Rank Stratification of Spaces of Quadrics and Moduli of Curves

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Abstract

In this thesis, we study varieties of singular quadrics containing a projective curve and effective divisors in $\overline{\mathcal{M}}_{g,n}$ defined via various constructions involving quadric hypersurfaces.

In Chapter 2, we compute the class of the effective divisor in $\overline{\mathcal{M}}_{g,n}$, which is defined as the locus of pointed curves $[C, p_1, \ldots, p_n]$ such that the image of C under the morphism induced by the linear series $|K_C(-p_1 - \cdots - p_n)|$ lies on a quadric hypersurface. Using this class, we show that the moduli spaces $\overline{\mathcal{M}}_{16,8}$ and $\overline{\mathcal{M}}_{17,8}$ are of general type.

In Chapter 3, we stratify the space of quadrics that contain a given curve in the projective space, using the ranks of the quadrics. We show, in a certain numerical range, that each stratum has the expected dimension if the curve is general in its Hilbert scheme. By incorporating the datum of the rank of quadrics, a similar construction as the one in Chapter 2 yields new divisors in $\overline{\mathcal{M}}_{g,n}$. We compute the class of these divisors and show that $\overline{\mathcal{M}}_{15,9}$ is of general type.

In Chapter 4, we present miscellaneous results, which are related with our main work in the previous chapters. Firstly, we consider divisors in $\overline{\mathcal{M}}_g$, which are defined as the failure locus of maximal rank conjecture for hypersurfaces of degree greater than two. We illustrate three examples of such divisors and compute their classes. Secondly, using the classical correspondence between rank 4 quadrics and pencils on curves, we show that the map that associates to a pair of pencils their tensor product in the Picard variety is surjective, when the curve is general and obvious numerical assumptions are satisfied. Finally, we use divisor classes, that are already known in the literature, to show that $\overline{\mathcal{M}}_{12,10}$ is of general type.

Zusammenfassung

In dieser Arbeit untersuchen wir Varietäten singulärer, quadratischer Hyperflächen, die eine projektive Kurve enthalten, und effektive Divisoren in $\overline{\mathcal{M}}_{g,n}$, die mittels verschiedener Eigenschaften von quadratischen Hyperflächen definiert werden.

In Kapitel 2 berechnen wir die Klasse des effektiven Divisors in $\overline{\mathcal{M}}_{g,n}$, der als der Ort von solchen markierten Kurven $[C, p_1, \ldots, p_n]$ definiert ist, dass das Bild von C unter der von $|K_C(-p_1 - \cdots - p_n)|$ induzierten Abbildung auf einer quadratischen Hyperfläche liegt. Mithilfe dieser Klasse zeigen wir, dass $\overline{\mathcal{M}}_{16,8}$ und $\overline{\mathcal{M}}_{17,8}$ Varietäten von allgemeinem Typ sind.

In Kapitel 3 stratifizieren wir den Raum von quadratischen Hyperflächen, die eine projektive Kurve enthalten, mithilfe des Rangs dieser Hyperflächen. Wir zeigen, dass jedes Stratum die erwartete Dimension hat, falls die Kurve ein allgemeines Element des Hilbertschemas ist. Mit Rücksicht auf Rang von quadratischen Hyperflächen, eine ähnliche Konstruktion wie in Kapitel 2 ergibt neue Divisoren in $\overline{\mathcal{M}}_{g,n}$. Wir berechnen die Klasse von diesen Divisoren und zeigen, dass $\overline{\mathcal{M}}_{15,9}$ von allgemeinem Typ ist.

In Kapitel 4 präsentieren wir unterschiedliche Resultate, die mit Themen von vorigen Kapiteln im Zusammenhang stehen. Zum Ersten betrachten wir Divisoren in $\overline{\mathcal{M}}_g$, die als die Orte von Kurven definiert sind, wo die maximale Rang Vermutung nicht gilt. Wir berechnen die Klasse von drei solchen Divisoren. Zweitens benutzen wir die klassische Korrespondenz zwischen Geradenbündel mit zwei Schnitten und quadratische Hyperflächen vom Rang 4 zu zeigen, dass jedes Geradenbündel als das Tensorprodukt von zwei Geradenbündeln mit zwei Schnitten geschrieben werden kann, falls die Kurve allgemein ist und eine gewisse numerische Bedingung erfüllt ist. Zuletzt benutzen wir Divisorklassen, die in der Literatur schon bekannt sind, zu zeigen, dass $\overline{\mathcal{M}}_{12,10}$ von allgemeinem Typ ist.

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Chapter 1

Introduction

1.1 Moduli space of curves

The moduli space of curves, which has been the subject of active research in mathematics for more than a century now, was first noticed by Riemann. In his famous paper [Rie57] in 1857, Riemann considered curves as abstract one dimensional complex manifolds and using an argument that relies on viewing curves as branched covers of $\mathbb{P}^1$, he concluded that curves of genus g vary continuously under the dependence of $3g-3$ parameters. He referred to this quantity as *moduli* and thus coined the term that we use today.

Although there was no formal treatment of the construction of the moduli space, it was not a major source of concern for the mathematicians of the time, since it was clear to them that such a space should exist. That had the rather odd consequence that many results about the moduli space of curves predate the proof of its existence. Almost a century after Riemann, the formal construction of the moduli space was achieved by Teichmuller [Tei82] in 1940 and by Mumford [FM82] in 1965 using analytical and algebraic methods, respectively.

In his work, Mumford constructed the moduli space of curves as a G.I.T. quotient of the Hilbert scheme of pluricanonical curves by the automorphism group of the projective space in consideration. Though it was a big achievement, there were two deficiencies of the resulting space, which were due to curves with automorphisms. Firstly, a local neighborhood of a point $[C] \in \mathcal{M}_g$ is isomorphic to the quotient of the space of first order deformations of C by the action of the automorphism group. Therefore, depending on the nature of this action, some curves with automorphisms give rise to finite quotient singularities in $\mathcal{M}_g$ (See Chapter XII of [ACG11] for a precise statement). Secondly, $\mathcal{M}_g$ was not the optimal answer to the moduli problem from a functorial perspective. Ideally, one would like to have a bijection between the set of families of curves parametrized by a scheme B and the set of maps $B \to \mathcal{M}_g$. This is the defining property of what one calls a *fine moduli space.* The variety $\mathcal{M}_g$, on the other hand, has this bijective correspondence only when $B = \mathrm{Spec}(\mathbb{C})$ and in general, there is the one sided correspondence that the family $X \to B$ gives rise to a moduli map $B \to \mathcal{M}_g$, but not vice versa, which makes $\mathcal{M}_g$ into a *coarse moduli space.*

These two deficiencies can be overcome by considering *stacks*, which are objects of a larger category than the category of schemes, where standard notions of geometry make sense. In their seminal work [DD69] in 1969, Deligne and Mumford showed that there is a smooth Deligne-Mumford stack, which is a fine moduli space to the moduli problem described above.

In the same paper a modular compactification of $\mathcal{M}_g$ was also described. In principle, limits of families of smooth curves can have a variety of different types of singularities, however, Deligne and Mumford showed that such families can always be modified so that they tend to *stable* curves, which are connected nodal curves with finite automorphism group. By incorporating all singular stable curves in $\mathcal{M}_g$, they have constructed the compact moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus g.

As will be discussed in the next section in more detail, singular stable curves are obtained by gluing various smooth curves at some marked points on them. For this and many other reasons, it is natural to consider the more general space $\mathcal{M}_{g,n}$, which parametrizes the isomorphism classes of genus g curves with an ordered set of n points. With minor modifications, the methods developed in [DD69] can be used to construct a Deligne-Mumford stack of n pointed genus g curves, as well as a compactification of it via stable pointed curves. Analogously, stable pointed curves are defined to be pointed nodal curves with finite automorphism group, where by an automorphism we mean an automorphism of the curve fixing the special points on it, that is, the marked points or the nodes.

1.2 Picard group of the moduli space

In this section we describe the natural divisor classes on $\overline{\mathcal{M}}_{g,n}$ and the structure of its Picard group. The results we will present clearly specialize to those on $\overline{\mathcal{M}}_g$ by setting $n = 0$.

The boundary of $\overline{\mathcal{M}}_{g,n}$ consists of irreducible components of codimension one that are denoted by Δ_{irr} and $\Delta_{i:S}$, where $0 \le i \le g$ and S is a subset of $\{1, \ldots, n\}$. The general element of Δ_{irr} is an n-pointed curve of (arithmetic) genus g, which only has a node as singularity. The general element of $\Delta_{i:S}$ is a reducible curve consisting of two components of genus $g - i$ and i, where the markings labeled by S lie on the genus i component. Since genus 0 curves have 3 dimensional automorphism group, one needs to have at least 3 special points on them so that the automorphism group of the pointed curve is finite. Therefore for the boundary components $\Delta_{0:S}$, we insist that $|S| \ge 2$.

Since the variety $\overline{\mathcal{M}}_{g,n}$ has only finite quotient singularities, Weil divisors on it are $\mathbb{Q}$-Cartier. Therefore, as for divisor classes, the primary object of study is the Picard group $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ with rational coefficients. The Picard groups of the moduli space and the moduli stack are different objects, but they are of course closely related. To explain this relationship we will follow the rather informal but comprehensible treatment of [HM98].

Definition 1.2.1. A rational divisor class on the moduli stack of n pointed genus g curves is an association γ to each family $\pi : \mathcal{X} \to B$ of stable pointed curves, of an element $\gamma(\pi) \in \mathrm{Pic}_{\mathbb{Q}}(B)$ such that if

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{\varphi} & B \end{array}$$

is a cartesian diagram then $\gamma(\pi') = \varphi^* (\gamma(\pi))$.

Given a rational divisor class Γ on $\overline{\mathcal{M}}_{g,n}$, a multiple $k\Gamma$ of it corresponds to a line bundle L on $\overline{\mathcal{M}}_{g,n}$. For any family $\pi : \mathcal{X} \to B$ of stable pointed curves the associated class of Γ on the stack is simply given as

$$\gamma(\pi) = \frac{1}{k} m(\pi)^*(L),$$

where $m(\pi) : B \to \overline{\mathcal{M}}_{g,n}$ is the moduli map induced by the family $\pi : \mathcal{X} \to B$. To go in the other direction, we fix a family $\pi : \mathcal{X} \to \Omega$ such that the induced map $m(\pi) : \Omega \to \overline{\mathcal{M}}_{g,n}$ is surjective and finite. Such families are known to exist [Loo92]. Now a class γ on the stack gives rise to an element $\gamma(\pi) \in \mathrm{Pic}_{\mathbb{Q}}(\Omega)$ and the associated class in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ is the pushforward of $\gamma(\pi)$ via $m(\pi)$ divided by the degree of this map.

Now that we have a dictionary between the classes of the moduli space and the moduli stack, we can define the natural divisor classes on $\overline{\mathcal{M}}_{g,n}$. To this end, we let $\pi : \mathcal{X} \to B$ be a family of stable curves with markings $\sigma_i : B \to \mathcal{X}$ for $i = 1, \ldots, n$. The pushforward of the dualizing sheaf of the map $\pi : \mathcal{X} \to B$ is a rank g vector bundle called the *Hodge* bundle and its first Chern class is denoted by

$$\lambda(\pi) = c_1 \left(\pi_* \omega_\pi\right).$$

Another natural class is obtained by pulling back the dualizing sheaf ω_π via the section $\sigma_i : B \to \mathcal{X}$ and taking its first Chern class. We denote this class by

$$\psi_i(\pi) = c_1 \left(\sigma_i^* \omega_\pi\right).$$

Finally we denote by δ_{irr} and $\delta_{i:S}$ the classes on the moduli stack that correspond to the boundary divisors Δ_{irr} and $\Delta_{i:S}$, respectively. In the rest of the thesis, by a divisor class on $\overline{\mathcal{M}}_{g,n}$, we will always mean the associated class on the moduli stack.

The fundamental result about the Picard group of $\overline{\mathcal{M}}_{g,n}$ is that it is generated by the natural divisor classes we presented above.

Theorem 1.2.2 ([Har83],[AC87])**.** *For $g \geq 3$, the Picard group $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ is freely generated by the classes λ, ψ_i with $i = 1, \ldots, n$, δ_{irr} and $\delta_{i:S}$ with $0 \leq i \leq [g/2]$ and $S \subseteq \{1, \ldots, n\}$.*

For $g \leq 2$, there are some nontrivial relations among these classes. We refer the interested reader to [AC98] for these relations.

1.3 Brill-Noether theory

As explained beautifully by Harris [Har10], in 20th century the study of algebraic curves underwent an analogous shift to the one in group theory. In 19th century mathematics, a group simply meant a subset of the set invertible matrices, which is closed under multiplication and inversion. With the introduction of the concept of abstract group in 20th century, groups and structure preserving maps to GL_n they admit, were studied separately, giving birth to what we call representation theory today. Analogously, in 19th century curves were considered as one dimensional subsets of the projective space defined by polynomial equations. In 20th century, however, the notion of abstract curve became central and the maps that a curve admits to the projective space became a separate subject of interest, which goes today under the name *Brill-Noether theory.*

Morphisms from a curve to a projective space are parametrized by linear series. Therefore, in a nutshell, Brill-Noether theory is the study of the following spaces: For a smooth curve C of genus g, we define

$$W^r_d(C) := \big\{L \in \mathrm{Pic}^d(C) \mid h^0(L) \geq r+1\big\},$$

and

$$G^r_d(C) := \big\{(L, V) \mid L \in \mathrm{Pic}^d(C), V \subseteq H^0(L), \dim V = r+1\big\}.$$

The first considerations of these spaces were done by Brill and Noether in their famous paper [BN74]. Based on a naive dimension count and concrete computations in low genus, they asserted that the dimension of the variety $G^r_d(C)$ is bounded from below by the *Brill-Noether number*

$$\rho(g, r, d) := g - (r+1)(g - d + r),$$

and is equal to that if C has general moduli. Although the truth of this statement was widely accepted in the mathematical community of the time, there was no actual proof of it until a century after the work of Brill and

Noether. The rigorous treatment of the first half of the statement was done independently by Kempf and in a joint work by Kleiman and Laksov.

Theorem 1.3.1 ([Kem71],[KL72]). *For a smooth curve C of genus g every component of $G^r_d(C)$ has dimension at least $\rho(g,r,d)$. In particular, $G^r_d(C) \neq \emptyset$ if $\rho(g,r,d) \geq 0$.*

Building up on the ideas developed by Castelnuovo and Severi over the years, Griffiths and Harris proved the remaining half of the statement in 1980.

Theorem 1.3.2 ([GH80]). *For a general curve C of genus g, the variety $G^r_d(C)$ is of pure dimension $\rho(g,r,d)$. In particular, $G^r_d(C) = \emptyset$ if $\rho(g,r,d) < 0$.*

The variety $W^r_d(C)$ is naturally defined as the degeneracy locus of a vector bundle morphism over the Picard variety $\mathrm{Pic}^d(C)$. By making use of the positivity properties of the bundles in question, Fulton and Lazarsfeld showed that $W^r_d(C)$ (and hence $G^r_d(C)$) is connected.

Theorem 1.3.3 ([FL81]). *Let C be a curve of genus g and $\rho(g,r,d) \geq 1$. Then the varieties $W^r_d(C)$ and $G^r_d(C)$ are connected.*

Clearly for $\rho(g,r,d) = 0$ one cannot talk about connectedness of these spaces, as for a general curve they are equal to a finite set of points by Theorem 1.3.2. The formula for the cardinality of $W^r_d(C)$ in the case $\rho(g,r,d) = 0$ was discovered by Castelnuovo.

Theorem 1.3.4 ([Cas89]). *When $\rho(g,r,d) = 0$ the number of g^r_d's on a general curve of genus g is equal to*

$$g! \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}.$$

From the determinantal description of the variety $W^r_d(C)$ it follows immediately that $W^{r+1}_d(C)$ lies in the singular locus of $W^r_d(C)$. By a deformation theoretic analysis of the tangent space of $W^r_d(C)$, one sees that a line bundle

$$L \in W^r_d(C) \setminus W^{r+1}_d(C)$$

is a smooth point of $W^r_d(C)$ if and only if the multiplication map

$$\mu_0(L) : H^0(L) \otimes H^0(K_C \otimes L^\vee) \to H^0(K_C) \tag{1.1}$$

is injective. These maps were conjectured by Petri [Pet25] to be injective and it was proven by Gieseker in 1982 that this is indeed the case for a general curve.

Theorem 1.3.5 ([Gie82]). *For a general curve C of genus g and every line bundle $L \in \mathrm{Pic}^d(C)$ the Petri map (1.1) is injective.*

This result coupled with Theorem 1.3.3 implies the following important corollary.

Corollary 1.3.6. *If $\rho(g,r,d) \geq 1$ then the varieties $G^r_d(C)$ and $W^r_d(C)$ are irreducible for a general curve C of genus g.*

The theorems 1.3.2 and 1.3.5 have now much easier proofs due to the theory of *limit linear series* developed by Eisenbud and Harris. We refer the reader to the papers [EH83b] and [EH83a] for the proofs of these results using limit linear series and to [EH86] for a general treatment of the theory. We also note that arguably the most elegant proof of Theorem 1.3.5 is due to Lazarsfeld [Laz86], who showed that the general hyperplane section of a general K3 surface satisfies the statement of Theorem 1.3.5.

