no guarantee that $\eta(\epsilon)$ is bounded from below, hence the constant from Lemma 2.22 might become large, so that no such $\Theta_0 < 1$ as in step (**E**) would exist. Hence, the perturbation space $\mathcal{J}(M, V_1 \cup V_2, J_1, \Theta_0)$ would be empty, although both constructions seem to produce similar geometric objects.

Previous discussion together with the proof of the preceeding theorem gives rise to a natural question. Namely, how does the construction of a Donaldson quadruple in the original setting work, since its existence is required in the proof of Theorem 1.3 (independence of a Donaldson pair) in [CM07] and Opshtein's theorem was not available. We make the following

Remark 4.23. Consider a symplectic manifold (M, ω). The existence of a Donaldson quadruple (V_0, J_0, V_1, J_1) as in [CM07] (i.e. $\mathrm{PD}([V_i]) = D_i[\omega]$, $\angle_m(V_0, V_1) > \eta$, $\theta(V_i, \omega, J_i) < \Theta_3$ and $\|J_0 - J_1\| < \Theta_3$ for $i = 0, 1$) should follow from Proposition[1] 2.3. However, in this approach the transversality parameter η depends on the first hypersurface V_0 (although it does not depend

[1]It is called the "stability property" in the original work.

on V_1 for any $D_1 \gg 0$), hence the existence of constants ($0 < \Theta_3 < \Theta_2 < \Theta_1 < \Theta_0 < 1$) ensuring that the space $\mathcal{J}(M, V_0 \cup V_1, J_0, \Theta_1)$ is nonempty and path-connected in $\mathcal{J}(M, V_0 \cup V_1, J_0, \Theta_0)$ is not obvious.
Indeed, Lemma 2.22 asserts $\Theta_2 > \frac{1}{\eta} \max\{\theta(V_0, \omega, J_0), \theta(V_1, \omega, J_0)\}$. We have $\theta(V_0, \omega, J_0) < \Theta_3$ and Lemma 2.14 implies

$$|\theta(V_1, \omega, J_0) - \theta(V_1, \omega, J_1)| < C\Theta_3^{1/4},$$

i.e. above difference depends on the Kähler angle of V_0, but making it smaller might decrease η making it impossible to find a Θ_2, s.t $\Theta_0 < 1$.
However, it is still possible to show its existence. Instead of J_0 we can take another $\bar{J}_0 \in \mathcal{J}_c(\omega)$ with $\bar{J}_0 TV \subset TV$ and measure wrt. $(\omega, \bar{J}_0)$. Then, $\theta(V_0, \omega, \bar{J}_0) = 0$ and by choosing D_1 large implies that $\theta(V_1, \omega, \bar{J}_0)$ might be chosen arbitrarily small. Combined with the fact that η (obtained from the application of 2.3 to $(V_0, \bar{J}_0)$) does not depend on D_1, it follows that $(V_0, \bar{J}_0, V_1, J_1)$ is a Donaldson quadruple, up to the fact that we have to choose $D_0 > D^*(\bar{J}_0, \omega, \Theta_0)$.
Observe that D^* depends on $\bar{J}_0$, which in turn depends on V_0, so on D_0! However, by Lemma 2.19 we can choose $\bar{J}_0$, such that $\left\|J_0 - \bar{J}_0\right\| < C\theta(V_0, \omega, J_0)$, hence by choosing D_0 sufficiently[1] small, we can (see discussion after Lemma 3.27) gain control over the difference $|D^*(J_0, \omega, \Theta_0) - D^*(\bar{J}_0, \omega, \Theta_0)|$, such that we can choose $D_0 > D^*(\bar{J}_0, \omega, \Theta_0)$ and Θ_3 to be small enough.

[1] Since we have for the Kähler angle $\theta(V_0) \sim 1/\sqrt{D_0}$, we actually assume $D_0 \gg 1/(1 - \Theta_0)$.

