K-theoretic methods in the representation theory of p-adic analytic groups

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Abstract

In chapter 3, we prove the following: Let p be a prime number such that $p \geq 5$. Let $G = H \times Z$, where H is a torsion free compact p-adic analytic group such that its Lie algebra is split semisimple over $\mathbb{Q}_p$ and $Z \cong \mathbb{Z}_p^n$, where $n \geq 0$. Let M be a finitely generated torsion module over the Iwasawa algebra Λ_G of G, such that it has no non-zero pseudo-null submodules. Let $q(M)$ denote the image of M in the quotient category mod-$\Lambda_G/\mathcal{C}^1_{\Lambda_G}$ via the quotient functor q, where $\mathcal{C}^1_{\Lambda_G}$ denotes the Serre-subcategory of pseudo-null Λ_G-modules of Λ_G-modules, mod-Λ_G. Then $q(M)$ is completely faithful if and only if M is Λ_Z-torsion free.

We denote by $\mathfrak{N}_H(G)$, the category of finitely generated Λ_G-modules that are also finitely generated as Λ_H-modules. In chapter 4, we prove the following theorem: Let G and p be as in chapter 3. Let M, $N \in \mathfrak{N}_H(G)$ such that they have no non-zero pseudo-null Λ_G-submodules and let $q(M)$ be completely faithful. If $[M] = [N]$ in $K_0(\mathfrak{N}_H(G))$ then $q(N)$ is also completely faithful.

Let now G be an arbitrary compact p-adic analytic group with no element of order p. Choose an open normal uniform pro-p subgroup H of G. Let K be a finite extension of $\mathbb{Q}_p$ such that it contains all the n-th roots of unity, where $n := |G/H|$. Define $K[[G]] := K \otimes_{\mathbb{Z}_p} \Lambda_G$. In chapter 5, we prove that $K_0(K[[G]]) \cong \mathbb{Z}^c$, where c is the number of conjugacy classes of G/H of order relative prime to p. We also prove that if $r \in p^{\mathbb{Q}}$ such that $1/p < r < 1$ and the absolute ramification index e of K satisfies that $r = p^{-m/e}$, for an appropriate $m \in \mathbb{N}$, then $K_0(D_{<r}(G, K))$ is isomorphic to $\mathbb{Z}^c$, where c is the number of conjugacy classes of G/H of order relative prime to p. Moreover, we prove that the canonical injection $K[[G]] \to D(G, K)$ induces an injective map $\mathbb{Z}^c \to K_0(D(G, K))$.

Contents

1 Introduction

Let us fix a prime number p. A p-adic analytic group is a p-adic manifold which is also a group, the group operations being analytic functions, i.e. locally given by formal power series with coefficients from $\mathbb{Q}_p$. p-adic analytic groups include a wide variety of classes of groups. To give an example, the class of linear algebraic groups over $\mathbb{Q}_p$ is included in the class of p-adic analytic groups, the group operations given locally by polynomials with coefficients from $\mathbb{Q}_p$. Certainly, this includes the general linear group $GL_n(\mathbb{Q}_p)$. This group has deep connections with the local Langlands correspondence. Another, and for us more important, example is the compact open subgroup $GL_n(\mathbb{Z}_p)$ of $GL_n(\mathbb{Q}_p)$.
In fact, if we are given a pro-p group G of finite rank, one characterization of G being p-adic analytic is that G is a closed subgroup of $GL_n(\mathbb{Z}_p)$ for some $n \geq 1$ (See Interlude A in [19]).
Michel Lazard in the 1960's proved a striking result in his famous paper, *Groups analytiques p-adiques* [25]. He characterized p-adic analytic groups in a completely group-theoretic manner, without using any 'analytic' machinery. More precisely, he proved that a topological group G is p-adic analytic if and only if it contains an open subgroup H which is a powerful finitely generated pro-p group. All the required properties on the subgroup in the theorem are defined in a completely group-theoretic fashion. Recently, this theorem has other useful variations, one of them is that the topological group G is p-adic analytic if and only if G has an open normal uniform pro-p subgroup H.
p-adic analytic groups have many connections to various fields of mathematics, especially number theory and arithmetic geomerty. They play important role in formulating question about arithmetic objects, related to elliptic curves. The background (and motivation) of Chapter 3 and Chapter 4, which lies in noncommutative Iwasawa theory for elliptic curves, serves as a concrete example: The arithmetic of elliptic curves and especially the conjectures of Birch and Swinnerton-Dyer have been lying in the centre of research in arithmetic geometry. The motivation to develop Iwasawa theory is that it could provide a powerful tool to attack various arithmetic questions, especially the above mentioned conjectures. The idea is to relate various arithmetic objects to complex L-functions via a so-called p-adic L-function. The main conjectures of Iwasawa theory provide one of the most competent general methods known at present for studying the mysterious relationship between purely arithmetic problems and the special values of complex L-functions, typified by the conjecture of Birch and Swinnerton-Dyer and its generalizations.
The Iwasawa theory for the field obtained by adjoining all p-power roots of unity to $\mathbb{Q}$ is now very well understood and complete. It seems natural to expect a precise analogue of this theory to exist for the field obtaind by adjoining to $\mathbb{Q}$ all the p-power division points on an elliptic curve E defined over $\mathbb{Q}$. When E

admits complex multiplication, i.e. the endomorphism ring of E is larger than the integers, this is known to be true. However, when E does not admit complex multiplication, very little is known.

In 2004, the authors in [17] formulated the main conjecture for Iwasawa theory for elliptic curves over $\mathbb{Q}$ without complex multiplication. We write $\mathbb{Q}^{\text{cyc}}$ for the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, and put $\Gamma = \text{Gal}(\mathbb{Q}^{\text{cyc}}/Q) \cong \mathbb{Z}_p$, i.e. the Galois group of the extension. Let E be an elliptic curve defined over $\mathbb{Q}$ and E_{p^∞} the group of all p-power division points on E. We define

$$F_\infty := \mathbb{Q}(E_{p_\infty}).$$

By the Weil pairing, $\mathbb{Q}(\mu_{p^\infty}) \subset F_\infty$, where μ_{p^∞} denotes the group of all p-power roots of unity. Hence F_∞ contains $\mathbb{Q}^{\text{cyc}}$ and we define G :=Gal$(F_\infty/\mathbb{Q})$ and H :=Gal$(F_\infty/\mathbb{Q}^{\text{cyc}})$ which is a normal subgroup of G. Then $G/H \cong \Gamma \cong \mathbb{Z}_p$. One of the celebrated theorems of Serre is that G is an open subgroup of $GL_2(\mathbb{Z}_p)$. Therefore, it is a 4 dimensional p-adic analytic group. Unfortunately, in general it is non-abelian. Let us consider the Iwasawa algebra over G, i.e. $\Lambda_G := \varprojlim_{N \triangleleft_o G} \mathbb{Z}_p[G/N]$ which is in general non-commutative. Let us denote by $X(E/F_\infty)$, the compact Pontrjagin dual of the Selmer group $S(E/F_\infty)$ of E over F_∞. Then $X(E/F_\infty)$ becomes a module over Λ_G, endowed with its natural Λ_G-module structure. The non-commutative property of Λ_G raises many obstacles in formulating the main conjectures in the case. One of the most problematic is to find the right definition of the characteristic element: There is a natural left and right Ore set S in Λ_G, defined by all the elements $f \in \Lambda_G$ such that $\Lambda_G/f\Lambda_G$ is a finitely generated Λ_H-module. Let $S^* := \cup_{n\geq 0} p^n S$. It was proven in [16], that S^* is also a left and right Ore set in Λ_G. Let Λ_{G,S^*} denote the localization of Λ_G at S^* and denote by $\mathfrak{M}_H(G)$, the category of finitely generated S^*-torsion Λ_G-modules. Mainly, as a consequence of Quillen's localization sequence in algebraic K-theory, we have a map

$$\partial_G : K_1(\Lambda_{G,S^*}) \to K_0(\mathfrak{M}_H(G))$$

where $K_0(\mathfrak{M}_H(G))$ denotes the Grothendieck group of the category $\mathfrak{M}_H(G)$ and K_1 the Whitehead group of the ring Λ_{G,S^*} (for precise definitions, see Chapter II. and III. in [31]). It was proven in [17] that ∂_G is surjective. We then define a characteristic element of a finitely generated S^*-torsion Λ_G-module M to be an inverse image of $[M] \in K_0(\mathfrak{M}_H(G))$ via ∂_G. The first conjecture in [17] states that under suitable assumptions on E and p, $X(E/F_\infty)$ is an object in $\mathfrak{M}_H(G)$. If we assume the first one to be true, then the second conjecure in [17] states that we can define a certain p-adic L-function $\mathcal{L}_E$ in $K_1(\Lambda_{G,S^*})$, attached to the elliptic curve E, interpolating special values of the complex L-functions. These two conjectures serve as a backbone of the main conjecture which states that $\mathcal{L}_E$ is in fact a characteristic element of $X(E/F_\infty)$. To attack these conjectures, it is rather natural to start with investigating the stucture of the (Λ_G-torsion) modules over the Iwasawa algebra Λ_G. In [16], the authors define pseudo-null modules over the Iwasawa algebra Λ_G, where G is a p-valued compact p-adic analytic group. They also prove an analog of the existing structure theorem in the commutative setting for finitely generated torsion modules over Λ_G up to pseudo-isomorphism in the non-commutative case. The category of pseudo-null modules $\mathcal{C}^1_{\Lambda_G}$ is a Serre subcategory of the category of modules over Λ_G. Hence there is a unique quotient category mod-$\Lambda_G/\mathcal{C}^1_{\Lambda_G}$ and a unique quotient functor

$$q : \text{mod-}\Lambda_G \to \text{mod-}\Lambda_G/\mathcal{C}^1_{\Lambda_G}.$$

Moreover, pseudo-null Λ_G-modules are contained in the category $\mathcal{C}^0_{\Lambda_G}$ of Λ_G-torsion modules. In [16], it was shown that in this quotient category, there are two basic 'building blocks', namely the completely faithful objects and the locally bounded objects. More precisely, if M is a Λ_G-torsion module then $q(M)$ decomposes uniquely as $q(M) = \mathcal{M}_0 \oplus \mathcal{M}_1$, where $\mathcal{M}_0$ is a completely faithful object and $\mathcal{M}_1$ is a locally bounded object in the quotient category. The authors in [16], with an eye on the GL_2 conjectures, also raise a number of questions concerning the structure of $X(E/F_\infty)$. Two of them which motivated our investigation:

1. Let Z be the center of G. Is $X(E/F_\infty)$ torsion-free over Λ_Z?
2. With some assumptions on the elliptic curve E, $X(E/F_\infty)$ is Λ_G-torsion. Is $q(X(E/F_\infty))$ completely faithful?

As for the first question, the author in [1] proved the following: Let $G = H \times Z$ where H is a torsion-free compact p-adic analytic group such that its Lie algebra is split semisimple over $\mathbb{Q}_p$ and $Z \cong \mathbb{Z}_p$. Let M be a finitely generated Λ_G-torsion module, which has no non-zero pseudo-null submodules. Then $q(M)$ is completely faithful if and only if M is Λ_Z-torsion free. We remark that the assumptions on G are fairly mild. Any open pro-p group of $GL_2(\mathbb{Z}_p)$ satisfies them, since we can take H to be $G \cap SL_2(\mathbb{Z}_p)$.
In Chapter 3, we give a generalized version of this:

Theorem 3.1.2: Let G be the group $H \times Z$, where H is a torsion free compact p-adic analytic group such that its Lie algebra is split semisimple over $\mathbb{Q}_p$ and $Z \cong \mathbb{Z}_p^n$, where $n \geq 0$. Let M be a finitely generated torsion Λ_G-module such that it has no non-zero pseudo-null submodules. Then $q(M)$ is completely faithful if and only if M is Λ_Z-torsion free.

This generalized theorem proved to be useful in one of the main results of [29].
As for the second question, let us denote by $\mathfrak{N}_H(G)$, the category of finitely generated Λ-modules that are finitely generated as Λ_H-modules. It is not always

true that $X(E/F_\infty)$ is finitely generated over Λ_H, but if we pose some suitable hypothesis on G, p, and E (see Proposition 7.1 in [17]), in fact it is. One more interesting connection between the category $\mathfrak{M}_H(G)$ (hence the first conjecture) and the objects of $\mathfrak{N}_H(G)$ was proven in [16]. Namely, that a finitely generated Λ_G-module M belongs to $\mathfrak{M}_H(G)$ if and only if $M/M(p)$ belongs to $\mathfrak{N}_H(G)$, where $M(p)$ denotes the p-primary submodule of M. So if the first conjecture is true, then $X(E/F_\infty)/X(E/F_\infty)(p)$ belongs to $\mathfrak{N}_H(G)$. In Chapter 4, we prove the following theorem:

Theorem 4.1.1: Let p be a prime number such that $p \geq 5$. Let H be a torsion-free compact p-adic analytic group whose Lie algebra $\mathcal{L}(H)$ is split semisimple over $\mathbb{Q}_p$ and let $G = H \times Z$ where $Z \cong \mathbb{Z}_p^n$ for some non-negative integer n. Let M, $N \in \mathfrak{N}_H(G)$ such that they have no non-zero pseudo-null Λ_G-submodules and let $q(M)$ be completely faithful. If $[M] = [N]$ in $K_0(\mathfrak{N}_H(G))$ then $q(N)$ is also completely faithful.

One interesting consequence this theorem is that the completely faithful property in the category $\mathfrak{N}_H(G)$ is "K_0-invariant". Therefore it brings us closer to answer the second question, since, for example if $X(E/F_\infty) \in \mathfrak{N}_H(G)$, it is now enough to prove that the one of the modules M in the class $[X(E/F_\infty)]$ satisfies that $q(M)$ is completely faithful. It also suggests that even the characteristic element of $X(E/F_\infty)$ might 'carry' the information about the completely faithful property. There are important examples when $X(E/F_\infty) \in \mathfrak{N}_H(G)$, see for example Proposition 7.2, Example 7.7, Proposition 7.8 in [17].

In the last chapter, we turn our attention towards other aspects of p-adic analytic groups. Namely, we investigate some questions connected to the module categories of distribution algebras of p-adic analytic groups. In a series of papers [38], [39], [41], [42], [43], [44], the authors develop and systematically study continuous and locally analytic representations of compact p-adic analytic groups. These representations include many interesting well-known representation types, for example when the group is the group of K-points of an algebraic group where K is a finite extension of $\mathbb{Q}_p$, then locally analytic representations include principal series representations, finite dimensional algebraic representations, and smooth representations. As in the classical representation theory of finite groups, it is convenient to find a suitable algebra and a suitable module category of which objects correspond to the representations of intereset. After finding a reasonable finitness condition for both continuous and locally analytic representations, called admissibility, it turns out that in the continuous case, the admissible continuous represenations correspond to finitely generated $K[[G]]$-modules, where $K[[G]] = K \otimes_{\mathbb{Z}_p} \Lambda_G$. The locally analytc case is somewhat more complex. Consider the K-Banach space

$C^{an}(G,K)$ of locally analytic function on G, i.e. those functions that locally given by convergent power series. Let $D(G,K)$ denote its dual space with the strong topology. $D(G,K)$ is called the locally analytic distribution algebra of G. In [38], the authors show that $D(G,K)$ is a Fréchet-Stein algebra: For a moment, let us assume that $G = H$ is a uniform pro-p group of dimension d and choose a minimal (ordered) topological generating set $h_1, \ldots, h_d$. Then there is a bijective global chart

$$\mathbb{Z}_p^d \xrightarrow{\sim} H$$
$$(x_1, \ldots, x_d) \mapsto (h_1^{x_1}, \ldots, h_d^{x_d}).$$

Putting $b_i := h_i - 1$, $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, $|\alpha| = \sum \alpha_i$ and $b^\alpha := b_1^{\alpha_1} \ldots b_d^{\alpha_d}$, one can identify $D(H,K)$ with all convergent power series

$$\sum_\alpha d_\alpha b^\alpha,\ d_\alpha \in K,\ \text{such that the set } \{|d_\alpha| r^{|\alpha|}\}$$

is bounded for all $0 < r < 1$. Let κ be 2, if $p = 2$ and let $\kappa = 1$, if $p > 2$. Then for any $r \in p^{\mathbb{Q}}$, $1/p \leq r < 1$, we have a multiplicative norm $|| \, ||_r$ on $D(H,K)$ given by

$$||\lambda||_r := \sup_\alpha |d_\alpha| r^{\kappa|\alpha|}.$$

Whenever we are given an arbitrary compact p-adic analytic group G, we can choose an open uniform normal subgroup H, with index $n := |G/H|$. Choose a set $g_1, \ldots, g_n$ of coset representatives of G/H. Then $D(G,K)$ is actually the crossed product of $D(H,K)$ and G/H. In paricular,

$$D(G,K) = \bigoplus_{k=1}^{n} D(H,K) g_k.$$

Hence, we define the norm on $D(G,K)$ with respect to a parameter r as the maximum norm, i.e.

$$||\mu||_r := \max(||\lambda_1||_r, \ldots, ||\lambda_n||_r)$$

where $\mu = \sum \lambda_k g_k$ is an arbitrary element of $D(G,K)$. Denote by $D_r(G,K)$ the completion of $D(G,K)$ with respect to $|| \, ||_r$. Fix a sequence of real numbers $1/p \leq r_1 \leq r_2 \leq \cdots < 1$ such that $r_i \in p^{\mathbb{Q}}$ and $r_i \to 1$ if $i \to \infty$. Then the distribution algebra is the projective limit of the Noetherian K-Banach algebras $D_{r_i}(G,K)$. Moreover, the maps $D_{r_i}(G,K) \to D_{r_j}(G,K)$, where $r_j < r_i$, are flat. This shows, by definition, that $D(G,K)$ is a Fréchet-Stein algebra. This also leads us to the right definition of admissibility. Indeed, following [38], we call a $D(G,K)$-module M coadmissible, if $M \cong \varprojlim_{r_i} M_{r_i}$, where M_{r_i} are finitely generated $D_{r_i}(G,K)$-modules such that there is an isomorphism

$$M_{r_{i-1}} \cong M_{r_i} \otimes_{D_{r_i}(G,K)} D_{r_{i-1}}(G,K)$$

for any positive integer i. A coadmissible module need not be finitely generated, neither is a finitely generated module always coadmissible. An easy example is the following: Consider an ideal I in $D(\mathbb{Z}_p, K)$ that is not closed (i.e. not finitely generated), then $D(\mathbb{Z}_p, K)/I$ is finitely generated, but not coadmissible. We call a locally analytic representation admissible if the corresponding $D(G, K)$-module is coadmissible. In [41], the authors show that the category of coadmissible modules is abelain. If we are given a (skeletally small) exact category $\mathcal{A}$, it often very useful to compute the Grothendieck group $K_0(\mathcal{A})$ to extract informations about objects themselfs. The most basic example is that if we look at the exact category of finitely generated projective modules over a ring R, then if the Grothendieck group of the category of finitely generated projective R-modules, denoted by $K_0(R)$, is isomorphic to $\mathbb{Z}$, then it shows that every finitely generated projective R-module is stably free. One can ask the most basic questions: What is $K_0(D(G, K))$ and $K_0(K[[G]])$? If G is a uniform pro-p group, in [38] the authors define another K-Banach algebra, the so-called algebra of bounded distributions, denoted by $D_{<r}(G, K)$, where $r \in p^{\mathbb{Q}}$, $1/p < r < 1$. They also show that if we have a sequence of parameters $1/p < r_1 \leq r_2 \leq \cdots \leq r_n \leq \cdots < 1$ such that $r_n \in p^{\mathbb{Q}}$ for all $n \in \mathbb{N}$, then

$$D(G, K) \cong \varprojlim_i D_{r_i}(G, K).$$

In Chapter 5, we define the algebra of bounded distributions for arbitrary compact p-adic analytic groups and for any $r \in p^{\mathbb{Q}}$ such that $1/p < r < 1$. This algebra is in many ways better suited for our purpose, i.e. to compute the Grothendieck group of $D(G, K)$. For example, without going into details right now, the graded 0-th part $\mathrm{gr}^0 D_{<r}(G, K)$ of the associated graded ring of $D_r(G, K)$ has many nice properties that $\mathrm{gr}^0 D_r(G, K)$ does not possess. A number of natural questions arise:

1. What is $K_0(D_{<r}(G, K))$ for an arbitrary r such that $r \in p^{\mathbb{Q}}$ and $1/p < r < 1$?

2. Does the projective limit commute with $K_0(\ )$, i.e. is it true that

$$\varprojlim_{r_i} K_0(D_{<r}(G, K)) \cong K_0(\varprojlim_{r_i} D_{<r_i}(G, K))?$$

Let G be an arbitrary p-adic analytic group with no element of order p. Choose an open normal subgroup H of G that is a uniform pro-p group. Then under a mild condition on the field K, i.e. that it contains all the n-th roots of unity, where $n = |G/H|$, we prove the following theorems:

Theorem 5.2.4: $K_0(K[[G]]) \cong \mathbb{Z}^c$, where c is the number of conjugacy classes of G/H of order relative prime to p.

Let us consider a fixed parameter $r \in p^{\mathbb{Q}}$ such that $1/p < r < 1$. Assume that K satisies that it has absolute ramification index e with the property that $r = p^{-m/e}$ for an appropriate $m \in \mathbb{N}$. Then

Theorem 5.5.1: $K_0(D_{<r}(G,K))$ is isomorphic to $\mathbb{Z}^c$, where c is the number of conjugacy classes of G/H of order relative prime to p.

Of course, if G is a uniform pro-p group, then as a consequence of this theorem, $K_0(D_{<r}(G,K)) \cong \mathbb{Z}$, i.e. every finitely generated projective $D_{<r}(G,K)$-module is stably free. As an other application of the previous theorem, we will get an injective map $\mathbb{Z}^c \to K_0(D(G,K))$. We very much suspect that this map is in fact an isomorphism. We also get some partial results on the Grothendieck group of $D_r(G,K)$.

2 Preliminaries

2.1 Ring theoretic notions

In this section we collect all the notions from category theory, K-theory and ring theory that come up throughout the thesis. We also build up all the tools that we use in our proofs.

2.1.1 Serre subcategories

Let $\mathcal{A}$ be an abelian category. We call a (non-empty) full subcategory $\mathcal{B} \subset \mathcal{A}$ **Serre-subcategory** if whenever there is an exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$ then $A, C \in \mathcal{B}$ if and only if $B \in \mathcal{B}$. The following lemma is trivial, but it is still very useful.

Lemma 2.1.1. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{B}$ be a Serre subcategory of $\mathcal{A}$. Then

(i) $0 \in Ob(\mathcal{B})$,

(ii) $\mathcal{B}$ is a strictly full subcategory of $\mathcal{A}$, i.e. it is closed under isomorphisms,

(iii) any subobject or quotient of an object in $\mathcal{B}$ is an object of $\mathcal{B}$, i.e. $\mathcal{B}$ is closed under subobjects and quotients.

Example 2.1.2. Let $\mathcal{A}, \mathcal{B}$ abelian categories and $F : \mathcal{A} \to \mathcal{B}$ an exact functor. The full subcategory of objects $A \in \mathcal{A}$ such that $F(A) = 0$ is a Serre subcategory of $\mathcal{A}$.

Proof. It follows from the definition □

We call the subcategory in the example above the **kernel of the functor** F. It is well-known that if $\mathcal{B}$ is a Serre-subcategory, we can form a quotient category $\mathcal{A}/\mathcal{B}$ characterized by the following universal property:

Proposition 2.1.3. Let $\mathcal{A}$ be an abelian category and $\mathcal{B} \subset \mathcal{A}$ a Serre subcategory. There exists an abelian category $\mathcal{A}/\mathcal{B}$ and an exact functor

$$q : \mathcal{A} \to \mathcal{A}/\mathcal{B}$$

which is essentially surjective and its kernel is $\mathcal{B}$. The category $\mathcal{A}/\mathcal{B}$ and the functor q are characterized by the following universal property: For any exact functor $G : \mathcal{A} \to \mathcal{C}$ such that $\mathcal{B} \subset \mathrm{Ker}(G)$ there exists a factorization $G = H \circ q$ with a unique exact functor $H : \mathcal{A}/\mathcal{B} \to \mathcal{C}$.

Proof. See Corollary 3.11 Chapter IV. in [33]. □

2.1.2 Pseudo-null modules, fractional ideals and c-ideals

The notion of pseudo-null modules is fundamental for one to have a nice structure theorem for finitely generated torsion modules over both commutative and non-commutative Iwasawa algebras. Let R be an associative ring with identity element. We denote the category of right R-modules by mod-R and unless stated otherwise an R-module will always mean a right R-module. For an arbitrary R-module L, denote by $E(L)$ the injective hull of L. Consider the minimal injective resolution of L, i.e.

$$0 \longrightarrow L \xrightarrow{\mu_0} E_0 \xrightarrow{\mu_1} E_1 \xrightarrow{\mu_2} \dots$$

where $E_0 = E(L)$ and $E_i = E(\mathrm{coker}(\mu_i))$.

Definition 2.1.4. Let M be an R-module. Then we denote by $\mathcal{C}_L^n$ the subcategory of mod-R in which the objects are modules $M \in$ mod-R such that $\mathrm{Hom}_R(M, E_0 \oplus E_1 \oplus \dots E_n) = 0$.

Lemma 2.1.5. An R-module M lies in $\mathcal{C}_L^n$ if and only if $\mathrm{Ext}_R^i(M', L) = 0$ for any R-submodule $M' \subseteq M$ and for all $i \leq n$.

Proof. See Lemma 1.1 in [16] □

Throughout this section we assume that R is a Noetherian domain.

Proposition 2.1.6.

$$\mathcal{C}_R^0 = \text{full subcategory of all torsion } R\text{-modules } M$$

Proof. It follows from the well-known theorem by Goldie (Theorem 2.3.6 in [28]) that R has a skewfield of fractions $Q(R)$. By Proposition 3.8, Chapter II in [47], $Q(R)$ is an injective R-module, hence $E(R) = Q(R)$. □

Definition 2.1.7. The objects of the subcategory $\mathcal{C}_R^1$ are called **pseudo-null** modules.

The category of pseudo-null modules is a full subcategory of mod-R. Moreover, it is a Serre subcategory which is easy to see from the definition and the existence of the long exact sequence of cohomology for an arbitrary short exact sequence of R-modules. It is also easy to see that any R-module has a largest unique submodule contained in $\mathcal{C}_R^1$. By Proposition 2.1.3, we have the quotient category mod-$R/\mathcal{C}_R^1$ and the quotient functor

$$q : \text{mod-}R \to \text{mod-}R/\mathcal{C}_R^1.$$

One important observation is that every pseudo-null module is automatically a torsion R-module. This follows from Lemma 2.1.5 and Proposition 2.1.6.

Definition 2.1.8. Let L be a right R-module such that $L \subseteq Q(R)$. Then it is called **fractional right ideal** if it is non-zero and there is a $q \in Q(R)$ such that $q \neq 0$ and $L \subseteq qR$.

One can define fractional left ideals similarly. If we have a fractional right ideal L, one defines its **inverse** by

$$L^{-1} := \{q \in Q(R) \mid qL \subseteq R\}$$

which is a fractional left ideal.

There is a similar definition of the inverse for fractional left ideals. Let us consider the dual of L, i.e. $L^* = \mathrm{Hom}_R(L, R)$. This is a left R-module and there is a natural isomorphism $u : L^{-1} \to L^*$ that sends an element $l \in L^{-1}$ to the right R-module homomorphism induced by left multiplication by l. The following elementary lemma is useful to compute L^{-1} in some special cases.

Lemma 2.1.9. Let R be a Noetherian domain and I be a non-zero right ideal of R. Then $I^{-1}/R \cong \mathrm{Ext}^1(R/I, R)$.

Proof. It follows from the long exact sequence of cohomology applied to the exact sequence $0 \to I \to R \to R/I \to 0$ and the fact that $L^{-1} \cong L^*$. □

Definition 2.1.10. Let I be a fractional right ideal. **The reflexive closure** of I is defined to be $\overline{I} := (I^{-1})^{-1}$. This is also a fractional right ideal and it contains I. I is called **reflexive** if it is the same as its reflexive closure, i.e. $I = \overline{I}$.

One can say equivalently that $I \to (I^*)^*$ is an isomorphism. The next proposition will be quite useful, since it shows the connection between ring extensions and reflexive closures.

Proposition 2.1.11. Let $R \hookrightarrow S$ be a ring extension such that R is Noetherian and S is flat as a left and right R-module. Then there is a natural isomorphism

$$\psi_M^i : S \otimes_R \operatorname{Ext}_R^i(M, R) \to \operatorname{Ext}_S^i(M \otimes_R S, S)$$

for all finitely generated right R-modules and all $i \geq 0$. A similar statement holds for left R-modules. If moreover S is a Noetherian domain, then

(i) $\overline{I \cdot S} = \overline{I} \cdot S$ for all right ideals I of R.

(ii) If moreover J is a reflexive right S-ideal, then $I \cap R$ is a reflexive right R-ideal.

Proof. See Proposition 1.2 in [9] □

Definition 2.1.12. Let L be a fractional right ideal which is also a fractional left ideal. We say that L is a **fractional c-ideal** if it is reflexive on both sides. L is called simply a **c-ideal** if $L \subseteq R$. If L is in addition a prime ideal, then we call it **prime c-ideal**.

Later we will be interested in prime c-ideals of Iwasawa algebras. In some cases it is possible to explicitly determine the structure of a proper c-ideal:

Proposition 2.1.13. Let R be a Noetherian domain and I be a proper c-ideal of R. Let $x \in R$ be an element such that x is non-zero, central in R. Assume moreover that R/xR is a domain and $x \in I$. Then $I = xR$.

Proof. See Lemma 2.2 in [1]. □

We turn our attention to a special class of rings, the so-called maximal orders. We will see that, if such a ring is given, there is a very nice way to determine all the fractional c-ideals of the ring, once the prime c-ideals are determined.

Definition 2.1.14. A Noetherian domain R is called **maximal order** in its skewfield of fractions $Q(R) = Q$ if whenever there is a subring S of Q containing R such that $aSb \subseteq R$ for some non-zero elements $a, b \in Q$, then $S = R$.

Lemma 2.1.15. The commutative maximal orders are the integrally closed domains.

Proof. See Lemma 5.3.3 in [28] □

Consider the set $G(R)$ of fractional c-ideals of R. Assano showed in [10] that $G(R)$ is an Abelian group with the following operations:

$$I \cdot J := \overline{IJ},\ I \to I^{-1}$$

Moreover, he proved the following theorem:

Theorem 2.1.16. $G(R)$ is a free Abelian group and the free generators of $G(R)$ are the prime c-ideals of R.

Proof. See II.1.8. and II.2.6. in [27] □

2.1.3 Completely faithful and locally bounded objects

Throughout this section, we assume that R is a Noetherian maximal order without zero divisors. Recall that the category $\mathcal{C}_R^1$ of pseudo-null R-modules is a Serre subcategory. Hence by Proposition 2.1.3, it makes sense to talk about the quotient category mod-$R/\mathcal{C}_R^1$ and moreover, we are given the quotient functor q : mod-$R \to$ mod-$R/\mathcal{C}_R^1$ which is exact. Completely faithful objects can be seen as one of the basic building blocks in the quotient category mod-$R/\mathcal{C}_R^1$, along with locally bounded objects. Moreover, completely faithful objects play important role in many questions regarding arithmetic objects related to elliptic curves.

