Hypersurfaces with defect and their densities over finite fields

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Summary

The first topic of this dissertation is the defect of projective hypersurfaces. This has been discussed by various authors, e.g., by Cheltsov [12], Cynk [16], Dimca [23], Kloosterman [45], Polizzi/Rapagnetta/Sabatino [59], Rams [61], and Werner [66]. However, the literature is mostly on hypersurfaces with at most ordinary double points as singularities and exclusively over the field of complex numbers. In many of these works, it is indicated that hypersurfaces with defect have a rather large singular locus. In the first chapter, this will be made precise and proven for hypersurfaces with arbitrary isolated singularities over a field of characteristic zero, and for certain classes of hypersurfaces in positive characteristic. Moreover, over a finite field, we give an estimate on the density of hypersurfaces without defect. Finally, it is shown that a non-factorial threefold hypersurface with isolated singularities always has defect.

Over the complex numbers, the classical theorem of Bertini asserts that a general member of a base-point-free linear system on a smooth projective variety is itself smooth. This result does not carry over to positive characteristic. A Bertini theorem over finite fields was given by Poonen [60], considering the density of smooth hypersurface sections in a smooth ambient variety. The second chapter deals with extending this Bertini theorem to a version for quasismooth hypersurfaces in simplicial toric varieties. Since the ambient space is possibly singular, some new phenonema occur. The main application is to show that hypersurfaces admitting a large singular locus compared to their degree have density zero. Furthermore, the chapter contains a Bertini irreducibility theorem for simplicial toric varieties generalizing work of Charles and Poonen [11].

The third chapter continues with density questions over finite fields. In the beginning, certain fibrations over smooth projective bases living in a weighted projective space are considered. The first result is a Bertini-type formula for smooth fibrations, giving back Poonen's theorem on smooth hypersurfaces. The final section deals with elliptic curves over a function field of a variety of dimension at least two. The techniques developed in the first two sections allow to produce a lower bound on the density of such curves with Mordell-Weil rank zero, improving an estimate of Kloosterman [44].

The precise mathematical statements of the main theorems are given in the overview sections in the beginning of each chapter. The results of the first two chapters have led to the articles [49], [50], and [51].

Chapter 1

Hypersurfaces with defect

1.1 Overview

Let k be a field of characteristic $p \geq 0$ and let $n \geq 3$ be an integer. A projective hypersurface $X \subseteq \mathbb{P}^n_k$ is said to have *defect* (see Definition 1.2.20) if

$$h^i(X) \neq h^i(\mathbb{P}^n_k) \quad \text{for some } i \in \{n, n+1, \ldots, 2n-2\},$$

where h^n denotes the n-th Betti number in a reasonable cohomology theory for k-varieties. Examples for such cohomology theories include algebraic de Rham cohomology (if $p = 0$) and étale cohomology. For a list of various applicable cohomology theories, see the beginning of the subsequent section.

In any of these theories, a hypersurface with defect is necessarily singular. Moreover, it seems that defect forces the hypersurface to have "many" singularities compared to their degree: For example, an important class of hypersurfaces with defect is formed by non-factorial hypersurfaces $X \subseteq \mathbb{P}^4$, see §1.5. By a result of Cheltsov [12], if such an X has at most ordinary double points as singularities, then the singular locus consists of at least $(\deg(X) - 1)^2$ nodes.

Another family of hypersurfaces of defect in $\mathbb{P}^n$ is given by cones over smooth hypersurfaces in $\mathbb{P}^{n-1}$, see Corollary 1.3.6 and the following examples. The vertex of the cone is a singularity with big Milnor number.

The aim of this chapter is to generalize this philosophy to arbitrary projective hypersurfaces over arbitrary fields. At first, several features of cohomology are studied and applied to projective hypersurfaces in §1.2. The main result of §1.3 is the following:

Theorem 1.1.1. *Let k be a field of characteristic zero. Suppose that $X \subseteq \mathbb{P}^n_k$, $n \geq 3$, is a hypersurface with defect in one of the cohomology theories given in §1.2.1. Denote by $\tau(X)$ the global Tjurina number of X. Then*

$$\tau(X) \geq \frac{\deg(X) - n + 1}{n^2 + n + 1}.$$

Moreover, if X has at most weighted homogeneous singularities, then

$$\tau(X) \geq \deg(X) - n + 1.$$

Of course, $\tau(X)$ will only be finite if X has at most isolated singularities. The main ingredient in the proof is a close inspection of the algebraic de Rham cohomology of hypersurface complements in the spirit of Griffiths [32] and Dimca [23].

The situation for positive characteristic fields is much more subtle. As explained in §1.3.5, there are some obstructions to extending the proof of Theorem 1.1.1. However, for hypersurfaces with mild singularities, a resolution of singularities approach motivated by [59] leads to:

Theorem 1.1.2. *Let k be an algebraically closed field of characteristic $\neq 2$. Let $X \subseteq \mathbb{P}^n_k$ be a hypersurface with defect in étale or rigid cohomology. Suppose further that X has a zero-dimensional singular locus $\Sigma = \Sigma_O \cup \Sigma_A$, where*

- *Σ_O is formed by $x \in \Sigma$ being ordinary multiple points of multiplicity m_x and*
- *Σ_A consists of $x \in \Sigma$ which are singular points of type A_{k_x}.*

Then

$$\sum_{x\in\Sigma_O} m_x + \sum_{x\in\Sigma_A} 2\left\lceil \frac{k_x}{2}\right\rceil \geq \deg(X).$$

The key is to show that the strict transform of a hypersurface with defect is a non-ample divisor in the total space of an embedded resolution. For details, see §1.4. We conjecture that the philospophy "defect implies many singularities" should extend to hypersurfaces with arbitrary isolated singularities in any positive characteristic.

As an application of Theorem 1.1.2 and the results of the second chapter, we prove in §2.2.6 a lower bound on the density of hypersurfaces without defect over a finite field:

Theorem 1.1.3 (Density of hypersurfaces with defect)**.** *Let q be an odd prime power. Then*

$$\lim_{d\to\infty} \frac{\#\{f\in k[x_0,\dots,x_n]_d \mid \{f=0\}\subseteq \mathbb{P}^n_{\mathbb{F}_q} \text{ has no defect}\}}{\#k[x_0,\dots,x_n]_d} \geq \frac{1}{\zeta_{\mathbb{P}^n}(n+3)} = \prod_{i=3}^{n+3}\left(1-q^{-i}\right).$$

In view of Theorems 1.1.1 and 2.1.7, it seems plausible that this limit is actually 1.

1.2 Cohomological preliminaries

1.2.1 Cohomology theories and Betti numbers

Let k be a field of characteristic $p \geq 0$. Depending on p, various cohomology theories for a separated scheme X of finite type over k are at disposal:

- ($p = 0$) algebraic de Rham cohomology $H^\bullet_{\mathrm{dR}}(X)$ in the sense of Hartshorne [37],
- ($p = 0$) algebraic Kähler-de Rham cohomology $H^\bullet_{\mathrm{KdR}}(X)$, i.e., the hypercohomology of the algebraic de Rham complex of X, motivated by [1],
- ($p \geq 0$) étale cohomology $H^\bullet_{\text{ét}}(X, \mathbb{Q}_\ell)$, where ℓ is a prime not equal to p [2],
- ($p > 0$) rigid cohomology $H^\bullet_{\mathrm{rig}}(X/K)$, where k is assumed to be perfect and K denotes the fraction field of the ring of Witt vectors over k [48].

If k is of characteristic zero, then the Lefschetz principle allows to embed the field of definition of X into the complex numbers $\mathbb{C}$. Therefore, performing a base change and passing over to the analytic space underlying X, one may in this case also consider

- ($p = 0$) singular cohomology $H^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C})$,
- ($p = 0$) analytic de Rham cohomology $H^\bullet_{\mathrm{dR}}(X^{\mathrm{an}})$ in the sense of [37, Chapter IV].

The cohomology groups arising from these theories are all finite-dimensional vector spaces over a field of characteristic zero: The fields in question are $k, k, \mathbb{Q}_\ell, K, \mathbb{C}, \mathbb{C}$ in the order of appearance. As usual, the dimension of these spaces will be called *Betti numbers* and will be denoted by $h^\bullet(-)$ with a suitable subscript.

Remark. Although algebraic (Kähler-)de Rham cohomology is well-defined in any characteristic, it may lead to infinite Betti numbers for non-proper schemes if $p \geq 0$. For example, the 0-th cohomology group of the de Rham complex of the univariate polynomial ring $k[x]$ contains the 0-forms $x^p, x^{2p}, x^{3p}, \ldots$. These are closed, but not exact. In particular, $H^0_{\mathrm{dR}}(\mathbb{A}^1_k)$ is infinite-dimensional.

There are various comparison theorems relating the Betti numbers of the different cohomology theories:

Theorem 1.2.1 (Comparison of Betti numbers)**.** *Let X be a separated k-scheme of finite type.*

(1) *Assume $p = 0$. Then*

$$h^\bullet_{\mathrm{dR}}(X) = h^\bullet_{\mathrm{dR}}(X^{\mathrm{an}}) = h^\bullet_{\mathrm{sing}}(X^{\mathrm{an}}) = h^\bullet_{\text{ét}}(X).$$

If X is smooth, then additionally $h^\bullet_{\mathrm{dR}}(X) = h^\bullet_{\mathrm{KdR}}(X)$.

(2) *Assume $p = 0$. If X is complete or has at most isolated singularities, then*

$$h^\bullet_{\mathrm{dR}}(X) \leq h^\bullet_{\mathrm{KdR}}(X).$$

(3) *Assume $p > 0$. If X is smooth, proper and defined over a finite field, then*

$$h^\bullet_{\text{ét}}(X) = h^\bullet_{\mathrm{rig}}(X).$$

(4) *Assume $p > 0$. Let R be a mixed characteristic discrete valuation ring with residue field k. Denote by K the field of fractions of R. Let $\mathcal{X}$ be a smooth proper R-scheme with special fiber $\mathcal{X}_s/k$ and generic fiber $\mathcal{X}_\eta/K$. Then*

$$h^\bullet_{\mathrm{rig}}(\mathcal{X}_s) = h^\bullet_{\mathrm{dR}}(\mathcal{X}_\eta) = h^\bullet_{\text{ét}}(\mathcal{X}_\eta) = h^\bullet_{\text{ét}}(\mathcal{X}_s).$$

Moreover, this holds for open subschemes of $\mathcal{X}$ which are complements of a relative simple normal crossings divisor.

Proof. (1) As mentioned before, we can assume that X is a scheme over $\mathbb{C}$. The first two equalities on the left are due to Hartshorne [37, Theorem IV.1.1], building on work of Grothendieck [34, Theorem 1′] and the classical de Rham theorem in the smooth case. The equality on the right was proven by Artin [2, Exposé XVI, Théorème 4.1]. In the smooth case, computing algebraic de Rham cohomology using the closed embedding $X \hookrightarrow X$ gives an isomorphism to Kähler-de Rham cohomology.

(2) See [8, Corollary 3.14].

(3) This is a consequence of the Katz-Messing comparison theorem [41, Corollary 1], since both étale and rigid cohomology are Weil cohomology theories by results of Deligne [20] and Kedlaya [42], respectively.

(4) The left equality comes from the Baldassarri-Chiarellotto comparison theorem [4, Corollary 2.6]. The identity in the middle is between Betti numbers of schemes in characteristic 0 and hence follows by (1).

The statement on the étale Betti numbers is mainly a consequence of proper-smooth base change [2, Exposé XVI, Corollaire 2.2]. If $\mathcal{X}$ is the complement of a simple normal crossings divisor, the assertion follows applying the upcoming Gysin sequence (Lemma 1.2.8) and the Mayer-Vietoris spectral sequence (Lemma 1.2.12). □

1.2.2 Toolbox

Let k be a field of characteristic $p \geq 0$. Choose a suitable cohomology theory $H^\bullet(-)$ from the list given at the beginning of §1.2.1 and denote its field of coefficients by K. Then $H^\bullet(-)$ is a contravariant functor from separated finite type k-schemes to finite-dimensional K-vector spaces, the morphisms being closed immersions and linear maps, respectively.

In the following, more standard tools needed in the sequel are collected. Most proofs are omitted, they can be found in [1,3,37] for the de Rham theories, [2,53] for étale cohomology, and [6,7,48] for rigid cohomology. Some results are unproven for algebraic Kähler-de Rham cohomology or rigid cohomology, these will be marked by ~~KdR~~ or ~~rig~~, respectively.

Affine and projective space

Example 1.2.2 (Betti numbers of affine and projective space)**.** Let $n \geq 1$ be an integer. Then

$$h^i(\mathbb{A}^n_k) = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0 \end{cases} \quad \text{and} \quad h^i(\mathbb{P}^n_k) = \begin{cases} 1 & \text{if } i \in \{0, 2, \ldots, 2n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Variants of cohomology

Let X be a separated k-scheme of finite type and let $Z \subseteq X$ be a closed subscheme.

Lemma 1.2.3 (Local cohomology). *There is a theory of local cohomology $H^\bullet_Z(X)$ of X with support in Z. More precisely, this is a contravariant functor from closed embeddings $Z \subseteq X$ of separated k-schemes to finite-dimensional K-vector spaces.*

Furthermore, $H^i_X(X) = H^i(X)$, and the natural map $H^\bullet_Z(X) \to H^\bullet_Z(U)$ is an isomorphism for any open subscheme $U \subseteq X$ containing Z.

Lemma 1.2.4 (Compact supports, ~~KdR~~). *There is a theory of cohomology $H^\bullet_c(X)$ of X with compact supports. More precisely, this is a functor from separated k-schemes to K-vector spaces of finite dimension, which is contravariant for proper morphisms and covariant for open immersions.*

Furthermore, if X is proper, then there is a natural isomorphism $H^\bullet_c(X) \cong H^\bullet(X)$. The dimension of $H^{2\dim X}_c(X)$ counts the number of top-dimensional geometrically irreducible components of X.

Lemma 1.2.5 (Poincaré duality, ~~KdR~~). *If X is smooth and irreducible, then there is a perfect pairing $H^i_Z(X) \times H^{2\dim X - i}_c(Z) \to K$. This pairing is compatible with the excision and Gysin sequences mentioned below.*

Lemma 1.2.6 (Künneth formula, ~~KdR~~). *For separated k-schemes X and Y, there is an isomorphism $H^i_c(X \times_{\operatorname{Spec} k} Y) \cong H^i_c(X) \otimes_K H^i_c(Y)$.*

Long exact sequences

Lemma 1.2.7 (Excision sequence). *There is a long exact sequence*

$$\cdots \to H^i(X) \to H^i(X \setminus Z) \to H^{i+1}_Z(X) \to H^{i+1}(X) \to \dots$$

Lemma 1.2.8 (Gysin sequence, ~~KdR~~). *There is a long exact sequence*

$$\cdots \to H^i_c(X) \to H^i_c(Z) \to H^{i+1}_c(X \setminus Z) \to H^{i+1}_c(X) \to \dots$$

Lemma 1.2.9 (Smooth Gysin sequence). *If X and Z are both smooth, X is irreducible and Z is of codimension r, then there is a long exact sequence*

$$\cdots \to H^i(X) \to H^i(X \setminus Z) \xrightarrow{\rho} H^{i+1-2r}(Z) \to H^{i+1}(X) \to \dots$$

The map ρ is called Poincaré residue map.

Remark. This is the Poincaré dual of the Gysin sequence in Lemma 1.2.8.

Lemma 1.2.10 (Proper birational morphisms, ~~KdR~~). *Suppose that X is proper over k. Further let $\pi : Y \to X$ be a proper birational morphism such that its restriction $\pi|_{Y \setminus E} : Y \setminus E \to X \setminus Z$ is an isomorphism for a closed subscheme $E \subseteq Y$. Then there is a long exact sequence*

$$\cdots \to H^i(X) \to H^i(Y) \oplus H^i(Z) \to H^i(E) \to H^{i+1}(X) \to \dots$$

Proof. See also [37, Theorem II.4.4] for algebraic de Rham cohomology and [44, Proposition 2.3] for étale cohomology. Since cohomology with compact support is contravariant with respect to proper morphisms, π induces a commutative ladder

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & H^i_c(X\setminus Z) & \longrightarrow & H^i_c(X) & \xrightarrow{\beta} & H^i_c(Z) & \longrightarrow & H^{i+1}_c(X\setminus Z) & \longrightarrow & \dots \\ & & \| & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} & & \| & & \\ \dots & \longrightarrow & H^i_c(Y\setminus E) & \longrightarrow & H^i_c(Y) & \xrightarrow{\delta} & H^i_c(E) & \longrightarrow & H^{i+1}_c(Y\setminus E) & \longrightarrow & \dots \end{array}$$

By diagram chasing, this yields a long exact sequence

$$\dots \to H^i_c(X) \xrightarrow{(\alpha,\beta)} H^i_c(Y)\oplus H^i_c(Z) \xrightarrow{\gamma-\delta} H^i_c(E) \to H^{i+1}_c(X) \to \dots$$

Since X, Y, Z, E are all proper, the compact support may be omitted. □

In order to keep the proof of the subsequent lemma short, we make the following technical definition:

Definition 1.2.11. A proper k-scheme X is *embeddable with respect to* $H^\bullet(-)$ if

- $H^\bullet(-) = H^\bullet_{\mathrm{dR}}(-)$ or $H^\bullet_{\mathrm{dR}}(-^{\mathrm{an}})$ and X admits a closed embedding into a smooth proper k-scheme Y, or
- $H^\bullet(-) = H^\bullet_{\mathrm{rig}}(-)$ and X admits a closed embedding into the closed fiber of a smooth formal $W(k)$-scheme $\mathscr{P}$, or
- $H^\bullet(-) = H^\bullet_{\mathrm{KdR}}(-)$, $H^\bullet_{\text{ét}}(-)$ or $H^\bullet_{\mathrm{sing}}(-^{\mathrm{an}})$.

Lemma 1.2.12 (Mayer-Vietoris for closed coverings)**.** *Let $X_1,\dots,X_r$ be proper separated k-schemes and set $X := X_1\cup\dots\cup X_r$. Assume that X is embeddable with respect to $H^\bullet(-)$. Then there is a spectral sequence*

$$E_1^{p,q} := \bigoplus_{1\le i_0<\dots<i_p\le r} H^q(X_{i_0}\cap\dots\cap X_{i_p}) \quad\Rightarrow\quad H^{p+q}(X).$$

In particular, if the triple intersections $X_j\cap X_k\cap X_\ell$ are empty for pairwise distinct j,k,ℓ, then there is a long exact sequence

$$\dots\to H^i(X)\to\bigoplus_{j=1}^r H^i(X_j)\to\bigoplus_{1\le j<k\le r} H^i(X_j\cap X_k)\to H^{i+1}(X)\to\dots$$

Proof. In the algebraic de Rham case, let $X\hookrightarrow Y$ be a closed immersion into a smooth proper k-scheme Y. Then there is an exact sequence of formally completed de Rham complexes

$$0\to\Omega^\bullet_Y/_X\to\bigoplus_{i=1}^r\Omega^\bullet_Y/_{X_i}\to\bigoplus_{1\le i_0<i_1\le r}\Omega^\bullet_Y/_{X_{i_0}\cap X_{i_1}}\to\dots\to\bigoplus_{1\le i_0<\dots<i_p\le r}\Omega^\bullet_Y/_{X_{i_0}\cap\dots\cap X_{i_p}}\to 0,$$

compare [37, Proposition II.4.1]. The Mayer-Vietoris spectral sequence now evolves as the standard E_1-spectral sequence for hypercohomology. Note that if the triple intersections are empty, the above sequence of formally completed de Rham complexes is short exact, and it remains to take the corresponding long exact sequence in hypercohomology.

The proof for analytic de Rham cohomology is analogous. For Kähler-de Rham cohomology, the proof is even simpler, as we can stick to the usual de Rham complexes. For rigid cohomology, a similar technique works: Embed X into the closed fiber of a smooth formal scheme $\mathscr{P}$. Apply hypercohomology to the exact sequence

$$0 \to \Omega^\bullet_{]X[_\mathscr{P}} \to \bigoplus_{i=1}^{r} \Omega^\bullet_{]X_i[_\mathscr{P}} \to \cdots \to \bigoplus_{1\leq i_0<\cdots<i_p\leq r} \Omega^\bullet_{]X_{i_0}\cap\cdots\cap X_{i_p}[_\mathscr{P}} \to 0.$$

For étale cohomology, let ι_I denote the inclusion of X_I into X for a subset $I \subseteq \{1,\ldots,r\}$ and apply $H^\bullet_{\text{ét}}(-)$ to

$$0 \to \mathbb{Q}_\ell \to \bigoplus_{j=1}^{r} \iota_{\{i\}_*}\mathbb{Q}_\ell \to \cdots \to \bigoplus_{1\leq i_0<\cdots<i_p\leq r} \iota_{\{i_0,\ldots,i_p\}_*}\mathbb{Q}_\ell \to 0.$$

This works as well for singular cohomology, taking the constant sheaves $\mathbb{C}_{X_I}$ instead of $\mathbb{Q}_\ell$. □

Cohomological dimension

Lemma 1.2.13 (Dimension vanishing). *$H^i_Z(X) = 0$ and $H^i_c(X) = 0$ for $i < 0$ and $i > 2\dim X$.*

A rather subtle topic is the vanishing for affine schemes:

Lemma 1.2.14 (Cohomological dimension of affines). *Suppose that X is an affine scheme.*

(1) (~~KdR~~, ~~rig~~) *$H^i(X) = 0$ for $i > \dim X$.*

(2) *If X is smooth, then $H^i(X) = 0$ for $i > \dim X$.*

(3) *Assume $p = 0$. If $X \subseteq \mathbb{A}^n$ is a hypersurface, then $H^i_{\text{KdR}}(X) = 0$ for $i \geq n$.*

(4) *If $X \subseteq \mathbb{A}^n$ is a hypersurface defined by a weighted homogeneous polynomial, then $H^i(X) = 0$ for $i \geq 1$.*

(5) *Assume $p > 0$. If $X \subseteq \mathbb{A}^n$ is a hypersurface with at most isolated weighted homogeneous singularities, then $H^i_{\text{rig}}(X) = 0$ for $n+1 \leq i \leq 2n-4$ and $h^{2n-3}_{\text{rig}}(X) = h^{2n-2}_{\text{rig}}(X)$.*

Proof. (1) In the analytic world, this follows by [8, Corollary 3.15]. Using the comparison of Theorem 1.2.1 (1), this carries over to algebraic de Rham and étale cohomology in characteristic 0. The statement for étale cohomology in positive characteristic is proven in [2, Exposé XIV, Corollaire 3.2].

(2) For Kähler-de Rham cohomology, this follows from (1). The statement for rigid cohomology follows since $H^i(X)$ is expressible as the cohomology of a complex of length $\dim X$ in the framework of Monsky-Washnitzer cohomology (see [7, Proposition 1.10]).

(3) Denote by R the coordinate ring of $\mathbb{A}^n_k$ and suppose that the hypersurface X is defined by $f \in R$. Consider the natural surjection
$$\Omega^\bullet_R \to \Omega^\bullet_R \otimes R/(f) \to \Omega^\bullet_{R/(f)}.$$
This is compatible with the exterior derivative d and gives thus a short exact sequence
$$0 \to \mathcal{K}^\bullet \to \Omega^\bullet_R \to \Omega^\bullet_{R/(f)} \to 0$$
of complexes. This yields in turn a long exact sequence in cohomology, which reads
$$\dots \to H^i(\mathcal{K}^\bullet) \to H^i_{\mathrm{dR}}(\mathbb{A}^n) \to H^i_{\mathrm{KdR}}(X) \to H^{i+1}(\mathcal{K}^\bullet) \to \dots$$
Since $\Omega^i_R = 0$ for $i > n$, we have $\mathcal{K}^i = 0$ and $\Omega^i_{R/(f)} = 0$ and thus $H^i_{\mathrm{KdR}}(X) = 0$ for $i > n$. Moreover, $H^n(\mathbb{A}^n) = 0$ and $\mathcal{K}^{n+1} = 0$ imply $H^n_{\mathrm{KdR}}(X) = 0$.

(4) In characteristic zero, this follows since cones are contractible. For $p > 0$, there is a nice proof using alterations in [57, Proposition 3.2.3].

(5) Let Σ denote the singular locus of X. By Poincaré duality (Lemma 1.2.5) on $\mathbb{A}^n \setminus \Sigma$, $h^i_{\mathrm{rig}}(\mathbb{A}^n \setminus \Sigma) = h^{2n-i}_{c,\mathrm{rig}}(\mathbb{A}^n \setminus \Sigma)$. The Gysin sequence (Lemma 1.2.8) for $\Sigma \subseteq \mathbb{A}^n$ reads
$$\dots \to H^{2n-i-1}_{c,\mathrm{rig}}(\Sigma) \to H^{2n-i}_{c,\mathrm{rig}}(\mathbb{A}^n \setminus \Sigma) \to H^{2n-i}_{c,\mathrm{rig}}(\mathbb{A}^n) \to H^{2n-i}_{c,\mathrm{rig}}(\Sigma) \to \dots$$
and using dimension vanishing (Lemma 1.2.13) on Σ and Poincaré duality on $\mathbb{A}^n$ we conclude
$$H^i_{\mathrm{rig}}(\mathbb{A}^n \setminus \Sigma) = 0, \quad 1 \le i \le 2n-2.$$
Moreover, we have $h^0_{c,\mathrm{rig}}(\mathbb{A}^n \setminus \Sigma) \le h^0_{c,\mathrm{rig}}(\mathbb{A}^n) = 0$ and hence
$$h^1_{c,\mathrm{rig}}(\mathbb{A}^n \setminus \Sigma) = h^0_{c,\mathrm{rig}}(\Sigma) = \#\Sigma.$$
Now consider the excision long exact sequence (Lemma 1.2.7) for the closed immersion $\Sigma \subseteq X$,
$$\dots \to H^i_{\Sigma,\mathrm{rig}}(X) \to H^i_{\mathrm{rig}}(X) \to H^i_{\mathrm{rig}}(X \setminus \Sigma) \to H^{i+1}_{\Sigma,\mathrm{rig}}(X) \to \dots$$
By the smooth Gysin sequence (Lemma 1.2.9) for $X \setminus \Sigma \subseteq \mathbb{A}^n \setminus \Sigma$,
$$\dots \to H^{i+1}_{\mathrm{rig}}(\mathbb{A}^n \setminus X) \to H^i_{\mathrm{rig}}(X \setminus \Sigma) \to H^{i+2}_{\mathrm{rig}}(\mathbb{A}^n \setminus \Sigma) \to H^{i+2}_{\mathrm{rig}}(\mathbb{A}^n \setminus X) \to \dots,$$
we have thus $H^i_{\mathrm{rig}}(X \setminus \Sigma) \cong H^{i+2}_{\mathrm{rig}}(\mathbb{A}^n \setminus \Sigma)$ for $i \ge n$, because $\mathbb{A}^n \setminus X$ is smooth and affine. Thus $H^i_{\mathrm{rig}}(X \setminus \Sigma) = 0$ for $n \le i \le 2n-4$, which implies in turn
$$H^i_{\mathrm{rig}}(X) \cong H^i_{\Sigma,\mathrm{rig}}(X), \quad n+1 \le i \le 2n-4.$$
There is an isomorphism $H^i_{\Sigma,\mathrm{rig}}(X) \cong \bigoplus_{x \in \Sigma} H^i_{\{x\},\mathrm{rig}}(X)$, see [7, Proposition 2.4 (ii)]. Since the local cohomology $H^i_{\{x\},\mathrm{rig}}(X)$ depends only on the contact equivalence class of (X, x) by [58, §1.2], it suffices to determine $H^i_{\{0\},\mathrm{rig}}(Z)$, where $Z \subseteq \mathbb{A}^n$ is defined by a weighted homogeneous polynomial. However, $H^i_{\{0\},\mathrm{rig}}(Z) = 0$ by [57, Corollary 3.2.6] for $n+1 \le i \le 2n-3$. This shows $H^i_{\mathrm{rig}}(X) = 0$ for $n+1 \le i \le 2n-4$.

For the two top degrees, there is a long exact sequence
$$0 \to H^{2n-3}_{\mathrm{rig}}(X) \to H^1_{c,\mathrm{rig}}(\mathbb{A}^n \setminus \Sigma) \to H^{2n-2}_{\Sigma,\mathrm{rig}}(X) \to H^{2n-2}_{\mathrm{rig}}(X) \to 0.$$
By [57, Proposition 3.2.1], $h^{2n-2}_{\Sigma,\mathrm{rig}}(X) = \#\Sigma$, so $h^{2n-3}_{\mathrm{rig}}(X) = h^{2n-2}_{\mathrm{rig}}(X)$. □

The Lefschetz hyperplane theorem

Lemma 1.2.15 (Lefschetz hyperplane theorem, ~~KdR~~). *Let Z be an effective ample divisor on X. Suppose that $X \setminus Z$ is smooth and irreducible. Then the natural restriction $H^i(X) \to H^i(Z)$ is an isomorphism for $i \leq \dim X - 2$ and injective for $i \leq \dim X - 1$.*

Proof. Let $m > 0$ be an integer such that mZ is very ample. Then $|mZ|$ gives an embedding of X into some projective space $\mathbb{P}^n$. In particular, on the image of X, the divisor mZ is cut out by a hyperplane $H \subseteq \mathbb{P}^n$. It follows that $X \setminus Z = X \setminus mZ = X \setminus (X \cap H)$ is smooth, irreducible and affine. By Lemma 1.2.14 (2), $H^i(X \setminus Z) = 0$ for $i > \dim X$. Applying Poincaré duality (Lemma 1.2.5), $H^i_c(X \setminus Z) = 0$ for $i < \dim X$. The assertion follows by looking at the Gysin sequence (Lemma 1.2.8) for $Z \subseteq X$,

$$\cdots \to H^i_c(X \setminus Z) \to H^i_c(X) \to H^i_c(Z) \to H^{i+1}_c(X \setminus Z) \to \dots$$

Observe that $H^i_c(X) \cong H^i(X)$ and $H^i_c(Z) \cong H^i(Z)$ since X and Z are both projective. □

1.2.3 Projective hypersurfaces

Notation 1.2.16. Let k be a field of characteristic $p \geq 0$ and let $H^\bullet(-)$ denote one of the cohomology theories from §1.2.1. Fix an integer $n \geq 3$. Let $\overline{X} \subseteq \mathbb{P}^n_k$ be a projective hypersurface.

Smooth hypersurfaces

Lemma 1.2.17 (Betti numbers of smooth projective hypersurfaces). *Suppose that $\overline{X}$ is smooth of degree d. Then*

(1) $h^i(\overline{X}) = h^i(\mathbb{P}^n)$ *for* $i \notin \{n-1, 2n\}$,

(2) $h^{2n}(\overline{X}) = 0$,

(3) $h^{n-1}(\overline{X}) = \dfrac{(d-1)^{n+1} + (-1)^{n-1}(d-1)}{d} + \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$

Remark. The Betti numbers of a smooth hypersurface $\overline{X} \subseteq \mathbb{P}^n$ depend thus only on its degree and on its dimension.

Proof. (1) First, the Lefschetz hyperplane theorem states that $h^i(\overline{X}) = h^i(\mathbb{P}^n)$ for $i \leq n-2$. Secondly, Poincaré duality implies that $h^i(\overline{X}) = h^{2n-2-i}(\overline{X})$ for all i, and this shows $h^i(\overline{X}) = h^i(\mathbb{P}^n)$ for $n \leq i \leq 2n-2$. Moreover, $h^{2n-1}(\mathbb{P}^n) = 0$ by Example 1.2.2 and $h^{2n-1}(\overline{X}) = 0$ for dimension reasons (Lemma 1.2.13).

(2) Clear by Lemma 1.2.13.

(3) This formula is well-known in characteristic 0, see e.g., [25, Exercise 5.3.7]. For $p > 0$, note that a smooth hypersurface $\overline{X}$ over k always lifts to a smooth hypersurface over the quotient field K of the ring of Witt vectors $W(k)$. By Theorem 1.2.1 (4), the étale or rigid Betti numbers of $\overline{X}$ are given by the Betti numbers of the lift to characteristic 0. □

Example 1.2.18 (Smooth quadrics)**.** Suppose that $d = 2$, so that $\overline{X}$ is a smooth quadric in $\mathbb{P}^n$. Then Lemma 1.2.17 (3) implies that

$$h^{n-1}(\overline{X}) = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Singular hypersurfaces

However, for singular hypersurfaces, Poincaré duality may fail. This makes the situation more complicated:

Lemma 1.2.19 (Betti numbers of singular projective hypersurfaces)**.** *Suppose that $\overline{X}$ is singular and let Σ denote the singular locus of $\overline{X}$. Then*

(1) (~~KdR~~) $h^i(\overline{X}) = h^i(\mathbb{P}^n)$ *for* $i \leq n-2$*,*

(2) (~~KdR~~) *If* $p = 0$*, then* $h^i(\overline{X}) = h^i(\mathbb{P}^n)$ *for* $n + \dim\Sigma + 1 \leq i \leq 2n-1$*,*

(3) (~~rig~~) *If* $\dim\Sigma = 0$*, then* $h^i(\overline{X}) = h^i(\mathbb{P}^n)$ *for* $n+1 \leq i \leq 2n-1$*.*

(4) *If* $\dim\Sigma = 0$ *and* $\overline{X}$ *has only weighted homogeneous singularities, then* $h^i_{\mathrm{rig}}(\overline{X}) = h^i_{\mathrm{rig}}(\mathbb{P}^n)$ *for* $n+1 \leq i \leq 2n-2$*.*

(5) $h^{2n}(\overline{X}) = 0$.

Proof. (1) Again, this follows from the Lefschetz hyperplane theorem (Lemma 1.2.15).

(2) Let $M := \{f = 1\} \subseteq \mathbb{A}^{n+1}$, where f is a defining polynomial of degree d for $\overline{X}$. M is a global Milnor fiber of $\overline{X}$. Since $H^i(M) = 0$ for $i \leq n - \dim\Sigma - 1$ by [40], see also [25, Corollary 6.2.22], it follows that $H^i(\mathbb{P}^n \setminus \overline{X}) = 0$ for $i \leq n - \dim\Sigma - 1$, as $\mathbb{P}^n \setminus \overline{X}$ is the quotient of M by an action of the d-th roots of unity. The claim follows now by the same Gysin sequence as in the proof of Lemma 1.2.15.

(3) By Bertini's theorem [38, Theorem II.8.18], after possibly extending the base field, there is a hyperplane $H \subseteq \mathbb{P}^n$ such that $\Sigma \cap H = \emptyset$ and $\overline{Y} := \overline{X} \cap H$ is a smooth hypersurface in $H \cong \mathbb{P}^{n-1}$. In particular, by Lemma 1.2.17 and Example 1.2.2,

$$h^i(\overline{Y}) = h^i(\mathbb{P}^{n-1}) = h^i(\mathbb{P}^n), \quad i \leq n-3.$$

Let $X := \overline{X} \setminus \overline{Y}$. This is a singular hypersurface in $\mathbb{A}^n$, so $H^i(X) = 0$ for $i \geq n$ by Lemma 1.2.14 (1) and (3). Using the excision long exact sequence for $\overline{Y} \subseteq \overline{X}$,

$$\cdots \to H^{i-1}(X) \to H^i_{\overline{Y}}(\overline{X}) \to H^i(\overline{X}) \to H^i(X) \to \ldots,$$

we obtain

$$H^i_{\overline{Y}}(\overline{X}) \cong H^i(\overline{X}), \quad i \geq n+1.$$

Since $\overline{Y}$ is a smooth closed subscheme of $\overline{X} \setminus \Sigma$, by the properties of local cohomology

$$h^i_{\overline{Y}}(\overline{X}) = h^i_{\overline{Y}}(\overline{X} \setminus \Sigma).$$

Suppose for the moment that the cohomology theory under consideration is not Kähler-de Rham cohomology. Then, using Poincaré duality (Lemma 1.2.5) on $\overline{X} \setminus \Sigma$,

$$h^i_{\overline{Y}}(\overline{X} \setminus \Sigma) = h^{2n-2-i}(\overline{Y})$$

and hence

$$h^i(\overline{X}) = h^{2n-2-i}(\overline{Y}) = h^{2n-2-i}(\mathbb{P}^n) = h^i(\mathbb{P}^n), \quad i \geq n+1.$$

In the case of Kähler-de Rham cohomology, it is sufficient to show

$$h^i_{\overline{Y},\mathrm{KdR}}(\overline{X} \setminus \Sigma) = h^i_{\overline{Y},\mathrm{dR}}(\overline{X} \setminus \Sigma), \quad i \geq n+1.$$

Note that $H^i_{\overline{Y}}(\overline{X} \setminus \Sigma)$ fits into a long exact excision sequence

$$\cdots \to H^{i-1}(\overline{X} \setminus \Sigma) \to H^{i-1}(X \setminus \Sigma) \to H^i_{\overline{Y}}(\overline{X} \setminus \Sigma) \to H^i(\overline{X} \setminus \Sigma) \to H^i(X \setminus \Sigma) \to \dots$$

in both Kähler-de Rham and algebraic de Rham cohomology. In particular, since $\overline{X} \setminus \Sigma$ and $X \setminus \Sigma$ are smooth,

$$\begin{aligned} &h^i_{\overline{Y},\mathrm{KdR}}(\overline{X} \setminus \Sigma) \\ &= \dim\ker\big(H^i(\overline{X} \setminus \Sigma) \to H^i(X \setminus \Sigma)\big) + \dim\operatorname{coker}\big(H^{i-1}(\overline{X} \setminus \Sigma) \to H^{i-1}(X \setminus \Sigma)\big) \\ &= h^i_{\overline{Y},\mathrm{dR}}(\overline{X} \setminus \Sigma), \quad i \geq n+1. \end{aligned}$$

(4) Let X and $\overline{Y}$ be as in the proof of (3). The cohomology of X is computed in Lemma 1.2.14 (5): $H^i_{\mathrm{rig}}(X) = 0$ for $n+1 \leq i \leq 2n-4$. With the long exact sequence

$$\cdots \to H^{i-1}_{\mathrm{rig}}(X) \to H^i_{\overline{Y},\mathrm{rig}}(\overline{X}) \to H^i_{\mathrm{rig}}(\overline{X}) \to H^i_{\mathrm{rig}}(X) \to \dots,$$

we obtain the identity $h^i_{\mathrm{rig}}(\overline{X}) = h^i_{\mathrm{rig}}(\mathbb{P}^n)$ for $n+2 \leq i \leq 2n-4$ as in (3). For the degrees $2n-3$ and $2n-4$, we obtain an exact seqeunce

$$\begin{aligned} 0 &\to H^{2n-3}_{\overline{Y},\mathrm{rig}}(\overline{X}) \to H^{2n-3}_{\mathrm{rig}}(\overline{X}) \to H^{2n-3}_{\mathrm{rig}}(X) \\ &\to H^{2n-2}_{\overline{Y},\mathrm{rig}}(\overline{X}) \to H^{2n-2}_{\mathrm{rig}}(\overline{X}) \to H^{2n-2}_{\mathrm{rig}}(X) \to H^{2n-1}_{\overline{Y},\mathrm{rig}}(\overline{X}). \end{aligned}$$

By the same trick as above, $h^i_{\overline{Y},\mathrm{rig}}(\overline{X}) = h^i(\mathbb{P}^n)$ for $i \geq 2n-3$. This yields the identity

$$h^{2n-3}_{\mathrm{rig}}(\overline{X}) - h^{2n-2}_{\mathrm{rig}}(\overline{X}) = h^{2n-3}_{\mathrm{rig}}(X) - h^{2n-2}_{\mathrm{rig}}(X) - 1 = -1$$

by Lemma 1.2.14 (5). Since $\overline{X}$ has only isolated singularities, it is geometrically irreducible and hence $h^{2n-2}_{\mathrm{rig}}(\overline{X}) = 1$ and consequently $h^{2n-3}_{\mathrm{rig}}(\overline{X}) = 1$.