A

Appendix

A.1 Complex line bundles

In this section we review some standard facts about complex line bundles. For a detailed treatment of the subject we refer to the book [Kob87].

Let M be a smooth n-dimensional manifold. We consider a smooth bundle $\pi : L \to M$ with the fiber diffeomorphic to $\mathbb{C}$. We recall the cocycle definition. Let $\{U_i\}$ be a good covering of M, i.e. the sets

$$U_i,\, U_{ij} := U_i \cap U_j \text{ and } U_{ijk} := U_i \cap U_j \cap U_k \text{ are contractible.}$$

Moreover, over each U_i we have trivialization $\psi_i : \pi^{-1} \to U_i \times \mathbb{C}$. Then, restricted to each U_{ij} the compositions $\psi_i \circ \psi_j^{-1}$ define cocycles $G_{ij}(x) \in GL(1, \mathbb{C})$ for any $x \in U_{ij}$. Such G_{ij} satisfy cocycle conditions, namely

$$G_{ij} \cdot G_{jk} = G_{ik} \text{ and } G_{ij} \cdot G_{ji} = 1.$$

In the case where all cocycles satisfy $G_{ij} \in U(1)$ we get a Hermitian structure on L. Denote the space of smooth sections of L by $\Gamma(L)$, then a connection on L is a map

$$\nabla : \Gamma(L) \longrightarrow \Gamma(L \otimes T^*M),$$

such that for any smooth function $f : M \to \mathbb{R}$ and any section $s \in \Gamma(L)$ we have

$$\nabla(f \cdot s) = s \otimes df + f \nabla s.$$

Let h be a Hermitian metric on L. A connection ∇ is called **Hermitian** if

$$Dh(s, s') = h(\nabla s, s') + h(s, \nabla s') \text{ for any } s, s' \in \Gamma(L).$$

Locally, over each U_i a connection can be represented by $\nabla = d + A_i$ for some $A_i \in \Omega_1(U_i, \mathbb{C})$. One can show that on the intersections U_{ij} one has $dA_i = dA_j$, hence exterior derivatives dA_i yield a globally defined complex valued 2-form on M, which is called a **connection 2-form** on L. Note that in the Hermitian case all A_i take values in $i\mathbb{R}$. A basic fact in the Chern-Weil theory is the following

Proposition A.1. *Given a section $s \in \Gamma(L)$ of a Hermitian line bundle L, assume it is transversal to the zero section and denote the zero locus of s by V. Let F be the curvature form and write $F = i\omega$ for some (closed) real valued 2-form ω. Then for any 2-cycle $S \subset M$ which intersects V transversely we have*

$$\frac{1}{2\pi}\int_S \omega = V \cdot S.$$

The next theorem is a fundamental fact which is also used in the theory of geometric quantization.

Theorem A.2. *Given a closed (real valued) 2-form $\omega \in \Omega_2(M)$. Assume that ω represents an integer class, i.e. $[\omega] \in H^2(M, \mathbb{Z})$. Then there exists a line bundle $\pi : L \to M$ together with a Hermitian connection ∇ whose curvature form is given by $-\frac{i}{2\pi}\omega$.*

Given two complex line bundles $L, L' \to M$ over the same base M. Then the tensor product $L \otimes L'$ is a well defined complex line bundle over M and we have the relation on the Chern classes:

Lemma A.3 (cf. Proposition 3.10 in [Hat09]).

$$c_1(L \otimes L') = c_1(L) + c_1(L').$$

Hence for $L^k := \underbrace{L \otimes .. \otimes L}_{k}$ above lemma yields $c_1(L^k) = k \cdot c_1(L)$.

A.2 Pseudocycles

Here we give a short account of definitions and results on pseudocycles from section 6.5 in [MS04] and rational pseudocycles defined in [CM07].
Given a smooth n-dimensional manifold M. A subset $A \subset M$ has **dimension at most** d (with $d \leq n$) if it is contained in the image of a smooth map $W \to M$ where W is a smooth manifold of the dimension less or equal to d.