We close this section by recording a commonly used consequence of Brill-Noether theory. We denote by $\mathrm{Hilb}_{g,r,d}$ the Hilbert scheme of curves in $\mathbb{P}^r$ having genus g and degree d. The variety $\mathrm{Hilb}_{g,r,d}$ is in general reducible and has irreducible components of unexpected dimension and moduli. However, in the nonnegative Brill-Noether range, there is a *unique* irreducible component $\mathcal{H}_{g,r,d}$ of expected dimension such that the forgetful rational map

$$\sigma : \mathcal{H}_{g,r,d} \dashrightarrow \mathcal{M}_g$$

is dominant. This is a direct consequence of Theorem 1.3.2 and Corollary 1.3.6 when $\rho(g,r,d) \geq 1$ and in the case $\rho(g,r,d) = 0$ it follows from a result of Eisenbud and Harris [EH87a].

1.4 Birational geometry of $\overline{\mathcal{M}}_{g,n}$

In this section we present the historical development of the study of birational geometry of $\overline{\mathcal{M}}_g$ and $\overline{\mathcal{M}}_{g,n}$. We start with giving some basic definitions.

Definition 1.4.1. A variety X with $\dim X = n$ is called

i) rational if there is a birational map $\mathbb{P}^n \dashrightarrow X$,

ii) unirational if there is a dominant rational map $\mathbb{P}^N \dashrightarrow X$ for some $N \geq n$,

iii) rationally connected if a general pair of points can be connected via a rational curve,

iv) uniruled if there is a rational curve passing through a general point of X.

Clearly, one has the implications i) $\Rightarrow$ ii) $\Rightarrow$ iii) $\Rightarrow$ iv). Contrary to the expectation one might naively have, it is a difficult problem to find counter examples for the converses of these implications. For instance, to exhibit a unirational variety that is not rational is referred to as the "Lüroth problem" in the literature and was solved by Clemens and Griffiths [CG72] in 1972.

Moreover, there is no known example of a rationally connected variety that is not unirational, which however is expected to exist.

The definitions i)-iv) single out varieties that are rather "simple" from a birational point of view, but many varieties do not enjoy these characteristics. There is a numerical invariant reflecting the birational complexity of a variety, which is called the Kodaira dimension. To define the Kodaira dimension we need some preliminary definitions.

Definition 1.4.2. Let X be an algebraic variety and L be a line bundle on it. The ring of sections of L is defined as

$$R(X, L) := \bigoplus_{d \geq 0} H^0(X, L^{\otimes d}).$$

The Iitaka dimension of L is defined as

$$\kappa(X, L) := \begin{cases} -\infty & \text{if} \quad R(X, L) = 0, \\ \dim \operatorname{Proj} R(X, L) & \text{otherwise.} \end{cases}$$

The line bundle L is called to be *big*, if $\kappa(X, L) = \dim X$.

A frequently used characterization for big line bundles is given by the following lemma.

Lemma 1.4.3 ([Laz04])**.** *A line bundle L on X is big if and only if there exists a positive integer m such that $L^{\otimes m} = A \otimes E$, where A is an ample line bundle and E is a line bundle associated to an effective divisor.*

Definition 1.4.4. The Kodaira dimension of a smooth variety X is defined to be $\kappa(X) := \kappa(X, K_X)$, where K_X denotes the canonical class of X. The variety X is called to be of *general type* if $\kappa(X) = \dim X$.

For singular varieties the Kodaira dimension is defined to be that of any desingularization of it. It is a standard fact that the Kodaira dimension is a birational invariant, therefore, the Kodaira dimension of a singular variety is well defined.

The birational study of $\mathcal{M}_g$ was initiated by Severi in his paper in 1915 [Sev15]. He considered plane models of curves of minimal degree with δ nodes as singularities and observed that these nodes can be chosen as general points in $\mathbb{P}^2$, when $g \leq 10$. Using this idea, he was able to construct a dominant map $\mathbb{P}^N \dashrightarrow \mathcal{M}_g$ and concluded that $\mathcal{M}_g$ is unirational for $g \leq 10$. Severi went on and conjectured that $\mathcal{M}_g$ is unirational for all genera.

Although Severi's idea was believed to be true by many mathematicians of the time, there was no improvement of his result for decades. In 1981 Sernesi [Ser81] has shown that $\mathcal{M}_{12}$ is unirational and later Chang and Ran proved that the same holds for $\mathcal{M}_{11}$ and $\mathcal{M}_{13}$ [CR84]. In two subsequent papers [CR86],[CR91], they have also shown that $\mathcal{M}_{15}$ and $\mathcal{M}_{16}$ have Kodaira dimension $-\infty$. In 2005, using a beautiful argument concerning Hurwitz spaces Verra managed to show that $\mathcal{M}_g$ is unirational for $11 \leq g \leq 14$ [Ver05]. He also showed, in a paper with Bruno, that $\mathcal{M}_{15}$ is rationally connected [BV05]. In 2010, Farkas remarked in his survey paper that using the BDPP theorem [BDPP13] the result of Cheng and Ran implies that $\mathcal{M}_{16}$ is uniruled [Far10]. For low genus it was shown that $\mathcal{M}_g$ is even rational. Results in this direction are chronologically due to Igusa [Igu60] for $g = 2$, Shepherd-Barron [She87],[She89] for $g = 4, 6$ and Katsylo [Kat92], [Kat96] for $g = 5, 3$.

The groundbreaking result in the birational classification problem of $\mathcal{M}_g$ was undoubtedly due to Harris and Mumford [HM82], where they have shown that $\overline{\mathcal{M}}_g$ is of general type for odd genus $g \geq 25$, disproving Severi's conjecture in the maximal possible way. Later, Harris proved the same result for even genus $g \geq 40$ [Har84] and with the advent of the theory of limit linear series, the proof of this result was immensely simplified and extended by Eisenbud and Harris to hold for all genera $g \geq 24$ [EH87b]. Later, Farkas proved that the same holds for $\mathcal{M}_{22}$ [Far10] and he showed that $\kappa(\mathcal{M}_{23}) \geq 2$ [Far00].

We will outline the method developed in [HM82] (and continued in [Har84] and [EH87b]) to determine the birational geometry of moduli spaces, since it will play an important role in our arguments in the following chapters. The Kodaira dimension of $\overline{\mathcal{M}}_g$ is by definition the Iitaka dimension of the canonical class of a desingularization of $\overline{\mathcal{M}}_g$. Harris and Mumford showed that any pluricanonical form defined on the smooth part $\overline{\mathcal{M}}_{g,reg}$ of $\overline{\mathcal{M}}_g$ extends to a form on a desingularization of $\overline{\mathcal{M}}_g$. Thus if $\nu : \widetilde{\mathcal{M}}_g \to \overline{\mathcal{M}}_g$ is a desingularization map then one has the isomorphism of pluricanonical sections, that is the map

$$\nu^* : H^0\left(\overline{\mathcal{M}}_{g,reg}, K^{\otimes k}_{\overline{\mathcal{M}}_{g,reg}}\right) \to H^0\left(\widetilde{\mathcal{M}}_g, K^{\otimes k}_{\widetilde{\mathcal{M}}_g}\right)$$

is an isomorphism. Therefore, the Kodaira dimension of $\overline{\mathcal{M}}_g$ is equal to the Iitaka dimension of the canonical class $K_{\overline{\mathcal{M}}_g}$. In the same paper, they also computed the class of $K_{\overline{\mathcal{M}}_g}$.

Theorem 1.4.5 ([HM82])**.** *For $g \geq 4$, the canonical class $K_{\overline{\mathcal{M}}_g}$ of $\overline{\mathcal{M}}_g$ is given by the following formula*

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta_0 - 3\delta_1 - 2\delta_2 - \cdots - 2\delta_{[g/2]}.$$

The class λ is known to be big, therefore finding a representative for the canonical class

$$K_{\overline{\mathcal{M}}_g} = c_1\lambda + c_2 E, \tag{1.2}$$

where E is an effective class and $c_1, c_2 > 0$, would imply by Lemma 1.4.3 that $K_{\overline{\mathcal{M}}_g}$ has maximal Iitaka dimension and hence $\overline{\mathcal{M}}_g$ is of general type. To carry out this argument one needs effective divisors on $\overline{\mathcal{M}}_g$, which capture the geometry of curves of that genus to a good extent. The effective divisors that served this purpose are characterized as the loci where the Brill-Noether theorem (Theorem 1.3.2) or the Gieseker-Petri theorem (Theorem 1.3.5) fails. More precisely, we have the following theorems.

Theorem 1.4.6 ([HM82],[EH87b]). *Let g, r, d be integers such that $g+1$ is composite and $\rho(g,r,d) = -1$. Then the locus of curves defined as*

$$\mathcal{M}^r_{g,d} := \left\{ [C] \in \mathcal{M}_g \mid W^r_d(C) \neq \emptyset \right\}$$

is a divisor and the class of its closure is given by the formula

$$\left[\overline{\mathcal{M}}^r_{g,d}\right] = c\left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{[g/2]} i(g-i)\delta_i \right),$$

where c is a positive rational number.

Theorem 1.4.7 ([Har84], [EH87b]). *Let $g = 2k-2$ for some positive integer k. Then the locus of curves defined as*

$$\mathcal{GP}_g := \left\{ [C] \in \mathcal{M}_g \mid \exists L = g^1_k \text{ with } \mathrm{Ker}(\mu_0(L)) \neq 0 \right\}$$

is a divisor and the class of its closure is given by the formula

$$\left[\overline{\mathcal{GP}}_g\right] = 2\frac{(2k-4)!}{k!(k-2)!}\left((6k^2+k-6)\lambda - k(k-1)\delta_0 - \sum_{i=1}^{[g/2]} b_i\delta_i \right),$$

where $b_1 = (2k-3)(3k-2)$, $b_2 = 3(k-2)(4k-3)$ and $b_i > b_{i-1}$ for $i \geq 3$.

Using these divisor classes, one can easily find positive rational numbers c_1, c_2 satisfying the equality (1.2) in the range $g \geq 24$. Our knowledge about the birational geometry of $\overline{\mathcal{M}}_g$ can be summarized in the following theorem.

Theorem 1.4.8. *The moduli space $\overline{\mathcal{M}}_g$ is*

- *rational for $2 \leq g \leq 6$,*
- *unirational for $7 \leq g \leq 14$,*
- *rationally connected for $g = 15$,*
- *uniruled for $g = 16$,*
- *of Kodaira dimension ≥ 2 for $g = 23$ and*
- *of general type for $g = 22$ and $g \geq 24$.*

The outlined strategy for determining the Kodaira dimension of $\overline{\mathcal{M}}_g$ can also be used for $\overline{\mathcal{M}}_{g,n}$, which was first implemented in the thesis of Logan. Using the expression for the class of $K_{\overline{\mathcal{M}}_g}$, he computed the canonical class of $\overline{\mathcal{M}}_{g,n}$.

Theorem 1.4.9 ([Log03]). *The canonical divisor of $\overline{\mathcal{M}}_{g,n}$ is given by the following formula*

$$K_{\overline{\mathcal{M}}_{g,n}} = 13\lambda - 2\delta_{irr} + \sum_{i=1}^{n} \psi_n - 2 \sum_{\substack{S\in P \\ |S|\geq 2}} \delta_{0:S} - 3 \sum_{S\in P} \delta_{1:S} - 2 \sum_{i=2}^{\lfloor g/2 \rfloor} \sum_{S\in P} \delta_{i:S},$$

where P denotes the power set of $\{1,\ldots,n\}$.

It is clear that to come up with a representative for $K_{\overline{\mathcal{M}}_{g,n}}$ as in (1.2) one has to find new divisors that also take the marked points into account. The divisor classes Logan used are summarized in the following theorem.

Theorem 1.4.10 ([Log03]). *Let $a_1,\ldots,a_n$ be nonnegative integers such that $\sum_{i=1}^{n} a_i = g$. The locus of pointed curves defined as*

$$\mathfrak{L}_{g:a_1,\ldots,a_n} := \left\{ [C,p_1,\ldots,p_n] \in \mathcal{M}_{g,n} \mid h^0\Big(\sum_{i=1}^{n} a_i p_i\Big) \geq 2 \right\}$$

is a divisor and the class of its closure is given by the formula

$$\left[\overline{\mathfrak{L}}_{g:a_1,\ldots,a_n}\right] = -\lambda + \sum_{i=1}^{n} \binom{a_i+1}{2} \psi_i + 0\cdot\delta_{irr} - \left(a_i a_j + \binom{a_i+1}{2}\right)\delta_{0:ij} - \ldots$$

where the omitted boundary coefficients are all less than 0.

We also note that Farkas computed numerous divisor classes on $\overline{\mathcal{M}}_{g,n}$ using a variety of different geometric constructions [Far09]. Our knowledge of the cases where $\overline{\mathcal{M}}_{g,n}$ is of general type can be summarized in the following theorem, which is mainly due to [Log03] and [Far09].

Theorem 1.4.11. *The moduli space $\overline{\mathcal{M}}_{g,n}$ is of general type for all $n \geq f(g)$, where $f(g)$ is as described in the following table.*

g	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$f(g)$	16	15	16	15	14	13	11	12	11	11	10	10	9	9	9	7	6	4

1.5 Maximal rank conjecture

One of the celebrated problems of the theory of algebraic curves is the maximal rank conjecture, which predicts that the natural restriction maps

$$H^0\big(\mathbb{P}^r, \mathcal{O}(m)\big) \to H^0\big(C, \mathcal{O}_C(m)\big)$$

are of maximal rank, that is, either injective or surjective for a general point $[C \subseteq \mathbb{P}^r] \in \mathcal{H}_{g,r,d}$ in the range $\rho(g,r,d) \geq 0$. The original formulation of the conjecture is due to Harris [Har82] and it amounts to showing that for every m the number of linearly independent hypersurfaces of degree m that contain C is the least possible.

We note that the assumption $\rho(g,r,d) \geq 0$ in the statement of the conjecture is necessary. In the negative Brill-Noether range there is no distinguished component of the Hilbert scheme like $\mathcal{H}_{g,r,d}$, for which the same question can be phrased. One could ask the question for all irreducible components of $\mathrm{Hilb}_{g,r,d}$ at once, whose general element is a smooth, irreducible, non degenerate curve, but then it can readily be answered in the negative. For instance, if $Q_{10,3,9}$ denotes the component of $\mathrm{Hilb}_{10,3,9}$ parametrizing curves of genus 10 that lie as a $(3,6)$ curve on a quadric surface then the maximal rank conjecture would suggest that the general $[C \subseteq \mathbb{P}^3] \in Q_{10,3,9}$ lies on a pencil of cubics. However, there are at least 4 independent cubics containing C, which are given as the union of a hyperplane with the quadric that C lies on.

Although the general conjecture is still open, it is known to hold in many different cases. The special case of canonically embedded curves is a classical result by Max Noether.

Theorem 1.5.1 (Max Noether's Theorem)**.** *For a non-hyperelliptic curve C the multiplication maps*

$$\mathrm{Sym}^k H^0(C, K_C) \to H^0(C, K_C^{\otimes k})$$

are surjective for $k \geq 1$.

The cases where the dimension of the projective space is equal to $3, 4$ or 5, as well as the case of nonspecial curves (i.e. $r = d - g$) has been verified by Ballico and Ellia in a series of papers (see [BE87b] and references therein). Later, Ballico and Fontanari [BF10] proved the conjecture in the range, where

$$\dim \mathrm{Sym}^2 H^0(C, L) \geq \dim H^0(C, L^{\otimes 2}).$$

More recently, using the theory of limit linear series and degenerating to a chain of elliptic curves Liu, Osserman, Teixidor and Zhang managed to systematically treat many other cases of the conjecture (See [LOTZ17] for a precise statement). We note that there is also a tropical proof of the case of quadrics by Jensen and Payne [JP16].

Maximal rank conjecture has beautiful connections with other problems in the theory of algebraic curves. In her paper [Voi92], Voisin confirmed that for pencils A of minimal degree, the map

$$H^0(K_C \otimes A^\vee) \otimes H^0(K_C \otimes A^\vee) \to H^0(K_C^{\otimes 2} \otimes (A^\vee)^{\otimes 2})$$

has maximal rank and using this, she was able to deduce the surjectivity of the Gaussian-Wahl map for generic curves. Farkas confirmed the conjecture when the Brill-Noether number is zero and the map

$$\phi : \mathrm{Sym}^2 H^0(C, L) \to H^0(C, L^{\otimes 2})$$

is expected to be an isomorphism. By considering the locus of curves, where ϕ fails to be an isomorphism, he obtained the first infinite family of divisors in $\overline{\mathcal{M}}_g$ violating the slope conjecture [Far09]. This result was also the motivation for us to consider maximal rank divisors in $\overline{\mathcal{M}}_{g,n}$ (See Chapter 2).