Definition 2.1.17. Let $\mathcal{M}$ be an object of mod-$R/\mathcal{C}_R^1$. The **annihilator ideal** of $\mathcal{M}$ is defined as follows:

$$\operatorname{ann}(\mathcal{M}) := \sum \{\operatorname{ann}_R(N) \mid q(N) \cong \mathcal{M}\}$$

$\mathcal{M}$ is said to be **completely faithful** if $\operatorname{ann}(\mathcal{L}) = 0$ for any non-zero subquotient object $\mathcal{L}$ of $\mathcal{M}$. It is called **locally bounded** if $\operatorname{ann}(\mathcal{N}) \neq 0$ for any subobject $\mathcal{N} \subseteq \mathcal{M}$.

The following two propositions will be used frequently. The first one provides a structure theorem for the images of torsion R-modules in terms of completely faithful and locally bounded objects.

Proposition 2.1.18. Any object $\mathcal{M}$ in the quotient category $\mathcal{C}_R^0/\mathcal{C}_R^1$ decomposes uniquely into a direct sum $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$ where $\mathcal{M}_0$ is a completely faithful and $\mathcal{M}_1$ is a locally bounded object.

Proof. See Proposition 5.1 (i) in [16] □

We will call an R-module M **bounded** if its annihilator (in the classical sense) is not zero, i.e. $\mathrm{ann}_R(M) \neq 0$.

Proposition 2.1.19. Let us assume that R is a Noetherian domain and a maximal order. Let M be a finitely generated bounded torsion R-module, and let M_0 be its maximal pseudo-null submodule. Then

(*i*) $\mathrm{ann}_R(M/M_0) = \mathrm{ann}(q(M))$,

(*ii*) $\mathrm{ann}(q(M))$ is a c-ideal.

Proof. See Lemma 5.3 (i) in [16]. □

Now that we have all the definitions in hand, we end this section by stating two more results. One gives an alternative description of pseudo-null modules in special cases, and the other gives a very nice characterization of the reflexive closure of a non-zero ideal in a unique factorization domain.

Proposition 2.1.20. Let R be a Noetherian domain and let M be a finitely generated R-module. Then

(*i*) M is pseudo-null if and only if $\mathrm{ann}_R(x)^{-1} = R$ for all $x \in M$.

(*ii*) If R is commutative then M is pseudo-null if and only if $\mathrm{ann}_R(M)^{-1} = R$.

Proof. See Proposition 1.3 in [9]. □

Proposition 2.1.21. Let R be a commutative unique factorization domain and I a non-zero ideal of R. Then $\overline{I} = xR$ for some $x \in R$ and xR/I is pseudo-null.

Proof. See Lemma 1.4 in [9]. □

Remark 2.1.22. It is worth mentioning that even more can be said in the situation of the last proposition. We state it, but the proper definitions will be given later in Section 2.3. With the assumptions of Proposition 2.1.21, if moreover R is a graded ring and I is a graded ideal, then x is a homogeneous element.

2.2 Compact p-adic analytic groups

We are mainly interested in various representation types of compact p-adic analytic groups and also arithmetic objects in connection with them. Hence it is rather necessary to start with introducing this notion and gather its main properties. Roughly speaking, a p-adic analytic group (also called p-adic Lie group) is a p-adic manifold with an additional group stucture, such that the group operations are analytic functions. The key objects to this are the so-called uniform pro-p groups. Moreover, we have the following theorem due to Lazard:

Theorem 2.2.1. *(Lazard:)* A topological group G has the structure of a p-adic analytic group if and only if G has an open subgroup which is a powerful pro-p group.

Proof. See Theorem 8.1 in [19]. □

As mentioned earlier, there is a useful variation to this theorem:

Theorem 2.2.2. A topological group G has the structure of a p-adic analytic group if and only if G has an open subgroup which is a uniform pro-p group.

Proof. See Theorem 8.32 in [19]. □

Definition 2.2.3. A **profinite group** G is a compact Hausdorff topological group whose open subgroups form a base for the neighbourhoods of the identity.

For example, a discrete group is profinite if and only if it is finite. Since G is compact, every open subgroup has finite index in G (the union of the cosets is an open cover for G). There is another description of profinite groups in terms of the inverse limit. Note that the family Λ of open normal subgroups of G form an inverse system $(G/N)_{N\in\Lambda}$ with the reverse inclusion and the maps being the natural epimorphisms $G/N \to G/M$ for $N \leq M$.

Proposition 2.2.4. If G is a profinite group then it is (topologically) isomorphic to $\varprojlim_{N \lhd_o G} G/N$.

Proof. See Proposition 1.3 in [19]. □

A subset X of a topological group G generates G topologically if $\overline{\langle X \rangle} = G$. We say that G is **finitely generated** if it is generated topologically by a finite subset.

Proposition 2.2.5. If a profinite group G is finitely generated then every open subgroup of G is also finitely generated.

Proof. See Proposition 1.7 in [19] □

From now on, p denotes a fixed prime number.

Definition 2.2.6. A profinite group G is called **pro-p group** if every open normal subgroup has index equal to some power of p.

A pro-p group is the analouge of a p-group among profinite groups. The most basic example for such a group is given by the p-adics:

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$$

As well as being historically the origin of the subject of studying p-adic analytic groups, it plays the role in pro-p groups analogous to that of the cyclic groups in abstract group theory. Basically, analytic pro-p groups are built up in a simple way from finitely many copies of $\mathbb{Z}_p$.

Proposition 2.2.7. A topological group G is pro-p if and only if it is (topologically) isomorphic to an inverse limit of finite p-groups.

Proof. See Propositions 1.12 in [19] □

We now define the so-called lower p-series of a pro-p group.

Definition 2.2.8. Let G be a pro-p group. Define

$$P_1(G) = G_1 = G \text{ and } P_{i+1}(G) = G_{i+1} = \overline{P_i(G)^p[P_i(G), G]}$$

The decreasing chain of subgroups $G \geq P_2(G) \geq \dots \geq P_k(G) \geq \dots$ is called the **lower p-series of G**.

These subgroups are **topologically characteristic** which means that they are invariant under all continuous automorphisms of G. Moreover, $P_i(G)$'s form a basis of open neighbourhoods for 1 in G.

Definition 2.2.9. Let G be a pro-p group. It is called **powerful** if $G/\overline{G^p}$ is abelian, or $G/\overline{G^4}$ is abelian, when $p = 2$. We say that G is **uniform** if it is powerful, finitely generated and $[G : P_2(G)] = [P_i(G) : P_{i+1}(G)]$ for all $i \geq 1$.

We collect some of the nice properties in one proposition that powerful and uniform pro-p groups enjoy. Whenever G is finitely generated, denote by $d(G)$ the minimal cardinality of a topological generating set of G.

Proposition 2.2.10. Let $G = \overline{\langle a_1, \dots, a_d \rangle}$ a finitely generated poweful pro-p group. Then

(*i*) $G_{i+k} = G_i^{p^k} = \{g^{p^k} \ : \ g \in G_i\}$ for all $k \geq 0, i \geq 1$.

(*ii*) The map $\varphi_k : G \to G$, $x \mapsto x^{p^k}$ induces a surjective homomorphism $G_i/G_{i+1} \to G_{i+k}/G_{i+k+1}$ for all i, k.

(*iii*) G is the product of its procyclic subgroups $\overline{\langle a_1 \rangle}, \dots, \overline{\langle a_d \rangle}$

(*iv*) When G is uniform, $\varphi_k : G \to G_{k+1}$ is a bijection (but not necessarily a group homomorphism), so every element of $x \in G_{k+1}$ has a unique p^k-th root in G.

(*v*) If G is uniform, so is G_i for all $i \geq 1$.

(*vi*) $d(H_1) = d(H_2)$ for any open uniform subgroups of G; this enables us to define the dimension of G to be $d(H)$ for any open uniform subgroup of G.

Proof. See Theorem 3.6, Proposition 3.7, Lemma 4.6 and 4.10 in [19]. □

For any pro-p group, $g \in G$ and $\lambda \in \mathbb{Z}_p$, one can define

$$g^\lambda = \lim_{n\to\infty} g^{s_n}$$

where $\lim_{n\to\infty} s_n = \lambda$. This limit exists, since the sequence (g^{s_n}) is Cauchy. Indeed, by Proposition 2.2.4, $G = \varprojlim_{N \lhd_o G} G/N$ and if $|G/N| = p^j$ for an open normal subgroup $N \lhd G$, there is an integer $n_0 \in \mathbb{N}$ such that

$$s_n \equiv s_m \pmod{p^j}$$

for all $n, m \geq n_0$. Hence $g^{s_n} \equiv g^{s_m} \pmod N$.

Theorem 2.2.11. Let $G = \overline{\langle a_1, \ldots, a_d \rangle}$ be a uniform pro-p group. Then the maping

$$(\lambda_1, \ldots, \lambda_d) \mapsto a_1^{\lambda_1} \ldots a_d^{\lambda_d}$$

from $\mathbb{Z}_p^d$ to G is a homeomorphism.

Definition 2.2.12. A $\mathbb{Z}_p$**-Lie algebra** is a free $\mathbb{Z}_p$-module L equipped with a $\mathbb{Z}_p$-billinear antisymmetric map $L \times L \to L$ satisfying the Jacobi identity $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$ for all $x, y, z \in L$. It is called **powerful** $\mathbb{Z}_p$**-Lie algebra** if in addition L has finite rank as a $\mathbb{Z}_p$-module and satisfies $[L, L] \subseteq pL$.

It is possible to define a $\mathbb{Z}_p$-Lie algebra stucture on a given uniform pro-p group as follows:

Theorem 2.2.13. Let G be a uniform pro-p group, $x, y \in G$. Let $[a, b] = a^{-1}b^{-1}ab$ if $a, b \in G$. Then the operations

$$x + y = \lim_{n\to\infty} (x^{p^n} y^{p^n})^{p^{-n}} \text{ and}$$

$$(x, y) = \lim_{n\to\infty} [x^{p^n}, y^{p^n}]^{p^{-2n}}$$

define the structure of a powerful $\mathbb{Z}_p$-Lie algebra on G, denoted by L_G. Moreover, $L_G \cong \mathbb{Z}_p^d$ as a $\mathbb{Z}_p$-module.

Proof. See Theorem 4.30 in [19]. □

It is possible to define a $\mathbb{Q}_p$-Lie algebra.

Definition 2.2.14. Let G be a uniform pro-p group. The $\mathbb{Q}_p$-Lie algebra $\mathcal{L}(G) = L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is called the **Lie algebra** of G.

If we have a powerful $\mathbb{Z}_p$-Lie algebra L, one can define, using the Campbell-Hausdorff formula, a binary operation $* : L \times L \to L$ which makes L a uniform pro-p group. In fact, it can be shown that there is a one-to-one correspondence between uniform pro-p groups and powerful $\mathbb{Z}_p$-Lie algebras. In fancier terms, one can say that there is an equivalence of categories between the category of uniform pro-p groups and powerful Lie-algebras over $\mathbb{Z}_p$. More precisely:

Theorem 2.2.15. The functors

$$G \mapsto L_G, \ L \mapsto (L, *)$$

give equivalences of categories between the category of uniform pro-p gorups and the category of $\mathbb{Z}_p$-Lie algebras.

2.3 Filtrations and gradings

One of the most powerful techniques to study ring-theoretic properties of a given ring is via filtrations and the associated graded rings attached to them. More precisely, the idea is that one defines a certain filtration on the object in question and then studies the associated graded object which is many times easier to understand but still preserves a lot of information about the original object. These techniques are important tools for studying both Iwasawa algebras and distributions algebras. In this section following [22], we build up the tools we use later.

Definition 2.3.1. The ring R is said to be a **filtered ring** (or $\mathbb{Z}$-filtered ring) if there is an ascending chain of additive subgroups of R, say $FR = \{F_nR, n \in \mathbb{Z}\}$, satisfying:

(i) $1 \in F_0R$

(ii) $F_nR \subseteq F_{n+1}R$ and

(iii) $F_nRF_mR \subseteq F_{n+m}R$ for all $n, m \in \mathbb{Z}$.

Note that if R is a filtered ring then F_0R is automatically a subring of R.

Remark 2.3.2. We could define filtration using a descending chain of additive subgroups of R analogously. In fact the filtrations we use in Chapter VI. will be descreasing filtrations (filtration with a descending chain of subgroups). That is not a problem since one can always reverse a descreasing filtration to get an increasing one.

Definition 2.3.3. Let R be a filtered ring with filtration FR. An R-module M is called a **filtered R-module** if there is an ascending chain of additive subgroups of M, say $FM = \{F_nM, n \in \mathbb{Z}\}$, satisfying:

(*i*) $F_nM \subseteq F_{n+1}M$ and

(*ii*) $F_mMF_nR \subseteq F_{n+m}M$ for all $n, m \in \mathbb{Z}$.

If R and S are filtered rings and M is an R-S-bimodule then M is said to be a **filtered R-S-bimodule** is there exists and ascending chain of additive subgroups of M as before, satisfying: $F_nM \subseteq F_{n+1}M$, $F_nRF_mM \subseteq F_{n+m}M$, $F_mMF_nS \subseteq F_{n+m}M$ for all $n, m \in \mathbb{Z}$.

Clearly any filtered ring is a filtered module over itself and also a filtered R-R-bimodule. We give some basic examples to filtered rings. An arbitrary ring R can be viewed as a filtered ring if we put the **trivial filtration** on it which is defined to be $F_nR = R$ for all $n \geq 0$ and $F_n = 0$ for any $n < 0$. Another example is the **I-adic filtration** on a ring which we will use very frequently. Let I be an ideal of R and define the I-aidc filtration to be $F_nR = R$ if $n \geq 0$ and $F_nR = I^{-n}$ for $n < 0$.

Definition 2.3.4. Let R be a filtered ring and M a filtered R-module.

(*i*) If $F_nM = 0$ for $n < 0$ then FM is called **positive filtration** and analogously one can define **negative filtration** with the property that $FM = M$ for $n \geq 1$; If there exists an $n_0 \in \mathbb{Z}$ such that $F_mM = 0$ for all $m < n_0$, then the filtration FM is called **discrete filtration**.

(*ii*) If $M = \bigcup F_nM$ then is called **exhaustive**.

(*iii*) If $\bigcap F_nM = 0$ then FM is called **separated**.

For example, the I-adic filtration defined above is a negative filtration.

Definition 2.3.5. Let R and S be filtered rings and $n \in \mathbb{Z}$. A ring homomorphism $f : R \to S$ is called **filtered ring homomorphism of degree** n, if $f(F_mR) \subseteq F_{n+m}S$ for all $m \in \mathbb{Z}$. In similar fashion, an R-module homomorphism $f : M \to N$ between two filtered R-modules M, N is a **filtered R-module homomorphism of degree** n, if $f(F_mM) \subseteq F_{n+m}N$.

It is rather convenient to regard these objects and morphisms in a category-theoretical manner: Let R be a filtered ring. We denote by **fil-R** the category in which the objects are the filtered R-modules and the morphisms are the filtered R-module homomorphisms of degree 0. These morphisms are simply called **filtered**

homomorphisms. We can define subobjects of an object in fil-R the following way: If $M \in$ fil-R and N is a submodule of M such that there is a filtration on N with the property that $F_nN \subseteq F_nM$ for all $n \in \mathbb{Z}$ then N is a **filtered submodule** of M, i.e. a subobject of M in the category fil-R. Any submodule N of a given filtered module M can be regarded as a filtered submodule of M by defining the filtration FN as follows: Let $F_nN = N \cap F_nM$, $n \in \mathbb{Z}$. Then N is a filtered submodule. The filtration obtained this way is called the **induced filtration**. It is clear that fil-R is an additive category and if f is a filtered homomorphism then Ker$f$ and Coker$f$ exist in fil-R. One defines the **quotient filtration** by $F_nM/N = F_nM + N/N$. One can easily check the following facts: monomorpisms and epimorphisms are just the injective resp. surjective morphisms, moreover arbitrary direct sums, directs products as well as inductive and inverse limits exist in fil-R (note that $F_n(\varinjlim M_i) = \varinjlim F_nM_i$, and $F_n(\varprojlim M_i) = \varprojlim F_nM_i$). We will use the following two basic functors:

Definition 2.3.6.

(*i*) The **forgetful functor** fil-$R \to$ mod-R is the functor that that associates a filtered module M with the R-module M by forgeting the filtration of M.

(*ii*) The **shift functor** $T(n)$: fil-$R \to$ fil-R, for any $n \in \mathbb{Z}$, is the functor that associates a filtered module M with filtration FM with the filtered module $T(n)(M)$ obtained by filtering the R-module M by defining $F_mT(n)(M)$ to be $F_{n+m}M$ for all $m \in \mathbb{Z}$.

Definition 2.3.7. Let R be a filtered ring. Let M be a filtered R-module with two filtrations, FM and $F'M$. We say that FM and $F'M$ are **topologically equivalent** if for every $n, m \in \mathbb{Z}$, there are $n_1, m_1 \in \mathbb{Z}$ such that $F'_{n_1}M \subseteq F_nM$ and $F_{m_1}M \subseteq F'_nM$. We say that they are **algebraically equivalent** if there is an integer $c \in \mathbb{Z}$ such that for all $n \in \mathbb{Z}$,

$$F_{n-c}M \subseteq F'_nM \subseteq F_{n+c}M.$$

When we use simply the term **equivalent**, we always mean algebraically equivalent.

From now on, all the filtrations are considered to be exhaustive. The elements of the filtration FM form a basis for open neighbourhoods at 0. Consider the natural topology generated by them. The sets of the form $x + F_nM$ will be a basis for the topology.

Definition 2.3.8. Let $M \in$ fil-R. The topology given by the sets of the form $x+F_nM, x \in M, n \in \mathbb{Z}$ as a base for the topology is called the **filtration topology** on M.

Note that a filtration on a module enables us to define analytical notions such as convergence and completion. It turns out to be very useful later.

Definition 2.3.9. Let R be a filtered ring and M be a filtered R-module. A sequence $(x_i)_{i>0}$ of elements of M is said to be **Cauchy** if for every integer $s \geq 0$ there is an integer $N(s) > 0$ such that $x_n - x_m \in F_{-s}M$ for all $n, m \geq N(s)$. It is enough to require that $x_{n+1} - x_n \in F_{-s}M$ for any $n \geq N(s)$. A sequence $(x_i)_{i>0}$ **converges** to an element $x \in M$ if there is an integer $N(s) > 0$ for every integer $s \geq 0$ such that $x_n - x \in F_{-s}M$ for all $n \geq N(s)$. If we assume that the filtration is separated, it follows that the filtration topology is Hausdorff. Hence every convergent sequence converges to a unique element.

Definition 2.3.10. Let R be a filtered ring. An object $M \in$ fil-R is said to be **complete** if every Cauchy-sequence converges to some element in M.

One can define the **completion** of a filtered module which always exits: Note that the quotient groups M/F_nM form an inverse system with the natural surjections. Hence we can take the projective limit $\widehat{M} = \varprojlim M/F_nM$.

Definition 2.3.11. We define $\widehat{M}$ to be the completion of M.

$\widehat{M}$ is a complete filtered R-module and it is easy to see that M is complete if and only if the natural map $M \to \widehat{M}$ given by $m \mapsto (m + F_nM)_{n\in\mathbb{Z}}$ is an isomorphism. Now we turn our attention to define a category with graded objects and graded morphisms. Later, we associate such a category to fil-R where R is a filtered ring.

Definition 2.3.12. Let R be a ring. Then R is a $\mathbb{Z}$**-graded ring** or simply **graded ring** if $R = \oplus_{i\in\mathbb{Z}} R_i$ where R_i are additive subgroups of R satisfying $R_iR_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}$. If $R_iR_j = R_{i+j}$ then it is said to be **strongly** $\mathbb{Z}$**-graded**.

Let R be a graded ring. We denote by **gr-**R the category in which the objects are graded R-modules and the morphisms are the graded morphisms of degree 0. The following lemma gives a characterization for a graded ring to be strongly graded.

Proposition 2.3.13. Let $R = \oplus_i R_i$ a $\mathbb{Z}$-graded ring. Then R is strongly graded if and only if $1 \in R_iR_{-i}$ for all $i \in \mathbb{Z}$.

Proof. It follows from the definition. □

An important characterization of strongly graded rings is stated in the following theorem, due to Dade.

Theorem 2.3.14. *(Dade)* Let R be a graded ring. Then R is strongly graded if and only if the functors $(\ )_0 :$ gr-$R \to$ mod-R_0 and $(- \otimes_{R_0} R) :$ mod-$R_0 \to$ gr-R form equivalences of categories.

Proof. See Proposition 4.17 in [22]. □

Definition 2.3.15. Let R be a graded ring. An R-module M is called **graded module** if there are additive subgroups M_i, $i \in \mathbb{Z}$, satisfying $M_i R_j \subseteq M_{i+j}$ such that $M = \oplus_i M_i$. If $M_i R_j = M_{i+j}$ then M is a **strongly graded module**.

An element of $h(R) = \cup R_i$ resp. $h(M) = \cup M_i$ is called **homogeneous element** of R resp. of M. If M is a graded R-module over a graded ring R, then it follows from the definition that every element can be written in a unique way as a sum of homogeneous elements. If $m = m_{i_1} + \dots m_{i_d}$ then the elements m_{i_j} are the **homogeneous components** of m.

Definition 2.3.16. Let M be a graded R-module. A submodule N of M is a **graded submodule** if $N = \oplus(M_i \cap N)$.

Definition 2.3.17. Let R, S be graded rings. A ring homomorphism $g : R \to S$ is said to be a **graded morphism of degree** n if $g(R_i) \subseteq S_{i+n}$ for all $i \in \mathbb{Z}$. An R-module homomorphism $f : M \to N$ between two graded R-modules M, N is said to be graded morphism of degree n if $f(M_i) \subseteq N_{i+n}$.

We define two basic functors that are the analouges of the functors that we defined in 2.3.6.

- (i) The **forgetful functor** which simply assigns for a graded module M the module M forgetting the graded structure.
- (ii) The **shift functor** $T(n) : R\text{-gr} \to R\text{-gr}$, associating to $M \in R\text{-gr}$ the graded module obtaind by defining on the R-module M a new grading given by $T(n)(M)_i = M_{i+n}$.

Let R be a filtered ring and M be a filtered R-module. We define the abelian groups:

$$\mathrm{gr}^{\cdot} R = \oplus_n F_n R / F_{n-1} R$$

$$\mathrm{gr}^{\cdot} M = \oplus_i F_n M / F_{n-1} M$$

Let $e_n : F_n M / F_{n-1} M \to \mathrm{gr}^{\cdot} M$ denote the canonical injection of $F_n M / F_{n-1} M$ into the direct sum. For any $x \in M$ define the **t**he **degree** of x, denoted by $\deg(x)$, to be the integer n such that $x \in F_M \setminus F_{n-1} M$.

Definition 2.3.18. We define the **principal symbol** of x to be $\sigma(x) = e_n(x + F_{n-1} M)$.

Definition 2.3.19. The abelian groups $\mathrm{gr}^{\cdot} R$ resp. $\mathrm{gr}^{\cdot} M$ with the multiplication given by $\sigma(x)\sigma(y) = e_{\deg(x)+\deg(y)}(xy)$ for $x, y \in R$ resp. $x \in R, y \in M$ is called the **associated graded ring** of R resp. the **associated graded module** of M.

Note that if $\sigma(x)\sigma(y) \neq 0$ the multiplication simplifies down to $\sigma(x)\sigma(y) = \sigma(xy)$. It is also very convenient that the associated graded modules behave well with respect to induced and quotient filtrations. In particular, one can easily check the following:

Lemma 2.3.20. Let R be a filtered ring, M a filtered R-module. Suppose that $0 \to N \to M \to M/N \to 0$ is an exact sequence of R-modules, where N, M/N are equipped with the induced and quotient filtrations, respectively. Then the sequence of $\mathrm{gr}^{\cdot} R$-modules $0 \to \mathrm{gr}^{\cdot} N \to \mathrm{gr}^{\cdot} M \to \mathrm{gr}^{\cdot} M/N \to 0$ is exact.

Proof. It is part of a more general theorem. See Theorem 4.2.4 (1) Chapter I. in [22]. □

One observes that the completion doesn't change the associated graded module, since $\widehat{M}/F_n\widehat{M} \cong M/F_nM$. Hence we get:

Lemma 2.3.21. If M is a filtered R-module then $\mathrm{gr}^{\cdot} M \cong \mathrm{gr}^{\cdot} \widehat{M}$.

Proof. See Corollary 3.4 Chapter I. in [22]. □

2.4 Zariskian Filtrations

As mentioned before, the idea behind developing these techniques is to associate to a ring of interest another ring that is simplier to investigate, yet it preserves enough information about the original object. The so-called Zariskian filtrations are particulary well-suited for this task.

Definition 2.4.1. Let R be a filtered ring. The **Rees ring** of R is defined to be

$$\widetilde{R} = \oplus F_n R$$

If we denote by e_n the canonical injection of F_nR into $\widetilde{R}$ then the multiplication in $\widetilde{R}$ is given by $e_n(x)e_m(y) = e_{n+m}(xy)$ for any $x \in F_nR$ and $y \in F_mR$.

Definition 2.4.2. Let $M \in$ fil-R with filtration FM. If there exist $m_1, \ldots, m_s \in M$, $k_1, \ldots k_s \in \mathbb{Z}$ such that for all $n \in \mathbb{Z}$

$$F_nM = \sum_{i=1}^{s} m_i F_{n-k_i} R$$

then FM is called a **good filtration** on M.

It is clear that if M has a good filtration FM then it is a finitely generated R-module. On the other hand, if M is finitely generated and $\{m_1, \dots, m_s\}$ is a generating set, then one can alwasy define a good filtration FM on M as follows: take $k_1, \dots, k_s \in \mathbb{Z}$ and put $F_nM = \sum_{i=1}^{s} m_i F_{n-k_i}R, n \in \mathbb{Z}$, then it is obvious that it is an exhaustive filtration and good. However, not all filtrations on a finitely generated module M are good. The next statement shows that in the case of complete filtered rings, one has a nice characterization of a separated filtration FM on an R-module M to be good.

Theorem 2.4.3. Let R be a complete filtered ring, M a filtered R-module with separated filtration FM. Then FM is good if and only if $\mathrm{gr}^{\cdot} M$ is finitely generated over $\mathrm{gr}^{\cdot} R$.

Proof. See Theorem 5.7, Chapter I in [22]. □

Definition 2.4.4. A filtered ring R is said to be a **left Zariski ring**, or FR a **left Zariskian filtration** if the Rees ring $\widetilde{R}$ of R associated with FR is left noetherian and $F_{-1}R$ is contained in the Jacobson radical $J(F_0R)$ of F_0R. Filtrations with the last condition are called **faithful filtrations**.

One can similary define right Zariskian rings and filtrations. Whenever R is both left and right Zariskian, we will simply say that R is **Zariskian**.

Definition 2.4.5. Let M be a filtered module over a filtered ring R. If for any finitely generated submodule $N = \sum u_i R$ of M, there is an integer $c \in \mathbb{Z}$ such that for all $n \in \mathbb{Z}$

$$F_nM \cap N \subseteq F_{n+c} u_i R$$

Then FM is said to have the (right) **Artin-Rees property**.

There are many characterizations of the Zariski property and we collect some of them in a theorem.

Theorem 2.4.6. *(Characterizations of the Zariski property:)* For a filtered ring R with filtration FR, the following are equivalent:

(a) R is a right Zariski ring;

(b) FR is separated, faithful, $\mathrm{gr}^{\cdot} R$ is right Noetherian, and every good filtration FM on $M \in$ fil-R has the Artin-Rees property;

(c) FR is separated, faithful, $\mathrm{gr}^{\cdot} R$ is right Noetherian, and FR has the Artin-Rees property;

(d) $\mathrm{gr}^{\cdot} R$ is right Noetherian and the completion $\widehat{R}$ of R with respect to the FR-topology on R is a faithfuly flat (left) module;

(e) $\text{gr}^{\cdot} R$ is right Noetherian and good filtrations in fil-R induce good filtrations on R-submodules and good filtrations are separated.

Proof. See Theorem 2.2 Chapter II. in [22] □

The commutative Zariski rings that appear in commutative algebra or algebraic geometry provide important examples. A commutative Zariski ring R is a commutative Noetherian ring with I-adic filtration where $I \subseteq J(R)$. We would like to emphasise that, in general, the connection between I-adic filtrations and Zariskian filtrations are deep. In a not-so-precise way, one could say that if FR is a Zariskian filtration on a ring R, the subring $F_0(R)$ with the induced filtration is "almost" $F_{-1}R$-adic rings. For more precise statement, see Lemma 2.1.4 and Corollary 2.1.5, Chapter II in [22]. In fact, we will have a perfect example for such a phenomena later in the theory of locally analytic representations.

Now we show some nice properties of Zariskian filtrations.

Proposition 2.4.7. Let R be a complete filtered ring such that $\text{gr}^{\cdot} R$ is Noetherian. Then R is Zariski.

Proof. See Proposition 2.2.1 in [22]. □

Note that a positively filtered ring R is always complete. Hence if in addition $\text{gr}^{\cdot} R$ is Noetherian, then R is Zariski. This provides a lot of interesting classes of rings as examples. We list some of them without any detail (for details, see Corollary 2.2.2 Chapter II. in [22]): Ordinary and skew polynomial rings, the universal enveloping algebra $U(g)$ of a fintie dimensional k-Lie algebra (where k is a field), derivation algebra $A[d]$ of a commutative k-algebra over a commutative ring k, the n-th Weyl algebra $A_n(k)$ over a field k with the Bernstein filtration, and many more.

Lemma 2.4.8. Let $M \in$ fil-R with good filtration FM. If N is an R-submodule of M with filtration induced by FM such that $\text{gr}^{\cdot} N = \text{gr}^{\cdot} M$, then $N = M$.

Proof. See Lemma 3.1.1 Chapter II in [22] □

The next theorem shows that many important ring-theoretical properties can be lifted from the associated graded ring to a Zariski ring.

Theorem 2.4.9. Let R be a Zariski ring with Zariskian filtration FR. Then

(a) If $\text{gr}^{\cdot} R$ is a domain then so is R.

(b) If $\text{gr}^{\cdot} R$ is prime then R is also prime.

(c) If $\text{gr}^{\cdot} R$ is a maximal order then then so is R.

(d) If $\mathrm{gr}^{\cdot} R$ is Auslander-Gorenstein then so is R

(e) If $\mathrm{gr}^{\cdot} R$ has finite global dimension then R also has finite global dimension. Moreover, gl.dim.$(R) \leq$ gl.dim.$(\mathrm{gr}^{\cdot} R)$.

(f) If $\mathrm{gr}^{\cdot} R$ has finite Krull dimension then so is R. Moreover, K.dim$R \leq$ K.dimgr$^{\cdot} R$.

(g) If $\mathrm{gr}^{\cdot} R$ is (semi)simple then so is R.