Definition A.4. *A d-**dimensional pseudocycle** in M is a smooth map $f : V \to M$ on a smooth oriented d-dimensional manifold V, such that $\overline{f(V)}$ is a compact set in M and* $\dim \Omega_f \leq \dim V - 2$.
*The **omega limit set** is given by*

$$\Omega_f := \bigcap_{K \subset V\, compact} \overline{f(V \backslash K)}.$$

*Any two d-dimensional pseudocycles $f : V \to M$ and $f' : V' \to M$ are **cobordant** if there exists a $(d+1)$-dimensional oriented manifold W with $\partial W = V \cup (-V')$ together with a smooth map $F : W \to M$, such that $F_{|V} = f$, $F_{|V'} = f'$ and $\dim \Omega_F \leq d-1$.*

Remark A.5. Clearly, for a fixed M one could define a group of bordism classes of pseudocycles in M. Graded by the dimension denote it by $\mathcal{H}_*(M)$. It was shown in [Par01] and [Sch99] for compact M that $\mathcal{H}_*(M)$ is naturally isomorphic to $H_*(M, \mathbb{Z})$. Such isomorphism fails if M is not compact. However, it was observed in [Zin08] that such isomorphism can be still established if one restricts to pseudocycles whose images are pre-compact sets in M.

Definition A.6. *A **rational pseudocycle** in M is a pseudocycle multiplied with a positive rational number. We denote it by lf for $f : V \to M$ a pseudocycle and $l \in \mathbb{Q}$.*
*Given two pseudocycles $f : V \to M$ and $f' : V' \to M$, the rational pseudocycles f and lf are **equal as currents** if there exists a covering map $\phi : V \to V'$ of degree l such that $f = f' \circ \phi$.*
*The equivalence relation on rational pseudocycles of M generated by equality as currents and cobordisms of pseudocycles is called **rational cobordism**.*

Recall that two pseudocycles f and f' are strongly transverse if $\Omega_{f'} \cap \overline{f(V)} = \emptyset$, $\Omega_{f'} \cap \overline{f(V)} = \emptyset$ and at any intersection point the intersection is transverse. It was shown in [MS04] (Lemma 6.5.5) that for strongly transversal pseudocycles f, f' of dimension k and $n-k$ respectively, there is a well-defined intersection number $f \cdot f'$. It depends only on the bordism classes of f and f'. Since geometric intersections are not affected by a rational weight and compositions with covering maps (after dividing by the degree of covering), this result carries over to rational pseudocycles.

Now observe that any smooth cycle, i.e. a smooth map $W \to M$ where W is a closed manifold, is of course a pseudocycle. A fundamental theorem of R. Thom states that for any homology class $\alpha \in H_*(M, \mathbb{Q})$ there exists an integer k such that $k\alpha$ is the fundamental class of a smooth closed submanifold of M. This fact makes perturbation theory available for rational cycles (representing classes in $H_*(M, \mathbb{Z})$) in order to achieve (strong) transversality. That allows to define intersection between (rational) pseudocycles and rational homology classes in M.

References

AF59. Aldo Andreotti and Theodore Frankel. The Lefschetz theorem on hyperplane sections. *Ann. of Math. (2)*, 69:713–717, 1959.

ALP94. Michèle Audin, François Lalonde, and Leonid Polterovich. Symplectic rigidity: Lagrangian submanifolds. In *Holomorphic curves in symplectic geometry*, volume 117 of *Progr. Math.*, pages 271–321. Birkhäuser, Basel, 1994.

AS08. Denis Auroux and Ivan Smith. Lefschetz pencils, branched covers and symplectic invariants. In *Symplectic 4-manifolds and algebraic surfaces*, volume 1938 of *Lecture Notes in Math.*, pages 1–53. Springer, Berlin, 2008.