For degree $n+1$, as in the proof of Lemma 1.2.14 (5), $H^{n+1}_{\Sigma,\mathrm{rig}}(\overline{X}) = H^{n+2}_{\Sigma,\mathrm{rig}}(\overline{X}) = 0$, so that

$$h^{n+1}_{\mathrm{rig}}(\overline{X}) = h^{n+1}_{\mathrm{rig}}(\overline{X} \setminus \Sigma) = h^{n-3}_{c,\mathrm{rig}}(\overline{X} \setminus \Sigma) = h^{n-3}_{c,\mathrm{rig}}(\mathbb{P}^n \setminus \Sigma) = h^{n-3}_{c,\mathrm{rig}}(\mathbb{P}^n) = h^{n+1}_{\mathrm{rig}}(\mathbb{P}^n),$$

using Poincaré duality, two Gysin sequences and $H^i_{c,\mathrm{rig}}(\mathbb{P}^n \setminus \overline{X}) = 0$ for $i < n$.

(5) This is due to dimension reasons (Lemma 1.2.13). □

Remark (~~KdR~~). If $\overline{X}$ has at most isolated singularities and $p = 0$, then the two interesting Betti numbers $h^{n-1}(\overline{X})$ and $h^n(\overline{X})$ are related by the formula

$$h^n(\overline{X}) - h^{n-1}(\overline{X}) = h^n(\mathbb{P}^n) - h^{n-1}(\overline{S}) + \mu,$$

where $\overline{S} \subseteq \mathbb{P}^n$ is a smooth hypersurface of the same degree as $\overline{X}$ and μ denotes the global Milnor number of $\overline{X}$ [22, Corollary 2.3].

1.2.4 Defect

We still follow Notation 1.2.16. Lemma 1.2.19 motivates the central definition of this chapter:

Definition 1.2.20 (Qualitative definition of defect)**.** $\overline{X}$ has *defect* if $h^i(\overline{X}) \neq h^i(\mathbb{P}^n)$ for some $i \in \{n, n+1, \ldots, 2n-2\}$.

Remarks.

- A hypersurface $\overline{X}$ with defect has to be singular by Lemma 1.2.17.
- If $\overline{X}$ has at most isolated singularities and if $H^\bullet(-)$ is not rigid cohomology, then

  $$X \text{ has defect} \quad \Leftrightarrow \quad h^n(X) \neq h^n(\mathbb{P}^n)$$

  by Lemma 1.2.19.

Example 1.2.21. Suppose that $\overline{X}$ is reducible. Since $h^{2n-2}(\mathbb{P}^n) = 1$, but $h^{2n-2}(\overline{X})$ counts the number of irreducible components, $\overline{X}$ has defect.

Example 1.2.22 (Schoen's quintic [63])**.** Let k be a field of characteristic 0 and let $\overline{X} \subseteq \mathbb{P}^4_k$ be the quintic hypersurface defined by $F \in k[x_0, \ldots, x_n]$, where

$$F = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5x_0x_1x_2x_3x_4.$$

Then $\overline{X}$ has precisely 125 ordinary double points as singularities, namely at

$$(\xi_0 : \xi_1 : \xi_2 : \xi_3 : \xi_4), \quad \text{where} \quad \xi_0\xi_1\xi_2\xi_3\xi_4 = 1 \quad \text{and} \quad \xi_i^5 = 1, \quad i = 0, \ldots, 4.$$

We will see in Example 1.3.18 that $h^4(\overline{X}) = 25$, so $\overline{X}$ has defect.

Further examples of hypersurfaces with defect will be given in the subsequent sections. These include:

- Projective cones over certain smooth projective hypersurfaces, see Corollary 1.3.6 and the succeeding examples.
- Non-factorial hypersurfaces with isolated singularities, see §1.5.

The upcoming section §1.3 deals with defect of hypersurfaces with isolated singularities in characteristic 0. In §1.4, for hypersurfaces with certain singularity types, defect is connected with Betti numbers of the strict transform in a resolution of singularities.

1.3 Defect in characteristic zero

Notation 1.3.1. Let k be a field of characteristic 0. Choose a suitable cohomology theory from the list in §1.2.1. Fix an integer $n \geq 3$. Let $\overline{X} \subseteq \mathbb{P}^n_k$ be a hypersurface with singular locus Σ. Assume that $\dim \Sigma = 0$. Let $F \in k[x_0, \dots, x_n]_d$ be a homogeneous polynomial of degree d defining $\overline{X}$. As in the proof of Lemma 1.2.19 (3), after possibly extending the base field and changing coordinates, we can assume that $\Sigma \subseteq \{x_0 \neq 0\}$ and that $\overline{Y} := \overline{X} \cap \{x_0 = 0\}$ is a smooth hypersurface in $\mathbb{P}^{n-1}_k$. Let $f := F(1, x_1, \dots, x_n)$ and denote the corresponding hypersurface in $\mathbb{A}^n_k$ by X.

In view of Lemma 1.2.19, we make the following definition:

Definition 1.3.2 (Quantitative definition of defect)**.** The *defect of* $\overline{X}$ is defined as

$$\delta(\overline{X}) := h^n(\overline{X}) - h^n(\mathbb{P}^n).$$

When indicating a specific cohomology theory, $\delta(\overline{X})$ carries the corresponding subscript. Note that Theorem 1.2.1 (1) implies $\delta_{\mathrm{dR}}(\overline{X}) = \delta_{\mathrm{dR}}(\overline{X}^{\mathrm{an}}) = \delta_{\mathrm{sing}}(\overline{X}^{\mathrm{an}}) = \delta_{\text{ét}}(\overline{X})$. A priori, $\delta_{\mathrm{dR}}(\overline{X})$ might not coincide with $\delta_{\mathrm{KdR}}(\overline{X})$. However, $\delta_{\mathrm{dR}}(\overline{X}) \leq \delta_{\mathrm{KdR}}(\overline{X})$ by Theorem 1.2.1 (2). As a consequence of Corollary 1.3.5, in fact $\delta_{\mathrm{dR}}(\overline{X}) = \delta_{\mathrm{KdR}}(\overline{X})$ holds.

Remarks. More rather elementary remarks:

- By Lemma 1.3.3, the defect of $\overline{X}$ is always non-negative.
- Defect depends only on the Betti numbers. Since our cohomology theories involved are compatible with field extensions, extending the base field k does no harm.
- In the literature on hypersurfaces with defect (e.g., [23, 25, 61, 66]), the notion "defect" is employed in the sense of defect of certain linear systems. However, as a consequence of the relation between defect and the de Rham cohomology of hypersurface complements (see Lemma 1.3.9), our definition agrees with the standard one as given in [23, p. 292]. In the case of nodal hypersurfaces, a linear system with defect will be described explicitly in the end of §1.3.4.
- The number $\delta(\overline{X})$ is the dimension of the n-th primitive cohomology of $\overline{X}$ [23, p. 291].

1.3.1 Defect and cokernels

Keep Notation 1.3.1.

Lemma 1.3.3. *Consider the long exact excision sequence*

$$\dots \to H^{n-1}(\overline{X} \setminus \Sigma) \xrightarrow{\alpha} H^n_\Sigma(\overline{X}) \to H^n(\overline{X}) \to H^n(\overline{X} \setminus \Sigma) \to H^{n+1}_\Sigma(\overline{X}) \to \dots$$

for $\Sigma \subseteq \overline{X}$*. Then* $\delta(\overline{X}) = \dim \operatorname{coker} \alpha$*.*

Remark. In particular, hypersurfaces with $H^n_\Sigma(\overline{X}) = 0$, e.g., smooth hypersurfaces, have no defect.

Proof. The proof consists of a few technical computations.

First assume that $n \geq 4$. The variety $\overline{X} \setminus \Sigma$ is a smooth closed subvariety of codimension one in $\mathbb{P}^n \setminus \Sigma$. The corresponding smooth Gysin sequence is

$$\cdots \to H^{n+1}(\mathbb{P}^n \setminus \overline{X}) \to H^n(\overline{X} \setminus \Sigma) \to H^{n+2}(\mathbb{P}^n \setminus \Sigma) \to H^{n+2}(\mathbb{P}^n \setminus \overline{X}) \to \dots$$

Since $\mathbb{P}^n \setminus \overline{X}$ is smooth and affine of dimension n, applying Lemma 1.2.14 (2) gives an isomorphism $H^n(\overline{X} \setminus \Sigma) \cong H^{n+2}(\mathbb{P}^n \setminus \Sigma)$.

The singular locus Σ is a closed subvariety of codimension n in $\mathbb{P}^n$. The associated Gysin sequence in algebraic de Rham cohomology reads

$$\cdots \to H^{n-2}_{\mathrm{dR}}(\Sigma) \to H^{n-2}_{\mathrm{dR}}(\mathbb{P}^n) \to H^{n-2}_{c,\mathrm{dR}}(\mathbb{P}^n \setminus \Sigma) \to H^{n-3}_{\mathrm{dR}}(\Sigma) \to \dots$$

Since $n \geq 4$, we can use $\dim \Sigma = 0$, Theorem 1.2.1 (1) and Poincaré duality to obtain

$$h^{n+2}(\mathbb{P}^n \setminus \Sigma) = h^{n+2}_{\mathrm{dR}}(\mathbb{P}^n \setminus \Sigma) = h^{n-2}_{c,\mathrm{dR}}(\mathbb{P}^n \setminus \Sigma) = h^{n-2}_{\mathrm{dR}}(\mathbb{P}^n) = h^{n-2}(\mathbb{P}^n).$$

Using the description of the Betti numbers of projective space (Example 1.2.2),

$$h^n(\overline{X} \setminus \Sigma) = h^{n+2}(\mathbb{P}^n \setminus \Sigma) = h^{n-2}(\mathbb{P}^n) = h^n(\mathbb{P}^n).$$

Since Σ lies inside the affine part $X \subseteq \overline{X}$, $H^{n+1}_\Sigma(X) = H^{n+1}_\Sigma(\overline{X})$ by the properties of local cohomology. Using the two Gysin sequences for $X \setminus \Sigma \subseteq \mathbb{A}^n \setminus \Sigma$ and $\Sigma \subseteq \mathbb{A}^n$ gives

$$h^n(X \setminus \Sigma) = h^{n+2}(\mathbb{A}^n \setminus \Sigma) = h^{n-2}(\mathbb{A}^n) = 0.$$

On the other hand, $H^{n+1}(X) = 0$ since X is an affine hypersurface of dimension $n-1$ (Lemma 1.2.14). The excision sequence for $\Sigma \subseteq X$ then yields $H^{n+1}_\Sigma(X) = 0$. Putting this together,

$$h^n(\overline{X}) = \dim \operatorname{coker} \alpha + h^n(\overline{X} \setminus \Sigma) = \dim \operatorname{coker} \alpha + h^n(\mathbb{P}^n).$$

This proves the assertion for $n \geq 4$.

In the case $n = 3$, the long exact sequence in the statement of the lemma gives

$$h^3(\overline{X}) = \dim \operatorname{coker} \alpha + h^3(\overline{X} \setminus \Sigma) - h^4_\Sigma(\overline{X}) + h^4(\overline{X}) - h^4(\overline{X} \setminus \Sigma),$$

where we used $H^5_\Sigma(\overline{X}) = 0$ (Lemma 1.2.13). Using Poincaré duality on $\overline{X} \setminus \Sigma$ and the compact support Gysin sequence

$$0 \to H^0_{c,\mathrm{dR}}(\overline{X} \setminus \Sigma) \to H^0_{\mathrm{dR}}(\overline{X}) \to H^0_{\mathrm{dR}}(\Sigma) \to H^1_{c,\mathrm{dR}}(\overline{X} \setminus \Sigma) \to H^1_{\mathrm{dR}}(X) = 0$$

we obtain

$$h^3(\overline{X} \setminus \Sigma) - h^4(\overline{X} \setminus \Sigma) = h^1_{c,\mathrm{dR}}(\overline{X} \setminus \Sigma) - h^0_{c,\mathrm{dR}}(\overline{X} \setminus \Sigma) = h^0_{\mathrm{dR}}(\Sigma) - h^0_{\mathrm{dR}}(\overline{X}).$$

Note that $h^0_{\mathrm{dR}}(\overline{X}) = 1$ and $h^4(\overline{X}) = 1$ by Lemma 1.2.19. Thus

$$h^3(\overline{X}) = \dim \operatorname{coker} \alpha + h^0_{\mathrm{dR}}(\Sigma) - h^4_\Sigma(\overline{X}).$$

Moreover,

$$h^4_\Sigma(\overline{X}) = h^4_\Sigma(X) = h^3(X \setminus \Sigma) = h^1_{c,\mathrm{dR}}(X \setminus \Sigma) = h^0_{\mathrm{dR}}(\Sigma),$$

using $h^4(X) = 0$ and the compact support Gysin sequences

$$\dots \to H^0_{c,\mathrm{dR}}(X) \to H^0_{\mathrm{dR}}(\Sigma) \to H^1_{c,\mathrm{dR}}(X \setminus \Sigma) \to H^1_{c,\mathrm{dR}}(X) \to \dots$$

and

$$0 = H^i_{c,\mathrm{dR}}(\mathbb{A}^n) \to H^i_{c,\mathrm{dR}}(X) \to H^{i+1}_{c,\mathrm{dR}}(\mathbb{A}^3 \setminus X) = 0, \quad i = 0, 1.$$

Consequently

$$\delta(\overline{X}) = h^3(\overline{X}) = \dim \operatorname{coker} \alpha. \qquad \square$$

The open immersion $X \hookrightarrow \overline{X}$ induces a commutative ladder

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & H^{n-1}(\overline{X}) & \longrightarrow & H^{n-1}(\overline{X} \setminus \Sigma) & \xrightarrow{\alpha} & H^n_\Sigma(\overline{X}) & \longrightarrow & H^n(\overline{X}) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & H^{n-1}(X) & \xrightarrow{\vartheta} & H^{n-1}(X \setminus \Sigma) & \longrightarrow & H^n_\Sigma(X) & \longrightarrow & H^n(X) & \longrightarrow & \dots \end{array}$$

of long exact sequences.

Lemma 1.3.4. *We have $\delta(\overline{X}) = \dim \operatorname{coker} \beta$, where*

$$\beta : H^{n-1}(\overline{X} \setminus \Sigma) \to H^{n-1}(X \setminus \Sigma)/\vartheta(H^{n-1}(X))$$

is the map induced by $X \setminus \Sigma \hookrightarrow \overline{X} \setminus \Sigma$.

Proof. As $H^n(X) = 0$ (Lemma 1.2.14), the natural map $H^{n-1}(X \setminus \Sigma) \to H^n_\Sigma(X)$ is surjective. Since its kernel is given by the image of ϑ,

$$H^{n-1}(X \setminus \Sigma)/\vartheta(H^{n-1}(X)) \to H^n_\Sigma(X)$$

is an isomorphism. The singular locus Σ lies inside the affine part X, so the natural map $H^n_\Sigma(\overline{X}) \to H^n_\Sigma(X)$ is an isomorphism as well. Therefore $\operatorname{coker} \beta \cong \operatorname{coker} \alpha$, which finishes the proof by the preceding Lemma 1.3.3. $\square$

We can now prove that the defect of $\overline{X}$ is the same number for algebraic de Rham and Kähler-de Rham cohomology:

Corollary 1.3.5. *Let $\overline{X} \subseteq \mathbb{P}^n$ be a projective hypersurface with at most isolated singularities. Then $\delta_{\mathrm{dR}}(\overline{X}) = \delta_{\mathrm{KdR}}(\overline{X})$. In particular $h^n_{\mathrm{dR}}(\overline{X}) = h^n_{\mathrm{KdR}}(\overline{X})$.*

Proof. By Theorem 1.2.1 (2), it remains to show the inequality $\delta_{\mathrm{dR}}(\overline{X}) \geq \delta_{\mathrm{KdR}}(\overline{X})$. To this end, note that the natural comparison map between algebraic and Kähler-de Rham cohomology yields a commutative diagram

$$\begin{array}{ccc} H^{n-1}_{\mathrm{dR}}(X) & \xrightarrow{\vartheta_{\mathrm{dR}}} & H^{n-1}_{\mathrm{dR}}(X \setminus \Sigma) \\ \downarrow & & \downarrow \simeq \\ H^{n-1}_{\mathrm{KdR}}(X) & \xrightarrow{\vartheta_{\mathrm{KdR}}} & H^{n-1}_{\mathrm{KdR}}(X \setminus \Sigma). \end{array}$$

This gives a surjection

$$H^{n-1}_{\mathrm{dR}}(X \setminus \Sigma)/\vartheta_{\mathrm{dR}}(H^{n-1}_{\mathrm{dR}}(X)) \twoheadrightarrow H^{n-1}_{\mathrm{KdR}}(X \setminus \Sigma)/\vartheta_{\mathrm{KdR}}(H^{n-1}_{\mathrm{KdR}}(X)).$$

If $\beta_{\mathrm{dR}}, \beta_{\mathrm{KdR}}$ denote the two versions of the map β of Lemma 1.3.4, then this gives rise to a surjection $\operatorname{coker} \beta_{\mathrm{dR}} \twoheadrightarrow \operatorname{coker} \beta_{\mathrm{KdR}}$. Hence

$$\delta_{\mathrm{dR}}(\overline{X}) = \dim \operatorname{coker} \beta_{\mathrm{dR}} \geq \dim \operatorname{coker} \beta_{\mathrm{KdR}} = \delta_{\mathrm{KdR}}(\overline{X}). \qquad \square$$

Defect of cones

Corollary 1.3.6. *Suppose that* $H^{n-1}(X) = 0$. *Then* $h^n(\overline{X}) = h^{n-2}(\overline{Y})$.

Proof. Following the proof of Lemma 1.2.19 (3), $H^{n-1}(X) = 0$ implies

$$h^n(\overline{X}) = h^n_{\overline{Y}}(\overline{X}) = h^{n-2}_{\mathrm{dR}}(\overline{Y}) = h^{n-2}(\mathbb{P}^n).$$ □

If X is defined by some weighted homogeneous polynomial, then $H^{n-1}(X) = 0$ in virtue of Lemma 1.2.14 (4). Thus Corollary 1.3.6 gives several examples of hypersurfaces with defect: In particular, if $\overline{X}$ is the cone over a smooth projective hypersurface $\overline{Y} \subseteq \mathbb{P}^{n-1}$, then

$$\delta(\overline{X}) = h^{n-2}(\overline{Y}) - h^n(\mathbb{P}^n),$$

and $h^{n-2}(\overline{Y})$ can be determined by means of Lemma 1.2.17 (3).

Example 1.3.7 (Cones over curves)**.** Let $\overline{X} \subseteq \mathbb{P}^3$ be the projective cone over a nonsingular plane curve $\overline{C} \subseteq \mathbb{P}^2$ of positive genus. Then $\overline{X}$ has defect, since $\delta(\overline{X}) = h^1(\overline{C}) > 0$.

Example 1.3.8 (Quadric cones)**.** Let $\overline{X} \subseteq \mathbb{P}^n$ be the projective cone over a smooth quadric hypersurface $\overline{Y} \subseteq \mathbb{P}^{n-1}$. Then

$$h^n(\overline{X}) = h^{n-2}(\overline{Y}) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

by Example 1.2.18. In particular, $\overline{X}$ has defect if and only if n is even.

Defect and cohomology of hypersurface complements

We finish this section with two more cohomological characterizations of defect: There is a big commutative diagram of smooth Gysin sequences

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^{n-2}(\mathbb{P}^{n-1} \setminus \overline{Y}) & \longrightarrow & H^{n-3}(\overline{Y}) & \longrightarrow & H^{n-1}(\mathbb{P}^{n-1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^n(\mathbb{P}^n \setminus \overline{X}) & \xrightarrow{\sigma} & H^{n-1}(\overline{X} \setminus \Sigma) & \longrightarrow & H^{n+1}(\mathbb{P}^n \setminus \Sigma) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^n(\mathbb{A}^n \setminus X) & \xrightarrow{\rho} & H^{n-1}(X \setminus \Sigma) & \longrightarrow & H^{n+1}(\mathbb{A}^n \setminus \Sigma) & \longrightarrow & \cdots,
\end{array}$$

where ρ is the Poincaré residue map.

Lemma 1.3.9. *We have* $\delta(\overline{X}) = \dim \operatorname{coker} \gamma$, *where*

$$\gamma : H^n(\mathbb{P}^n \setminus \overline{X}) \to H^n(\mathbb{A}^n \setminus X)/\rho^{-1}(\vartheta(H^{n-1}(X))) \cong H^n_\Sigma(X)$$

is the map induced by the open immersion $\mathbb{A}^n \setminus X \hookrightarrow \mathbb{P}^n \setminus \overline{X}$.

Proof. One checks that $H^n(\mathbb{A}^n \setminus \Sigma) = H^{n+1}(\mathbb{A}^n \setminus \Sigma) = 0$, so ρ is an isomorphism. We obtain a commutative diagram

$$\begin{array}{ccc} H^n(\mathbb{P}^n \setminus \overline{X}) & \xrightarrow{\sigma} & H^{n-1}(\overline{X} \setminus \Sigma) \\ \gamma \downarrow & & \downarrow \beta \\ H^n(\mathbb{A}^n \setminus X)/\rho^{-1}(\vartheta(H^{n-1}(X))) & \xrightarrow[\rho]{\simeq} & H^{n-1}(X \setminus \Sigma)/\vartheta(H^{n-1}(X)), \end{array}$$

where β is as in Lemma 1.3.4. In particular, $\delta(\overline{X}) = \dim \operatorname{coker} \beta$.

We claim that $\dim \operatorname{coker} \gamma = \dim \operatorname{coker} \beta$, or equivalently, $\dim \operatorname{im} \beta = \dim \operatorname{im}(\beta \circ \sigma)$. If n is even, then the map σ is surjective, since $H^{n+1}(\mathbb{P}^n \setminus \Sigma) = 0$. Otherwise, if n is odd, then the maps $H^{n-3}(\overline{Y}) \to H^{n-1}(\mathbb{P}^{n-1})$ and $H^{n-1}(\mathbb{P}^{n-1}) \to H^{n+1}(\mathbb{P}^n \setminus \Sigma)$ in the big commutative diagram are isomorphisms and the assertion follows by a diagram chase. □

Finally, there is another characterization of defect involving hypersurface complements:

Lemma 1.3.10. $\delta(\overline{X}) = h^{n-1}(\mathbb{P}^n \setminus \overline{X}) = h^{n-1}(\mathbb{A}^n \setminus X)$.

Proof. Assume for the moment that the cohomology theory under consideration is not Kähler-de Rham cohomology. Consider the Gysin sequence for $\overline{X} \subseteq \mathbb{P}^n$

$$\cdots \to H^n_c(\mathbb{P}^n \setminus \overline{X}) \to H^n(\mathbb{P}^n) \to H^n(\overline{X}) \to H^{n+1}_c(\mathbb{P}^n \setminus \overline{X}) \to H^{n+1}(\mathbb{P}^n) \to \dots$$

If n is even, then $H^{n+1}(\mathbb{P}^n) = 0$ (Lemma 1.2.19) and the restriction map $H^n(\mathbb{P}^n) \to H^n(\overline{X})$ is injective. Thus

$$\delta(\overline{X}) = h^n(\overline{X}) - h^n(\mathbb{P}^n) = \dim \operatorname{coker}(H^n(\mathbb{P}^n) \to H^n(\overline{X})) = h^{n+1}_c(\mathbb{P}^n \setminus \overline{X}).$$

For odd n, the restriction map $H^{n+1}(\mathbb{P}^n) \to H^{n+1}(\overline{X})$ is injective and hence

$$\delta(\overline{X}) = h^n(\overline{X}) = h^{n+1}_c(\mathbb{P}^n \setminus \overline{X}).$$

Using Poincaré duality on $\mathbb{P}^n \setminus \overline{X}$,

$$\delta(\overline{X}) = h^{n-1}(\mathbb{P}^n \setminus \overline{X}).$$

Now take the smooth Gysin sequences for $\mathbb{P}^{n-1} \setminus \overline{Y} \subseteq \mathbb{P}^n \setminus \overline{X}$ and $\overline{Y} \subseteq \mathbb{P}^{n-1}$:

$$\cdots \to H^{n-3}(\mathbb{P}^{n-1} \setminus \overline{Y}) \to H^{n-1}(\mathbb{P}^n \setminus \overline{X}) \to H^{n-1}(\mathbb{A}^n \setminus X) \to H^{n-2}(\mathbb{P}^{n-1} \setminus \overline{Y}) \to \dots$$

$$\cdots \to H^{n-3}(\mathbb{P}^{n-1} \setminus \overline{Y}) \to H^{n-4}(\overline{Y}) \to H^{n-2}(\mathbb{P}^{n-1}) \to H^{n-2}(\mathbb{P}^{n-1} \setminus \overline{Y}) \to \dots$$

By Lemma 1.2.17, $H^{n-3}(\mathbb{P}^{n-1} \setminus \overline{Y}) = H^{n-2}(\mathbb{P}^{n-1} \setminus \overline{Y}) = 0$, so that

$$\delta(\overline{X}) = h^{n-1}(\mathbb{P}^n \setminus \overline{X}) = h^{n-1}(\mathbb{A}^n \setminus X).$$

For Kähler-de Rham cohomology, apply Corollary 1.3.5. □

1.3.2 De Rham cohomology of hypersurface complements

Keep Notation 1.3.1. Suppose further that $H^\bullet(-) = H^\bullet_{\mathrm{KdR}}(-)$.

Explicit description

In view of Lemma 1.3.9, the defect of $\overline{X}$ may be approached by investigating the top-dimensional cohomology of the hypersurface complements $\mathbb{P}^n \setminus \overline{X}$ and $\mathbb{A}^n \setminus X$. Fortunately, these spaces can be explicitly described. Both varieties in question are smooth and affine of dimension n, so their n-th algebraic de Rham cohomology is just a quotient of the module of n-forms on their coordinate rings. More precisely:

Lemma 1.3.11 (Explicit description of de Rham cohomology of hypersurface complements)**.**

(1) *$H^n(\mathbb{P}^n \setminus \overline{X})$ is generated by*

$$\left\{ \frac{G\Omega}{F^j} \,\middle|\, G \in k[x_0, \ldots, x_n]_{jd-n-1}, j \geq 0 \right\},$$

where

$$\Omega := \sum_{i=0}^{n} (-1)^i x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

(2) *$H^n(\mathbb{A}^n \setminus X)$ is generated by*

$$\left\{ \frac{g\omega}{f^j} \,\middle|\, g \in k[x_1, \ldots, x_n], j \geq 0 \right\},$$

where $\omega := dx_1 \wedge \cdots \wedge dx_n$.

(3) *The natural restriction map is given by*

$$H^n(\mathbb{P}^n \setminus \overline{X}) \to H^n(\mathbb{A}^n \setminus X), \quad \left[\frac{G\Omega}{F^j} \right] \to \left[\frac{g\omega}{f^j} \right],$$

where g is the dehomogenization of G.

Proof. (2) and (3) are immediate. For (1), see [25, Chapter 6]. □

The pole-order filtration

Definition 1.3.12 (Pole-order filtration)**.**

- The *pole-order filtration on $H^n(\mathbb{P}^n \setminus \overline{X})$* is defined as

$$P^j H^n(\mathbb{P}^n \setminus \overline{X}) := \left\{ \left[\frac{G\Omega}{F^j} \right] \in H^n(\mathbb{P}^n \setminus \overline{X}) \,\middle|\, G \in k[x_0, \ldots, x_n]_{jd-n-1} \right\}, \quad j \geq 0.$$

- The *pole-order filtration on $H^n_\Sigma(X)$* is defined as

$$P^j H^n_\Sigma(X) := \left\{ \left[\frac{g\omega}{f^j} \right] \in H^n_\Sigma(X) \,\middle|\, G \in k[x_1, \ldots, x_n] \right\}, \quad j \geq 0,$$

where $H^n_\Sigma(X)$ is realized as a quotient of $H^n(\mathbb{A}^n \setminus X)$ as in Lemma 1.3.4.

These are ascending filtrations, as can be seen by extending the fractions with F or f, respectively. Note that this filtration is slightly different to the one given in Dimca's article [23, §1]. In any case, the pole-order filtration gives rise to the j-th graded objects

$$\begin{aligned} \operatorname{Gr}_P^j H^n(\mathbb{P}^n \setminus \overline{X}) &:= P^j H^n(\mathbb{P}^n \setminus \overline{X}) / P^{j-1} H^n(\mathbb{P}^n \setminus \overline{X}), && j \geq 0 \\ \operatorname{Gr}_P^j H^n_\Sigma(X) &:= P^j H^n_\Sigma(X) / P^{j-1} H^n_\Sigma(X) && j \geq 0, \end{aligned}$$

with the convention $P^{-1} := \{0\}$. The natural restriction

$$\gamma : H^n(\mathbb{P}^n \setminus \overline{X}) \to H^n_\Sigma(X)$$

induces maps $\operatorname{Gr}_P^j(\gamma)$ on the corresponding graded parts. In view of Lemma 1.3.9, there is an immediate corollary:

Corollary 1.3.13. *$\overline{X}$ has defect if and only if there is an integer $j \geq 0$ such that $\operatorname{Gr}_P^j(\gamma)$ is not surjective.*

Reduction of the pole order

Set $S := k[x_0, \dots, x_n]$ and $R := k[x_1, \dots, x_n]$. The explicit description of the cohomology groups given in Lemma 1.3.11 yields a commutative diagram

$$\begin{array}{ccc} S_{jd-n-1} & \longrightarrow & R \\ \big\downarrow & & \big\downarrow{\scriptstyle \varphi_j} \\ \operatorname{Gr}_P^j H^n(\mathbb{P}^n \setminus X) & \xrightarrow{\operatorname{Gr}_P^j(\gamma)} & \operatorname{Gr}_P^j H^n_\Sigma(X) \end{array}$$

for any $j \geq 0$ with surjective vertical arrows and the horizontal arrows being the natural restriction maps. The top right corner can actually be made smaller:

Lemma 1.3.14 (Reduction of the pole order)**.** *Let $\varphi_j : R \to \operatorname{Gr}_P^j H^n_\Sigma(X)$ be as in the above diagram. Let $J(f)$ denote the ideal in R spanned by the partial derivatives of f. Then:*

(1) *For $j \geq 2$, the map φ_j factors through $R/((f) + J(f))$.*

(2) *For $j = 1$, the map φ_1 factors through $R/((f) + J(f)^3)$.*

(3) $\operatorname{Gr}_P^0 H^n(\mathbb{P}^n \setminus \overline{X}) = \operatorname{Gr}_P^0 H^n_\Sigma(X) = 0.$

Proof.

(1) If $g \in (f) + J(f)$, then there are polynomials $h_0, \dots, h_n$ such that

$$g = h_0 f + \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i}.$$

The class of $h_0 f \omega / f^j$ vanishes in the graded object Gr_P^j by definition. As an application of the quotient rule, one computes that

$$d\left((-1)^i \frac{h_i}{f^{j-1}} dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n\right) = (j-1) \cdot h_i \cdot \frac{\partial f}{\partial x_i} \cdot \frac{\omega}{f^j} - \frac{\partial h_i}{\partial x_i} \cdot \frac{\omega}{f^{j-1}}. \tag{1.1}$$

Hence if $j \geq 2$, the cohomology class of $h_i \frac{\partial f}{\partial x_i} \omega / f^j$ can be rewritten as the class of a differential form with lower pole order. However, these classes vanish in the graded object Gr_P^j.

(3) For $j = 0$ observe at first that $\operatorname{Gr}_P^0 H^n(\mathbb{P}^n \setminus \overline{X})$ is generated by $S_{n-1} = 0$. If $h \in R$ is any polynomial, then the relation (1.1) for $j = 1$ shows that all forms of the type $\frac{\partial h}{\partial x_i}\omega$ vanish in $\operatorname{Gr}_P^0 H^n_\Sigma(X)$. Integrating symbolically, any form can be written in this way.

(2) If $j = 1$, the pole-order reduction trick as above does not apply anymore. In the notation of Lemma 1.3.9, put $V := \rho^{-1}(\vartheta(H^{n-1}(X)))$, so that

$$H^n(\mathbb{A}^n \setminus X)/V \cong H^{n-1}(X \setminus \Sigma)/\vartheta(H^{n-1}(X)) \cong H^n_\Sigma(X),$$

the first isomorphism being induced by the Poincaré residue ρ.

Let $\eta \in \Omega^{n-1}_R$ be a global $(n-1)$-form. Then the class of η in $\Omega^{n-1}_{R/(f)}$ lies in the kernel of $d : \Omega^{n-1}_{R/(f)} \to \Omega^n_{R/(f)}$ if and only if $d\eta = f\xi + \zeta \wedge df$ for some $\xi \in \Omega^n_R, \zeta \in \Omega^{n-1}_R$. Such an η defines a cohomology class in $H^{n-1}(X)$. Restricting to the open subscheme $X \setminus \Sigma$ via ϑ and applying the inverse of the Poincaré residue map ρ (see e.g., [36, Theorem III.8.3]) induces a map

$$\rho^{-1} \circ \vartheta : W := \{\eta \in \Omega^{n-1}_R \mid \exists \xi \in \Omega^n_R, \zeta \in \Omega^{n-1}_R : d\eta = f\xi + \zeta \wedge df\} \to V,$$
$$\eta \mapsto \left[\frac{\eta \wedge df}{f}\right].$$

In particular, all forms inside the image of this map will vanish in $\operatorname{Gr}_P^1 H^n_\Sigma(X)$.

We will now give a description of differential forms in terms of polynomials: Write

$$\eta := \sum_{i=1}^n (-1)^i h_i \cdot dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n, \quad h_i \in R.$$

Then

$$\eta \wedge df = \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i}\omega \quad \text{and} \quad d\eta = \sum_{i=1}^n \frac{\partial h_i}{\partial x_i}.$$

A polynomial $g \in (f) + J(f)^3$ can be written as

$$g = fh' + \sum_{i,j,k=1}^n h_{ijk} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\frac{\partial f}{\partial x_k} = fh' + \sum_{i=1}^n \left(\sum_{j,k=1}^n h_{ijk}\frac{\partial f}{\partial x_j}\frac{\partial f}{\partial x_k}\right)\frac{\partial f}{\partial x_i}, \quad h', h_{ijk} \in R$$

and by the product rule,

$$\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sum_{j,k=1}^n h_{ijk}\frac{\partial f}{\partial x_j}\frac{\partial f}{\partial x_k}\right) \in J(f).$$

Thus if $(g - fh')\omega = \eta \wedge df$ as above, then $d\eta = \zeta \wedge df$ for some ζ and hence $\eta \in W$. In particular, inside $H^n(\mathbb{A}^n \setminus X)$,

$$\left[\frac{g\omega}{f}\right] = \left[\frac{(g - fh')\omega}{f}\right] + \left[\frac{fh'\omega}{f}\right] = \left[\frac{\eta \wedge df}{f}\right] + [h'\omega] = [\rho^{-1}(\vartheta(\eta))] + 0 \in V.$$

Consequently, the map φ_1 factors through $R/((f) + J(f)^3)$. □

1.3.3 Defect and Tjurina number

Theorem 1.3.15 (Defect ⇒ high Tjurina number)**.** *Let $\tau := \dim_k k[x_1, \ldots, x_n]/((f) + J(f))$ be the global Tjurina number of $\overline{X}$. If $\overline{X}$ has defect, then*

$$\tau \geq \frac{d - n + 1}{n^2 + n + 1}.$$

Moreover, if the map $\mathrm{Gr}_P^j(\gamma)$ is not surjective for some $j \geq 2$, then

$$\tau \geq jd - n + 1.$$

Proof. Since $\overline{X}$ has defect, there is an integer $j \geq 0$ such that $\mathrm{Gr}_P^j(\gamma)$ is not surjective by Corollary 1.3.13. Using Lemma 1.3.14 (3), $j \geq 1$.

Assume first that $j \geq 2$. Then the non-surjectivity of $\mathrm{Gr}_P^j(\gamma)$ for some j implies the non-surjectivity of the natural restriction map $S_{jd-n-1} \to T(f)$, where $T(f) := R/((f) + J(f))$ denotes the global Tjurina algebra of f. Since $\dim \Sigma = 0$, $T(f)$ is a finite-dimensional k-algebra. Applying Poonen's trick ([60, Lemma 2.1(b)], see also Lemma 2.2.9) shows that the image of S_i in $T(f)$ stricty increases with i until it fills the whole space. In particular, the restriction map has to be surjective for $i \geq \tau - 1$. From that one infers that $jd - n - 1 \leq \tau - 2$, whence $\tau \geq jd - n + 1$.

If $j = 1$, then the same argument shows that $\dim_k R/((f) + J(f)^3) \geq d - n + 1$. Using the exact sequences of $T(f)$-modules

$$0 \to J(f)^i/J(f)^{i+1} \to R/((f) + J(f)^{i+1}) \to R/((f) + J(f)^i) \to 0$$

for $i = 1, 2$, we obtain

$$\dim_k R/((f) + J(f)^3) = \dim_k J(f)^2/J(f)^3 + \dim_k J(f)/J(f)^2 + \dim_k T(f).$$

Since $J(f)^i/J(f)^{i+1}$ can be generated by n^i elements, it has length at most n^i as $T(f)$-module. Thus

$$d - n + 1 \leq \dim_K R/((f) + J(f)^3) \leq n^2\tau + n\tau + \tau = (n^2 + n + 1) \cdot \tau. \qquad \square$$

1.3.4 Local computations

Weighted homogeneous singularities

If the singularities of $\overline{X}$ happen to be weighted homogeneous, then the methods of Dimca [24] improve the bound of Theorem 1.3.15:

Lemma 1.3.16. *In the situation of Lemma 1.3.14, suppose that $\overline{X}$ has only weighted homogeneous singularities. Then the natural map*

$$\varphi_1 : R \to \mathrm{Gr}_P^1 H_\Sigma^n(X)$$

factors through $R/((f) + J(f))$.