1.6 Singular quadrics and their parameter spaces

Quadric hypersurfaces are one of the simplest and most fundamental objects in algebraic geometry. Their structure is completely governed by the defining quadratic polynomial and hence by the underlying bilinear map. That reduces many questions concerning the geometry of a quadric to a linear algebra problem, which one can deal with rather easily. We refer the reader to the beautiful treatment in [Har92] for the basics of quadric hypersurfaces. We record here some properties of the parameter spaces of singular quadrics.

Theorem 1.6.1. *Let $Q_k(\mathbb{P}^r)$ be the variety of quadrics of rank at most k in $\mathbb{P}^r$. We have that*

i) The codimension of $Q_k(\mathbb{P}^r)$ in the projective space $|\mathcal{O}_{\mathbb{P}^r}(2)|$ of all quadrics is equal to $\binom{r+2-k}{2}$.

ii) The singular locus of the variety $Q_k(\mathbb{P}^r)$ is equal to $Q_{k-1}(\mathbb{P}^r)$.

Another important property of these varieties is their degrees as subvarieties of the projective space $|\mathcal{O}_{\mathbb{P}^r}(2)|$.

Theorem 1.6.2 ([HT84])**.** *The degree of $Q_k(\mathbb{P}^r)$ inside the projective space $|\mathcal{O}_{\mathbb{P}^r}(2)|$ is given by the following formula*

$$\deg\left(Q_k(\mathbb{P}^r)\right) = \prod_{t=0}^{r-k} \frac{\binom{r+t+1}{r-k-t+1}}{\binom{2t+1}{t}}.$$

As already pointed out in the previous section, the maximal rank conjecture holds for quadrics. Bringing the rank stratification of $|\mathcal{O}_{\mathbb{P}^r}(2)|$ into the picture, one can pose a similar problem from a refined perspective. To this end, we let C be a curve of genus g and ℓ be a base point free g^r_d on it. We denote by $Q_k(C,\ell)$ the projective variety of quadrics of rank at most k containing $\varphi(C)$, where

$$\varphi : C \to \mathbb{P}^r$$

is the map induced by the linear series ℓ. This variety is clearly equal to the intersection of the degree 2 piece of the ideal sheaf $|I_2(C,\ell)|$ and $Q_k(\mathbb{P}^r)$. Under "normal" circumstances one would expect this intersection to be dimensionally transverse. Therefore, we suggest the following:

Conjecture 1.6.3. Let C be a general curve of genus g and ℓ a general g^r_d on it, where $\rho(g,r,d) \geq 0$. For every $3 \leq k \leq r+1$, the variety $Q_k(C,\ell)$ is of pure dimension $q(g,r,d,k)$, where

$$q(g,r,d,k) := \binom{r+2}{2} - \binom{r-k+2}{2} - 2d + g - 2.$$

In particular $Q_k(C,\ell) = \emptyset$ if $q(g,r,d,k) < 0$.

The rank 4 case of this problem can be readily reduced to a problem in Brill-Noether theory using the classical correspondence between rank 4 quadrics and pencil pairs. We state that as a lemma for later use.

Lemma 1.6.4 ([AM67])**.** *Let $C \subseteq \mathbb{P}^r$ be a smooth curve. There is a one to one correspondence between rank 4 quadrics $Q \subseteq \mathbb{P}^r$ containing C and the data (ℓ_1, ℓ_2, F) such that*

$$|\ell_1 + \ell_2 + F| = |\mathcal{O}_C(1)|,$$

where ℓ_1 and ℓ_2 are base point free pencils cut out by the rulings of Q and F is an effective divisor supported at the singular locus of Q. Moreover, one has the same correspondence for rank 3 quadrics with the additional assumption that $\ell_1 = \ell_2$.

There is a similar correspondence in the rank 5 and rank 6 cases. Since the Plücker embedding of the Grassmanian $\mathrm{Gr}(2,4)$ is a smooth quadric in $\mathbb{P}^5$, if a curve C lies on a rank 6 quadric then the pullbacks of the tautological bundles on $\mathrm{Gr}(2,4)$ to C yield a rank 2 vector bundle with 4 sections. There is also a converse to that statement. We refer the reader to [BV96] for details as well as an analysis of the rank 5 case. The implications of this correspondence in rank two Brill-Noether theory has already been explored by Farkas and Ortega [FO11]. We also note that these vector bundles arising from quadrics of rank $3,4,5$ and 6 are particular instances of spinor bundles studied by Ottaviani [Ott88].

In the statement of Conjecture 1.6.3 the generality assumptions on the curve and on the linear series are both necessary. As was observed by Zamora [Zam99], for bicanonical curves of genus 3 the expected dimension of $Q_4(C, K_C^{\otimes 2})$ is equal to 3, but it manifestly has an irreducible component of dimension 4, which parametrizes rank 4 quadrics corresponding to pencil pairs $(K_C, V_1), (K_C, V_2)$ with $V_1, V_2 \subseteq H^0(K_C)$. On the other hand, by choosing the curve to be of special gonality (for instance hyperelliptic), one can find a plethora of counter examples in the rank 4 case.

Considering loci of curves where $Q_k(C, \ell)$ has unexpected behavior, one can construct interesting cycles in moduli spaces. In a recent paper [FR17], Farkas and Rimányi used this idea to obtain various divisor classes in the moduli space of curves and K3 surfaces. A crucial part of their work was to find a general formula for the fundamental classes of loci defined by existence of singular quadrics. The formulas they discovered will be one of the essential ingredients in our divisor class computations in Chapter 3, therefore, we state them here.

Theorem 1.6.5 ([FR17])**.** *Let X be an algebraic variety and suppose that there is a morphism of vector bundles*

$$\phi : \mathrm{Sym}^2\mathcal{E} \to \mathcal{F}$$

on X, where $\mathcal{E}$ and $\mathcal{F}$ are vector bundles of ranks e and f, respectively. For $k \leq e$, we define the subvariety

$$\Sigma^k_{e,f}(\phi) := \big\{x \in X \mid \exists\, 0 \neq q \in \mathrm{Ker}(\phi(x)) \text{ with } \mathrm{rk}(q) \leq k\big\}.$$

If $f = \binom{e+1}{2} - \binom{e-k+1}{2}$ then $\Sigma^k_{e,f}(\phi)$ is expected to be of codimension one in X and its virtual class is given by the formula

$$\Big[\Sigma^k_{e,f}(\phi)\Big] = A^k_e \cdot \left(c_1(\mathcal{F}) - \frac{2f}{e} \cdot c_1(\mathcal{E})\right) \in H^2(X, \mathbb{Q}),$$

where

$$A^k_e := \prod_{t=0}^{e-k-1} \frac{\binom{e+t}{e-k-t}}{\binom{2t+1}{t}}.$$

Note that the coefficient A^k_e is equal to the degree of the variety $Q_k(\mathbb{P}^{e-1})$, as stated in Theorem 1.6.2.

Adopting the terminology in [FR17], we call a pencil P of quadrics in $\mathbb{P}^r$ *degenerate* if the intersection

$$P \cap Q_r(\mathbb{P}^r)$$

is not transverse. The second formula we quote concerns the fundamental classes of loci, where such a pencil is degenerate.

Theorem 1.6.6 ([FR17]). *Let X be an algebraic variety and*

$$\phi : \mathrm{Sym}^2(\mathcal{E}) \to \mathcal{F}$$

be a morphism of vector bundles over X where $\mathrm{rk}(\mathcal{E}) = e$ *and* $\mathrm{rk}(\mathcal{F}) = \binom{e+1}{2} - 2$. *The class of the virtual divisor*

$$\mathfrak{Dp}(\phi) := \{x \in X \mid \mathrm{Ker}(\phi(x)) \text{ is a degenerate pencil}\}$$

is given by the formula

$$[\mathfrak{Dp}(\phi)] = (e-1) \cdot \big(e \cdot c_1(\mathcal{F}) - (e^2 + e - 4) \cdot c_1(\mathcal{E})\big) \in H^2(X, \mathbb{Q}).$$

1.7 Outline of results

In Chapter 2, we study divisors in $\overline{\mathcal{M}}_{g,n}$, which are defined as the loci of pointed curves $[C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n}$ such that $\varphi(C) \subseteq \mathbb{P}^{g-n-1}$ lies in a quadric hypersurface, where

$$\varphi : C \to \mathbb{P}^{g-n-1}$$

is the map induced by the linear series $|K_C(-\sum_{i=1}^n p_i)|$. This condition singles out a codimension one locus when

$$\big(g(t), n(t)\big) = \left(\frac{t^2 + 5t + 10}{2}, \frac{t^2 + 3t + 2}{2}\right) \text{ for } t \in \mathbb{N}. \tag{1.3}$$

Therefore for every $t \in \mathbb{N}$, we obtain a divisor $\mathfrak{Quad}_{g(t),n(t)}$ in $\mathcal{M}_{g(t),n(t)}$. In our main result in Chapter 2, we compute the class of the closure of this divisor in $\overline{\mathcal{M}}_{g(t),n(t)}$.

Theorem 1.7.1. *The class of the divisor $\overline{\mathfrak{Quad}}_{g(t),n(t)}$ is given by the following formula:*

$$\big[\overline{\mathfrak{Quad}}_{g(t),n(t)}\big] = (8-t) \cdot \lambda + t \cdot \sum_{j=1}^{n(t)} \psi_j - \delta_{irr} - \sum_{i,s \geq 0} b_{i:s}(t) \cdot \sum_{|S|=s} \delta_{i:S}$$

where

$$b_{0:s}(t) = \frac{s}{2}(st + s + t - 1) \text{ for } s \geq 2,$$

$$b_{1:0}(t) = t + 4, \quad b_{1:s}(t) = \frac{1}{2}(s^2 t + s^2 - st + s + 6) \text{ for } s \geq 1,$$

$$\text{and } b_{i:s}(t) \geq 1 \text{ for } 2 \leq i \leq g(t)/2 \text{ and } 0 \leq s \leq n(t).$$

Pulling back the divisor $\overline{\mathfrak{Quad}}_{17,10}$ to $\overline{\mathcal{M}}_{16,8}$ and $\overline{\mathcal{M}}_{17,8}$ by the well known cluthing maps, we obtain new divisor classes in these moduli spaces and show that they are of general type.

Theorem 1.7.2. *The moduli spaces $\overline{\mathcal{M}}_{16,8}$ and $\overline{\mathcal{M}}_{17,8}$ are of general type.*

In Chapter 3, we confirm the Conjecture 1.6.3 in the range $g - d + r \leq 1$.

Theorem 1.7.3. *Let C be a general curve of genus g and ℓ be a general g^r_d on C where $g - d + r \leq 1$ and the Brill-Noether number $\rho(g, r, d)$ is nonnegative. Then the variety $Q_k(C, \ell)$ is of pure dimension $q(g, r, d, k)$. In particular, $Q_k(C, \ell) = \emptyset$ if $q(g, r, d, k) < 0$.*

We use Theorem 1.7.3 to construct new divisors on $\overline{\mathcal{M}}_{g,n}$ as follows. We fix integers g, n, k such that $4 \leq k \leq g - n$ and

$$q(g, g-n-1, 2g-2-n, k) = -1,$$

and define the locus

$$\mathfrak{Quad}^k_{g,n} = \left\{ [C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n} \mid q \in I_2\Big(C, K_C\Big(-\sum_{i=1}^{n} p_i\Big)\Big),\ \mathrm{rk}(q) \leq k \right\}.$$

It follows from Theorem 1.7.3 that $\mathfrak{Quad}^k_{g,n}$ is proper closed subset of $\mathcal{M}_{g,n}$. In Theorem 1.7.4 we compute the class of its closure in $\overline{\mathcal{M}}_{g,n}$.

Theorem 1.7.4. *The class of the divisor $\overline{\mathfrak{Quad}}^k_{g,n}$ is given by the following formula:*

$$\left[\overline{\mathfrak{Quad}}^k_{g,n}\right] = \alpha^k_{g,n} \cdot \left(a \cdot \lambda + c \cdot \sum_{j=1}^{n} \psi_j - b_{irr} \cdot \delta_{irr} - \sum_{i,s \geq 0} b_{i:s} \cdot \sum_{|S|=s} \delta_{i:S} \right),$$

where

$$\alpha^k_{g,n} = \prod_{t=0}^{g-n-k-1} \frac{\binom{g-n+t}{g-n-k-t}}{\binom{2t+1}{t}}, \quad a = \frac{7g-9n+6}{g-n}, \quad c = \frac{g+n-6}{g-n}, \quad b_{irr} = 1,$$

and all other coefficients are ≥ 1. For $k = 4$, we can further compute that

$$b_{0:s} = \frac{s(gs - 3s + n - 3)}{g - n}.$$

There is another locus in $\overline{\mathcal{M}}_{15,8}$ defined by a degenerate pencil condition, which is expected to be of codimension one. In Theorem 1.7.5, we check using *Macaulay* that it is indeed a divisor and compute its class.

Theorem 1.7.5. *The locus of pointed curves defined as*

$$\mathfrak{D}_{15,8} := \left\{ [C, p_1, \ldots, p_8] \in \mathcal{M}_{15,8} \mid I_2\Big(C, K_C\Big(-\sum_{j=1}^{8} p_j\Big)\Big) \text{ is degenerate} \right\}$$

is a divisor and the class of its closure is given by the following formula

$$[\overline{\mathfrak{D}}_{15,8}] = 6 \cdot \Big(39 \cdot \lambda + 17 \cdot \psi - b_{irr} \cdot \delta_{irr} - \sum_{i,s \geq 0} b_{i:s} \cdot \sum_{|S|=s} \delta_{i:S}\Big),$$

where $b_{irr}, b_{i:s} \geq 7$ *for all* $i, s \geq 0$.

Using the pullback of this divisor to $\overline{\mathcal{M}}_{15,9}$, we show that this moduli space is of general type as well.

Theorem 1.7.6. *The moduli space* $\overline{\mathcal{M}}_{15,9}$ *is of general type.*

In Chapter 4 we collect some miscellaneous results, which were side outcomes of our studies that led to this thesis. Our first result concerns the class computation of two maximal rank divisors defined by the existence of an octic and quintic hypersurface. More precisely, we have the following theorem.

Theorem 1.7.7. *Let* $\widetilde{\mathcal{M}}_g$ *be the partial compactification* $\mathcal{M}_g \cup \Delta_0 \cup \Delta_1$*. Then the loci defined as*

$$\mathfrak{D}_{28} := \big\{[C] \in \mathcal{M}_{28} \mid \exists L \in W^3_{24}(C) \text{ such that } I_8(C, L) \neq 0\big\},$$

and

$$\mathfrak{D}_{35} := \big\{[C] \in \mathcal{M}_{35} \mid \exists L \in W^4_{32}(C) \text{ such that } I_5(C, L) \neq 0\big\}$$

are divisors and the class of their closures in $\widetilde{\mathcal{M}}_g$ *are as follows:*

$$[\overline{\mathfrak{D}}_{28}] = N(28, 3, 24) \cdot \left(\frac{41633}{39} \cdot \lambda - \frac{19376}{117} \cdot \delta_0 - \frac{11957}{13} \cdot \delta_1\right),$$

and

$$[\overline{\mathfrak{D}}_{35}] = N(35, 4, 32) \cdot \left(\frac{10415}{17} \cdot \lambda - \frac{1640}{17} \cdot \delta_0 - 545 \cdot \delta_1\right),$$

where we denote by $N(g, r, d)$ *the number of* g^r_d*'s on a general curve of genus* g *in the case* $\rho(g, r, d) = 0$.

Next, using the correspondence of rank 4 quadrics and pencils, we obtain the following result on the behavior of multiplication maps on Brill-Noether varieties.

Theorem 1.7.8. *Let* C *be a general curve of genus* $g \geq 2$ *and* d_1, d_2 *intergers satisfying*

i) $2(d_1 + d_2) - 3g - 4 \geq 0$,

ii) $\rho(g, 1, d_i) \geq 0$ *for* $i = 1, 2$.

Then the multiplication map

$$\begin{aligned}\mu_{d_1,d_2} : W^1_{d_1}(C) \times W^1_{d_2}(C) &\to \mathrm{Pic}^{d_1+d_2}(C)\\ (L_1, L_2) &\mapsto L_1 \otimes L_2\end{aligned}$$

is surjective.

We conclude this chapter by noting that the divisor classes computed by Farkas and Verra in the papers [FV14] and [FV13] can be used to show that $\overline{\mathcal{M}}_{12,10}$ is of general type.

Theorem 1.7.9. *The moduli space* $\overline{\mathcal{M}}_{12,10}$ *is of general type.*

All these results except for the ones in Chapter 4 are published in arXiv. The content of Chapter 2 and Chapter 3 can be found in the papers [Kad17a] and [Kad17b], respectively, with minor differences in the presentation.