Proof. See Theorem 3.1.4, Proposition 3.2.4, Lemma 3.2.7, Theorem 3.2.11, Corollary 3.1.2, Corollary 3.1.3, all of them in Chapter I, and Theorem 3.3.1 Chapter III in [22]. □

2.5 The Grothendieck group of rings and categories

One of the main tools that we will use is the theory of algebraic K-groups. We will make use of both the ungraded and the graded versions of the Grothendieck group of rings and categories.
There are several ways to construct the Grothendieck group of a mathematical object. We begin with the group completion version, because it has been the most historically important. After giving the applications to rings, we describe the Grothendieck group of an exact category.
Let R be a ring. The set $P(R)$ of isomorphism classes of finitely generated projecitve R-modules, together with the direct sum $\oplus$ and identity 0, forms an abelian monoid. One can define the **group completion**, denoted by $M^{-1}M$, of any abelian monoid M the following way: $M^{-1}M$ is an abelian group with a monoid map $[\;] : M \to M^{-1}M$ and if we have another abelian group A and a monoid map $\alpha : M \to A$, there is a unique abelian group homomorphism $\widetilde{\alpha} : M^{-1}M \to A$ such that $\widetilde{\alpha}([m]) = \alpha(m)$ for all $m \in M$. The usual standard construction of a universal object also works here: We generate the free abelian group on symbols $[m]$ for all $m \in M$. Then we factor out by the subgroup $S(M)$ generatd by the relations $[m+n] = [m] - [n]$. We have a natural monoid map $[\;] : M \to M^{-1}M, m \mapsto [m]$ and one can easily check that $M^{-1}M$ satisfies the universal property above. Thus the group completion is a functor from abelian monoids to abelian groups. The most basic example is to take $M = \mathbb{N}$. Then $M^{-1}M = \mathbb{Z}$. Another interesting example of the set of finite dimensional representations over the complex numbers of a finite group G, denoted by $\mathrm{Rep}_{\mathbb{C}}(G)$, which form an abelian monoid with the direct sum $\oplus$. By Maschke's Theorem, $\mathbb{C}[G]$ is semisimple and $\mathrm{Rep}_{\mathbb{C}}(G) \cong \mathbb{N}^r$, where r is the number of conjugacy classes of G. Therefore the group completion $\mathrm{Rep}_{\mathbb{C}}(G)^{-1}\mathrm{Rep}_{\mathbb{C}}(G)$ is isomorphic to $\mathbb{Z}^r$.

Definition 2.5.1. Let R be a ring. Thne the **Grothendieck group** of R, denoted by $K_0(R)$, is the group completion $P^{-1}P$ of $P(R)$.

Let $f : R \to S$ be a ring homomorphism between two rings, R and S. The extension of scalars gives us a monoid map $\otimes_R S : P(R) \to P(S)$. Hence, by the universal property, one has a group homomorphism $f^* : K_0(R) \to K_0(S)$. Therefore K_0 is a functor from the category of rings to the category of abelian groups.

Lemma 2.5.2. Let R be a ring. If $P, Q \in P(R)$ then the following conditions are equivalent:

(*i*) $[P] = [Q]$ in $K_0(R)$;

(*ii*) $P \oplus D \cong Q \oplus D$ for some $D \in P(R)$;

(*iii*) $P \oplus R^t \cong Q \oplus R^t$ for some $t \in \mathbb{N}$.

Proof. Straightforward □

If P, Q are as in lemma above, they are said to be **stably isomorphic**. The following proposition characterizes injective and surjective homomorphisms between Grothendieck groups. Injectivity is evident, surjectivity is a little more complicated.

Proposition 2.5.3. Let R, S be rings and $f : R \to S$ a ring homomorphism. Then the induced group homomorphism $f^* : K_0(R) \to K_0(S)$ is

(*i*) injective if and only if $P \otimes_R S$ being stably isomorphic to $Q \otimes_R S$ implies that P is stably isomorphic to Q;

(*ii*) surjective if and only if given $Q \in P(S)$, there exists a $P \in P(R)$ and $n \in \mathbb{N}$ such that $P \otimes_R S \cong Q \oplus S^n$.

Proof. See 12.1.8 in [28]. □

The next lemma is obvious from the fact that K_0 is a functor, but we will state it since this observation comes in handy quite often.

Lemma 2.5.4. If there are homomorphisms $f : R \to S$ and $g : S \to R$ such that $g \circ f = id_R$ then $g^* \circ f^*$ is the identity on $K_0(R)$ and so $K_0(R)$ is a direct summand of $K_0(S)$.

Proof. See Proposition 12.1.9 in [28]. □

The Grothendieck group has the nice property that whenever two rings are Morita equivalent, then their Grothendieck groups are isomorphic.

Lemma 2.5.5. Let R and S be two rings. If R and S are Morita equivalent the $K_0(R) \cong K_0(S)$.

Proof. See Corollary 2.7.1 Chapter II in [51]. □

Suppose that $f : R \to S$ is a ring map. There are two important maps between their associated Grothendieck groups, namely the **base change map**, denoted by f^*, and the **transfer map**, denoted by f_*. We have already defined the first one above, but for the second one to make sense, we need to assume in addition that S is fintely generated projective R-module. Then there is a forgetful functor from $P(S)$ to $P(R)$; it is represented by S, an R-S-bimodule because it sends Q to $Q \otimes_S S$. The induced map $f_* : K_0(S) \to K_0(R)$ is called the transfer map.

Another very useful observation comes basically from idempotent lifting:

Proposition 2.5.6. Let R be a ring and I a nilpotent, or more generally a complete ideal in R (i.e. R is an I-adic ring). Then

$$K_0(R/I) \cong K_0(R)$$

Proof. It is Lemma 2.2., Chapter II in [51]. □

Now we turn our attention to the generalization of the Grothendieck group from rings to skeletaly small exact categories. Recall that a category is called small if the class of objects of $\mathcal{A}$ forms a set and it is called skeletaly small if it is equivalent to a small category. There is an obvious set-theoretic difficulty in defining $K_0(\mathcal{A})$ when $\mathcal{A}$ is not skeletaly small.
The natural notion of exact sequence in an exact category enables us to generalize the classical definition of the Grothendieck group. Most of the time, we will deal with an even more special type of categories, namely abelian categories. However, the category of finitely generated projective modules over a ring R is only exact, by virtue of its embedding in the category of R-modules.

Definition 2.5.7. Let $\mathcal{A}$ be a small exact category. Then the **Grothendieck group** $K_0(\mathcal{A})$ of $\mathcal{A}$ is the abelian group having one generator $[A]$ for each object in $\mathcal{A}$ and a relation $[A] = [A_1] + [A_2]$ for every short exact sequence

$$0 \to A_1 \to A \to A_2 \to 0$$

in $\mathcal{A}$.

Lemma 2.5.8. The following easy identities hold in $K_0(\mathcal{A})$:

(a) $[0] = 0$;

(b) $A \cong A'$ then $[A] = [A']$;

(c) $[A \oplus A'] = [A] + [A']$

We cannot take the Grothendieck group of all R-modules, because it is not skeletaly small. Let us now suppose that R is Noetherian and consider the category mod-R of all finitely generated R-modules. By the noetherian property, mod-R is an abelian category and we write $G_0(R)$ for $K_0(\text{mod-}R)$. We mention at this point that there is a definition of G_0 for non-Noetherian rings, but we will only deal with Noetherian rings, so we leave it out. The new definition of Grothendieck group is indeed a generalization of our previous definition since $P(R)$ is a small exact subcategory of mod-R and every short exact sequence with projective modules splits.

Lemma 2.5.9. Let $\mathcal{A}$ be a small abelian category. If $[A_1] = [A_2]$ in $K_0(\mathcal{A})$ then there are short exact sequences in $\mathcal{A}$

$$0 \longrightarrow C \longrightarrow K \longrightarrow D \longrightarrow 0$$

$$0 \longrightarrow C \longrightarrow L \longrightarrow D \longrightarrow 0$$

such that $A_1 \oplus K = A_2 \oplus L$.

Proof. It is a special case of a more general statement. See Ex. 7.2 in [51]. □

We now turn our attention to important theorems which provide powerful tools for us to investigate certain module categories later on.

Theorem 2.5.10. *(Devissage Theorem)* Let $\mathcal{B} \subset \mathcal{A}$ small abelian categories. Suppose that

(i) $\mathcal{B}$ is an exact abelian subcategory of $\mathcal{A}$, closed in $\mathcal{A}$ under subobjects and quotients,

(ii) Every object A of $\mathcal{A}$ has a finite filtration

$$A = A_0 \supset A_1 \supset \cdots \supset A_n = 0$$

with all quotients A_i/A_{i+1} in $\mathcal{B}$.

Then the inclusion functor $\mathcal{B} \subset \mathcal{A}$ is exact and induces an isomorphism

$$K_0(\mathcal{B}) \cong K_0(\mathcal{A})$$

Proof. See [51] Chapter II.,Theorem 6.3. □

Example 2.5.11. Let R be a Noetherian ring and s a central element in R. Denote by mod_s-R the abelian subcategory of mod-R consisting of finitely generated R-modules M such that $Ms^n = 0$ for some $n \in \mathbb{N}$. That is, modules such that the chain of submodules

$$M \supset Ms \supset Ms^2 \supset \dots$$

is finite. By Devissage, $K_0(\text{mod-}R) \cong G_0(R/sR)$. More generally, suppose that we are given an ideal $I \subset R$. Let mod_I-R be the abelian subcategory of mod-R consisting of finitely generated R-modules such that the filtration $M \supset MI \supset MI^2 \supset \dots$ is finite, i.e. such that $MI^n = 0$ for some n. Again by Devissage,

$$K_0(\text{mod}_I\text{-}R) \cong G_0(R/I)$$

Theorem 2.5.12. *(Localization theorem)* Let $\mathcal{A}$ be a small abelian category, and $\mathcal{B}$ a Serre subcategory of $\mathcal{A}$. Then the following sequence is exact:

$$K_0(\mathcal{B}) \to K_0(\mathcal{A}) \to K_0(\mathcal{A}/\mathcal{B}) \to 0$$

Proof. See [51] Chapter II., Theorem 6.4. □

Example 2.5.13. Let R be a Noetherian ring and S a central multiplicative set in R. Denote by S-tors the subcategory of finitely generated S-torsion modules. There is a natural equivalence between mod-$S^{-1}R$ and the quotient category mod-R/S-tors. Moreover, S-tors is a Serre subcateory. Then the localization sequence becomes:

$$K_0(S\text{-tors}) \to G_0(R) \to G_0(S^{-1}R) \to 0.$$

Example 2.5.14. Let $s \in R$ a central non-zero divisior. Then $S = \{1, s, s^2, \dots\}$ is the central multiplicative set. Using Devissage Theorem 2.5.10 on mod_s-$R \subset$ S-tors and the Localization Theorem, we get the following exact sequence:

$$G_0(R/sR) \to G_0(R) \to G_0(R[1/s]) \to 0$$

We now turn to a classical result and application of the Localization Theorem: The Fundamental Theorem for G_0 of a Noetherian ring R. Via the ring map $\pi : R[t] \to R$ sending t to 0, we have an inclusion map mod-$R \subset$ mod-$R[t]$ and hence a transfer map $\pi_* : G_0(R) \to G_0(R[t])$. By the Localization Theorem, we have the following exact sequence:

$$G_0(R) \to G_0(R[t]) \to G_0(R[t, t^{-1}]) \to 0$$

The first map is π_* and we denote the second map by j_*. Given an R-module M, the exact sequence of $R[t]$-modules

$$0 \to M[t] \to M[t] \to M \to 0$$

shows that in $G_0(R[t])$

$$\pi_*([M]) = [M] = [M[t]] - [M[t]] = 0.$$

Thus $\pi_* = 0$, meaning that the second map j_* is an isomorphism. This was the easy part of the following result:

Theorem 2.5.15. *(Fundamental Theorem for G_0-theory of rings)* Let R be a Noetherian ring. The inclusions $R \hookrightarrow R[t] \hookrightarrow R[t, t^{-1}]$ induce isomorphisms

$$G_0(R) \cong G_0(R[t]) \cong G_0(R[t, t^{-1}])$$

If one assumes in addition that R is regular, i.e. every module has finite projective dimension (note that it is not equivalent to assuming that R has finite global dimension), we have a stronger result:

Theorem 2.5.16. *(Fundamental Theorem for K_0 of regular rings:)* If R is a regular Noetherian ring, then $G_0(R) \cong K_0(R)$. Moreover,

$$K_0(R) \cong K_0(R[t]) \cong K_0(R[t, t^{-1}])$$

Proof. See Theorem 7.8 in [51]. □

To end this section, we introduce the graded version of the Grothendieck group which will be very useful for us later.

Definition 2.5.17. Let R be a graded ring. Then the **graded Grothendieck group**, denoted by $K_{0g}(R)$, is the group completion of the abelian monoid $P_{gr}(R)$, formed by the graded isomorphism classes of graded projective modules and the direct sum as addition operation.

2.6 Pseudocompact rings

Definition 2.6.1. Let R be a complete Hausdorff topological ring which admits a system of neighborhoods of 0 consisting of two sided ideals I for which R/I is an Artin ring. Then we call R a **pseudocompact ring**. A complete Hausdorff topological ring A will be called a **pseudocompact algebra** over R if:

(i) A is an R-algebra in the usual sense,

(ii) A admits a system of neighborhoods of 0 consisting of two sided ideals I such that A/I has finite length as R-modules.

Let R be a pseudocompact ring and A a pseudocompact R-algebra.

Definition 2.6.2. An A-module is a **pseudocompact module**, if it is the inverse limit of A-modules of finite length.

Pseudocompact rings include a wide variety of classes of rings, for example complete discrete valuation rings. Pseudocompact algebras include, for example, completed group algebras (see section 2.7). The homological aspects of pseudocompact algebras were studied by Brumer in [12], who computed for example the homological dimension of a completed group algebra over a profinite group G with coefficients from a pseudocompact ring R. He also showed that the category of pseudocompact A-modules $\mathcal{P}$ is dual to the category of discrete A-module $\mathcal{D}$ where A is a pseudocompact algebra over some pseudocompact ring. We give a precise statement:

Proposition 2.6.3. There are functors S and T that define a duality between $\mathcal{P}$ and $\mathcal{D}$. Moreover, their composition is naturally equivalent to the identity functor on the respective category.

Proof. See Proposition 2.3 in [12] □

Now we state the result about the homological dimension of complete group algebras:

Theorem 2.6.4. *(Brumer)* Let R be a pseudocompact ring and G an arbitrary profinite group. Then

$$\text{gl.dim} R[[G]] = \text{gl.dim} R + \text{cd}_R G$$

where cd_R denotes the cohomological dimension of G over R.

Proof. See Theorem 4.1 in [12] □

2.7 Iwasawa algebras and completed group algebras

In the recent years, there has been great intereset in noncommutative Iwasawa algebras, which are certain completions of group algebras, for they have many deep connections to number theorey and arithmetic geomerty. Their definition and fundamental properties were established by Michel Lazard (see [25]) in 1965, but after that, they were litte studied. Interest in them has been revived by developments in number theory over the past decades. One of their application lies in the study of a very important arithmetic object in number theory, namely the Selmer group of an elliptic curve over a number field, moreover the GL_2 conjectures for elliptic curves without complex multiplication in [17], which gives the main motivation for many of the results in this thesis. Hence, we devote this section to establish all the necessary notions and collect all the results in connection with non-commutative Iwasawa algebras which will be used later.

We need to start with some fundamental definitions. The notion of a group ring is well known. Given a ring R and a group G, the group ring $R[G]$ is defined to be a free right R-module with elements of G as a basis and with multiplication given by $(gr)(hs) = (gh)(rs)$ together with billinearity. In fact $R[G]$ has the following universal property: given a ring S, a ring homomorphism $\phi : R \to S$ and a group homomorphism ξ from G to the group of units of S such that

$$\phi(r)\xi(g) = \xi(g)\phi(r), \qquad r \in R,\ g \in G$$

then there exists a homomorphism $\eta : R[G] \to S$ such that $\eta(r) = \phi(r)$ and $\eta(g) = \xi(g)$.
We extend this idea by allowing the group to have some action on the ring of scalars in order to get a more general notion, more precisely:

Definition 2.7.1. Let R be a ring, G a group and φ a homomorphism $\varphi : G \to \mathrm{Aut}(R)$. Let us denote the image of $r \in R$ under $\varphi(g)$ by r^g. The **skew group ring** $R\#G$ is defined to be the free right R-module with elements of G as a basis as before but the multiplication is defined by

$$(gr)(hs) = (gh)(r^h s)$$

The skew group ring contains G as a subgroup in its group of units, and R as a subring. When $\varphi(g) = 1$ for all $g \in G$, we get the ordinary group ring.

Example 2.7.2. There is a connection with semidirect products of groups. Let N, H be groups and $\varphi : H \to \mathrm{Aut}(N)$ a group homomorphism. As in the definition, we write $\varphi(h)(n) = n^h$. The corresponding semidirect product G is $N \times H$ with multiplication $(f, n)(h, m) = (fh, n^h m)$. In fact, G being a semidirect product is equivalent to there being a split short exact sequence

$$1 \to N \to G \to H \to 1.$$

One can extend φ to a homomorphism $\varphi : H \to \mathrm{Aut}(R[N])$ by letting H act trivialy on R. This way the ordinary group ring can be identified with $RH\#N$.

An even more general notion, the so-called crossed product, is what we will need later.

Definition 2.7.3. Let R be a ring and G a group. Let S be a ring containing R and a set of units $\overline{G} = \{\overline{g} \mid g \in G\}$ isomorphic to G as a set such that

(i) S is a free right R-module with basis $\overline{G}$ and $\overline{1}_G = 1_S$,

(ii) for all $g_1, g_2 \in G$ $\overline{g_1}R = R\overline{g_1}$ and $\overline{g_1}\ \overline{g_2}R = \overline{g_1 g_2}R$.

Then S is called a **crossed product** and we denote such a ring by $R * G$. If we require more in (ii), namely that $\overline{g}r = r\overline{g}$ for all $r \in R$ and $g \in G$, then S is called a **twisted group ring**.

The next lemma shows the connection between subgroups of G and subrings of $R * G$.

Lemma 2.7.4. Let $X \subseteq G$ be a set of representatives of the cosets of G modulo some subgroup H. Then $R * G$ is freely generated as an $R * H$-module by $\overline{X}$. If $H = N$ is a normal subgroup of G, then $R * G = (R * N) * (G/N)$

Proof. See Lemma 1.5.9 in [28]. □

Example 2.7.5. This extends Example 2.7.2 to nonsplit extentions, namely if $1 \to N \to G \to H \to 1$ is a short exact sequence, then (ii) above shows that $R[G] \cong R[N] * H$ and likewise $R\#G \cong R\#N * H$.

Now we turn our attention to define Iwasawa algebras.

Completed group algebras

Now we can define Iwasawa algebras, however it is more convenient to begin with a more general class of rings, since we will use them later. First, let K be any finite extention of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_K$, a finite extention of $\mathbb{Z}_p$. Fix a prime element π of $\mathcal{O}_K$ and let k be the residue field of $\mathcal{O}_K$.

Definition 2.7.6. Let G be a profinite group. The completed group algebra of G with coefficient in $\mathcal{O}_K$ is defined to be the inverse limit

$$\mathcal{O}_K[[G]] := \varprojlim_{N \lhd_o G} \mathcal{O}_K[G/N]$$

as N runs over all the open normal subgroups of G. Similarly one can define

$$k[[G]] = \varprojlim_{N \lhd_o G} k[G/N]$$

If $K = \mathbb{Q}_p$ then the first ring in the definition is called the **Iwasawa algebra** of G, denoted by Λ_G. We denote the second ring, which is the epimorphic image of the first one, by Ω_G. Whenever G is finite, both rings in the definition become just ordinary group rings. In fact, there is a natural embedding of G into both $\mathcal{O}_K[[G]]$ and $k[[G]]$, since by the Hausdorff property of the topology on G, there always exists an open normal subgroup such that $g \notin N$ for any $g \in G$. So we can define the embedding to be the map $g \mapsto (gN)_{N \lhd_o G}$. We begin to investigate these rings and collect their ring-theoretic properties. For the moment, we allow more general rings to be the coefficient rings of completed group algebras: Let $\mathcal{O}$ be a commutative local ring with maximal ideal m, such that it is complete in its m-adic topology. Let us, moreover, assume that $k = \mathcal{O}/m$ is finite of characteristic p and G be a profinite group.

Definition 2.7.7. The kernel of the canonical epimorhism

$$\mathcal{O}[[G]] \twoheadrightarrow \mathcal{O}$$

is called the **augmentation ideal** and denoted by $I(G)$.

Theorem 2.7.8. Let $\mathcal{O}$ be as above.

(i) Then the following are equivalent:

  (a) $\mathcal{O}[[G]]$ is semi-local.

  (b) $|G/G_p| < \infty$ where G_p is the pro-p sylow subgroup of G.

(ii) $\mathcal{O}[[G]]$ is local if and only if G is a pro-p group. In this case the maximal ideal of $\mathcal{O}[[G]]$ is $m\mathcal{O}[[G]] + I(G)$ where m is the maximal ideal of $\mathcal{O}[[G]]$ and $I(G)$ is the augmentation ideal.

Proof. See Proposition 5.2.16 in [30]. □

The next result is due to Brumer 2.6.4.

Theorem 2.7.9. Let be a compact p-adic analytic group of dimension d. Then both $k[[G]]$ and $\mathcal{O}[[G]]$ have finite global dimension if and only if G has no element of order p. In this case

$$\text{gl.dim}(\mathcal{O}[[G]]) = d + 1 \qquad \text{gl.dim}(k[[G]]) = d$$

An important application is the ring of interegs $\mathcal{O}_K$ of some finite extention K of $\mathbb{Q}_p$. Let us now assume that G is a compact p-adic analytic group. By Theorem of Lazard 2.2.2 in Section 2.2, every p-adic analytic group contains an open uniform pro-p group H. When G is uniform, the completed group algebras with coefficients in $\mathcal{O}_K$ enjoy many nice properties.

Lemma 2.7.10. Let $H \subseteq G$ be any open subgroup. Then both $\mathcal{O}_K[[G]]$ and $k[[G]]$ are free right modules over the algebra $\mathcal{O}_K[[H]]$ and $k[[H]]$, respectively, If $H = N$ is an open normal subgroup of G, then both rings $\mathcal{O}_K[[G]], k[[G]]$ become crossed products of $\mathcal{O}_K[[N]]$ and $k[[N]]$ respectively, by G/N, i.e.

$$\mathcal{O}_K[[G]] = \mathcal{O}_K[[N]] * G/N$$

$$k[[G]] = k[[N]] * G/N$$

Proof. See Lemma 2.6.2 in [2]. □

Now this last lemma indicates that $\mathcal{O}_K[[G]]$ is closely related to $\mathcal{O}_K[[H]]$. As a consequence, it is often enough to consider completed group algebra with coefficients in $\mathcal{O}_K$ over uniform pro-p groups.

Proposition 2.7.11. Let G be a compact p-adic analytic group.

(*i*) The ring $\mathcal{O}_K[[G]]$ is always semiprime.

(*ii*) $k[[G]]$ and $\mathcal{O}_K[[G]]$ is prime if and only if has no non-trivial finite normal subgroups.

(*iii*) $k[[G]]$ is semiprime if and only if G has no non-trivial finite normal subgroups of order divisible by p.

(*iv*) $\mathcal{O}_K[[G]]$ and $k[[G]]$ domains if and only if G is torsion-free.

Proof. The proof of (*i*), (*ii*) and (*iii*) are essentialy the same as that of Proposition 2.5 in [3] and Theorem 4.2 in [6]. Similarly, the proof of (*iv*) is the same as that of Theorem 4.3 in [6]. □

Proposition 2.7.12. Let G be a compact p-adic analytic group. Then the rings $\mathcal{O}_K[[G]]$ and $k[[G]]$ are Auslander-Gorenstein. In particular, both rings are Noetherian.

Proof. See Proposition 2.4 in [3]. □

Now we recall an important result. It is called the Topological Nakayama Lemma. If G is a pro-p group, by Theorem 3.2.5, the Iwasawa algbera Λ_G is local. Let us denote by $\mathcal{M}$ the unique maximal ideal of Λ_G

Lemma 2.7.13. *(Topological Nakayama Lemma)* Let G be a pro-p group and let M be a compact Λ_G-module. Then M is generated by $m_1, \ldots, m_n$ if and only if $m_i + M\mathcal{M},\ i = 1, \ldots, n$ generate $M/M\mathcal{M}$ as an $\mathbb{F}_p$-vector space.

Proof. See Lemma 1.1 in [49]. □

We turn our attention to Iwasawa algebras over compact p-adic analytic groups. We assume again that G is uniform. In this case, every element of Λ_G can be written as a unique power series in finite number of variables. We make this more precise in the following statement.

Theorem 2.7.14. Let G be a uniform pro-p group with topological generating set $\{a_1, \ldots, a_d\}$. Let $J_0 = \ker(\mathbb{Z}_p[G] \to \mathbb{F}_p)$, i.e. the ideal $I(G) + p\mathbb{Z}_p[G]$. Let $b_i = a_i - 1 \in \mathbb{Z}_p[G]$. Then (*i*) Λ_G is isomorphic to the completion of $\mathbb{Z}_p[G]$ with respect to the J_0-adic filtration.

(*ii*) Each element can be written uniquely as a convergent power series

$$\sum_{\alpha\in\mathbb{N}^d} \lambda_\alpha b^\alpha$$

where $\lambda_\alpha \in \mathbb{F}_p$, $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ and $b^\alpha = b_1^{\alpha_1} \dots b_d^{\alpha_d}$.

Proof. See Theorem 7.1 and 7.20 in [19]. □

In fact, the topology of Λ_G is given by a certain norm. Moreover, Λ_G is the completion of the ordinary group ring $\mathbb{Z}_p[G]$ with respect to this norm.

Theorem 2.7.15. Let G be a uniform pro-p group and $c = \sum \lambda_\alpha b^\alpha$ be an element of Λ_G. Then the norm on Λ_G is

$$||c|| = \sup_\alpha\{p^{-|\alpha|}|\lambda_\alpha|\}$$

Proof. See Theorem 7.21 in [19]. □

There is a natural filtration given by

$$F_k = \{c \in \mathbb{Z}_p[[G]] \mid ||c|| \leq p^{-k}\}$$

This filtration is a refinement of the J-adic filtration where J is the unique maximal ideal of Λ_G. As emphasised before, in passing from the filtered ring to the associated graded ring, one loses a certain amount of information. The advantage is that the associated graded ring is easier to understand. In fact, the associated graded ring of both Λ_G and Ω_G is well-understood.

Theorem 2.7.16. Let G be a uniform pro-p group of dimension d. The associated graded ring of $\mathbb{Z}_p[[G]]$ with respect to the filtration $F\mathbb{Z}_p[[G]]$ is isomorphic to a polynomial ring in $d+1$ variables over $\mathbb{F}_p$, where d is the dimension of G, i.e

$$\mathrm{gr}^{\cdot}\mathbb{Z}_p[[G]] \cong \mathbb{F}_p[X_0, \dots, X_d]$$

Proof. See Theorem 7.22 in [19] □

We state the "$\mathbb{F}_p$" version of the previous results.

Theorem 2.7.17. Let G be a uniform pro-p group with topological generating set $\{a_1, \dots, a_d\}$. Let $\overline{J_0} = \ker(\mathbb{F}_p[G] \to \mathbb{F}_p)$, i.e. the augmentation ideal of $\mathbb{F}_p[G]$. Let $b_i = a_i - 1 \in \mathbb{F}[G]$. Then (*i*) Ω_G is isomorphic to the completion of $\mathbb{F}_p[G]$ with respect to the $\overline{J_0}$-adic filtration.

(*ii*) Each element can be written uniquely as a convergent power series

$$\sum \lambda_\alpha b^\alpha$$

where $\lambda_\alpha \in \mathbb{F}_p$, $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ and $b^\alpha = b_1^{\alpha_1} \ldots b_d^{\alpha_d}$.

(*iii*) Ω_G is a local ring with unique maximal ideal $\overline{J} = \ker(\Omega_G \to \mathbb{F}_p)$.

(*iv*) The associated graded ring with respect to the $\overline{J}$-adic filtration is isomorphic to a polynomial algebra in d variables over $\mathbb{F}_p$, i.e.

$$\mathrm{gr}^{\cdot}\,\Omega_G \cong \mathbb{F}_p[X_1, \ldots, X_d]$$

Proof. See Theorem 7.23 in [19]. □

To finish this section we give one more theorem.

Proposition 2.7.18. Let G be a torsion-free compact p-adic analytic group. Then Λ_G is a maximal order.

Proof. See [1] Theorem 4.1. □

2.8 Algebras of p-adic distributions

We turn our attention to define the algebras of continuous and locally analytic distributions and collect their properties. Throughout this section, we assume that K is a finite extension of $\mathbb{Q}_p$. Once more, fix a prime element p.

Definition 2.8.1. Let V be a K-vector space. We say that V is a **locally convex vector space** if it is equipped with a locally convex topology, i.e. there is a family of seminorms $\{q_i\}_{i \in I}$ such that the basis of neighbourhoods for 0 is given by

$$V(q_{i_1}, \ldots, q_{i_n}, \varepsilon) := \{v \in V \mid q_{i_j}(v) < \varepsilon\} \tag{1}$$

where $i_j \in I$.

Definition 2.8.2. Let V be a K-vector space. A **lattice** L in V is an $\mathcal{O}_K$-module such that for any vector $v \in V$ there is a non-zero $a \in K^*$ such that $av \in L$.

Definition 2.8.3. A locally convex K-vector space V is called **barrelled** if every closed lattice of V is open.

The advantage of barreled vector spaces is that we have the Banach-Steinhaus theorem. We remark that there is an alternative description of locally convex vector spaces via families of lattices [40].
If the space is finite dimensional, there is a very simple description of Hausdorff and locally convex vector spaces:

Proposition 2.8.4. Let V be an n dimensional K-vector space. The only Hausdorff and locally convex topology on V is given by the maximum norm, i.e. $||(v_1,\ldots,v_n)|| = \mathrm{max}_i |v_i|$.

The next general class of locally convex vector spaces that we are interested in is formed by the metrizable ones, i.e. those whose topology can be defined by a norm.

Proposition 2.8.5. Let V be a Hausdorff and locally convex K-vector space. The following as sertions are equivalent:

(i) V is metrizable;

(ii) the topology of V can be defined by a countable family of seminorms.

Proof. See Proposition 5.1 in [40] □

Definition 2.8.6. A locally convex K-vector space V is called **Fréchet-space** if it is metrizable and complete (with respect to the metric that defines the topology).

Banach spaces are basic examples for Fréchet-spaces. It is clear from the definition and the previous proposition that any countable projective limit of Banach-spaces is a Fréchet-space.

Definition 2.8.7. Let A be an associative unital K-algebra such that the underlying K-vector space is a Fréchet-space and the algebra multiplication is continuous. Then A is called **Fréchet-algebra**.

2.8.1 Fréchet-Stein algebras

Consider a continuous seminorm q on a Fréchet-algebra A. It induces a norm on the quotient space $A/\{a \mid q(a) = 0\}$. The completion will be a K-Banach space and we will denote it by A_q. Clearly, we have a natural continuous map $A \to A_q$ with dense image. Moreover, if two continuous seminorms $q_1 \leq q_2$ are given, then the identitiy on A extends naturally to a continuous, in fact norm decreasing, map

$\phi^{q_1}_{q_2} : A_{q_2} \to A_{q_1}$ such that

$$\begin{array}{ccc} & & A_{q_2} \\ & \nearrow & \\ A & & \big\downarrow \phi^{q_1}_{q_2} \\ & \searrow & \\ & & A_{q_1} \end{array} \tag{2}$$

commutes. Now, if we have a family of seminorms $q_1 \leq q_2 \leq \cdots \leq q_i \leq \ldots$ then it defines a Fréchet-topology on A. With the maps $\phi^{q_1}_{q_2}$, the A_{q_i} form an inverse system. By density of A in each A_{q_i} and the commutativity of the diagram above,

$$A \cong \varprojlim_{i \in \mathbb{N}} A_{q_i}$$

as locally convex K-vector spaces. We say that a continuous seminorm q on A is an **algebra seminorm** if the algebra multiplication on A is continuous with respect to the seminorm, i.e. for any $a, b \in A$, $q(ab) \leq cq(a)q(b)$ where $c \in \mathbb{R}$ such that $c > 0$. Clearly, this way the quotient and hence the completion will also be an algebra, the later will be a K-Banach algebra. The maps defined in (2) will be algebra homomorphisms. In this case, the isomorphism

$$A \cong \varprojlim_{i \in \mathbb{N}} A_{q_i}$$

will be an isomorphism of Fréchet-algebras.