Proof. By the Lefschetz principle, assume that $k \subseteq \mathbb{C}$ and use analytic de Rham cohomology. As in [24, Section 3], the map $\operatorname{Gr}^1_P(\gamma)$ can be described as the natural restriction

$$\operatorname{Gr}^1_P(\gamma) : \operatorname{Gr}^1_P H^n(\mathbb{P}^n \setminus X) \to \bigoplus_{x \in \Sigma} \operatorname{Gr}^1_{P_x} H^n(\Omega^\bullet_{f,x}),$$

where $\Omega^\bullet_{f,x}$ denotes the localization of the holomorphic de Rham complex $\Omega^\bullet_{\mathbb{C}^n,x}$ with respect to f, and P_x is the corresponding local pole-order filtration. In particular, for any $x \in \Sigma$ there is a natural surjection

$$\varphi_x : \mathcal{O}_{\mathbb{C}^n,x} \to \operatorname{Gr}^1_{P_x} H^n(\Omega^\bullet_{f,x}), \quad g \mapsto \left[\frac{g}{f}\, dx_1 \wedge \cdots \wedge dx_n\right].$$

Suppose now that the singularity of $\overline{X}$ at x is contact-equivalent to a weighted homogeneous singularity. Then there is a biholomorphic coordinate change ψ sending x to $(0,\ldots,0)$ such that $f' = \psi(f)$ is a weighted homogeneous polynomial. Moreover, ψ induces an isomorphism of the local Tjurina algebras of f at x and f' at 0, respectively.

Take a polynomial $h \in (f) + J(f)$. Under the natural map

$$R \to H^n(\Omega^\bullet_{f,x}) \xrightarrow{\simeq} H^n(\Omega^\bullet_{f',0})$$

induced by ψ, the class $[h/f\, dx_1 \wedge \cdots \wedge dx_n]$ is sent to some $[h'/f' dx'_1 \wedge \cdots \wedge dx'_n]$ with h' lying in the analytic ideal $(f') + J(f') \subseteq \mathcal{O}_{\mathbb{C}^n,0}$. However, the calculation [24, Example 3.6] shows that $[h'/f' dx'_1 \wedge \cdots \wedge dx'_n] = 0$. □

Applying the same methods as in the proof of Theorem 1.3.15, this yields:

Corollary 1.3.17. *Suppose that $\overline{X}$ has at most weighted homogeneous singularities. If $\overline{X}$ has defect, then $\tau \geq d - n + 1$.*

Nodal hypersurfaces

A well-known application is the following ([23, Proposition 3.3], [66]): Suppose that $k = \mathbb{C}$ and $\overline{X}$ has at most ordinary double points as singularities. The polynomial $f' = x_1^2 + \cdots + x_n^2$ is weighted homogeneous of degree 2 with respect to the weights $(1,\ldots,1)$. The the local cohomology piece $\operatorname{Gr}^j_{P_0} H^n(\Omega^\bullet_{f',0})$ is hence spanned by homogeneous forms of degree $2j - n$ in the Tjurina algebra of f'. Therefore $\operatorname{Gr}^j_{P_0} H^n(\Omega^\bullet_{f',0}) = 0$ for $j \neq \frac{n}{2}$. This has the following consequences:

- $\overline{X}$ cannot have defect if n is odd.
- If n is even, then $\overline{X}$ has defect if and only if the map $\operatorname{Gr}^{n/2}_P(\gamma)$ is not surjective. In this case, $\tau \geq \frac{dn}{2} - n + 1$ by the second part of Theorem 1.3.15.
- If n is even, then $H^n(\Omega^\bullet_{f',0})$ is one-dimensional. Moreover

$$\begin{aligned}\delta(\overline{X}) &= \dim \operatorname{coker}\left(\operatorname{Gr}^{n/2}_P H^n(\mathbb{P}^n \setminus \overline{X}) \xrightarrow{\operatorname{Gr}^{n/2}_P(\gamma)} \operatorname{Gr}^{n/2}_P H^n_\Sigma(X)\right) \\ &= \dim \operatorname{coker}\left(\mathbb{C}[x_0,\ldots,x_n]_{dn/2-n-1} \to \mathbb{C}^{\#\Sigma}, \quad h \mapsto (h(x))_{x\in\Sigma}\right).\end{aligned}$$

If $I_\Sigma = \sqrt{J(F)}$ denotes the ideal of Σ inside the homogeneous coordinate ring $\mathbb{C}[x_0, \ldots, x_n]$, then the dimension of the image of $\mathbb{C}[x_0, \ldots, x_n]_{dn/2-n-1}$ in $\mathbb{C}^{\#\Sigma}$ is precisely the Hilbert function of $\mathbb{C}[x_0, \ldots, x_n]/I_\Sigma$ evaluated at $dn/2 - n - 1$. In particular, the defect $\delta(\overline{X})$ is the difference between the Hilbert polynomial and the Hilbert function of I_Σ at $\frac{dn}{2} - n - 1$.

Another formulation is the following: The number $\delta(\overline{X})$ is the defect of the linear system of homogeneous polynomials in $n+1$ variables of degree $dn/2 - n - 1$ passing through Σ. In general, one would expect that vanishing at $\#\Sigma$ points is a codimension $\#\Sigma$ condition in $\mathbb{C}[x_0, \ldots, x_n]_{dn/2-n-1}$. If $\overline{X}$ has defect, then more forms than expected pass through the points in Σ, which indicates that the nodes of $\overline{X}$ are in a special position.

Example 1.3.18. Let $\overline{X} \subseteq \mathbb{P}^4$ be the quintic from Example 1.2.22. Let I_Σ denote the ideal of the 125 nodes on $\overline{X}$. One computes that

$$\dim_{\mathbb{C}}(\mathbb{C}[x_0, \ldots, x_n]/I_\Sigma)_5 = 101,$$

so $\delta(\overline{X}) = 125 - 101 = 24$ and $h^4(\overline{X}) = 25$. In $\mathbb{P}^4$, 125 points in general position determine a unique quintic. However, the position of the singularities of $\overline{X}$ is special: There is a 24-dimensional family of quintics passing through the nodes of $\overline{X}$.

Remarks. Let $\overline{X} \subseteq \mathbb{P}^n$ be a nodal hypersurface.

- One can actually show that if $\overline{X}$ has defect and $\dim \overline{X} = 3$, then $\tau \geq (d-1)^2$, see [12] or [45, Theorem 4.1]. The latter proof carries over to higher dimensions.
- For even n, it is conjectured in [45] that $\tau \geq (d-1)^{n/2}$.

Proof of Theorem 1.1.1

Proof. The statements for algebraic de Rham and Kähler-de Rham cohomology are immediate consequences of Theorem 1.3.15 and Corollary 1.3.17. The version for the other cohomology theories follow from Theorem 1.2.1 (1) and the Lefschetz principle. □

1.3.5 Remarks on positive characteristic

The main difficulty in generalizing the proof of Theorem 1.1.1 to positive characteristic is an explicit description of the cohomology groups involved. Although there is a similar description for hypersurface complements in rigid cohomology as in Lemma 1.3.11 − replacing polynomials by overconvergent power series − the rigid cohomology of singular varieties remains a rather mysterious object. Currently, it is not even known whether $H^n_{\mathrm{rig}}(X) = 0$ holds for an arbitrary singular affine hypersurface $X \subseteq \mathbb{A}^n$.

A field k of positive characteristic admits a ring of Witt vectors $W(k)$ with quotient field K. Let $F \in W(k)[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree d with coefficients in $W(k)$. Then $F = 0$ defines a $W(k)$-scheme $\mathcal{X}$. Its generic fiber is hence the hypersurface $\mathcal{X}_\eta := \{F = 0\} \subseteq \mathbb{P}^n_K$. The special fiber $\mathcal{X}_s$ is a hypersurface in $\mathbb{P}^n_k$ defined by reducing F modulo p. Both the rigid cohomology of $\mathcal{X}_s$ and the algebraic de Rham cohomology of $\mathcal{X}_\eta$ take values in K, and there is a natural cospecialization map relating them [3, §6.7, §6.8]. This map is an isomorphism when $\mathcal{X}$ is smooth [4, Corollary 2.6].

For singular $\mathcal{X}$, this is no longer true:

Example 1.3.19. Consider $\mathcal{X} = \{F = 0\}$, where $F = px_0^2 + x_1x_2 + x_3x_4 \in \mathbb{Z}_p[x_0, \ldots, x_4]$. The corresponding generic fiber $\mathcal{X}_\eta$ is a smooth hypersurface in $\mathbb{P}^4_{\mathbb{Q}_p}$ and hence $h^4_{\mathrm{dR}}(\mathcal{X}_\eta) = 1$ by Lemma 1.2.17. On the other hand, $h^4_{\mathrm{rig}}(\mathcal{X}_s) > 1$, see Examples 1.3.8 and 1.5.2. If instead we choose $F = x_1x_2 + x_3x_4$ as a defining polynomial for $\mathcal{X}$, the special fiber does not change, but the generic fiber has defect as well.

This motivates the following question:

Question 1.3.20. Let $X \subseteq \mathbb{P}^n_k$ be a hypersurface with $h^n_{\mathrm{rig}}(\overline{X}) \neq h^n_{\mathrm{rig}}(\mathbb{P}^n_k)$. Does X admit a lift $\mathcal{X} \subseteq \mathbb{P}^n_{W(k)}$ such that the generic fiber $\mathcal{X}_\eta \subseteq \mathbb{P}^n_K$ has defect in algebraic de Rham cohomology?

If this question had an affirmative answer, then we could use the results of Section 1.3:

Corollary 1.3.21. *Let $X \subseteq \mathbb{P}^n_k$ be a hypersurface of degree d with global Tjurina number τ admitting a lift with defect. Then*

$$\tau \geq \frac{d - n + 1}{n^2 + n + 1}.$$

Proof. By assumption, we can lift X to a $W(k)$-scheme $\mathcal{X}$ such that the generic fiber $\mathcal{X}_\eta$ is a hypersurface in $\mathbb{P}^n_K$ with defect. If τ_η denotes the global Tjurina number of $\mathcal{X}_\eta$, then

$$\tau_\eta \geq \frac{d - n + 1}{n^2 + n + 1}$$

by Theorem 1.1.1. Define $M := W(k)[x_1, \ldots, x_n]/((f) + J(f))$. Applying Nakayama's lemma,

$$\tau = \dim_k M \otimes_{W(k)} k \geq \dim_K M \otimes_{W(k)} K = \tau_\eta \geq \frac{d - n + 1}{n^2 + n + 1}. \qquad \square$$

However, Question 1.3.20 seems to be very delicate. By [65, Theorem 1.1], there are surfaces $S \subseteq \mathbb{P}^4$ that do not lift to characteristic zero. Such surfaces cannot be complete intersections, so no hypersurface X containing S can be factorial. In particular, if such an X is defined over $\overline{\mathbb{F}_p}$, then X will have defect by Theorem 1.5.1. On the other hand, it is well possible that every lift of X is factorial, as S is not liftable.

1.4 Defect via resolutions of singularities

1.4.1 Ordinary multiple points and A_k singularities

In this section, we relate defect of hypersurfaces to the number of singularities following the ideas presented in [59]. Let K be an algebraically closed field of characteristic $p \neq 2$. Choose a cohomology theory $H^\bullet$ from the list of §1.2.1 different from Kähler-de Rham cohomology.

For a positive integer $n \geq 3$, let $X \subseteq \mathbb{P}^n_K$ be an irreducible hypersurface of degree d with at most isolated singularities.

Definition 1.4.1 (Quantitative definition of defect in arbitrary characteristic)**.** The *defect of X* is defined as

$$\delta(X) := h^n(X) - h^n(\mathbb{P}^n).$$

Suppose further that the singular points of X belong to the following classes:

- *Ordinary multiple points.* A point x is an *ordinary multiple point of multiplicity m* if the projectivized tangent cone at x is the cone over a smooth degree m hypersurface in $\mathbb{P}^{n-1}$ for some $m \geq 2$.
- *A_k singularities.* These are points whose completed local ring is isomorphic to

  $$K[[x_1, \ldots, x_n]]/(x_1^{k+1} + x_2^2 + \cdots + x_n^2)$$

  for some $k \geq 1$.

Note that an ordinary double point is an A_1 singularity, and this is the only common member of both families.

Let Σ_O be the set of ordinary multiple points in X of multiplicity ≥ 3, and denote by m_x the multiplicity of a point $x \in \Sigma_O$. Similarly, define Σ_A to be the union of all A_k points in X for $k \geq 1$, and for an A_k singularity $x \in \Sigma_A$ let $\mu_x := k$ and $r_x := \lceil k/2 \rceil$.

The embedded resolution

The advantage of restricting to these two classes of singularities is the very explicit nature of a resolution of singularities:

Proposition 1.4.2. *Let X be as above. Then there is an embedded resolution of singularities*

$$\pi : (Y \subseteq P) \to (X \subseteq \mathbb{P}^n)$$

such that P is a smooth n-fold obtained from $\mathbb{P}^n$ by a finite sequence of blowups in points. More precisely:

(1) *P is obtained by $\mathbb{P}^n$ as a sequence of*

$$s := \#\Sigma_O + \sum_{x \in \Sigma_A} r_x$$

blowups in points.

(2) *As a divisor on P, the strict transform Y of X is linearly equivalent to*

$$dH - \sum_{x \in \Sigma_O} m_x \mathcal{D}_x - \sum_{x \in \Sigma_A} \sum_{i=1}^{r_x} 2i \cdot \mathcal{E}_{x,i}$$

where

- *H is the pullback of a hyperplane,*
- *$\mathcal{D}_x \cong \mathbb{P}^{n-1}$ and $D_x := Y \cap \mathcal{D}_x$ is a smooth degree m_x hypersurface in $\mathbb{P}^{n-1}$,*
- *$\mathcal{E}_{x,i}$ is obtained from $\mathbb{P}^{n-1}$ by $r_x - i$ blowups in points and $E_{x,i} := Y \cap \mathcal{E}_{x,i}$ is isomorphic to the blowup at the vertex of the cone over a smooth quadric in $\mathbb{P}^{n-2}$ for $i = 1, \dots, r_x - 1$.*
- *$\mathcal{E}_{x,r_x} \cong \mathbb{P}^{n-1}$ and $E_{x,r_x} := Y \cap \mathcal{E}_{x,r_x}$ is isomorphic to a smooth quadric in $\mathbb{P}^{n-1}$ if k is odd,*
- *$\mathcal{E}_{x,r_x} \cong \mathbb{P}^{n-1}$ and $E_{x,r_x} := Y \cap \mathcal{E}_{x,r_x}$ is isomorphic to the cone over a smooth quadric in $\mathbb{P}^{n-2}$ if k is even.*
- *$E_{x,i} \cap E_{x,j} = \emptyset$ unless $|i-j| \leq 1$ and $E_{x,i} \cap E_{x,i+1}$ is isomorphic to a smooth quadric in $\mathbb{P}^{n-2}$ for $i = 1, \dots, r_x - 1$.*
- *$E_{x,i} \cap E_{x,j} \cap E_{x,k} = \emptyset$ for pairwise distinct i, j, k.*

Proof. See [59] for the case of ordinary multiple points and [17], [62] for details on resolving A_k singularities. □

Vanishing of local cohomology

Before computing Betti numbers of the resolution, we remark that if n happens to be odd, then A_k singularities do not contribute to defect:

Lemma 1.4.3. *If n is odd, then $H^n_{\Sigma_A}(X) = 0$. In particular, if X has at most A_k singularities, then $\delta(X) = 0$.*

Proof. The "in particular" statement follows from Lemma 1.3.3. The proof works in positive characteristic as well due to the vanishing of the n-th local cohomology.

The result $H^n_{\Sigma_A}(X) = 0$ is well-known in characteristic zero, see [23, Examples 1.9]. Hence assume that the chosen cohomology theory is étale or rigid cohomology in positive characteristic. The space $H^n_{\Sigma_A}(X)$ decomposes into the direct sum $\bigoplus_{x \in \Sigma_A} H^n_{\{x\}}(X)$. Moreover, $H^n_{\{x\}}(X)$ depends only on (X, x) up to contact equivalence, see [58, §1.2].

Thus we are left with computing $H^n_{\{0\}}(Z)$ for the variety $Z = \{x_1^k + x_2^2 + \cdots + x_n^2 = 0\} \subseteq \mathbb{A}^n$. The affine hypersurface Z is defined by the vanishing of a weighted homogeneous polynomial. Applying Lemma 1.2.14 (4), $H^i(Z) = 0$ for $i \geq 1$. Considering the excision long exact sequence

$$\cdots \to H^{n-1}(Z) \to H^{n-1}(Z \setminus \{0\}) \to H^n_{\{0\}}(Z) \to H^n(Z) \to \dots,$$

it hence suffices to show that $H^{n-1}(Z \setminus \{0\}) = 0$. To this end, use the smooth Gysin sequence for $\overline{Z} \setminus Z \subseteq \overline{Z} \setminus \{0\}$, where $\overline{Z}$ denotes the projective closure of Z in $\mathbb{P}^n$:

$$\cdots \to H^{n-3}(\overline{Z} \setminus Z) \to H^{n-1}(\overline{Z} \setminus \{0\}) \to H^{n-1}(Z \setminus \{0\}) \to H^{n-2}(\overline{Z} \setminus Z) \to \dots$$

The variety $\overline{Z} \setminus Z$ is either a smooth quadric in $\mathbb{P}^{n-1}$ ($k = 1$) or a hyperplane of multiplicity k ($k \geq 2$). In both cases, $H^{n-2}(\overline{Z} \setminus Z) = 0$ by Lemma 1.2.17 (3) and Example 1.2.18.

It remains to show that $H^{n-1}(\overline{Z} \setminus \{0\}) = 0$. By the description given in Proposition 1.4.2, the resolution of singularities of $\overline{Z}$ lifts to characteristic 0. Using the comparison from Theorem 1.2.1 (4), we can reduce to the known de Rham cohomology case. □

Remarks.

- The n-th local rigid cohomology of A_k singularities in $\mathbb{A}^n$ is computed using complements of hypersurfaces in weighted projective spaces in [57, §4.3.3].
- Ordinary multiple points of multiplicity ≥ 3 can cause defect on even-dimensional hypersurfaces: Let $X \subseteq \mathbb{P}^3$ be the projective cone over a smooth plane curve C of degree $m \geq 3$. Then $h^3(X) = h^1(C) = (m-1)(m-2) > 0$ (Example 1.3.7), so X has defect.

1.4.2 Defect and Betti numbers of the embedded resolution

We will now give a cohomological criterion for defect using the embedded resolution of singularities π from Proposition 1.4.2.

Betti numbers of P

First, we need the Betti numbers of P, which is obtained by s successive blowups.

Lemma 1.4.4. *We have*

$$h^i(P) = \begin{cases} s+1 & \text{if } i \in \{2, \ldots, 2n-2\}, \\ 1 & \text{if } i \in \{0, 2n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $P_0 := \mathbb{P}^n$ and for $j = 1, \ldots, s$ denote by P_j the blowup of P_{j-1} in a point. By Lemma 1.2.10, there is an exact sequence

$$\cdots \to H^i(P_j) \to H^i(P_{j+1}) \oplus H^i(\{\text{point}\}) \to H^i(\mathbb{P}^{n-1}) \to H^{i+1}(P_j) \to \ldots$$

Using the Betti numbers of projective space (Example 1.2.2), the claim follows by induction. □

Betti numbers of the exceptional divisor

The next step is to compute some Betti numbers of the exceptional divisor E associated to the resolution $\pi|_Y : Y \to X$, i.e.,

$$E := Y \cap \left(\sum_{x \in \Sigma_O} D_x + \sum_{x \in \Sigma_A} \sum_{i=1}^{r_x} E_{x,i} \right).$$

Lemma 1.4.5. *Suppose that n is even. Then $h^{n-1}(E) = 0$ and $h^n(E) = s$.*

Proof. E is the disjoint union of the divisors D_x, $x \in \Sigma_O$, and $E_x = \sum_{i=1}^{r_x} E_{x,i}$, $x \in \Sigma_A$. Hence we can treat each singularity type separately.

(1) D_x for $x \in \Sigma_O$. By the description given in Proposition 1.4.2, D_x is isomorphic to a smooth degree m_x hypersurface in $\mathbb{P}^{n-1}$. Hence by Lemma 1.2.17 (1), $h^i(D_x) = h^i(\mathbb{P}^{n-1})$ for $i \notin \{n-2, 2n-2\}$. In particular $h^{n-1}(D_x) = 0$ and $h^n(D_x) = 1$.

(2) E_x for $x \in \Sigma_A$.

Let Q be a smooth quadric in $\mathbb{P}^{n-2}$, let C be the cone over Q in $\mathbb{P}^{n-1}$ and denote by B the blowup of C in its vertex. Further let S be a smooth quadric in $\mathbb{P}^{n-1}$. Using Lemma 1.2.17 along with Examples 1.2.18 and 1.3.8,

$$h^i(Q) = h^i(C) = h^i(S) = 0 \quad \text{for all odd } i \geq n-1.$$

Moreover, using the exact sequence

$$\cdots \to H^i(C) \to H^i(B) \oplus H^i(\{\text{point}\}) \to H^i(\mathbb{P}^{n-2}) \to H^{i+1}(C) \to \dots \tag{1.2}$$

from Lemma 1.2.10, $h^i(B) = 0$ when $i \geq n-1$ is odd.

Since there are no triple intersections between the components of E_x, Lemma 1.2.12 yields a long exact Mayer-Vietoris sequence

$$\cdots \to H^q(E_x) \to \bigoplus_i H^q(E_{x,i}) \xrightarrow{d_q} \bigoplus_{i<j} H^q(E_{x,i} \cap E_{x,j}) \to H^{q+1}(E_x) \to \dots$$

We claim that the maps d_{n-2} and d_n are surjective. Assuming this, we immediately have $h^{n-1}(E_x) = 0$ by the description given in Proposition 1.4.2.

In the case $n = 4$, the $E_{x,i}$ are irreducible surfaces, so $h^4(E_x) = \sum_{i=1}^{r_x} h^4(E_{x,i}) = r_x$. For $n \geq 6$, one computes $h^n(Q) = h^n(S) = 1$ by Lemma 1.2.17, $h^n(C) = 1$ by Lemma 1.2.19 and thus $h^n(B) = 2$ by (1.2). Therefore

$$h^n(E_x) = (r_x - 1) \cdot h^n(E_{x,i}) + h^n(E_{x,r_x}) - (r_x - 1) \cdot h^n(E_{x,i} \cap E_{x,j}) = r_x.$$

It remains to prove the surjectivity of

$$\bigoplus_i H^q(E_{x,i}) \xrightarrow{d_q} \bigoplus_{i<j} H^q(E_{x,i} \cap E_{x,j})$$

for $q = n-2, n$. Since $E_{x,i} \cap E_{x,j}$ is empty unless $|i - j| = 1$, this would follow from the surjectivity of all the maps

$$H^q(E_{x,i}) \to H^q(E_{x,i} \cap E_{x,i+1}), \quad i = 1, \dots, r_x - 1.$$

However, the intersection $E_{x,i} \cap E_{x,i+1} \cong Q$ is a smooth quadric inside the exceptional divisor $F \cong \mathbb{P}^{n-2}$ of the blowup of C at its vertex. In particular, the restriction morphism $H^q(F) \to H^q(Q)$ is surjective for $q = n-2, n$. Moreover, using $h^{q+1}(C) = 0$ in the long exact sequence (1.2), we obtain that the map $H^q(B) \to H^q(F)$ is surjective and so is the composition

$$H^q(E_{x,i}) \xrightarrow{\simeq} H^q(B) \to H^q(F) \to H^q(Q) \xrightarrow{\simeq} H^q(E_{x,i} \cap E_{x,i+1}).$$

Summing up,

$$h^{n-1}(E) = 0 \quad \text{and } h^n(E) = \sum_{x\in\Sigma_O} 1 + \sum_{x\in\Sigma_A} r_x = s. \qquad \square$$

Lemma 1.4.6. *Suppose that n is odd. Then $h^n(E) = 0$ and $h^n(X) \le h^n(Y)$.*

Proof. The proof that $h^n(E) = 0$ is analogous to the proof of $h^{n-1}(E) = 0$ in Lemma 1.4.5.

It remains to show the inequality $h^n(X) \le h^n(Y)$. First blow up the ordinary multiple points successively. This gives a partial resolution $\psi : Y_O \to X$ with Y_O having at most A_k singularities. The morphism ψ comes from an embedded resolution $P_O \to \mathbb{P}^n$, and thus there is the following commutative diagram by Lemma 1.2.10:

$$\begin{array}{ccccccccc} H^{n-1}(P_O) & \longrightarrow & \bigoplus_{x\in\Sigma_O} H^{n-1}(\mathcal{D}_x) & \longrightarrow & H^n(\mathbb{P}^n) & & & & \\ \downarrow & & \downarrow & & \downarrow & & & & \\ H^{n-1}(Y_O) & \longrightarrow & \bigoplus_{x\in\Sigma_O} H^{n-1}(D_x) & \longrightarrow & H^n(X) & \longrightarrow & H^n(Y_O) & \longrightarrow & H^n(E). \end{array}$$

Since $\mathcal{D}_x \cong \mathbb{P}^{n-1}$, and D_x is a smooth hypersurface therein, the natural restriction map $H^{n-1}(\mathcal{D}_x) \to H^{n-1}(D_x)$ is an isomorphism by the Lefschetz hyperplane theorem. Together with $H^n(\mathbb{P}^n) = 0$ this implies that $H^{n-1}(Y_O) \to \bigoplus_{x\in\Sigma_O} H^{n-1}(D_x)$ is surjective and thus $H^n(X) \cong H^n(Y_O)$.

Let $D_O := \sum_{x\in\Sigma_O} D_x$. Then

$$H^n_{\psi^{-1}(\Sigma_A)}(Y_O) \cong H^n_{\psi^{-1}(\Sigma_A)}(Y_O \setminus D_O) \cong H^n_{\Sigma_A}(X \setminus \Sigma_O) = H^n_{\Sigma_A}(X) = 0$$

by Lemma 1.4.3. Resolving Y_O, we obtain our smooth hypersurface Y in P with the exceptional divisor $E_A \cong \sum_{x\in\Sigma_A} \sum_{i=1}^{r_x} E_{x,i}$. This resolution gives a commutative diagram

$$\begin{array}{ccccc} H^n_{\psi^{-1}(\Sigma_A)}(Y_O) & \longrightarrow & H^n(Y_O) & \longrightarrow & H^n(Y_O \setminus \psi^{-1}(\Sigma_A)) \\ & & \downarrow & & \| \\ & & H^n(Y) & \longrightarrow & H^n(Y \setminus E_A) \end{array}$$

It follows that $H^n(Y_O) \to H^n(Y_O \setminus \psi^{-1}(\Sigma_A)) \to H^n(Y \setminus E_A)$ is injective. This implies that the map $H^n(Y_O) \to H^n(Y)$ is injective as well. Consequently, $h^n(X) = h^n(Y_O) \le h^n(Y)$. $\square$

Defect via Betti numbers of the strict transform

With the two lemmata above, we obtain a simple formula for the defect of X:

Proposition 1.4.7. *The defect of X may be computed as follows:*

$$\delta(X) = \begin{cases} h^n(Y) - s - 1 & \text{if } n \text{ is even,} \\ h^n(Y) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Applying Lemma 1.2.10 to $\pi|_Y : Y \to X$, there is a long exact sequence

$$\cdots \to H^{n-1}(Y) \to H^{n-1}(E) \to H^n(X) \to H^n(Y) \to H^n(E) \to H^{n+1}(X) \to \dots$$

Suppose first that n is even. Using Lemma 1.4.5, $h^{n-1}(E) = 0$ and $h^n(E) = s$. Applying Lemma 1.2.19, we have $h^{n+1}(X) = 0$. It follows that $h^n(Y) = h^n(X) + s$. If n is odd, then inserting $h^n(E) = 0$ into the above long exact sequence implies $h^n(X) \ge h^n(Y)$. On the other hand, $h^n(X) \le h^n(Y)$ by Lemma 1.4.6, so that $h^n(X) = h^n(Y)$. $\square$

1.4.3 Ampleness of the strict transform

We keep the notation of the previous subsection. If the strict transform Y of X happens to be an ample divisor in P, then the Lefschetz hyperplane theorem 1.2.15 shows that the restriction map

$$H^{n-2}(P) \to H^{n-2}(Y)$$

is an isomorphism. Applying Poincaré duality on Y, $h^{n-2}(P) = h^n(Y)$. Hence we have the following corollary of Proposition 1.4.7 and Lemma 1.4.4:

Corollary 1.4.8. *Suppose that Y is ample in P. Then $\delta(X) = 0$.*

Finally, we can relate ampleness of Y to the number of singularities of X.

Lemma 1.4.9. *Suppose that*

$$\sum_{x \in \Sigma_O} m_x + \sum_{x \in \Sigma_A} 2r_x < d.$$

Then Y is ample in P.

Proof. This is a variant of [59, Theorem 4.1]. By Proposition 1.4.2, inside $\mathrm{Pic}(P)$,

$$\begin{aligned} Y &= dH - \sum_{x \in \Sigma_O} m_x D_x - \sum_{x \in \Sigma_A} \sum_{i=1}^{r_x} 2i \cdot \mathcal{E}_{x,i} \\ &= dH - \sum_{x \in \Sigma_O} m_x D_x - 2 \sum_{x \in \Sigma_A} \sum_{i=1}^{r_x} \underbrace{\sum_{j=i}^{r_x} \mathcal{E}_{x,j}}_{=: \widetilde{\mathcal{E}_{x,i}}} \\ &= \left(d - \sum_{x \in \Sigma_O} m_x - \sum_{x \in \Sigma_A} 2r_x \right) H + \sum_{x \in \Sigma_O} m_x (H - D_x) + 2 \sum_{x \in \Sigma_A} \sum_{i=1}^{r_x} (H - \widetilde{\mathcal{E}_{x,i}}). \end{aligned}$$

Since H is the pullback of a hyperplane, the linear system $|H|$ has no base points. Using the hypothesis,

$$\left| \left(d - \sum_{x \in \Sigma_O} m_x - \sum_{x \in \Sigma_A} 2r_x \right) H \right|$$

is base-point free as well. If $x \in X$ is a singular point, then x is scheme-theoretically cut out by hyperplanes. In particular, its ideal sheaf twisted by $\mathcal{O}(1)$ is globally generated, and so are the pullbacks $\mathcal{O}_P(H - D_x)$ and $\mathcal{O}_P\left(H - \widetilde{\mathcal{E}_{x,1}}\right)$, respectively. Similarly, $\mathcal{O}_P\left(H - \widetilde{\mathcal{E}_{x,i}}\right)$ is globally generated for any i. In total, $\mathcal{O}_P(Y)$ is a globally generated invertible sheaf on P.

It follows that if $C \subseteq P$ is an irreducible curve, then $Y.C \geq 0$. In order to show that Y is ample, it suffices to show that such an intersection $Y.C$ is always positive. If $\pi_* C$ is a curve on $\mathbb{P}^n$, then by the projection formula $H.C = (\pi_* C).\mathcal{O}(1) > 0$, thus $Y.C > 0$.

If C is contracted by π, then $H.C = 0$ again by the projection formula. By base-point freeness of $|H - D_x|$ and $|H - \widetilde{\mathcal{E}_{x,i}}|$, all the intersection numbers $D_x.C$ and $\widetilde{\mathcal{E}_{x,i}}.C$ are hence nonpositive. The Picard group of P is spanned by H, the $\mathcal{D}_x$ and the $\widetilde{\mathcal{E}_{x,i}}$. Since P is projective, there must be integers $h, d_x, e_{x,i}$ such that the divisor

$$A := hH + \sum_{x \in \Sigma_O} d_x \mathcal{D}_x + \sum_{x \in \Sigma_A} e_{x,i} \widetilde{\mathcal{E}_{x,i}}$$

is ample and thus $A.C > 0$. In particular, at least one of the intersection products $D_x.C$ or $\widetilde{\mathcal{E}_{x,i}}.C$ is nonzero and hence strictly negative. This implies $Y.C > 0$. □

Remark. This proof does not carry over to singular points of type D_k or E_k. For $n = 4$, the standard embedded resolution of these singularities has the property that the s exceptional divisors of the resolution $P \to \mathbb{P}^n$ break into several components when intersecting with the strict transform Y of X. In particular, $h^4(E) \geq s + 1 = h^2(P)$. But then by Lemma 1.2.10

$$h^4(Y) \geq h^4(X) + h^4(E) \geq h^4(X) + h^2(P) \geq 1 + h^2(P),$$

thus $h^4(Y) = h^2(Y) \neq h^2(P)$. Consequently, Y cannot be ample in P in virtue of the Lefschetz hyperplane theorem (Lemma 1.2.15).

However, in case that the ground field is of characteristic zero, the Hodge numbers of resolutions of hypersurfaces with at most ADE singularities were investigated by Rams [61, §4]. In this way, the defect $\delta(\overline{X})$ can be computed as the defect of a linear system similar to that in the case of nodal hypersurfaces constructed in §1.3.4.

Proof of Theorem 1.1.2

Proof. Suppose that X has defect. Let $\pi : (Y \subseteq P) \to (X \subseteq \mathbb{P}^n)$ be the embedded resolution from Proposition 1.4.2. By Corollary 1.4.8, Y cannot be ample in P. Now Lemma 1.4.9 implies that

$$\sum_{x \in \Sigma_O} m_x + \sum_{x \in \Sigma_A} 2r_x \geq d.$$ □

Note that if the chosen cohomology theory is Kähler-de Rham cohomology − which was excluded in the beginning of §1.4 − then X has also defect in algebraic de Rham cohomology by Corollary 1.3.5.

1.5 Factorial hypersurfaces

Let k be a field and let $X \subseteq \mathbb{P}^4_k$ be a hypersurface defined by a homogeneous polynomial $f \in k[x_0, \dots, x_4]$. Denote by $\operatorname{Pic}(X)$ resp. $\operatorname{Cl}(X)$ the group of Cartier resp. Weil divisors modulo linear equivalence.

X is *factorial* if the homgeneous coordinate ring $k[x_0, \dots, x_4]/(f)$ is a unique factorization domain. By [38, Exercise II.6.3], X is factorial if and only if the natural map $\operatorname{Pic}(X) \to \operatorname{Cl}(X)$ is an isomorphism, i.e., if and only if every Weil divisor on X is linearly equivalent to a Cartier divisor.

Furthermore, X is called $\mathbb{Q}$-*factorial* if the map $\operatorname{Pic}(X) \to \operatorname{Cl}(X)$ becomes an isomorphism after tensoring with $\mathbb{Q}$, i.e., if every Weil divisor on X is linearly equivalent to a $\mathbb{Q}$-Cartier divisor.

Theorem 1.5.1. *Suppose $k \subseteq \overline{\mathbb{F}_p}$. Let $X \subseteq \mathbb{P}^4_k$ be a hypersurface with at most isolated singularities. If $h^4_{\text{ét}}(X) = 1$ or $h^4_{\text{rig}}(X) = 1$, then X is factorial.*

Remarks.

- The corresponding statement in characteristic zero is shown in [59, Proposition 3.2]. Although it is stated only for hypersurfaces with at most ordinary multiple points as singularities, the proof remains valid for arbitrary isolated singularities. However, the proof requires transcendental methods.
- Over any field k, a projective hypersurface $X \subseteq \mathbb{P}^n_k$ is factorial if its singular locus has codimension ≥ 4 in X [59, Proposition 2.7]. In particular, if $X \subseteq \mathbb{P}^n_k$ has at most isolated singularities, then X is factorial if $n \geq 5$.

Example 1.5.2. Let $X = \{f = 0\} \subseteq \mathbb{P}^4_k$ for $f = x_1x_2 + x_3x_4$. Then the homogeneous coordinate ring $S := k[x_0, \dots, x_4]/(x_1x_2 + x_3x_4)$ is not a unique factorization domain, since $x_1x_2 = -x_3x_4$ in S. Moreover, $\{x_1x_2 = -x_3x_4 = 0\} \subseteq X$ is a Weil divisor which is not locally principal. By Theorem 1.5.1, $h^4(\overline{X}) > 1$. In fact, since X is the cone over a smooth quadric in $\mathbb{P}^3$, Example 1.3.8 shows $h^4(X) = 2$.

Let $X \subseteq \mathbb{P}^4_k$ be a hypersurface defined over $k = \overline{\mathbb{F}_p}$ with zero-dimensional singular locus Σ. Since X is a threefold, [13] provides a resolution of singularities $\pi : Y \to X$. Denote by E the exceptional divisor and by s the number of its irreducible components.

Lemma 1.5.3. $\operatorname{rk}\operatorname{Cl}(X) = \operatorname{rk}\operatorname{Pic}(Y) - s$.

Proof. Since Σ has codimension 3 in X,

$$\operatorname{Cl}(X) \cong \operatorname{Cl}(X \setminus \Sigma) \cong \operatorname{Cl}(Y \setminus E).$$

Let $E_1, \dots, E_s$ denote the irreducible components of the exceptional divisor E. Then there is a standard exact sequence [38, Proposition 6.5]

$$\bigoplus_{i=1}^{s} \mathbb{Z} \cdot E_i \to \operatorname{Cl}(Y) \to \operatorname{Cl}(Y \setminus E) \to 0.$$

This sequence is also exact on the left: Suppose $\sum_{i=1}^{s} a_i[E_i] = 0 \in \operatorname{Cl}(Y)$ for $a_1, \dots, a_s \in \mathbb{Z}$. If $H \subseteq Y$ is a general hyperplane, then $D := \sum_{i=1}^{s} a_i(E_i \cap H)$ is linearly equivalent to 0 as a divisor on the surface $Y \cap H$. However, as in [31, Example 2.4.4], D has negative self-intersection, contradicting that $[D] = 0 \in \operatorname{Cl}(Y \cap H)$. $\square$

Lemma 1.5.4. *For both étale and rigid cohomology,*

$$h^4(Y) - s \leq h^4(X).$$

Proof. Since $H^4(\Sigma) = 0$ as $\dim \Sigma = 0$, this follows from the long exact sequence

$$\cdots \to H^4(X) \to H^4(Y) \oplus H^4(\Sigma) \to H^4(E) \to \ldots$$

from Lemma 1.2.10. □

In order to compare Picard rank and Betti numbers, we need the following result on the étale cycle class map:

Lemma 1.5.5. *Let Z be a smooth projective variety over K. Then the étale cycle class map*

$$\mathrm{Pic}(Z) \otimes \mathbb{Q}_\ell \to H^2(Z, \mathbb{Q}_\ell(1))$$

is injective.