Chapter 2

Maximal rank divisors on $\overline{\mathcal{M}}_{g,n}$

2.1 Introduction

In this chapter, we use a construction analogous to [Far09] to obtain new divisor classes on $\overline{\mathcal{M}}_{g,n}$, which are singled out as the failure locus of the maximal rank conjecture. More precisely, we consider a map of vector bundles $\phi : E \to F$ over $\mathcal{M}_{g,n}$, which restricts at a moduli point $x = [C, p_1, \ldots, p_n]$ to the multiplication map

$$\operatorname{Sym}^2\Big(H^0\big(K_C(-p_1 - \cdots - p_n)\big)\Big) \xrightarrow{\phi(x)} H^0\big(K_C^{\otimes 2}(-2p_1 - \cdots - 2p_n)\big). \quad (2.1)$$

We consider pairs

$$\big(g(t), n(t)\big) = \left(\frac{t^2 + 5t + 10}{2}, \frac{t^2 + 3t + 2}{2}\right) \text{ for } t \in \mathbb{N}, \quad (2.2)$$

in which case the dimensions of both sides in (2.1) are equal. Since the maximal rank conjecture is known to hold for quadrics (see Section 1.5), the locus where the vector bundle map ϕ fails to be an isomorphism is a divisor $\mathfrak{Quad}_{g(t),n(t)}$ in $\mathcal{M}_{g(t),n(t)}$. By taking its closure we obtain a divisor in $\overline{\mathcal{M}}_{g(t),n(t)}$ for every $t \in \mathbb{N}$. The sequence of the pairs $g(t), n(t)$ has the following pattern:

t	0	1	2	3	4	5	6	7	8	9	10	...
g	5	8	12	17	23	30	38	47	57	68	80	...
n	1	3	6	10	15	21	28	36	45	55	66	...

In Section 2.2, we extend the determinantal structure of the locus $\mathfrak{Quad}_{g(t),n(t)}$ over the boundary divisors of $\overline{\mathcal{M}}_{g,n}$ and thereby obtain a modular characterization of the points in the closure $\overline{\mathfrak{Quad}}_{g(t),n(t)}$. This determinantal condition breaks down over the boundary components $\Delta_{i:S}$ for $i \geq 2$, enabling us to compute the class of $\overline{\mathfrak{Quad}}_{g(t),n(t)}$ up to positive multiples of these boundary classes. Summarizing, we obtain the following result.

Theorem 2.1.1. *The class of the divisor* $\overline{\mathfrak{Quad}}_{g(t),n(t)}$ *is given by the following formula:*

$$\left[\overline{\mathfrak{Quad}}_{g(t),n(t)}\right] = (8-t)\cdot\lambda + t\cdot\sum_{j=1}^{n(t)}\psi_j - \delta_{irr} - \sum_{i,s\geq 0} b_{i:s}(t)\cdot\sum_{|S|=s}\delta_{i:S}$$

where

$$b_{0:s}(t) = \frac{s}{2}(st+s+t-1) \text{ for } s \geq 2,$$

$$b_{1:0}(t) = t+4, \quad b_{1:s}(t) = \frac{1}{2}(s^2t+s^2-st+s+6) \text{ for } s \geq 1,$$

$$\text{and } b_{i:s}(t) \geq 1 \text{ for } 2 \leq i \leq g(t)/2 \text{ and } 0 \leq s \leq n(t).$$

In section 2.3, we use Theorem 2.1.1 to study the birational geometry of $\overline{\mathcal{M}}_{g,n}$. As we have discussed in Section 1.4 in length, there are still quite a few cases of g, n, where the birational type of $\overline{\mathcal{M}}_{g,n}$ is not known. Using the divisor $\overline{\mathfrak{Quad}}_{g(t),n(t)}$ we treat two unknown cases of this problem.

Theorem 2.1.2. *The moduli spaces* $\overline{\mathcal{M}}_{16,8}$ *and* $\overline{\mathcal{M}}_{17,8}$ *are of general type.*

We note that for $t = 0$ (and only for this case) $\overline{\mathfrak{Quad}}_{g(t),n(t)}$ specializes to a well known divisor: For a pointed curve $[C,p] \in \mathcal{M}_{5,1}$, the linear system $|K_C(-p)|$ maps the curve to $\mathbb{P}^3$ and the existence of a quadric containing it is equivalent to the existence of a g^1_3 by Lemma 1.6.4. Therefore, $\overline{\mathfrak{Quad}}_{5,1}$ is the pullback of the Brill-Noether divisor $\overline{\mathcal{M}}^1_{5,3}$ (see Theorem 1.4.6) to $\overline{\mathcal{M}}_{5,1}$.

2.2 The class of $\overline{\mathfrak{Quad}}_{g(t),n(t)}$

As we already pointed out in the previous section, the first step towards computing the class of $\overline{\mathfrak{Quad}}_{g(t),n(t)}$ is to extend its determinantal structure over the boundary. To this end, we let

$$\pi : \overline{\mathcal{M}}_{g(t),n(t)+1} \to \overline{\mathcal{M}}_{g(t),n(t)}$$

be the map that forgets the last marked point and $\mathscr{L}$ be the relative cotangent line bundle. That is, $\mathscr{L}$ is naturally isomorphic to the dualizing sheaf ω_C when restricted to the fiber $\pi^{-1}\left([C,p_1,\ldots,p_{n(t)}]\right)$. We let ϕ denote the natural multiplication map

$$\mathrm{Sym}^2\left(\pi_*\mathscr{L}\left(-\sum_{j=1}^{n(t)}\delta_{0:\{j,n(t)+1\}}\right)\right) \xrightarrow{\phi} \pi_*\mathscr{L}^{\otimes 2}\left(-2\cdot\sum_{j=1}^{n(t)}\delta_{0:\{j,n(t)+1\}}\right). \quad (2.3)$$

Over a moduli point $[C, p_1, \ldots, p_{n(t)}]$ where C is a smooth curve, the map ϕ restricts to the map (2.1) and therefore extends the degeneracy locus structure to the boundary.

Note that if we have

$$i < s \quad \text{or} \quad g(t) - i < n(t) - s,$$

the evaluation map

$$\pi_* \mathscr{L} \xrightarrow{ev} \pi_* \left(\mathscr{L}|_{\sum_{j=1}^{n(t)} \delta_{0:\{j,n(t)+1\}}} \right)$$

fails to be surjective over $\Delta_{i:S}$ (Here and in what follows we set $s := |S|$). We will deal with such boundary components later, for now we restrict our attention to the partial compactification $\widetilde{\mathcal{M}}_{g(t),n(t)}$, which we define as the union of $\mathcal{M}_{g(t),n(t)}$ together with boundary divisors $\Delta_{i:S}$, where

$$s \leq i \quad \text{and} \quad n(t) - s \leq g(t) - i.$$

The sheaves in (2.3) are locally free over $\widetilde{\mathcal{M}}_{g(t),n(t)}$ away from loci of codimension at least 2. Therefore the first degeneracy locus $D_1(\phi)$ contains the divisor $\overline{\mathfrak{Quad}}_{g(t),n(t)} \cap \widetilde{\mathcal{M}}_{g(t),n(t)}$. We use Grothendieck-Riemann-Roch formula to compute its class and obtain the following result:

Theorem 2.2.1. *The coefficients of λ, ψ_i and δ_{irr} in $\overline{\mathfrak{Quad}}_{g(t),n(t)}$ are $8-t, t$ and -1, respectively, Moreover, $b_{i:s}(t) \geq 1$ whenever $s \leq i$ and $n(t)-s \leq g(t)-i$.*

Proof. On $\widetilde{\mathcal{M}}_{g(t),n(t)}$ we have the exact sequence

$$0 \to \pi_* \left(\mathscr{L}(-\sum_{j=1}^{n(t)} \delta_{0:\{j,n(t)+1\}}) \right) \to \pi_* \mathscr{L} \xrightarrow{ev} \pi_* \left(\mathscr{L}|_{\sum_{j=1}^{n(t)} \delta_{0:\{j,n(t)+1\}}} \right) \to$$

$$\to R^1 \pi_* \left(\mathscr{L}(-\sum_{j=1}^{n(t)} \delta_{0:\{j,n(t)+1\}}) \right) \to R^1 \pi_* \mathscr{L} \to 0.$$

It is easy to see that the evaluation map ev is surjective in codimension 2 in the range $s \leq i$ and $n(t) - s \leq g(t) - i$. Since $R^1 \pi_* \mathscr{L} \cong \mathcal{O}$, it follows that

$$R^1 \pi_* \left(\mathscr{L}(-\sum_{j=1}^{n(t)} \delta_{0:\{j,n(t)+1\}}) \right)$$

is isomorphic to $\mathcal{O}$ in codimension 2. Since the rank of $\pi_* \mathscr{L}(-\sum_{j=1}^{n(t)} \delta_{0:\{j,n(t)+1\}})$ is equal to $g(t) - n(t) = t + 4$, we have that

$$c_1 \left(\mathrm{Sym}^2 \left(\pi_* \mathscr{L}(-\sum_{j=1}^{n(t)} \delta_{0:\{j,n(t)+1\}}) \right) \right) = (t+5) \cdot c_1 \left(\pi_* \mathscr{L}(-\sum_{j=1}^{n(t)} \delta_{0:\{j,n(t)+1\}}) \right).$$

From the exact sequence above, it follows immediately that

$$c_1\left(\pi_*\mathscr{L}(-\sum_{j=1}^{n(t)}\delta_{0:\{j,n(t)+1\}})\right)=\lambda-\sum_{j=1}^{n(t)}\psi_j.$$

We use Grothendieck-Riemann-Roch formula to compute that

$$c_1\left(\pi_*\mathscr{L}^{\otimes 2}(-2\cdot\sum_{j=1}^{n(t)}\delta_{0:\{j,n(t)+1\}})\right)=13\lambda-5\cdot\sum_{j=1}^{n(t)}\psi_j-\delta,$$

where δ denotes the class of the whole boundary. By Porteous formula we get that

$$[D_1(\phi)]=(8-t)\cdot\lambda+t\cdot\sum_{j=1}^{n(t)}\psi_j-\delta.$$

The class $[D_1(\phi)]$ is equal to the sum of $\overline{\mathfrak{Quad}}_{g(t),n(t)}$ and positive multiples of the boundary components, over which the map (2.3) is degenerate. Thus we obtain the bound $b_{i:s}(t)\geq 1$ whenever $s\leq i$ and $n(t)-s\leq g(t)-i$. In Theorem 2.2.7, we will prove that (2.3) is generically non-degenerate over Δ_{irr}, which will imply that the coefficient of δ_{irr} is equal to -1. □

To obtain a bound for $b_{i:s}(t)$ in the case when $i<s$ or $g(t)-i<n(t)-s$, we extend the sheaves in (2.3) as vector bundles over $\overline{\mathcal{M}}_{g(t),n(t)}$ as follows:
We let

$$\mathscr{L}':=\mathscr{L}\left(-\sum_{j=1}^{n(t)}\delta_{0:\{j,n(t)+1\}}+\sum_{\substack{0\leq i\leq g(t)\\ i<s\\ |S|=s}}(i-s-1)\cdot\delta_{i:S\cup\{n(t)+1\}}\right),$$

and consider the natural map

$$\mathrm{Sym}^2\big(\pi_*\mathscr{L}'\big)\xrightarrow{\phi'}\pi_*\big(\mathscr{L}'^{\otimes 2}\big).\tag{2.4}$$

Using Grauert's Theorem it can easily be confirmed that the dimension of fibers of $\pi_*\mathscr{L}'$ and $\pi_*\left(\mathscr{L}'^{\otimes 2}\right)$ stay constant over an open subset of $\overline{\mathcal{M}}_{g(t),n(t)}$, whose complement has codimension at least 2. Therefore in codimension 2, the map ϕ' is a map of vector bundles and is an extension of ϕ.

To compute the class of the degeneracy locus $[D_1(\phi')]$, we will intersect it with simple test curves, whose intersection with the generators of $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ we already know. To this end, we let

$$[D,q',\{p_j\mid j\in S\}]\in\mathcal{M}_{i,S\cup\{q'\}}$$

and

$$[C, \{p_j \mid j \in S^c\}] \in \mathcal{M}_{g(t)-i,S^c}$$

be general pointed curves and define the test curve $T_{i:S}$ as follows:

$$T_{i:S} := \{[C \cup_{q\sim q'} D, p_1, \ldots, p_{n(t)}]\}_{q\in C},$$

that is, the point of attachment moves on the curve C. The intersection of $T_{i:S}$ with the standard divisor classes of $\overline{\mathcal{M}}_{g,n}$ can be computed using Lemma 1.4 in [AC98]. We note them here for readers convenience:

i) $T_{i:S} \cdot \psi_j = 1$ if $j \in S^c$,

ii) $T_{i:S} \cdot \delta_{i:S\cup\{j\}} = 1$ if $j \in S^c$,

iii) $T_{i:S} \cdot \delta_{i:S} = -(2(g(t) - i) - 2 + n(t) - s)$,

and the intersection of $T_{i:S}$ with all other generators of $\mathrm{Pic}(\overline{\mathcal{M}}_{g,n})$ equals zero.

Lemma 2.2.2. *For $0 \le i \le g(t)$ and $i < s$, we have the following intersection numbers:*

$$\begin{aligned}
&T_{i:S} \cdot c_1\big(\pi_*\mathscr{L}'\big) = -(i-s)\big((i-s-1)(g-i-1)+n-s\big),\\
&T_{i:S} \cdot c_1\Big(\pi_*\big(\mathscr{L}'^{\otimes 2}\big)\Big) = -2\Big(i^2(4g+6s+1) + i\big(-g(6s+5)+3n-2s^2+5\big)\Big)\\
&-2s\big(g(2s+3)-2n-3\big)+8i^3.
\end{aligned}$$

Proof. The fiber of the bundle $\pi_*\mathscr{L}'$ over $[C \cup_{q\sim q'} D, p_1, \ldots, p_{n(t)}] \in T_{i:S}$ is equal to the sections of

$$H^0\left(K_C + (i-s)q - \sum_{j\in S^c} p_j\right) \oplus H^0\left(K_D + (2-i+s)q' - \sum_{j\in S} p_j)\right) \quad (2.5)$$

that are compatible at the node $q \sim q'$. To prove the lemma we need to globalize this fibral description. To this end, we define the clutching maps

$$\begin{aligned}
\eta_{g-i} &: \overline{\mathcal{M}}_{g-i,S^c\cup\{n+1,0\}} \times \overline{\mathcal{M}}_{i,S\cup\{0\}} \to \overline{\mathcal{M}}_{g,n+1},\\
\eta_i &: \overline{\mathcal{M}}_{g-i,S^c\cup\{0\}} \times \overline{\mathcal{M}}_{i,S\cup\{n+1,0\}} \to \overline{\mathcal{M}}_{g,n+1},
\end{aligned}$$

which are defined as the maps that identify the points with the labels 0. Clearly they map onto the boundary divisors $\Delta_{i:S}$ and $\Delta_{i:S\cup\{n+1\}}$, respectively. These boundary divisors intersect at the locus where the point with the label $n+1$ hits the node and this locus is isomorphic to the image of the clutching map

$$\eta_\Sigma : \overline{\mathcal{M}}_{g-i,S^c\cup\{0\}} \times \overline{\mathcal{M}}_{0,\{0,n+1,-1\}} \times \overline{\mathcal{M}}_{i,S\cup\{-1\}} \to \overline{\mathcal{M}}_{g,n+1},$$

which identifies the points with labels 0 and -1, respectively. We also have maps from the domains of these three clutching maps to

$$\overline{\mathcal{M}}_{g-i,S^c\cup\{0\}} \times \overline{\mathcal{M}}_{i,S\cup\{0\}},$$

which are defined as the maps that forget the point with label $n+1$. We denote these maps by π_{g-i}, π_i nad π_Σ, respectively. In what follows (by abuse of notation) we denote by $\pi_*\mathscr{L}'$ the pullback of its restriction to $\Delta_{i:S} \subseteq \overline{\mathcal{M}}_{g,n}$ under the clutching map

$$\overline{\mathcal{M}}_{g-i,S^c\cup\{0\}} \times \overline{\mathcal{M}}_{i,S\cup\{0\}} \to \Delta_{i:S}.$$

The bundle $\pi_*\mathscr{L}'$ sits in the following exact sequence

$$0 \to \pi_*\mathscr{L}' \to \pi_{g-i_*}\big(\eta_{g-i}^*\mathscr{L}'\big) \oplus \pi_{i*}\big(\eta_i^*\mathscr{L}'\big) \to \pi_{\Sigma*}\big(\eta_\Sigma^*\mathscr{L}'\big) \to 0.$$

Therefore, we have that

$$c_1\big(\pi_*\mathscr{L}'\big) = c_1\Big(\pi_{g-i_*}\big(\eta_{g-i}^*\mathscr{L}'\big)\Big) + c_1\Big(\pi_{i*}\big(\eta_i^*\mathscr{L}'\big)\Big) - c_1\Big(\pi_{\Sigma*}\big(\eta_\Sigma^*\mathscr{L}'\big)\Big). \quad (2.6)$$

We need to compute the intersection number of these Chern classes with the test curve $T_{i:S}$. These classes are elements of

$$\mathrm{Pic}_{\mathbb{Q}}\big(\overline{\mathcal{M}}_{g-i,S^c\cup\{0\}}\big) \oplus \mathrm{Pic}_{\mathbb{Q}}\big(\overline{\mathcal{M}}_{i,S\cup\{0\}}\big),$$

and classes belonging to the second direct summand clearly have 0 intersection with the test curve $T_{i:S}$. Therefore it suffices to compute $\mathrm{Pic}_{\mathbb{Q}}\big(\overline{\mathcal{M}}_{g-i,S^c\cup\{0\}}\big)$ part of the Chern classes appearing in the formula (2.6) and their intersection with the test curve $T'_{i:S} \subseteq \overline{\mathcal{M}}_{g-i,S^c\cup\{0\}}$, which is defined by fixing a general element of $\overline{\mathcal{M}}_{g-i,S^c\cup\{0\}}$ and letting the point with label 0 vary on the curve.