Aud94. Michèle Audin. Symplectic and almost complex manifolds. In *Holomorphic curves in symplectic geometry*, volume 117 of *Progr. Math.*, pages 41–74. Birkhäuser, Basel, 1994. With an appendix by P. Gauduchon.

Aur97. D. Auroux. Asymptotically holomorphic families of symplectic submanifolds. *Geom. Funct. Anal.*, 7(6):971–995, 1997.

Aur00. Denis Auroux. Symplectic 4-manifolds as branched coverings of $\mathbf{CP}^2$. *Invent. Math.*, 139(3):551–602, 2000.

Aur02. D. Auroux. A remark about Donaldson's construction of symplectic submanifolds. *J. Symplectic Geom.*, 1(3):647–658, 2002.

Beh97. K. Behrend. Gromov-Witten invariants in algebraic geometry. *Invent. Math.*, 127(3):601–617, 1997.

BLY94. Jean-Pierre Bourguignon, Peter Li, and Shing-Tung Yau. Upper bound for the first eigenvalue of algebraic submanifolds. *Comment. Math. Helv.*, 69(2):199–207, 1994.

Bot59. Raoul Bott. On a theorem of Lefschetz. *Michigan Math. J.*, 6:211–216, 1959.

BP90. Larry Bates and Georg Peschke. A remarkable symplectic structure. *J. Differential Geom.*, 32(2):533–538, 1990.

Bry88. Jean-Luc Brylinski. A differential complex for Poisson manifolds. *J. Differential Geom.*, 28(1):93–114, 1988.

BT00. I. K. Babenko and I. A. Taĭmanov. On nonformal simply connected symplectic manifolds. *Sibirsk. Mat. Zh.*, 41(2):253–269, i, 2000.

BU00. David Borthwick and Alejandro Uribe. Nearly Kählerian embeddings of symplectic manifolds. *Asian J. Math.*, 4(3):599–620, 2000.

Buh13. Lev Buhovsky. Unboundedness of the first eigenvalue of the Laplacian in the symplectic category. *J. Topol. Anal.*, 5(1):13–56, 2013.

Che70. Jeff Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*, pages 195–199. Princeton Univ. Press, Princeton, N. J., 1970.

CM07. Kai Cieliebak and Klaus Mohnke. Symplectic hypersurfaces and transversality in Gromov-Witten theory. *J. Symplectic Geom.*, 5(3):281–356, 2007.

DGMS75. Pierre Deligne, Phillip Griffiths, John Morgan, and Dennis Sullivan. Real homotopy theory of Kähler manifolds. *Invent. Math.*, 29(3):245–274, 1975.

Don96. S. K. Donaldson. Symplectic submanifolds and almost-complex geometry. *J. Differential Geom.*, 44(4):666–705, 1996.

Don99. S. K. Donaldson. Lefschetz pencils on symplectic manifolds. *J. Differential Geom.*, 53(2):205–236, 1999.

Dră99. Tedi Drăghici. The Kähler cone versus the symplectic cone. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)*, 42(90)(1):41–49, 1999.

Eli89. Y. Eliashberg. Classification of overtwisted contact structures on 3-manifolds. *Invent. Math.*, 98(3):623–637, 1989.

Eli93. Yakov Eliashberg. Classification of contact structures on $\mathbf{R}^3$. *Internat. Math. Res. Notices*, (3):87–91, 1993.

Eva12. Jonathan David Evans. The infimum of the Nijenhuis energy. *Math. Res. Lett.*, 19(2):383–388, 2012.

Flo88. Andreas Floer. The unregularized gradient flow of the symplectic action. *Comm. Pure Appl. Math.*, 41(6):775–813, 1988.

FM05. Marisa Fernández and Vicente Muñoz. Formality of Donaldson submanifolds. *Math. Z.*, 250(1):149–175, 2005.

FO99. Kenji Fukaya and Kaoru Ono. Arnold conjecture and Gromov-Witten invariant. *Topology*, 38(5):933–1048, 1999.