Proof. Let ℓ be a prime not equal to p. The étale cycle class map tensored with $\mathbb{Q}_\ell$ factors as

$$\mathrm{Pic}(Z) \otimes \mathbb{Q}_\ell \xrightarrow{\alpha} \mathrm{NS}(Z) \otimes \mathbb{Q}_\ell \xrightarrow{\beta} H^2(Z, \mathbb{Q}_\ell(1)),$$

where $\mathrm{NS}(Z)$ denotes the Néron-Severi group of Z. As in [53, pp. 216–217], one obtains that β is is injective. The kernel of α is precisely $\mathrm{Pic}^0(Z) \otimes \mathbb{Q}_\ell$. But since $k = \overline{\mathbb{F}_p}$, the group $\mathrm{Pic}^0(Z)$ is torsion [43, Lemma 2.16]. Hence α is injective as well. □

Corollary 1.5.6. *For both étale and rigid cohomology, we have* $\mathrm{rk}\,\mathrm{Cl}(X) \leq h^4(X)$. *In particular, if $h^4(X) = 1$, then X is $\mathbb{Q}$-factorial.*

Proof. By Lemma 1.5.5, $\mathrm{rk}\,\mathrm{Pic}(Y) \leq h^2_{\text{ét}}(Y, \mathbb{Q}_\ell(1)) = h^2_{\text{ét}}(Y, \mathbb{Q}_\ell)$. As étale and rigid cohomology are both Weil cohomologies and Y is defined over some finite field, applying Theorem 1.2.1 (3) yields $h^2_{\text{ét}}(Y) = h^2_{\text{rig}}(Y)$. Now Poincaré duality on Y gives $h^2(Y) = h^4(Y)$. Thus, with the help of Lemma 1.5.3 and Lemma 1.5.4,

$$\mathrm{rk}\,\mathrm{Cl}(X) = \mathrm{rk}\,\mathrm{Pic}(Y) - s \leq h^4(Y) - s \leq h^4(X). \qquad \square$$

Remark. Note that the inequality $\mathrm{rk}\,\mathrm{Cl}(X) \leq h^4_c(X)$ is true for an arbitrary normal threefold X which can be defined over $\overline{\mathbb{F}_p}$, not only for hypersurfaces.

Finally, we need to proceed from $\mathbb{Q}$-factoriality to factoriality.

Lemma 1.5.7. *If X is $\mathbb{Q}$-factorial, then X is factorial.*

Proof. We follow the proof of [59, Proposition 2.15]. Since X is normal and Cohen-Macaulay, the proof of [39, Proposition 2.15] generalizes and gives an exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Cl}(X) \to \bigoplus_{x \in \Sigma} \mathrm{Cl}(\mathcal{O}_{X,x}).$$

In particular, there is an injection

$$\mathrm{Cl}(X)/\,\mathrm{Pic}(X) \hookrightarrow \bigoplus_{x \in \Sigma} \mathrm{Cl}(\mathcal{O}_{X,x}).$$

By hypothesis, $\mathrm{Cl}(X)/\,\mathrm{Pic}(X)$ is a torsion group. Fix $x \in \Sigma$. By [18, Corollary 2.10], the Picard group of the punctured spectrum U_x of $\mathcal{O}_{X,x}$ is torsion-free. Since X has only isolated singularities, $\mathrm{Pic}(U_x) \cong \mathrm{Cl}(\mathcal{O}_{X,x})$, see [29, Proposition 18.10]. Consequently, $\mathrm{Cl}(X)/\,\mathrm{Pic}(X)$ is a torsion subgroup of a torsion-free group and hence trivial. Thus X is factorial. □

Proof of Theorem 1.5.1. Since étale and rigid cohomology behave well with respect to base change, $X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{\mathbb{F}_p}$ is factorial by Corollary 1.5.6 and Lemma 1.5.7. In other words, if S denotes the homogeneous coordinate ring of X, then $S \otimes_k \overline{\mathbb{F}_p}$ is factorial. But this implies that S and hence X are factorial. □

Chapter 2

Bertini theorems for simplicial toric varieties over finite fields

2.1 Overview

2.1.1 Simplicial toric varieties

In the upcoming sections, the following variety will serve as an ambient space:

Notation 2.1.1. Let $k = \mathbb{F}_q$ be a finite field of characteristic p. Let $\mathbb{P}$ be a toric k-variety of dimension n with torus T and fan Δ. Assume further that

- $\mathbb{P}$ is projective and normal,
- $\mathbb{P}$ is simplicial, i.e., all the cones in Δ are cones over simplices,
- $\mathbb{P}$ is split, i.e., the torus T is k-isomorphic to $\mathbb{G}_m^n$,
- $\mathbb{P}$ has no p-torsion in the Weil divisor class group $\mathrm{Cl}(\mathbb{P})$.

Remark. Note that $\mathbb{P}$ is not necessarily smooth. In particular, Weil divisors do not need to be Cartier.

The last two conditions in Notation 2.1.1 are only included because $\mathbb{F}_q$ is neither algebraically closed nor of characteristic zero. They ensure that the usual machinery of simplicial toric varieties is available for finite fields. The subtleties of constructing toric varieties over arbitrary C_1 fields are discussed in [35, §2].

For example, a toric variety $\mathbb{P}$ as in Notation 2.1.1 gives rise to a homogeneous coordinate ring ([14], [15, §5.2, §5.3]). This is a polynomial ring, which is graded by $\mathrm{Cl}(\mathbb{P})$ and whose number of variables is given by the number m of one-dimensional cones in the fan Δ.

Notation 2.1.2. Let $S = k[x_1, \dots, x_m]$ denote the homogeneous coordinate ring of $\mathbb{P}$. For a Weil divisor $D \in \mathrm{Cl}(\mathbb{P})$ on $\mathbb{P}$, denote by S_D the degree $[D]$ part of S.

Lemma 2.1.3. *For any $D \in \mathrm{Cl}(\mathbb{P})$, there is a natural isomorphism $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D)) \cong S_D$ compatible with multiplications.*

Proof. See [14, Prop. 1.1]. □

Moreover, consider the group $G := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(\mathbb{P}), \mathbb{G}_m)$. This group acts on affine m-space $\mathbb{A}^m$ via

$$G \times \mathbb{A}^m \to \mathbb{A}^m, \quad g \cdot (a_1, \dots, a_m) := (g(\deg(x_1)) \cdot a_1, \dots, g(\deg(x_m)) \cdot a_m).$$

This makes sense as $x_1, \dots, x_m$ are homogeneous elements of S with respect to the $\mathrm{Cl}(\mathbb{P})$-grading.

Theorem 2.1.4 (Quotient construction)**.** *There is a closed subscheme $B \subseteq \mathbb{A}^m$ such that $\mathbb{P}$ arises as a geometric quotient $\pi : \mathbb{A}^m \setminus B \to \mathbb{P}$ by the above group action.*

Proof. This is essentially [14, Theorem 2.1]. For carrying over the proof to positive characteristic, we first need to replace $\mathbb{C}^*$ by $\mathbb{G}_m$. This causes no problems, as the torus acting on $\mathbb{P}$ was assumed to be split in Notation 2.1.1, see also [27, §2] for this arithmetic reformulation.

The remaining step is to show that the group G is linearly reductive, so that the geometric invariant theory is still applicable. Since the divisor class group $\mathrm{Cl}(\mathbb{P})$ is a finitely generated group of rank $m-n$ [30, §3.4], it decomposes as $\mathrm{Cl}(\mathbb{P}) \cong \mathbb{Z}^{m-n} \times H$, where H is a finite group. By the last assumption in Notation 2.1.1, the order of H is not divisible by p. Consequently, $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(\mathbb{P}), \mathbb{G}_m) \cong \mathbb{G}_m^{m-n} \times \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{G}_m)$ is the product of a torus and a finite group of order not divisible by p and is hence linearly reductive [56]. □

Example 2.1.5. The main source of examples for varieties as in Notation 2.1.1 will be weighted projective spaces $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_n)$ for positive integers $w_0, \ldots, w_n$. They are defined as $\operatorname{Proj} k[x_0, \ldots, x_n]$, where $\deg(x_i) = w_i$ for all i.

The divisor class group of such a weighted projective space $\mathbb{P}$ is always isomorpic to $\mathbb{Z}$ [15, Example 5.1.14], and hence $G = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Cl}(\mathbb{P}), \mathbb{G}_m) \cong \mathbb{G}_m$. As homogeneous coordinate ring, we get back $k[x_0, \ldots, x_n]$ with the grading by the w_i. The action of G on $\mathbb{A}^{n+1}$ is simply

$$\mathbb{G}_m \times \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}, \quad g \cdot (a_0, \ldots, a_n) := (g^{w_0} \cdot a_0, \ldots, g^{w_n} \cdot a_n),$$

and $\mathbb{P}$ is the quotient by this group action. In particular, closed points in $\mathbb{P}$, i.e., $\mathbb{G}_m$-orbits, may be represented by (weighted) homogeneous coordinates $(a_0 : \cdots : a_n)$.

Unless all weights are 1, weighted projective spaces are singular. More precisely, their singular locus is given by

$$\{(a_0 : \cdots : a_n) \mid \gcd(\{w_i \mid a_i \neq 0, i = 0, \ldots, n\}) > 1\}.$$

These are precisely the points where the $\mathbb{G}_m$-action is not free. In order to ensure that $\mathbb{P}$ is normal, one hence requires that

$$\gcd(w_0, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n) = 1 \quad \text{for all } i,$$

since otherwise $\mathbb{P}$ would contain a singular locus of codimension one.

2.1.2 Main results

The Bertini theorem on quasismoothness

The main result of §2.2 is the following:

Theorem 2.1.6. *Let k and $\mathbb{P}$ be as in Notation 2.1.1. Fix a Weil divisor D and an ample Cartier divisor E on $\mathbb{P}$. Let $X \subseteq \mathbb{P}$ be any quasismooth subscheme such that the intersection of X with the singular locus of $\mathbb{P}$ is zero-dimensional. Then*

$$\lim_{d \to \infty} \frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \mid X \cap \{f = 0\} \text{ is quasismooth}\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))} = \prod_{P \in X \text{ closed}} \left(1 - q^{-\nu_D(P)}\right),$$

where $\nu_D(P)$ is a non-negative integer depending on P and D with the property that $\nu_D(P)$ equals $\deg P \cdot (\dim X + 1)$ if X is smooth at P.

Remarks.

- If $\mathbb{P} = \mathbb{P}^n$, $D = 0$ and E is a hyperplane, this recovers Poonen's result [60, Theorem 1.1]:

  $$\lim_{d \to \infty} \frac{\#\{f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \mid X \cap \{f = 0\} \text{ is smooth}\}}{\#H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))} = \frac{1}{\zeta_X(\dim X + 1)}.$$

  Here, ζ_X denotes the Hasse-Weil zeta function

  $$\zeta_X(s) = \prod_{P \in X \text{ closed}} \left(1 - q^{-s \deg P}\right)^{-1} = \exp\left(\sum_{r=1}^{\infty} \#X(\mathbb{F}_{q^r}) \cdot \frac{q^{-rs}}{r}\right), \quad s \in \mathbb{C},$$

  which converges for $\operatorname{Re}(s) > \dim X$.

- In [28], Poonen's formula is generalized to a semiample setting. In the special case that X is a smooth toric variety, the result [28, Theorem 1.1] implies our Theorem 2.1.6.
- For a precise definition of the number $\nu_D(P)$, its properties and its visualization, we refer to Definition 2.2.10 and the following pages.
- The formula in Theorem 2.1.6 is in particular valid if $\nu_D(P) = 0$ for some closed point $P \in Y$. In this case, both sides of the equation are zero. Moreover, $X \cap \{f = 0\}$ fails to be quasismooth for all $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))$ and all $d \geq 0$, see Corollary 2.2.16. For a situation where $\nu_D(P) = 0$ occurs, see Examples 2.2.14 and 2.2.17. However, if X is smooth or D is Cartier, then $\nu_D(P)$ is always positive by Lemma 2.2.12.
- If the intersection of X with the singular locus of $\mathbb{P}$ is of positive dimension, then Theorem 2.1.6 may fail, see Lemma 2.2.18 and Example 2.2.20.

The proof uses a modified version of Poonen's closed point sieve: The closed points of X are divided into low, medium and high degree points and it is shown that the impact of the latter two is negligible. The main difference to Poonen is that we have to cope with the singularities coming from the ambient space $\mathbb{P}$.

At first, the notion of quasismoothness will be discussed. After an extensive study of restriction maps to zero-dimensional subschemes and the numbers $\nu_D(P)$ in §2.2.2, Poonen's strategy is adapted to prove Theorem 2.1.6 in §2.2.3.

The subsequent section §2.2.4 contains some direct corollaries of Theorem 2.1.6.

In §2.2.5 we give formulas for the density of quasismooth hypersurfaces with an upper bound on the number of singular points and the length of the singular schemes, respectively. Finally, for smooth toric varieties, we show that hypersurfaces of degree d whose singular scheme has length growing with d form a set of density zero:

Theorem 2.1.7. *In the situation of Theorem 2.1.6, suppose that $\mathbb{P}$ is smooth. Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ be a function with $\lim_{d\to\infty} g(d) = \infty$. Then*

$$\lim_{d\to\infty} \frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \mid \operatorname{length}(\Sigma(f)) < g(d)\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))} = 1.$$

The last subsection 2.2.6 deals with the density of hypersurfaces in $\mathbb{P}^n$ over a finite field with defect. The proof uses the results of §1.4. As a byproduct, we give an estimate of the density of hypersurfaces with at most A_k singularities.

The Bertini theorem on geometric irreducibility

Let k and $\mathbb{P}$ be as in Notation 2.1.1. Fix a Weil divisor D and an ample Cartier divisor E on $\mathbb{P}$. The content of §2.3 is an extension of the main result of [11] to simplicial toric varieties. Although many results can almost be transferred word by word, the singularities of $\mathbb{P}$ require a few changes.

Notation 2.1.8. Let X be a scheme of finite type over $\overline{k}$, $Y \subseteq X$ a subscheme. Let further $\varphi : X \to \mathbb{P}_{\overline{k}}$ be a $\overline{k}$-morphism.

- Y is called *horizontal* if $\dim \overline{\varphi(Y)} \geq 1$ and $\overline{\varphi(Y)}$ is not contained in $(\mathbb{P}_{\overline{k}})^{\mathrm{sing}}$,
- $\operatorname{Irr} Y$ denotes the set of irreducible components of Y,

- $\operatorname{Irr}_{\mathrm{horiz}} Y$ is the set of all horizontal irreducible components of Y,
- Y_{horiz} denotes the union of all horizontal irreducible components of Y.

For sections $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))$, where d is an integer, define $X_f := \varphi^{-1}(\{f = 0\})$. Moreover, the singular resp. smooth locus of a scheme X will be denoted by X^{sing} and X^{sm}, respectively.

Theorem 2.1.9. *Suppose that $\varphi : X \to \mathbb{P}_{\overline{k}}$ is a $\overline{k}$-morphism such that for each $C \in \operatorname{Irr} X$ holds $\dim \overline{\varphi(C)} \geq 2$ and $\dim \overline{\varphi(C)} \cap (\mathbb{P}_{\overline{k}})^{\mathrm{sing}} \leq \dim \overline{\varphi(C)} - 2$. Then*

$$\lim_{d\to\infty} \frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \mid \operatorname{Irr} X \to \operatorname{Irr}_{\mathrm{horiz}} X_f, C \mapsto (C \cap X_f)_{\mathrm{horiz}} \text{ is a bijection}\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))} = 1.$$

Remarks.

- Suppose that $D = 0$ and E is a very ample Cartier divisor defining a closed immersion $i : \mathbb{P} \hookrightarrow \mathbb{P}^n$. This gives a linear map
$$i^* : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(dE)),$$
which is surjective for $d \gg 0$. In particular, Theorem 2.1.9 is obtained from [11, Theorem 1.6], as $\{i^*(g) = 0\} = i^{-1}(\{g = 0\})$ for $g \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. Moreover, this holds for any projective variety $\mathbb{P}$ over $\mathbb{F}_q$, and the conditions on $(\mathbb{P}_{\overline{k}})^{\mathrm{sing}}$ may be dropped as well.
- The codimension two condition on the intersection with the singular locus is necessary if D is not trivial, see Example 2.3.12.

If X and φ satisfy the hypotheses of Theorem 2.1.9 and moreover φ is an immersion, then Theorem 2.1.9 implies a Bertini theorem for geometrically irreducible hypersurfaces:

Corollary 2.1.10. *Let X be a geometrically irreducible subscheme of $\mathbb{P}$ over k. If $\dim X \geq 2$ and $\dim \overline{X} \cap \mathbb{P}^{\mathrm{sing}} \leq \dim X - 2$, then*

$$\lim_{d\to\infty} \frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \mid X \cap \{f = 0\} \text{ is geometrically irreducible}\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))} = 1.$$

In particular, this holds for $X = \mathbb{P}$.

Proof. Any irreducible component of $X \cap \{f = 0\}$ is horizontal, since

$$\dim X \cap \{f = 0\} \geq \dim X - 1$$

and the singular locus of $\mathbb{P}$ has codimension ≥ 2 in X. □

Remark. In contrast to the quasismoothness result (Theorem 2.1.6), it is hence true that 100% of all hypersurfaces are geometrically irreducible. Moreover, note that the hypotheses on the singular locus are milder.

2.2 Quasismooth hypersurfaces

Let k and $\mathbb{P}$ be as in Notation 2.1.1. Denote by π the quotient map of Theorem 2.1.4.

2.2.1 Quasismoothness

Definition 2.2.1 (Quasismoothness)**.** Let $X \subseteq \mathbb{P}$ be a subscheme and $P \in X$ be a closed point.

- X is called *quasismooth at* P if $\pi^{-1}(X)$ is smooth at all points in $\pi^{-1}(P)$.
- X is called *quasismooth* if $\pi^{-1}(X)$ is smooth.

Remark. In particular, $\mathbb{P}$ is quasismooth.

Lemma 2.2.2. *Let $X \subseteq \mathbb{P}$ be a subscheme and $P \in X$ be a closed point.*

(1) *If X is smooth at P, then X is quasismooth at P.*

(2) *If X is quasismooth at P and $\mathbb{P}$ is smooth at P, then X is smooth at P. In other words, if X is quasismooth, then its singular locus is contained in the singular locus of $\mathbb{P}$.*

(3) *If $\pi^{-1}(X)$ is smooth at some point in $\pi^{-1}(P)$, then X is quasismooth at P.*

Proof.

(1) Pick a point $Q \in \pi^{-1}(P)$ and consider the standard exact sequence [38, Prop. II.8.11]

$$\pi^*\Omega_X \otimes \kappa(Q) \to \Omega_{\pi^{-1}(X)} \otimes \kappa(Q) \to \Omega_{\pi^{-1}(X)/X} \otimes \kappa(Q) \to 0, \tag{2.1}$$

where $\kappa(Q)$ denotes the residue field of Q.

As mentioned in the proof of Theorem 2.1.4, the group $G = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(\mathbb{P}), \mathbb{G}_m)$ is linearly reductive. It acts on $\mathbb{P}$ and hence on X with finite isotropy groups [15, Exercise 5.1.11]. This means that the fibers of the quotient map π are themselves linearly reductive group schemes, being quotients of the G by a finite group scheme. By [56], there is no p-torsion, so the fibers of π are smooth group schemes. Hence the vector space $\Omega_{\pi^{-1}(X)/X} \otimes \kappa(Q)$ is of dimension $\dim G = \dim \pi^{-1}(X) - \dim X$.

Moreover, as X is smooth at P, $\pi^*\Omega_X \otimes \kappa(Q)$ has dimension $\dim X$. This implies

$$\dim \Omega_{\pi^{-1}(X)} \otimes \kappa(Q) \leq \dim X + \dim \pi^{-1}(X) - \dim X = \dim \pi^{-1}(X),$$

thus $\pi^{-1}(X)$ is smooth at Q.

(2) Let $Q \in \pi^{-1}(P)$. Since $\mathbb{P}$ is smooth at P, the quotient map π induces a flat homomorphism of local rings $\mathcal{O}_{X,P} \to \mathcal{O}_{\pi^{-1}(X),Q}$ [55, Prop. 0.9]. Since $\Omega_{\pi^{-1}(X)/X,Q} \otimes \kappa(Q)$ is of dimension $\dim \pi^{-1}(X) - \dim X$, it follows that π is smooth at Q [64, Tag 01V9]. As a consequence, the sequence (2.1) is exact [64, Tag 02K4] and hence X is smooth at P.

(3) Let $Q, Q' \in \pi^{-1}(P)$. The translation maps from Q to Q' and vice versa give isomorphisms $\Omega_{\pi^{-1}(X),Q} \otimes \overline{k} \cong \Omega_{\pi^{-1}(X),Q'} \otimes \overline{k}$. □

Example 2.2.3. If $\mathbb{P}$ is a weighted projective space, then quasismoothness of a subscheme X means smoothness of the affine quasicone of X. This can be effectively tested by means of the Jacobian criterion. For an ordinary projective space, Lemma 2.2.2 (2) simply states the Jacobian criterion for varieties in projective space is indeed a test for smoothness.

Quasismooth hypersurfaces

Let X be a quasismooth subscheme of $\mathbb{P}$. Pick a $\mathrm{Cl}(\mathbb{P})$-homogeneous polynomial of the homogeneous coordinate ring S. Then quasismoothness of the hypersurface $X \cap \{f = 0\}$ is still local on X: If P is a closed point of X, we pull back the first-order infinitesimal neighborhood of all points in $\pi^{-1}(P)$. More precisely, we have the following:

Lemma 2.2.4. *Let $X \subseteq \mathbb{P}$ be a quasismooth subscheme, $P \in X$ a closed point. Then there is a closed subscheme $X_P \subseteq \mathbb{P}$ such that for all Weil divisors D on X and $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D))$ we have*

$$X \cap \{f = 0\} \text{ is quasismooth at } P \quad \Leftrightarrow \quad \varphi_{P,D}(f) \neq 0,$$

where $\varphi_{P,D}$ is the natural restriction map

$$\varphi_{P,D}: \quad H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D)) \to H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P}).$$

Proof. Let S be the homogeneous coordinate ring of $\mathbb{P}$ and $\pi : \mathbb{A}^m \setminus B \to \mathbb{P}$ the map from the quotient construction in Theorem 2.1.4. Note that S is the coordinate ring of $\mathbb{A}^m$, so for any $Q \in \pi^{-1}(P)$, there are natural maps

$$\vartheta_Q: \quad S \to \mathcal{O}_{\pi^{-1}(X),Q} \to \mathcal{O}_{\pi^{-1}(X),Q}/\mathfrak{m}_Q^2,$$

where $\mathfrak{m}_Q$ is the maximal ideal of the local ring $\mathcal{O}_{\pi^{-1}(X),Q}$ of $\pi^{-1}(X)$ at Q. Denote by I_P the largest homogeneous ideal of S contained in $\bigcap_{Q\in\pi^{-1}(P)} \ker \vartheta_Q$ with respect to the grading given by $\mathrm{Cl}(\mathbb{P})$. Via the toric version of the ideal-variety correspondence [14, Prop. 2.4], I_P defines a closed subscheme X_P of $\mathbb{P}$.

Let D be a Weil divisor on X. For $f \in S_D$, the intersection $X \cap \{f = 0\}$ is not quasismooth at P if and only if there is a point $Q \in \pi^{-1}(P)$ such that $\vartheta_Q(f) = 0$. By Lemma 2.2.2 (3), this is equivalent to $\vartheta_Q(f) = 0$ for all $Q \in \pi^{-1}(P)$, which is in turn equivalent to $f \in I_P \cap S_D$. In other words, f lies in $\ker \varphi_{P,D}$, after applying the isomorphism $S_D \cong H^0(X, \mathcal{O}_{\mathbb{P}}(D))$ from Lemma 2.1.3. □

Example 2.2.5. Suppose $X = \mathbb{P}$. Let $P \in \mathbb{P}$ be a closed point. By definition, the ideal I_P inside the homogeneous coordinate ring $S = K[x_1, \ldots, x_m]$ is generated by all homogeneous polynomials $f \in S$ such that

$$f(Q) = \frac{\partial f}{\partial x_1}(Q) = \cdots = \frac{\partial f}{\partial x_m}(Q) = 0$$

for all $Q \in \pi^{-1}(P)$. The quasismoothness of the hypersurface $\{f = 0\}$ can hence be checked with the usual Jacobian criterion on $\mathbb{A}^m \setminus B$.

When X is closed in $\mathbb{P}$, there is a more accessible algebraic interpretation of the ideal I_P:

Lemma 2.2.6. *Let $X \subseteq \mathbb{P}$ be a closed quasismooth subscheme, cut out by a $\mathrm{Cl}(\mathbb{P})$-homogeneous ideal J_X. Let $P \in X$ be a closed point and denote by $\mathfrak{p}$ the prime ideal of S defining P in $\mathbb{P}$. Then*

$$I_P = J_X + \mathfrak{p}^{(2)},$$

where $\mathfrak{p}^{(2)}$ denotes the symbolic square of $\mathfrak{p}$.

Proof. As seen in the proof of Lemma 2.2.2 (2), the fiber $\pi^{-1}(P)$ is a smooth group scheme. This implies that $S/(J_X + \mathfrak{p})$ is a regular ring. Since X is quasismooth, S/J_X is a regular ring as well. The statement is now an application of [26, Corollary 1]. □

2.2.2 Restricting sections to zero-dimensional subschemes

Let k and $\mathbb{P}$ be as above. Further let $X \subseteq \mathbb{P}$ be a quasismooth subscheme. Fix a Weil divisor D and an ample Cartier divisor E on $\mathbb{P}$.

The goal is to determine the proportion of sections of $D + dE$ having a quasismooth intersection with X as $d \to \infty$. In view of Lemma 2.2.4, we will take a closer look at the k-vector spaces $H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P})$ and the map $\varphi_{P,D}$.

Surjectivity of $\varphi_{P,D}$

Let Z be a zero-dimensional subscheme of $\mathbb{P}$ and denote the corresponding closed immersion by $i : Z \hookrightarrow \mathbb{P}$. Then there is an associated surjective map $\mathcal{O}_{\mathbb{P}} \twoheadrightarrow i_*\mathcal{O}_Z$ of sheaves. Tensoring with $\mathcal{O}_{\mathbb{P}}(D)$, taking the long exact sequence in cohomology, and applying the projection formula, this yields a natural map on global sections

$$\varphi_Z : H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D)) \to H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z).$$

This way, we recover $\varphi_{P,D}$ if Z equals the scheme X_P. Tensoring with $\mathcal{O}_{\mathbb{P}}(D + dE)$ instead of $\mathcal{O}_{\mathbb{P}}(D)$, we obtain

$$\varphi_{Z,d} : H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \to H^0(Z, \mathcal{O}_{\mathbb{P}}(D + dE)|_Z) \cong H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z).$$

The last isomorphism comes from the fact that

$$\mathcal{O}_{\mathbb{P}}(D + dE) \cong \mathcal{O}_{\mathbb{P}}(D) \otimes \mathcal{O}_{\mathbb{P}}(E)^{\otimes d},$$

since $\mathbb{P}$ is normal, and that $\mathcal{O}_{\mathbb{P}}(E)$ is locally free of rank one, as E is Cartier.

We see that $\varphi_{Z,d}$ is surjective if $H^1(\mathbb{P}, \mathcal{K} \otimes \mathcal{O}_{\mathbb{P}}(dE))$ vanishes, where $\mathcal{K}$ is the kernel of the surjection $\mathcal{O}_{\mathbb{P}}(D) \to \mathcal{O}_{\mathbb{P}}(D)|_Z$. Since $\mathcal{K}$ is a coherent sheaf on the projective variety $\mathbb{P}$ and E is ample, we have the following result by Serre vanishing [38, Theorem II.5.3]:

Lemma 2.2.7. *For any zero-dimensional subscheme $Z \subseteq \mathbb{P}$ exists an integer d_Z such that the natural map*

$$\varphi_{Z,d} : H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \to H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z)$$

is surjective for all $d \geq d_Z$.

The remainder of this subsection is devoted to an improvement of this result. In order to achieve this, we need to have a look at multiplication of sections on toric varieties. Define $\operatorname{reg}_E(D)$ to be the smallest integer $\ell \geq 1$ such that

$$H^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + (d - i)E)) = 0 \quad \text{for all } d \geq \ell \text{ and } i \geq 1.$$

The number $\operatorname{reg}_E(D)$ exists and coincides with the Castelnuovo-Mumford regularity of the sheaf $\mathcal{O}_{\mathbb{P}}(D)$ with respect to the ample line bundle $\mathcal{O}_{\mathbb{P}}(E)$ on $\mathbb{P}$.

Lemma 2.2.8. *The natural multiplication map*

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(E)) \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + (d + 1)E))$$

is surjective for all $d \geq \operatorname{reg}_E(D)$.

Proof. See [54, Theorem 2]. □

We will now give an enhanced version of Lemma 2.2.7:

Lemma 2.2.9. *For all zero-dimensional subschemes Z the map $\varphi_{Z,d}$ is surjective whenever*

$$d \geq \dim_k H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z) + \mathrm{reg}_E(D) - 1.$$

Proof. Let Z be a zero-dimensional subscheme of $\mathbb{P}$. Since cohomology commutes with flat base change, we can check the surjectivity of the map $\varphi_{Z,d}$ after a base change to some field extension. Thus we can w.l.o.g. assume the existence of a section $f_0 \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(E)) \cong S_E$ defined over k satisfying $\{f_0 = 0\} \cap Z = \emptyset$. Choose elements $f_1, \ldots, f_s \in S_E$ such that $\{f_0, \ldots, f_s\}$ forms a k-basis of S_E.

By Lemma 2.2.8, we have surjective multiplication maps

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + \ell E)) \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(E))^{\otimes d-\ell} \twoheadrightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)),$$

whenever $d \geq \ell := \mathrm{reg}_E(D)$. By Lemma 2.1.3, these are compatible with the isomorphisms $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-)) \cong S_{[-]}$. Identify now $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(E))^{\otimes(d-\ell)}$ with the space of homogeneous polynomials in $f_0, \ldots, f_s$ of degree $d - \ell$. Homogenization via f_0 yields an isomorphism

$$S_{D+\ell E} \otimes k[f_1, \ldots, f_s]_{\leq d-\ell} \cong S_{D+\ell E} \otimes k[f_0, \ldots, f_s]_{d-\ell}$$

and we thus obtain a surjective k-linear map

$$\begin{aligned} S_{D+\ell E} \otimes k[f_1, \ldots, f_s]_{\leq d-\ell} &\twoheadrightarrow S_{D+\ell E} \otimes S_E^{\otimes(d-\ell)} \\ &\xrightarrow{\simeq} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + \ell E)) \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(E))^{\otimes(d-\ell)} \\ &\twoheadrightarrow H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)), \end{aligned}$$

the last map being the multiplication.

Consider now the composition

$$\vartheta_d : S_{D+\ell E} \otimes k[f_1, \ldots, f_s]_{\leq d-\ell} \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \xrightarrow{\varphi_{Z,d}} H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z).$$

The linear map ϑ_d becomes surjective for large enough d by Lemma 2.2.7. Furthermore, if ϑ_d is surjective, then so is $\varphi_{Z,d}$. Define the subspaces

$$B_j := \vartheta_d \left(S_{D+\ell E} \otimes k[f_1, \ldots, f_s]_{\leq j} \right), \quad j = -1, \ldots, d - \ell.$$

This yields an ascending chain of subspaces $\{0\} = B_{-1} \subseteq B_0 \subseteq \ldots$ of $H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z)$, thus for some $j \geq -1$ holds $B_j = B_{j+1}$. Then, if $[f_i]$ denotes the image of f_i in $H^0(Z, \mathcal{O}_Z)$, we obtain

$$B_{j+2} = \sum_{i=1}^{s} [f_i] \cdot B_{j+1} = \sum_{i=1}^{s} [f_i] \cdot B_j = B_{j+1}.$$

A fortiori, $B_r = B_j$ for $r \geq j$. But ϑ_d is eventually surjective, so as soon as $B_j = B_{j+1}$, it must be the whole of $H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z)$. In particular, the dimension of B_j grows with every step until it reaches $\dim_k H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z)$. This means that ϑ_d and hence $\varphi_{Z,d}$ are surjective whenever

$$d - \ell \geq \dim_k H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z) - 1.$$ □

The numbers $\nu_D(P)$

Definition 2.2.10. With the same notation as above, define

$$\nu_D(P) := \dim_k H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P}).$$

In the case $X = \mathbb{P}$, a general recipe to compute $\nu_D(P)$ is the following: Let $\pi : \mathbb{A}^m \setminus B \to \mathbb{P}$ denote the quotient map from Theorem 2.1.4. Pick a closed point $P \in \mathbb{P}$. By Lemma 2.2.4, a section $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))$ lies in the kernel of

$$\varphi_{X_P,d} : H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \to H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P})$$

if and only if $\{f = 0\} \subseteq \mathbb{P}$ is not quasismooth at P, i.e., if and only if the hypersurface $\{f = 0\} \subseteq \mathbb{A}^m \setminus B$ is not smooth at some point $Q \in \pi^{-1}(P)$. The latter condition can be tested with the Jacobian criterion and gives therefore a description of $\ker \varphi_{X_P,d}$. Since $\varphi_{X_P,d}$ is surjective for $d \gg 0$ by Lemma 2.2.7, this computes the number $\nu_D(P)$ as the codimension of $\ker \varphi_{X_P,d}$ in $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))$ for large enough d.

An alternative description stems from Lemma 2.2.6: Suppose that $X \subseteq \mathbb{P}$ is a closed quasismooth subscheme with ideal J_X inside S. Pick a closed point $P \in X$ and let $\mathfrak{p}$ denote the corresponding prime ideal in S. Then, for $d \gg 0$, $\nu_D(P)$ equals the k-dimension of the degree $(D + dE)$-part of the $\mathrm{Cl}(\mathbb{P})$-graded S-module $S/(J_X + \mathfrak{p}^{(2)})$.

Example 2.2.11. Let $\mathbb{P}$ be the weighted projective space $\mathbb{P}(1, \ldots, 1, 2)$ of dimension n with the coordinates $x_0, \ldots, x_n$. Furthermore, let $X = \mathbb{P}$, $D = V(x_n)$ and $E = V(x_0)$. Then $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))$ corresponds to the space of weighted homogeneous polynomials in the variables $x_0, \ldots, x_n$ of degree $2d + 1$. Such a polynomial f can be written as

$$f = \sum_{i=0}^{d} x_n^i \cdot f_i(x_0, \ldots, x_{n-1}), \quad f_i \text{ homogeneous of degree } 2(d - i) + 1.$$

If $Q \in \mathbb{A}^{n+1} \setminus \{0\}$ lies over the singular point $P = (0 : \cdots : 0 : 1)$, then one computes that both f and $\frac{\partial f}{\partial x_n}$ always vanish at Q. Moreover, the partial derivatives $\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_{n-1}}$ vanish simultaneously at Q if and only if $f_d = 0$. Thus f lies in $\ker \varphi_{X_P,d}$ if and only if $f_d = 0$. Since f_d is a linear homogeneous polynomial in n variables, this is a codimension n condition. Hence $\nu_D(P) = n$.

Alternatively, let $\mathfrak{p} = (x_0, \ldots, x_{n-1})$ be the prime ideal of $S = k[x_0, \ldots, x_n]$ corresponding to $P = (0 : \cdots : 0 : 1)$. One checks that $\mathfrak{p}^{(2)} = \mathfrak{p}^2$, so

$$\nu_D(P) = \lim_{d \to \infty} \dim_k (S/\mathfrak{p}^2)_{2d+1} = n,$$

as $(S/\mathfrak{p}^2)_{2d+1}$ is spanned by the classes of $x_0 x_n^d, x_1 x_n^d, \ldots, x_{n-1} x_n^d$.

For more computations of $\nu_D(P)$, see Example 2.2.17 and the graphics in Example 2.2.14. The following lemma summarizes some general properties of the number $\nu_D(P)$:

Lemma 2.2.12 (Properties of $\nu_D(P)$)**.** *Let $X \subseteq \mathbb{P}$ is a quasismooth subscheme of $\mathbb{P}$ and let P be a closed point of X.*

(1) *$\nu_D(P)$ is divisible by* $\deg P$.

(2) *If D is Cartier, then $\nu_D(P) \geq \deg P$.*

(3) *If $\mathbb{P}$ is smooth at P, then $\nu_D(P) = \deg P \cdot (\dim X + 1)$.*

(4) *In general, $\nu_D(P) \leq \deg P \cdot (\dim X + 1)$.*

Proof. Recall that in the proof of Lemma 2.2.4, X_P was defined by the homogeneous ideal I_P, which was the largest homogeneous ideal contained in $\bigcap_{Q \in \pi^{-1}(P)} \ker(S \to \mathcal{O}_{\pi^{-1}(Y),Q}/\mathfrak{m}_Q^2)$.

(1) Let $\kappa(P)$ be the residue field of P. Since k is perfect, the field extension $\kappa(P)/k$ is separable. Suppose that $P_1, \ldots, P_{\deg P}$ are the $\deg P$ distinct points lying over P. Denote by $\mathbb{P}'$, X' and D' the respective base changes of $\mathbb{P}$, X and D to $\kappa(P)$. Then

$$H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P}) \otimes_k \kappa(P) \cong \bigoplus_{i=1}^{\deg P} H^0(\mathbb{P}'_{P_i}, \mathcal{O}_{\mathbb{P}'}(D')|_{X'_{P_i}}),$$

where all the direct summands on the right-hand side have the same dimension over $\kappa(P)$.

(2) If D is Cartier, then $\mathcal{O}_{\mathbb{P}}(D)$ is locally free and hence

$$H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P}) \cong H^0(X_P, \mathcal{O}_{X_P}).$$

Since the latter space is of positive dimension, (1) yields the estimate $\nu_D(P) \geq \deg P$.