Using the formula

$$c_1(\mathscr{L}) = \psi_{n+1} - \sum_{j=1}^{n} \delta_{0:\{j,n+1\}},$$

we first compute that

$$\begin{aligned} c_1(\mathscr{L}') = \ & \psi_{n+1} - 2\sum_{j=1}^{n} \delta_{0:\{j,n+1\}} + \\ & + (i-s-1)\cdot\delta_{i:S\cup\{n+1\}} + \sum_{j\in S^c} (i-s-2)\cdot\delta_{i:S\cup\{j,n+1\}} + \dots \end{aligned}$$

(Here the "dots" denote the classes, which have 0 intersection with the test curve $T_{i:S}$ and hence are irrelevant to our computation.)

Using the pullback formulas in [AC98] and the fact that the map π_Σ is an isomorphism, we compute that

$$c_1\Big(\pi_{\Sigma *}\big(\eta_\Sigma^*\mathscr{L}'\big)\Big) = -(i-s-1)\cdot\psi_0 + \\ + \sum_{j\in S^c}(i-s-2)\cdot\delta_{0:\{0,j\}} + \cdots \in \operatorname{Pic}_{\mathbb{Q}}\big(\overline{\mathcal{M}}_{g-i,S^c\cup\{0\}}\big).$$

To compute $c_1\Big(\pi_{i*}\big(\eta_i^*\mathscr{L}'\big)\Big)$, we observe that

$$c_1\big(\eta_i^*\mathscr{L}'\big) = -(i-s-1)\cdot\psi_0 + \sum_{j\in S^c}(i-s-2)\cdot\delta_{0:\{0,j\}} + \cdots \in \operatorname{Pic}_{\mathbb{Q}}\big(\overline{\mathcal{M}}_{g-i,S^c\cup\{0\}}\big).$$

The restriction of the bundle $\pi_{i*}\big(\eta_i^*\mathscr{L}'\big)$ to $\overline{\mathcal{M}}_{g-i,S^c\cup\{0\}}$ is the twist of a trivial bundle by this class. Therefore,

$$c_1\Big(\pi_{i*}\big(\eta_i^*\mathscr{L}'\big)\Big) = \operatorname{rank}\Big(\pi_{i*}\big(\eta_i^*\mathscr{L}'\big)\Big)\cdot c_1\big(\eta_i^*\mathscr{L}'\big).$$

From the fibral description (2.5), it is easy to see that $\operatorname{rank}\Big(\pi_{i*}\big(\eta_i^*\mathscr{L}'\big)\Big) = 1$. Therefore, we have that

$$T'_{i:S}\cdot c_1\big(\pi_*\mathscr{L}'\big) = T'_{i:S}\cdot c_1\Big(\pi_{g-i_*}\big(\eta_{g-i}^*\mathscr{L}'\big)\Big).$$

To compute this last quantity, we use Grothendieck-Riemann-Roch formula. First we compute that

$$c_1\big(\eta_{g-i}^*\mathscr{L}'\big) = \psi_{n+1} - 2\sum_{j\in S^c}\delta_{0:\{j,n+1\}} + (i-s-1)\cdot\delta_{0:\{0,n+1\}} + \\ + \sum_{j\in S^c}(i-s-2)\cdot\delta_{0:\{0,j,n+1\}} + \cdots \in \operatorname{Pic}_{\mathbb{Q}}\big(\overline{\mathcal{M}}_{g-i,S^c\cup\{n+1,0\}}\big).$$

As in the proof of Theorem 2.2.1, one can show that

$$c_1\Big(R^1\pi_{g-i_*}\big(\eta_{g-i}^*\mathscr{L}'\big)\Big) = 0.$$

Then a standard Grothendieck-Riemann-Roch computation yields that

$$T'_{i:S}\cdot c_1\Big(\pi_{g-i_*}\big(\eta_{g-i}^*\mathscr{L}'\big)\Big) = -(i-s)\big((i-s-1)(g-i-1)+n-s\big).$$

The computation of $T_{i:S}\cdot c_1\Big(\pi_*\big(\mathscr{L}'^{\otimes 2}\big)\Big)$ is done in the exact same way and we skip these details. □

Theorem 2.2.3. *We have that $b_{i:s}(t) \geq 1$ for $0 \leq i \leq g(t)$ and $0 \leq s \leq n(t)$.*

Proof. In Theorem 2.2.1 we have already shown that $b_{i:s}(t) \geq 1$ whenever $s \leq i$ and $n(t) - s \leq g(t) - i$. To deal with the remaining cases, we assume that $i < s$. We consider the degeneracy locus of the map (2.4) and write the relation

$$[D_1(\phi')] = \left[\overline{\mathfrak{Quad}}_{g(t),n(t)}\right] + \sum d_{i:s}(t) \cdot \delta_{i:S},$$

where $d_{i:s}(t) \geq 0$. By intersecting both sides of this equality with the test curve $T_{i:S}$, we obtain the relation

$$T_{i:S} \cdot [D_1(\phi')] = \big(2g(t) - 2i - 2 + n(t) - s\big)\tilde{b}_{i:s}(t) - \big(n(t) - s\big)\tilde{b}_{i:s+1}(t) + \big(n(t) - s\big)t,$$

where $\tilde{b}_{i:s}(t) := b_{i:s}(t) - d_{i:s}(t)$. Since $b_{i:s}(t) \geq \tilde{b}_{i:s}(t)$, it suffices to prove that $\tilde{b}_{i:s}(t) \geq 1$. Using Lemma 2.2.2 we solve this equation and obtain that

$$\tilde{b}_{i:s}(t) = \frac{1}{2}\big(i^2(t-3) - i(2s(t-1) + t - 5) + s(st + s + t - 1)\big). \tag{2.7}$$

It is elementary to check that this quantity is always greater than 1. □

The vector bundle map (2.4) is degenerate over most of the boundary divisors in $\overline{\mathcal{M}}_{g(t),n(t)}$, but it is actually generically non degenerate over $\Delta_{0:S}$. To see this, first note that the fiber of (2.4) over a general element of the test curve $T_{0:S}$ has the form

$$\mathrm{Sym}^2\left(H^0\left(K_C\Big(-s \cdot q - \sum_{j \in S^c} p_j\Big)\right)\right) \xrightarrow{\phi'} H^0\left(K_C^{\otimes 2}\Big(-2s \cdot q - 2\sum_{j \in S^c} p_j\Big)\right).$$

In Theorem 2.2.5 we will prove that this map is an isomorphism if the pointed curve

$$[C, q, \{p_j \mid j \in S^c\}] \in \mathcal{M}_{g(t), S^c \cup \{q\}}$$

is general. We first state a lemma known as "Lemme d'Horace", which we will be using in the proof of Theorem 2.2.5.

Lemma 2.2.4. *Let $H \subseteq \mathbb{P}^r$ be a hyperplane and $X, Y \subseteq \mathbb{P}^r$ be reduced subschemes such that $Y \subseteq H$ and no irreducible component of X lies in H. Then for any integer $m \geq 1$, one has a short exact sequence of ideal sheaves*

$$0 \to \mathcal{I}_{X/\mathbb{P}^r}(m-1) \to \mathcal{I}_{X \cup Y/\mathbb{P}^r}(m) \to \mathcal{I}_{(X \cup Y) \cap H/H}(m) \to 0.$$

Proof. See [Hir81]. □

Theorem 2.2.5. *Let C be a general curve of genus $g(t)$ and $p_1, \ldots, p_k$ general points on C. Let $a_1, \ldots, a_k$ be natural numbers such that $\sum_{j=1}^k a_j = n(t)$. Then the multiplication map*

$$\mathrm{Sym}^2 H^0\left(K_C\Big(-\sum_{j=1}^k a_j p_j\Big)\right) \to H^0\left(K_C^{\otimes 2}\Big(-\sum_{j=1}^k 2a_j p_j\Big)\right)$$

is an isomorphism.

Proof. Clearly, it is sufficient to prove the theorem for the special case where all points come together, i.e. it suffices to find a pointed curve $[C, p] \in \mathcal{M}_{g(t),1}$ such that

$$\mathrm{Sym}^2 H^0\Big(K_C\big(-n(t)p\big)\Big) \to H^0\Big(K_C^{\otimes 2}\big(-2n(t)p\big)\Big)$$

is an isomorphism.

We prove this using degeneration. Let C' be a genus $g(t)-2$ curve and q_1, q_2, q_3, p general points on it. We consider the stable pointed curve $[X, p] \in \overline{\mathcal{M}}_{g(t),1}$, which we obtain by gluing C' with a rational curve R' at the points q_1, q_2, q_3. We have the short exact sequence

$$0 \to \omega_X\big(-n(t)p\big) \to \omega_{\tilde{X}}\big(-n(t)p\big) \to \mathbb{C}_{q_1} \oplus \mathbb{C}_{q_2} \oplus \mathbb{C}_{q_3} \to 0,$$

where $\tilde{X}$ is the normalization of X and the right most map is the difference of the residues of the differentials on C' and R'. Therefore, the sections of $H^0\Big(\omega_X\big(-n(t)p\big)\Big)$ is equal to the kernel of the map

$$H^0\Big(K_{C'}\big(q_1+q_2+q_3-n(t)p\big)\Big) \oplus H^0\Big(K_{R'}\big(q_1+q_2+q_3\big)\Big) \xrightarrow{\varphi} \mathbb{C}_{q_1} \oplus \mathbb{C}_{q_2} \oplus \mathbb{C}_{q_3}.$$

The restriction of $\mathrm{Ker}(\varphi)$ to C' is equal to $H^0\Big(K_{C'}\big(q_1+q_2+q_3-n(t)p\big)\Big)$, since for any section of $H^0\Big(K_{C'}\big(q_1+q_2+q_3-n(t)p\big)\Big)$ with residues $\lambda_1, \lambda_2, \lambda_3$ at q_1, q_2, q_3, one can find an element of $H^0\Big(K_{R'}\big(q_1+q_2+q_3\big)\Big)$ having residues $-\lambda_1, -\lambda_2, -\lambda_3$ at these points, so that these sections glue to give a section of $H^0\Big(\omega_X\big(-n(t)p\big)\Big)$. Similarly, the sections of $\mathrm{Ker}(\varphi)$ restrict on R' to $H^0\Big(K_{R'}\big(q_1+q_2+q_3\big)\Big)$.

Therefore, the line bundle $\omega_X\big(-n(t)p\big)$ gives a map to the projective space, whose image consists of the image of $C' \to \mathbb{P}^r$ under the linear system

$$\big|K_{C'}\big(q_1+q_2+q_3-n(t)p\big)\big|$$

(that is, $r = \dim\big|K_{C'}\big(q_1+q_2+q_3-n(t)p\big)\big| = g(t)-n(t)-1$) and the 3-secant line $\overline{q_1, q_2, q_3}$ embedded by the linear system $|K_{R'}(q_1+q_2+q_3)|$.

Since a quadric in $\mathbb{P}^r$ containing C' automatically contains the 3-secant line, to prove the theorem it is sufficient to show that the map

$$\mathrm{Sym}^2 H^0\Big(K_{C'}\big(q_1+q_2+q_3-n(t)p\big)\Big) \to H^0\Big(K_{C'}^{\otimes 2}\big(2q_1+2q_2+2q_3-n(t)p\big)\Big)$$

is an isomorphism.

In order to show that we degenerate further and consider the following stable curve: We let R'' be a rational curve with $r+2$ marked points on it, which are labeled as $q_1, q_2, s_1, \ldots, s_r$. Let C'' be a curve of genus $g(t)-r-1$

with marked points $q_3, p, s_1, \dots, s_r$. We let $[X, q_1, q_2, q_3, p] \in \overline{\mathcal{M}}_{g(t)-2,4}$ be the stable curve, which we obtain by gluing C'' with R'' at the marked points with the same label. Along the same lines of reasoning as above, we observe that the line bundle $\omega_X\big(q_1 + q_2 + q_3 - n(t)p\big)$ induces a map $X \to \mathbb{P}^r$, whose image can be described as follows:

The image of R'' is a rational normal curve in $\mathbb{P}^r$ embedded via

$$|K_{R''}(q_1 + q_2 + s_1 + \cdots + s_r)|$$

and C'' is embedded to the hyperplane $H := \mathrm{Span}\{s_1, \dots, s_r\}$ via the linear series

$$\big|K_{C''}\big(q_3 + s_1 + \cdots + s_r - n(t)p\big)\big|.$$

Since C'' lies in the hyperplane and $C'' \cap R'' = \{s_1, \dots, s_r\}$, by Lemma 2.2.4,

$$H^1\big(\mathcal{I}_{X/\mathbb{P}^r}(2)\big) = H^1\big(\mathcal{I}_{C''/H}(2)\big).$$

That is, the original problem is now reduced to finding a general pointed curve $[C, p, q_1, \dots, q_{r+1}]$ of genus $g(t) - r - 1$ such that

$$\mathrm{Sym}^2 H^0\left(K_C\Big(\sum_{j=1}^{r+1} q_j - n(t)p\Big)\right) \to H^0\left(K_C^{\otimes 2}\Big(2\sum_{j=1}^{r+1} q_j - 2n(t)p\Big)\right)$$

is an isomorphism.

Note that as opposed to the first degeneration, the latter one reduces the dimension of the projective space in consideration. Using this degeneration successively (that is, in the next step we consider a pointed curve $[C''', p, q_3, \dots, q_{r+1}, s_1, \dots, s_{r-1}]$ of genus $g(t) - 2r + 1$ glued to a pointed rational curve $[R''', q_1, q_2, s_1, \dots, s_{r-1}]$ at the points with label s_j), we can reduce the question to a question in $\mathbb{P}^3$. Precisely, to prove the theorem it suffices to show that the map

$$\mathrm{Sym}^2 H^0\left(K_C\Big(\sum_{j=1}^{n(t)+2} q_j - n(t)p\Big)\right) \to H^0\left(K_C^{\otimes 2}\Big(2\sum_{j=1}^{n(t)+2} q_j - 2n(t)p\Big)\right)$$

is an isomorphism for a general pointed curve $[C, q_1, \dots, q_{n(t)+2}, p]$, where the genus of C is 3. (That the number of the points q_j is $n(t) + 2$ and the genus is 3 can be computed using the formulas in (2.2) and the fact that we need precisely t such degenerations, since $r = g(t) - n(t) - 1 = t + 3$).

To prove this final statement we can specialize to the case where $q_j = p$ for $j = 4, \dots, n(t) + 2$ and show that

$$\mathrm{Sym}^2 H^0\big(K_C(q_1 + q_2 + q_3 - p)\big) \to H^0\big(K_C^{\otimes 2}(2q_1 + 2q_2 + 2q_3 - 2p)\big) \qquad (2.8)$$

is an isomorphism for a general pointed genus 3 curve $[C, q_1, q_2, q_3, p]$. This statement, which can be confirmed also directly, is true by [GL86], since $\deg\big(K_C(q_1 + q_2 + q_3 - p)\big) = 6$ and it is equal to $2g(C) + 1 - \mathrm{Cliff(C)}$ if C is not hyperelliptic. Hence, (2.8) is an isomorphism, if C is not hyperelliptic. □

Corollary 2.2.6. *We have that* $b_{0:s}(t) = \frac{s}{2}(st + s + t - 1)$ *for* $s \geq 2$.

Proof. Setting $i = 0$ in (2.7) we obtain the claimed formula. □

Using the same methods as in the proof of Theorem 2.2.5, we prove the following theorem, which finishes the computation of the coefficient of δ_{irr}:

Theorem 2.2.7. *The vector bundle map (2.3) is generically nondegenerate over* Δ_{irr}.