Definition 2.8.8. A K-Fréchet-algebra is called **K-Fréchet-Stein algebra** if there is a sequence $q_1 \leq q_2 \leq \cdots \leq q_i \leq \ldots$ of algebra seminorms on A which define the Fréchet-topology such that

(i) A_{q_i} is (right) Noetherian,

(ii) A_{q_i} is a flat $A_{q_{i+1}}$-module (via the transition map) for any $i \in I$.

2.8.2 Continuous and locally analytic representations

For the sake of completness, we briefly recall the how the continuous and locally analytic representations of a p-adic analytic group are defined. However, apart from the continuous and locally analytic distribution algebra (in fact, we will use a nice decription of them, explained in the next section), we will not use anything from this section directly. Since it benefits us little to do everything precisely, we refer the kind reader to other sources for precise definitions and treatment of the following.

Consider the space of continuous K-valued functions, denoted by $C(G, K)$. We define $D^c(G, K)$ to be the continuous dual of $C(G, K)$ equipped with the bounded-weak topology (see Chapter 7. in[40]). Since G is compact and a locally $\mathbb{Q}_p$-analytic group, it can be seen that since K is a finite extension of $\mathbb{Q}_p$, $D^c(G, K) = K[[G]] = K \otimes_{\mathcal{O}_K} \mathcal{O}_K[[G]]$ (see Chapter 12 in [40]).

Definition 2.8.9. Let V be a K-Banach space. A K-Banach space (or continuous) representation on V is a G-action by continuous linear automorphisms such that the map $G \times V \to V$ giving the action is continuous.

Denote the category of K-Banach space representations of G by $\mathrm{Ban}_G(K)$. There are some pathologies that exist, if we consider general K-Banach space representations. For example, there exist non-isomorphic irreducible K-Banach space representations V and W of G and there is a non-zero G-equivariant continuous linear map $V \to W$. By Proposition 7.1 in [36], the continuous action of G on V extends to a separately continuous $D^c(G, K)$-module action and G-equivariant continuous linear maps extend to $D^c(G, K)$-module homomorphisms. It is more useful to consider, not the space V, but its dual V' which is also a $D^c(G, K)$-module. Indeed, let $\mathcal{M}(\mathcal{O}_K[[G]])$ denote the category of continuous $\mathcal{O}_K[[G]]$-modules such that the underlying $\mathcal{O}_K$-module lies in $\mathcal{M}(\mathcal{O}_K)$, the category of linear-topological compact and torsionfree $\mathcal{O}_K$-modules. Let $\mathcal{M}(\mathcal{O}_K[[G]])_{\mathbb{Q}}$ denote the additive category whose objects are the objects of $\mathcal{M}(\mathcal{O}_K[[G]])$ such that

$$\mathrm{Hom}_{\mathcal{M}(\mathcal{O}_K[[G]])_{\mathbb{Q}}}(A, B) := \mathrm{Hom}_{\mathcal{M}(\mathcal{O}_K[[G]])}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$$

Then we have the following anti-equivalence of categories:

Theorem 2.8.10. The functor

$$\mathrm{Ban}_G(K) \to \mathcal{M}(\mathcal{O}_K[[G]])_{\mathbb{Q}}$$

$$V \mapsto V'$$

is an anti-equivalence of categories.

Proof. See Theorem 8.3 in [36]. □

In order to avoid the above mentioned pathologies, we need to impose an additional finitness condition on our Banach space representations. Let V be a K-Banach space representation of G. Recall from Theorem 2.7.12 that $\mathcal{O}_K[[G]]$ and hence $K[[G]] = K \otimes_{\mathcal{O}_K} \mathcal{O}_K[[G]]$ are both Noetherian. Therefore a natural finitness condition we can impose is the following:

Definition 2.8.11. A K-Banach space representation V of G is **admissible** if its dual V' is finitely generated as a $K[[G]]$-module.

We denote by $\mathrm{Ban}_G^a(K)$, the category of admissible K-Banach space representations. Let $\mathrm{mod}_{fg}K[[G]]$ denote the category of finitely generated $K[[G]]$-modules. Then we have the following equivalence of categories:

Theorem 2.8.12. The functor

$$\mathrm{Ban}_G^a(K) \to \mathrm{mod}_{fg}K[[G]]$$

$$V \mapsto V'$$

is an anti-equivalence of categories.

There is a similar story with the locally analytic representations of G, but it is a more complicated. Let $U \subseteq K^d$ an open subset and V a K-Banach space. The norm of an element $x \in U$ is the maixmum of the norms of its coordinates, we denote by $|| \ ||_V$ the norm on V. We call a function $f : U \to V$ locally analytic if for any point $x_0 \in U$, there exists a closed polydisk $B_r(x_0) := \{x \in U \ : \ ||x|| \leq r\}$ such that

$$f(x) = \sum_\alpha v_\alpha (x - x_0)^\alpha \text{ with } v_\alpha \in V \text{ and } \lim_{|\alpha| \to \infty} r^{|\alpha|} ||v_\alpha||_V \to 0$$

where $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, $|\alpha| := \alpha_1 + \cdots + \alpha_d$, $(x-a)^\alpha := (x_1 - a_1)^{\alpha_1} \ldots (x_d - a_d)^{\alpha_d}$. G is a $\mathbb{Q}_p$-manifold of dimension d for some $d \in \mathbb{N}_0$, hence it makes sense to talk about locally analytic K-valued functions on G, since for each point $g \in G$, we can find an open neighbourhood of g, homeomorphic to some closed polydisk of $\mathbb{Q}_p^d$. Consider the K-vector space $C^{an}(G,K) \subseteq C(G,K)$ of locally analytic K-valued functions on G. We denote by $D(G,K) := C^{an}(G,K)'_b$ the dual of the vector space $C^{an}(G,K)$ with the strong topology (see Chapter 7 in [40]).

Definition 2.8.13. A locally analytic representation of G is an action of G on a locally convex barrelled K-vector space V such that, for each $v \in V$, the map $g \mapsto gv$ belongs to $C^{an}(G,V)$, i.e. the locally analytic, V-valued functons on G.

We denote the category of locally analytic representations of G by $\mathrm{Rep}_G(K)$. If V is an arbitrary locally convex K-vector space, locally analytic V-valued functions on G are complicated to define and we would need a lot of machinery in order to do so. We refer to [40], [41], for details. However, we remark that when V is a K-Banach space we already defined locally analytic V-valued functions above.
As in the Banach space represenation case, we want to have a reasonable theory and avoid certain pathologies. So we need some finitness condition. We have to find something else than what we had in the case of Banach space representations since the algebra $D(G,K)$ is in general not Noetherian. By Propostition 17.1 in [40], if V is a locally analytic representation of G, then the G action extends

to a separately continuous $D(G,K)$-module structure on V and G-equivariant continuous linear maps extend to $D(G,K)$-module homomorphisms. Moreover, in the proof of Corollary 3.3 in [41], it was shown that V carries a separately continuous $D(G,K)$-structure if and only if V'_b does, where V'_b denotes the dual of V equipped with the strong topology.
In [38], the authors show that the definition of the so-called coadmissible modules gives the right finitness condition that we need. Fix a Fréchet-Stein algebra A with a family of algebra seminorms $(q_i)_{i\in\mathbb{N}}$.

Definition 2.8.14. A **coherent sheaf** for $(A,(q_i))$ is a family $(M_i)_{i\in\mathbb{N}}$ of modules, where M_i is a A_{q_i}-module for all $i\in\mathbb{N}$, and there is an isomorphism

$$A_{q_i}\otimes_{A_{q_{i+1}}} M_i \cong M_{i+1}$$

for any $i\in\mathbb{N}$. For any coherent sheaf $(M_i)_i$, the A-module of global sections is defined by

$$\Gamma(M_n) := \varprojlim_n M_n.$$

Then an A-module M is called **coadmissible** if it is isomorphic to the module of global sections of some coherent sheaf.

The next proposition shows that the category of coadmissible modules, denoted by $\mathcal{C}_A$, is an abelian category.

Proposition 2.8.15.

- *(i)* The direct sum of two coadmissible modules is coadmissible;
- *(ii)* the (co)kernel and (co)image of any A-linear map between coadmissible A-modules is coadmissible;
- *(iii)* The sum of two coadmissible submodules of a coadmissible A-module is coadmissible;
- *(iv)* any finitely generated submodule of a coadmissible A-module is coadmissible;
- *(v)* any finitely presented A-module is coadmissible.

Proof. See Corollary 3.4 in [38] □

Corollary 2.8.16. $\mathcal{C}_A$ is abelian subcategory of mod-A.

Proof. See Corollary 3.5 in [38]. □

At this point, we do not know if $D(G,K)$ is a Fréchet-Stein algebra, but it is. We show the connection between the category of admissible locally analytic representations of G and coadmissible $D(G,K)$-modules with the following theorem:

Theorem 2.8.17. The functor

$$\mathrm{Rep}_G^a(K) \to \mathcal{C}_{D(G,K)}$$
$$V \mapsto V_b'$$

is an anti-equivalence of categories.

Proof. See Theorem 20.1 in [40]. □

2.8.3 $K[[G]]$ and $D(G,K)$

Let $\kappa = 1$, if p is odd and $\kappa = 2$, if p is even. Let G be a uniform pro-p group. Let us fix a minimal (ordered) topological basis $h_1, \dots, h_d$ for G. Then there is a bijective global chart

$$\mathbb{Z}_p^d \xrightarrow{\sim} H$$
$$(x_1, \dots, x_d) \mapsto (h_1^{x_1}, \dots, h_d^{x_d}).$$

Putting $b_i := h_i - 1$, $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, $|\alpha| = \sum \alpha_i$ and $b^\alpha := b_1^{\alpha_1} \dots b_d^{\alpha_d}$, one can identify $D(H,K)$ with all convergent power series

$$\sum_\alpha d_\alpha b^\alpha, \ d_\alpha \in K, \text{ such that the set } \{|d_\alpha| r^{|\alpha|}\}$$

is bounded for all $0 < r < 1$. Moreover, the Fréchet-topology on $D(G,K)$ is defined by the family of norms

$$||\lambda||_r := \sup_{\alpha \in \mathbb{N}_0^d} |d_\alpha| r^{\kappa|\alpha|}$$

for $0 < r < 1$. Since G is compact, by Proposition 2.3 in [41], $D(G,K)$ is a Fréchet-algebra with multiplication given by the convolution product and identity element the Dirac delta distribution δ_1. We embedd the group ring $\mathbb{Z}_p[G]$ into $D(G,K)$ by viewing a group element $g \in G$ as the Dirac delta distribution δ_g. If we assume that $1/p \le r < 1$ then the norm $|| \, ||_r$ on $D(G,K)$ is submultiplicative. Hence we can define a (decreasing) filtration on $D(G,K)$.

$$F_r^s D(G,K) := \{\lambda \in D(G,K) \ : \ ||\lambda||_r \le p^{-s}\}$$
$$F_r^{s+} D(G,K) := \{\lambda \in D(G,K) \ : \ ||\lambda||_r < p^{-s}\}.$$

Then

$$\mathrm{gr}^\cdot D(G,K) := \bigoplus_s F_r^s D(G,K)/F_r^{s+} D(G,K)$$

is the associated graded ring. If $r \in p^{\mathbb{Q}}$, this filtration is quasi-integral, meaining that there exists an $n_0 \in \mathbb{N}$ such that $\{s \in \mathbb{R} \ : \ \mathrm{gr}^s D(G,K) \neq 0\} \subseteq 1/n_0\mathbb{Z}$. We let $D_r(G,K)$ denote the completion of $D(G,K)$ with respect to the norm $|| \, ||_r$. As a K-Banach space $D_r(G,K)$ is given by all series

$$\lambda = \sum d_\alpha b^\alpha$$

such that $d_\alpha \in K$ and $|d_\alpha| r^{|\alpha|} \to 0$ as $|\alpha| \to \infty$. When G is abelian, these are just the rigid-analytic K-valued functions on the d dimensional closed polydisk with radius r. We introduce an even larger K-Banach space $D_{<r}(G,K)$ given by all series

$$\lambda = \sum d_\alpha b^\alpha$$

such that $d_\alpha \in K$ and the set $\{|d_\alpha| r^{|\alpha|}\}_\alpha$ is bounded. On both $D_r(G,K)$ and $D_{<r}(G,K)$, the norm continues to be given by

$$||\lambda||_r := \mathrm{sup}_\alpha |d_\alpha| r^{\kappa|\alpha|}$$

where $\lambda = \sum_\alpha d_\alpha b^\alpha$ is an element of $D_r(G,K)$, resp. $D_{<r}(G,K)$. By Proposition 4.2 in [38], the multiplication on $D(G,K)$ extends to both $D_r(G,K)$ and $D_{<r}(G,K)$, which makes $D_r(G,K)$ a K-Banach algebra. $D_{<r'}(G,K)$ is also a K-Banach algebra if $1/p < r'$. We get a system of K-Banach spaces

$$\cdots \subseteq D_r(G,K) \subseteq D_{<r}(G,K) \subseteq D_{r'}(G,K) \subseteq D_{<r'}(G,K) \subseteq \cdots \subseteq D_{1/p}(G,K)$$

with $1/p \leq r < r' < 1$ and

$$D(G,K) = \varprojlim_r D_r(G,K) = \varprojlim_r D_{<r}(G,K).$$

On $R = D_r(G,K)$, resp. $D_{<r}(G,K)$, we again have, for any $1/p \leq r < 1$, the filtration

$$F_r^s R := \{\lambda \in R \ : \ ||\lambda||_r \leq p^{-s}\} \tag{3}$$
$$F_r^{s+} R := \{\lambda \in R \ : \ ||\lambda||_r < p^{-s}\}$$

and associated graded ring

$$\mathrm{gr}^{\cdot} R := \bigoplus \mathrm{gr}^n R, \text{ where } \mathrm{gr}^n R := F_r^n R / F_r^{n+} R.$$

Theorem 2.8.18. Let G be a uniform pro-p group. For $1/p \leq r < 1$ and $r \in p^{\mathbb{Q}}$ the ring $\mathrm{gr}^{\cdot} D_r(G,K)$ is a polynomial ring over $\mathrm{gr}^{\cdot} K$ in the principal symbols $\sigma(b_i)$ for $i = 1, \ldots, d$. Moreover, $D_r(G,K)$ is a Noetherian integral domain.

Proof. See Theorem 4.5 in [38]. □

Theorem 2.8.19. Assume that G is a uniform pro-p group and $1/p < r < 1$ and $r \in p^{\mathbb{Q}}$. Then

(*i*) the natural inclusions

$$\mathbb{Z}_p[[G]] \hookrightarrow K[[G]] \hookrightarrow D_r(G, K)$$

are flat,

(*ii*) $D_{<r}(G, K)$ is Noetherian and the natural inclusion $D_r(G, K) \hookrightarrow D_{<r}(G, K)$ is flat,

(*iii*) $D_{<r}(G, K) \hookrightarrow D_{r'}(G, K)$ is flat.

Proof. See Proposition 4.7, Lemma 4.8 in [38] and Theorem 4.9 in [38]. □

Theorem 2.8.20. Let G be a compact p-adic analytic group.

(*i*) The natural inclusion

$$K[[G]] \hookrightarrow D(G, K)$$

is faithfully flat.

(*ii*) Then $D(G, K)$ is a Fréchet-Stein algebra.

(*iii*) gl.dim.$D_r(G, K) \leq d$ where $d := \dim(G)$

Proof. See Theorem 5.1, Theorem 5.2 and Theorem 8.9 in [38]. □

2.9 Tools from modular represenation theory

For the moment, G is an arbitrary finite group.

Definition 2.9.1. Let G be an arbitrary finite group of exponent n and let F be an arbitrary field. Then F is called a **splitting field of** G if for any simple $F[G]$-module V, $\mathrm{End}_{F[G]}(V) \cong F$.

Following Serre, we say that an arbitrary field F is **sufficiently large (relative to** G**)** if F contains all the n-th roots of unity where $n = |G|$.

Remark 2.9.2. If char$F = 0$, then F is sufficiently large relative to G if and only if F contains a cyclotomic field of n-th roots of unity. On the other hand, if char$F = p > 0$, write $n = mp^a$ where $p \nmid m$. Then in $F[X]$ we have

$$x^n - 1 = (x^m - 1)^{p^a},$$

and thus F contains the n-th roots unity if and only if F contains the m-th roots of unity. The polynomial $x^m - 1$ is separable over F, and its roots form a cyclic group $\langle \omega \rangle$ of order m, generated by a primitive m-th root of unity.

Theorem 2.9.3. If the field F is sufficiently large relative to G, then F is a splitting field for G and all its subgroups.

Proof. See Theorem (17.1) in [18]. □

Definition 2.9.4. A p-**modular system** (K, R, k) consists of a discrete valuation ring R, its quotient field K, and residue field k of characteristic p.

Certainly, if K is a finite extension of $\mathbb{Q}_p$, then $(K, \mathcal{O}_K, k)$ is a p-modular system.

Theorem 2.9.5. Let (K, R, k) be a p-modular system and assume that $\text{char} K = 0$. If K is sufficiently large relative to G then k is also sufficiently large relative to G, and both K and k are splitting fields for G.

Proof. See Corollary (17.2) in [18]. □

Definition 2.9.6. We say that a conjugacy class of G is p-**regular** if its order is relative prime to p.

We compute the Grothendieck group of the group algebra $k[G]$.

Lemma 2.9.7. Let (K, R, k) be a p-modular system. Assume that G is a finite group of exponent n and that K is sufficiently large relative to G. Then the Grothendieck group of $k[G]$ is $\mathbb{Z}^c$ where c is the number of p-regular conjugacy classes of G.

Proof. By Theorem 2.9.5, k is a splitting field for G. Hence by Theorem 2.8 Chapter III. in [20], the number of non-isomorphic simple modules is equal to the number of p-regular conjugacy classes of G, i.e. the classes with order relative prime to p. By Theorem 7.1 in [26] there is a one-to-one correspondence between the isomorphism classes of indecomposable projective modules and the isomorphism classes of simple modules. Using the fact that $k[G]$ is semiperfect, it follows from Proposition (16.7) in [18], that the Grothendieck group $K_0(k[G]) \cong \mathbb{Z}^c$. □

Let G be again an arbitrary finite group and (K, R, k) a p-modular system. Denote by m the unique maximal ideal of R. For simplicity, let us assume that R is complete m-adically (it is not really necessary, see the discussion before Proposition (16.7) in [15]). By Theorem (18.2) in [18], there is an isomorphism $\rho : K_0(R[G]) \to K_0(k[G])$ induced by sending $[P]$ to $[P/Pm]$, where P is an arbitrary finitely generated projective R-module. Moreover, its inverse $\rho^{-1} : K_0(k[G]) \to K_0(R[G])$ is given by sending $[Q]$, a finitely generated projective $k[G]$-module, to the class $[P]$, where P is the projective cover of Q as a $R[G]$-module. Note that the projective cover exists since $R[G]$ is semiprime, which is easy to see from Proposition 1.2.1 (*iii*) in [37] and the definition of semiprimenes.

There is also a homomorphism $\kappa : K_0(R[G]) \to K_0(K[G])$ induced by the assigment $P \mapsto P \otimes_R K$, where P is a finitely generated R-module. We can define a homomorphism (which is part of the so-called Cartan-Brauer triangle)

$$e_G : K_0(k[G]) \xrightarrow{\rho^{-1}} K_0(R[G]) \xrightarrow{\kappa} K_0(K[G]) \tag{4}$$

(See for example (18.2) in [18]).

Proposition 2.9.8. The homomorphism e_G is injective.

Proof. See Corollary (18.15) in [18]. □

Corollary 2.9.9. The homomorphism κ is also injective.

2.10 Additional tools from ring theory

We will briefly mention some additional tools we will use. First, suppose R is a commutative ring. The support of an R-module M, denoted by $\mathrm{Supp}_R(M)$, is the set of prime ideals $P \subseteq R$ such that the localized module $M_P \neq 0$. The following proposition is well-known.

Proposition 2.10.1. If M is finitely generated then $\mathrm{Supp}_R(M)$ is exactly the set of prime ideals containing $\mathrm{ann}_R(M)$.

We will also need the following observation:

Proposition 2.10.2. *M is torsion-free over R if and only if M has no non-zero R-submodule $N \subseteq M$ such that $\mathrm{ann}_R(N) \neq 0$.*

Proof. The one direction is trivial. For the only if part, let us assume that $0 \neq m \in M$ is an R-torsion element, i.e. there exists an element $0 \neq r \in R$ such that $mr = 0$. Then by commutativity, $xr'r = xrr' = 0$ for any $r' \in R$. Hence the cyclic R-module is a non-zero torsion R-submodule of M. □

We use the usual notation for the set of all prime ideals of a ring R by $\mathrm{Spec}(R)$. It is also well known that the nilradical is the set of nilpotent elements and also the intersection of all prime ideals of R.

2.10.1 Domains and rings that dominate them

It is a natural question to ask that whenever a right Noetherian ring is given, does it have zero-divisors? A major tool in the investigation of this question is the significant result, due to Walker, that can be used for a wide class of rings and it gives a necessary and sufficient condition for a right Noetherian local ring to be a domain. It will be one of our essential tool.

Definition 2.10.3. Let R be a ring and M an R-module. An element $m \in M$ is called **singular element** of M if the right ideal $\text{ann}(m)$ is an essential submodule of R. The set of all singular elements of M is denoted by $\mathcal{Z}(M)$. If we consider R as a right R-module, denoted by R_R, the set of singular elements $\mathcal{Z}(R_R)$ of R_R will be called the **singular right ideal** of R.

Theorem 2.10.4. *(Walker)* Let R be a right Noetherian local ring such that every non zero right ideal has finite homological dimension. Then R is a domain if and only if the singular right ideal of R is zero.

Proof. See Theorem 2.9 in [50]. □

The following important result is due to Chevalley. It gives some partial answer to the question: what rings lie between a commutative Noetherian domain and its field of fractions? More precisely, it states that if a commutative local Noetherian domain with field of fractions $Q(R)$ is given then there is always an intermediate ring S between R and $Q(R)$ which is S is a discrete valuation ring.

Definition 2.10.5. Let (R, m_R) be a commutative local ring with maximal ideal m_R and field of fractions $Q(R)$. We say that a local ring (S, m_S) **dominates** R if R is a subring of S and $m_R = m_S \cap R$ or equivalently the inclusion $R \hookrightarrow S$ is a local homomorphism. S **birationally dominates** R if moreover S is contained in the field of fractions of R, i.e. $S \subset Q(R)$.

Theorem 2.10.6. *(Chevalley)* Let (R, m_R) be a commutative Noetherian local domain. Then there exists a discrete valuation ring S that birationally dominates it.

Proof. See Theorem 2.2 and 2.3 in [14]. □

3 Reflexive ideals, centres of skewfields, characterization of the completely faithful property

3.1 The statement

In [1], Ardakov proved the following theorem:

Theorem 3.1.1. Let $p \geq 5$ and let H be a compact p-adic analytic group without torsion element, whose Lie algebra $\mathcal{L}(H)$ over $\mathbb{Q}_p$ is split semisimple. Moreover, let $G = H \times Z$ where $Z \cong \mathbb{Z}_p$ and let M be a finitely generated torsion Λ_G-module which has no non-zero pseudo-null submodules. Then $q(M)$ is completely faithful if and only if M is torsion-free over Λ_Z.

Proof. See Theorem 1.3 in [1]. □

In the next section, we prove a more general version of this theorem:

Theorem 3.1.2. Let $p \geq 5$ and let $G = H \times Z$, where H is a compact p-adic analytic group such that it is torsion-free and its Lie algebra $\mathcal{L}(H)$ over $\mathbb{Q}_p$ is split semisimple and let $Z \cong \mathbb{Z}_p^n$ for some integer $n \geq 0$. Let M be a finitely generated torsion Λ_G-module such that it has no non-zero pseudo-null submodules. Then $q(M)$ is completely faithful if and only if M is Λ_Z torsion-free.

3.2 The proof of the statement

Proposition 2.7.8 states that whenever G is a pro-p group, the Iwasawa algebra Λ_G is a local ring with maximal ideal $\mathcal{M} = I(G) + (p)$ where $I(G)$ is the augmentation ideal. The group H is pro-p (since it is torsion-free) and normal is G. Let $w_{H,G} = \ker(\Lambda_G \to \Omega_{G/H})$ and take its prime radical $I_H = \sqrt{w_{H,G}}$, i.e. the intersection of all the prime ideals of Λ_G that contain $w_{G,H}$. By Theorem G in [5] (or by Proposition 2.4 together with Proposition 2.6 in [17]), the ideal I_H is a localizable ideal in Λ_G, meaning that the set

$$S = \{s \in \Lambda_G \mid \text{s is regular mod } I_H\} \tag{5}$$

is a two-sided Ore set in Λ_G. We choose a minimal topological generating set $g_1, \ldots, g_n$ for Z. Denote by Z_i the subgroup of Z generated by $g_1, \ldots, g_{i-1}, g_{i+1}, g_n$. Consider the prime radical I_{G_i} of the ideal $\ker(\Lambda_G \to \Omega_{G/G_i})$ where $G_i := H \times Z_i$. The group G_i is a normal subgroup of G such that $G/G_i \cong \mathbb{Z}_p$ for all $i = 1, \ldots, n$. Define

$$S_{G_i} = \{s \in \Lambda_G \mid \text{s is regular mod } I_{G_i}\}.$$

Now By Proposition 2.7.11 (ii), the rings $\Omega_{G/H}$ and Ω_{G/G_i} $(i = 1, \ldots, n)$ are prime. So in particular, they are semiprime. Hence the ideals $\ker(\Lambda_G \to \Omega_{G/H})$ and $\ker(\Lambda_G \to \Omega_{G/G_i})$, $(i = 1, \ldots, n)$, are semiprime ideals. Therefore, by the fact the the prime radical of an ideal is the smallest semiprime ideal that contains the ideal we deduce that $I_H = \ker(\Lambda_G \to \Omega_{G/H})$ and $I_{G_i} = \ker(\Lambda_G \to \Omega_{G/G_i})$. Since for a fixed index i, $I_{G_i} = (H - 1, z_1, \ldots, z_{i-1}, z_{i+1}, z_n)$, where $z_j = g_j - 1$ for all $j = 1, \ldots, n$, it is easy to see that

$$\bigcap_{i=1}^{n} I_{G_i} = I_H.$$

Proposition 3.2.1. Let G be of the form as in Theorem 3.1.2 and let I a non-zero prime c-ideal of Λ_G. Then $I \cap \Lambda_Z \neq 0$.

Before giving a proof we present a few technical lemmas.

Lemma 3.2.2. If $I \cap S = \emptyset$ then $I = (p)$

Proof. Proposition 3.4 and Theorem 4.2 in [1] together imply that the localized ideal $I_{G,H}$ of I in $\Lambda_{G,H}$ is generated by p. It follows that p is in I, by well the known connection between the localized ideal and the ideal itself: I is the intersection of the localized ideal $I_{G,H}$ and Λ_G. Note that p is a central non-zero divisor in Λ_G such that $\Lambda_G/p\Lambda_G = \Omega_G$ is a domain. Hence by Proposition 2.1.13, $I = p\Lambda_G$. □

Note that if $n = 0$, then $G \cong H$. This implies, by Proposition 4.4 in [5], that the only prime c-ideal of $\Lambda_G = \Lambda_H$ is (p). Hence $p \in I \cap \Lambda_Z$. So the Proposition holds in this case. The other case is that $I \cap S \neq \emptyset$. Our assumption implies that $n \geq 1$.

Lemma 3.2.3. Let us assume that $I \cap S \neq \emptyset$. Then there exists a subgroup $\overline{Z}$ of Z, isomorphic to $\mathbb{Z}_p^{n-1}$, such that the Λ_G-module Λ_G/I is finitely generated as a module over the subalgebra $\Lambda_{\overline{G}} \subset \Lambda_G$ where $\overline{G} = H \times \overline{Z}$.

Proof. We prove that there is an index $i_0 \in \{1, \ldots, n\}$ such that $I \cap S_{G_i} \neq \emptyset$. Let $0 \neq \lambda \in I \cap S$, i.e. $\lambda \in I$ has the property that the image, denoted by $\overline{\lambda_H}$, of λ in $\Omega_{G/H}$ is regular. We note that $\Omega_{G/H}$ is a domain since $\Omega_{G/H} \cong \mathbb{F}_p[[z_1, \ldots, z_n]]$. Let us assume that $I \cap I_{G_i} = \emptyset$ for all $i = 1, \ldots, n$. The fact that $\Omega_{G/G_i} \cong \mathbb{F}_p[[z_i]]$ implies that Ω_{G/G_i} is a domain for all $i = 1, \ldots, n$. Therefore, the assumption that $I \cap I_{G_i} = \emptyset$ implies that $\lambda \in I_{G_i}$ for all $i = 1, \ldots, n$. Hence $\lambda \in I_H = \cap_{i=1}^{n} I_{G_i}$ which is a contradiction by the assumption that $\lambda \in S$. It follows that there is an index i_0 such that $\lambda \in S_{G_{i_0}}$. The fact that $G/G_{i_0} \cong \mathbb{Z}_p$ allows us to use Proposition 2.6 and 2.3 in [17], which shows that Λ_G/I is finitely generated as a $\Lambda_{G_{i_0}}$-module where $G_{i_0} = H \times Z_{i_0}$. Hence $\overline{Z} = Z_{i_0}$ is the subgroup of Z satisying the requirements of the lemma. □

3.2.1 Reflexive ideals and skewfield of fractions

In order to proceed, we need to prove an analogous result in connection with complete group algebras over complete discrete valuation rings that are not necesseraly finite extentions of $\mathbb{Z}_p$, but at least they birationally dominate it (see Definition 2.10.5). The result is similar to the well-known finite case (see Proposition 2.7.8), but more general.

Let $\mathcal{O}$ be a discrete valuation ring with maximal ideal $\mathfrak{M}$ and G a profinite group. The ring $\mathcal{O}$ is an $\mathfrak{M}$-adic ring hence the ideals

$$\mathfrak{M}^n\mathcal{O}[[G]] + I(N)$$

form a fundamental system of neighbourhoods for $0 \in \mathcal{O}[[G]]$ where N runs through the open normal subgroups of G.

Definition 3.2.4. Define $\mathrm{Rad}(\mathcal{O}[[G]])$ to be the inverse limit of the Jacobson radicals (the intersection of all maximal right ideals) of $\mathcal{O}/\mathfrak{M}^n[G/N]$.

It is easy to see that $\mathrm{Rad}(\mathcal{O}[[G]])$ is the intersection of all open maximal right ideals of $\mathcal{O}[[G]]$.

Proposition 3.2.5. Let G be a pro-p group and let $\mathcal{O}$ be a complete discrete valuation ring with maximal ideal $\mathfrak{M} = (\pi)$ (where π is a prime element in $\mathcal{O}$) such that $\mathbb{Z}_p \subseteq \mathcal{O}$ and $(p) = \mathbb{Z}_p \cap \mathfrak{M}$. Then $\mathcal{O}[[G]]$ is local.