(3) Let $\mathcal{O}_P$ be the local ring of X at P with maximal ideal $\mathfrak{m}_P$. Since $\mathcal{O}_{\mathbb{P}}(D)$ is invertible when restricted to the smooth locus, we get a honest restriction map $\rho : S \to \mathcal{O}_P$. By Lemmas 2.2.4 and 2.2.2 (2),

$$\begin{aligned} f \in I_P &\Leftrightarrow X \cap \{f = 0\} \text{ quasismooth at } P \\ &\Leftrightarrow X \cap \{f = 0\} \text{ smooth at } P \\ &\Leftrightarrow \rho(f) \in \mathfrak{m}_P^2. \end{aligned}$$

Since X is smooth at P, the k-dimension of $\mathcal{O}_P/\mathfrak{m}_P^2$ equals $\deg P \cdot (\dim X + 1)$.

(4) Pick a point $Q \in \pi^{-1}(P)$ of the same degree as P. As the restriction map $\varphi_{X_P,d}$ is eventually surjective for large enough d by Lemma 2.2.7, $H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P})$ has the same dimension as $(S/I_P)_{D+dE}$ for all $d \gg 0$. But the latter space injects into $\mathcal{O}_{\pi^{-1}(X),Q}/\mathfrak{m}_Q^2$, which has dimension $\deg Q \cdot (\dim \pi^{-1}(X) + 1)$, as $\pi^{-1}(X)$ is smooth at Q. The image of $(S/I_P)_{D+dE}$ is contained in the invariant part under the group action in the quotient construction, which has codimension $\dim \pi^{-1}(X) - \dim X$ by the reasoning in the proof of Lemma 2.2.2 (1). Consequently,

$$\nu_D(P) \leq \deg Q \cdot (\dim X + 1) = \deg P \cdot (\dim X + 1). \qquad \square$$

Corollary 2.2.13. *Suppose that P is a closed point of the quasismooth subscheme $X \subseteq \mathbb{P}$. Let d be a positive integer such that*

$$\deg P \leq \frac{d - \operatorname{reg}_E(D) + 1}{\dim X + 1}.$$

Then the map $\varphi_{X_P,d} : H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \to H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P})$ is surjective.

Proof. Lemma 2.2.12 (4) gives the bound

$$\nu_D(P) \leq \deg P \cdot (\dim X + 1) \leq d - \operatorname{reg}_E(D) + 1.$$

Thus $\varphi_{X_P,d}$ is surjective, as $d \geq \nu_D(P) + \operatorname{reg}_E(D) - 1$ due to Lemma 2.2.9. $\square$

Visualization of $\nu_D(P)$

The values of $P \mapsto \nu_D(P)/\deg P$ may be visualized on the fan Δ of $\mathbb{P}$. There is a correspondence between cones $\sigma \in \Delta$ and standard open affines $U_\sigma \subseteq \mathbb{P}$ [15, Theorem 3.2.6]: Let $\rho_1, \ldots, \rho_m$ denote the one-dimensional cones in Δ. Then U_σ is the spectrum of the homogeneous localization of the homogeneous coordinate ring $S = k[x_1, \ldots, x_m]$ at $\prod_{\rho_i \not\subseteq \sigma} x_i$. However, we will take a different interpretation of the cones: Each cone σ will instead by labeled by the closed subvariety $\bigcap_{\rho_i \not\subseteq \sigma} \{x_i = 0\}$.

Example 2.2.14. Consider the weighted projective space $\mathbb{P} = \mathbb{P}(1,2,3,6)$ with coordinates x_0, x_1, x_2, x_3. These build the four one-dimensional cones $\rho_0, \rho_1, \rho_2, \rho_3$ of the fan of $\mathbb{P}$. The divisor class group of $\mathbb{P}$ is isomorphic to $\mathbb{Z}$. The sheaf $\mathcal{O}_{\mathbb{P}}(\ell)$ is invertible if and only if ℓ is divisible by 6. If ℓ is positive, then $\mathcal{O}_{\mathbb{P}}(\ell)$ is also ample, see [21].

The values of $P \mapsto \nu_\ell(P)/\deg(P)$ on the fan of $\mathbb{P}$ with the above labeling are sketched in Figure 2.1 below. The one-dimensional cones are $\rho_0, \rho_1, \rho_2, \rho_3$ in counter-clockwise order, starting with ρ_0 pointing downward. With the above labeling, these stand for the points $(1 : 0 : 0 : 0)$, $(0 : 1 : 0 : 0)$, $(0 : 0 : 1 : 0)$, $(0 : 0 : 0 : 1)$ respectively. The two-dimensional cones label points with at least two coordinates being zero. The interior of the three-dimensional cones is not drawn: They mark only points P with a single zero coordinate, which are all smooth points of $\mathbb{P}$ and hence $\nu_\ell(P)/\deg P = 4$ by Lemma 2.2.12 (3).

Observe that $\nu_\ell(0 : 0 : 0 : 1) = 0$ when $\ell \equiv 4$ or $5 \mod 6$. Furthermore, the function $P \mapsto \nu_\ell(P)/\deg P$ is lower semicontinuous on $\mathbb{P}$.

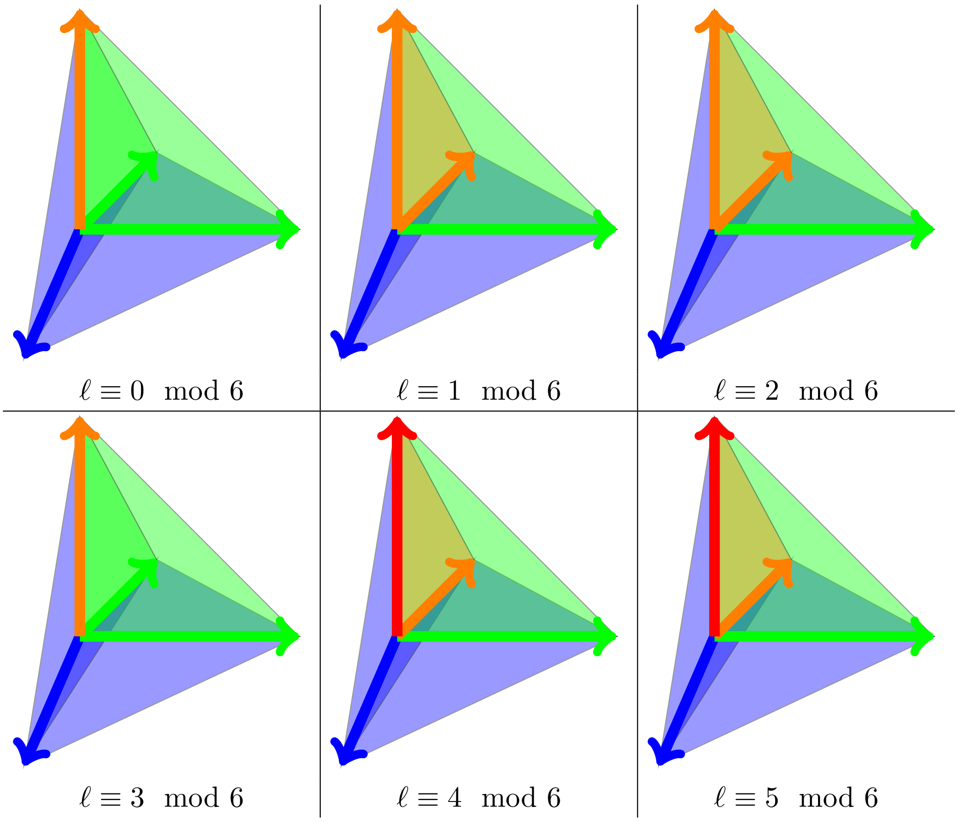

Figure 2.1: Values of $\nu_\ell(P)/\deg P$ on $\mathbb{P} = \mathbb{P}(1,2,3,6)$: red: 0, orange: 1, green: 2, blue: 4

2.2.3 Sieving closed points

Let k and $\mathbb{P}$ be as in Notation 2.1.1. Let further $X \subseteq \mathbb{P}$ be a quasismooth subscheme. Fix a Weil divisor D and an ample Cartier divisor E on $\mathbb{P}$.

Low degree points

Lemma 2.2.15 (Low degree points). *For $r \geq 1$, let $X_{<r}$ be the set of closed points of X of degree less than r. Then there is a positive integer d_r such that for all $d \geq d_r$ holds*

$$\frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \mid X \cap \{f=0\} \text{ is quasismooth at all } P \in X_{<r}\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))} = \prod_{P \in X_{<r}} \left(1 - q^{-\nu_D(P)}\right).$$

Proof. Let Z be the union of all schemes X_P for $P \in X_{<r}$. By Lemma 2.2.4, a hypersurface defined by $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))$ is quasismooth at all $P \in X_{<r}$ if and only if $\varphi_{Z,d}(f)$ vanishes nowhere, where $\varphi_{Z,d}$ denotes the composition

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \to H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z) \xrightarrow{\simeq} \prod_{P \in X_{<r}} H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P}).$$

According to Lemma 2.2.7, there is a constant d_r such that for all $d \geq d_r$, the map $\varphi_{Z,d}$ is surjective. The fibers of a surjective linear map between finite vector spaces have all the same cardinality, hence

$$\begin{aligned}
&\frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \mid X \cap \{f=0\} \text{ is quasismooth at all } P \in X_{<r}\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))} \\
&= \frac{\#\varphi_{Z,d}^{-1}\left(\prod_{P \in X_{<r}} \left(H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P}) \setminus \{0\}\right)\right)}{\#\varphi_{Z,d}^{-1}\left(\prod_{P \in X_{<r}} H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P})\right)} \\
&= \prod_{P \in X_{<r}} \left(1 - \frac{1}{\#H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P})}\right) \\
&= \prod_{P \in X_{<r}} \left(1 - q^{-\nu_D(P)}\right).
\end{aligned}$$

□

Corollary 2.2.16. *If $\nu_D(P) = 0$ for some closed point P of X, then $X \cap \{f=0\}$ is not quasismooth at P for all $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))$ and all $d \geq 0$.*

Proof. Let $P \in X$ be a closed point with $\nu_D(P) = 0$. In particular, $H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P}) = 0$. Then the map $\varphi_{X_P,d}$ is surjective for all $d \geq 0$ for trivial reasons. Repeating the computation in the proof of Lemma 2.2.15 above shows that

$$\frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \mid X \cap \{f=0\} \text{ is quasismooth at } P\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))} = 0.$$

□

Example 2.2.17. It is possible that $\nu_D(P) = 0$, as observed in Example 2.2.14. In this case, Corollary 2.2.16 states that no hypersurface of degree $D + dE$ has quasismooth intersection with X.

An even easier example where this occurs is the following: Consider the n-dimensional weighted projective space

$$X = \mathbb{P} = \mathbb{P}(1, \ldots, 1, w)$$

of dimension n, where $w \geq 3$. Choose a Weil divisor D_ℓ corresponding to $\mathcal{O}_X(\ell)$, where $\ell \in \{0, \ldots, w-1\}$. D_ℓ is not Cartier if $\ell \neq 0$. However, the sheaf $\mathcal{O}_X(w)$ is ample and invertible [21]. The only singular point of $\mathbb{P}$ is $P = (0 : \cdots : 0 : 1)$ in weighted homogeneous coordinates. All other points Q have $\nu_{D_\ell}(Q) = \deg Q \cdot (n+1)$ by Lemma 2.2.12 (3).

To compute $\nu_{D_\ell}(P)$, write a weighted homogeneous polynomial $f \in S_{dw+\ell}$ as

$$f = \sum_{i=0}^{d} x_n^i \cdot f_i(x_0, \ldots, x_{n-1}), \quad f_i \text{ homogeneous of degree } (d-i)w + \ell.$$

If $\ell = 1$, then $f(P) = 0$, and f is not quasismooth at P if and only if $f_d = 0$. As f_d is a linear homogeneous polynomial in n variables, this is a codimension n condition, thus $\nu_{D_1}(P) = n$, compare Example 2.2.11. With a similar computation, one obtains for $\ell = 0$ that $\nu_{D_0}(P) = 1$.

However, if $\ell \geq 2$, then f and all its partial derivatives automatically vanish at P. So the surjective map $\varphi_{X_P,d}$ is the zero map, and consequently $\nu_{D_\ell}(P) = 0$.

Medium degree points

As seen in the previous example, low values of $\nu_D(P)$ should better be avoided. For $i \geq 0$, define

$$\beta_i := \dim \overline{\{P \in X \text{ closed} \mid \nu_D(P) = i \deg P\}}.$$

Lemma 2.2.18 (Medium degree points)**.** *Fix an integer $r \geq 1$. Let $X_{r,sd}$ be the set of closed points P of X with $r \leq \deg P \leq sd$, where*

$$s := \frac{1}{\operatorname{reg}_E(D) \cdot (\dim X + 1)}.$$

(1) *If $\beta_i < i$ for all $i = 0, \ldots, \dim X$, then*

$$\lim_{r\to\infty} \lim_{d\to\infty} \frac{\#\left\{ f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \;\middle|\; \begin{array}{c} X \cap \{f=0\} \text{ is not quasismooth} \\ \text{at some } P \in X_{r,sd} \end{array} \right\}}{\# H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))} = 0.$$

(2) *Otherwise*

$$\lim_{d\to\infty} \frac{\#\left\{ f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \;\middle|\; \begin{array}{c} X \cap \{f=0\} \text{ is not quasismooth} \\ \text{at some } P \in X_{r,sd} \end{array} \right\}}{\# H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))} = 1.$$

Proof. (1) Let d be a positive integer such that $d \geq \ell := \operatorname{reg}_E(D)$. Since $\ell \geq 1$, we have the inequalities $d \cdot (1-\ell) \leq \ell \cdot (1-\ell)$ and thus

$$d \leq d\ell + \ell \cdot (1-\ell) = \ell \cdot (d - \ell + 1).$$

Hence, for $P \in X_{r,sd}$,

$$\deg P \leq \frac{d}{\ell \cdot (\dim X + 1)} \leq \frac{d - \ell + 1}{\dim X + 1},$$

so the map $\varphi_{X_P,d}$ is surjective by Corollary 2.2.13. Following the proof of Lemma 2.2.15, one finds that

$$\frac{\#\left\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \,\middle|\, X \cap \{f=0\} \text{ is not quasismooth at } P\right\}}{\# H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))} = q^{-\nu_D(P)}.$$

Hence we get the estimate

$$\begin{aligned} &\frac{\#\left\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \,\middle|\, X \cap \{f=0\} \text{ is not quasismooth at some } P \in X_{r,sd}\right\}}{\# H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))} \\ &\leq \sum_{e=r}^{sd} \sum_{P \in X \colon \deg P = e} q^{-\nu_D(P)} \\ &\leq \sum_{e=r}^{sd} \sum_{i=0}^{\dim X+1} \sum_{P \in X \colon \deg P = e, \nu_D(P) = ei} q^{-ei}. \end{aligned}$$

Using the Lang-Weil bound [47, Theorem 1], there is a constant $L > 0$ such that

$$\#\{P \in X \mid \deg P = e, \nu_D(P) = ei\} \leq L q^{e\beta_i}.$$

Hence

$$\begin{aligned} &\frac{\#\left\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \,\middle|\, X \cap \{f=0\} \text{ is not quasismooth at some } P \in X_{r,sd}\right\}}{\# H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))} \\ &\leq \sum_{e=r}^{sd} \sum_{i=0}^{\dim X+1} L q^{-e(i-\beta_i)} \leq \sum_{i=0}^{\dim X+1} \sum_{e \geq 0} L q^{-(e+r)(i-\beta_i)} \\ &= \sum_{i=0}^{\dim X+1} L q^{-r(i-\beta_i)} \frac{1}{1-q^{\beta_i - i}}. \end{aligned}$$

If $\beta_i < i$, this becomes arbitrarily small as $r \to \infty$.

(2) Otherwise, choose an integer $i \in \{0, \ldots, \dim X\}$ and a subscheme $Y \subseteq X$ of dimension at least i such that for every closed point $P \in Y$ holds $\nu_D(P) = i \deg P$. For any integer $t \geq 0$, denote by $Y_{r,t}$ the finite set of closed points of Y whose degree lies between r and t. Further define for integers $d, t \geq 0$ the rational number

$$a_{d,t} := \frac{\#\left\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \,\middle|\, X \cap \{f=0\} \text{ is not quasism. at some } P \in Y_{r,t}\right\}}{\# H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))}.$$

By the techniques of Lemma 2.2.15,

$$\begin{aligned} \lim_{d\to\infty} a_{d,t} &= 1 - \prod_{P \in Y_{r,t} \text{ closed}} \left(1 - q^{-\nu_D(P)}\right) \\ &= 1 - \prod_{P \in Y_{r,t} \text{ closed}} \left(1 - q^{-i \deg P}\right) \\ &= 1 - \prod_{P \in Y_{<r}} \left(1 - q^{-i \deg P}\right)^{-1} \cdot \prod_{P \in Y_{\leq t}} \left(1 - q^{-i \deg P}\right). \end{aligned}$$

The latter product vanishes if $i = 0$. Otherwise, we can use the standard power series expansion for the Hasse-Weil zeta function to obtain

$$\prod_{P \in Y_{\leq t}} \left(1 - q^{-i \deg P}\right) = \exp\left(-\sum_{e=1}^{t} \#Y(\mathbb{F}_{q^e}) \frac{q^{-ei}}{e}\right).$$

The Lang-Weil estimate [47, Theorem 1] gives a constant $M > 0$ depending on Y such that $\#Y(\mathbb{F}_{q^e}) \geq M q^{e \dim Y}$. Since $\dim Y \geq i$, the sum inside the exponential diverges to ∞ as $t \to \infty$ and therefore

$$\lim_{t\to\infty} \lim_{d\to\infty} a_{d,t} = 1.$$

Applying the succeeding Lemma 2.2.19 to the sequence $(a_{d,st})_{d,t\in\mathbb{N}}$ yields

$$\lim_{d\to\infty} a_{d,sd} = 1. \qquad \square$$

Lemma 2.2.19. *Let $(a_{i,j})_{i,j\in\mathbb{N}} \subseteq \mathbb{R}$ be a double sequence and let $c \in \mathbb{R}$. Suppose that*

- *$a_{i,j} \leq c$ for all i, j,*
- *$\lim_{j\to\infty} \lim_{i\to\infty} a_{i,j} = c$ and*
- *$a_{i,j} \leq a_{i,j'}$ for all i and $j \leq j'$.*

Then the diagonal sequence $(a_{i,i})_{i\in\mathbb{N}}$ converges and $\lim_{i\to\infty} a_{i,i} = c$.

Proof. Let $\varepsilon > 0$. Then there is a number j_ϵ such that $c - \varepsilon \leq \lim_{i\to\infty} a_{i,j_\varepsilon}$. Using the inequality $a_{i,j_\varepsilon} \leq a_{i,i}$ for $i \geq j_\varepsilon$ shows

$$c - \varepsilon \leq \lim_{i\to\infty} a_{i,j_\varepsilon} \leq \liminf_{i\to\infty} a_{i,i} \leq c. \qquad \square$$

Remark. The condition $\beta_i < i$ is automatically satisfied if X is smooth. It is still true if X has only finitely many singularities, provided that no point P has $\nu_D(P) = 0$. We have already seen in Corollary 2.2.16 that the latter condition is necessary for having quasismooth intersections at all.

Example 2.2.20. An example where the second case of Lemma 2.2.18 applies in a non-trivial fashion is given by the weighted projective space $X = \mathbb{P} = \mathbb{P}(1, 2, 3, 6)$. Denote the coordinates by x_0, x_1, x_2, x_3. Pick divisors D and E such that $\mathcal{O}_{\mathbb{P}}(D) \cong \mathcal{O}_{\mathbb{P}}(1)$ and $\mathcal{O}_{\mathbb{P}}(E) \cong \mathcal{O}_{\mathbb{P}}(6)$. As pictured in Example 2.2.14, $\nu_D(P) = 1 \cdot \deg P$ for any point $P \in H := \{x_0 = x_1 = 0\}$, implying $\beta_1 \geq \dim H = 1$. In contrast to Example 2.2.17, there is no point $P \in \mathbb{P}(1, 2, 3, 6)$ with $\nu_D(P) = 0$. However, the hypersurfaces of degree $6d + 1$ which are not quasismooth at some point in H still form a set of density one by Lemma 2.2.18 (2).

High degree points

We need two preparatorial lemmas.

Lemma 2.2.21. *Let $\ell := \operatorname{reg}_E(D)$.*

(1) *Suppose that $\mathbb{P}$ is smooth at the closed point P. Then, for $d \geq \ell$,*

$$\frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \mid f(P) = 0\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))} \leq q^{-\min(d-\ell, \deg P)}.$$

(2) *Let $V \subseteq \mathbb{P}$, $\dim V \geq 1$, be a subscheme which intersects the singular locus of $\mathbb{P}$ in finitely many points only. Then*

$$\frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \mid V \subseteq \{f = 0\}\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))} \leq q^{\ell-d}.$$

Proof. Let Z be the closed subscheme corresponding to the maximal ideal at P. Since $\mathbb{P}$ is smooth at P, we have $H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z) \cong H^0(Z, \mathcal{O}_Z)$, and the k-dimension of this vector space equals $\deg P$. By the proof of Lemma 2.2.9, the dimension of the image of the evaluation map

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \xrightarrow{\varphi_{Z,d}} H^0(Z, \mathcal{O}_Z)$$

is at least $\min(d-\ell, \deg P)$. This proves (1). For (2), pick a point $P \in V$ contained in the smooth locus of $\mathbb{P}$ such that $\deg P \geq d - \ell$. □

Note that the condition on smoothness is essential: Examples 2.2.17 and 2.2.20 indicate that the fractions in question can be equal to one in the non-smooth case.

We need one more technical result. Let W be a Weil divisor on $\mathbb{P}$ and let $f \in S_W$ be a homogeneous polynomial of degree W with respect to the grading given by the class group $\mathrm{Cl}(\mathbb{P})$. Since $S_W \subseteq k[x_1, \ldots, x_m]$, the polynomial f carries a degree $\deg_{\mathrm{std}}(f)$ with respect to the standard grading on the polynomial ring $k[x_1, \ldots, x_m]$. Define

$$\delta(W) := \max\{\deg_{\mathrm{std}}(f) \mid f \in S_W\}.$$

Lemma 2.2.22. *The quantity $\delta(D + dE)$ grows linearly in d.*

Proof. By Lemma 2.2.8, the natural multiplication map

$$S_{D+\ell E} \otimes S_E^{\otimes(d-\ell)} \to S_{D+dE}$$

is surjective for $d \geq \ell := \mathrm{reg}_E(D)$. Consequently,

$$\delta(D+dE) = \delta(D+\ell E) + (d-\ell) \cdot \delta(E), \quad d \geq \ell.$$

In particular, $\delta(D + dE)$ grows linearly in d. □

Lemma 2.2.23 (High degree points)**.** *Fix a rational number $s > 0$ and denote by $X_{>sd}$ the set of closed points of X of degree $> sd$. Suppose that X meets the singular locus of $\mathbb{P}$ only in finitely many points. Then*

$$\limsup_{d\to\infty} \frac{\#\left\{ f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \,\middle|\, \begin{array}{c} X \cap \{f=0\} \text{ is not quasismooth} \\ \text{at some } P \in X_{>sd} \end{array} \right\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))} = 0.$$

Proof. The proof will be divided into six steps. The strategy is as follows: We give first a global proof for $X = \mathbb{P}$. We choose an open cover of $\mathbb{P}$ such that on each open, a hypersurface fails to be quasismooth if $\dim \mathbb{P}$ many derivations vanish. Then we draw sections of $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))$ uniformly at random and compute that the probability that the locus where all derivations vanish contains a point of high degree. Applying Poonen's trick of decoupling derivatives, we show that this probability becomes arbitrarily small as $d \to \infty$. The last step is to generalize the proof to arbitrary quasismooth subschemes $X \subseteq \mathbb{P}$ with finitely many singular points.

Step 1. Testing quasismoothness with $n = \dim \mathbb{P}$ many derivations.

Let $f \in S = k[x_1, \ldots, x_m]$ be homogeneous with respect to the Cl($\mathbb{P}$)-grading. Then, by the definition of quasismoothness, $\{f = 0\}$ is not quasismooth at $P \in \mathbb{P}$ if and only if

$$f(P) = \frac{\partial f}{\partial x_1}(P) = \cdots = \frac{\partial f}{\partial x_m}(P) = 0.$$

In fact, even more is true: Let σ be an n-dimensional cone in the simplicial fan Δ associated to $\mathbb{P}$. The homogeneous coordinate ring S has a variable x_i for each one-dimensional cone $\rho_i \in \Delta$, where $i = 1, \ldots, m$. Define $U_\sigma \subseteq X$ to be the open affine subvariety given by the homogeneous localization at $\prod_{\rho_i \not\subseteq \sigma} x_i$. Renumbering the variables, assume w.l.o.g. that $\prod_{\rho_i \not\subseteq \sigma} x_i = x_{n+1} \cdots x_m$. By [5, Lemma 3.6], if $P \in U_\sigma$, then $\{f = 0\}$ is not quasismooth at $P \in \mathbb{P}$ if and only if

$$f(P) = \frac{\partial f}{\partial x_1}(P) = \cdots = \frac{\partial f}{\partial x_n}(P) = 0.$$

$\mathbb{P}$ can be covered with finitely many such sets U_σ, and quasismoothness may be tested with n derivations on each U_σ. So we may w.l.o.g. restrict our search for non-quasismooth points of high degree to $U_\sigma = \{x_{n+1} \ldots x_m \neq 0\} \subseteq \mathbb{P}$.

Step 2. Drawing sections at random.

Let D_i be the divisor corresponding to $V(x_i)$, so that x_i is a global section of $\mathcal{O}_{\mathbb{P}}(D_i)$, where $i = 1, \ldots, n$. Set $D_0 := 0 \in$ Div($\mathbb{P}$). For $i = 0, \ldots, n$ and $b = 0, \ldots, q-1$, pick a divisor $B_{i,b}$ such that $q \cdot B_{i,b} \leq D + bE - D_i$, where q is the cardinality of the ground field $k = \mathbb{F}_q$. Now fix an integer $d \geq 1$ and write $d = \lfloor d/q \rfloor \cdot q + b$. Define $C_i := B_{i,b} + \lfloor d/q \rfloor \cdot E$. There is a natural multiplication map

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(C_i)) \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE - D_i)), \quad g \mapsto g^q.$$

In order to see this, choose $g \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(C_i))$. Then

$$\operatorname{div}(g^q) = q \cdot \operatorname{div}(g) \geq q \cdot (-C_i) \geq -q \left\lfloor \frac{d}{q} \right\rfloor E - (D + bE - D_i) = -(D + dE - D_i),$$

hence $g^q \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE - D_i))$. Note that for all $g \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(C_i))$,

$$\frac{\partial g^q}{\partial x_j} = 0, \quad i = 0, \ldots, n, \quad j = 1, \ldots, n.$$

Combine these maps to

$$\begin{array}{rcl} & H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) & \\ & \oplus & \\ \psi: & \bigoplus_{i=1}^n H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(C_i)) & \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)), \\ & \oplus & \\ & H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(C_0)) & \end{array}$$

$$(f_0, g_1, \dots, g_n, h) \mapsto f_0 + \sum_{i=1}^n g_i^q \cdot x_i + h^q.$$

This map is $\mathbb{F}_q$-linear and surjective, hence we can compute densities on the left-hand side.

Step 3. Decoupling of derivatives.

For $f = \psi(f_0, g_1, \dots, g_n, h)$, define the subsets

$$W_i := \left\{ \frac{\partial f}{\partial x_1} = \dots = \frac{\partial f}{\partial x_i} = 0 \right\} \subseteq \{x_{n+1} \cdots x_m \neq 0\}, \quad i = 0, \dots, n.$$

Note that W_0 is n-dimensional and for $i \geq 0$, W_i does not depend on $g_{i+1}, \dots, g_n$ and h: Indeed,

$$\begin{aligned} \frac{\partial f}{\partial x_i} &= \frac{\partial f_0}{\partial x_i} + \sum_{j=1}^m \frac{\partial x_j}{\partial x_i} \cdot g_j^q + \sum_{j=1}^m \underbrace{\frac{\partial g_j^q}{\partial x_i}}_{=0} \cdot x_j + \underbrace{\frac{\partial h^q}{\partial x_i}}_{=0} \\ &= \frac{\partial f_0}{\partial x_i} + g_i^q, \quad i = 1, \dots, n. \end{aligned}$$

Step 4. For $0 \leq i \leq n-1$, conditioned on a choice of $f_0, g_1, \dots, g_i$ for which $\dim W_i \leq n - i$, the probability that $\dim W_{i+1} \leq n - i - 1$ is $1 - o(1)$ as $d \to \infty$.

If $\dim W_i = n - i$, the number of $(n-i)$-dimensional k-irreducible components of W_i is bounded from above by the number of $(m-i)$-dimensional k-irreducible components of $\pi^{-1}(W_i)$, where $\pi : \mathbb{A}^m \setminus B \to \mathbb{P}$ is the quotient map from Theorem 2.1.4. Applying Bézout's theorem for affine space, this quantity is bounded by $O(\delta^i)$, where $\delta = \deg_{\mathrm{std}}(f)$ is the degree of $f \in k[x_1, \dots, x_m]$ with respect to the standard grading.

Let V be such an $(n-i)$-dimensional component of W. Define

$$G_V^{\mathrm{bad}} := \left\{ g_{i+1} \in H^0(X, \mathcal{O}_X(C_{i+1})) \,\middle|\, V \subseteq \left\{ \frac{\partial \psi(f_0, g_1, \dots, g_{i+1}, *)}{\partial x_{i+1}} = 0 \right\} \right\}.$$

Suppose that $G_V^{\mathrm{bad}} \neq \emptyset$. If $g, g' \in G_V$, then $g^q - (g')^q = (g - g')^q$ vanishes identically on $V \subseteq W_i$. So $g - g'$ must vanish identically on V. Hence there is a bijection

$$G_V^{\mathrm{bad}} \leftrightarrow \{g \in H^0(X, \mathcal{O}_X(C_{i+1})) \mid V \subseteq \{g = 0\}\}.$$

Recall that $C_{i+1} = B_{i+1,b} + \lfloor d/q \rfloor \cdot E$, where $k = \lfloor d/q \rfloor \cdot q + b$. Using Lemma 2.2.21,

$$\frac{\# G_V^{\mathrm{bad}}}{\# H^0(X, \mathcal{O}_X(C_{i+1}))} = O(q^{-\lfloor d/q \rfloor}).$$

Since there are at most $O(\delta^i)$ such components V, and this number grows like $O(d^i)$ in virtue of Lemma 2.2.22, the probability that W_{i+1} has dimension greater than $n-i-1$ is

$$O(d^i q^{-\lfloor d/q \rfloor}) = o(1) \quad \text{as } d \to \infty.$$

Step 5. Conditioned on a choice of $f_0, g_1, \ldots, g_n$ for which W_n is finite, the probability that $W_n \cap \{f = 0\}$ contains a point of degree $> sd$ is $o(1)$ as $d \to \infty$.

We can follow the lines of the previous step: The number of points in W_n is $O(d^n)$ again by Bézout's theorem and Lemma 2.2.22. Pick $P \in W_n$ and let

$$H_P^{\text{bad}} := \{h \in H^0(X, \mathcal{O}_X(C_0)) \mid \psi(f_0, g_1, \ldots, g_n, h)(P) = 0\}.$$

Another application of Lemma 2.2.21 yields that for all large enough d and $\deg P > sd$ either

$$\frac{\# H_P^{\text{bad}}}{\# H^0(X, \mathcal{O}_X(C_0))} = O(q^{-\lfloor d/q \rfloor})$$

or P is a singular point of $\mathbb{P}$. The latter possibility can be ruled out since $\mathbb{P}$ contains only finitely many singular points by hypothesis and $\deg P > sd$. As a consequence, the probability that $W_n \cap \{f = 0\}$ contains a point of degree $> sd$ is

$$O(d^n q^{-\lfloor d/q \rfloor}) = o(1) \quad \text{as } d \to \infty.$$

Putting everything together, the probability that a hypersurface $\{f = 0\}$, determined by choosing $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))$ at random via ψ, is not quasismooth at some point in $P \in \{x_{n+1} \cdots x_m \neq 0\}$ of degree $> sd$ is $o(1)$ as $d \to \infty$. This proves the lemma in the case $X = \mathbb{P}$.

Step 6. Proof for general X.

Following the strategy of the proof of [60, Lemma 2.6], we can restrict to an open affine subset U of the smooth locus $\mathbb{P}^{\text{sm}}$ of $\mathbb{P}$. We can find coordinates $t_1, \ldots, t_n \in \mathcal{O}_U(U)$ defining $X \cap \mathbb{P}^{\text{sm}}$ locally by $t_{m+1} = \cdots = t_n = 0$, where $m = \dim X$. Moreover, there are derivations $d_1, \ldots, d_m : \mathcal{O}_U(U) \to \mathcal{O}_U(U)$ such that for $f \in \mathcal{O}_U(U)$ and $P \in X \cap U$,

$$\begin{aligned} X \cap \{f = 0\} \text{ is not quasismooth at } P &\Leftrightarrow X \cap \{f = 0\} \text{ is not smooth at } P \\ &\Leftrightarrow f(P) = d_1(f) = \cdots = d_m(f) = 0. \end{aligned}$$

For $i = 1, \ldots, m$, the coordinate t_i may be considered as element of $k(\mathbb{P}) \cong k(U)$, and therefore $t_i \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(-\text{div}(t_i)))$. This allows us to draw sections as in Step 2, replacing D_i by $-\text{div}(t_i)$. Restricting elements of $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))$ to U, the rest of the proof can be carried out analogously to the case $X = \mathbb{P}$. □

Proof of Theorem 2.1.6

Proof of Theorem 2.1.6. If X happens to be zero-dimensional, then the assertion is a direct consequence of Lemma 2.2.15. Otherwise, as in [60, §2.4], the theorem follows from Lemmas 2.2.15, 2.2.18 and 2.2.23 as $r \to \infty$. □

2.2.4 First applications

First examples

We list some easily obtained consequences of Theorem 2.1.6:

Example 2.2.24. Let $d_1, d_2, e_1, e_2 \in \mathbb{Z}$, $e_1, e_2 > 0$. On the smooth toric variety $\mathbb{P} = \mathbb{P}^m \times \mathbb{P}^n$, $\mathcal{O}_{\mathbb{P}^m \times \mathbb{P}^n}(e_1, e_2)$ is an ample invertible sheaf. By Theorem 2.1.6, as $d \to \infty$, the probability that a hypersurface of bidegree $(d_1 + de_1, d_2 + de_2)$ in $\mathbb{P}^m \times \mathbb{P}^n$ is smooth equals

$$\zeta_{\mathbb{P}^m \times \mathbb{P}^n}(m + n + 1)^{-1} = \prod_{i=0}^{m} \prod_{j=0}^{n} (1 - q^{i+j-m-n-1}),$$

as computed in [28, Example 4.3].

Example 2.2.25. Consider the weighted projective space $\mathbb{P} = \mathbb{P}(1, \ldots, 1, w)$ of dimension n, where $w, \ell \in \mathbb{Z}$, $w \geq 1$, $0 \leq \ell \leq w - 1$. Its Hasse-Weil zeta function agrees with the one of standard projective space. As $d \to \infty$, the probability that a hypersurface of degree $dw + \ell$ is quasismooth in $\mathbb{P}$ therefore equals

$$\begin{cases} 0 & \text{if } \ell \geq 2, \\ (1 - q^{-1}) \cdots (1 - q^{-n+1}) \cdot (1 - q^{-n})^2 & \text{if } \ell = 1, \\ (1 - q^{-1})^2 \cdot (1 - q^{-2}) \cdots (1 - q^{-n}) & \text{if } \ell = 0. \end{cases}$$

This follows from the computations in Examples 2.2.11 and 2.2.17. Moreover, as seen in Example 2.2.17, in the case $\ell \geq 2$, every hypersurface passes through $(0 : \cdots : 0 : 1)$ and is not quasismooth at this point.

Taylor conditions

As in [60, Theorem 1.2], there is an extended version of Theorem 2.1.6:

Theorem 2.2.26. *Let k and $\mathbb{P}$ be as in Notation 2.1.1. Fix a Weil divisor D and an ample Cartier divisor E on $\mathbb{P}$. Let $X \subseteq \mathbb{P}$ be a quasismooth subscheme such that X meets the singular locus of $\mathbb{P}$ only in finitely many points. Let further Z be a zero-dimensional subscheme of X and fix a subset $T \subseteq H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z)$. Then*

$$\lim_{d\to\infty} \frac{\# \left\{ f \in H^0(X, \mathcal{O}_X(D + dE)) \,\middle|\, \begin{array}{c} (X \setminus (X \cap Z)) \cap \{f = 0\} \text{ is quasismooth} \\ \text{and } \varphi_{Z,d}(f) \in T \end{array} \right\}}{\# H^0(X, \mathcal{O}_{\mathbb{P}}(D + dE))}$$
$$= \frac{\# T}{\# H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z)} \cdot \prod_{P \in X \setminus (X \cap Z) \text{ closed}} \left(1 - q^{-\nu_D(P)}\right),$$

where $\varphi_{Z,d}$ is the map as defined in subsection 2.2.2.

Proof. Since the set of sections in question is a subset of

$$\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \mid (X \setminus (X \cap Z)) \cap \{f = 0\} \text{ is quasismooth}\},$$

we can apply Lemma 2.2.18 and 2.2.23. It suffices thus to modify the statement on low degree points.

Let Z' be the union of our Z with the zero-dimensional subscheme Z used in the proof of Lemma 2.2.15. Then a section $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))$ is quasismooth at all P in $(X \setminus (X \cap Z))_{<r}$ and $\varphi_{Z,d} \in T$ if and only if f lies in the preimage of

$$T \times \prod_{P \in (X \setminus (X \cap Z))_{<r}} \left(H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P}) \setminus \{0\}\right)$$

under the composition

$$\begin{aligned}\varphi_{Z',d} :& H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \to H^0(Z', \mathcal{O}_{\mathbb{P}}(D)|_{Z'}) \\ &\xrightarrow{\simeq} H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z) \times \prod_{P \in (X \setminus (X \cap Z))_{<r}} H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P}).\end{aligned}$$

In virtue of Lemma 2.2.7, this map becomes surjective for all sufficiently large d. Hence we can derive the formula given in the theorem. □

As an application, let Z be the zero-dimensional subscheme of all $\mathbb{F}_q$-rational points of $\mathbb{P}$. Assume that no closed point $P \in \mathbb{P}$ has $\nu_D(P) = 0$. Then $T := H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z) \setminus \{0\}$ is non-empty and

$$\begin{aligned}&\lim_{d\to\infty} \frac{\#\left\{ f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \,\middle|\, \begin{array}{c} (\mathbb{P} \setminus Z) \cap \{f = 0\} \text{ is quasismooth} \\ \text{and } \{f = 0\} \text{ has no } \mathbb{F}_q\text{-rational points} \end{array} \right\}}{\# H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))} \\ &= \frac{\# T}{\# H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z)} \cdot \prod_{P \in \mathbb{P} \setminus Z \text{ closed}} \left(1 - q^{-\nu_D(P)}\right) \\ &> 0.\end{aligned}$$

In particular, for $d \gg 0$ exist quasismooth sections of $D + dE$ without $\mathbb{F}_q$-rational points.