Proof. To prove the theorem it is sufficient to exhibit a curve C of genus $g(t)-1$ with marked points $q_1, q_2, p_1, \ldots, p_{n(t)}$ such that the image of C under the map given by the linear system

$$\left|K_C(q_1 + q_2 - p_1 - \cdots - p_{n(t)})\right|$$

does not lie on any quadric. To prove this we will use the same type of degenerations that we used in the proof of Theorem 2.2.5. We let

$$[X, q_1, q_2, p_1, \ldots, p_{n(t)}] \in \overline{\mathcal{M}}_{g(t)-1,n(t)+2}$$

be the stable curve which we obtain by gluing a pointed rational curve $[R, q_1, q_2, s_1, \ldots, s_r]$ with a genus $g(t) - r$ curve $[C, s_1, \ldots, s_r, p_1, \ldots, p_{n(t)}]$ at the points with the same labels s_j (As before $r = g(t) - n(t) - 1 = t + 3$). The image of X under $\left|\omega_X(q_1 + q_2 - p_1 - \cdots - p_{n(t)})\right|$ is again the union of the rational curve R embedded to $\mathbb{P}^r$ via

$$|K_R(q_1 + q_2 + s_1 + \cdots + s_r)|$$

with the curve C embedded to the hyperplane $H := \mathrm{Span}\{s_1, \ldots, s_r\}$ via

$$\left|K_C(s_1 + \cdots + s_r - p_1 - \cdots - p_{n(t)})\right|.$$

By Lemma 2.2.4, we obtain $H^1\big(\mathcal{I}_{X/\mathbb{P}^r}(2)\big) = H^1\big(\mathcal{I}_{C/H}(2)\big)$, which again reduces the problem showing that $C \subseteq H$ does not lie on any quadrics. As in the proof of Theorem 2.2.5, we keep degenerating in this manner until the question is reduced to proving that for a general pointed genus 4 curve $[C, q_1, \ldots, q_{n(t)+1}, p_1, \ldots, p_{n(t)}]$ the image of C under the linear system

$$\left|K_C(q_1 + \cdots + q_{n(t)+1} - p_1 - \cdots - p_{n(t)})\right|$$

does not lie on any quadrics. Specializing to the case where $q_j = p_j$ for $j = 1, \ldots, n(t) - 1$ reduces our problem to finding a pointed genus 4 curve $[C, q_1, q_2, p]$ such that the image of C under $|K_C(q_1 + q_2 - p)|$ does not lie on any quadrics, which we already know, since this is the $t = 0$ case of our problem and in that particular case C lies on a quadric only if it is Brill-Noether special as we indicated earlier in the introduction. □

Remark 2.2.8. With the same methods used in the proofs above, one can prove a finer version of the maximal rank conjecture in the case $g - r + d = 1$. Namely, one can prove that for a general curve C of genus g and general points $p_1, \ldots, p_k$ the map

$$\mathrm{Sym}^2 H^0\Big(K_C\Big(-\sum_{j=1}^{k} a_j p_j\Big)\Big) \to H^0\Big(K_C^{\otimes 2}\Big(-\sum_{j=1}^{k} 2a_j p_j\Big)\Big)$$

is of maximal rank for any choice of natural numbers $a_1, \ldots, a_k$. We did not modify the proof to cover also these cases only because it would complicate the numerology in the proof further and we will not need this fact in what follows.

We have proven that the vector bundle map (2.4) is generically non degenerate over $\Delta_{0:S}$, but this is no longer true over $\Delta_{1:S}$. In order to compute the coefficients $b_{1:s}(t)$ precisely (rather than just giving a lower bound for it), one needs a finer analysis of the limit points of $\overline{\mathfrak{Quad}}_{g(t),n(t)}$ inside the boundary of $\overline{\mathcal{M}}_{g,n}$. We will use limit linear series to carry out this analysis. The limiting behaviour of very similar multiplication maps over moduli spaces has been successfully studied using limit linear series in the papers [EH83a], [FP05] and [Far06]. Here we will adapt the ideas developed in these papers to our situation. We start with some definitions.

Definition 2.2.9. Given a pointed smooth curve $[C, p]$ and a line bundle L on it, we define the vector space $W_k(p, L)$ of symmetric tensors of L with vanishing order $\geq k$ at p as follows: We let $\big(a_0^L(p), \ldots, a_r^L(p)\big)$ be the vanishing sequence of L at p and $\{\sigma_0, \sigma_1, \ldots, \sigma_r\} \subseteq H^0(L)$ be a basis such that

$$\mathrm{ord}_p(\sigma_i) = a_i^L(p).$$

Then we define

$$W_k(p, L) := \mathrm{Span}\big\{\sigma_i \sigma_j \mid a_i^L(p) + a_j^L(p) \geq k\big\} \subseteq \mathrm{Sym}^2 H^0(L).$$

Moreover, for a symmetric tensor $\rho \in \mathrm{Sym}^2 H^0(L)$ we define its order of vanishing at p as $\mathrm{ord}_p(\rho) = k$ if $\rho \in W_k(p, L) \setminus W_{k+1}(p, L)$.

Lemma 2.2.10. *The definition of $W_k(p, L)$ is independent of the chosen basis.*

Proof. If we let $\{\sigma'_0, \sigma'_1, \ldots, \sigma'_r\} \subseteq H^0(L)$ be another basis with the property that $\mathrm{ord}_p(\sigma'_i) = a_i^L(p)$ then clearly

$$\sigma'_i = \sum_{\ell=i}^{r} \lambda_\ell \sigma_\ell, \quad \lambda_\ell \in \mathbb{C}.$$

Therefore $\sigma'_i \sigma'_j$ can be written as a linear combination of symmetric tensors $\sigma_m \sigma_n$ where $m \geq i$ and $n \geq j$. □

Using very similar ideas as in [EH86], we construct locally a space of "limit quadrics", which coincides with $\mathfrak{Quad}_{g(t),n(t)}$ in the smooth locus of $\overline{\mathcal{M}}_{g(t),n(t)}$ and has a concrete geometric description for its elements in the boundary.

Theorem 2.2.11. *For $\emptyset \neq S \subseteq \{1,\dots,n(t)\}$, let $[E,q',\{p_j \mid j \in S\}]$ be a general pointed genus one curve and $[C,\{p_j \mid j \in S^c\}]$ a general pointed genus $g(t)-1$ curve. We let $q \in C$ and fix the nodal curve*

$$X_0 := [C \cup_{q\sim q'} E, p_1, \dots, p_{n(t)}] \in \overline{\mathcal{M}}_{g(t),n(t)}.$$

We further let

$$\pi : X \to B, \quad \sigma_j : B \to X \text{ for } j = 1,\dots,n(t)$$

be the versal deformation space of $[X_0, p_1, \dots, p_{n(t)}]$ with $\pi^{-1}(0) = X_0$ and $\sigma_j(0) = p_j$. Then there exists a scheme $\mathcal{Q} \subseteq B$, which is cut out by the following geometric conditions:

If $b \in \mathcal{Q}$ and X_b is smooth then the multiplication map

$$\mathrm{Sym}^2 H^0\Big(K_{X_b}\Big(-\sum_{j=1}^{n(t)} \sigma_j(b)\Big)\Big) \to H^0\Big(K_{X_b}^{\otimes 2}\Big(-2\sum_{j=1}^{n(t)} \sigma_j(b)\Big)\Big) \tag{2.9}$$

is not an isomorphism. If $b \in \mathcal{Q}$ and X_b is a singular curve obtained by gluing $[C', q, \{\sigma_j(b) \mid j \in S^c\}]$ and $[E', q', \{\sigma_j(b) \mid j \in S\}]$ at the marked points q and q' then the map

$$W_3\Big(q, K_{C'}\Big(-(s-1)q - \sum_{j\in S^c} \sigma_j(b)\Big)\Big) \to H^0\Big(K_{C'}^{\otimes 2}\Big(-(2s+1)q - 2\sum_{j\in S^c} \sigma_j(b)\Big)\Big) \tag{2.10}$$

is not an isomorphism.

Moreover, every irreducible component of $\mathcal{Q}$ has dimension $\geq \dim B - 1$.

Proof. We let $\Delta \subseteq B$ be the locus where the node q of X_0 is not smoothed and let $\mathscr{C}_q$ and $\mathscr{E}_q$ be the components of $\pi^{-1}(\Delta)$ containing $C \setminus q$ and $E \setminus q'$, respectively. By shrinking the base B, if necessary, we can assume that $\mathcal{O}_X(\mathscr{C}_q + \mathscr{E}_q) \cong \mathcal{O}_X$. We let

$$L_C := \omega_\pi\Big(-s \cdot \mathscr{E}_q - \sum_{j=1}^{n(t)} \sigma_j(B)\Big),$$

and

$$L_E := L_C\big(-(t+3)\cdot \mathscr{E}_q\big). \tag{2.11}$$

Note that the twists for the bundles are chosen in such a way that over X_0 the space of sections can be identified as follows:

$$H^0(L_C|_{X_0}) = H^0\Big(K_C\Big(-(s-1)q - \sum_{j\in S^c} p_j\Big)\Big)$$

and

$$H^0(L_E|_{X_0}) = H^0\Big(\mathcal{O}_E\Big((s+t+4)q' - \sum_{j\in S} p_j\Big)\Big).$$

We let $F_C \to B$ and $F_E \to B$ be the bundle of projective frames of the vector bundles $\pi_* L_C$ and $\pi_* L_E$ and we consider

$$F := F_C \times_B F_E.$$

The space F parametrizes the data

$$\Big[b, \{\sigma_j^C\}_{j=0}^{t+3}, \{\sigma_j^E\}_{j=0}^{t+3}\Big],$$

where $\{\sigma_j^C\}_{j=0}^{t+3}$ and $\{\sigma_j^E\}_{j=0}^{t+3}$ are ordered bases of the fibers of $\pi_* L_C$ and $\pi_* L_E$ at $b \in B$ up to scalars. We fix sections $\tau_C \in \mathcal{O}_X(\mathscr{C}_q)$ and $\tau_E \in \mathcal{O}_X(\mathscr{E}_q)$ that only vanish on $\mathscr{C}_q$ and $\mathscr{E}_q$, respectively. We denote by $\tilde{\sigma}_j^C$ and $\tilde{\sigma}_j^E$ the tautological bundles on F, whose fibers over each point are the 1-dimensional vector spaces corresponding to the frame with the same symbol. We define a subscheme $F' \subseteq F$ subject to the conditions

$$\tilde{\sigma}_j^C \cdot \tau_C^{t+3-j} = \tilde{\sigma}_j^E \cdot \tau_E^j \tag{2.12}$$

as sections of the bundles

$$\pi_* L_C\big((t+3-j)\cdot \mathscr{C}_q\big) \cong \pi_* L_E\big(j \cdot \mathscr{E}_q\big),$$

where the isomorphism is induced by the equality (2.11) and the isomorphism $\mathcal{O}_X(\mathscr{C}_q + \mathscr{E}_q) \cong \mathcal{O}_X$. The resulting space F' parametrizes the data

$$\Big[b, \{\sigma_j^C\}_{j=0}^{t+3}, \{\sigma_j^E\}_{j=0}^{t+3}\Big],$$

where σ_j^C and σ_j^E are identified if $b \in B \setminus \Delta$ and if $b \in \Delta$ then

$$ord_q(\sigma_j^C) \geq j \text{ and } ord_{q'}(\sigma_j^E) \geq t+3-j.$$

Note by (2.12) that the section $\tilde{\sigma}_j^C$ vanishes at least j times along $\mathscr{E}_q$. Thus, we have an injective map

$$\tilde{\sigma}_j^C \hookrightarrow \pi_* L_C(-j \cdot \mathscr{E}_q).$$

Similarly, we have that

$$\tilde{\sigma}_j^E \hookrightarrow \pi_* L_E\big(-(t+3-j)\cdot \mathscr{C}_q\big).$$

Therefore for $j+k \geq 3$, we have maps

$$\tilde{\sigma}_j^C \otimes \tilde{\sigma}_k^C \hookrightarrow \pi_* L_C(-j\cdot\mathscr{E}_q) \otimes \pi_* L_C(-k\cdot\mathscr{E}_q) \to \pi_* L_C^{\otimes 2}(-(j+k)\cdot\mathscr{E}_q) \to \pi_* L_C^{\otimes 2}(-3\cdot\mathscr{E}_q),$$

where the middle map is the usual multiplication map and the last map is induced by multiplying sections with τ_E^{j+k-3}. On the other hand, for $j+k<3$, we have maps

$$\begin{aligned}&\tilde{\sigma}_j^E \otimes \tilde{\sigma}_k^E \hookrightarrow \pi_* L_E(-(t+3-j)\cdot \mathscr{C}_q) \otimes \pi_* L_E(-(t+3-k)\cdot \mathscr{C}_q)\\ &\to \pi_* L_E^{\otimes 2}(-(2t+6-j-k)\cdot \mathscr{C}_q) \to \pi_* L_E^{\otimes 2}(-(2t+3)\cdot \mathscr{C}_q) \cong \pi_* L_C^{\otimes 2}(-3\cdot \mathscr{E}_q),\end{aligned}$$

where similarly the last map is multiplying sections with τ_C^{3-j-k} and the isomorphism is induced by (2.11). Next, we define the vector bundle

$$S := \left(\bigoplus_{\substack{j+k\geq 3\\ k\geq j}} \tilde{\sigma}_j^C \otimes \tilde{\sigma}_k^C \right) \oplus \left(\bigoplus_{\substack{j+k<3\\ k\geq j}} \tilde{\sigma}_j^E \otimes \tilde{\sigma}_k^E \right),$$

and consider the vector bundle map

$$\phi : S \to \pi_* L_C^{\otimes 2}(-3\cdot \mathscr{E}_q),$$

which at fibers is the map that takes the quadratic polynomials given by the individual direct summands of S and evaluates their sum under the multiplication map. Note that due to the identifications (2.12) the fiber of this vector bundle map at a point $b \in B\setminus \Delta$ is the map in (2.9). Next, we describe the fiber over $b=0$. First note that the fiber of $\pi_* L_C^{\otimes 2}(-3\cdot \mathscr{E}_q)$ over 0 is identified by the vector subspace of sections in

$$H^0\Bigg(K_C^{\otimes 2}\Big(-(2s+1)q - 2\sum_{j\in S^c}\sigma_j(b)\Big)\Bigg) \oplus H^0\Bigg(\mathcal{O}_E\Big((2s+5)q' - 2\sum_{j\in S}\sigma_j(b)\Big)\Bigg)$$

that are compatible at the node $q\sim q'$.

The direct summands $\tilde{\sigma}_j^E \otimes \tilde{\sigma}_k^E$ in S are multiplied by nontrivial powers of τ_C (since $3-j-k>0$) as described above. Therefore, the sections of $\pi_* L_C^{\otimes 2}(-3\cdot \mathscr{E}_q)$ that are in the image of the map

$$\tilde{\sigma}_j^E \otimes \tilde{\sigma}_k^E \to \pi_* L_C^{\otimes 2}(-3\cdot \mathscr{E}_q)$$

restrict to zero on C and on E they restrict to sections of

$$H^0\Bigg(\mathcal{O}_E\Big((2s+5)q' - 2\sum_{j\in S}\sigma_j(b)\Big)\Bigg),$$

that vanish at q'. Arguing in the same way, we observe that for $j+k>3$, sections that are in the image of

$$\tilde{\sigma}_j^C \otimes \tilde{\sigma}_k^C \to \pi_* L_C^{\otimes 2}(-3\cdot \mathscr{E}_q)$$

restrict to zero on E and on C they restrict to sections of

$$H^0\Big(K_C^{\otimes 2}\Big(-(2s+1)q - 2\sum_{j\in S^c}\sigma_j(b)\Big)\Big),$$

that vanish at q. The images of the remaining direct summands, $\tilde{\sigma}_0^C \otimes \tilde{\sigma}_3^C$ and $\tilde{\sigma}_1^C \otimes \tilde{\sigma}_2^C$ in $\pi_* L_C^{\otimes 2}(-3\cdot \mathscr{E}_q)$ restrict to sections on E and C that are compatible at the node $q \sim q'$.

It is elementary to observe that the fiber of ϕ at $0 \in B$ always surjects onto the sections on E. Therefore, ϕ fails to be an isomorphism over $0 \in B$ if and only if the map (2.10) is not an isomorphism.

We define $\tilde{\mathcal{Q}} \subseteq F'$ as the locus where the map ϕ fails to be an isomorphism and let $\mathcal{Q}$ be the image of $\tilde{\mathcal{Q}}$ under the morphism $F \to B$.