Proof. Let us take an open maximal right ideal $\mathcal{M}$ of $\mathcal{O}[[G]]$. It follows that the quotient $M = \mathcal{O}[[G]]/\mathcal{M}$ is a discrete $\mathcal{O}[[G]]$-module with the quotient topology. Take an arbitrary non-zero element $m \in M$ and consider the submodule $L = m\mathcal{O}[[G]] \subseteq M$. It is a discrete module with the subspace topology. Then $\mathrm{ann}_{\mathcal{O}[[G]]}(L)$ is an open ideal in $\mathcal{O}[[G]]$. Therefore, since $\mathrm{ann}_{\mathcal{O}[[G]]}(L)$ is a neighbourhood of 0, there is an integer $k \in \mathbb{N}$ and an open normal subgroup N of G such that L is a $\mathcal{O}/\mathfrak{M}^k[G/N]$-module. Applying this argument, it follows that there is a integer t and an open normal subgroup N of G such that the cyclic $\mathcal{O}[[G]]$-module M is an $\mathcal{O}/\mathfrak{M}^t[G/N]$-module. Hence $p^t \in \mathrm{ann}_{\mathcal{O}[[G]]}(M)$.
There is an natural number s, such that $p = \pi^s u$ where u is a unit in $\mathcal{O}[[G]]$. It follows that

$$M\mathfrak{M}^{s+t} = 0 \tag{6}$$

But $\mathcal{M}$ was maximal hence M is simple. The set $\mathfrak{M} \subset \mathcal{O}[[G]]$ is central in $\mathcal{O}[[G]]$. Hence $M\mathfrak{M}$ is an $\mathcal{O}[[G]]$-submodule of M. Assume that it is a non-zero submodule. Then it must be isomorphic to M by the fact that M is simple. But that is impossible by (6). It implies that $M\mathfrak{M} = 0$. Therefore $\mathfrak{M} \subseteq \mathcal{M}$. But it is true for any open maximal right ideal of $\mathcal{O}[[G]]$ hence $\mathfrak{M} \subseteq \mathrm{Rad}(\mathcal{O}[[G]])$.
Take any element $g \in G$ and any open normal subgroup $N \lhd_o G$. Since G is pro-p, it follows that there is an $n \in \mathbb{N}$ such that $g^{p^n} \in N$. Hence the image of $g - 1$ is nilpotent in $\mathcal{O}/\mathfrak{M}[G/N]$. By definition, it means that $(g - 1)$ is contained in $\mathrm{Rad}(\mathcal{O}[[G]])$. These elements are the generators of the agumentation ideal. Hence $\mathfrak{M}\mathcal{O}[[G]] + I(G) \subseteq \mathrm{Rad}(\mathcal{O}[[G]])$. $\mathrm{Rad}(\mathcal{O}[[G]]) \subseteq \mathfrak{M}\mathcal{O}[[G]] + I(G)$ is trivial since the later is an open maximal ideal in $\mathcal{O}[[G]]$. Now we see that the radical equals to a maximal ideal and hence $\mathcal{O}[[G]]$ is local. $\square$

Proposition 3.2.6. Let $G = H \times Z$ where H is a torsion free compact p-adic analytic group and $Z \cong \mathbb{Z}_p^n$ such that $n \geq 0$. Let I be a prime c-ideal in Λ_G such that $I_Z = I \cap \Lambda_Z \neq 0$. Then I_Z is a principal reflexive prime ideal in Λ_Z generated by a prime element f and I is just $f\Lambda_G$.

Proof. To prove that I_Z is a prime ideal we need to show that if $ab \in I_Z$ where $a, b \in \Lambda_Z$ then a or b is in I_Z. But $ab\Lambda_G = a\Lambda_G b \subseteq I$ since $b \in \Lambda_Z$ is a central

element in Λ_G and I is an ideal of Λ_G. Hence by the assumption that I is prime in Λ_G implies that a or b is in I. Then a or b is in $I \cap \Lambda_Z$.
$I \cap \Lambda_Z$ is reflexive by Proposition 2.1.11 (*ii*). Moreover, Λ_Z is a UFD (it is a power series ring in n variables over $\mathbb{Z}_p$), so one can apply Lemma 2.1.21 to show that $I \cap \Lambda_Z$ is principal. Hence it contains a prime element of Λ_Z. Thus, by Lemma 2.1.13, it is generated by a prime element. Let us denote this prime element by $f \in \Lambda_Z$.

Lemma 3.2.7. $\Lambda_G/f\Lambda_G$ is a domain.

Proof. The ring $\Lambda_Z/f\Lambda_Z$, which we will denote by R, is a commutative local ring with a unique maximal ideal, denoted by $\mathfrak{M}_R$. First, we use Theorem 2.10.6, due to Chevalley. It implies that there is a discrete valuation ring S with maximal ideal $\mathfrak{M}_S$ such that $\mathfrak{M}_R = \mathfrak{M}_S \cap R$. Now we complete S to get a complete discrete valuation ring $\widehat{S}$. By Remark 0.1 *ii.* in [45], this ring is a commutative pseudocompact ring in the $\widehat{\mathfrak{M}}_S$-adic topology, since it is $\widehat{\mathfrak{M}}_S$-adically complete and the quotient $\widehat{S}/\widehat{\mathfrak{M}}_S \cong k$ is artinian, where k is some field extension (it can be infinite) of $\mathbb{F}_p$. Observe that $\Lambda_G = \varprojlim_{N \lhd_o H} \Lambda_Z[H/N]$. So $\Lambda_G/f\Lambda_G = \varprojlim_{N \lhd_o H} \Lambda_Z/f\Lambda_Z[H/N] = R[[H]]$ since Λ_Z is central. Now by the inclusions $R \subseteq S \subseteq \widehat{S}$ we have

$$0 \to R[H/N] \hookrightarrow S[H/N] \tag{7}$$

$$0 \to S[H/N] \hookrightarrow \widehat{S}[H/N] \tag{8}$$

for any open normal subgroup N of H. But the projective limit functor is left exact. Hence we get the following:

$$0 \to R[[H]] \hookrightarrow S[[H]] \tag{9}$$

$$0 \to S[[H]] \hookrightarrow \widehat{S}[[H]]. \tag{10}$$

So if we prove that the ring $\widehat{S}[[H]]$ is a domain, we are done. For that, we apply Theorem 2.10.4, due to Walker. But first, we check that $\widehat{S}[[H]]$ has all the properties that the theorem requires.

Lemma 3.2.8. The ring $\widehat{S}[[H]]$ is a domain.

Proof. Requirement 1 : $\widehat{S}[[H]]$ is Noetherian. H is a p-adic analytic group which means that it has an open uniform subgroup N of dimension d. It is enough to prove that $\widehat{S}[[N]]$ is Noetherian, since $\widehat{S}[[H]]$ is a free module over $\widehat{S}[[N]]$ with rank $|H/N|$ (in fact, $\widehat{S}[[H]]$ is actually a crossed product of $\widehat{S}[[N]]$ and the quotient group

H/N). First, we will prove a quite general result in connection with (completed) group rings over a field (possibly infinite) of characteristic p. The special case of this result, i.e. when k is finite can be found in many textbooks.

Theorem 3.2.9. Let k be a field of characteristic p and G a uniform pro-p group of dimension d. Consider the completed group ring $k[[G]]$ and the filtration with respect to its maximal ideal which is the augmentation ideal. Then the associated graded ring of $k[[G]]$ is isomorphic to the polynomial ring over k in d variables.

Proof. First, we let G be a more general group, namely a powerful pro-p group of dimension d. Fix a topological generating set $\{a_1, \dots, a_d\}$ for G and let $b_i = a_i - 1$. We have already defined the the lower p-series $G_1 = G \geq \dots \geq G_i \geq \dots$ in G in Definition 2.2.8. Let I_i be the kernel of the map $\pi_i : k[G] \to k[G/G_i]$. Note that I_i equals $k(G_i - 1)$. Let us define the following set $T_i = \{\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d \mid \alpha_j < p^{i-1}, j = 1, \dots, d\}$.

Lemma 3.2.10. Let $u_1, \dots, u_r \in G$ and put $v_i = u_i - 1$. Then for any $\beta \in \mathbb{N}^r$

$$u^\beta = \sum_{\alpha \in \mathbb{N}^r} \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_r}{\alpha_r} v^\alpha$$

$$v^\beta = \sum_{\alpha \in \mathbb{N}^r} (-1)^{|\beta|-|\alpha|} \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_r}{\alpha_r} u^\alpha$$

where $u^\alpha := u_1^{\alpha_1} \cdots u_r^{\alpha_r}$ and v^β is defined analogously.

Proof. See Lemma 7.8 in [19] □

Proposition 3.2.11. Let k and G be as above. Then we have the following:

(*i*)
$$k[G] = I_i + \sum_{\alpha \in T_i} k b^\alpha$$
where $b = b_1^{\alpha_1} \cdots b_d^{\alpha_d}$

(*ii*) If G is in addition uniform then
$$k[G] = I_i \oplus \bigoplus_{\alpha \in T_i} k b^\alpha$$

(*iii*) $b^\alpha \in I_i$ for each $\alpha \in \mathbb{N}^d \setminus T_i$.

Proof. (*i*) Proposition 3.7 in [19] states that each element of G/G_i can be written in the form $a_1^{\alpha_1} \dots a_d^{\alpha_d} G_i$ with $0 \leq \alpha_j < p^{i-1}$ for $j = 1, \dots, d$. Hence the images $\{\pi_i(a^\alpha) \mid \alpha \in T_i\}$ generate $k[G/G_i]$ as a k-module (vector space). The previous lemma shows that $\{\pi_i(b^\alpha) \mid \alpha \in T_i\}$ generates exactly the same module.
(*ii*) Suppose that G is uniform. Then $|G/G_i| = p^{(i-1)d}$. So $\pi_i(k[G]) = k[G/G_i]$ is a free k-module of rank $p^{(i-1)d}$. Since $p^{(i-1)d} = |T_i|$ it follows that the generating set $\{\pi_i(b^\alpha) \mid \alpha \in T_i\}$ is now actually a basis for this module. So we have (*ii*).
(*iii*) Let $\alpha \in \mathbb{N}^d \setminus T_i$. Then $\alpha_j > p^{i-1}$ for some j, so b^α has a factor of the form

$$b_j^{p^{i-1}} = (a_j - 1)^{p^{i-1}} = a_j^{p^{i-1}} - 1$$

As $a_j^{p^{i-1}} \in G_i$ it follows that $b_j^{p^{i-1}} \in (G_i - 1)k = I_i$. □

Now $I_1 = I(G)$ is the augmentation ideal which is a maximal ideal of $k[G]$. Let $I_0 := k[G]$. It is easy to check from the definition that the ideals I_i, $i \geq 0$ form a filtration of $k[G]$. Consider the filtration with respect to the maximal ideal I_1. Theorem 3.6 in [19] states that $G_i = G_{i-1}^p = \{x^p \mid x \in G_{i-1}\}$. Using this, it is clear that $I_i^p = I_{i+1}$ for any $i \geq 1$ so we have

$$I_1 \supset I_1^2 \supset \dots \supset I_1^p = I_2 \supset \dots$$

Hence it is indeed a refinement of the filtration by the ideals I_i. Assume now that G is uniform. By Proposition 3.2.11 (*ii*) and (*iii*), it follows that the images of b^α in the graded ring are free generators of $\mathrm{gr}^{\cdot}(k[G])$ as a k-module. Hence the images $x_i = b_i + I_1^2$ generate the associated graded ring as a k-algebra and they are free generators. We prove that this k-algebra is commutative. We have to show that $b_i b_j - b_j b_i \in I_1^2$. Now

$$(g_i - 1)(g_j - 1) - (g_j - 1)(g_i - 1) = g_i g_j - g_j g_i = [g_i, g_j] - 1.$$

G is assumed to be uniform therefore, by definition, $[G, G] \subseteq G^p$. But again by Theorem 3.6 in [19], $G^p = G_1^p = G_2$. So $[g_i, g_j] - 1 \in k(G_2 - 1) = I_2 \subseteq I_1^2$. Hence $\mathrm{gr}(k[G]) \cong k[x_1, \dots, x_d]$ where $x_i = b_i + I_1^2$. But $k[G]$ is dense in its completion with respect to its maximal ideal. This completion is $k[[G]]$. Therefore the associated graded ring of $k[G]$ and $k[[G]]$ are isomorphic. It follows that the graded ring of $k[[G]]$ is also a polynomial ring. □

In the light of the previous theorem and Proposition 2.4.7, it is clear that the filtration on $k[[G]]$ by its maximal ideal is a Zariskian filtration since $k[[G]]$ is complete with respect to its filtration and the associated graded ring is Noetherian. Hence by Theorem 2.4.9 (*d*), $k[[G]]$ is an Auslander-Gorenstein ring. In particular, it is Noetherian.

Theorem 3.2.12. Let R be a ring and $a \in R$ is a normal element in the Jacobson radical of R. Assume that the quotient R/aR is Auslander-Gorenstein (Auslander-regular) then R is also Auslander-Gorenstein (Auslander-regular).

Proof. See Theorem 2.2 in [15]. □

We finish the proof of Lemma 3.2.8. Let π be a prime element of the complete DVR $\widehat{S}$ that generates the maximal ideal. It is certainly a normal element in $\widehat{S}[[H]]$ since it is central. The quotient ring $\widehat{S}[[H]]/\pi\widehat{S}[[H]]$ is isomorphic to $k[[G]]$ where k is the residue field of $\widehat{S}$, i.e. $k = \widehat{S}/(\pi)$. The field k is a possibly infinte extention of $\mathbb{F}_p$ since $\mathbb{Z}_p \subseteq \widehat{S}$ and $(\pi) \cap \mathbb{Z}_p = (p)$ by the properties of $\widehat{S}$. By Theorem 3.2.9, $k[[G]]$ is Auslander-regular. Hence by Theorem 3.2.12 $\widehat{S}[[H]]$, is also Auslander-regular. In particular, it is Noetherian.
So we proved one of the requirements of Walker's Theorem.
Requirement 2 : $\widehat{S}[[H]]$ is local. That was Proposition 3.2.5.
Requirement 3: $\widehat{S}[[H]]$ has finite global dimension. We have already showed above that the ring $\widehat{S}$ is pseudocompact. H is pro-p, hence we can use Theorem 2.6.4, due to Brumer.

Now that $\widehat{S}[[H]]$ has all the properties we can use Theorem 2.10.4. The theorem states that $\widehat{S}[[H]]$ is a domain if and only if the singular right ideal of $\widehat{S}[[H]]$ (Definition 2.10.3) is zero.
We claim that in order to prove that the singular right ideal of $\widehat{S}[[H]]$ is zero it is enough to prove that $\widehat{S}[[H]]$ is semiprime. The reason is the following: A semiprime ring that satisfies the ascending chain condition on annihilators of elements has zero singular right ideal by Corollary 7.19 in [24]. The ring $\widehat{S}[[H]]$ is Noetherian. So the only thing to prove is the following:

Lemma 3.2.13. The ring $\widehat{S}[[H]]$ is semiprime.

Proof. First, let K be any field of characteristic 0 and G any finite group. Consider the group algebra $K[G]$. Let us define for an arbitrary element $x = \sum k_g g \in K[G]$ the *trace* of x by $\operatorname{tr}(x) = x_1$ (the coefficient corresponding to the identity element). Lemma 2.1.2 in [32] states that if the element x is nilpotent then $\operatorname{tr}(x) = 0$. Let R be a commutative domain such that its field of fractions Q is of characteristic zero. We can embedd R into Q. It is clear that the lemma remains valid for $R[G]$ via this embedding. We claim that the group algebra $R[G]$ is always semiprime. We use the following definition of a ring being semiprime: If $x \in R[G]$ is an element such that $xR[G]x \subseteq (0)$ then $x \in (0)$, i.e. $x = 0$. Consider a non-zero element $x = \sum x_g g \in R[G]$. If $xR[G]x \subseteq (0)$ then it follows that $x^2 = 0$, i.e. it is nilpotent. Moreover, for an arbitrary element $g \in G$, the element xg^{-1} is also nilpotent since

$(xg^{-1})^2 = xg^{-1}xg^{-1} = (xg^{-1}x)g^{-1}$ and $xg^{-1}x \in xR[G]x \subseteq (0)$, hence $(xg^{-1})^2 = 0$. So by Lemma 2.1.2 in [32],

$$\operatorname{tr}(xg^{-1}) = x_g = 0.$$

This is true for any g hence $x = 0$. Therefore $R[G]$ is semiprime. Now let G be a profinite group and consider the completed group ring $R[[G]]$. Let us assume that x is a non-zero element of $R[[G]]$ such that $xR[[G]]x \subseteq (0)$. x is non-zero hence there is an open normal subgroup $U \triangleleft G$ such that the image $\overline{x}$ of x in $R[G/U]$ is not-zero. But the assumption that $\overline{x}R[G/U]\overline{x} \subseteq (0)$ remains valid in $R[G/U]$. By the semiprime property of the group algebra $R[G/U]$, it follows that $\overline{x} = 0$. But that implies that $x \in \ker(R[[G]] \to R[G/U])$ which is a contradiction since we assumed that the image of x in $R[G/U]$ is not zero. Certainly, we can apply this argument to our situation and so it follows that $\widehat{S}[[H]]$ is semiprime. □

We are done with the proof of Lemma 3.2.8 since, as we have pointed out above, the last lemma implies that that the ring $\widehat{S}[[H]]$ is a domain. □

So by the tower of inclusions $\Lambda_Z/f\Lambda_Z[[H]] \subseteq S[[H]] \subseteq \widehat{S}[[H]]$ and Lemma 3.2.8 we conlude that $\Lambda_Z/f\Lambda_Z[[H]]$ is a domain. Hence we are done with Lemma 3.2.7. □

Now we can use Proposition 2.1.13 and Lemma 3.2.7 to conclude. □

We turn our attention to the proof of Proposition 3.2.1. We will use an inductive argument on the dimension of Z. Recall that the group of interest has the form $G = H \times Z$ where H is torsion free and its Lie algebra $\mathcal{L}(H)$ is split semisimple, the group Z has the property that $Z \cong \mathbb{Z}_p^n$. If $n = 0$ then $\Lambda_Z = \mathbb{Z}_p$. As noted before, the statement then follows from Theorem 4.4 in [1] since the only prime c-ideal of $\Lambda_G = \Lambda_H$ is $I = (p)$. Then it is certainly true that $I \cap \mathbb{Z}_p \neq 0$ since p is a non-zero element of the intersection. In the previous section we built up all the necessary tools to proceed. Now we apply induction on the dimension of Z which we denoted by n. Let us suppose that the statement of Proposition 3.2.1 holds for an arbitrary natural number n. More precisely, if $G = H \times Z$ where H is as above and $Z \cong \mathbb{Z}_p^n$ and I is a proper prime c-ideal then I has the property that $I_Z = I \cap \Lambda_Z \neq \emptyset$. We prove that the statement holds for $n+1$ if it holds for n. For an arbitrary profinite group G', let us denote the skewfield of fractions (if it exists) of the Iwasawa algebra $\Lambda_{G'}$ by $Q(G')$ and let us denote the center of $Q(G')$ by $Z(Q(G'))$.

Proposition 3.2.14. Let us consider the subgroup $\overline{G} := H \times \overline{Z}$ of G where $\overline{Z} \leq Z$ such that $\overline{Z} \cong \mathbb{Z}_p^n$. Then $Z(Q(\overline{G}))$ equals $Q(\overline{Z})$.

Proof. The inclusion that we need to show is $Z(Q(\overline{G})) \subseteq Q(\overline{Z})$. The other inclusion is clear since $\Lambda_{\overline{Z}}$ is central in Λ_G. Choose and fix a topological generating

set $\{g_1, \ldots, g_n\}$ for $\overline{Z}$. Consider an arbitrary element q that is in the center of $Q(\Lambda_{\overline{G}})$. By definition, the right $\Lambda_{\overline{G}}$-module $q\Lambda_{\overline{G}}$ is a fractional right ideal. It is easy to check, again from the definitions, that $(q\Lambda_{\overline{G}})^{-1} = \Lambda_{\overline{G}}q^{-1}$ and the same way that $(\Lambda_{\overline{G}}q^{-1})^{-1} = q\Lambda_{\overline{G}}$. Hence $q\Lambda_{\overline{G}}$ is reflexive as a right $\Lambda_{\overline{G}}$-module. One proves analogously that the left fractional ideal $\Lambda_{\overline{G}}q$ is also reflexive. We assumed that $q \in Z(Q(\overline{G}))$. Hence it follows that $q\Lambda_{\overline{G}} = \Lambda_{\overline{G}}q$. Therefore $q\Lambda_{\overline{G}}$ is a fractional left and right ideal and it is reflexive on both sides, i.e. it is a fractional c-ideal.
Observe that since $\overline{G} = H \times \overline{Z}$ and $\overline{Z} \cong \mathbb{Z}_p^n$, we are able to use our induction hypothesis. Hence if I is a proper prime c-ideal in $\Lambda_{\overline{G}}$ then $I \cap \Lambda_{\overline{Z}} \neq \emptyset$. By Proposition 3.2.6, it follows that $I = f\Lambda_{\overline{G}}$ where f is a prime element in $\Lambda_{\overline{Z}}$. Note that $\overline{G}$ is a pro-p group hence $\Lambda_{\overline{G}}$ is a maximal order. Then by the Theorem of Asano 2.1.16, the fractional c-ideals of $\Lambda_{\overline{G}}$ can be written as a product of prime c-ideals (and their inverses) of $\Lambda_{\overline{G}}$. It easily follows that q can be written as $q = \frac{f_1}{f_2}h$ where f_1, f_2 are products of prime element of $\Lambda_{\overline{Z}}$ and $h \in \Lambda_{\overline{G}}$. Our assumption that $q \in Z(Q(\overline{G}))$ and the fact that f_1, f_2 are central elements in Λ_G together imply that $h \in Z(\Lambda_{\overline{G}})$. The center $Z(\Lambda_{\overline{G}})$ is just $\mathbb{Z}_p[[z_1, \ldots, z_n]] = \Lambda_{\overline{Z}}$ where $z_j = g_j - 1$ for all $j = 1, \ldots, n$, by Corollary A in [4]. But then

$$q = \frac{f_1}{f_2}h \in Q(\Lambda_{\overline{Z}})$$

since $f_1, f_2, h \in \Lambda_{\overline{Z}}$. Hence we are done. □

Now we can finish the proof of Proposition 3.2.1. Recall the following: Lemma 3.2.3 states that if $I \cap S \neq \emptyset$ then there is a subgroup $\overline{Z}$ of Z such that $Z \cong \mathbb{Z}_p^n$ such that the Λ_G-module Λ_G/I is finitely generated over the subalgebra $\Lambda_{\overline{G}}$ where $\overline{G} = H \times \overline{Z}$. If $I \cap \Lambda_{\overline{G}} \neq 0$ then by the induction hypothesis $I \cap \Lambda_{\overline{Z}} \neq 0$. Hence we are done since $\Lambda_{\overline{Z}} \subset \Lambda_Z$. If $I \cap \Lambda_{\overline{G}} = 0$ then we need some extra argument.

Proposition 3.2.15. Let us assume that $I \cap \Lambda_{\overline{G}} = 0$. Then $\Lambda_Z \cap I \neq 0$

Proof. Choose and fix a minimal set $\{g_1, \ldots, g_{n+1}\}$ of topological generators ofr Z. Recall that from the proof of Lemma 3.2.3 that there is an index $i_0 \in \{1, \ldots, n+1\}$ such that $\overline{Z}$ is generated by $g_1, \ldots, g_{i_0-1}, g_{i_0+1}, \ldots, g_{n+1}$. Therefore, putting $z_j = g_j - 1$ for all $j = 1, \ldots, n+1$ as usually, it follows that $\Lambda_G \cong \Lambda_{\overline{G}}[[z_{i_0}]]$. Consider the increasing chain of finitely generated $\Lambda_{\overline{G}}$-modules

$$\Lambda_{\overline{G}} = A_0 \subset A_1 \subset A_2 \ldots \tag{11}$$

where $A_i = \bigoplus_{k=0}^{i} \Lambda_{\overline{G}} z_{i_0}^k$. The $\Lambda_{\overline{G}}$-module Λ_G/I is finitely generated by our assumption. Therefore it is Noetherian as a $\Lambda_{\overline{G}}$-module. Hence the image of the chain $A_0 \subset \ldots$ in Λ_G/I must stabilize by the Noetherian property. So $I \cap A_n \neq 0$

for some n. Let us consider the minimal such n. Note that $I \cap \Lambda_{\overline{G}} = \emptyset$, so we have a non-zero polynomial

$$a = a_n z_{i_0}^n + \cdots + a_0 \in I. \tag{12}$$

By minimality of n, a_n is non-constant. The algebra $\Lambda_{\overline{G}}$ is a domain. Hence it has a skewfield of fractions $Q(\overline{G})$ by Goldie's Theorem (Theorem 5.4 in [28]). $Q(\overline{G})$ is the localization of $\Lambda_{\overline{G}}$ at the two-sided Ore set $T = \Lambda_{\overline{G}} \setminus \{0\}$.

Lemma 3.2.16. The multiplicatively closed set T is a left and right Ore set in $\Lambda_{\overline{G}}[z_{i_0}]$.

Proof. The set T has the left and right Ore condition in $\Lambda_{\overline{G}}$. Consider arbitrary elements $f = \sum_{j=0}^{k} b_j z_{i_0}^j \in \Lambda_{\overline{G}}[z_{i_0}]$ and $t \in T$. We only prove that T has the right Ore condition in $\Lambda_{\overline{G}}[z_{i_0}]$, i.e. that there exist elements $g \in \Lambda_{\overline{G}}[z_{i_0}]$ and $t' \in T$ such that

$$ft' = tg \tag{13}$$

One proves the left Ore condition analogously, using that T has the left Ore condition in $\Lambda_{\overline{G}}$.
By Lemma 2.1.18 in [28] there exist elements $c_0, c_1, \ldots, c_k \in \Lambda_{\overline{G}}$ and $t' \in S$ such that $b_0 t' = tc_0, b_1 t' = tc_1, \ldots, b_k t' = tc_k$. Considering the elemens $g = c_0 + c_1 z_{i_0} + \ldots c_k z_{i_0}^k$ and $t' \in S$, one checks easily that they satisfy (13). $\square$

By the previous lemma, we can localize $\Lambda_{\overline{G}}[z_{i_0}]$ at T. The localized ring will be the polynomial ring $Q(\overline{G})[z_{i_0}]$. Denote by $(I \cap \Lambda_{\overline{G}}[z_{i_0}])_T$ is the localization of the non-zero two-sided ideal $I \cap \Lambda_{\overline{G}}[z_{i_0}] \lhd \Lambda_{\overline{G}}[z_{i_0}]$ at T. It is a two-sided ideal in in the localized ring, i.e. $Q(H)[z_{i_0}]$, by Proposition 2.1.16 in [28]. Therefore if we multiply the polynomial a in (12) with a_n from the left, we see that $a_n^{-1} a \in (I \cap \Lambda_{\overline{G}}[z_{i_0}])_T$. Consider an element $u \in Q(\overline{G})$ and look at the commutator $[u, a_n^{-1} a]$. It has strictly smaller degree than n and it is still in the ideal $(I \cap \Lambda_{\overline{G}}[z_{i_0}])_T$. So with clearing the common denominator we get an element which is in $I \cap A_{n-1}$. It must be zero by minimality of n. But it means that

$$a_n^{-1} a_i \in \ Z(Q(\overline{G})) \text{ for all } i < n \tag{14}$$

Now consider an arbitrary element q from the center of $Q(\overline{G})$, i.e. $q \in Z(Q(\overline{G}))$. Observe that since $\overline{G} = H \times \overline{Z} \cong H \times \mathbb{Z}_p^n$ hence we can use Proposition 3.2.14. Hence $a_n^{-1} a_i \in Z(Q(\overline{Z}))$ for all $i < n$. It means that there are elements $f_{1,i}, f_{2,i} \in Q(\overline{Z})$ for all $i = 1, \ldots, n-1$ such that $a_n^{-1} a_i = \frac{f_{1,i}}{f_{2,i}}$. Therefore clearing the common denominator it follows that $f_{2,1} \ldots f_{2,n-1} a_n^{-1} a \in \Lambda_{\overline{Z}}[[z_{i_0}]] = \Lambda_Z$. Λ_Z is central so $f_{2,1} \cdots f_{2,n-1} a = a_n (f_{2,1} \cdots f_{2,n-1} a_n^{-1} a) \in I$ and moreover

$$a_n \Lambda_G (f_{2,1} \ldots f_{2,n-1} a_n^{-1} a) = (a_n f_{2,1} \ldots f_{2,n-1} a_n^{-1} a) \Lambda_G \subset I$$

By our assumption that I is a prime ideal, a_n or $f_{2,1} \dots f_{2,n-1} a_n^{-1} a$ is in I. But $\Lambda_{\overline{G}} \cap I = 0$ hence a_n is not in I. So $(f_{2,1} \dots f_{2,n-1} a_n^{-1} a) \in I$ but $(f_{2,1} \dots f_{2,n-1} a_n^{-1} a) \in \Lambda_Z$. Therefore $I \cap \Lambda_Z \neq \emptyset$. □

That completes the proof of Proposition 3.2.1.

We would like to emphasize an important consequence of Proposition 3.2.6 and Proposition 3.2.1.

Corollary 3.2.17. The prime c-ideals of Λ_G are the ideals $f\Lambda_G$ where $f \in \Lambda_Z$ and f is a prime element of Λ_Z.

Now we are ready to prove Theorem 3.1.2.

3.2.2 Proof of Theorem 3.1.2

Proof. First by Proposition 2.7.18 Λ_G is a maximal order. By Proposition 4.1.1. in [16] and the fact that M is Λ_G-torsion, $q(M) = M_0 \oplus M_1$ where M_0 is a completely faithful object and M_1 is a locally bounded object.
Let us suppose that $q(M)$ is not completely faithful, i.e. M_1 is non-zero object in the quotient category. Now, M_1 is a subobject of $q(M)$, so we can find a non-zero submodule T of M such that $q(T) \cong M_1$ by the properties of quotient categories. Since Λ_G is Noetherian, T is finitely generated. Let us denote the maximal pseudo-null submodule of M and T by M_o and T_o, respectively. T_o is a submodule of $M_o = 0$. Then by Lemma 2.5 in [34], $\mathrm{ann}_{\Lambda_G}(T) = \mathrm{ann}(q(T))$. M_1 is locally bounded, so T is a Λ_G-torsion bounded object in $\mathrm{mod}(\Lambda_G)$. Therefore, by Lemma 4.3 (i) in [16] $\mathrm{ann}_{\Lambda_G}(T)$ is a non-zero prime c-ideal. Hence, by Proposition 3.2.1, Proposition 3.2.6 and Theorem 2.1.16 there is a non-zero element $x = f_1 \dots f_k f_{k+1}^{-1} \dots f_n^{-1} \in Q(Z)$ contained in the ideal $\mathrm{ann}_{\Lambda_G}(T)$. Clearing the denominator of x, we get an element $y \in \Lambda_Z$ such that $y \in \mathrm{ann}_{\Lambda_G}(T)$, which means that T is a non-zero Λ_Z-torsion submodule of M.
Denote by N the Λ_Z-torsion submodule of M. Let us suppose that $N \neq 0$. Since Λ_Z is central, N is a Λ_G-submodule of M. Hence, $q(N)$ is a subobject of $q(M)$ since M has no non-zero pseudo-null submodules. But then $\mathrm{ann}(q(N)) \neq 0$, hence $q(M)$ cannot be completely faithful. □

4 K_0-invariance of completely faithful objects

4.1 The statement

Let p be a prime number such that $p \geq 5$. Let H be a torsion-free compact p-adic analytic group whose Lie algebra $\mathcal{L}(H)$ is split semisimple over $\mathbb{Q}_p$. Let $G = H \times Z$

where $Z \cong \mathbb{Z}_p^n$ for some $n \in \mathbb{N}_0$. We will denote by $\mathfrak{N}_H(G)$ the abelian category of all finitely generated Λ_G-modules that are finitely generated as Λ_H-modules. In this section, we aim to prove the following result:

Theorem 4.1.1. *Let p be a prime number such that $p \geq 5$. Let H be a torsion-free compact p-adic analytic group whose Lie algebra $\mathcal{L}(H)$ is split semisimple over $\mathbb{Q}_p$ and let $G = H \times Z$ where $Z \cong \mathbb{Z}_p^n$ for some non-negative integer n. Let M, $N \in \mathfrak{N}_H(G)$ such that they have no non-zero pseudo-null Λ_G-submodules and let $q(M)$ be completely faithful. If $[M] = [N]$ in $K_0(\mathfrak{N}_H(G))$ then $q(N)$ is also completely faithful.*

Before presenting the proof, we need to make an observation about the objects of the category $\mathfrak{N}_H(G)$.