Singularities of positive dimension

Corollary 2.2.27. *With the notation of Theorem 2.1.6, denote by* $\mathrm{NQS}(f)$ *the locus where the intersection* $X \cap \{f = 0\}$ *is not quasismooth. Then*

$$\limsup_{d\to\infty} \frac{\#\{f \in H^0(P, \mathcal{O}_{\mathbb{P}}(D + dE)) \mid \dim \mathrm{NQS}(f) \geq 1\}}{\# H^0(P, \mathcal{O}_{\mathbb{P}}(D + dE))} = 0.$$

Proof. This follows immediately from Lemma 2.2.23, as such an f produces non-quasismooth points in $X \cap \{f = 0\}$ of arbitrarily large degree. □

2.2.5 Variations on the number of singularities

Allowing a finite number of singularities

The following theorem deals with the density of hypersurfaces with a bound on the number of non-quasismooth points:

Theorem 2.2.28. *In the situation of Theorem 2.1.6, suppose further that for any closed point $P \in X$ holds $\nu_D(P) > 0$. Choose an integer $s \geq 1$. Then*

$$\lim_{d\to\infty} \frac{\#\left\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \,\middle|\, X \cap \{f=0\} \textit{ is quasismooth except for } < s \textit{ points}\right\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))}$$
$$= \prod_{P \in X \textit{ closed}} (1 - q^{-\nu_D(P)}) \cdot \sum_{J \subseteq Y, \#J < s} \prod_{P \in J} \frac{1}{q^{\nu_D(P)} - 1}.$$

Proof. Again, we can apply the strategy for medium and high degree points without big changes. So we take a look at low degree points. Fix an integer $r \geq 1$ and let $X_{<r}$ be the set of closed points of U of degree less than r. Denote again by Z the union of all X_P for $P \in X_{<r}$.

Recall that for $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))$, the intersection $X \cap \{f=0\}$ is quasismooth at all points in $X_{<r}$ if and only if all entries $\varphi_{Z,d}(f)$ are non-zero, where $\varphi_{Z,d}$ is the composition

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \to H^0(Z, \mathcal{O}_{\mathbb{P}}(D)|_Z) \cong \prod_{P \in X_{<r}} H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P})$$

as in the proof of Lemma 2.2.15. In particular, the intersection $X \cap \{f=0\}$ is quasismooth at all points in $X_{<r}$ except for less than s points if and only if less than s entries of $\varphi_{Z,d}(f)$ are zero.

Fix an enumeration $X_{<r} = \{P_1, \dots, P_t\}$. If $0 \leq i < s$, then the number of elements in $\prod_{P \in X_{<r}} H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P})$ where precisely i entries are zero is given by

$$\sum_{1 \leq j_1 < \cdots < j_i \leq t} \prod_{\ell \in \{1,\dots,t\} \setminus \{j_1,\dots,j_i\}} \left(\#H^0(X_{P_\ell}, \mathcal{O}_{\mathbb{P}}(D)|_{X_{P_\ell}}) - 1\right).$$

Hence $X \cap \{f=0\}$ is quasismooth at all points $X_{<r}$ except for less than s points if and only if f lies in the preimage of

$$\sum_{i=0}^{s-1} \sum_{1 \leq j_1 < \cdots < j_i \leq t} \prod_{\ell \in \{1,\dots,t\} \setminus \{j_1,\dots,j_i\}} \left(q^{\nu_D(P_\ell)} - 1\right)$$

elements under $\varphi_{Z,d}$. By Lemma 2.2.7, for any r exists an integer d_r such that $\varphi_{Z,d}$ is surjective for $d \geq d_r$. Thus for large enough d, the fibers of $\varphi_{Z,d}$ have the same cardinality. Hence

$$\frac{\#\left\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \,\middle|\, \begin{array}{c} X \cap \{f=0\} \text{ is quasismooth at all points} \\ \text{in } X_{<r} \text{ with } < s \text{ exceptions} \end{array}\right\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))}$$
$$= \frac{\sum_{i=0}^{s-1} \sum_{1 \leq j_1 < \cdots < j_i \leq t} \prod_{\ell \in \{1,\dots,t\} \setminus \{j_1,\dots,j_i\}} \left(q^{\nu_D(P_\ell)} - 1\right)}{\prod_{\ell=1}^{t} q^{\nu_D(P_\ell)}}$$
$$= \sum_{i=0}^{s-1} \sum_{1 \leq j_1 < \cdots < j_i \leq t} \prod_{\ell \in \{1,\dots,t\} \setminus \{j_1,\dots,j_i\}} \left(1 - q^{-\nu_D(P_\ell)}\right) \prod_{\ell=1}^{i} q^{-\nu_{P_{j_\ell}}(D)}$$
$$= \prod_{\ell=1}^{t} \left(1 - q^{-\nu_D(P_\ell)}\right) \cdot \sum_{i=0}^{s-1} \sum_{1 \leq j_1 < \cdots < j_i \leq t} \prod_{\ell=1}^{i} \frac{q^{-\nu_D(P_{j_\ell})}}{1 - q^{-\nu_D(P_{j_\ell})}}$$
$$= \prod_{P \in X_{<r}} (1 - q^{-\nu_D(P)}) \cdot \sum_{J \subseteq X_{<r}, \#J < s} \prod_{P \in J} \frac{1}{q^{\nu_D(P)} - 1}.$$

It remains to show that

$$\sum_{J\subseteq X_{<r},\#J<s}\prod_{P\in J}\frac{1}{q^{\nu_D(P)}-1}$$

converges as $r\to\infty$. To this end, note that this is an increasing sequence as r grows. So it suffices to give an absolute upper bound. Since

$$\begin{aligned}\sum_{J\subseteq X_{<r},\#J<s}\prod_{P\in J}\frac{1}{q^{\nu_D(P)}-1} &= \sum_{i=0}^{s-1}\sum_{\{P_1,\dots,P_i\}\subseteq X_{<r}}\frac{1}{q^{\nu_D(P_1)}-1}\cdots\frac{1}{q^{\nu_D(P_i)}-1}\\ &\leq \sum_{i=0}^{s-1}\left(\sum_{P\in X_{<r}}\frac{1}{q^{\nu_D(P)}-1}\right)^i,\end{aligned}$$

it suffices to bound $\sum_{P\in X_{<r}}(q^{\nu_D(P)}-1)^{-1}$. By Lemma 2.2.12, we have for all $P\in X$ that $\nu_D(P)\leq\deg P\cdot(\dim X+1)$ and $\nu_D(P)\geq\deg P$. Analogously to the proof of Lemma 2.2.18,

$$\begin{aligned}\sum_{P\in X_{<r}}\frac{1}{q^{\nu_D(P)}-1} &\leq \sum_{e=1}^{r-1}\sum_{i=1}^{\dim X+1}\frac{\#\{P\in X\mid\deg P=e,\nu_D(P)=ei\}}{q^{ei}-1}\\ &\leq \sum_{i=1}^{\dim X+1}\sum_{e=1}^{r-1}\frac{L\cdot q^{e(i-1)}}{q^{ei}-1},\end{aligned}$$

for some constant L not depending on r. Therefore

$$\sum_{P\in X_{<r}}\frac{1}{q^{\nu_D(P)}-1}\leq L\cdot\sum_{i=1}^{\dim X+1}\sum_{e=1}^{\infty}\frac{1}{q^e-q^{-e(i-1)}}.$$

Since $\sum_{e=1}^{\infty}(q^e-q^{-e(i-1)})^{-1}$ exists for $i\geq 1$, the expression on the left-hand side is bounded from above. Thus the desired limit exists. □

Example 2.2.29. For $X=\mathbb{P}=\mathbb{P}^2$, the density of plane curves with at most one singular point is given by

$$\frac{1}{\zeta_{\mathbb{P}^2}(3)}\cdot\left(1+\sum_{P\in\mathbb{P}^2\text{ closed}}\frac{1}{q^{3\deg P}-1}\right).$$

For $q=5$, this quantity is about 0.96984.

We investigate now the density of hypersurfaces of degree d whose number of singularities is bounded in terms of an increasing function in d.

Lemma 2.2.30.

(1) *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers. Then for any $n\in\mathbb{N}$,*

$$\sum_{J\subseteq\{1,\dots,n\}}\prod_{j\in J}a_j=\prod_{j=1}^{n}(a_j+1).$$

(2) *Under the hypotheses of Theorem 2.2.28,*

$$\lim_{s\to\infty} \sum_{J\subseteq X, \#J<s} \prod_{P\in J} \frac{1}{q^{\nu_D(P)}-1} = \prod_{P\in X \text{ closed}} \frac{1}{1-q^{-\nu_D(P)}}.$$

Proof. Part (1) is easy. For (2), part (1) implies for any integer $r \geq 1$ the identity

$$\lim_{s\to\infty} \sum_{J\subseteq X_{<r}, \#J<s} \prod_{P\in J} \frac{1}{q^{\nu_D(P)}-1} = \sum_{J\subseteq X_{<r}} \prod_{P\in J} \frac{1}{q^{\nu_D(P)}-1} = \prod_{P\in X_{<r}} \frac{1}{1-q^{-\nu_D(P)}}.$$

Taking limits,

$$\lim_{r\to\infty} \lim_{s\to\infty} \sum_{J\subseteq X_{<r}, \#J<s} \prod_{P\in J} \frac{1}{q^{\nu_D(P)}-1} = \lim_{r\to\infty} \prod_{P\in X_{<r}} \frac{1}{1-q^{-\nu_D(P)}}.$$

Since the double sequence

$$\left(\sum_{J\subseteq X_{<r}, \#J<s} \prod_{P\in J} \frac{1}{q^{\nu_D(P)}-1} \right)_{r,s}$$

is increasing and bounded, the iterated limits may be interchanged. □

Corollary 2.2.31. *Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ be a function with $\lim_{d\to\infty} g(d) = \infty$. Then, under the hypotheses of Theorem 2.2.28,*

$$\lim_{d\to\infty} \frac{\#\left\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \,\middle|\, X \cap \{f=0\} \text{ is quasismooth except for } < g(d) \text{ points}\right\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))} = 1.$$

Proof. For integers $d \geq 0$, $s \geq 1$ define

$$a_{d,s} := \frac{\#\left\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE)) \,\middle|\, X \cap \{f=0\} \text{ is quasismooth except for } < s \text{ points}\right\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D+dE))}.$$

Due to Theorem 2.2.28 and Lemma 2.2.30,

$$\lim_{s\to\infty} \lim_{d\to\infty} a_{d,s} = 1.$$

The claim follows now by applying Lemma 2.2.19 to the sequence $(a_{d,g(s)})_{d,s}$. □

Length of the singular scheme

As a final application, we show an analogue of Corollary 2.2.31 for lengths of singular schemes of hypersurfaces on smooth toric varieties. Let $f \in S = k[x_1, \ldots, x_m]$ be a $\mathrm{Cl}(\mathbb{P})$-homogeneous polynomial. Endow the singular locus $\Sigma(f)$ of the hypersurface $\{f=0\}$ with the scheme structure given by the ideal $\left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m}\right)$.

Pick a closed point $P \in \mathbb{P}$ with local ring $\mathcal{O}_P$ and maximal ideal $\mathfrak{m}_P$. Since $\mathbb{P}$ is smooth, we have a natural restriction map $S \to \mathcal{O}_P$. Define

$$\mathrm{length}_P(\Sigma(f)) := \dim_k \mathcal{O}_P \Big/ \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_m}\right).$$

Then

$$\operatorname{length}(\Sigma(f)) = \sum_{P \in \mathbb{P} \text{ closed}} \operatorname{length}_P(\Sigma(f)).$$

Suppose that $\{f = 0\}$ has at most isolated hypersurface singularities, i.e. the global Tjurina number $\operatorname{length}(\Sigma(f))$ is finite. Note that due to positive characteristic, the global Milnor number need not be finite. However, isolated hypersurface singularities are finitely determined [9, Theorem 3]. In particular, the number $\operatorname{length}_P(\Sigma(f))$ depends only on the Taylor expansion of f up to some degree. More precisely, for each integer $b \geq 0$ exists an $e_0 \geq 0$ such that for all integers $e \geq e_0$, we find a set $B_{P,b,e} \subseteq \mathcal{O}_P/\mathfrak{m}_P^e$ with the property that $\operatorname{length}_P(\Sigma(f)) = b$ if and only if f lies in the preimage of $B_{P,b,e}$ under the natural map $S \to \mathcal{O}_P/\mathfrak{m}_P^e$. Write

$$\mu_P(b) := \frac{\# B_{P,b,e}}{\# \mathcal{O}_P/\mathfrak{m}_P^e}.$$

This is the local probability for a random hypersurface to have a singularity with local Tjurina number b at P. The quantity $\mu_P(b)$ does not depend on the choice of e due to finite determinacy. For example,

$$\mu_P(0) = \frac{\#((\mathcal{O}_P/\mathfrak{m}_P^2) \setminus \{0\})}{\# \mathcal{O}_P/\mathfrak{m}_P^2} = 1 - q^{-\deg P(\dim \mathbb{P}+1)}.$$

We can now derive a result similar to Theorem 2.2.28:

Theorem 2.2.32. *In the situation of Theorem 2.1.6, suppose further that $\mathbb{P}$ is smooth. Choose an integer $s \geq 1$ and let*

$$B_s := \left\{ (b_P)_{P \in \mathbb{P} \text{ closed}} \;\middle|\; b_P \in \{0, 1, \dots, s\} \text{ for all } P \in \mathbb{P} \text{ closed and } \sum_{P \in \mathbb{P} \text{ closed}} b_P < s \right\}.$$

Then

$$\lim_{d \to \infty} \frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \mid \operatorname{length}(\Sigma(f)) < s\}}{\# H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))} = \frac{1}{\zeta_{\mathbb{P}}(\dim \mathbb{P} + 1)} \cdot \sum_{b \in B_s} \prod_{P \in \mathbb{P} \text{ closed}} \frac{\mu_P(b_P)}{\mu_P(0)}.$$

Proof. In view of Corollary 2.2.27, we can restrict to hypersurfaces with at most isolated hypersurface singularities. It is sufficient to perform the low degree computation and show convergence, the strategy for medium and high degree points being the same as previously.

Fix an $r \geq 1$ and let $\mathbb{P}_{<r} = \{P_1, \dots, P_t\}$ be the set of closed points of $\mathbb{P}$ of degree $< r$. Let $(b_1, \dots, b_t)$ be a sequence of non-negative integers satisfying $b_1 + \cdots + b_t = s$. Fix an integer e being large enough to test whether $\operatorname{length}_{P_i}(\Sigma(f)) = b_i$ for all $i \in \{1, \dots, t\}$. The ideals $\mathfrak{m}_{\mathbb{P},P_i}^e$, $i = 1, \dots, t$, define zero-dimensional subschemes of $\mathbb{P}$, let Z denote their union. Then the natural map

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \to H^0(Z, \mathcal{O}_Z) \xrightarrow{\simeq} \prod_{i=1}^{t} \mathcal{O}_{\mathbb{P},P_i}/\mathfrak{m}_{\mathbb{P},P_i}^e$$

becomes surjective for large enough d due to Lemma 2.2.7. Hence, imitating the proof of Lemma 2.2.15,

$$\frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \mid \mathrm{length}_{P_i}(\Sigma(f)) = b_i,\ i = 1, \ldots, t\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))} = \prod_{i=1}^{t} \mu_{P_i}(b_i), \quad d \gg 0.$$

Consequently,

$$\begin{aligned}
&\frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \mid \mathrm{length}(\Sigma(f)) < s\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))} \\
&= \sum_{(b_1,\ldots,b_t)\colon \sum_{i=1}^t b_i < s} \prod_{i=1}^{t} \mu_{P_i}(b_i) \\
&= \prod_{i=1}^{t} \mu_{P_i}(0) \cdot \sum_{(b_1,\ldots,b_t)\colon \sum_{i=1}^t b_i < s} \prod_{i=1}^{t} \frac{\mu_{P_i}(b_i)}{\mu_{P_i}(0)} \\
&= \prod_{P \in \mathbb{P}_{<r}} (1 - q^{-\deg P(\dim \mathbb{P}+1)}) \cdot \sum_{(b_P)_{P \in \mathbb{P}_{<r}}\colon \sum_P b_P < s} \prod_{P \in \mathbb{P}_{<r}} \frac{\mu_P(b_P)}{\mu_P(0)}
\end{aligned}$$

for $d \gg 0$. The convergence of this expression follows from Theorem 2.2.28, as hypersurfaces f with $\mathrm{length}(\Sigma(f)) < s$ have less than s singular points. □

Example 2.2.33 (Plane curves with at most a single node)**.** For $X = \mathbb{P}^2$, one finds

$$\mu_P(1) = q^{-3\deg P} - q^{-4\deg P}, \quad P \in \mathbb{P}^2 \text{ closed}.$$

The density of plane curves with at most one ordinary double point as a singularity is therefore given by

$$\frac{1}{\zeta_{\mathbb{P}^2}(3)} \cdot \left(1 + \sum_{P \in \mathbb{P}^2 \text{ closed}} \frac{1}{q^{\deg P} + q^{2\deg P} + q^{3\deg P}}\right).$$

For $q = 5$, this quantity is about 0.93113.

Proof of Theorem 2.1.7

Proof. Applying a similar strategy as in the proofs of Lemma 2.2.30 and Corollary 2.2.31, it suffices to show that

$$\lim_{s \to \infty} \sum_{\sum_P b_P < s} \prod_{P \in \mathbb{P}_{<r}} \frac{\mu_P(b_P)}{\mu_P(0)} = \prod_{P \in \mathbb{P}_{<r}} \frac{1}{1 - q^{-\deg P(\dim \mathbb{P}+1)}}, \quad r \geq 1,$$

or equivalently,

$$\lim_{s \to \infty} \sum_{(b_P)_{P \in \mathbb{P}_{<r}}\colon \sum_P b_P < s} \prod_{P \in \mathbb{P}_{<r}} \mu_P(b_P) = 1, \quad r \geq 1.$$

This follows easily from the fact that

$$\sum_{b \geq 0} \mu_P(b) = 1$$

for all closed points $P \in \mathbb{P}$, which is a consequence of Corollary 2.2.27. □

2.2.6 Density of hypersurfaces without defect

Let $k = \mathbb{F}_q$ be a finite field of characteristic $\neq 2$. Fix an integer $n \geq 3$. By [60, Theorem 1.1] or Theorem 2.1.6,

$$\lim_{d\to\infty} \frac{\#\{f \in k[x_0,\dots,x_n]_d \mid \{f=0\} \subseteq \mathbb{P}^n_k \text{ is smooth}\}}{\#k[x_0,\dots,x_n]_d} = \frac{1}{\zeta_{\mathbb{P}^n_k}(n+1)}.$$

This limit is smaller than 1, so that a "random" hypersurface is smooth with a probability strictly less than 100%. However, it is true that hypersurfaces with few singularities compared to the degree form a set of density 1 by Theorem 2.1.7: For any constant $c > 0$,

$$\lim_{d\to\infty} \frac{\#\{f \in k[x_0,\dots,x_n]_d \mid \tau(f) \leq c \cdot d\}}{\#k[x_0,\dots,x_n]_d} = 1,$$

where $\tau(f)$ is the global Tjurina number of the hypersurface $\{f = 0\}$.

If Theorem 1.1.1 held over finite fields, then this would imply that hypersurfaces without defect form a set of density 1. However, so far, we can only use the restricted singularity types from Theorem 1.1.2 and obtain:

Theorem 2.2.34 (= Theorem 1.1.3). *Let q be an odd prime power. Then*

$$\lim_{d\to\infty} \frac{\#\{f \in k[x_0,\dots,x_n]_d \mid \{f=0\} \subseteq \mathbb{P}^n_{\mathbb{F}_q} \text{ has no defect}\}}{\#k[x_0,\dots,x_n]_d} \geq \frac{1}{\zeta_{\mathbb{P}^n}(n+3)} = \prod_{i=3}^{n+2} \left(1-q^{-i}\right).$$

The proof requires two lemmata:

Lemma 2.2.35. *The number $p_{n,q}$ of quadratic forms in n variables of rank $\geq n-1$ over a field with q elements equals*

$$p_{n,q} = \prod_{i=1}^{\lfloor (n-1)/2 \rfloor} \frac{q^{2i}}{q^{2i}-1} \prod_{i=0}^{n-2}(q^{n-i}-1) - \prod_{i=1}^{\lfloor n/2 \rfloor} \frac{q^{2i}}{q^{2i}-1} \prod_{i=0}^{n-1}(q^{n-i}-1).$$

Moreover,

$$q^{\frac{n(n+1)}{2}}(1-q^{-2}) \leq p_{n,q} \leq q^{\frac{n(n+1)}{2}}(1-q^{-3}).$$

Proof. The formula for $p_{n,q}$ can be found in [52, Theorem 2]. Suppose first that n is even. Then

$$\begin{aligned} p_{n,q} &= \left(1 + \frac{q^n}{q^n-1}\cdot(q-1)\right) \prod_{i=1}^{n/2-1} \frac{q^{2i}}{q^{2i}-1} \prod_{i=0}^{n-2}(q^{n-i}-1) \\ &= \frac{q^{n+1}-1}{q^n-1} \cdot \prod_{i=1}^{n/2-1} \frac{q^{2i}}{q^{2i}-1} \prod_{i=0}^{n-2}(q^{n-i}-1) \\ &= \prod_{i=1}^{n/2-1} q^{2i} \cdot \prod_{i=0}^{n/2-1} (q^{n+1-2i}-1) \end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=0}^{n/2-1} (q^{n+1} - q^{2i}) \\
&= q^{\frac{n(n+1)}{2}} \prod_{i=0}^{n/2-1} (1 - q^{2i-n-1}) \\
&= q^{\frac{n(n+1)}{2}} (q^{-n-1}; q^2)_{n/2},
\end{aligned}
$$

where we used the notation for the q-Pochhammer symbol. It is clear that $(q^{-n-1}; q^2)_{n/2}$ is a decreasing sequence bounded above from $1 - q^{-3}$. Induction on $q \geq 2$ shows the inequality

$$\prod_{i=3}^{n} (1 - q^{-i}) \geq 1 - q^{-2} + q^{-n},$$

whence

$$(q^{-n-1}; q^2)_{n/2} \geq \prod_{i=3}^{\infty} (1 - q^{-3}) \geq 1 - q^{-2}.$$

For odd n, we can reduce to the even case by observing that $p_{n,q} = q^n \cdot p_{n-1,q}$. □

Lemma 2.2.36 (Local probability for A_k singularities). *Let $P \in \mathbb{A}^n$ be a closed point with residue field $\kappa(P)$. Fix a positive integer d and choose a polynomial $f \in \mathbb{F}_q[x_1, \ldots, x_n]_{\leq d}$ uniformly at random. Then the probability that $\{f = 0\}$ has at most an A_k singularity for some $k \geq 1$ in x is at least*

$$1 - \#\kappa(P)^{-n-3}.$$

Proof. Let $\mathcal{O}_P$ be the local ring of $\mathbb{A}^n$ at P and denote by $\mathfrak{m}_P$ its maximal ideal. Let

$$[f] = f_0 + f_1 + f_2 \in \mathcal{O}_P/\mathfrak{m}_P^3 \quad \text{with } \deg f_i = i, \quad i = 0, 1, 2,$$

be the 2-jet of f at P. Define X to be the hypersurface $\{f = 0\} \subseteq \mathbb{A}^n$. Then:

(1) X does not pass through $P \Leftrightarrow f_0 \neq 0$,

(2) X is smooth at $P \Leftrightarrow f_0 = 0$ and $f_1 \neq 0$,

(3) X has an ordinary double point at $P \Leftrightarrow f_0 = 0$, $f_1 = 0$ and f_2 is a quadratic form of rank n,

(4) X has an A_k singularity for some $k \geq 2$ at $P \Leftrightarrow f_0 = 0$, $f_1 = 0$ and f_2 is a quadratic form of rank $n - 1$.

The vector space $\mathcal{O}_P/\mathfrak{m}_P^3$ has dimension $1 + n + \frac{n(n+1)}{2}$ over $\kappa(P)$. Let $r := \#\kappa(P)$. Thus the probability that X has at most an A_k singularity at P equals

$$\frac{(r-1)r^{n+n(n+1)/2} + (r^n - 1)r^{n(n+1)/2} + p_{n,r}}{r^{1+n+n(n+1)/2}} = 1 - \frac{r^{n(n+1)/2} - p_{n,r}}{r^{1+n+n(n+1)/2}}.$$

where $p_{n,r}$ is the number of quadratic forms in n variables of rank $\geq n - 1$ over $\kappa(P)$. The bounds from Lemma 2.2.35 give

$$1 - r^{-n-4} \geq 1 - \frac{r^{n(n+1)/2} - p_{n,r}}{r^{1+n+n(n+1)/2}} \geq 1 - r^{-n-3}.$$ □

Proof of Theorem 1.1.3. For a property $\mathcal{P}$ of projective hypersurfaces defined by polynomials $f \in k[x_0, \ldots, x_n]_d$, write

$$\mu(\mathcal{P}) := \lim_{d\to\infty} \frac{\#\{f \in k[x_0, \ldots, x_n]_d \mid \{f = 0\} \subseteq \mathbb{P}^n_k \text{ satisfies } \mathcal{P}\}}{\#k[x_0, \ldots, x_n]_d}.$$

By Theorems 1.1.2 and 2.1.7, there is a constant $c > 0$ such that

$$\mu(\text{defect and at most } A_k \text{ singularities}) \leq \mu(\tau(f) > c \cdot d) = 0.$$

Moreover, combining the Lemma 2.2.36 with [60, Theorem 1.3],

$$\begin{aligned}\mu(\text{defect and worse than } A_k \text{ singularities}) &\leq \mu(\text{worse than } A_k \text{ singularities}) \\ &\leq 1 - \frac{1}{\zeta_{\mathbb{P}^n}(n+3)}.\end{aligned}$$

Putting this together,

$$\begin{aligned}\mu(\text{no defect}) &= 1 - \mu(\text{defect}) \\ &= 1 - \mu(\text{defect and at most } A_k \text{ sing.}) - \mu(\text{defect and worse than } A_k \text{ sing.}) \\ &\geq \frac{1}{\zeta_{\mathbb{P}^n}(n+3)},\end{aligned}$$

which completes the proof. □

Remark. In view of Theorem 1.1.2, we could have added the contribution of ordinary multiple points. The probability for a hypersurface to have a singularity at a point P and this being an ordinary multiple point of multiplicity ≥ 3, equals

$$\sum_{d\geq 3} \#\{f \in \kappa(P)[x_0, \ldots, x_n]_d \mid \{f = 0\} \text{ is smooth}\} \cdot \#\kappa(P)^{-\binom{n+d}{d}}.$$

This turns out to be small compared to the local density of at most A_k singularities and we do not expect this to bring a substantial improvement to the bound given in Lemma 2.2.36.

2.3 Geometrically irreducible hypersurfaces

Let $k = \mathbb{F}_q$ and $\mathbb{P}$ be as in Notation 2.1.1. Fix a Weil divisor D and an ample Cartier divisor E on $\mathbb{P}$. Recall Notation 2.1.8.

2.3.1 Tools

From now on, a property $\mathcal{P}$ of sections $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))$ for some positive integer d, is said to hold for f in a set of density 1 if

$$\lim_{d\to\infty} \frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \mid f \text{ satisfies } \mathcal{P}\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))} = 1.$$

Lemma 2.3.1. *Let X be either*

- *a subscheme of $\mathbb{P}$ over k such that $\dim X \setminus (X \cap \mathbb{P}^{\mathrm{sing}}) \geq 1$, or*
- *a subscheme of $\mathbb{P}_{\overline{k}}$ over $\overline{k}$ such that $\dim X \setminus (X \cap (\mathbb{P}_{\overline{k}})^{\mathrm{sing}}) \geq 1$.*

Then for f in a set of density 1, f does not vanish on X.

Proof. Replacing X by its image under the natural map $\mathbb{P}_{\overline{k}} \to \mathbb{P}_k$, assume that X is defined over k. By Lemma 2.2.21 (2),

$$\begin{aligned}
&\frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \mid X \subseteq \{f = 0\}\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))} \\
&\quad \leq \frac{\#\{f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \mid X \setminus (X \cap \mathbb{P}^{\mathrm{sing}}) \subseteq \{f = 0\}\}}{\#H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE))} \\
&\quad \leq q^{\mathrm{reg}_E(D) - d} \xrightarrow{d\to\infty} 0.
\end{aligned}$$

□

Lemma 2.3.2. *Let $X \subseteq \mathbb{P}$ (or $\mathbb{P}_{\overline{k}}$) be a subscheme over k (or $\overline{k}$) such that $\dim X \geq 1$. Then for f in a set of density 1, $X \cap \{f = 0\} \neq \emptyset$.*

Proof. Assume again that X is defined over k. Fix a positive integer r and denote by $X_{<r}$ the set of closed points of X whose degree is smaller than r. Analogously to the proof of Lemma 2.2.15, the density of sections f such that $\{f = 0\} \cap X_{<r}$ is empty equals

$$\prod_{P \in X_{<r}} \left(1 - q^{-\mu_D(P)}\right),$$

where $\mu_D(P) := \dim_k H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P})$ and X_P is the subscheme of X corresponding to the maximal ideal $\mathfrak{m}_P$.

If $\mu_D(P) = 0$ for some point $P \in X_{<r}$, then the above product equals zero. Otherwise choose a positive integer m such that mD is Cartier. This is possible since $\mathbb{P}$ is simplicial and hence $\mathbb{Q}$-factorial (see e.g., [15, Proposition 4.2.7]). Using that the sheaf $\mathcal{O}_{\mathbb{P}}(mD)|_{X_P}$ is invertible and hence locally isomorphic to $\mathcal{O}_{X_P}$, there is an injective map

$$H^0(X_P, \mathcal{O}_{\mathbb{P}}(D)|_{X_P}) \to H^0(X_P, \mathcal{O}_{\mathbb{P}}(mD)|_{X_P}) \cong H^0(X_P, \mathcal{O}_{X_P}), \quad g \mapsto g^m,$$

thus $0 < \mu_D(P) \leq \deg P$. As $\deg P$ necessarily divides $\mu_D(P)$, this implies $\mu_D(P) = \deg P$. Observe that the situation here is similar, but less complicated than in Lemma 2.2.12.

In particular, the density of f such that $\{f = 0\} \cap X_{<r} = \emptyset$ equals

$$\prod_{P \in X_{<r}} \left(1 - q^{-\deg P}\right) = \frac{1}{\zeta_{X_{<r}}(1)}.$$

This diverges to 0 as $r \to \infty$, since $\dim X \geq 1$. □

Lemma 2.3.3. *Let X be a $\overline{k}$-scheme of finite type and let $\varphi : X \to \mathbb{P}_{\overline{k}}$ be a $\overline{k}$-morphism such that $\dim \overline{\varphi(C)} \geq 2$ for all $C \in \operatorname{Irr} X$. Let U be a dense open subscheme of X. Then for f in a set of density 1, the map*

$$\operatorname{Irr}_{\text{horiz}} X_f \to \operatorname{Irr}_{\text{horiz}} U_f, \quad C \mapsto C \cap U,$$

is bijective.

Proof. If every $C \in \operatorname{Irr}_{\text{horiz}} X_f$ meets U, the above map is clearly bijective with its inverse given by taking the closure in X_f.

There is nothing to show if $\operatorname{Irr}_{\text{horiz}}(X \setminus U) = \emptyset$. Otherwise, let $C \in \operatorname{Irr}_{\text{horiz}}(X \setminus U)$. Since $\overline{\varphi(C)}$ is of dimension ≥ 1 and is not contained in $(\mathbb{P}_{\overline{k}})^{\text{sing}}$, Lemma 2.3.1 states that the set of f vanishing on $\overline{\varphi(C)}$ has density 0. Excluding these f, every $C \in \operatorname{Irr}_{\text{horiz}} X_f$ meets U, because otherwise $C \in \operatorname{Irr}_{\text{horiz}}(X \setminus U)$ and $f(\overline{\varphi(C)}) = 0$.

□

Lemma 2.3.4. *Let X and φ be as in Theorem 2.1.9 and assume further that X is a smooth $\overline{k}$-scheme. Let $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \setminus \{0\}$ for some $d \geq 0$. Then C_f contains a horizontal component for any $C \in \operatorname{Irr} X$. Moreover, the following are equivalent:*

(1) *There is a bijection* $\operatorname{Irr} X \to \operatorname{Irr}_{\text{horiz}} X_f$, $C \mapsto (C_f)_{\text{horiz}}$.

(2) *For every $C \in \operatorname{Irr} X$, the scheme $(C_f)_{\text{horiz}}$ is irreducible.*

Proof. Let $C \in \operatorname{Irr} X$. Then $\dim \overline{\varphi(C)} \geq 2$ and thus

$$\dim \overline{\varphi(C_f)} = \dim \overline{\varphi(C \cap \varphi^{-1}(\{f = 0\}))} = \dim \overline{\varphi(C) \cap \{f = 0\}} \geq \dim \overline{\varphi(C)} - 1 \geq 1.$$

In particular, C_f has an irreducible component C' such that the codimension of $\overline{\varphi(C')}$ in $\overline{\varphi(C)}$ is ≤ 1. By hypothesis, the singular locus of $\mathbb{P}_{\overline{k}}$ has codimension ≥ 2 in $\overline{\varphi(C)}$. Thus $\overline{\varphi(C')}$ is not contained in $(\mathbb{P}_{\overline{k}})^{\text{sing}}$ and hence C' is horizontal.

Concerning the "moverover" part, (1) ⇒ (2) is obvious. For (2) ⇒ (1), note that the map is defined and surjective. By smoothness of X, the components of X do not intersect, so the map is also injective. □

Lemma 2.3.5. *Let X be a smooth subscheme of $\mathbb{P}_{\overline{k}}$ such that $X \cap (\mathbb{P}_{\overline{k}})^{\text{sing}}$ is finite. For f in a set of density 1, the singular locus $(X_f)^{\text{sing}}$ is finite.*

Proof. In view of Corollary 2.2.27, the difficulty comes from the larger fields involved. Splitting X into orbits under the action of the absolute Galois group of k, we can follow the proof of [11, Lemma 3.5] to obtain a covering of $X \cap (\mathbb{P}_{\overline{k}})^{\text{sm}}$ by finitely many open subschemes U and global derivations $D_1, \ldots, D_m : \mathcal{O}_U(U) \to \mathcal{O}_U(U)$ such that

$$P \in U \cap (X_f)^{\text{sing}} \Rightarrow f(P) = D_1(f)(P) = \cdots = D_m(f)(P) = 0.$$

Proceeding as in the proof of Lemma 2.2.23, $U \cap \{D_1(f) = \cdots = D_m(f)\}$ is finite with probability $1 - o(1)$ as $d \to \infty$. □

2.3.2 Surfaces

Proposition 2.3.6. *Let X be a 2-dimensional closed integral subscheme of $\mathbb{P}$ such that $X \cap \mathbb{P}^{\text{sing}}$ is finite. For f in a set of density 1, there is a bijection $\operatorname{Irr} X_{\overline{k}} \to \operatorname{Irr}(X_f)_{\overline{k}}$ sending C to C_f.*

Proof. The natural restriction map

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + dE)) \to H^0(X, \mathcal{O}_{\mathbb{P}}(D + dE)|_X)$$

is surjective for sufficiently large d by a Serre vanishing argument similar to the one given in Lemma 2.2.7. Therefore densities may be calculated by counting elements X_f living in $\mathbb{P}H^0(X, \mathcal{O}_{\mathbb{P}}(D + dE)|_X)$, which are Weil divisors on X for f in a set of density 1 in virtue of Lemma 2.3.1. The restriction of X_f to $X \cap \mathbb{P}^{\text{sm}}$ is a Cartier divisor. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities of X. Taking the pullback under π and taking the closure gives a Cartier divisor on $\widetilde{X}$.

Step 1. For f in as set of density 1, the divisor X_f is irreducible.

Similar to [11, Proposition 4.1], one computes that for any positive constant d_0, the number of reducible X_f is at most

$$q^{\frac{d^2 E.E}{2} - \frac{d_0 d}{2} + O(d)}.$$

It remains to determine $\#H^0(X, \mathcal{O}_{\mathbb{P}}(D + dE)|_X)$. Let C be an effective Cartier divisor on X. Then there is an exact sequence of sheaves

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0.$$

In particular, by tensoring with the d-th tensor power of the invertible sheaf $\mathcal{L} := \mathcal{O}_{\mathbb{P}}(E)|_X$, for the Euler characteristic χ holds

$$\chi(\mathcal{O}_X(C) \otimes \mathcal{L}^{\otimes d}) = \chi(\mathcal{L}^{\otimes d}) + \chi(\mathcal{O}_C(C) \otimes \mathcal{L}^{\otimes d}).$$

Since $\mathcal{O}_C(C) \otimes \mathcal{L}^{\otimes d}$ is supported on a codimension 1 subscheme of X, the leading terms of the Hilbert polynomials $\chi(\mathcal{O}_X(C) \otimes \mathcal{L}^{\otimes d})$ and $\chi(\mathcal{L}^{\otimes d})$ coincide.

Pick now ℓ large enough such that $\mathcal{O}_{\mathbb{P}}(D+\ell E)$ is globally generated. This allows to choose a section $g \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(D + \ell E))$ which does not vanish on X. Further choose a positive integer m such that mD is Cartier. Then there is a chain of injective maps

$$H^0(X, \mathcal{O}_{\mathbb{P}}(dE)|_X) \to H^0(X, \mathcal{O}_{\mathbb{P}}(D + (d + \ell)E)|_X) \to \cdots \to H^0(X, \mathcal{O}_{\mathbb{P}}(mD + (d + \ell m)E)|_X)$$

induced by multiplication with g. As a consequence, Serre vanishing yields for $d \gg 0$

$$\chi(\mathcal{L}^{\otimes d}) \leq \chi(\mathcal{O}_{\mathbb{P}}(D + \ell E)|_X \otimes \mathcal{L}^{\otimes d}) \leq \chi(\mathcal{O}_{\mathbb{P}}(mD + \ell mE)|_X \otimes \mathcal{L}^{\otimes d}).$$

But $\mathcal{O}_{\mathbb{P}}(mD + \ell mE)|_X$ is the sheaf of an effective Cartier divisor, so by the previous, the leading terms of these three Hilbert polynomials agree. Thus

$$\#H^0(X, \mathcal{O}_{\mathbb{P}}(D + dE)|_X) = q^{\chi(\mathcal{L}^{\otimes d}) + O(d)} = q^{\frac{d^2 E.E}{2} + O(d)}, \quad d \gg 0.$$

Choosing d_0 large enough, we obtain that the density of reducible X_f is 0.