To estimate the dimension of $\mathcal{Q}$, first observe that the fibers of F are isomorphic to two copies of the projective linear group of a vector space of dimension $t+4$. Therefore,

$$\dim F = \dim B + 2(t+3)(t+4).$$

Each of the conditions in (2.12) is a single equation on the elements of a projective bundle with fibers isomorphic to $\mathbb{P}^{t+3}$. Therefore, each of them impose $t+3$ conditions. The determinantal condition on ϕ clearly imposes (at most) one condition. Thus, we have the estimate that every irreducible component of $\tilde{\mathcal{Q}}$ has dimension at least

$$\dim F - (t+3)(t+4) - 1 = \dim B + (t+3)(t+4) - 1.$$

To finish the proof, we need to show that the fiber dimension of $\tilde{\mathcal{Q}} \to B$ is at most $(t+3)(t+4)$. This is clear over $b \in B \setminus \Delta$, since in this case the frames are identified and the fiber of $F' \to B$ is isomorphic to a single copy of $\mathbb{P}GL_{t+4}$. Over $b \in \Delta$, we have the same estimate on the fiber dimension, because by the generality of the pointed elliptic curve $[E, q', \{p_i \mid i \in S\}]$, we have that

$$H^0\Big(\mathcal{O}_E\Big(s\cdot q' - \sum_{j\in S} p_j\Big) = 0\Big),$$

which forces $ord_{q'}(\sigma_j^E) = t+3-j$ for all j. Similarly, by the generality of the pointed curve $[C, \{p_j \mid j \in S^c\}]$, we have that $ord_q(\sigma_j^C) = j$ for all j (We are disregarding the case where $s = n(t)$ and q is a Weierstrass point of C, because it plays no role in the dimension count). An elementary dimension count now shows that the possible frames $\{\sigma_j^C\}_{j=0}^{t+3}, \{\sigma_j^E\}_{j=0}^{t+3}$ subject to conditions

$$ord_q(\sigma_j^C) + ord_{q'}(\sigma_{t+3-j}^E) = t+3,$$

depend on $(t+3)(t+4)$ parameters. □

Note that by Theorem 2.2.11, we have the necessary condition that if the pointed nodal curve $[X_0, p_1, \ldots, p_{n(t)}] \in \overline{\mathfrak{Quad}}_{g(t),n(t)}$ then the map (2.10) fails to be an isomorphism. To show that this is also sufficient, one has to rule out the possibility that an irreducible component of $\mathcal{Q}$ lies in the boundary. Since we have already shown that every irreducible component of $\mathcal{Q}$ has dimension at least $\dim \overline{\mathcal{M}}_{g(t),n(t)} - 1$, we can exclude this possibility by checking that the map (2.10) is generically nondegenerate over the boundary divisors $\Delta_{1:S}$. This is the content of the next theorem.

Theorem 2.2.12. *For $\emptyset \neq S \subseteq \{1, \ldots, n(t)\}$ and a general pointed genus $g(t) - 1$ curve $[C, q, \{p_j \mid j \in S^c\}] \in \mathcal{M}_{g(t)-1, S^c \cup \{q\}}$ the map*

$$W_3\Big(q, K_C\Big(- \sum_{j \in S^c} p_j - (s-1)q\Big)\Big) \to H^0\Big(K_C^{\otimes 2}\Big(- 2\sum_{j \in S^c} p_j - (2s+1)q\Big)\Big)$$

is an isomorphism.

Proof. Clearly it is sufficient to specialize to the case where the points $p_j = q$ for all $j \in S^c$ and prove that the map

$$W_3\Big(q, K_C\big(- (n(t) - 1)q\big)\Big) \to H^0\Big(K_C^{\otimes 2}\big(- 2(n(t) - 1)q\big)\Big)$$

is an isomorphism for a general element $[C, q] \in \mathcal{M}_{g(t)-1,1}$.

To prove this statement, we follow the same steps of successive degenerations as in the proof of Theorem 2.2.5, which (skipping the details) reduces the question to prove that there exists a pointed curve $[C, q, q_1, q_2, q_3, q_4] \in \mathcal{M}_{2,5}$ such that

$$W_3\big(q, K_C(q_1 + q_2 + q_3 + q_4 - q)\big) \to H^0\big(K_C^{\otimes 2}(2q_1 + 2q_2 + 2q_3 + 2q_4 - 2q)\big)$$

is an isomorphism.

To see this, note that if we choose the points $q, q_1, q_2, q_3, q_4 \in C$ to be general, then the image of C under $|K_C(q_1 + q_2 + q_3 + q_4 - q)|$ is contained in a unique (rank 4) quadric, which correspond to the pencils $|K_C|$ and $|q_1 + q_2 + q_3 + q_4 - q|$, both of which have the vanishing type $(0, 1)$ at the point q. That is, the tangent space of the quadric has multiplicity 2 at q and therefore the quadric is not an element of $W_3\big(q, K_C(q_1 + q_2 + q_3 + q_4 - q)\big)$. □

Corollary 2.2.13. *We have that $b_{1:0}(t) = t + 4$ and $b_{1:1}(t) = 4$.*

Proof. We consider the gluing map

$$\nu : \overline{\mathcal{M}}_{1,2} \to \overline{\mathcal{M}}_{g(t),n(t)}$$

that attaches a general pointed genus $g(t) - 1$ curve $[C, q, p_1, \ldots, p_{n(t)-1}]$ to $[E, p, q'] \in \overline{\mathcal{M}}_{1,2}$ by identifying the points q and q'. By Theorem 2.2.12, we have that

$$\nu^*\Big(\big[\overline{\mathfrak{Quad}}_{g(t),n(t)}\big] \Big) = 0.$$

Thus, we get the relation

$$(8-t)\cdot\lambda-\delta_{irr}+t\cdot\psi_p+b_{1:1}(t)\cdot\psi_{q'}-b_{1:0}(t)\cdot\delta_{0:\{p,q'\}}=0 \tag{2.13}$$

in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{1,2})$. Among the classes $\lambda,\psi_p,\psi_{q'},\delta_{irr},\delta_{0:\{p,q'\}}$, we have the following relations (see [AC98]):

$$12\lambda=\delta_{irr}\quad\text{and}\quad\psi_p=\psi_{q'}=\lambda+\delta_{0:\{p,q'\}}.$$

Using these, we can rewrite the relation (2.13) as

$$\big(b_{1:1}(t)-4\big)\cdot\lambda+\big(b_{1:1}(t)-b_{1:0}(t)+t\big)\cdot\delta_{0:\{p,q'\}}=0,$$

from which the statement clearly follows. □

Corollary 2.2.14. *We have that* $b_{1:s}(t)=\frac{1}{2}(s^2t+s^2-st+s+6)$ *for* $s\geq 1$.

Proof. Intersecting the test curve $T_{1:S}$ with the class of $\overline{\mathfrak{Quad}}_{g(t),n(t)}$ we obtain the relation

$$T_{1:S}\cdot\big[\overline{\mathfrak{Quad}}_{g(t),n(t)}\big]=t\big(n(t)-s\big)+\big(2g(t)-4+n(t)-s\big)b_{1:s}(t)-\big(n(t)-s\big)b_{1:s+1}(t). \tag{2.14}$$

The construction of the space $\mathcal{Q}$ in Theorem (2.2.11) can be carried out with obvious modifications in the special case, where $q\in C$ and one of the marked points $p_i\in C$ come together. This enables us to compute the left hand side of the equation (2.14):

We consider the maps

$$\begin{array}{ccc} & C\times C & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ C & & C \end{array}$$

and let $\Delta:=\{(p,p)\in C\times C\mid p\in C\}$ and

$$L:={\pi_1}^*\bigg(K_C\Big(-\sum_{j\in S^c}p_j\Big)\bigg)\otimes\mathcal{O}\big(-(s-1)\Delta\big).$$

The intersection number $T_{1:S}\cdot\big[\overline{\mathfrak{Quad}}_{g(t),n(t)}\big]$ is equal to the class of the degeneracy locus of the vector bundle map

$$W_3(\pi_{2*}L)\xrightarrow{\theta}\pi_{2*}\big(L^{\otimes 2}\otimes\mathcal{O}(-3\Delta)\big).$$

The bundle $W_3(\pi_{2*}L)$ sits naturally in the following exact sequences:

$$0\to\mathrm{Sym}^2\pi_{2*}\big(L(-2\Delta)\big)\to W'\to\pi_{2*}\big(L(-2\Delta)\big)\otimes\pi_{2*}\big(L\otimes\mathcal{I}_\Delta/\mathcal{I}_\Delta^2\big)\to 0,$$

and

$$0 \to W' \to W_3(\pi_{2*}L) \to \pi_{2*}\big(L(-3\Delta)\big) \otimes \pi_{2*}\big(L \otimes \mathcal{O}_\Delta\big) \to 0.$$

Using these exact sequences and applying Grothendieck-Riemann-Roch and Porteous formulas, we compute

$$\begin{aligned}[D_1(\theta)] =& \frac{1}{2}\big(s^2(t^3+6t^2+13t+8) - 2s(t^3+4t^2+4t-3) + \\ &+ t^3+8t^2+29t+34\big).\end{aligned}$$

Using this equation we solve the recurrence relation (2.14) and obtain that

$$b_{1:s}(t) = \frac{1}{2}(s^2t + s^2 - st + s + 6).$$

□

2.3 Kodaira dimensions of $\overline{\mathcal{M}}_{16,8}$ and $\overline{\mathcal{M}}_{17,8}$

In this section we explore the birational theoretic consequences of Theorem 2.1.1. We follow the strategy of Harris and Mumford explained in Section 1.4. Recall that the canonical class of $\overline{\mathcal{M}}_{g,n}$ was computed by Logan (Theorem 1.4.9) and it has the following expression.

$$K_{\overline{\mathcal{M}}_{g,n}} = 13\lambda - 2\delta_{irr} + \sum_{i=1}^{n}\psi_n - 2\sum_{\substack{S\in P\\ |S|\geq 2}}\delta_{0:S} - 3\sum_{S\in P}\delta_{1:S} - 2\sum_{i=2}^{\lfloor g/2\rfloor}\sum_{S\in P}\delta_{i:S},$$

where P denotes the power set of $\{1,\dots,n\}$.

Proof of Theorem 2.1.2. We consider the map

$$\nu_{i,j} : \overline{\mathcal{M}}_{16,8} \to \overline{\mathcal{M}}_{17,10}$$

that attaches an elliptic curve with two marked points to the point labeled by i and a rational curve with two marked points to the point labeled by j.

We want to pullback the divisor $\overline{\mathfrak{Quad}}_{17,10}$ via this gluing map. To ensure that we obtain an effective divisor on $\overline{\mathcal{M}}_{16,8}$ this way, one needs to check that the map

$$W_3\Big(p_i, K_C\Big(-p_i - 2p_j - \sum_{\ell\neq i,j}^{8} p_\ell\Big)\Big) \to H^0\Big(K_C^{\otimes 2}\Big(-2p_i - 4p_j - 2\sum_{\ell\neq i,j}^{8} p_\ell\Big)\Big)$$

is an isomorphism for a general element $[C, p_1, \dots, p_8] \in \mathcal{M}_{16,8}$, so that the image of $\nu_{i,j}$ is not contained in $\overline{\mathfrak{Quad}}_{17,10}$. This clearly follows from Theorem 2.2.12.

Using the pullback formulas in [AC98], we compute

$$\nu_{i,j}^*\Big(\left[\overline{\mathfrak{Quad}}_{17,10}\right]\Big) = 5\cdot\lambda + 3\cdot\sum_{\ell\neq i,j}\psi_\ell - \delta_{irr} + 9\cdot\psi_i + 10\cdot\psi_j - \dots$$

We compute this pullback for every choice of markings $\{i,j\}\subseteq\{1,\dots,8\}$ and take the average of the resulting divisors to obtain the effective class

$$\mathfrak{Q}_{16,8} = 40\cdot\lambda + 37\cdot\sum_{i=1}^{8}\psi_i - 8\cdot\delta_{irr} - \dots$$

Next, we consider the pullback of the effective divisor $\mathfrak{Z}_{16}\subseteq\overline{\mathcal{M}}_{16}$, which is defined as the closure of the locus of curves $[C]\in\mathcal{M}_{16}$ that are contained in a quadric under the map given by a g^7_{21}. The class of $\mathfrak{Z}_{16}$ is computed in [Far06]:

$$[\mathfrak{Z}_{16}] = 407\cdot\lambda - 61\cdot\delta_{irr} - \dots$$

Using these two classes, the canonical class of $\mathcal{M}_{16,8}$ can be written as

$$K_{\overline{\mathcal{M}}_{16,8}} = \frac{13}{272}\sum_{j=1}^{8}\psi_j + \frac{7}{272}\mathfrak{Q}_{16,8} + \frac{1}{34}\left[\mathfrak{Z}_{16}\right] + E,$$

where E an effective divisor supported on $\overline{\mathcal{M}}_{16,8}\setminus(\mathcal{M}_{16,8}\cup\Delta_{irr})$. Since the class $\sum_{j=1}^{8}\psi_j$ is *big*, the result follows.

To obtain an analogous description for $K_{\overline{\mathcal{M}}_{17,8}}$, we consider the map

$$\nu_{i,j}:\overline{\mathcal{M}}_{17,8}\to\overline{\mathcal{M}}_{17,10}$$

that attaches a rational curve with two marked points to each of the points labeled by i and j. By Theorem 2.2.5, we have that

$$\nu_{i,j}\Big(\overline{\mathcal{M}}_{17,8}\Big)\not\subseteq\overline{\mathfrak{Quad}}_{17,10}.$$

Therefore, the class

$$\nu_{i,j}^*\Big(\left[\overline{\mathfrak{Quad}}_{17,10}\right]\Big) = 5\cdot\lambda + 3\cdot\sum_{\ell\neq i,j}\psi_\ell - \delta_{irr} + 10\cdot\psi_i + 10\cdot\psi_j - \dots$$

is an effective divisor on $\overline{\mathcal{M}}_{17,8}$. As before, we apply this procedure for every $\{i,j\}\subseteq\{1,\dots,8\}$ and take the average to obtain the divisor class

$$\mathfrak{Q}_{17,8} = 20\cdot\lambda + 19\cdot\sum_{i=1}^{8}\psi_i - 4\cdot\delta_{irr} - \dots$$

The pullback of the Brill-Noether divisor on $\overline{\mathcal{M}}_{17}$ to $\overline{\mathcal{M}}_{17,8}$ has the class

$$\left[\overline{\mathcal{M}}^1_{17,9}\right] = 20\cdot\lambda - 3\cdot\delta_{irr} - \dots$$

Using these, we can write

$$K_{\overline{\mathcal{M}}_{17,8}} = \frac{1}{20}\sum_{j=1}^{8}\psi_j + \frac{1}{20}\mathfrak{Q}_{17,8} + \frac{3}{5}\left[\overline{\mathcal{M}}^1_{17,9}\right] + E,$$

where as before E is an effective divisor supported on $\overline{\mathcal{M}}_{17,8}\backslash(\mathcal{M}_{17,8}\cup\Delta_{irr})$. $\square$

Chapter 3

Variety of singular quadrics containing a projective curve

3.1 Introduction

In this chapter we turn to the problem we posed in Conjecture 1.6.3. We recall that for a projective curve $C \subseteq \mathbb{P}^r$, the variety $Q_k(C, \mathbb{P}^r)$ is defined to be the intersection of the degree 2 piece $\big|I_2\big(C, \mathcal{O}_C(1)\big)\big|$ of the ideal sheaf $\mathcal{I}_{C/\mathbb{P}^r}$ with the variety $Q_k(\mathbb{P}^r)$. This intersection takes place in the projective space $|\mathcal{O}_{\mathbb{P}^r}(2)|$ of all quadrics. Therefore, one immediately obtains that

$$\dim Q_k(C, \mathbb{P}^r) \geq q(g, r, d, k), \tag{3.1}$$

where

$$q(g, r, d, k) := \binom{r+2}{2} - \binom{r-k+2}{2} - 2d + g - 2.$$

Note that this holds for *all* projective curves $C \subseteq \mathbb{P}^r$. Next, we let $\mathcal{H}_{g,r,d}$ be the unique component of the Hilbert scheme of curves (note that we are in the range $\rho(g, r, d) \geq 0$) and $\mathcal{Q}_{g,r,d,k}$ be the parameter space of pairs

$$[C \subseteq \mathbb{P}^r, Q] \in \mathcal{H}_{g,r,d} \times Q_k(\mathbb{P}^r)$$

such that Q is a quadric of rank at most k and C lies on Q. We consider the forgetful map

$$\pi : \mathcal{Q}_{g,r,d,k} \to \mathcal{H}_{g,r,d}.$$

By the upper semicontinuity of fiber dimension of π, to show that (3.1) is an equality for general $C \subseteq \mathbb{P}^r$, it suffices to exhibit a *single* element of $\mathcal{H}_{g,r,d}$, whose fiber has dimension $q(g, r, d, k)$. We construct these curves by attaching suitable secant lines to either a rational normal curve or to a canonical curve in $\mathbb{P}^r$. In Section 3.2 we carry out this outlined strategy and obtain the following result.

Theorem 3.1.1. *Let C be a general curve of genus g and ℓ be a general g^r_d on C where $g-d+r \leq 1$ and the Brill-Noether number $\rho(g,r,d)$ is nonnegative. Then the variety $Q_k(C,\ell)$ is of pure dimension $q(g,r,d,k)$. In particular, $Q_k(C,\ell)=\emptyset$ if $q(g,r,d,k)<0$.*

In Section 3.3, we continue our search for divisor classes in $\overline{\mathcal{M}}_{g,n}$ by considering loci where Theorem 3.1.1 fails. To this end, we let g,n,k be integers such that $4 \leq k \leq g-n$ and

$$q(g, g-n-1, 2g-2-n, k) = -1,$$

and define the locus

$$\mathfrak{Quad}^k_{g,n} = \left\{ [C,p_1,\dots,p_n] \in \mathcal{M}_{g,n} \mid 0 \neq q \in I_2\Big(C, K_C\Big(-\sum_{j=1}^{n} p_j\Big)\Big), \mathrm{rk}(q) \leq k \right\}$$

Clearly Theorem 3.1.1 implies that this locus is proper closed subset of $\mathcal{M}_{g,n}$. However, since it is not defined as a degeneracy locus of vector bundles over $\mathcal{M}_{g,n}$ (as the divisor $\overline{\mathfrak{Quad}}_{g(t),n(t)}$ in Chapter 2), it can have irreducible components of high codimension. In Theorem 3.1.2, we compute its class and show that it has at least one divisorial component.