Proposition 4.1.2. *Let us assume that $M \in \mathfrak{N}_H(G)$. Then*

(*i*) *M is a Λ_G-torsion module.*

(*ii*) *The following are equivalent:*

   (a) *M has no non-zero pseudo-null Λ_G-submodules.*

   (b) *M is Λ_H torsion-free.*

Proof. (*i*): Proposition 3.1 in [45] states that whenever L is a Λ_G-module such that $L \in \mathfrak{N}_H(G)$ then $\mathrm{Hom}_{\Lambda_G}(L, \Lambda_G) = 0$. The algebras Λ_G and Λ_H are Noetherian and L is finitely generated over both Λ_G and Λ_H. Hence it follows that L is Noetherian as a Λ_G- and also as a Λ_H-module. It means that any Λ_G-submodule $L' \subseteq L$ is also finitely generated as a Λ_G- and also as a Λ_H-module, i.e. $L' \in \mathfrak{N}_H(G)$. Applying Proposition 3.1 in [45] again to L', it follows that $\mathrm{Hom}_{\Lambda_G}(L', \Lambda_G) = 0$. Now we apply this argument for $L = M$. By Proposition 2.1.6, M is a Λ_G-torsion module.
(*ii*): This is Proposition 5.4 in [48] applied to our situation. □

Remark 4.1.3. One example for a group of the form in the statement of Theorem 4.1.1 is the following: Consider Γ_1 which is the first inertia subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ i.e.

$$\Gamma_1 = \{\gamma \in \mathrm{GL}_n(\mathbb{Z}_p) | \gamma \equiv 1 \mod (p)\}$$

In this case $G = Z \times H$ where $Z \cong \mathbb{Z}_p$ is the centre of G and H is an open subgroup of $\mathrm{SL}_n(\mathbb{Z}_p)$ that is normal in G.

4.2 The proof of the statement

Proof. By the assumption that $M, N \in \mathfrak{N}_H(G)$, Proposition 4.1.2 (*i*) implies that both M and N are Λ_G-torsion modules. This property and the second assumption, namely that neither M nor N has no non-zero pseudo-null Λ_G-submodules, together assure us that we are in the situation of Theorem 3.1.2. Hence it is enough to prove that N is Λ_Z torsion-free. Note that by Proposition 2.10.2, it suffices to show that $\mathrm{ann}_{\Lambda_Z}(N') = 0$ for all non-zero N' Λ_Z-submodule of N.

Lemma 4.2.1. It is enough to show that $\mathrm{ann}_{\Lambda_Z}(N') = 0$ for all non-zero Λ_G-submodules $N' \subseteq N$.

Proof. Let us assume that $\mathrm{ann}_{\Lambda_Z}(N') = 0$ for all non-zero Λ_G-submodules $N' \subseteq N$ and that there is a non-zero Λ_Z-submodule $N' \subseteq N$. Choose a generating set $\{n'_1, \dots\}$ of N' as a Λ_Z-module. Consider the module $\overline{N'}$ generated by the same set of elements $\{n'_1, \dots\}$ as a Λ_G-module. The subalgebra Λ_Z is central in Λ_G. Therefore if there is a non-zero element $\lambda \in \Lambda_Z$ such that $\lambda \in \mathrm{ann}_{\Lambda_Z}(N')$ then it still annihilates all the elements of $\overline{N'}$ because it annihilates all the generators. Hence there is a Λ_G-submodule of M such that $\mathrm{ann}_{\Lambda_Z}(\overline{N'}) \neq 0$ which is a contradiction. □

So let us suppose that there is a non-zero Λ_G-submodule $N' \subseteq N$ such that $\mathrm{ann}_{\Lambda_Z}(N') \neq 0$. Let $P \in \mathrm{Supp}_{\Lambda_Z}(N')$ be an arbitrary prime ideal of Λ_Z from the support of N' as a Λ_Z-module. Then P contains $\mathrm{ann}_{\Lambda_Z}(N')$. So if $\mathrm{Supp}_{\Lambda_Z}(N')$ was $\mathrm{Spec}(\Lambda_Z)$ then, by the fact that the nilradical of Λ_Z is zero, it would follow that $\mathrm{ann}_{\Lambda_Z}(N')$ is zero. Hence our assumption on N' means that there is a $P \in \mathrm{Spec}(\Lambda_Z)$ prime ideal such that $N'_P = 0$. By Proposition 2.7.11 (*i*) and Proposition 2.7.12, the algebra Λ_H is semiprime and Noetherian. Hence by Theorem 2.1.15 in [28], it has finite uniform dimension. Using Proposition 2.7.11 (*iv*), we see that the ideal (0) is prime, so we can localize Λ_H at the (0) ideal. Thus Theorem 2.3.6 in [28], which is due to Goldie, implies that after localization we get a skewfield which we will denote by $Q(H)$. Now recall that by Lemma 2.5.9 we have short exact sequences

$$\begin{array}{c} 0 \longrightarrow C \longrightarrow K \longrightarrow D \longrightarrow 0 \\ 0 \longrightarrow C \longrightarrow L \longrightarrow D \longrightarrow 0 \end{array} \tag{15}$$

such that all modules in the short exact sequences are objects of the category $\mathfrak{N}_H(G)$ and

$$M \oplus K = N \oplus L. \tag{16}$$

If T is an arbitrary Λ_G-module such that $T \in \mathfrak{N}_H(G)$, then after localization at (0) we get a finite dimensional vector space $Q(T)$ over $Q(H)$. It is well-known that localization is exact and commutes with finite direct sums. Hence after localizing Λ_H at the prime ideal (0), we still have the localized versions of the exact sequences in (15) and the equation in (16) but this time with finite dimensional $Q(H)$-vector spaces, i.e.

$$\begin{aligned} 0 \longrightarrow Q(C) \longrightarrow Q(K) \longrightarrow Q(D) \longrightarrow 0 \\ 0 \longrightarrow Q(C) \longrightarrow Q(L) \longrightarrow Q(D) \longrightarrow 0 \end{aligned} \tag{17}$$

and

$$Q(M) \oplus Q(K) = Q(N) \oplus Q(L). \tag{18}$$

Moreover, Λ_Z is central in Λ_G which means that they naturally inherit the commuting Λ_Z-action from the non-localized modules. So now we can localize Λ_Z at the prime ideal P and get

$$\begin{aligned} 0 \longrightarrow Q(C)_P \longrightarrow Q(K)_P \longrightarrow Q(D)_P \longrightarrow 0 \\ 0 \longrightarrow Q(C)_P \longrightarrow Q(L)_P \longrightarrow Q(D)_P \longrightarrow 0 \end{aligned} \tag{19}$$

such that $Q(M)_P \oplus Q(K)_P = Q(N)_P \oplus Q(L)_P$.

Lemma 4.2.2. Let V be a finite dimensional vector space over $Q(H)$ with a commuting Λ_Z action on it and let P be an arbitrary prime ideal of Λ_Z. Then V_P is also finite dimensional over $Q(H)$ where V_P denotes the localized module of V aP. Moreover, $\dim_{Q(H)} V_p \leq \dim_{Q(H)} V$.

Proof. Let $S = (\Lambda_Z \setminus P) \subseteq \Lambda_Z$. Denote by V^{tor} the S-torsion part of V. It is a Λ_Z-submodule of V since Λ_Z is a Noetherian domain. We know that the algebra Λ_Z is central in Λ_G and the set S is multiplicatively closed. These properties enable us to prove that V^{tor} is also a $Q(H)$-subspace of V: By definition, for any two elements $v_1, v_2 \in V^{tor}$ there are elements $s_1, s_2 \in S$ such that $v_1 s_1 = v_2 s_2 = 0$. Then

$$(v_1 + v_2)s_1 s_2 = v_1 s_1 s_2 + v_2 s_1 s_2 = (v_1 s_1)s_2 + (v_2 s_2)s_1 = 0 \tag{20}$$

and for any $\lambda \in \Lambda_H$ and any $v \in V^{tor}$ such that the element $s \in S$ annihilates v, i.e. $vs = 0$ we have

$$v\lambda s = vs\lambda = 0 \tag{21}$$

Hence V^{tor} is a $Q(H)$-subspace of V. By the construction of localization, the localized Λ_Z-module is zero, i.e.

$$V_P^{tor} = 0 \tag{22}$$

Note that since both V and V^{tor} are finite dimensional, V/V^{tors} is also finite dimensional with dimension $\dim(V/V^{tor}) = \dim(V) - \dim(V^{tor})$. Let $\overline{v} \in V/V^{tor}$ be a non-zero element of the quotient. If there is an element $s \in S$ such that $\overline{v}s = 0$, then $vs \in V^{tor}$. But the later implies that there exists an element $s_1 \in S$ such that $(vs)s_1 = v(ss_1) = 0$. The set S is multiplicatively closed. Hence $ss_1 \in S$ which implies that $v \in V^{tor}$. But that cannot happen because $\overline{v}$, which is the image of v, is non-zero in the quotient. This argument shows that the quotient V/V^{tor} is S torsion-free.
We have a short exact sequence of vector spaces over $Q(H)$ with a commuting Λ_Z-action on them:

$$0 \longrightarrow V^{tor} \longrightarrow V \longrightarrow V/V^{tor} \longrightarrow 0 \tag{23}$$

After localizing this sequence at P, (22) implies that

$$V_P \cong (V/V^{tor})_P \tag{24}$$

as $Q(H)$-vector spaces with a commuting Λ_Z-action on them. The later is true because Λ_Z is central in Λ_G. We will prove that

$$(V/V^{tor})_P \cong V/V^{tor}$$

as $Q(H)$-vector spaces. We can consider the localization of a Λ_Z-module as tensoring it over Λ_Z by $(\Lambda_Z)_P$. Consider any element $\sum_{i=1}^n (\overline{v}_i \otimes_{\Lambda_Z} \frac{r_i}{s_i}) \in (V/V^{tor})_P = V/V^{tor} \otimes_{\Lambda_Z} (\Lambda_Z)_P$. Let $S_i := s_1 \cdots s_{i-1} \cdot s_{i+1} \cdots s_n$ and $s := s_1 \cdots s_n$. Then

$$\sum (\overline{v}_i \otimes_{\Lambda_Z} \frac{r_i}{s_i}) = \sum (\overline{v}_i \otimes_{\Lambda_Z} \frac{(r_i S_i)}{s}) = \sum (\overline{v}_i (r_i S_i) \otimes_{\Lambda_Z} \frac{1}{s}) =$$

$$= ((\sum \overline{v}_i r_i S_i) \otimes_{\Lambda_Z} \frac{1}{s}) = (\overline{v} \otimes_{\Lambda_Z} \frac{1}{s}) \tag{25}$$

Hence the elements of $(V/V^{tor})_P$ are of the form $(\overline{v} \otimes \frac{1}{s})$ where $\overline{v} \in V/V^{tor}$ and $s \in S$. Let us observe that (22) implies that multiplication with an arbitrary element $s \in S$ is an injective linear transformation φ_s on the finite dimensional vector space V/V^{tor}. Since V/V^{tor} is finite dimensional, it implies that φ_s is automatically an automorphism. Hence by surjectivity, every $\overline{v} \in V/V^{tor}$ can be written $\overline{v} = \overline{w}s$ for some $\overline{w} \in V/V^{tor}$. Together with (25), this implies that any element $(\overline{v} \otimes_{\Lambda_Z} \frac{1}{s}) \in (V/V^{tor})_P$ is actually of the form $(\overline{w} \otimes_{\Lambda_Z} 1)$ where $\overline{w}$ is the preimage of $\overline{v}$ with respect to the linear transformation φ_s, i.e. $\overline{v} = \varphi_s(\overline{w}) = \overline{w}s$. We proved that V/V^{tor} is S torsion-free. Therefore, the natural map $V/V^{tor} \to (V/V^{tor})_P$, $\overline{v} \mapsto \overline{v} \otimes 1$, which is $Q(H)$-linear, is injective. -by the fact that every element has the form $(\overline{w} \otimes_{\Lambda_Z} 1)$ for some $\overline{w} \in V/V^{tor}$, it is also surjective. Hence it is an isomorphism of $Q(H)$-vector spaces and V/V^{tor} is finite dimensional with dimension $\leq \dim_{Q(H)} V$. □

By Proposition 4.1.2 (*ii*), all the vector spaces in (17) and equation in (18) are finite dimensional since every module in (15) and in (16) is an object of the category $\mathfrak{N}_H(G)$. Also recall that our initial assumption on the submodule $N' \subseteq N$ was that $N'_P = 0$. From this, as a consequence of Λ_Z being central in Λ_G, we deduce that $Q(N')_P = 0$. We use the exact sequences in 17 and the equation (18) again. Note that after localization of the module N at the (0) ideal in Λ_H, the localization $Q(N')$ of the Λ_G-submodule N' becomes a non-trivial $Q(H)$-subspace of $Q(N)$. Hence by the assumption on N', after localization at P, the dimension of $Q(N)$ strictly decreases, i.e. $\dim_{Q(H)} Q(N)_P < \dim_{Q(H)} Q(N)$. The vector spaces $Q(K)_P$ and $Q(L)_P$ have the same dimension by 19. Then it follows from the equation in (19) that the dimension of $Q(M)$ must also decrease after localization at P, i.e.

$$\dim_{Q(H)} Q(M)_P < \dim_{Q(H)} Q(M) \tag{26}$$

Lemma 4.2.3. Let us suppose that an arbitrary Λ_G-module L is Λ_H-torsion free. Then L is torsion-free over Λ_Z if and only if $Q(L)$ is torsion-free over Λ_Z

Proof. Let us denote this time by S the multiplicatively closed set $\Lambda_H \setminus \{0\} = S$. Let us suppose first that $Q(L)$ is torsion-free over Λ_Z. Once more, Λ_Z is central in Λ_G. Hence if l is a non-zero Λ_Z-torsion element then all the elements l/s are Λ_Z-torsion elements of $Q(L)$. They are not zero in $Q(L)$ because L is torsion-free over Λ_H by our assumption. So we get non-zero Λ_Z-torsion elements in $Q(L)$. In fact, the Λ_Z-torsion submodule of $Q(L)$ is the localization of the Λ_Z-torsion submodule of L.
The other direction can be proved the following way: let us suppose that L is torsion-free over Λ_Z and assume indirectly that there is a Λ_Z-torsion part of $Q(L)$. It means that there exists at least one non-zero element $\frac{l}{s}Q(L)$ and an element $z \in \Lambda_Z$ such that $\frac{l}{s}z = 0$ in $Q(L)$. By the construction of localization, there are elements $s_1, s_2 \in S$ such that $(lzs_1 - 0s)s_2 = lzs_1s_2 = 0$ in L. Hence $ls_1s_2z = 0$ because Λ_Z is central in Λ_G. But L is torsion-free over Λ_H hence z annihilates the element $(ls_1s_2) \in L$. But that cannot be since L is torsion-free over Λ_Z by our assumption. □

Now we are ready to finish the proof of Theorem 4.1.1. Recall that by Theorem 3.1.1 we see that M is Λ_Z-torsion free. Hence by our initial assumption on M in the statement of Theorem 3.1.2 and by Lemma 4.2.3, $Q(M)$ has the same property. The natural map

$$\varphi : Q(M) \hookrightarrow Q(M)_P$$

is therefore injective since the kernel of this map consists of Λ_Z-torsion elements in $Q(M)$. Recall that we have the inequality 26, i.e. $\dim_{Q(H)} Q(M)_P < \dim_{Q(H)} Q(M)$. But that cannot happen by the injectivity of φ. $\square$

5 The Grothendieck group of algebras of continuous and locally analytic distributions

In this chapter, we switch from right modules to left modules. If we say module we always mean a left module. The reason for it is that the authors in [38] use left modules. Also various structures with groups and rings will emerge throughout the chapter, e.g. group rings, skew group rings and their notations suggest that we should use left modules.

5.1 The Grothendieck group of $k[G/H]$

Recall that G is an arbitrary compact p-adic analytic group with no element of order p. We choose an open uniform pro-p normal subgroup H of G. So the quotient group G/H is finite with $n := |G/H|$. Let K be a finite extension of $\mathbb{Q}_p$. Hence $(K, \mathcal{O}_K, k)$ is a p-modular system.

Assumption: From now on, we *always* assume that K is sufficiently large for the group G/H (in the sense of Section 2.9).

We need to compute the Grothendieck group of $k[G/H]$ in order to get results for the Grothendieck group of $D(G, K)$. Recall that we defined p-regular conjugacy classes of a finite group G to be those conjugacy classes that have order relative prime to p.

Lemma 5.1.1. *The Grothendieck group of $k[G/H]$ is isomorphic to $\mathbb{Z}^c$, where c is the number of p-regular conjugacy classes of G/H.*

Proof. By Theorem 2.9.5, k is a splitting field for G/H. Hence Lemma 2.9.7 implies the statement. □

5.2 The Grothendieck group of the algebra of continuous distrubutions

Throughout this chapter, we assume that G is a compact p-adic analytic group such that it has no element of order p. Recall that by the localization theorem (Theorem 2.5.12), we have the following exact sequence:

$$K_0(\pi\text{-tors}) \xrightarrow{\varphi} G_0(\mathcal{O}_K[[G]]) \xrightarrow{\xi} G_0(K[[G]]) \longrightarrow 0 \qquad (27)$$

where O_K is the ring of integers in K. In this section we prove that the map ξ in (27) is injective. We will call an arbitrary finitely generated $\mathcal{O}[[G]]$-module M **strict π-torsion module** if $M\pi = 0$.

Lemma 5.2.1. Let M be a strict π-torsion $\mathcal{O}[[G]]$-module. The image of the class $[M]$ of M in $G_0(\mathcal{O}_K[[[G]])$ with respect to φ is zero.

Proof. The image of a π-torsion module is itself since the map φ is induced by the natural inclusion π-tors $\subset$ mod-$\mathcal{O}_K[[G]]$. First, we investigate the case when M has global dimension 0 as a $k[[G]] = \mathcal{O}_K[[G]]/\mathcal{O}_K[[G]]\pi$-module. In this case $M = P$ is a finitely generated projective $k[[G]]$-module. Then there exists a finitely generated projective $k[[G]]$-module Q such that $P \oplus Q \cong k[[G]]^l$ for some natural number l. Recall that the ring $\mathcal{O}_K[[G]]$ is π-adically complete. Therefore, by the property that idempotents can be lifted via the ideal generated by π (see Proposition 1.5.7 in [37]), there exists a projective $\mathcal{O}_K[[G]]$-module $\overline{P}$ such that $\overline{P}/\overline{P}\pi \cong P$ as $k[[G]]$-modules. We have an exact sequence of $\mathcal{O}_K[[G]]$-modules

$$0 \longrightarrow \overline{P} \xrightarrow{\cdot\pi} \overline{P} \longrightarrow \overline{P}/\overline{P}\pi \longrightarrow 0$$

The cokernel $\overline{P}/\overline{P}\pi$ and P are isomorphic mod π. But it means that their classes are the same in $G_0(\mathcal{O}_K[[G]])$, i.e. $[\overline{P}/\overline{P}\pi] = [P]$. But the class $[\overline{P}/\overline{P}\pi]$ equals the element $[\overline{P}] - [\overline{P}] = 0$. So we are done with the case when P has projective dimension 0 as a $k[[G]]$-module.
Recall that $k[[G]]$ has finite global dimension whenever G has no element of order p. That was our initial assumption on G. It means that any $k[[G]]$-module M has a finite projective resolution with projective modules P_i. By the definition of the Grothendieck group, $[M] = \sum(-1)^{i+1}[P_i]$. Hence using what we proved above, it follows that the image of $[M]$ is 0. □

Lemma 5.2.2. Let us assume that M is an arbitrary π-torsion $\mathcal{O}_K[[G]]$-module. The image of its class in $G_0(\mathcal{O}_K[[G]])$ is zero.

Proof. We can identify the group $K_0(\pi\text{-tors})$ in (27) with $G_0(k[[G]])$. The reason for that is the following: for any finitely generated π-torsion module M, there exist a positive integer n, such that $M\pi^n = 0$. So there is a filtration $M \supset M\pi \supset M\pi^2 \supset \cdots \supset M\pi^{n-1} \supset 0$ such that all the quotients $M\pi^i/M\pi^{i+1}$ are naturaly $k[[G]]$-modules. Hence the class $[M]$ is equal to the element $\sum[M\pi^i/M\pi^{i+1}]$ in $G_0(\mathcal{O}_K[[G]])$. Note that all the modules $M\pi^i/M\pi^{i+1}$ are finitely generated strict π-torsion modules. Hence, by Lemma 5.2.1, the images of the classes of these modules in $G_0(\mathcal{O}_K[[G]])$ are zero which implies our statement. □

Proposition 5.2.3. The Grothendieck group of $\mathcal{O}_K[[G]]$ is isomorphic to $\mathbb{Z}^c$.

Proof. Choose an open normal uniform pro-p subgroup H of G. The fact that $K_0(k[[G]]) \cong \mathbb{Z}^c$ then follows from that $\mathcal{O}_K[[G]]$ is complete with respect to the ideal $m\mathcal{O}_K[[G]] + I(H)$ which is the kernel of the projection $\mathcal{O}_K[[G]] \to k[G/H]$, by Proposition 3.3 (*b*) in [8]. Hence by Proposition 2.5.6, $K_0(\mathcal{O}_K[[G]]) \cong K_0(k[G/H])$. Now we use Lemma 5.1.1. □

Now comes the main theorem of this section:

Theorem 5.2.4. Let G be a compact p-adic analytic group and assume in addition that it has no element of order p. Then $K_0(K[[G]]) \cong \mathbb{Z}^c$.

Proof. By Proposition 5.2.3, $K_0(\mathcal{O}_K[[G]]) \cong \mathbb{Z}^c$. By Lemma 5.2.2, the homomorphism $\xi : G_0(\mathcal{O}_K[[G]]) \to G_0(K[[G]])$ is an isomorphism. The algebra $K[[G]]$ is just the localization of $\mathcal{O}_K[[G]]$ at the uniformizer element π and hence its global dimension of bounded above by the global dimension of $\mathcal{O}_K[[G]]$. By our assumption, the global dimension of $\mathcal{O}_K[[G]]$ is finite. Then it follows from Theorem 2.5.16 that $G_0(\mathcal{O}_K[[G]]) \cong K_0(\mathcal{O}_K[[G]])$ and $G_0(K[[G]]) \cong K_0(K[[G]])$. It means that ξ induces an isomorphism between $K_0(\mathcal{O}_K[[G]])$ and $K_0(K[[G]])$. $\square$

5.3 Algebras of p-adic distributions

5.3.1 Distribution algebras over compact p-adic analytic groups

Recall that G is a compact p-adic analytic group. We choose an open normal uniform pro-p subgroup H of G. It follows that G/H is a finite group of exponent n. We also have a p-regular system $(K, \mathcal{O}_K, k)$. By Proposition 2.1 in [35], H has a p-valuation with the property that for any set of (ordered) minimal topological generators $\{h_1, \dots, h_d\}$ of H

$$\omega(h_1) = \omega(h_2) = \cdots = \omega(h_d) = 1.$$

Choose and fix a set of representatives $X := \{g_1, \dots, g_n\}$ of the cosets of G/H. By the definition of the crossed product and the properties of the Dirac delta distributions, it is easy to check that the algebra $D(G, K)$ is the crossed product of $D(H, K)$ and the group G/H with the mapping $g_i \mapsto \delta_{g_i}$, where δ_i are the dirac delta distributions (we remark at this point that if it does not lead to any confusion, we will still denote the image δ_g of an arbitrary group element $g \in G$ by the group element itself). Hence by definition, it means that every element $\mu \in D(G, K)$ can be written as $\mu = \sum_{g_i} \lambda_i g_i$. In [38], Section 5, the authors define a function on $D(G, K)$:

$$q_r(\mu) := \max_i(||\lambda_1||_r, \dots, ||\lambda_n||_r)$$

and they also show the following facts: q_r is a continuous norm on $D(G, K)$ and it is the extension of the norm $||\ ||_r$ on $D(H, K)$. The multiplication in $D(G, K)$ is continuous with respect to q_r. The completion $D_r(G, K)$ of $D(G, K)$ with respect to q_r contains $D_r(H, K)$ and $D_r(G, K)$ is the crossed product of $D_r(H, K)$ and G/H with the mapping $g_i \mapsto \delta_{g_i}$.

Proposition 5.3.1. Let us assume that $r \in p^{\mathbb{Q}}$ and $1/p < r < 1$. Then the norm q_r on $D(G,K)$ is submultiplicative.

Proof. Let $\mu_1 = \sum_{i=1}^{n} \lambda_i(\mu_1) g_i$ and $\mu_2 = \sum_{j=1}^{n} \lambda_j(\mu_2) g_j$ be two arbitrary elements of $D(G,K)$. Let $y := \mu_1\mu_2$. Then

$$y = \sum_{i,j} \lambda_i(\mu_1) g_i \lambda_j(\mu_2) g_j = \sum_{i,j} \lambda_i(\mu_1)(g_i \lambda_j(\mu_2) g_i^{-1}) g_i g_j$$

The product $g_i g_j$ is in the coset of some coset representative $g_k \in X$ and hence there is an element $h_{i,j} \in H$ such that $g_i g_j = h_{i,j} g_k$. Then

$$y = \sum_k (\sum_{g_i g_j \in H g_k} \lambda_i(\mu_1)(g_i \lambda_j(\mu_2) g_i^{-1}) h_{i,j}) g_k$$

By definition, the norm of y is equal to the maximum of the norms of the coefficients. The coefficient of g_k is

$$\theta_k := \sum_{g_i g_j \in H g_k} \lambda_i(\mu_1)(g_i \lambda_j(\mu_2) g_i^{-1}) h_{i,j}.$$

Using the ultrametric property of the norm, we get that

$$||\theta_k||_r \leq \max_{g_i g_j \in H g_k} (||\lambda_i(\mu_1)(g_i \lambda_j(\mu_2) g_i^{-1}) h_{i,j}||_r).$$

But we know that $||g_i \lambda_j(\mu_2) g_i^{-1}||_r = ||\lambda_j(\mu_2)||_r$ for all $i, j = 1, \ldots, n$. Moreover, by Theorem 4.5(i), in [38] the norm is multiplicative on $D(H,K)$ and for any $h \in H$, $||h-1||_r < 1$. Hence $||h||_r = ||(h-1)+1||_r = \max\{1, ||h-1||_r\} = 1$. Therefore,

$$q_r(\mu_1\mu_2) = q_r(y) \leq \max_{i,j} ||\lambda_i(\mu_1)||_r ||\lambda_j(\mu_2)||_r) = q_r(\mu_1) q_r(\mu_2)$$

□

Corollary 5.3.2. The norm q_r is submultiplicative on $D_r(G,K)$.

Proof. Since q_r continuously extends to $D_r(G,K)$ from $D(G,K)$, it follows from Proposition 5.3.1. □

We will use the same notation for the norm induced by r on $D(H,K)$ and its extension onto $D(G,K)$ (i.e. we drop the notation $q_r(\ )$).

Definition 5.3.3. We define the following abelian subgroups of $D_r(G,K)$:

$$F_r^s D_r(G,K) := \{\mu \in D_r(G,K) \ : \ ||\mu||_r \leq p^{-s}\}$$

$$F_r^{s+} D_r(G,K) := \{\mu \in D_r(G,K) \ : \ ||\mu||_r < p^{-s}\}$$

This is analogous to the uniform case (see (3)). By Proposition 5.3.2, these subgroups form a filtration on $D_r(G,K)$, with associated graded ring

$$\mathrm{gr}^{\cdot} D_r(G,K) := \bigoplus \mathrm{gr}^n D_r(G,K)$$

$$\text{where } \mathrm{gr}^n D_r(G,K) := F_r^n D_r(G,K)/F_r^{n+} D_r(G,K).$$

Obviously, this filtration is the extension of the filtration on $D_r(H,K)$ in (3). Moreover, by the definition of $|| \, ||_r$,

$$F_r^s D_r(G,K) = \bigoplus_i F_r^s D_r(H,K) g_i$$

$$F_r^{s+} D_r(G,K) = \bigoplus_i F_r^{s+} D_r(H,K) g_i.$$

Let us fix a minimal (ordered) topological generating set $h_1, \dots, h_d$ for H. Let $r \in p^{\mathbb{Q}}$ such that $1/p < r < 1$. We describe the left action of the images of the coset representatives from X (the dirac delta distributions) in $D_r(G,K)$ on the subalgebra $D_r(H,K) \subseteq D_r(G,K)$. This action is trivial on any $c \in K$. For an arbitrary element $\lambda = \sum_\alpha d_\alpha b^\alpha \in D_r(H,K)$ and arbitrary coset representative $g_k \in X$,

$$g_k \lambda = (g_k \lambda g_k^{-1} g_k) = \sum_\alpha d_\alpha (g_k b^\alpha g_k^{-1}) g_k.$$

Since

$$g_k b^\alpha g_k^{-1} = g_k b_1^{\alpha_1} g_k^{-1} g_k b_2^{\alpha_2} g_k^{-1} \dots g_k b_d^{\alpha_d} g_k^{-1} = g_k b_1 g_k^{-1} g_k b_1 g_k^{-1} \dots g_k b_d g_k^{-1},$$

the left action of the elements g_k is determined by the conjugation of the topological generators h_j of the open normal subgroup H by the coset representatives. Since H is a normal subgroup of G, $g_k b_i g_k^{-1} \in D_r(H,K)$ and hence $g_k \lambda g_k^{-1} \in D_r(H,K)$. For a fixed k, the map induced by the conjugation by g_k is a ring endomorphism of $D_r(H,K)$. We will denote it by $\phi_{g_k} : D_r(H,K) \to D_r(H,K)$. Moreover, it is a ring automorphism since the endomomorphism $\phi_{g_k^{-1}}$ is clearly the inverse of ϕ_{g_k}. Hence, by the definition of the skew group ring (see Definition 2.7.1), it is clear that $D_r(G,K)$ is almost a skew group ring of $D_r(H,K)$ and G/H such that $(\lambda g_i)(\mu g_j) = \lambda \phi_{g_i}(\mu) g_i g_j$, where $\lambda, \mu \in D_r(H,K)$ and $g_i, g_j \in X$. The only thing missing is that map

$$G/H \to D_r(G,K), g_i \mapsto \delta_{g_i}$$

doesn't always respect to group structure of the quotient group G/H. The problem is that $g_i g_j$ is not necessarily an element of X. We know that $g_i g_j = h_{ij} g_k$ for some $h_{ij} \in H$ and a coset representative $g_k \in X$. Of course, in G/H they are the same elements, but it is not necessarily true that $\delta_{g_ig_j} = \delta_{h_{ij}g_k}$. However, we show that if we pass to the associated graded ring of $D_r(G,K)$, it is no longer a problem, meaning that $\sigma(\delta_{g_ig_j}) = \sigma(\delta_{h_{ij}g_k})$, where σ denotes the principal symbol, defined in 2.3.18. Hence we get a skew group ring $\mathrm{gr}^{\cdot} D_r(G,K) = \mathrm{gr}^{\cdot} D_r(H,K)\#G/H$ with the left action of the images of the principal symbols $\sigma(\delta_{g_i})$ (which we still denote by the group element g_i, if it does not cause any confusion): Certainly, since $D_r(G,K)$ is a free $D_r(H,K)$-module, $g_1,\dots,g_n$ being the free generating set, $\mathrm{gr}^{\cdot} D_r(G,K)$ will be a free $\mathrm{gr}^{\cdot} D_r(H,K)$-module, $\sigma(g_1),\dots,\sigma(g_n)$ being the free generators. Moreover, the multiplication given by

$$(\sigma(\lambda)\sigma(g_i))(\sigma(\mu)\sigma(g_j)) = \sigma(\lambda)(\sigma(g_i)^{-1}\sigma(\mu)\sigma(g_i))\sigma(g_i)\sigma(g_j) \tag{28}$$

where $\lambda,\mu \in D_r(H,K)$ and $g_i, g_j \in X$ are arbitrary.