Step 2. For f in a set of density 1, there is a bijection $\operatorname{Irr} X_{\overline{k}} \to \operatorname{Irr}(X_f)_{\overline{k}}$, $C \mapsto C_f$.

Select an f such that the map $\operatorname{Irr} X_{\overline{k}} \to \operatorname{Irr}(X_f)_{\overline{k}}$ is not bijective. Let $Y \in \operatorname{Irr} \widetilde{X}_{\overline{k}}$ and consider it as an element of $\operatorname{Irr} \widetilde{X}_{k'}$, where k' is the field of definition of Y. Interpreting Y as a k-scheme via $Y \to \operatorname{Spec} k' \to \operatorname{Spec} k$, the maps $Y \to \widetilde{X} \to X$ are birational k-morphisms. In particular $Y_{\overline{k}}$ and $X_{\overline{k}}$ share a common smooth dense open subscheme. Thus by Lemma 2.3.3, the map

$$\operatorname{Irr}(Y_{\overline{k}}) \to \operatorname{Irr}_{\mathrm{horiz}}((Y_f)_{\overline{k}}), \quad C \mapsto (C_f)_{\mathrm{horiz}}$$

is not bijective for f in a set of density 1. It follows by Lemma 2.3.4 that there is a component $C \in \operatorname{Irr}(Y_{\overline{k}})$ such that $(C_f)_{\mathrm{horiz}}$ is not irreducible. This implies that $(Y_f)_{\mathrm{horiz}}$ is not geometrically irreducible.

In other words, there is a map from the set $\mathcal{X}$ of X_f such that $\operatorname{Irr} X_{\overline{k}} \to \operatorname{Irr}(X_f)_{\overline{k}}$ fails to be bijective to the set $\mathcal{Y}$ of irreducible and geometrically reducible schemes Z_{horiz} for some $Z \in \mathbb{P}H^0(Y, \pi^*\mathcal{O}_{\mathbb{P}}(D + dE)|_Y)$, sending X_f to $(Y_f)_{\mathrm{horiz}}$. This map depends on the choice of Y. However, it is injective, as X_f as a divisor is determined by a dense open subscheme.

It is now sufficient to show

$$\frac{\#\mathcal{Y}}{\#H^0(X, \mathcal{O}_{\mathbb{P}}(D + dE)|_X)} \xrightarrow{d\to\infty} 0,$$

and this can be done as in [11, Proposition 4.1]. □

2.3.3 Induction

Lemma 2.3.7. *Let $X \subseteq \mathbb{P}_{\overline{k}}$ be a smooth irreducible subscheme of dimension $m \geq 3$. Suppose that $\dim \overline{X} \cap (\mathbb{P}_{\overline{k}})^{\mathrm{sing}} \leq m - 2$. Then:*

(1) *There exists a hypersurface $J \subseteq \mathbb{P}$ defined over k such that*
   - *$J \cap X$ is irreducible,*
   - *$\dim J \cap X = m - 1$,*
   - *$\dim J \cap (\overline{X} \setminus X) \leq m - 2$,*
   - *$\dim J \cap \overline{X} \cap (\mathbb{P}_{\overline{k}})^{\mathrm{sing}} \leq m - 3$.*

(2) *For any J as in* (1), *there is a density* 1 *set of f for which the implication*

$$(J \cap X)_f \textit{ irreducible } \Rightarrow X_f \textit{ irreducible}$$

holds.

Proof. (1) Pick a positive integer k and choose sections $h_i \in H^0(\mathbb{P}, kE)$, $i = 0, \ldots, m$, such that $\dim V(h_0, \ldots, h_r) \cap \overline{X} = m - r - 1$ for $r = 0, \ldots, m-1$ and $V(h_0, \ldots, h_m) \cap \overline{X} = \emptyset$. This is possible since kE has no base points for $k \gg 0$. The sections $h_0, \ldots, h_m$ give rise to a map

$$\pi : \overline{X} \to \mathbb{P}^m_{\mathbb{F}}, \quad P \mapsto (h_0(P) : \cdots : h_m(P)).$$

The fiber over $(0 : \cdots : 0 : 1)$ is zero-dimensional, therefore π is a generically finite dominant morphism. Define

$$\begin{aligned} Z := &\{P \in \mathbb{P}^m_k \mid \operatorname{codim}_{\overline{X}} \pi^{-1}(P) = 1\} \\ &\cup \{\pi(C) \subseteq \mathbb{P}^m_k \mid C \in \operatorname{Irr}(\overline{X} \setminus X) \cup \operatorname{Irr}(\overline{X} \cap (\mathbb{P}_{\overline{k}})^{\mathrm{sing}}), \dim \pi(C) = 0\}. \end{aligned}$$

Then Z is finite. By [60, Theorem 1.2, Proposition 2.7], Lemma 2.3.1 and [11, Lemma 5.2], there is a positive density of homogeneous polynomials $g \in k[x_1, \dots, x_m]$ such that

- $\{g = 0\}$ is geometrically integral,
- $\{g = 0\} \cap Z = \emptyset$,
- $\pi(C) \not\subseteq \{g = 0\}$ for any $C \in \operatorname{Irr}(\overline{X} \setminus X) \cup \operatorname{Irr}(\overline{X} \cap (\mathbb{P}_{\overline{k}})^{\text{sing}})$ with $\dim \pi(C) \geq 1$,
- $X \cap \pi^{-1}(\{g = 0\})$ is irreducible of dimension $m - 1$.

Pick such a g and set $J := g(h_0, \dots, h_m) \in H^0(\mathbb{P}, k \deg g \cdot E)$. Then:

- $J \cap X = X \cap \pi^{-1}(\{g = 0\})$,
- J contains no irreducible component of $\overline{X} \setminus X$ or $\overline{X} \cap (\mathbb{P}_{\overline{k}})^{\text{sing}}$, whence

$$\dim J \cap (\overline{X} \setminus X) \leq m - 2 \quad \text{and} \quad \dim J \cap X \cap (\mathbb{P}_{\overline{k}})^{\text{sing}} \leq m - 3.$$

(2) Similar to [11, Lemma 5.3], if $(J \cap X)_f$ is irreducible and X_f happen to be reducible, then $X_f = V_1 \cup V_2$ for certain subschemes V_1, V_2 such that $V_1 \not\subseteq V_2$, $V_2 \not\subseteq V_1$ and $\dim V_1, \dim V_2 \geq m - 1$. Moreover, for $i = 1, 2$, $J \cap \overline{V_i}$ is nonempty of dimension $\geq m - 2$. For f in a set of density 1, Lemma 2.3.1 implies that

$$\dim J \cap (\overline{V}_i \setminus V_i) \leq \dim J \cap (\overline{X} \setminus X) \cap X_f \leq m - 3.$$

This implies that $J \cap V_i$ is of dimension $\geq m - 2$. Using that $(J \cap X)_f$ is irreducible, we can assume w.l.o.g. that $J \cap V_1 \subseteq J \cap V_2$. As a consequence,

$$m - 2 \leq \dim J \cap V_1 \leq \dim J \cap V_1 \cap V_2 \leq \dim J \cap (X_f)^{\text{sing}}.$$

Let $U := X \cap \mathbb{P}^{\text{sm}}$. Clearly $(X_f)^{\text{sing}} \subseteq (U_f)^{\text{sing}} \cup (X \cap (\mathbb{P}_{\overline{k}})^{\text{sing}})$. By Lemma 2.3.5, $(U_f)^{\text{sing}}$ is finite for f in a set of density 1, as U is smooth and does not meet $(\mathbb{P}_{\overline{k}})^{\text{sing}}$. In particular, for these f,

$$\begin{aligned} \dim J \cap (X_f)^{\text{sing}} &\leq \max\{\dim J \cap (U_f)^{\text{sing}}, \dim J \cap X \cap (\mathbb{P}_{\overline{k}})^{\text{sing}}\} \\ &\leq \max\{0, \dim J \cap \overline{X} \cap (\mathbb{P}_{\overline{k}})^{\text{sing}}\} \\ &\leq m - 3. \end{aligned}$$

This leads to the contradiction

$$m - 2 \leq \dim J \cap (X_f)^{\text{sing}} \leq m - 3.$$

Thus for f in a set of density 1, $(J \cap X)_f$ irreducible implies X_f irreducible. □

Proposition 2.3.8. *Let X be an irreducible subscheme of $\mathbb{P}$ of dimension $m \geq 2$ and suppose that $\dim \overline{X} \cap \mathbb{P}_{\text{sing}} \leq m - 2$. For f in a set of density 1, there is a bijection $\operatorname{Irr} X_{\overline{k}} \to \operatorname{Irr}(X_f)_{\overline{k}}$ sending C to C_f.*

Proof. We may assume that X is reduced. For surfaces note that $\overline{X} \cap \mathbb{P}^{\mathrm{sing}}$ is finite, thus the assertion for $\overline{X}$ follows from Proposition 2.3.6. Now Lemma 2.3.3 allows to proceed to X.

For $m \geq 3$, we can assume that X is smooth by Lemma 2.3.3. Pick an irreducible component $C \in \operatorname{Irr} X_{\overline{k}}$. Then C is a smooth irreducible subscheme of $\mathbb{P}_{\overline{k}}$ of dimension $m \geq 3$ and

$$\dim \overline{C} \cap (\mathbb{P}_{\overline{k}})^{\mathrm{sing}} \leq \dim \overline{X} \cap \mathbb{P}^{\mathrm{sing}} \leq m-2.$$

Lemma 2.3.7 applied to C produces a hypersurface $J \subseteq \mathbb{P}$ defined over k such that $J \cap C$ is irreducible of dimension $m-1$ and

$$\dim J \cap \overline{C} \cap (\mathbb{P}_{\overline{k}})^{\mathrm{sing}} \leq m-3.$$

Using the map $C \hookrightarrow X_{\overline{k}} \to X$, this means that $J \cap X$ is irreducible of dimension $m-1$ as well and

$$\dim J \cap \overline{X} \cap (\mathbb{P}_{\overline{k}})^{\mathrm{sing}} \leq m-3.$$

Performing induction on $J \cap X$ shows that for f in a set of density 1, $(J \cap C)_f$ is irreducible for any $C \in \operatorname{Irr} X_{\overline{k}}$. For a possibly smaller set of density 1, this implies that C_f is irreducible by part (2) of Lemma 2.3.7. Moreover every C_f is horizontal, since $\dim C_f \geq m-1$, whereas

$$\dim C_f \cap (\mathbb{P}_{\overline{k}})_{\mathrm{sing}} \leq \dim \overline{X} \cap (\mathbb{P}_{\overline{k}})_{\mathrm{sing}} \leq m-2.$$

Finally Lemma 2.3.4 yields a bijection

$$\operatorname{Irr} X_{\overline{k}} \xrightarrow{\sim} \operatorname{Irr}_{\mathrm{horiz}}(X_f)_{\overline{k}} \xrightarrow{\sim} \operatorname{Irr}(X_f)_{\mathbb{F}}, \quad C \mapsto C_f.$$

□

2.3.4 Final steps of the proof

Lemma 2.3.9. *Let X and Y be irreducible finite type $\overline{k}$-schemes. Suppose that $X \xrightarrow{\pi} Y \xrightarrow{\psi} \mathbb{P}_{\overline{k}}$ are morphisms such that π is finite and étale, ψ has relative dimension s at each point and $\dim \overline{\psi(Y)} \geq 2$. Then for f in a set of density 1, the implication*

$$Y_f \text{ irreducible } \Rightarrow X_f \text{ irreducible}$$

holds.

Proof. Following the proof of [11, Lemma 5.1], we only need to adjust the density estimate for f such that $\{f = 0\}$ misses at least $(c' + o(1))r^{me}/e$ points of $\overline{\psi(Y)}$ with residue field of size at most r^e, for fixed $c' > 0$, $e, m, r \in \mathbb{N}$, $m \geq 2$. As in the proof of Lemma 2.3.2, this density either equals zero or is bounded from above by

$$(1 - r^{-e})^{(c'+o(1))r^{me}/e}.$$

As $e \to \infty$, this quantity goes to zero due to $m \geq 2$. □

Lemma 2.3.10. *Let X and Y be irreducible finite type $\overline{k}$-schemes with morphisms $X \xrightarrow{\pi} Y \xrightarrow{\psi} \mathbb{P}_{\overline{k}}$ such that π is dominant, $\dim \overline{\psi(Y)} \geq 2$ and $\dim \overline{\psi(Y)} \cap (\mathbb{P}_{\overline{k}})^{\mathrm{sing}} \leq \dim \overline{\psi(Y)} - 2$. Then for f in a set of density 1, the implication*

$$(Y_f)_{\mathrm{horiz}} \text{ irreducible } \Rightarrow (X_f)_{\mathrm{horiz}} \text{ irreducible}$$

holds.

Proof. As in [11, Lemma 5.2]. □

Proposition 2.3.11. *Let X be a k-scheme of finite type. Let $\varphi : X \to \mathbb{P}$ be a morphism such that $\dim \overline{\varphi(C)} \geq 2$ and $\dim \overline{\varphi(C)} \cap (\mathbb{P}_{\overline{k}})^{\mathrm{sing}} \leq \dim \overline{\varphi(C)} - 2$ for each $C \in \operatorname{Irr} X$. Then for f in a set of density 1, there is a bijection $\operatorname{Irr} X_{\overline{k}} \to \operatorname{Irr}_{\mathrm{horiz}}(X_f)_{\overline{k}}$ sending C to $(C_f)_{\mathrm{horiz}}$.*

Proof. By Lemma 2.3.3, we may again assume that X is reduced and smooth, so its irreducible components are disjoint. W.l.o.g. we can thus further suppose that X is irreducible. Let $C \in \operatorname{Irr} X_{\overline{k}}$, then $\overline{\varphi(C)}$ is an irreducible component of $\overline{\varphi(X)}_{\overline{k}}$. By Proposition 2.3.8, $\overline{\varphi(C)}_f$ is irreducible for f in a set of density 1. Applying Lemma 2.3.10 to $C \to \overline{\varphi(C)} \hookrightarrow \mathbb{P}_{\overline{k}}$ shows that $(C_f)_{\mathrm{horiz}}$ is irreducible. Together with Lemma 2.3.4, this implies the existence of a bijection $\operatorname{Irr} X_{\overline{k}} \to \operatorname{Irr}_{\mathrm{horiz}}(X_f)_{\overline{k}}$ sending C to $(C_f)_{\mathrm{horiz}}$. □

Proof of Theorem 2.1.9. The strategy is to proceed to a finite field of definition for φ, X and its components, and then to apply several base change arguments. The details are as in [11, Theorem 1.6], adjusting the notion of horizontal components. □

2.3.5 Sharpness of the codimension condition

The following example illustrates that the codimension condition in Theorem 2.1.9 is essential:

Example 2.3.12. Consider the weighted projective space $\mathbb{P} = \mathbb{P}(1,2,3,6)$ with coordinates x_0, x_1, x_2, x_3. Let $X := \{x_0 = 0\} \subseteq \mathbb{P}_{\overline{k}}$ and let $\varphi : X \hookrightarrow \mathbb{P}_{\overline{k}}$ be the inclusion. X is an irreducible surface in $\mathbb{P}_{\overline{k}}$ and

$$X \cap (\mathbb{P}_{\overline{k}})^{\mathrm{sing}} = \{x_0 = x_1 = 0\} \cup \{x_0 = x_2 = 0\}$$

is one-dimensional. For $d \geq 1$, let f be a weighted homogeneous polynomial of degree $6d+1$. One finds that f can be written as

$$f = x_1^2 x_2 \sum_{i=0}^{d-1} \sum_{j=0}^{d-1-i} c_{ij} x_1^{3i} x_2^{2j} x_3^{d-i-j-1} + \text{ terms divisible by } x_0, \quad c_{ij} \in k.$$

Thus if $X \cap \{f = 0\}$ is irreducible, then f lies in a subspace of $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(6d+1))$ of codimension

$$\sum_{i=0}^{d-1} (d-1-i) = \frac{d(d-1)}{2}.$$

As a consequence, the fraction of $f \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(6d+1))$ such that $X \cap \{f = 0\}$ is irreducible is at most $q^{-d(d-1)/2}$. In particular, the density of f for which the map $\operatorname{Irr} X \to \operatorname{Irr} X_f$ is a bijection is bounded from above by $\lim_{d \to \infty} q^{-d(d-1)/2} = 0$.

Chapter 3

Densities of smooth fibrations

3.1 Overview

Let $k = \mathbb{F}_q$ be a finite field. In this chapter, we construct a certain class of fibrations and ask again for the density of smooth members.

3.1.1 Set-up

Let X be a smooth projective variety over k. Let $\mathcal{L}$ be a very ample line bundle giving a closed immersion $i : X \hookrightarrow \mathbb{P}^N$. Fix integer weights $w_0, \dots, w_m \geq 1$. Then for any integer $d \geq 1$, there is a rational map

$$\pi_d : \mathbb{P}_d := \mathbb{P}(dw_0, \dots, dw_m, 1, \dots, 1) \dashrightarrow \mathbb{P}^N, \quad (x_0 : \dots : x_m : z_0 : \dots : z_N) \mapsto (z_0 : \dots : z_N).$$

Let $H := \{z_0 = \dots = z_N = 0\} \subseteq \mathbb{P}_d$. Then the restriction $\pi_d|_{\mathbb{P}_d \setminus H} : \mathbb{P}_d \setminus H \to \mathbb{P}^N$ is a morphism. Define $Y_d := \pi_d|_{\mathbb{P}_d \setminus H}^{-1}(i(X))$. The singular locus $(\mathbb{P}_d)^{\mathrm{sing}}$ of $\mathbb{P}_d$ is contained in H, thus $Y_d \cap (\mathbb{P}_d)^{\mathrm{sing}} = \emptyset$ and hence Y_d is a smooth quasiprojective variety.

Choose degrees $0 < e_1 \leq \dots \leq e_n$ and set $e_0 := 0$, $e := e_n$. Fix further weighted homogeneous polynomials

$$g_i \in H^0(\mathbb{P}(w_0, \dots, w_m), \mathcal{O}(e - e_i)), \quad i = 0, \dots, n.$$

Moreover, suppose that $g_n \neq 0$. There are natural embeddings

$$H^0(X, \mathcal{L}^{\otimes de_i}) \hookrightarrow H^0(Y_d, \mathcal{O}_{\mathbb{P}_d}(de_i)|_{Y_d}), \quad i = 0, \dots, n$$

and, by change of the grading,

$$H^0(\mathbb{P}(w_0, \dots, w_m), \mathcal{O}(e - e_i)) \hookrightarrow H^0(\mathbb{P}_d, \mathcal{O}_{\mathbb{P}_d}(de - de_i)), \quad d \geq 1, \quad i = 0, \dots, n.$$

This allows to define the following maps:

$$\alpha_d : \bigoplus_{i=1}^{n} H^0(X, \mathcal{L}^{\otimes de_i}) \to H^0(Y_d, \mathcal{O}_{\mathbb{P}_d}(de)|_{Y_d}), \quad (a_1, \dots, a_n) \mapsto g_0 + \sum_{i=1}^{n} a_i g_i, \quad d \geq 1.$$

The goal is to answer the following question: As d tends to infinity, what is the density of $a \in \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i})$ such that the hypersurface $\{\alpha_d(a) = 0\} \subseteq Y_d$ is smooth? The answer depends on the choice of the polynomials $g_0, \dots, g_n$:

Definition 3.1.1 (Discriminant locus)**.** For an $(n+1)$-tuple $(g_0, \dots, g_n)$ of weighted homogeneous polynomials as above, define the *discriminant locus* as

$$\Delta(g_0, \dots, g_n) := \left\{ (a_1, \dots, a_n) \in \mathbb{A}^n \;\middle|\; \left\{ g_0 + \sum_{i=1}^{n} a_i g_i = 0 \right\} \subseteq \mathbb{A}^{m+1} \text{ is not smooth} \right\}.$$

Definition 3.1.2 (admissible)**.** An $(n+1)$-tuple $(g_0, \dots, g_n) \in \bigoplus_{i=1}^n H^0(\mathbb{P}(w_0, \dots, w_m), \mathcal{O}(e - e_i))$ as above is *admissible* if $\dim \Delta(g_0, \dots, g_n) < n$.

Remarks. Let p denote the characteristic of k.

- The admissibility condition is not superfluous: Take for example $n = 1$, $e = e_1 = p$, $g_0 = x_0^p$, $g_1 = 1$. Then $\{x_0^p + a_n = 0\}$ is singular for every choice of a_n.

- However, if $p \nmid e$, then for a choice $(a_1, \dots, a_n) = (0, \dots, 0, \lambda)$ with $\lambda \in \overline{k}^\times$, the polynomial $g_0 + \sum_{i=1}^n a_i g_i = g_0 + \lambda g_n$ defines a smooth hypersurface in $\mathbb{A}^{m+1}$: Indeed, if all partial derivatives vanish, then so does g_0 by the (weighted) Euler identity, but λg_n is always nonzero. As a consequence, $p \nmid e$ implies that $(g_0, \dots, g_n)$ is admissible.

3.1.2 Main results

Densities of smooth fibrations

Theorem 3.1.3 (Bertini smoothness theorem for fibrations)**.** *With the notation of §3.1, suppose that $(g_0, \dots, g_n)$ is admissible with discriminant locus $\Delta := \Delta(g_0, \dots, g_n)$. Then*

$$\lim_{d\to\infty} \frac{\#\{a \in \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i}) \mid \{\alpha_d(a) = 0\} \subseteq Y_d \textit{ is smooth}\}}{\# \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i})} = \prod_{P\in X \textit{ closed}} \left(1 - \frac{\#\Delta(\kappa(P))}{\#\kappa(P)^{\dim X + n}}\right),$$

where $\kappa(P)$ denotes the residue field of P and $\#\Delta(\kappa(P))$ is the number of $\kappa(P)$-rational points of the scheme Δ.

Remarks.

- The product converges as $(g_0, \dots, g_n)$ is admissible.
- The theorem is valid for smooth quasiprojective X as well. The only difference is that the very ample line bundle $\mathcal{L}$ and its sections live on $\overline{X}$ instead.
- The theorem recovers Poonen's Bertini theorem over finite fields, see Example 3.2.4.
- When $\Delta(K) = \#K^r$ for some positive integer r and all finite field extensions K/k, then the right-hand side of the formula simplifies to $\zeta_X(\dim X + n - r)^{-1}$. This occurs for certain fibrations in (hyper)elliptic curves, see Examples 3.2.5 and 3.2.6.
- Another Bertini smoothness theorem on fibrations in a different setting, but in a similar spirit, is given in [28].

Theorem 3.1.3 is shown in §3.2.1. The proof bears many similarities to the Bertini theorem on quasismoothness of §2.2. There is also an analogon to Theorem 2.1.7:

Theorem 3.1.4. *In the situation of Theorem 3.1.3, suppose that $(g_0, \dots, g_n)$ is strongly admissible. Let $g : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ be a function with $\lim_{d\to\infty} g(d) = \infty$. Then*

$$\lim_{d\to\infty} \frac{\#\{a \in \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i}) \mid \operatorname{length}(\Sigma(a)) < g(d)\}}{\# \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i})} = 1.$$

For the proof and the definition of "strongly admissible", we refer to §3.2.3.

Ranks of elliptic n-folds

The main application is a particular instance of the previous construction: Let X be a smooth and geometrically irreducible projective variety of dimension ≥ 2 and let $\mathcal{L}$ be a very ample line bundle on X.

For an arbitrary elliptic curve E over $k(X)$, denote by $r(E)$ the rank of the Mordell-Weil group of $E/k(X)$. This is finite by the Lang-Néron theorem [46]. Furthermore, define the *height* $h(E)$ as the minimal integer $d \geq 1$ such that E can be defined by a Weierstrass equation

$$x_1^2 = x_0^3 + a_1 x_0 + a_2 \quad \text{with } (a_1, a_2) \in H^0(X, \mathcal{L}^{\otimes 4d}) \oplus H^0(X, \mathcal{L}^{\otimes 6d})$$

in the weighted projective space $\mathbb{P}(2d, 3d, 1, \ldots, 1)$. Define $\mathcal{E}_d$ as the set of elliptic curves over $k(X)$ with height $\leq d$. This set is finite, since it is parametrized by a subset of the finite-dimensional $\mathbb{F}_q$-vector space $H^0(X, \mathcal{L}^{\otimes 4d}) \oplus H^0(X, \mathcal{L}^{\otimes 6d})$.

The following is the summary of the results presented in §3.3:

Theorem 3.1.5 (Density of elliptic n-folds with rank 0)**.** *With the above notation, let p denote the characteristic of k. Then*

$$\liminf_{d\to\infty} \frac{\#\{E \in \mathcal{E}_d \mid r(E) = 0\}}{\#\mathcal{E}_d} \begin{cases} \geq \zeta_X(\dim X + 3)^{-1} & \text{if } \dim X = 2 \text{ and } p \geq 3, \\ \geq \zeta_X(\dim X + 1)^{-1} & \text{if } \dim X = 2 \text{ and } p = 2, \\ = 1 & \text{if } \dim X \geq 3. \end{cases}.$$

Theorem 3.1.5 improves a bound obtained by Kloosterman [44, Theorem 1.1] in the case $\dim X = 2$. The first main insight for the proof is a connection between the rank of an elliptic curve parametrized by $a \in H^0(X, \mathcal{L}^{\otimes 4d}) \oplus H^0(X, \mathcal{L}^{\otimes 6d})$ and the factoriality of the corresponding hypersurface $\{\alpha_d(a) = 0\}$. Using the methods from the chapter on hypersurfaces with defect, this forces the hypersurface to have many singularities compared to its degree, and we can finish by Theorem 3.1.4.

Remarks.

- If the bound on the density of hypersurfaces without defect (Theorem 1.1.3) can be improved, then this will give a new lower bound for the density of elliptic threefolds with rank 0 as well. In particular, it is likely that this density actually equals 1.
- Theorem 3.1.5 shows that when ordered by height, the average rank of elliptic n-folds is 0 if $\dim X \geq 3$. In contrast, over the rational numbers, it is assumed that elliptic curves with rank 0 or 1 form a set of density $\frac{1}{2}$, respectively.
- The methods presented here do not carry over to the case $\dim X = 1$. However, there is an upper bound of $\frac{3}{2} + O(q^{-1})$ on the average rank of elliptic curves over $\mathbb{F}_q(t)$ by de Jong (see [19, Corollary 1.3]).

3.2 Fibrations in weighted projective space

3.2.1 Sieving

Keep the notation of §3.1. Let $(g_0, \ldots, g_n)$ be admissible with discriminant locus Δ. Set $\ell := \dim X$. The sieving process is similar to that in §2.2.3.

Low degree points

Lemma 3.2.1 (Low degree points). *Fix an integer $r \geq 1$ and denote by $X_{<r}$ the set of closed points of X of degree less than r. Then there is a constant d_r such that for all $d \geq d_r$ holds*

$$\frac{\#\{a \in \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes kd_i}) \mid \{\alpha_d(a) = 0\} \text{ is smooth at all points in } \pi_d^{-1}(P) \text{ for all } P \in X_{<r}\}}{\# \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i})}$$
$$= \prod_{P \in X_{<r}} \left(1 - \frac{\#\Delta(\kappa(P))}{\#\kappa(P)^{\dim X + n}}\right).$$

Proof. Let $P \in X$ be a closed point and pick a point $R \in \pi_d^{-1}(P)$. The homomorphism $\mathcal{O}_{X,P} \hookrightarrow \mathcal{O}_{Y_d,R}$ of local rings is flat, so we can extend a regular sequence $(t_1, \ldots, t_\ell)$ at P to a regular sequence $(t_1, \ldots, t_\ell, y_0, \ldots, y_m)$ at R. Denote by $\mathfrak{m}_P$ the maximal ideal of P and by $\kappa(P)$ its residue field. Let X_P be the subscheme of X corresponding to the closed immersion $\mathfrak{m}_P^2 \hookrightarrow \mathcal{O}_X$.

Further define the subset $S_P \subseteq \kappa(P)^{(\ell+1)n}$ given by all $a_{ij} \in \kappa(P)^{(\ell+1)n}$, $i = 1, \ldots, n$, $j = 0, \ldots, \ell$, such that

$$\exists\, Q \in \mathbb{A}^{m+1}(\overline{k}) : \quad \begin{aligned} & g_0(Q) + \sum_{i=1}^n a_{i0} \cdot g_i(Q) = 0 & & \\ & \frac{\partial g_0}{\partial y_j}(Q) + \sum_{i=1}^n a_{i0} \cdot \frac{\partial g_i}{\partial y_j}(Q) = 0, & & j = 0, \ldots, m \\ & \sum_{i=1}^n g_i(Q) \cdot a_{ij} = 0, & & j = 1, \ldots, \ell. \end{aligned} \tag{3.1}$$

With the natural restriction map (compare §2.2.2)

$$\varphi_{P,d} : \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i}) \to \bigoplus_{i=1}^n H^0(X_P, \mathcal{O}_{X_P}) \cong \bigoplus_{i=1}^n \kappa(P)^{\ell+1},$$
$$a_i \mapsto \left(a_i(P), \frac{\partial a_i}{\partial t_1}(P), \ldots, \frac{\partial a_n}{\partial t_\ell}(P)\right), \quad i = 1, \ldots, n,$$

the following holds:

$$\{\alpha_d(a) = 0\} \text{ is not smooth at some } R \in \pi^{-1}(P) \quad \Leftrightarrow \quad \varphi_{P,d}(a) \in S_P.$$

Observe that S_P does depend on $g_0, \ldots, g_n$ and $\kappa(P)$, but not on X. The size of S_P can be determined as follows: Since $g_n \in k^\times$, the polynomials $g_0, \ldots, g_n$ do not vanish simultaneously. Hence for any choice of Q the third row of equations in the system (3.1) is satisfied by precisely $\#\kappa(P)^{\ell(n-1)}$ solutions (a_{ij}), $i = 1, \ldots, n$, $j = 1, \ldots, \ell$. Moreover, the first two rows have precisely $\#\Delta(\kappa(P))$ solutions (a_{i0}), $i = 1, \ldots, n$. Therefore

$$\#S_P = \#\Delta(\kappa(P)) \cdot \#\kappa(P)^{\ell(n-1)}.$$

Finally, the density computation: Let Z denote the union of the subschemes X_P for all $P \in X_{<r}$. We obtain a map

$$\varphi_{Z,d} : \bigoplus_{i=1}^{n} H^0(X, \mathcal{L}^{\otimes de_i}) \to \bigoplus_{i=1}^{n} H^0(Z, (\mathcal{L}|_Z)^{\otimes de_i}) \cong \bigoplus_{i=1}^{n} \bigoplus_{P \in X_{<r}} H^0(X_P, \mathcal{O}_{X_P}),$$

whose restriction to the component belonging to a specific $P \in X_{<r}$ agrees with $\varphi_{P,d}$. By Lemma 2.2.7, there is a constant d_r such that $\varphi_{Z,d}$ is surjective whenever $d \geq d_r$. The proof can be finished as in Lemma 2.2.15:

$$\begin{aligned} &\frac{\#\{a \in \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i}) \mid \{\alpha_d(a) = 0\} \text{ smooth at all points in } \pi_d^{-1}(P) \text{ for all } P \in X_{<r}\}}{\# \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i})} \\ &\quad \frac{\#\{a \in \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i}) \mid \varphi_{P,d}(a) \notin S_P \text{ for all } P \in X_{<r}\}}{\# \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i})} \\ &\quad = \prod_{P \in X_{<r}} \left(1 - \frac{\#\Delta(\kappa(P))}{\#\kappa(P)^{\ell+n}}\right). \end{aligned}$$

□

Medium degree points

Lemma 3.2.2 (Medium degree points). *Fix an integer $r \geq 1$ and let*

$$s := \frac{1}{\dim X + 1}.$$

If $X_{r,sd}$ denotes the set of closed points P of X with $r \leq \deg P \leq sd$, then

$$\lim_{r \to \infty} \lim_{d \to \infty} \frac{\# \left\{ a \in \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i}) \,\middle|\, \begin{array}{l} \exists\, P \in X_{r,sd} : \{\alpha_d(a) = 0\} \textit{ is not} \\ \textit{smooth at some point in } \pi_d^{-1}(P) \end{array} \right\}}{\# \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i})} = 0.$$

Proof. Let $P \in X_{r,sd}$. Then $de_i \geq d \geq \deg P(\ell + 1)$ for all i, thus the map $\varphi_{P,d}$ constructed in the proof of Lemma 3.2.1 is surjective due to Corollary 2.2.13. Following the proof of Lemma 3.2.1,

$$\begin{aligned} &\frac{\#\{a \in \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i}) \mid \{\alpha_d(a) = 0\} \text{ is not smooth at some point in } \pi_d^{-1}(P)\}}{\# \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i})} \\ &\quad = \frac{1}{\#\kappa(P)^{\ell+n-e}}. \end{aligned}$$

Thus we obtain the estimate

$$\begin{aligned}
&\frac{\#\left\{a \in \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i}) \,\middle|\, \begin{array}{l}\exists\, P \in X_{r,sd} : \{\alpha_d(a) = 0\} \text{ is not smooth} \\ \text{at some point in } \pi_d^{-1}(P)\end{array}\right\}}{\# \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i})} \\
&\leq \sum_{P \in X_{r,sd}} \frac{\#\Delta(\kappa(P))}{\#\kappa(P)^{\ell+n}} \\
&\leq \sum_{i=r}^{sd} \frac{\#X(\mathbb{F}_{q^i}) \cdot \#\Delta(\mathbb{F}_{q^i})}{q^{i(\ell+n)}}
\end{aligned}$$

Let L_1 for X and L_2 for Δ be Lang-Weil bounds [47, Theorem 1]. Set $\delta := \dim \Delta$ and note that $\delta < n$ due to admissibility. Then

$$\begin{aligned}
\sum_{i=r}^{sd} \frac{\#X(\mathbb{F}_{q^i}) \cdot \#\Delta(\mathbb{F}_{q^i})}{q^{i(\ell+n)}} &\leq \sum_{i=r}^{sd} \frac{L_1 q^{i\ell} \cdot L_2 q^{i\delta}}{q^{i(\ell+n)}} \\
&\leq L_1 L_2 \sum_{i=r}^{\infty} q^{i(n-\delta)} \\
&\leq \frac{L_1 L_2}{q^{r(n-\delta)}} \cdot \frac{1}{1 - q^{n-\delta}},
\end{aligned}$$

which tends to zero as $r \to \infty$. □

High degree points

Lemma 3.2.3 (High degree points)**.** *Fix a rational number $s > 0$. Then*

$$\lim_{d \to \infty} \frac{\#\left\{a \in \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i}) \,\middle|\, \begin{array}{l}\exists\, P \in X, \deg P > sd : \{\alpha_d(a) = 0\} \textit{ is not} \\ \textit{smooth at some point in } \pi_d^{-1}(P)\end{array}\right\}}{\# \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i})} = 0.$$

Proof. The proof consists of three steps.

Step 1. Choosing coordinates and drawing sections at random.

Fix $(a_1, \ldots, a_{n-1}) \in \bigoplus_{i=1}^{n-1} H^0(X, \mathcal{L}^{\otimes kd_i})$. Replacing X by a small affine open U, find a local system of parameters $t_1, \ldots, t_\ell \in A$ as in the proof of [60, Lemma 2.6], where A is the coordinate ring of $\mathbb{A}^N$. As in the proof of Lemma 3.2.1, extend $(t_1, \ldots, t_\ell)$ to a local system of paramaters $(t_1, \ldots, t_\ell, y_0, \ldots, y_m)$ of $\pi_d^{-1}(U)$. Then the parameters $t_1, \ldots, t_\ell$ yield derivations $D_1, \ldots, D_\ell : A \to A$ satisfying $D_i(t_j) = s \cdot \delta_{ij}$ for some section s not vanishing on U.

Moreover, for any $a_n \in H^0(X, \mathcal{L}^{\otimes de_n})$, the hypersurface $\{\alpha_d(a) = 0\}$ fails to be smooth at a point $R = (Q, P) \in \pi_d^{-1}(P)$ if and only if

$$\begin{aligned}
g_0(Q) + \sum_{i=1}^n a_i(P) \cdot g_i(Q) &= 0 \\
\sum_{i=1}^n g_i(Q) \cdot D_j(a_i)(P) &= 0, \quad j = 1, \ldots, \ell. \qquad (3.2) \\
\frac{\partial g_0}{\partial y_j}(Q) + \sum_{i=1}^n a_i(P) \cdot \frac{\partial g_i}{\partial y_j}(Q) &= 0, \quad j = 0, \ldots, m
\end{aligned}$$

Let $\tau_l := \deg t_l$, $l = 1, \ldots, \ell$. Consider the surjective linear map

$$\psi : A_{\leq de} \oplus \bigoplus_{i=1}^{\ell} A_{\leq \frac{de-\tau_i}{q}} \oplus A_{\leq \frac{de}{q}} \to A_{\leq de}, \quad (f_0, \ldots, f_{\ell+1}) \mapsto f_0 + \sum_{i=1}^{\ell} f_i^q \cdot t_i + f_{\ell+1}^q.$$

Since $A_{\leq de} \cong H^0(\mathbb{P}^N, \mathcal{O}(de)) \to H^0(X, \mathcal{L}^{\otimes de})$ is surjective for large enough d by Serre vanishing, the map ψ can be used to draw an element $a_n \in H^0(X, \mathcal{L}^{\otimes de})$ at random. Define

$$W_0 := \pi_d^{-1}(U),$$
$$W_l := W_0 \cap \bigcap_{j=1}^{l} \left\{ \sum_{i=1}^{n} g_i \cdot D_j(a_i) = 0 \right\}, \quad l = 1, \ldots, \ell.$$

Note that if a_n is drawn via ψ, then W_l depends only on $f_0, \ldots, f_l$, $l = 0, \ldots, \ell$, because

$$D_j(a_n) = D_j(f_0) + f_j^q, \quad 1 \leq j \leq l.$$

Moreover, $\dim \pi_d(W_0) = \dim U = \ell$.