Ger13. A. Gerstenberger. Geometric transversality in higher genus gromov-witten theory. *arXiv:1309.1426*, 2013.

Gro85. M. Gromov. Pseudoholomorphic curves in symplectic manifolds. *Invent. Math.*, 82(2):307–347, 1985.

Gro86. Mikhael Gromov. *Partial differential relations*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1986.

GT83. David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

Hat09. Allen Hatcher. Vector bundles and k-theory, 2009.

Hof08. H. Hofer. Polyfolds and a general fredholm theory. *arXiv:0809.3753*, 2008.

How66. Alan Howard. On the homotopy groups of an affine algebraic hypersurface. *Ann. of Math. (2)*, 84:197–216, 1966.

IP03. Eleny-Nicoleta Ionel and Thomas H. Parker. Relative Gromov-Witten invariants. *Ann. of Math. (2)*, 157(1):45–96, 2003.

IP13. E.-N. Ionel and T. Parker. A natural gromov-witten virtual fundamental class. *arXiv:1302.3472*, 2013.

Knu83. Finn F. Knudsen. The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$. *Math. Scand.*, 52(2):161–199, 1983.

Kob87. Shoshichi Kobayashi. *Differential geometry of complex vector bundles*, volume 15 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1987. Kanô Memorial Lectures, 5.

Kru03. Boris S. Kruglikov. Non-existence of higher-dimensional pseudoholomorphic submanifolds. *Manuscripta Math.*, 111(1):51–69, 2003.

Kut12. Sergii Kutsak. Essential manifolds with extra structures. *Topology Appl.*, 159(10-11):2635–2641, 2012.

LT98a. Jun Li and Gang Tian. Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties. *J. Amer. Math. Soc.*, 11(1):119–174, 1998.

LT98b. Jun Li and Gang Tian. Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds. In *Topics in symplectic* 4*-manifolds (Irvine, CA, 1996)*, First Int. Press Lect. Ser., I, pages 47–83. Int. Press, Cambridge, MA, 1998.

LT99. Jun Li and Gang Tian. Comparison of algebraic and symplectic Gromov-Witten invariants. *Asian J. Math.*, 3(3):689–728, 1999.

LZ09. Tian-Jun Li and Weiyi Zhang. Comparing tamed and compatible symplectic cones and cohomological properties of almost complex manifolds. *Comm. Anal. Geom.*, 17(4):651–683, 2009.

Mat95. Olivier Mathieu. Harmonic cohomology classes of symplectic manifolds. *Comment. Math. Helv.*, 70(1):1–9, 1995.

McD84. Dusa McDuff. Examples of simply-connected symplectic non-Kählerian manifolds. *J. Differential Geom.*, 20(1):267–277, 1984.

McL14. Mark McLean. Symplectic invariance of uniruled affine varieties and log Kodaira dimension. *Duke Math. J.*, 163(10):1929–1964, 2014.

MM08. Xiaonan Ma and George Marinescu. Generalized Bergman kernels on symplectic manifolds. *Adv. Math.*, 217(4):1756–1815, 2008.

Moh03. J.-P. Mohsen. Ph.d. thesis. *unpublished*, 2003.

Mor01. Shigeyuki Morita. *Geometry of differential forms*, volume 201 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2001. Translated from the two-volume Japanese original (1997, 1998) by Teruko Nagase and Katsumi Nomizu, Iwanami Series in Modern Mathematics.

MPS02. V. Muñoz, F. Presas, and I. Sols. Almost holomorphic embeddings in Grassmannians with applications to singular symplectic submanifolds. *J. Reine Angew. Math.*, 547:149–189, 2002.

MS94. Dusa McDuff and Dietmar Salamon. *J-holomorphic curves and quantum cohomology*, volume 6 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1994.

MS98. Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

MS04. Dusa McDuff and Dietmar A. Salamon. *J-holomorphic curves and symplectic topology*, volume 52 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.