Lemma 5.3.4. Let G be a compact p-adic analytic group and r a parameter such that $r \in p^{\mathbb{Q}}$ and $1/p < r < 1$. Then $\mathrm{gr}^{\cdot} D_r(G,K)$ is isomorphic to the skew group ring $\mathrm{gr}^{\cdot} D_r(G,K)\#G/H$, with multiplication defined in (28).

Proof. The only thing we need to check is that the elements of a fixed coset Hg_k, where g_k is an element of the fixed set of representatives, are mapped to the same element in $\mathrm{gr}^{\cdot} D_r(G,K)$. We have

$$||hg_k - g_k||_r = ||(h-1)g_k||_r \leq ||h-1||_r||g_k||_r.$$

It is well-known that for an arbitrary element $h \in H$ the norm $||h-1||_r < 1$. Clearly $||g_k||_r = 1$. So $|||hg_k - g_k||_r < 1$ which shows that in the associated graded ring all the elements in one particular coset are mapped to the same element. □

Corollary 5.3.5. With the notations $\epsilon_0 := \sigma(\pi)$ (the uniformizer element of $\mathcal{O}_K$) and $x_i := \sigma(b_i)$, the associated graded ring of $D_r(G,K)$ with respect to the filtration defined in (5.3.3) is isomorphic to

$$k[\epsilon_0,\epsilon_0^{-1}][x_1,\dots,x_d]\#G/H.$$

Proof. Recall that $\mathrm{gr}^{\cdot} K \cong k[\epsilon_0,\epsilon_0^{-1}]$. Therefore, using the previous lemma, the statement follows from Theorem 2.8.18. □

Recall that in [38], for uniform pro-p groups, the authors define $D_{<r}(H,K)$, which is given by all series

$$\sum_{\alpha} d_\alpha b^\alpha \text{ with } d_\alpha \in K \text{ and such that } \{|d_\alpha| r^\alpha\} \text{ is bounded.}$$

For an arbitrary $r \in p^{\mathbb{Q}}$ such that $1/p < r < 1$, we know that $D_{<r}(H,K) \subseteq D_{1/p}(H,K)$. We define the algebra $D_{<r}(G,K)$ (inside of $D_{1/p}(G,K)$) to be the crossed product of $D_{<r}(H,K)$ and the group G/H, with the map of sets

$$G/H \to D_{<r}(G,K),\ g_i \mapsto \delta_{g_i}.$$

Hence the elements of $D_{<r}(G,K)$ are of the form $\mu = \sum \lambda_i g_i$ such that $\lambda_i \in D_{<r}(H,K)$. On $D_{<r}(G,K)$ the norm continues to be given by

$$||\mu||_r := \max_i(||\lambda_1||_r, \ldots, ||\lambda_n||_r).$$

Analogously to the uniform case, if $1/p < r < 1$, then $D_{<r}(G,K) \subseteq D_{1/p}(G,K)$ and $D_{<r}(G,K)$ is multiplicatively closed in $D_{1/p}(G,K)$ since $g_i\lambda = (g_i\lambda g_i^{-1})g_i$ and $g_i\lambda g_i^{-1}$ is certainly in $D_{<r}(H,K)$, where $g_i \in X$ and $\lambda \in D_{<r}(H,K)$. Moreover, $D_{<r}(G,K)$ is still a K-Banach space since it is a finitely generated free module over a Noetherian K-Banach algebra $D_{<r}(H,K)$, equipped with the maximum norm. Hence $D_{<r}(G,K)$ is a K-Banach algebra for all $r \in p^{\mathbb{Q}}$ such that $1/p < r < 1$. The norm is still submultiplicative on $D_{<r}(G,K)$, the proof is the very same as of Proposition 5.3.1. Hence, $||\ ||_r$ on $D_{<r}(G,K)$ induces a filtration

$$F_r^s D_{<r}(G,K) := \{\mu \in D_{<r}(G,K) :\ ||\mu||_r \leq p^{-s}\} \tag{29}$$

$$F_r^{s+} D_{<r}(G,K) := \{\mu \in D_{<r}(G,K) :\ ||\mu||_r < p^{-s}\}$$

for which $D_{<r}(G,K)$ is complete, since it is a K-Banach algebra with respect the norm $||\ ||_r$. For a fixed parameter r, we will often use the following assumption:

K has absolute ramification index e with the property that

$$r = p^{-m/e} \text{ for an appropriate } m \in \mathbb{N}. \tag{E}$$

Remark 5.3.6.

(a) Before we proceed, we need to justify that we can use the techniques that we introduced in Sections 2.3 and 2.4. Let us assume that G is a uniform pro-p group for a moment. If $r = p^{a/b} \in p^{\mathbb{Q}},\ 1/p \leq r < 1$ is fixed, then in [38] the authors state, that the filtration on $D_r(G,K)$ is quasi-integral, meaning that

$$\{s \in \mathbb{R}\ :\ \mathrm{gr}^s D_r(G,K) \neq 0\} \subseteq 1/n_0\mathbb{Z}$$

for some positive real number n_0. We try to explain what this exactly means and how we should think of the filtrations on $D_{<r}(G,K)$ and $D_r(G,K)$. By the definition of the norm on $D_r(G,K)$ and $D_{<r}(G,K)$, it is enough to investigate one of the algebras and the same will apply to the other one. So let us consider

$D_r(G,K)$. The possible values of $|| \ ||_r$ are rational powers of p: $|d_\alpha| = |\pi|^n$ or 0, where $n \in \mathbb{Z}$, and $|\pi| = p^{-1/e}$, where e denotes the absolute ramification index of K. Thus, $|d_\alpha| r^{|\alpha|} = p^{-n/e+|\alpha|a/b} = p^{t/[b,e]}$, where $[e,b]$ denotes the least common multiple of b and e. Now $[b,e] = t_e e = t_b b$. for some natural numbers t_e, t_b. Hence, $t = -nt_e + |\alpha| a t_b$. Observe that t_e is relative prime to at_b, since t_e is the product of powers of primes that must divide b and a is relative prime to b. Certainly, $(t_e, t_b) = 1$, since $[b,e]$ is the least common multiple of b and e. Hence if we choose $-n \in \mathbb{Z}$ and $|\alpha| \in \mathbb{Z}$ to be the Bézout coefficients, we get that $t = 1$. Consider only those abelian subgroups where the filtration "jumps", i.e. where $F_r^{s+} D_r(G,K) \subset F_r^s D_r(G,K)$. By the above, it happens if s is some integer multiple of $1/[e,b]$, i.e. $\{s \in \mathbb{R} \ : \ \mathrm{gr}^s D_r(G,K) \neq 0\} = (1/[b,e])\mathbb{Z}$. So after rescaling and reversing the filtration, we can really think of the filtrations as increasing $\mathbb{Z}$-filtrations on $D_{<r}(G,K)$ and $D_r(G,K)$. When K satisfies (E), the possible values of $|| \ ||_r$ lie in $|\pi|^{\mathbb{Z}} \cup \{0\}$ since in that case $b = e = [b,e]$. So we get that the jumps in the filtration happen if $s = t\frac{1}{e}$ for all $t \in \mathbb{Z}$. Of course, $F_r^{\frac{t}{e}+} D_r(G,K) = F_r^{\frac{t-1}{e}} D_r(G,K)$, for all $t \in \mathbb{Z}$.

(*b*) In the light of the last remark, we can make another observation. Let us assume that K satisfies (E). If $\sigma(x)$ is a homogeneous element of degree $t \in |\pi|^{\mathbb{Z}} \cup \{0\}$, then $\sigma(x)$ can be uniquely written as the product of a homogeneous element of degree 0 and $\sigma(\pi)^t$, where π is a prime element of K. Since every element in $\mathrm{gr}^{\cdot} D_{<r}(G,K)$ can be uniquely written as the sum of homogeneous elements, it follows that $\mathrm{gr}^{\cdot} D_{<r}(G,K) = \mathrm{gr}^0 D_{<r}(G,K)[\epsilon_0, \epsilon_0^{-1}]$, where $\epsilon_0 := \sigma(\pi)$. Analogously, $\mathrm{gr}^{\cdot} D_r(G,K) = \mathrm{gr}^0 D_r(G,K)[\epsilon_0, \epsilon_0^{-1}]$. Moreover, in the proof of Lemma 4.8 in [38], both $\mathrm{gr}^0 D_r(G,K)$ and $\mathrm{gr}^0 D_{<r}(G,K)$ were computed for uniform pro-p groups and for K that satisfies (E). More precisely, if G is a uniform pro-p group, then $\mathrm{gr}^0 D_{<r}(G,K) \cong k[[u_1, \ldots, u_d]]$ and $\mathrm{gr}^0 D_r(G,K) \cong k[u_1, \ldots, u_d]$, where d is the dimension of G, $u_i := \sigma(b_i/\pi^m)$ for all $i = 1, \ldots, d$. So if G is any p-adic analytic group and K satisfies (E), then after choosing an open normal uniform pro-p subgroup H of G, $\mathrm{gr}^0 D_{<r}(G,K) \cong k[[u_1, \ldots, u_d]]\#G/H$ and $\mathrm{gr}^0 D_r(G,K) \cong k[u_1, \ldots, u_d]\#G/H$. Since ϵ_0 is central in both $\mathrm{gr}^{\cdot} D_r(G,K)$ and $\mathrm{gr}^{\cdot} D_{<r}(G,K)$, we see that

$$\mathrm{gr}^{\cdot} D_{<r}(G,K) \cong k[\epsilon_0, \epsilon_0^{-1}][[u_1, \ldots, u_d]]\#G/H \cong k[\epsilon_0, \epsilon_0^{-1}][[x_1, \ldots, x_d]]\#G/H$$

where $x_i := \sigma(b_i)$ for all $i = 1, \ldots, d$.

Proposition 5.3.7. Let us suppose that K satisfies (E). Choose an open normal uniform pro-p subgroup H of G. Then the global dimensions of $D_{<r}(G,K)$ is finite and it is less than or equal to d where d is the dimension of H.

Proof. Let us first assume that G is in addition uniform. By part (*b*) of the previous remark,

$$\mathrm{gr}^0 D_{<r}(G,K) \cong k[[u_1,\ldots,u_d]]$$

where $u_i = \sigma(b_i/\pi^m)$ for all $i = 1,\ldots,d$. This implies that gl.dim.$\mathrm{gr}^0 D_{<r}(G,K)$ is finite and equals to d. Observe that by Lemma 2.1.4, Chapter II in [22], $F_r^0 D_{<r}(G,K)$ is a Zariski ring with respect to the filtration induced by the filtration on $D_{<r}(G,K)$. Hence, by Theorem 2.4.9 (d), gl.dim.$F_r^0 D_{<r}(G,K) \leq d$. Note that $D_{<r}(G,K)$ is just the localization of $F_r^0 D_{<r}(G,K)$ at π, where π is a prime element of K. Thus by Corollary 7.4.3 in [28], gl.dim.$D_{<r}(G,K) \leq d$.
For general G, note that, by construction, $D_{<r}(G,K)$ satisfies the assumptions of Lemma 8.8 in [38]. Then the statement follows from Lemma 8.8. □

We state two more useful observations:

Proposition 5.3.8. Let G be a compact p-adic analytic group. Then

(*i*) for any $r \in p^{\mathbb{Q}}$, $1/p \leq r < 1$, the natural inclusion $K[[G]] \hookrightarrow D_r(G,K)$ is flat.

(*ii*) For any $r \in p^{\mathbb{Q}}$, $1/p < r < 1$, the map $K[[G]] \hookrightarrow D_{<r}(G,K)$ is flat.

Proof. Choose an open normal uniform pro-p subgroup H of G. Then, $D_r(G,K) \cong D_r(H,K) \otimes_{K[[H]]} K[[G]]$ and $D_{<r}(G,K) \cong D_{<r}(H,K) \otimes_{K[[H]]} K[[G]]$ as bimodules, so by Proposition 4.7 in [38], the first assertion follows. By Lemma 4.8 in [38], combined with the first assertion, the seconds assertion also follows. □

As mentioned in the intorduction, our motivation is to be able to compute the Grothendieck group of $D(G,K)$. Let $r' < r \in p^{\mathbb{Q}}$ such that $1/p \leq r < r' < 1$. To sum up this section, altogether we obtained a system of K-Banach spaces

$$\cdots \subseteq D_r(G,K) \subseteq D_{<r}(G,K) \subseteq D_{r'}(G,K) \subseteq D_{<r'}(G,K) \subseteq \cdots \subseteq D_{1/p}(G,K).$$

Since the projective limit commutes with finite direct sums,

$$D(G,K) \cong \varprojlim_r D_r(G,K) \cong \varprojlim_r D_{<r}(G,K).$$

It is more practical to consider the objects of second projective limit, so that is what we are going to do, but we get some partial results on the objects of the first projective limit.

5.4 The Grothendieck group of $F_r^0 D_{<r}(G, K)$

5.4.1 The global dimension of $F_r^0 D_r(G, K)$ and $F_r^0 D_{<r}(G, K)$

In this section we will investigate the global dimension of the rings $F_r^0 D_{<r}(G, K)$ and $F_r^0 D_r(G, K)$. The statements and proofs are the same for the two rings. In order to avoid very long expressions, we use the following notations: we fix R to be the ring $F_r^0 D_r(G, K)$ resp. $F_r^0 D_{<r}(G, K)$ and let S denote the ring $D_r(G, K)$ resp. $D_{<r}(G, K)$. Fix a parameter $r \in p^{\mathbb{Q}}$ such that $1/p < r < 1$. We assume that K satisfies (E).

Lemma 5.4.1. Let M be a submodule of a finitely generated filtered-free R-module F with the induced filtration. Then

$$\mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot}(M) \cong \mathrm{gr}^{\cdot}(S \otimes_R M)$$

as graded $\mathrm{gr}^{\cdot} S$-modules.

Proof. The ring S is the localization of R at π, the ring $\mathrm{gr}^{\cdot} S$ is the localization of $\mathrm{gr}^{\cdot} R$ at $\sigma(\pi)$, where π is a prime element of K. Hence the scalar extensions with respect to $R \hookrightarrow S$ and $\mathrm{gr}^{\cdot} R \hookrightarrow \mathrm{gr}^{\cdot} S$ are flat. By the assumption on M, the natural inclusion induces an injection $M \hookrightarrow F$ which is strict. Therefore, if we put the tensor product filtration onto the modules $S \otimes_R M$ and $S \otimes_R F$, the induced map $S \otimes_R M \hookrightarrow S \otimes_R F$ is also a strict injective homomorphism. By the flatness property of S over R and Theorem 4.2.4 (1), Chapter I in [22], the sequence

$$0 \to \mathrm{gr}^{\cdot}(S \otimes_R M) \hookrightarrow \mathrm{gr}^{\cdot}(S \otimes_R F) \tag{30}$$

is exact. Theorem 4.2.4 (1), Chapter I in [22] and the flatness property of the ring $\mathrm{gr}^{\cdot} S$ over $\mathrm{gr}^{\cdot} R$ together imply that the sequence

$$0 \to \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} M \hookrightarrow \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} M. \tag{31}$$

is exact.
Observe that for any filtered ring R, whenever N is an any filtrered right R-module and L is any filtered left R-module, we may define a surjective graded homomorphism

$$\xi_{N,L} : \mathrm{gr}^{\cdot} N \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} L \to \mathrm{gr}^{\cdot}(N \otimes_R L)$$

given by $x_{(s)} \otimes y_{(t)} \mapsto (x \otimes y)_{s+t}$ where $x_{(s)} = \sigma(x)$ and $y_{(t)} = \sigma(y)$ for $x \in F_r^s N - F_r^{s-1} N$ and $y \in F_r^t L - F_r^{t-1} L$. Using the sequences (30), (31) and the construction of the graded epimorphism above, we get the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{gr}^{\cdot}(S \otimes_R M) & \hookrightarrow & \mathrm{gr}^{\cdot}(S \otimes_R F) \\ & & \uparrow \xi_{S,M} & & \uparrow \xi_{S,F} \\ 0 & \longrightarrow & \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} M & \hookrightarrow & \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} F \end{array}$$

By our assumption, F is a filtered-free R-module. Hence, by Lemma 6.14, Chapter I in [22], $\xi_{S,F}$ is an isomorphism. It follows that the map $\xi_{S,M}$ is injective. □

Recall that the induced filtration on the subring $R \subset S$ is the following:

$$F_r^{n/e} R := R \cap F_r^{n/e} S, \ n \in \mathbb{Z}. \tag{32}$$

By Lemma 2.1.4, Chapter II. in [22], R is also a Zariski ring with respect to the filtration given in (32). The associated graded ring is just the non-negative part of $\mathrm{gr}^{\cdot} S$. Moreover,

$$\mathrm{gr}^{\cdot} R = \mathrm{gr}^0 R \oplus I$$

where $I = \bigoplus_{n=1}^{\infty} \mathrm{gr}^{n/e} R = \bigoplus_{n=1}^{\infty} \mathrm{gr}^{n/e} S$. The ideal I is a graded ideal, meaning that it is generated by homogenous elements. We have a tower of inclusions $\mathrm{gr}^0 R \subset \mathrm{gr}^{\cdot} R \subset \mathrm{gr}^{\cdot} S$.

Lemma 5.4.2. $\mathrm{gr}^{\cdot} S$ is a faithfully flat $\mathrm{gr}^0 R$-module.

Proof. Recall that Remark 5.3.6 (b) implies that $\mathrm{gr}^{\cdot} S \cong \mathrm{gr}^0 S[\epsilon_0, \epsilon_0^{-1}]$ where π is a prime element of K and $\epsilon_0 = \sigma(\pi)$. By definition, $\mathrm{gr}^0 S = \mathrm{gr}^0 R$. It follows that

$$\mathrm{gr}^{\cdot} R = (\mathrm{gr}^0 R)[\epsilon_0].$$

Therefore, both $\mathrm{gr}^{\cdot} R$ and $\mathrm{gr}^{\cdot} S$ are faithfully flat $\mathrm{gr}^0 R$-modules. □

Let M be a non-zero finitely generated filtered R-module such that the filtration on M is a good filtration. R is a Zariski ring hence $M \neq 0$ implies that $\mathrm{gr}^{\cdot} M \neq 0$ (Lemma 1.2.9 (5), Chapter II in [22]). For simplicitly reasons, at this point we start denoting by $F_r^i S$ the subgroups $F_r^{i/e} S$ of S for all $i \in \mathbb{Z}$. So in particular, $F_r^i R$ denotes the subgroup $F_r^{i/e} R$ for all $i \geq 0$. Hence $\mathrm{gr}^i R = \mathrm{gr}^{i/e} R$ for all $i \geq 0$. Since R is complete with respect to its filtration, by Theorem 5.7, Chapter I in [22], $\mathrm{gr}^{\cdot} M$ is finitely generated as a graded $\mathrm{gr}^{\cdot} R$-module. Moreover, if $\mathrm{gr}^{\cdot}(M) = \sum_{i=1}^t \mathrm{gr}^{\cdot} R \sigma(u_i)$ where $u_i \in F^{k_i} M - F^{k_i+1} M$ for some $k_i \in \mathbb{Z}$ (recall that $\sigma(u_i)$ denotes the principal symbol of u_i) then $M = \sum_{i=1}^t R u_i$ and $F^n M = \sum_{i=1}^t F_r^{n-k_i} R u_i$. Choose and fix such a generating set $u_1, \ldots, u_t$ for M. Let n_0 be the minimal integer such that $\mathrm{gr}^{n_0} M \neq 0$ or equivalently $F_r^{n_0} M \supset F_r^{n_0+1} M$. Let us assume that $u_1, \ldots, u_t \in F_r^{n_0} M - F_r^{n_0+1} M$ (i.e. $k_1 = k_2 = \cdots = k_t = n_0$) or equivalently $\sigma(u_1), \ldots, \sigma(u_t) \in \mathrm{gr}^{n_0} M$. Then the following is true:

Lemma 5.4.3. $Q(\mathrm{gr}^{\cdot} M) := \mathrm{gr}^{\cdot} M / I \mathrm{gr}^{\cdot} M$ is equal to $M/F_r^1 RM$ as $\mathrm{gr}^0 R$-modules.

Proof. As we have seen above, $M \neq 0$ implies that $\mathrm{gr}^{\cdot} M$ is not zero. By the assumption on M, namely that $k_1 = k_2 = \cdots = k_t = n_0$, it follows that $F^{n_0} M = \sum F_r^0 Ru_i = \sum Ru_i = M$ and $F^{n_0+1} M = \sum F_r^{n_0+1-n_0} Ru_i = \sum F_r^1 Ru_i$. It is also clear that $F_r^1 RM = \sum F_r^1 Ru_i$. Now from this we see that $M/F_r^1 RM = F_r^{n_0} M / F_r^{n_0+1} M = \mathrm{gr}^{n_0} M$. On the other hand, $\mathrm{gr}^{\cdot} M = \sum \mathrm{gr}^{\cdot} R\sigma(u_i) = \bigoplus_{k \geq n_0} \mathrm{gr}^k M$. Clearly, $I \mathrm{gr}^{\cdot} M = \sum I\sigma(u_i) \subseteq \bigoplus_{k > n_0} \mathrm{gr}^k M$ and for a fixed $s \geq 1$, $\mathrm{gr}^{n_0+s} M = \sum \mathrm{gr}^s R\sigma(u_i) \subseteq \sum I\sigma(u_i)$. Hence $I \mathrm{gr}^{\cdot} M = \bigoplus_{k > n_0} \mathrm{gr}^k M$. It follows that

$$\mathrm{gr}^{\cdot} M / I \mathrm{gr}^{\cdot} M = \mathrm{gr}^{n_0} M.$$

□

Consider the functor $Q(T) = T/IT$ where T is an arbitrary graded $\mathrm{gr}^{\cdot} R$-module. By Theorem 12.2.8 in [28], if T is a finitely generated projective graded $\mathrm{gr}^{\cdot} R$-module then $\mathrm{gr}^{\cdot} R \otimes_{\mathrm{gr}^0 R} Q(T) \cong T$ as graded $\mathrm{gr}^{\cdot} R$-modules.

Lemma 5.4.4. Let $J \subseteq R$ be a left ideal of R equipped with a good filtration. Let us suppose that $a_1, \ldots, a_t$ are homogeneous generators of $\mathrm{gr}^{\cdot} J$. Let $n_0 \in \mathbb{N}_0$ be the minimal index such that $\mathrm{gr}^{n_0} J \neq 0$ and assume that all the homogeneous generators are of degree n_0. Then $\mathrm{gr}^{\cdot} J$ is isomorphic to $\mathrm{gr}^{\cdot} R \otimes_{\mathrm{gr}^0 R} Q(\mathrm{gr}^{\cdot} J)$ as graded $\mathrm{gr}^{\cdot} R$-modules.

Proof. Since $\mathrm{gr}^{\cdot} J = \sum \mathrm{gr}^{\cdot} Ra_i$ and $\mathrm{gr}^n J = \sum \mathrm{gr}^{n-n_0} Ra_i$, by the assumption on the generators, it follows that $\mathrm{gr}^{\cdot} J = \mathrm{gr}^{n_0} J \oplus I \mathrm{gr}^{\cdot} J = \sum \mathrm{gr}^0 Ra_i \oplus \sum Ia_i$ where $I = \bigoplus_{k>0} \mathrm{gr}^k R$. Hence in particular, we have a $\mathrm{gr}^0 R$-module isomorphism $\mathrm{gr}^{n_0} J \to \mathrm{gr}^{\cdot} J / I \mathrm{gr}^{\cdot} J$, induced by $a_i \mapsto \overline{a_i}$ for all $i = 1, \ldots, t$, where $\overline{a_i}$ denotes the image of a_i in $\mathrm{gr}^{\cdot} J / I \mathrm{gr}^{\cdot} J$ for all $i = 1, \ldots, t$ and this map has a section induced by $\overline{a_i} \mapsto a_i$. The ring $\mathrm{gr}^0 R$ is naturally a graded ring with the grading that is concentrated in degree zero (i.e. $\mathrm{gr}^0(\mathrm{gr}^0 R) = \mathrm{gr}^0 R$ and $\mathrm{gr}^i(\mathrm{gr}^0 R) = 0$ for all $i \neq 0$). Then the $\mathrm{gr}^0 R$-module $\mathrm{gr}^{\cdot} J / I \mathrm{gr}^{\cdot} J$ is naturally a graded $\mathrm{gr}^0 R$-module, concentrated in degree n_0, with the quotient grading. We put the grading that is concentrated in degree n_0 onto the $\mathrm{gr}^0 R$-module $\mathrm{gr}^{n_0} J$. So the above isomorphism $\mathrm{gr}^{n_0} J \cong \mathrm{gr}^{\cdot} J / I \mathrm{gr}^{\cdot} J$ is graded is a graded isomorphism. We put the tensor product grading onto both $\mathrm{gr}^{\cdot} R \otimes_{\mathrm{gr}^0 R} \mathrm{gr}^{\cdot} J / I \mathrm{gr}^{\cdot} J$ and $\mathrm{gr}^{\cdot} R \otimes_{\mathrm{gr}^0 R} \mathrm{gr}^{n_0} J$. As $\mathrm{gr}^{\cdot} R$ is a faithfully flat $\mathrm{gr}^0 R$-module, it follows, by the above, that $\mathrm{gr}^{\cdot} R \otimes_{\mathrm{gr}^0 R} \mathrm{gr}^{\cdot} J / I \mathrm{gr}^{\cdot} J$ and $\mathrm{gr}^{\cdot} R \otimes_{\mathrm{gr}^0 R} \mathrm{gr}^{n_0} J$ are isomorphic as graded $\mathrm{gr}^{\cdot} R$-modules. We show that $\mathrm{gr}^{\cdot} R \otimes_{\mathrm{gr}^0 R} \mathrm{gr}^{n_0} J \cong \mathrm{gr}^{\cdot} J$ as graded $\mathrm{gr}^{\cdot} R$-modules and that implies the lemma. Since $\mathrm{gr}^{\cdot} R = \bigoplus \mathrm{gr}^n R$, it is enough to see that $\mathrm{gr}^n R \otimes_{\mathrm{gr}^0 R} \mathrm{gr}^{n_0} J \cong \mathrm{gr}^{n_0+n} J$ as $\mathrm{gr}^0 R$-modules. Note that assumption (E) on K implies that $F_r^n R = (\pi)^n$ where π is a prime element of K: It follows from the fact that assumption (E) implies that the possible values of $|| \, ||_r$ lie in $|\pi|^{\mathbb{Z}} \cup \{0\}$. Assume that an arbitrary element $\lambda \in R$ lies in $\lambda \in F_r^k - F_r^{k+1} R$

where $k \in \mathbb{N}$. Then it can be written as $\pi^k(\pi^{-k}\lambda)$ where $\mu = (\pi^{-k}\lambda) \in F_r^0 R - F_r^1 R$ and $k > 0$. So any element x of $\mathrm{gr}^n R$ can be written as $\sigma(\pi)^n y$ where $y \in \mathrm{gr}^0 R$. Therefore, $\mathrm{gr}^n R \otimes_{\mathrm{gr}^0 R} \mathrm{gr}^{n_0} J$ consists of elements of the form $\sigma(\pi)^n \otimes m$ where $m \in \mathrm{gr}^{n_0} J$. Consider the $\mathrm{gr}^0 R$-submodule $\sigma(\pi)^n \mathrm{gr}^{n_0} J$ of $\mathrm{gr}^{\cdot} R$. We have a surjective $\mathrm{gr}^0 R$-module homomorphism $\mathrm{gr}^n R \otimes_{\mathrm{gr}^0 R} \mathrm{gr}^{n_0} J \to \sigma(\pi)^n \mathrm{gr}^{n_0} J$ given by the mapping $\sigma(\pi)^n \otimes m \mapsto \sigma(\pi)^n m$. It is also injective since $\mathrm{gr}^{\cdot} R$ is an integral domain. Note that by the assumption on J, $\mathrm{gr}^{n_0+n} J = \sum \mathrm{gr}^n R a_i$ for all $n \geq 0$ and hence, by the above, $\mathrm{gr}^{n_0+n} J = \sigma(\pi)^n \mathrm{gr}^{n_0} J$.
So we get a $\mathrm{gr}^{\cdot} R$-module isomorphism $\mathrm{gr}^{\cdot} R \otimes_{\mathrm{gr}^0 R} \mathrm{gr}^{n_0} J \to \mathrm{gr}^{\cdot} J$ and it is a graded isomorphism since we have put the tensor product grading onto $\mathrm{gr}^{\cdot} R \otimes_{\mathrm{gr}^0 R} \mathrm{gr}^{n_0} J$. Since $\mathrm{gr}^{\cdot} R \otimes_{\mathrm{gr}^0 R} \mathrm{gr}^{\cdot} J / I \mathrm{gr}^{\cdot} J \cong \mathrm{gr}^{\cdot} R \otimes_{\mathrm{gr}^0 R} \mathrm{gr}^{n_0} J$ as graded $\mathrm{gr}^{\cdot} R$-modules, we are done. □

Lemma 5.4.5. Let P be a finitely generated filtered projective R-module. Then $\mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^0 R} Q(\mathrm{gr}^{\cdot} P) \cong \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} P$.

Proof. By Theorem 12.2.8 in [28], we have a graded isomorphisms

$$\mathrm{gr}^{\cdot} R \otimes_{\mathrm{gr}^0 R} Q(P) \to \mathrm{gr}^{\cdot} P$$

We use again that $\mathrm{gr}^{\cdot} S$ is a flat graded $\mathrm{gr}^{\cdot} R$-module. So we get that

$$\mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} R \otimes_{\mathrm{gr}^0 R} Q(\mathrm{gr}^{\cdot} P) \cong \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} P$$

is also a graded isomorphism. Now by the well known associativity property of the tensor product, $\mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^0 R} Q(\mathrm{gr}^{\cdot} P) \cong (\mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} R) \otimes_{\mathrm{gr}^0 R} Q(\mathrm{gr}^{\cdot} P) \cong \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} (\mathrm{gr}^{\cdot} R \otimes_{\mathrm{gr}^0 R} Q(\mathrm{gr}^{\cdot} P))$ as left $\mathrm{gr}^{\cdot} S$-modules. Hence we are done. □

Remark 5.4.6. The same proof shows that if we are in the situation of Lemma 5.4.4, then $\mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^0 R} J \cong \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} J$

Choose an open normal uniform pro-p subgroup H of G.