Step 2. For $l = 0, \ldots, \ell - 1$, conditioned on a choice of $f_0, \ldots, f_l$ such that $\dim \pi_d(W_l) \leq \ell - l$, the probability that $\dim \pi_d(W_{l+1}) \leq \ell - l - 1$ is $1 - o(1)$ as $d \to \infty$.

If $\dim \pi_d(W_l) = \ell - l$, the number of $(\ell - l)$-dimensional k-irreducible components of $\pi_d(W_l)$ is bounded from above by

$$\deg \overline{U} \cdot (de - \tau_1) \cdots (de - \tau_l) = O(d^l)$$

in virtue of Bézout's theorem.

Let V be such an $(\ell - l)$-dimensional component of $\pi_d(W_l)$. Define

$$G_V^{\mathrm{bad}} := \left\{ f_{l+1} \in A_{\leq \frac{de-\tau_{l+1}}{q}} \;\middle|\; V \subseteq \pi_d \left(\left\{ \sum_{i=1}^{n-1} g_i D_{l+1}(a_i) + g_n D_{l+1}(\psi(f_0, \ldots, f_{l+1}, *)) = 0 \right\} \right) \right\}$$

and suppose that $G_V^{\mathrm{bad}} \neq \emptyset$. Observe that

$$\sum_{i=1}^{n} g_i \cdot D_{l+1}(\psi(f_0, \ldots, f_{l+1}, *)) = \sum_{i=1}^{n-1} g_i a_i + g_n D_{l+1}(f_0) + g_n f_{l+1}^q,$$

so that if $f_{l+1}, f'_{l+1} \in G_V$, then $g_n(f_{l+1} - f'_{l+1})^q$ vanishes identically on V. Since $g_n \in k^\times$ and $k = \mathbb{F}_q$, this means that $f_{l+1} - f'_{l+1}$ must vanish identically on V. Hence there is a bijection

$$G_V^{\mathrm{bad}} \leftrightarrow \{g \in A_{\leq \frac{de-\tau_{l+1}}{q}} \mid V \subseteq \{g = 0\}\}.$$

Using Lemma 2.2.21 (2) (for $\mathbb{P} = \mathbb{P}^n \supseteq V$),

$$\frac{\# G_V^{\mathrm{bad}}}{\# A_{\leq \frac{de-\tau_{l+1}}{q}}} = O(q^{-d}).$$

Since there are at most $O(d^l)$ such components V, the probability that $\pi_d(W_{l+1})$ has dimension greater than $\ell - l - 1$ is

$$O(d^l q^{-d}) = o(1) \quad \text{as } d \to \infty.$$

Step 3. Conditioned on a choice of $f_0, \dots, f_\ell$ such that $\dim \pi_d(W_\ell)$ is finite, the probability that

$$Y := \pi_d \left(W_\ell \cap \left\{ g_0 + \sum_{i=1}^{n} a_i g_i = 0 \right\} \right)$$

contains no point of degree $> sd$ is $1 - o(1)$ as $d \to \infty$.

By another application of Bézout's theorem, there are at most $O(d^\ell)$ points in $\pi_d(W_\ell)$. Let $P \in \pi_d(W_\ell)$ and define

$$H_P^{\text{bad}} := \left\{ f_{\ell+1} \in A_{\le \frac{de}{q}} \;\middle|\; P \in \pi_d \left(\left\{ \sum_{i=1}^{n-1} g_i a_i + g_n \cdot \psi(f_0, f_1, \dots, f_{\ell+1}) = 0 \right\} \right) \right\}.$$

Again, if H_P^{bad} happens to be non-empty, then there is a bijection

$$H_P^{\text{bad}} \leftrightarrow \{h \in A_{\le \frac{de}{q}} \mid h(P) = 0\}.$$

By Lemma 2.2.21 (1),

$$\frac{\# H_P^{\text{bad}}}{\# A_{\le \frac{de}{q}}} = O(q^{-\min(d-1, \deg P)}).$$

If P is of degree $> sd$, then this quantity is $O(q^{-d})$ as $d \to \infty$. In total, the probability that Y contains a point of degree $> sd$ is therefore

$$O(d^\ell q^{-d}) = o(1) \quad \text{as } d \to \infty.$$

We finish the proof as in [60, Lemma 2.6]: Pick arbitrary $a_1, \dots, a_{n-1}$ and choose a_n at random via the map ψ. Then the probability that there is no point $R = (Q, P) \in \pi_d^{-1}(P)$ with $\deg P > sd$ satisfying the first two rows of (3.2) is $1 - o(1)$ as $d \to \infty$. In particular, there is no point P of high degree satisfying all three rows with probability at least $1 - o(1)$ as $d \to \infty$. But for such a P, satisfying (3.2) is equivalent to $\{\alpha_d(a) = 0\}$ having a singular point in $\pi_d^{-1}(P)$. ☐

Proof of Theorem 3.1.3

Proof. This works as in the proof of Theorem 2.1.6. ☐

3.2.2 Examples

Example 3.2.4 (Poonen's Bertini theorem)**.** Pick:

m	w_0	n	e_1	g_0	g_1
0	1	1	1	0	1

The discriminant locus is then

$$\Delta = \big\{ \{a \in \mathbb{A}^1 \mid \{a = 0\} \subseteq \mathbb{A}^1 \text{ not smooth} \big\} = \{0\}.$$

Theorem 3.1.3 hence states

$$\lim_{d \to \infty} \frac{\#\{a \in H^0(X, \mathcal{L}^{\otimes d}) \mid \{\alpha_d(a) = 0\} \subseteq \mathbb{P}(d, 1, \dots, 1)\} \text{ is smooth}\}}{\# H^0(X, \mathcal{L}^{\otimes d})} = \frac{1}{\zeta_X(\dim X + 1)}.$$

Since the restriction $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d)) \to H^0(X, \mathcal{L}^{\otimes d})$ is surjective for $d \gg 0$, this is equivalent to

$$\lim_{d\to\infty} \frac{\#\{f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^N}(d)) \mid \pi_d^{-1}(X \cap \{f=0\}) \subseteq \mathbb{P}(d,1,\ldots,1)\} \text{ smooth}\}}{\#H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))}$$
$$= \frac{1}{\zeta_X(\dim X + 1)}.$$

Finally, since smoothness of $X \cap \{f = 0\}$ and $\pi_d^{-1}(X \cap \{f = 0\})$ are equivalent, we obtain Poonen's theorem [60, Theorem 1.1]:

$$\lim_{d\to\infty} \frac{\#\{f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^N}(d)) \mid X \cap \{f=0\} \subseteq \mathbb{P}^N\} \text{ is smooth}\}}{\#H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d))} = \frac{1}{\zeta_X(\dim X + 1)}.$$

Example 3.2.5 (Elliptic curves)**.** Assume that k has characteristic ≥ 5. Make the following choices:

m	w_0	w_1	n	e_1	e_2	g_0	g_1	g_2
1	2	3	2	4	6	$x_1^2 - x_0^3$	$-x_0$	-1

The discriminant locus is

$$\Delta = \left\{ \{(a_1, a_2) \in \mathbb{A}^2 \mid \{x_1^2 = x_0^3 + a_1 x_0 + a_2\} \subseteq \mathbb{A}^2 \text{ not smooth} \right\}$$
$$= \left\{ \{(a_1, a_2) \in \mathbb{A}^2 \mid 4a_1^3 + 27a_2^2 = 0 \right\},$$

thus $\#\Delta(K) = \#K$ for any finite field extension K/k. Now Theorem 3.1.3 implies that

$$\lim_{d\to\infty} \frac{\#\{(a_1, a_2) \in H^0(X, \mathcal{L}^{\otimes 4d}) \oplus H^0(X, \mathcal{L}^{\otimes 6d}) \mid \{x_1^2 - x_0^3 - a_1 x - a_2 = 0\} \text{ is smooth}\}}{\#H^0(X, \mathcal{L}^{\otimes 4d}) \oplus H^0(X, \mathcal{L}^{\otimes 6d})}$$

equals $1/\zeta_X(\dim X + 1)$.

Remark. The result is also true in characteristic 2 and 3 with the appropriate changes of the Weierstrass equations.

Example 3.2.6 (Hyperelliptic curves)**.** Let $g \geq 1$ be an integer. Assume that k has characteristic $\neq 2$. Choose

m	w_0	w_1	n	e_1	e_2	$\ldots$	e_n	g_0	g_1	g_2	$\ldots$	g_n
1	2	$2g+1$	$2g+1$	2	4	$\ldots$	$4g+2$	$x_1^2 - x_0^{2g+1}$	$-x_0^{2g}$	$-x_0^{2g-1}$	$\ldots$	-1

The plane curve $\{x_1^2 - x_0^{2g+1} - \sum_{i=0}^{2g} a_i x_0^i = 0\} \subseteq \mathbb{A}^2$ is singular if and only if the monic polynomial $x_0^{2g+1} + \sum_{i=0}^{2g} a_{2g-i} x_0^i \in k[x_0]$ is not squarefree. For any finite field K, the number of non-squarefree monic polynomials of degree $2g+1$ in $K[x]$ is given by by $\#K^{2g}$. Thus $\#\Delta(K) = \#K^{2g}$ and $(g_0, \ldots, g_n)$ is admissible. Applying Theorem 3.1.3 yields that

$$\lim_{d\to\infty} \frac{\#\{(a_0, \ldots, a_{2g}) \in \bigoplus_{i=0}^{2g} H^0(X, \mathcal{L}^{\otimes 2d(i+1)}) \mid \{x_1^2 - x_0^{2g+1} - \sum_{i=0}^{2g} a_{2g-i} x_0^i = 0\} \text{ smooth}\}}{\# \bigoplus_{i=0}^{2g} H^0(X, \mathcal{L}^{\otimes 2d(i+1)})}$$

equals $1/\zeta_X(\dim X + 1)$.

3.2.3 The singular scheme

Definition 3.2.7 (strongly admissible)**.** Let $(g_0, \dots, g_n)$ be an $(n+1)$-tuple as in §3.1. Define

$$\Delta^{\mathrm{pos}}(g_0,\dots,g_n) := \left\{ (a_1,\dots,a_n) \in \mathbb{A}^n \;\middle|\; \left\{ g_0 + \sum_{i=1}^{n} a_i g_i = 0 \right\} \subseteq \mathbb{A}^{m+1} \begin{array}{l}\text{has a singular}\\ \text{locus of dim. } \geq 1\end{array} \right\}.$$

The tuple $(g_0, \dots, g_n)$ is called *strongly admissible* if it is admissible and $\Delta^{\mathrm{pos}}(g_0,\dots,g_n) = \emptyset$.

Example 3.2.8.

- In Examples 3.2.5 and 3.2.6, $\Delta^{\mathrm{pos}}(g_0,\dots,g_n)$ is empty because the fibers of π_d are reduced curves.
- The pair $(g_0, g_1) = (0,1)$ defined in Example 3.2.4 is not strongly admissible, since $\Delta^{\mathrm{pos}}(0,1) = \Delta(0,1)$ is the point $\{0\}$. Indeed, if $X \cap \{f = 0\}$ is singular at $x \in X$ for a polynomial $f \in k[x_0,\dots,x_n]_d$, then $\pi_d^{-1}(X \cap \{f=0\})$ is singular at every point in $\pi_d^{-1}(x) \cong \mathbb{A}^1$.

Dimension of the singular scheme

For $a \in \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i})$, define $\Sigma(a)$ as the subscheme of X supported at the singular locus of $\{\alpha_d(a) = 0\}$ with the scheme structure at a singular point coming from the ideal spanned by a local equation of $\{\alpha_d(a) = 0\}$ and its partial derivatives. Let $\sigma(a)$ denote the dimension of $\Sigma(a)$.

Corollary 3.2.9. *Keep the notations from §3.1.*

(1) *If $(g_0,\dots,g_n)$ is admissible, then*

$$\lim_{d\to\infty} \frac{\#\{a \in \bigoplus_{i=1}^n H^0(X,\mathcal{L}^{\otimes de_i}) \mid \sigma(a) \leq m+1\}}{\# \bigoplus_{i=1}^n H^0(X,\mathcal{L}^{\otimes de_i})} = 1.$$

(2) *If $(g_0,\dots,g_n)$ is strongly admissible, then*

$$\lim_{d\to\infty} \frac{\#\{a \in \bigoplus_{i=1}^n H^0(X,\mathcal{L}^{\otimes de_i}) \mid \sigma(a) = 0\}}{\# \bigoplus_{i=1}^n H^0(X,\mathcal{L}^{\otimes de_i})} = 1.$$

Proof.

(1) Suppose that $\sigma(a) > m+1$ for some $a \in \bigoplus_{i=1}^n H^0(X,\mathcal{L}^{\otimes de_i})$. Since the dimension of the fibers of π_d is $m+1$, the dimension of $\pi_d(\Sigma(a))$ must be positive.

(2) Suppose that $\sigma(a) \geq 1$ for some $a \in \bigoplus_{i=1}^n H^0(X,\mathcal{L}^{\otimes de_i})$. Observe that no fiber $\pi_d^{-1}(x)$ contains infinitely many singular points of $\{\alpha_d(a) = 0\}$, as $(g_0,\dots,g_n)$ is strongly admissible.

In both cases, there must be point $x \in X$ of arbitrary high degree such that $\{\alpha_d(a) = 0\}$ is singular at some point in $\pi_d^{-1}(x)$. But then a belongs to a set of density 0 by Lemma 3.2.3. □

Length of the singular scheme

Theorem 3.2.10. *In the situation of Theorem 3.1.3, suppose that* $(g_0, \dots, g_n)$ *is strongly admissible. Choose an integer* $s \geq 1$ *and let*

$$B_s := \left\{ (b_P)_{P \in X \text{ closed}} \;\middle|\; b_P \in \{0, 1, \dots, s\} \text{ for all } P \in X \text{ closed and } \sum_{P \in X \text{ closed}} b_P < s \right\}.$$

Then

$$\lim_{d \to \infty} \frac{\#\{a \in \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i}) \mid \operatorname{length}(\Sigma(a)) < s\}}{\# \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i})}$$
$$= \prod_{P \in X \text{ closed}} \left(1 - \frac{\#\Delta(\kappa(P))}{\#\kappa(P)^{\dim X + n}}\right) \cdot \sum_{(b_P) \in B_s} \prod_{P \in X \text{ closed}} \frac{\mu_P(b_P)}{\mu_P(0)},$$

where μ_P *is defined in the proof below.*

Proof. The philosophy is as in the proof of Theorem 2.2.32. By Corollary 3.2.9 (2), we can assume that $\{\alpha_d(a) = 0\}$ has at most isolated hypersurface singularities. In particular, the length of the scheme $\Sigma(a)$ is finite. It is sufficient to perform the low degree computation and show convergence, the strategy for medium and high degree points being as in Lemmas 3.2.2 and 3.2.3.

For $R \in Y_d$, denote by $\mathcal{O}_R$ the local ring at R and by $\mathfrak{m}_R$ its maximal ideal, analogously for $x \in x$. Define

$$\operatorname{length}_R(\Sigma(a)) := \dim_k \mathcal{O}_R \Big/ \left(f, \frac{\partial(\alpha_d(a))}{\partial y_0}, \dots, \frac{\partial(\alpha_d(a))}{\partial y_m}, \frac{\partial(\alpha_d(a))}{\partial t_1}, \dots, \frac{\partial(\alpha_d(a))}{\partial t_\ell} \right),$$

where $y_0, \dots, y_m, t_1, \dots, t_\ell$ are local coordinates around y as in the proof of Lemma 3.2.1.

For each integer $b \geq 0$, there is an integer $e_b \geq 0$ such that for all $e \geq 0$ exists a subset $B'_{R,b,e} \subseteq \mathcal{O}_R/\mathfrak{m}_R^e$ such that

$$\operatorname{length}_R \Sigma(a) = b \quad \Leftrightarrow \quad \varphi_{R,e}(a) \in B'_{R,b,e},$$

where $\varphi_{R,e}$ denotes the map

$$\bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i}) \to \mathcal{O}_P/\mathfrak{m}_P^e \xrightarrow{\pi_d^\#} \mathcal{O}_R/\mathfrak{m}_R^e.$$

Let $P \in X$ and set

$$C_{P,b} := \left\{ (c_R)_{R \in \pi_d^{-1}(x)} \;\middle|\; c_R \geq 0 \text{ and } \sum_{R \in \pi_d^{-1}(P)} c_R = b \right\}.$$

Define

$$B_{P,b,e} := \bigcup_{(c_R) \in C_{P,b}} \bigcap_{R \in \pi_d^{-1}(P)} (\pi_d^\#)^{-1}(B'_{R,c_R,e}) \subseteq \mathcal{O}_P/\mathfrak{m}_P^e.$$

By finite determinacy [9, Theorem 3], the number

$$\mu_P(b) := \frac{\#B_{P,b,e}}{\#\mathcal{O}_P/\mathfrak{m}_P^e}$$

is independent of e. Moreover,

$$\mu_P(0) = 1 - \frac{\#\Delta(\kappa(P))}{\#\kappa(P)^{\dim X+n}},$$

since $B_{P,0,2}$ is the set S_P of the proof of Lemma 3.2.1.

Fix an integer $r \geq 1$ and let $X_{<r} = \{P_1, \ldots, P_t\}$ be the set of closed points of X of degree $< r$. Let $(b_1, \ldots, b_t)$ be a sequence of non-negative integers satisfying $b_1 + \cdots + b_t = s$. Choose an integer e large enough to test whether

$$\mathrm{length}_P(\Sigma(a)) := \sum_{R \in \pi^{-1}(P)} \mathrm{length}_R(\Sigma(a)) = b_i \quad \text{for all } R \in \pi_d^{-1}(P_i), \quad i \in \{1, \ldots, t\}.$$

The ideals $\mathfrak{m}_{P_i}^e$, $i = 1, \ldots, t$, define a zero-dimensional subscheme of X, let Z denote their union. Consider the restriction

$$\varphi_{Z,e} : \bigoplus_{i=1}^{n} H^0(X, \mathcal{L}^{\otimes de_i}) \to H^0(Z, \mathcal{O}_Z) \cong \prod_{i=1}^{t} \mathcal{O}_{P_i}/\mathfrak{m}_{P_i}^e.$$

Then $\varphi_{Z,e}$ becomes surjective for large enough d due to Lemma 2.2.7. Hence, imitating the proof of Lemma 2.2.15,

$$\lim_{d\to\infty} \frac{\#\{a \in \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i}) \mid \mathrm{length}_{P_i}(\Sigma(a)) = b_i, i = 1, \ldots, t\}}{\# \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i})} = \prod_{i=1}^{t} \mu_x(b_i), \quad d \gg 0.$$

Consequently,

$$\begin{aligned} &\frac{\#\{a \in \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i}) \mid \sum_{P \in X_{<r}} \mathrm{length}_P(\Sigma(a)) < s\}}{\# \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes de_i})} \\ &= \prod_{P \in X_{<r}} \left(1 - \frac{\#\Delta(\kappa(P))}{\#\kappa(P)^{\dim X+n}}\right) \cdot \sum_{(b_P)_{P \in X_{<r}} : \sum_P b_P < s} \prod_{P \in X_{<r}} \frac{\mu_P(b_P)}{\mu_P(0)} \end{aligned}$$

for $d \gg 0$. For the convergence, use the proof of Theorem 2.2.32. □

Proof of Theorem 3.1.4

Proof. This is analogous to the proof of Theorem 2.1.7. □

3.3 Ranks of elliptic n-folds

3.3.1 Elliptic n-folds and hypersurfaces in weighted projective space

Let X be a smooth and geometrically irreducible variety of dimension ≥ 2 over $k = \mathbb{F}_q$. Assume that k has characteristic $p \geq 5$. Let $\mathcal{L}$ be an ample line bundle on X. We will use the notation and set-up of §3.1 with the same data as in Example 3.2.5:

Consider for $d \geq 1$ and $a = (a_1, a_2) \in H^0(X, \mathcal{L}^{\otimes 4d}) \oplus H^0(X, \mathcal{L}^{\otimes 6d})$ the hypersurface

$$W_{a,d} := \{\alpha_d(a) = 0\} = \{x_1^2 = x_0^3 + a_1 x_0 + a_2\} \subseteq Y_d,$$

where $Y_d = \pi_d|_{\mathbb{P}_d \setminus H}^{-1}(i(X))$. As shown in Example 3.2.5,

$$\lim_{d\to\infty} \frac{\#\{a \in H^0(X, \mathcal{L}^{\otimes 4d}) \oplus H^0(X, \mathcal{L}^{\otimes 6d}) \mid W_{a,d} \text{ is smooth}\}}{\# H^0(X, \mathcal{L}^{\otimes 4d}) \oplus H^0(X, \mathcal{L}^{\otimes 6d})} = \frac{1}{\zeta_X(\dim X + 1)}.$$

The fibers of the morphism $\pi_d|_{W_{a,d}} : W_{a,d} \to X$ are cubic plane curves with a point at infinity. If some fiber is smooth, then $x_1^2 = x_0^3 + a_1 x_0 + a_2$ is a Weierstrass equation for an elliptic curve $E_{a,d}$ over the function field $k(X)$.

Proposition 3.3.1. *With the above notation,*

$$r(E_{a,d}) \leq \operatorname{rk} \operatorname{Cl}(W_{a,d}) - \operatorname{rk} \operatorname{Pic}(W_{a,d}).$$

Proof. By [44, Prop. 2.6], there is an injective map

$$E_{a,d}(\overline{k}(X)) \otimes_{\mathbb{Z}} \mathbb{Q} \to (\operatorname{Cl}(W'_{a,d}) / \operatorname{Pic}(W'_{a,d})) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $W'_{a,d}$ is the hypersurface

$$\{y^2 z = x^3 + a_1 x z^2 + a_2 z^3\} \subseteq \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}^{\otimes(-2d)}) \oplus \mathcal{L}^{\otimes(-3d)})$$

for certain sections x, y, z, see [44, §2]. Moreover, with $Z := W'_{a,d} \cap \{z = 0\}$, there is a natural isomorphism

$$W'_{a,d} \setminus Z \to \overline{W_{a,d}} \setminus \{(1 : 1 : 0 : \cdots : 0)\} = W_{a,d}.$$

by the proof of [44, Prop. 2.4]. This induces an isomorphism of the corresponding Picard groups and Weil divisor class groups, respectively. Using the exact sequence

$$\mathbb{Z} \cdot [Z] \to \operatorname{Cl}(W'_{a,d}) \to \operatorname{Cl}(W_{a,d} \setminus Z) \to 0,$$

we obtain

$$\operatorname{rk} \operatorname{Cl}(W'_{a,d}) \leq \operatorname{rk} \operatorname{Cl}(W_{a_d}) + 1$$

and hence

$$r(E_{a,d}) \leq \operatorname{rk} E_{a,d}(\overline{k}(X)) \leq \operatorname{rk} \operatorname{Cl}(W'_{a,d}) - \operatorname{rk} \operatorname{Pic}(W'_{a,d}) \leq \operatorname{rk} \operatorname{Cl}(W_{a,d}) + 1 - \operatorname{rk} \operatorname{Pic}(W'_{a,d}).$$

It remains to show that $\operatorname{rk} \operatorname{Pic}(W'_{a,d}) \geq \operatorname{rk} \operatorname{Pic}(W_{a,d}) + 1$. This follows since $\operatorname{Pic}(W'_{a,d})$ surjects onto $\operatorname{Pic}(W'_{a,d} \setminus Z) \cong \operatorname{Pic}(W_{a,d})$ and additionally contains the class of the Cartier divisor Z. □

Lemma 3.3.2 ([44, Corollary 2.8]). *Suppose $W_{a,d}$ is smooth in codimension 3. Then $r(E_{a,d}) = 0$.*

Proof. If $W_{a,d}$ is smooth in codimension 3, then by [33, Exposé XI, Corollaire 3.14], $W_{a,d}$ is locally factorial and hence by [38, Corollary II.6.16], $\operatorname{rk}\operatorname{Cl}(W_{a,d}) = \operatorname{rk}\operatorname{Pic}(W_{a,d})$. Applying Proposition 3.3.1 finishes the proof. □

Combining Example 3.2.5 and Lemma 3.3.2 yields the main theorem of [44]:

Theorem 3.3.3 ([44, Theorem 1.1]).

$$\liminf_{d\to\infty} \frac{\#\{E \in \mathcal{E}_d \mid r(E) = 0\}}{\#\mathcal{E}_d} \geq \frac{1}{\zeta_X(\dim X + 1)}.$$

In fact, even more is true:

Theorem 3.3.4 ([44, Theorem 1.1]). *If $\dim X \geq 3$, then*

$$\lim_{d\to\infty} \frac{\#\{E \in \mathcal{E}_d \mid r(E) = 0\}}{\#\mathcal{E}_d} = 1.$$

Proof. If $\dim X \geq 3$, then $\dim W_{a,d} \geq 4$. Therefore, if the singular locus of $W_{a,d}$ is at most 0-dimensional, then $\dim W_{a,d}$ is smooth in codimension 3 and hence $r(E_{a,d}) = 0$. However, this follows from Corollary 3.2.9 (2). □

Remark. The theorems 3.3.3 and 3.3.4 are valid in characteristic 2 and 3, changing the Weierstrass equations accordingly.

3.3.2 Rank distribution of elliptic threefolds

Keep the notation of the previous subsection. The reasoning of Theorem 3.3.4 does not work in the case $\dim X = 2$. However, using the machinery of the previous chapters, it is still possible to improve the bound of Theorem 3.3.3:

Theorem 3.3.5 (Density of elliptic threefolds with rank 0). *Suppose $\dim X = 2$. Then*

$$\liminf_{d\to\infty} \frac{\#\{E \in \mathcal{E}_d \mid r(E) = 0\}}{\#\mathcal{E}_d} \geq \frac{1}{\zeta_X(\dim X + 3)}.$$

The first lemma is a recollection of resolutions of hypersurfaces with defect:

Lemma 3.3.6 (High Mordell-Weil rank ⇒ many singularities). *If $r(E_{a,d}) \geq 1$ and $W_{a,d}$ has at most A_k singularities, then*

$$\sum_{k\geq 1} 2\left\lceil \frac{k}{2} \right\rceil \cdot \#\{A_k \text{ singularities of } W_{a,d}\} \geq 6d.$$

Proof. Suppose that the elliptic curve $E_{a,d}$ has rank $r(E_{a,d}) \geq 1$. Then Proposition 3.3.1 implies that $\operatorname{rk}\operatorname{Cl}(W_{a,d}) > \operatorname{rk}\operatorname{Pic}(W_{a,d})$. Since $W_{a,d}$ is a threefold, there is a resolution of singularities, and Corollary 1.5.6 yields that

$$h^4_c(W_{a,d}) > \operatorname{rk}\operatorname{Pic}(W_{a,d})$$

in étale cohomology, say.

Assume now that $W_{a,d}$ has at most A_k singularities. Resolve these with an embedded resolution of singularities $\pi : (\widetilde{W} \subseteq \widetilde{Y}) \to (\overline{W_{a,d}} \subseteq \overline{Y_d})$ as in §1.4. Both $\widetilde{W}$ and $\widetilde{Y}$ still contain the point $R := (1 : 1 : 0 : \cdots : 0)$, so $\widetilde{Y}$ and $\widetilde{W}$ cannot be expected to be smooth. However, we can still apply the proofs of Lemma 1.4.4 and Proposition 1.4.7, so that

$$h^4(\widetilde{W}) - h^4(\widetilde{Y}) = h^4(\overline{W_{a,d}}) - h^4(\overline{Y_d}),$$

provided that $H^5(\overline{W_{a,d}}) = 0$. This follows as in Lemma 1.2.19.

Moreover, the fibration $\pi_d|_{Y_d} : Y_d \to X$ is trivial, so $Y_d \cong \mathbb{A}^2 \times X$. Applying the Künneth formula (Lemma 1.2.6), $h^4_c(Y_d) = h^4_c(\mathbb{A}^2) \cdot h^0_c(X) = 1$ and analogously $h^2_c(Y_d) = 0$. By the Gysin sequence (Lemma 1.2.8)

$$\cdots \to H^i_c(Y_d) \to H^i(\overline{Y}_d) \to H^i(\overline{Y}_d \setminus Y_d) \to H^{i+1}_c(Y_d) \to \ldots,$$

$h^4(\overline{Y}_d) = h^4_c(Y_d) = 1$ and $h^2(\overline{Y}_d) = h^2(\overline{Y}_d \setminus Y_d) = 1$ as $\overline{Y}_d \setminus Y_d \cong \mathbb{P}(2d, 3d) \cong \mathbb{P}^1$ by [21, Proposition 1.3]. In particular $h^2(\widetilde{Y}) = h^4(\widetilde{Y})$.

The variety $\widetilde{W}$ has its only singularity at the point R, and this is a finite quotient singularity. This means that there is a smooth projective variety V and a finite group G such that $\widetilde{W}$ is the quotient of V by G. The quotient map $\psi : V \to \widetilde{W}$ induces a commutative ladder

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & H^i_R(\widetilde{W}) & \longrightarrow & H^i(\widetilde{W}) & \longrightarrow & H^i(\widetilde{W} \setminus \{R\}) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & H^i_{\psi^{-1}(R)}(V) & \longrightarrow & H^i(V) & \longrightarrow & H^i(V \setminus \psi^{-1}(R)) & \longrightarrow & \ldots
\end{array}$$

By Poincaré duality (Lemma 1.2.5), $H^i_{\psi^{-1}(R)}(V) \cong H^{6-i}_c(\psi^{-1}(R))^\vee = 0$ for $i \neq 6$. Moreover,

$$H^i(\widetilde{W}) = H^i_c(\widetilde{W}) = H^i_c(V)^G = H^i(V)^G$$

using that V and $\widetilde{W}$ are both projective, and [10, Lemma 2.3]. In particular, the composition $H^i(\widetilde{W}) \to H^i(V) \to H^i(V \setminus \psi^{-1}(R))$ is injective for $i \neq 6$, and so is the restriction map $H^i(\widetilde{W}) \to H^i(\widetilde{W} \setminus \{R\})$ by commutativity. With Poincaré duality on the smooth threefold $\widetilde{W} \setminus \{R\}$,

$$h^i(\widetilde{W}) \leq h^i(\widetilde{W} \setminus \{R\}) = h^{6-i}_c(\widetilde{W} \setminus \{R\}) = h^{6-i}_c(\widetilde{W}) = h^{6-i}(\widetilde{W}), \quad i \neq 6.$$

In particular, $h^2(\widetilde{W}) \leq h^4(\widetilde{W}) \leq h^2(\widetilde{W})$ and thus $h^2(\widetilde{W}) = h^4(\widetilde{W})$. Putting all this together,

$$h^2(\widetilde{W}) - h^2(\widetilde{Y}) = h^4(\widetilde{W}) - h^4(\widetilde{Y}) = h^4_c(W_{a,d}) - 1 > \operatorname{rk}\operatorname{Pic}(W_{a,d}) - 1 \geq 0.$$

Since $\widetilde{Y} \setminus \widetilde{W}$ does not contain the point R, it is smooth and irreducible. The Lefschetz hyperplane theorem (Lemma 1.2.15) shows that $\widetilde{W}$ is not ample in $\widetilde{Y}$. Let H' be an ample effective divisor on $\mathbb{P}_d$ with $\mathcal{O}_{\mathbb{P}_d}(H') \cong \mathcal{O}_{\mathbb{P}_d}(1)$. Then, in $\mathrm{Cl}(Y_d)$, $W_{a,d}$ is linearly equivalent to $6d \cdot H'|_{Y_d}$. We can now repeat the intersection theory argument in Lemma 1.4.9, replacing the hyperplane H by the pullback of H' to $\widetilde{Y}$. Consequently

$$\sum_{k \geq 1} 2 \left\lceil \frac{k}{2} \right\rceil \cdot \#\{A_k \text{ singularities of } W_{a,d}\} \geq 6d. \qquad \square$$

The second lemma is the analogue of Lemma 2.2.36:

Lemma 3.3.7 (Local probability of A_k singularities)**.**

$$\lim_{d\to\infty} \frac{\#\{a \in H^0(X,\mathcal{L}^{\otimes 4d}) \oplus H^0(X,\mathcal{L}^{\otimes 6d}) \mid W_{a,d} \textit{ has at most } A_k \textit{ sing.}\}}{\# H^0(X,\mathcal{L}^{\otimes 4d}) \oplus H^0(X,\mathcal{L}^{\otimes 6d})} \geq \frac{1}{\zeta_X(\dim X + 3)}.$$

Proof. By Corollary 3.2.9 (2), we can assume that $W_{a,d}$ has at most isolated singularities. It hence suffices to adjust the estimate on low degree points. Set $\ell := \dim X$. As in the proof of Lemma 2.2.36, a singular point $R \in W_{a,d}$ is an A_k singularity if and only if the rank of the Hessian matrix $H(R)$ of $\{x_1^2 = x_0^3 + a_1 x_0 + a_2\}$ is at least $\ell + 1$.

Let $R \in (W_{a,d})^{\mathrm{sing}}$. Set $P := \pi_d(R)$ and write $R = (x_0 : x_1 : P)$ by slight abuse of notation. Since R is a singularity of $W_{a,d}$,

$$\begin{aligned} x_1^2 &= x_0^3 + a_1(P)x_0 + a_2(P) \\ 0 &= 3x_0^2 + a_1(P) \\ 2x_1 &= 0. \end{aligned}$$

Since k is of characteristic ≥ 5, this means

$$x_1 = 0, \quad x_0 = \begin{cases} -\frac{3}{2} \cdot \frac{a_2(P)}{a_1(P)} & \text{if } a_1(P) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

This means that the point R is already determined by P.

Choose local coordinates $(t_1, \ldots, t_\ell)$ around P. The Hessian matrix $H(R)$ is then given by

$$\begin{pmatrix} 6x_0 & 0 & \frac{\partial a_1}{\partial t_1}(P) & \ldots & \frac{\partial a_1}{\partial t_\ell}(P) \\ 0 & -2 & 0 & \ldots & 0 \\ \frac{\partial a_1}{\partial t_1}(P) & 0 & \frac{\partial^2 a_1}{\partial t_1^2}(P)x_0 + \frac{\partial^2 a_2}{\partial t_1^2}(P) & \ldots & \frac{\partial^2 a_1}{\partial t_1 \partial t_\ell}(P)x_0 + \frac{\partial^2 a_2}{\partial t_1 \partial t_\ell}(P) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial a_1}{\partial t_\ell}(P) & 0 & \frac{\partial^2 a_1}{\partial t_1 \partial t_\ell}(P)x_0 + \frac{\partial^2 a_2}{\partial t_1 \partial t_\ell}(P) & \ldots & \frac{\partial^2 a_1}{\partial t_\ell^2}(P)x_0 + \frac{\partial^2 a_2}{\partial t_\ell^2}(P), \end{pmatrix}$$

and depends only on P.

Observe that the linear map

$$\begin{aligned} \varphi : H^0(X,\mathcal{L}^{\otimes 4d}) \oplus H^0(X,\mathcal{L}^{\otimes 6d}) &\to (\kappa(x) \oplus \kappa(x)^\ell \oplus \kappa(x)^{\ell(\ell+1)/2})^2, \\ (a_1, a_2) &\mapsto \left(a_i(P), \frac{\partial a_i}{\partial t_j}(P), \frac{\partial^2 a_i}{\partial t_{j_1} \partial t_{j_2}}(P) \right)_{i=1,2} \end{aligned}$$

is surjective for $d \gg 0$ by Lemma 2.2.7. Omitting the second row and column, $H(R)$ defines a symmetric $(\ell+1)\times(\ell+1)$-matrix over $\kappa(P)$. Since the linear map φ is eventually surjective, all symmetric $(\ell+1)\times(\ell+1)$-matrices over $\kappa(x)$ are obtained in this way if d is large enough. In particular,

$$\lim_{d\to\infty} \frac{\#\{a \in \bigoplus_{i=1}^n H^0(X,\mathcal{L}^{\otimes de_i}) \mid \operatorname{rk} H(R) \geq \ell + 1\}}{\# \bigoplus_{i=1}^n H^0(X,\mathcal{L}^{\otimes de_i})} = \frac{p_{\ell,\#\kappa(P)}}{\#\kappa(P)^{\ell(\ell+1)/2}},$$

where $p_{\ell,\#\kappa(P)}$ denotes the number of quadratic forms in ℓ variables of rank $\geq \ell - 1$ over $\kappa(P)$.

Let $P \in X$ be arbitary. The probability that $W_{a,d}$ has a singularity worse than type A_k at some point $R \in \pi^{-1}(P)$, given that R is a singular point of $W_{a,d}$ is thus

$$\#\kappa(P)^{-3} \leq \frac{\#\kappa(P)^{\ell(\ell+1)/2} - p_{\ell,\#\kappa(P)}}{\#\kappa(P)^{\ell(\ell+1)/2}} \leq \#\kappa(P)^{-2}$$

by the estimates of Lemma 2.2.35. The probability that $W_{a,d}$ has a singularity at a point $R \in \pi_d^{-1}(P)$ is

$$\frac{\#\Delta(\kappa(P))}{\#\kappa(P)^{\ell+2}} = \#\kappa(P)^{-\ell-1}$$

by Lemma 3.2.1 and as $\Delta \cong \mathbb{A}^1$ (see Example 3.2.5). Using the formula for conditional probability, $W_{a,d}$ has a singularity at a point $R \in \pi_d^{-1}(P)$ worse than A_k with probability at most $\#\kappa(P)^{-\ell-3}$. Consequently,

$$\begin{aligned}
&\lim_{d\to\infty} \frac{\#\{a \in H^0(X, \mathcal{L}^{\otimes 4d}) \oplus H^0(X, \mathcal{L}^{\otimes 6d}) \mid W_{a,d} \text{ has at most } A_k \text{ singularities}\}}{\#H^0(X, \mathcal{L}^{\otimes 4d}) \oplus H^0(X, \mathcal{L}^{\otimes 6d})} \\
&\quad \geq \prod_{x \in X \text{ closed}} \left(1 - \#\kappa(P)^{-\ell-3}\right) \\
&\quad = \frac{1}{\zeta_X(\dim X + 3)}.
\end{aligned}$$

□

Proof of Theorem 3.3.5. Combine Theorem 3.1.4, Lemma 3.3.6 and Lemma 3.3.7 analogously to the proof of Theorem 1.1.3. □

Remark. The same bound is valid in characteristic 3, taking into account the change of the Weierstrass equation to $x_1^2 = x_0^3 + a_1x_0^2 + a_2x_0 + a_3$.

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Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014, angegebenen Hilfsmittel angefertigt habe.

Berlin, den 25.10.2016

Niels Lindner