MT93. Dusa McDuff and Lisa Traynor. The 4-dimensional symplectic camel and related results. In *Symplectic geometry*, volume 192 of *London Math. Soc. Lecture Note Ser.*, pages 169–182. Cambridge Univ. Press, Cambridge, 1993.

Nic11. Liviu Nicolaescu. *An invitation to Morse theory.* Universitext. Springer, New York, second edition, 2011.

Ops13. Emmanuel Opshtein. Singular polarizations and ellipsoid packings. *Int. Math. Res. Not. IMRN*, (11):2568–2600, 2013.

Pao01. Roberto Paoletti. Symplectic submanifolds in special position. *Israel J. Math.*, 123:241–251, 2001.

Par01. J. Pardon. Pseudohomology and homology. *arXiv:math/0111223*, 2001.

Par14. J. Pardon. An algebraic approach to virtual fundamental cycles on moduli spaces of j-holomorphic curves. *arXiv:1309.2370*, 2014.

Pol98. Leonid Polterovich. Symplectic aspects of the first eigenvalue. *J. Reine Angew. Math.*, 502:1–17, 1998.

RT95. Yongbin Ruan and Gang Tian. A mathematical theory of quantum cohomology. *J. Differential Geom.*, 42(2):259–367, 1995.

Sal97. D. A. Salamon. Lectures on floer homology. *IAS/Park City Graduate Summer School*, 1997.

Sal12. D. Salamon. Uniqueness of symplectic structures. *arXiv:1211.2940*, 2012.

Sch99. Matthias Schwarz. Equivalences for Morse homology. In *Geometry and topology in dynamics (Winston-Salem, NC, 1998/San Antonio, TX, 1999)*, volume 246 of *Contemp. Math.*, pages 197–216. Amer. Math. Soc., Providence, RI, 1999.

Sie99a. Bernd Siebert. Algebraic and symplectic Gromov-Witten invariants coincide. *Ann. Inst. Fourier (Grenoble)*, 49(6):1743–1795, 1999.

Sie99b. Bernd Siebert. Symplectic Gromov-Witten invariants. In *New trends in algebraic geometry (Warwick, 1996)*, volume 264 of *London Math. Soc. Lecture Note Ser.*, pages 375–424. Cambridge Univ. Press, Cambridge, 1999.

Tau95. Clifford Henry Taubes. The Seiberg-Witten and Gromov invariants. *Math. Res. Lett.*, 2(2):221–238, 1995.

Thu76. W. P. Thurston. Some simple examples of symplectic manifolds. *Proc. Amer. Math. Soc.*, 55(2):467–468, 1976.

Tis77. David Tischler. Closed 2-forms and an embedding theorem for symplectic manifolds. *J. Differential Geometry*, 12(2):229–235, 1977.

TW11. Valentino Tosatti and Ben Weinkove. The Calabi-Yau equation, symplectic forms and almost complex structures. In *Geometry and analysis. No. 1*, volume 17 of *Adv. Lect. Math. (ALM)*, pages 475–493. Int. Press, Somerville, MA, 2011.

Voi02. Claire Voisin. A counterexample to the Hodge conjecture extended to Kähler varieties. *Int. Math. Res. Not.*, (20):1057–1075, 2002.

Voi04. Claire Voisin. On the homotopy types of compact Kähler and complex projective manifolds. *Invent. Math.*, 157(2):329–343, 2004.

War83. Frank W. Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition.

Wen13. C. Wendl. Lectures on holomorphic curves in symplectic and contact geometry. *arXiv:1011.1690*, 2013.

Wen14. Chris Wendl. Contact hypersurfaces in uniruled symplectic manifolds always separate. *J. Lond. Math. Soc. (2)*, 89(3):832–852, 2014.

Zin08. Aleksey Zinger. Pseudocycles and integral homology. *Trans. Amer. Math. Soc.*, 360(5):2741–2765, 2008.