Theorem 5.4.7. The global dimension of R is finite and

$$\text{gl. dim.}(R) \leq \text{gl.dim.}(S) + 1 \leq d + 1$$

where d denotes the dimension of H.

Proof. We show that the projective dimension of any non-zero left ideal $J \subseteq R$ is less than or equal to d. Let J be an arbitrary left ideal of R. Since R is Noetherian, J is finitely generated. Choose and fix a minimal set of generators $u_1, \ldots, u_t$ for J. Consider the free R-module $\tilde{F} = \bigoplus_{i=1}^t R\tilde{u}_i$ of rank t and the surjective R-module homomorphism $f : \tilde{F} \to J$, given by $\tilde{u}_i \mapsto u_i$ for all $i = 1, \ldots, t$. We

put the filtration onto $\tilde{F}$ that has no shifts, i.e. $F^n\tilde{F} = \bigoplus_{i=1}^{t} F_r^n R\tilde{u}_i$, $n \in \mathbb{Z}$, (this filtration can also be seen as the filtration defined by the maximum norm on $\tilde{F}$ induced by $||\cdot||_r$) and the quotient filtration onto J (this filtration can be regarded as the filtration given by the residue norm induced by the norm on $\tilde{F}$). This filtration on J is a good filtration. It follows that f a strict filtered R-module homomorphism. Hence it induces a graded homomorphism, $\mathrm{gr}^{\cdot} f : \mathrm{gr}^{\cdot}\tilde{F} \to \mathrm{gr}^{\cdot} J$, given by $\sigma(\tilde{u}_i) \mapsto \sigma(u_i)$ for all $i = 1, \dots, t$. Since $u_1, \dots, u_t$ is a minimal set of generators (therefore $\sigma(u_i) \neq 0$ for all i otherwise we would get a proper subset of $\{u_1, \dots, u_t\}$ with less than t elements which generate J, by Theorem 5.7 Chapter I in [22]), by the fact that $\mathrm{gr} f$ is a graded homomoprhism of degree 0, it follows that the degrees of the elements $\sigma(u_i)$ are the same for all $i = 1, \dots, t$, and thus the filtration on J has the property that all the generators have the same minimal degree just as what we required in Lemma 5.4.4.
In general, if we are in the situation of Lemma 5.4.4 (which obviously applies to J by the above), i.e. there are generators $u_1, \dots, u_t$ of J and a minimal integer n_0 for J such that $u_i \in F^{n_0}J - F^{n_0+1}J$, $F^nJ = \sum F_r^{n-n_0} Ru_i$ and $\mathrm{gr}^{\cdot} J = \sum_{i=1}^{t} \mathrm{gr}^{\cdot} R\sigma(u_i)$, $\mathrm{gr}^n J = \sum \mathrm{gr}^{n-n_0} R\sigma(u_i)$, we observe the following:
By Lemma 5.4.3, $Q(\mathrm{gr}^{\cdot} J) \cong J/F_r^1 RJ$. If we prove that $Q(\mathrm{gr}^{\cdot} J)$ has finite global dimension as a $\mathrm{gr}^0 R$-module then the quotient module $J/F_r^1 RJ$ also has finite projective dimension as a $\mathrm{gr}^0 R$-module.

Lemma 5.4.8. If $J/F_r^1 RJ$ has finite projective dimension as a $\mathrm{gr}^0 R$-module then J has finite projective dimension as an R-module.

Proof. As before we note that from assumption (E) on K it follows that $F_r^1 R = (\pi)$ where π is a prime element of K.
Observe that R is an integral domain and π is a regular central non-unit in R. Moreover, it is in the Jacobson radical of R since $||\pi||_r < 1$ and R is complete. Therefore, π is regular on any right ideal of R. Since $\mathrm{gr}^0 R = R/F_r^1 R = R/(\pi)$, it follows from Proposition 7.3.6, (b) in [28] that

$$\mathrm{pd}_{\mathrm{gr}^0 R}(J/F_r^1 RJ) = \mathrm{pd}_R(J).$$

□

By Corollary 6.3 (i) in [22], there is a strict exact sequence of filtered R-modules

$$0 \to M_0 \to F_0 \to J \to 0.$$

Hence we may consider a minimal filtered-free resolution of J

$$\dots \to F_n \to \dots F_0 \to J \to 0.$$

with all the maps being strict homomorphims. Recall that, by Proposition 5.3.7 and Theorem 2.8.20, the global dimensions of both $D_{<r}(G,K)$ and $D_r(G,K)$ are finite and they are less than or equal to d. So let us assume that $\mathrm{pd}_S(S \otimes_R J) = j$ $(j \leq d)$. Consider the strict exact sequence of filtered R-modules

$$0 \longrightarrow M_{j-1} \longrightarrow F_{j-1} \longrightarrow \ldots \longrightarrow F_0 \xrightarrow{\phi_0} J \longrightarrow 0. \tag{33}$$

Note that the quotient filtration with respect to the surjection ϕ_0 is the same as the original filtration on J. Apply the functor $(S \otimes_R -)$ to the resolution. By the flatness property of S over R and that $R \to S$ is a filtered injection of rings, we get a strict exact sequence of filtered S-modules with the tensor product filtrations on them:

$$0 \longrightarrow S \otimes_R M_{j-1} \longrightarrow \ldots \longrightarrow S \otimes_R F_0 \longrightarrow S \otimes_R J \longrightarrow 0. \tag{34}$$

where $S \otimes_R F_i$ is filtered free S-modules of finite rank for all $i = 0, \ldots, j-1$. By Schanuel's lemma (Lemma 1.1.6 in [37]), $S \otimes_R M_{j-1}$ is projective. It follows that it is a filtered projective S-module: By Theorem 5.4.7, Chapter I in [22], there are S-module generators $v_1, \ldots, v_m$ and integers $k_1, \ldots, k_m$ such that $S \otimes_R M_{j-1} = \sum_{i=1}^m Sv_i$, $F^n(S \otimes_R M_{j-1}) = \sum_{i=1}^m F_r^{n-k_i} Sv_i$, $n \in \mathbb{Z}$, $\mathrm{gr}^{\cdot}(S \otimes_R M_{j-1}) = \sum_{i=1}^m \mathrm{gr}^{\cdot} S\sigma(v_i)$, $\mathrm{gr}^n(S \otimes_R M_{j-1}) = \sum_{i=1}^m \mathrm{gr}^{n-k_i} S\sigma(v_i)$, $n \in \mathbb{Z}$. We have a filtered S-module surjection $\alpha : F_j \to S \otimes_R M_{j-1}$ where F_j is a free S-module of finite rank equipped with the filtration

$$F^n F_j := \bigoplus_{i=1}^m F^{n-k_i} S.$$

Since $S \otimes_R M_{j-1}$ is projective there is an S-module section β to α. Let $w_i = \beta(v_i)$ in F_j. Let k_0 be the minimum of the integers n_i such that $w_i \in F^{n_i} F_j - F^{n_i+1} F_j$. Then

$$\beta(F^n(S \otimes_R M_{j-1})) = \sum_{i=1}^m F_r^{n-k_i} S\beta(v_i) = \sum_{i=1}^m F_r^{n-k_i} Sw_i \subseteq F^{n-k_i+k_0} F_j.$$

Let k be equal to the minimum of the integers $-k_i + k_0$. Then $F^{n-k_i+k_0} F_j \subseteq F^{n+k} F_j$. It follows that β is a filtered S-module homomorphism of degree k. Then $\mathrm{gr}^{\cdot}\beta$ is a graded $\mathrm{gr}^{\cdot}S$-module homomorphism of degree k and moreover, it is easy to see that $\mathrm{gr}^{\cdot}\alpha \circ \mathrm{gr}^{\cdot}\beta$ is the identity on $\mathrm{gr}^{\cdot}(S \otimes_R M_{j-1})$. It follows that $\mathrm{gr}^{\cdot}(S \otimes_R M_{j-1})$ is a direct summand of $\mathrm{gr}^{\cdot} F_j$. Now by Corollary 4.1.3 (a), Chapter I in [22], $\mathrm{gr}^{\cdot}(S \otimes_R M_{j-1})$ is a graded projective module. Therefore, by Theorem 7.1.10, Chapter I in [22], $S \otimes_R M_{j-1}$ is indeed a filtered projective S-module.

By construction, all the maps in 34 are strict morphisms. We apply $\mathrm{gr}^{\cdot}(\ )$ to 34. By Theorem 4.2.4 (1), Chapter I. in [22], we get an exact sequence of graded modules

$$0 \longrightarrow \mathrm{gr}^{\cdot}(S \otimes_R M_{j-1}) \longrightarrow \ldots \longrightarrow \mathrm{gr}^{\cdot}(S \otimes_R F_0) \longrightarrow \mathrm{gr}^{\cdot}(S \otimes_R J) \longrightarrow 0$$

which is a graded projective resolution of $\mathrm{gr}^{\cdot}(S \otimes_R J)$ since filtered projective (free) modules are mapped to graded projective (free) modules via the fuctor $\mathrm{gr}^{\cdot}(\ )$. Using Lemma 5.4.1, we have graded isomorphisms $\mathrm{gr}^{\cdot}(S \otimes_R M_{j-1}) \cong \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} M_{j-1}$, $\mathrm{gr}^{\cdot}(S \otimes_R F_i) \cong \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} F_i$ for all $i = 1, \ldots, j-1$ and $\mathrm{gr}^{\cdot}(S \otimes_R J) \cong \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} J$. Hence we get the following graded exact sequence:

$$0 \longrightarrow \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} M_{j-1} \longrightarrow \ldots \longrightarrow \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} J \longrightarrow 0\ . \tag{35}$$

By the isomorphism $\mathrm{gr}^{\cdot}(S \otimes_R M_{j-1}) \cong \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} M_{j-1}$, the later module is also projective, hence by 7.6.6 in [28], it is graded projective. Obviously, $\mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} F_i$ are graded free $\mathrm{gr}^{\cdot} S$-modules of finite rank for all $i = 1, \ldots, j-1$.
By Remark 5.4.6, we have a graded isomorphism

$$\mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^0 R} Q(\mathrm{gr}^{\cdot} J) \to \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^{\cdot} R} \mathrm{gr}^{\cdot} J$$

Using Lemma 5.4.5 repeatedly, we get the following exact sequence:

$$0 \longrightarrow \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^0 R} Q(\mathrm{gr}^{\cdot} M_{j-1}) \longrightarrow \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^0 R} Q(\mathrm{gr}^{\cdot} F_{j-1}) \longrightarrow \ldots \tag{36}$$

$$\ldots \longrightarrow \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^0 R} Q(\mathrm{gr}^{\cdot} F_0) \longrightarrow \mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^0 R} Q(\mathrm{gr}^{\cdot} J) \longrightarrow 0.$$

Now this shows that $\mathrm{pd}_{\mathrm{gr}^{\cdot} S}(\mathrm{gr}^{\cdot} S \otimes_{\mathrm{gr}^0 R} Q(\mathrm{gr}^{\cdot} J)) \leq j \leq d$. The ring $\mathrm{gr}^0 R$ is a $\mathrm{gr}^0 R$-bimodule direct summand of $\mathrm{gr}^{\cdot} S$ and $\mathrm{gr}^{\cdot} S$ is a free $\mathrm{gr}^{\cdot} R$-module. Theorem 7.2.8 (i) in [28] implies that $\mathrm{pd}_{\mathrm{gr}^0 R}(Q(\mathrm{gr}^{\cdot} J)) \leq d$. Therefore, by Lemma 5.4.8, $\mathrm{pd}_R(J) \leq d$.
For an arbitrary cyclic R-module M, we have a short exact sequence

$$0 \to J \to R \to M \to 0$$

where J is the annihilator of M. J is a left ideal of R. By 7.1.6. in [28], whenever there is a short exact sequence of R-modules

$$0 \to A \to B \to C \to 0,$$

if two have finite projective dimension so does the third. Moreover,

$$\mathrm{pd}(A) = \max\{\mathrm{pd}(A), \mathrm{pd}(A)\}$$

unless $\mathrm{pd}(B) < \mathrm{pd}(C)$ in which case $\mathrm{pd}(C) = 1+\mathrm{pd}(A)$. Hence $\mathrm{pd}_R(M) \leq \mathrm{pd}_R(J) + 1$. But $\mathrm{pd}_R(J) \leq d$. Therefore, $\mathrm{pd}_R(M) \leq d + 1$. By Section 7.1.8 in [28], it is enough to compute the projective dimension of cyclic modules since $\mathrm{gl.dim.}(R) = \sup\{\mathrm{pd}(M)\ :\ M \text{ is a cyclic } R\text{-module}\}$.

□

5.4.2 K_0 of $\mathrm{gr}^0 D_{<r}(G,K)$ and $F_r^0 D_{<r}(G,K)$

Recall that the subalgebra $F_r^0 D_{<r}(G,K) \subset D_{<r}(G,K)$, equipped with the induced filtration, is a filtered subalgebra of $D_{<r}(G,K)$. When we pass to their associated graded rings, we see that $\mathrm{gr}^{\cdot} F_r^0 D_{<r}(G,K)$ is the non-negative part of $\mathrm{gr}^{\cdot} D_{<r}(G,K)$. Choose an open normal uniform pro-p subgroup H of G. We fix a parameter $r \in p^{\mathbb{Q}}$ such that $1/p < r < 1$. Assume that K satisfies (E). Recall that in Remark 5.3.6 (b), it was shown that the quotient ring

$$F_r^0 D_{<r}(H,K)/F_r^{0+} D_{<r}(H,K) = \mathrm{gr}^0 F_r^0 D_{<r}(H,K)$$

is isomorphic to

$$\mathrm{gr}^0 F_r^0 D_{<r}(G,K) \cong k[[u_1,\ldots,u_d]]\#G/H.$$

We define the following set:

$$\mathcal{I}_k := \sum_{i=1}^{|G/H|} I g_i \subset k[[u_1,\ldots,u_d]]\#G/H \tag{37}$$

where $I = (u_1,\ldots,u_d)$ is the maximal ideal of $k[[u_1,\ldots,u_d]]$.

Lemma 5.4.9. The set $\mathcal{I}_k$ is an ideal in $k[[u_1,\ldots,u_d]]\#G/H$.

Proof. Every element $\mu \in k[[u_1,\ldots,u_d]]\#G/H$ can be uniquely written in the form $\mu = \sum f_i g_i$ where $f_i \in k[[u_1,\ldots,u_d]]$. Since $k[[u_1,\ldots,u_d]]\#G/H$ is a skew group ring, it is enough to show that if we multiply μ from the left by the image of any coset representative $g_t \in X$, the element $g_t\mu$ lies in Ig_t. For that it suffices to show that $g_t u_j \in Ig_t$ for any $i \in \{1,\ldots,d\}$ and any $t \in \{1,\ldots,|G/H|\}$. For a fixed $t \in \{1,\ldots,|G/H|\}$, $g_t u_j = (g_t u_j g_t^{-1})g_t$ and the conjugation by g_t is an automorphism of $k[[u_1,\ldots,u_d]]$. So denote the element $g_t u_j g_t^{-1}$ by x, which is a power series. It is well-known that an arbitrary element of a power series ring over some commutative ring is invertible if and only if its constant term is invertible in the coefficient ring. In our situation, it means that if the constant term is non-zero then the power series is invertible. The image of a non-invertible element with respect to a ring automorphism cannot be invertible. Hence the constant term of x is zero. Therefore $x \in I$. It implies that $g_t u_j = x g_t \in Ig_t$. □

Lemma 5.4.10. The filtration on $k[[u_1,\ldots,u_d]]\#G/H$ induced by the ideal $\mathcal{I}_k$ is separated, i.e. $\cap_j \mathcal{I}_k^j = 0$.

Proof. It is well-known that the I-adic filtration on $k[[u_1,\ldots,u_d]]$ is separated, i.e. $\cap_j I^j = 0$. We state that $\mathcal{I}_k^j = \sum_i I^j g_i$ and that implies the statement of the lemma. We prove it by induction on the natural number j. If $j = 1$, then the statement follows from the definition of $\mathcal{I}_k$. Let us assume the the statement is true for an arbitrary natural number j. We show that then it is true for $j+1$. The equality $\mathcal{I}_k^{j+1} = \mathcal{I}_k^j \mathcal{I}_k$ enables us to write an arbitrary element $\mu \in \mathcal{I}_k^{j+1}$ as $\mu = \sum_{i=1}^m \mu_i \nu_i$ where $\mu_i \in \mathcal{I}_k^j$ and $\nu_i \in \mathcal{I}_k$, for all $i = 1, \ldots, m$. By the induction hypothesis, $\mu_i = \sum f_{i,s} g_s$ where $f_{i,s} \in I^j$ for all $i = 1, \ldots, m$ and $s = 1, \ldots, |G/H|$. Similarly, $\nu_i = \sum h_{i,t} g_t$ where $h_{i,t} \in I$ for all $i = 1, \ldots, m$ and $t = 1, \ldots, |G/H|$. It is clear that it is enough to show that all the products $\mu_i \nu_i$ lie in $\mathcal{I}_k^{j+1}$. Moreover, it is enough to show that $\mu_1 \nu_1$ lies in $\mathcal{I}_k^{j+1}$ since the proof is the same for all the other products. So to make the expressions more simple, we will denote $f_{1,s}$ by f_s and $h_{1,t}$ by h_t for all $s, t = 1, \ldots, |G/H|$.

$$\mu_1 \nu_1 = \sum f_s g_s \sum h_t g_t = \sum_{s=1}^{|G/H|} f_s (\sum_{t=1}^{|G/H|} g_s h_t g_t).$$

In the proof of Lemma 5.4.9, we have seen that $g_s h_t = x_t g_s$, where $x_t \in I$ for all $s, t = 1, \ldots, |G/H|$. Hence $\mu_1 \nu_1 = \sum (f_s x_t) g_s g_t$. $f_s x_t$ is clearly an element of I^{j+1}. Since $k[[u_1, \ldots, u_d]] \# G/H$ is a skew group ring, we get that $\mu_1 \nu_1 \in \sum_{n=1}^{|G/H|} I^{j+1} g_n$. $\square$

Lemma 5.4.11. The algebra $k[[u_1, \ldots, u_d]] \# G/H$ is complete with respect to the filtration induced by the ideal $\mathcal{I}_k$, i.e.

$$k[[u_1, \ldots, u_d]] \# G/H \cong \varprojlim_j (k[[u_1, \ldots, u_d]] \# G/H) / \mathcal{I}_k^j$$

Proof. The algebra $k[[u_1, \ldots, u_d]]$ is complete with respect to the filtration induced by I. In the proof of the previous lemma we showed that $\mathcal{I}_k^j = \sum I^j g_i$ for any $j \geq 1$. Hence the image $\overline{\mu}_j$ of an arbitrary element $\mu = \sum f_i g_i \in k[[u_1, \ldots, u_d]] \# G/H$ in $(k[[u_1, \ldots, u_d]] \# G/H) / \mathcal{I}_k^j$ equals $\sum \overline{f}_{ij} g_i$ where $\overline{f}_{ij}$ is the image of f_i in $k[[u_1, \ldots, u_d]] / I^j$. By the completness of $k[[u_1, \ldots, u_d]]$, we have that $f_i = \varprojlim_j \overline{f}_{ij}$. Hence $\mu = \varprojlim_j \sum \overline{f}_{ij} g_i$. $\square$

Theorem 5.4.12. The group $K_0(\mathrm{gr}^0 D_{<r}(G, K))$ is isomorphic to $\mathbb{Z}^c$ where c is the number of p-regular conjugacy classes of G/H.

Proof. There is a short exact sequence

$$0 \to I_K \to D_{<r}(G, K) \to K[G/H] \to 0 \tag{38}$$

induced by the group homomorphism $G \to G/H$ (it means that every element $h \in H$ is sent to 1). Note that the action of G/H on K is trivial, thus we can write $K[G/H]$, instead of $K \# G/H$. Moreover, if $D_{<r}(G, K)$ is equipped with the filtration induced by the norm $|| \ ||_r$, we can equip the ideal I_K with the induced

filtration. Also we equip $K[G/H]$ with the quotient filtration which is of course the same filtration as if we equipped $K[G/H]$ with the filtration induced by the norm which we define by putting the norm $|| \, ||_r$ on K and then considering the maximum norm on $K[G/H]$. This way (38) becomes strict. Apply $\mathrm{gr}^{\cdot}(\ )$ to (38). We get the following sequence:

$$0 \to \mathrm{gr}^{\cdot} I_K \to \mathrm{gr}^{\cdot} D_{<r}(G,K) \to (\mathrm{gr}^{\cdot} K)[G/H] \to 0 \tag{39}$$

Let M be an arbitrary graded $\mathrm{gr}^{\cdot} D_{<r}(G,K)$-module and N any $\mathrm{gr}^0 D_{<r}(G,K)$-module. By Proposition 2.3.13 and Theorem 2.3.14, the functors

$$(\ )_0 : \text{gr-gr}^{\cdot} D_{<r}(G,K) \to \text{mod-gr}^0 D_{<r}(G,K), \ \ M \mapsto \mathrm{gr}^0 M$$

$$(\mathrm{gr}^{\cdot} D_{<r}(G,K) \otimes_{\mathrm{gr}^0 D_{<r}(G,K)} -) : \text{mod-gr}^0 D_{<r}(G,K) \to \text{gr-gr}^{\cdot} D_{<r}(G,K),$$
$$N \mapsto \mathrm{gr}^{\cdot} D_{<r}(G,K) \otimes_{\mathrm{gr}^0 D_{<r}(G,K)} N$$

are equivalences of categories. Hence if we apply $(\ )_0$ to (39), the sequence

$$0 \to \mathrm{gr}^0 I_K \to \mathrm{gr}^0 D_{<r}(G,K) \to k[G/H] \to 0 \tag{40}$$

is exact. It is easy to see then that the ideal $\mathcal{I}_k$ defined in is isomorphic to $\mathrm{gr}^0 I_K$. Hence $\mathrm{gr}^0 D_r(G,K)$ is complete with respect to the filtration induced by $\mathrm{gr}^0 I_K$.

Therefore, using Proposition 2.5.6 and Lemma 5.1.1,

$$K_0(\mathrm{gr}^0 D_{<r}(G,K)) \cong K_0((\mathrm{gr}^0 D_{<r}(G,K))/\mathrm{gr}^0 I_K) \cong K_0(k[G/H]) \cong \mathbb{Z}^c.$$

□

Proposition 5.4.13. The $K_0(F_r^0 D_{<r}(G,K))$ is isomorphic to $\mathbb{Z}^c$ where c is the number of p-regular conjugacy classes of G/H.

Proof. Note that the algebra $F_r^0 D_{<r}(G,K)$ is complete with respect to the ideal $F_r^{0+} D_{<r}(G,K)$: By Proposition 2.1.6 Chapter II in [22], the $F_r^{0+} D_{<r}(G,K)$-adic filtration is topologically equivalent to the induced filtration on $F_r^0 D_{<r}(G,K)$ by the filtration on $D_{<r}(G,K)$. But $F_r^0 D_{<r}(G,K)$ is complete with respect to the induced filtration. Using Theorem 5.4.12 and Proposition 2.5.6 we get that

$$\mathbb{Z}^c \cong K_0(\mathrm{gr}^0 F_r^0 D_{<r}(G,K)) = K_0((F_r^0 D_{<r}(G,K))/F_r^{0+} D_{<r}(G,K)) \cong$$

$$\cong K_0(F_r^0 D_{<r}(G,K))$$

□

5.5 The Grothendieck group of $D_{<r}(G, K)$

Theorem 5.5.1. Let G be a compact p-adic analytic group with no element of order p. Let $r \in p^{\mathbb{Q}}$, $1/p < r < 1$ and let us assume that K satisfies (E). Then the Grothendieck group of $D_{<r}(G, K)$ is isomorphic to $\mathbb{Z}^c$.

Choose an open normal uniform pro-p group H of G. We begin the proof by finding a surjective map $\mathbb{Z}^c \to K_0(D_{<r}(G, K))$.

5.5.1 Surjectivity

Theorem 5.5.2. There is a surjective map $\mathbb{Z}^c \to K_0(D_{<r}(G, K))$, where c is the number of p-regular conjugacy classes of G/H.

Proof. Let π be a prime element of K. By Theorem 2.5.12, we have the following exact sequence of abelian groups:

$$K_0(\pi\text{-tors}) \to G_0(F_r^0 D_{<r}(G, K)) \to G_0(D_{<r}(G, K)) \to 0$$

where π-tors denotes the categroy of π-torsion $F_r^0 D_{<r}(G, K)$-modules. By Theorem 5.4.7 and Proposition 5.3.7, the above sequence induces a surjective group homomorphism $K_0(F_r^0 D_{<r}(G, K)) \to K_0(D_{<r}(G, K))$. By Proposition 5.4.13, $K_0(F_r^0 D_{<r}(G, K)) \cong \mathbb{Z}^c$, where c is the number of conjugacy classes of G/H relative prime to p. The statement then follows.

□

5.5.2 Injectivity

As mentioned in the introduction, the motivation of Chapter 5 is to be able to compute the Grothendieck group of $D(G, K)$. In this section, we take a step towards it. We prove that the group homomorphism $K_0(K[[G]]) \to K_0(D_{<r}(G, K))$ induced by the natural injection of rings $K[[G]] \to D_{<r}(G, K)$ is injective. This has a nice consequence, namely that there is a natural injective homomorphism $\mathbb{Z}^c \hookrightarrow K_0(D_{<r}(G, K))$ and more importantly, it implies that there is an injective group homomorphism $\mathbb{Z}^c \hookrightarrow K_0(D(G, K))$. We will see what this homomorphism exactly is. We very much suspect that it is in fact an isomorphism. Let us denote by d the dimension of H.

Theorem 5.5.3. There is an injective map $\mathbb{Z}^c \hookrightarrow K_0(D_{<r}(G, K))$, where c is the number of p-regular conjugacy classes of G/H.

Proof. Let us denote by I_0 the kernel of the ring homomorphism $\varphi_0 : K[[G]] \to K[G/H]$ induced by the surjective group homomorphism $G \to G/H$. Then the kernel I_0 is generated by the elements b_i for $i = 1, \ldots, d$. One may look at the algebra $K[G/H]$ as the distribution algebra of G/H, which is a compact p-adic analytic group of dimension 0. Obviously, in this case all the algebras $K[[G/H]]$, $D(G/H,K)$, $D_r(G/H,K)$, $D_{<r}(G/H,K)$ are the same, the group algebra $K[G/H]$. In the previous section we denoted by I_K the kernel of the surjection $\varphi_K : D_{<r}(G,K) \to K[G/H]$, induced by the group homomorphism $G \to G/H$. As before, I_K is generated by b_i for all $i = 1,\dots,d$. It is easy to see that I_K is the scalar extension of I_0 via the canonical flat ring map $K[[G]] \hookrightarrow D_{<r}(G,K)$. Hence we have the following commutative diagram:

$$\begin{array}{ccc} K[[G]] & \longrightarrow & D_{<r}(G,K) \\ \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_r} \\ K[G/H] & \xrightarrow{=} & K[G/H] \end{array} \tag{41}$$

Lemma 5.5.4. $K_0(\varphi_0) : K_0(K[[G]]) \hookrightarrow K_0(K[G/H])$ is injective.

Proof. It is easy to see that the diagram

$$\begin{array}{ccc} \mathcal{O}_K[[G]] & \longrightarrow & K[[G]] \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{O}_K[G/H] & \longrightarrow & K[G/H] \end{array}$$

is commutative where the horizontal maps are the natural inclusions the the vertical maps are the natural surjections. Hence it induces a diagram

$$\begin{array}{ccc} K_0(\mathcal{O}_K[[G]]) & \longrightarrow & K_0(K[[G]]) \\ \downarrow & & \downarrow \\ K_0(\mathcal{O}_K[G/H]) & \longrightarrow & K_0(K[G/H]). \end{array}$$

By Theorem 5.2.4 the upper horizontal map is an isomorphism.By Proposition 3.3 (b) in [8], $\mathcal{O}_K[[G]]$ is complete with respect to the augmentation ideal $I(H)$: It is the kernel of the map $\mathcal{O}_K[[G]] \to \mathcal{O}_K[G/H]$). Actually $I(H)$ is in the Jacobson radical of $\mathcal{O}_K[[G]]$, which is contained in the radical, the intersection of all open maximal left ideals of $\mathcal{O}_K[[G]]$. The Iwasawa algebra is complete with respect to the filtration induced by the radical, by Corollary 5.2.19 in [30]. Hence by Proposition 2.5.6, the vertical map on the left hand side is also an isomorphism. By Corollary 2.9.9, the lower horizontal map is injective. Hence the vertical map on the right hand side must be injective. $\square$

Diagram (41) induces a commutative diagram after applying $K_0(\ )$:

$$\begin{array}{ccc} K_0(K[[G]]) & \longrightarrow & K_0(D_{<r}(G,K)) \ . \\ \Big\downarrow{\scriptstyle K_0(\varphi_0)} & & \Big\downarrow{\scriptstyle K_0(\varphi_r)} \\ K_0(K[G/H]) & \xrightarrow{=} & K_0(K[G/H]) \end{array}$$

By Lemma 5.5.4, $K_0(\varphi_0)$ is injective. Hence the upper horizontal map must also be injective, by commutativity.

Remark 5.5.5. Note that for injectivity, we did not need assumption (E).

Corollary 5.5.6. The map $K_0(K[[G]]) \to K_0(D(G,K))$ induced by the natural inclusion $K[[G]] \to D(G,K)$ is injective. Hence we have an injective map $\mathbb{Z}^c \to K_0(D(G,K))$

Proof. It is an easy consequence of the fact that the natural inclusion $K[[G]] \hookrightarrow D_{<r}(G,K)$ factorizes through $D(G,K)$ since know that $D(G,K) \subset D_{<r}(G,K)$ for all $r^{\mathbb{Q}}$ such that $1/p < r < 1$. Then we can use that $K_0(\ )$ is a functor to get the desired injective map. □

Now the proof of Theorem 5.5.1: Hence the theorem follows from the well-known structure theorem for finitely generated modules over PID's since by Theorem 5.5.3 and by Theorem 5.5.2, we have an injective map $\mathbb{Z}^c \hookrightarrow K_0(D_{<r}(G,K))$ and a surjective map $\mathbb{Z}^c \to K_0(D_{<r}(G,K))$. Hence $\mathbb{Z}^c \cong K_0(D_{<r}(G,K))$. □

Corollary 5.5.7. Let G be a compact p-adic analytic group. Let $r \in p^{\mathbb{Q}}$, $1/p < r < 1$ and assume that K satisies (E). Then there is an injective map $\mathbb{Z}^c \to D_r(G,K)$.

Proof. The map $K_0(K[[G]]) \hookrightarrow K_0(D_{<r}(G,K))$ factorizes through $K_0(D_r(G,K))$. It follows that $\mathbb{Z}^c \hookrightarrow K_0(D_r(G,K))$. Hence using that K_0 is a functor, we get the injective map. □

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Selbstständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß §7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, den 04.10.2016 Tamás Csige