Theorem 3.1.2. *The class of the divisor $\overline{\mathfrak{Quad}}^k_{g,n}$ is given by the following formula:*

$$\left[\overline{\mathfrak{Quad}}^k_{g,n}\right] = \alpha^k_{g,n} \cdot \left(a\cdot\lambda + c\cdot \sum_{j=1}^{n} \psi_j - b_{irr}\cdot\delta_{irr} - \sum_{i,s\geq 0} b_{i:s}\cdot \sum_{|S|=s} \delta_{i:S}\right),$$

where

$$\alpha^k_{g,n} = \prod_{t=0}^{g-n-k-1} \frac{\binom{g-n+t}{g-n-k-t}}{\binom{2t+1}{t}}, \quad a = \frac{7g-9n+6}{g-n}, \quad c = \frac{g+n-6}{g-n}, \quad b_{irr} = 1,$$

and all other coefficients are ≥ 1. For $k=4$, we can further compute that

$$b_{0:s} = \frac{s(gs-3s+n-3)}{g-n}.$$

Note that the quantity $\alpha^k_{g,n}$ is equal to the degree of the variety $Q_k(\mathbb{P}^{g-n-1})$ stated in Theorem 1.6.2. If we specialize to the case of smooth quadrics (i.e. $k=g-n$) then $\alpha^k_{g,n}=1$ and we recover the formula in Theorem 2.1.1.

We do not obtain any result on the Kodaira dimension of $\overline{\mathcal{M}}_{g,n}$ using the divisor $\overline{\mathfrak{Quad}}^k_{g,n}$. That was somewhat expected, since this divisor has in general many irreducible components and some of them are not intrinsic to $\overline{\mathcal{M}}_{g,n}$, but come as pullbacks from other moduli spaces. For instance, in the case $k=4$

one of the components of $\overline{\mathfrak{Quad}}^k_{g,n}$ is the pullback of the Brill-Noether divisor $\overline{\mathcal{M}}^1_{g,[\frac{g+1}{2}]}$ from $\overline{\mathcal{M}}_g$. It would be interesting to know the class of the individual components of $\overline{\mathfrak{Quad}}^k_{g,n}$ and this could potentially have some birational theoretic consequences (See Conjecture 6.2 in [FR17] for a related problem concerning Gieseker-Petri divisors).

In Section 3.3 we consider a divisor in $\overline{\mathcal{M}}_{15,8}$ defined via a degenerate pencil condition (see Section 1.6), and compute its class.

Theorem 3.1.3. *The locus of pointed curves defined as*

$$\mathfrak{D}_{15,8} := \left\{ [C, p_1, \ldots, p_8] \in \mathcal{M}_{15,8} \mid I_2\Big(C, K_C\Big(-\sum_{j=1}^{8} p_j\Big)\Big) \text{ is degenerate} \right\}$$

is a divisor and the class of its closure is given by the following formula

$$\big[\overline{\mathfrak{D}}_{15,8}\big] = 6 \cdot \Big(39 \cdot \lambda + 17 \cdot \psi - b_{irr} \cdot \delta_{irr} - \sum_{i,s \geq 0} b_{i:s} \cdot \sum_{|S|=s} \delta_{i:S}\Big),$$

where $b_{irr}, b_{i:s} \geq 7$ for all $i, s \geq 0$.

To show that the locus $\mathfrak{D}_{15,8}$ is a divisor, one has to show that $\mathfrak{D}_{15,8} \neq \mathcal{M}_{15,8}$. We do this using the computer program *Macaulay*. The code we use can be found in the appendix of the thesis.

In Section 3.4, we use the pullback of this divisor to $\overline{\mathcal{M}}_{15,9}$ to show that this moduli space is of general type.

Corollary 3.1.4. *The moduli space $\overline{\mathcal{M}}_{15,9}$ is of general type.*

Notation

In what follows, we will denote by $Q_k(C, \mathbb{P}^r)$ the variety of quadrics that have rank at most k and contain C, if the embedding $C \hookrightarrow \mathbb{P}^r$ is clear from the context. We will write $Q_k(C, L)$ when the g^r_d is complete, i.e $g^r_d = (L, V)$ where $V = H^0(C, L)$.

3.2 Main theorem

This section is devoted to the proof of Theorem 3.1.1. The proof relies on an inductive argument, whose initial step is the case of rational normal curves when $g - d + r = 0$ and the case of canonical curves when $g - d + r = 1$. In the following lemmas, we confirm Theorem 3.1.1 for these two cases.

Lemma 3.2.1. *Let C be a general curve of genus g. Then $Q_k(C, K_C)$ is of pure dimension $q(g, g-1, 2g-2, k)$ for all $k \geq 3$.*

Proof. Let C be a general curve of genus g. For $k = 3$, the expected dimension of $Q_k(C, K_C)$ is equal to

$$q(g, g-1, 2g-2, 3) = -1. \tag{3.2}$$

Therefore we need to show that there are no rank 3 quadrics containing the canonical model of C. If a rank 3 quadric Q contains C then by Lemma 1.6.4 the ruling of Q cuts out a pencil A such that

$$K_C = A^{\otimes 2} \otimes \mathcal{O}_C(F),$$

where F is a divisor supported in $C \cap \mathrm{Sing}(Q)$. It follows from the base point free pencil trick (see [ACGH85]) that the Petri map

$$\mu : H^0(A + F) \otimes H^0(A) \to H^0(K_C)$$

has at least one dimensional kernel, which by Theorem 1.3.5 cannot happen for a general curve C.

For $k = 4$, we need to show that

$$\dim Q_4(C, K_C) = q(g, g-1, 2g-2, 4) = g - 4.$$

Using Lemma 1.6.4 we can estimate the dimension of $Q_4(C, K_C)$ as follows: To give an element of $Q_4(C, K_C)$, one has to specify a pencil A of degree a and a 2-dimensional space of sections of $H^0(K_C - A)$. Since C is general, we can use the Brill-Noether theorem to count the parameters that these choices depend on:

$$\dim Q_4(C, K_C) = \dim W^1_a(C) + \dim \mathrm{Gr}(2, g - a + 1) = g - 4.$$

That finishes the proof for $k = 4$.

To deal with the case $k \geq 5$, we let $\widetilde{\mathcal{H}}_g$ be the locus of curves in $\mathcal{H}_{g,g-1,2g-2}$, for which we have that $\dim Q_4(C, K_C) = g - 4$. We define the incidence variety

$$I_4 := \big\{(Q, [C \subseteq \mathbb{P}^{g-1}]) \mid C \subseteq Q\big\} \subseteq Q_4(\mathbb{P}^{g-1}) \times \widetilde{\mathcal{H}}_g.$$

Using the projection map $I_4 \to \widetilde{\mathcal{H}}_g$, we compute that

$$\dim I_4 = 3g - 3 + g^2 - 1 + g - 4 = g^2 + 4g - 8.$$

Since $\dim Q_4(\mathbb{P}^{g-1}) = 4g - 7$, the dimension of the general fiber of the other projection map

$$I_4 \to Q_4(\mathbb{P}^{g-1})$$

is equal to $g^2 - 1$. Since all rank 4 quadrics are projectively equivalent, we conclude that they all contain $g^2 - 1$ dimensional family of canonical curves.

Next we consider the incidence variety

$$I := \big\{(Q, [C \subseteq \mathbb{P}^{g-1}]) \mid C \subseteq Q\big\} \subseteq |\mathcal{O}_{\mathbb{P}^{g-1}}(2)| \times \widetilde{\mathcal{H}}_g.$$

Since canonical curves are projectively normal, I is a projective bundle over $\widetilde{\mathcal{H}}_g$. Thus we obtain that I is irreducible of dimension

$$\dim I = \frac{1}{2}(3g^2 + g - 4).$$

The projection map

$$I \to |\mathcal{O}_{\mathbb{P}^{g-1}}(2)|$$

is clearly dominant and thus has relative dimension $g^2 - 1$ over an open set of $|\mathcal{O}_{\mathbb{P}^{g-1}}(2)|$. By the discussion in the preceeding paragraph, quadrics of rank 4 lie in this open set. Since for every $k \geq 5$ the variety $Q_k(\mathbb{P}^{g-1})$ contains $Q_4(\mathbb{P}^{g-1})$, we can find quadrics of arbitrary rank, which lie in this open set and hence contain a $g^2 - 1$ dimensional family of canonical curves. By projective equivalence, this applies to *all* quadrics of rank $k \geq 4$.

Finally we restrict ourselves to quadrics of rank at most k and consider the incidence variety

$$I_k := \big\{(Q, [C \subseteq \mathbb{P}^{g-1}]) \mid C \subseteq Q\big\} \subseteq Q_k(\mathbb{P}^{g-1}) \times \widetilde{\mathcal{H}}_g.$$

By the above discussion, we know the relative dimension of the projection map $I_k \to Q_k(\mathbb{P}^{g-1})$. Using this, we compute

$$\dim I_k = \binom{g+1}{2} - \binom{g+1-k}{2} + g^2 - 2.$$

Hence the dimension of the general fiber of $I_k \to \widetilde{\mathcal{H}}_g$ is equal to

$$\binom{g+1}{2} - \binom{g+1-k}{2} - 3g + 2.$$

which is the same as $q(g, g-1, 2g-2, k)$. □

Lemma 3.2.2. *For any $k \geq 3$ and any rational normal curve $\Gamma \subseteq \mathbb{P}^r$, the variety $Q_k(\Gamma, \mathbb{P}^r)$ has the expected dimension $q(0, r, r, k)$.*

Proof. First we confirm the case $k = 3$, that is, we show that for a rational normal curve $\Gamma \subseteq \mathbb{P}^r$, we have that

$$\dim Q_3(\Gamma, \mathbb{P}^r) = q(0, r, r, 3).$$

The elements of $Q_3(\Gamma, \mathbb{P}^r)$ are in one to one correspondence with the data (A, F) such that

$$A^{\otimes 2} \otimes \mathcal{O}_\Gamma(F) = \mathcal{O}_\Gamma(1),$$

where A is a pencil and F is a divisor supported on the singular locus of the associated quadric. If we let $x = \deg(F)$ then the parameter count for the pairs (A, F) yields

$$\dim \mathrm{Gr}\Big(2, \frac{r-x}{2} + 1\Big) + x = r - 2.$$

Since $q(0, r, r, 3) = r - 2$, that finishes the proof of the case $k = 3$. The rest of the proof is analogous to the proof of Lemma 3.2.1, we leave these details to the reader. □

In the proof of the next proposition we will need the following lemma from [BE89].

Lemma 3.2.3. *Let g, r, d be integers such that $\rho(g, r, d) \geq 0$ and*

$$0 \leq g \leq d - r + \left\lfloor \frac{d - r - 2}{r - 2} \right\rfloor .$$

If $C \in \mathcal{H}_{d,g,r}$ and ℓ is a 2-secant line of C then $C \cup \ell \in \mathcal{H}_{d+1,g+1,r}$.

Proof. See Lemma 2.2 in [BE89]. □

Proposition 3.2.4. *The statement of Theorem 3.1.1 holds whenever we have $\rho(g, r, d) \geq 0$ and $0 \leq g - d + r \leq 1$.*

Proof. We fix $r \geq 3$ and apply induction on g. Clearly, in the case $g - d + r = 0$ the minimal value for the genus is 0, whereas in the case $g - d + r = 1$ (since $\rho(g, r, d) = 0$) the minimal value for the genus is $r + 1$. These are the cases of rational normal curves and canonical curves respectively and in Lemmas 3.2.1 and 3.2.2, the statement of the Theorem 3.1.1 was confirmed for these two cases, establishing the base step of the induction.

For the inductive step, we let $[C \subseteq \mathbb{P}^r] \in \mathcal{H}_{d,g,r}$ such that

$$\dim Q_k(C, \mathbb{P}^r) = q(g, r, d, k)$$

for all $k \geq 3$. Let ℓ be a general 2-secant line of C and consider the nodal curve $X := C \cup \ell$. By Lemma 3.2.3, we have that $[X \subseteq \mathbb{P}^r] \in \mathcal{H}_{d+1,g+1,r}$. Since

$$q(g + 1, r, d + 1, k) = q(g, r, d, k) - 1,$$

all we need to show is that to contain the secant line ℓ imposes a nontrivial condition on the variety $Q_k(C, \mathbb{P}^r)$. This follows from the fact that the secant variety of a non degenerate curve does not lie in *any* quadric [Cat01]. □

The only remaining case is the case of incomplete embeddings (i.e. the case where $g - d + r < 0$), which will be treated in the next proposition. We first make a simple observation that we will use in the proof of the proposition.

Lemma 3.2.5. *Let C be a smooth curve of genus g and $\ell = (L, V)$ a very ample g^r_d on the curve. Let $W^\vee$ be the kernel of the map $H^0(L)^\vee \to V^\vee$ induced by the inclusion $V \subseteq H^0(L)$. Consider the following commutative diagram*

$$
\begin{array}{ccc}
C & \xrightarrow{|L|} & \mathbb{P}(H^0(L)^\vee) \\
& \underset{|\ell|}{\searrow} & \downarrow \pi \\
& & \mathbb{P}(V^\vee)
\end{array}
$$

where π is the projection with center $\mathbb{P}(W^\vee)$. We have that

$$\dim Q_k(C,\ell) = \dim Q_k(C,L)[\mathbb{P}(W^\vee)],$$

where

$$Q_k(C,L)[\mathbb{P}(W^\vee)] := \left\{Q \in Q_k(C,L) \mid \mathbb{P}(W^\vee) \subseteq \mathrm{Sing}(Q)\right\}.$$

Proof. There is an evident map $Q_k(C,L)[\mathbb{P}(W^\vee)] \to Q_k(C,\ell)$ defined by projecting quadrics by π. The inverse of this map is given by assigning Q to the cone over Q with vertex $\mathbb{P}(W^\vee)$. □

Proposition 3.2.6. *Theorem 3.1.1 holds in the range $g-d+r<0$.*

Proof. We fix integers g, r, d, k such that $g-d+r<0$ and we let C be a general curve of genus g. The variety $G^r_d(C)$ is irreducible and sits over the Picard variety $\mathrm{Pic}^d(C)$ as a Grassmann bundle. Therefore a general g^r_d on C is simply a general point on the Grassmannian

$$\mathrm{Gr}\left(h^0(L)-r-1, H^0(L)\right)$$

for a general line bundle $L \in \mathrm{Pic}^d(C)$.

We fix a general line bundle $L \in \mathrm{Pic}^d(C)$. By Proposition 3.2.4, we have that

$$\dim Q_k(C,L) = q(g, d-g, d, k).$$

We consider the incidence correspondence

$$I := \left\{(Q,\Lambda) \mid \Lambda \subseteq \mathrm{Sing}Q\right\} \subseteq Q_k(C,L) \times \mathrm{Gr}(d-g-r, d-g+1),$$

and the projection maps

$$
\begin{array}{ccc}
& I & \\
\overset{\pi_1}{\swarrow} & & \overset{\pi_2}{\searrow} \\
Q_k(C,L) & & \mathrm{Gr}(d-g-r, d-g+1)
\end{array}
$$

The singular locus of a rank k quadric has dimension $d-g-k$ and therefore the fiber dimension of π_1 over the set of quadrics of rank exactly k is equal to the dimension of the Grassmannian $\mathrm{Gr}(d-g-r, d-g-k+1)$. Therefore, we have that

$$\dim I \geq q(g, d-g, d, k) + (d-g-r)(r+1-k).$$

In fact, this is an equality: If Z is a component of I with dimension strictly greater than that number, we must have that

$$\pi_1(Z) \subseteq Q_{k-1}(C, L).$$

Let k' be the smallest integer such that $\pi_1(Z) \subseteq Q_{k'}(C, L)$. Then a general element of $\pi_1(Z)$ is a quadric of rank k' and by the same dimension count we get that

$$\dim Z = q(g, d-g, d, k') + (d-g-r)(r+1-k'),$$

which is strictly smaller than $q(g, d-g, d, k) + (d-g-r)(r+1-k)$. Therefore we conclude that

$$\dim I = q(g, d-g, d, k) + (d-g-r)(r+1-k).$$

Now there are two cases to consider. First, if $q(g, r, d, k) \geq 0$ then the map π_2 is surjective, for if $\Lambda \in \operatorname{Gr}(d-g-r, d-g+1)$ then

$$\pi_2^{-1}(\Lambda) = Q_k(C, L)[\Lambda],$$

and by Lemma 3.2.5 we have that

$$\dim Q_k(C, L)[\Lambda] = \dim Q_k(C, \mathbb{P}^r) \geq q(g, r, d, k) \geq 0.$$

Therefore, the fiber dimension of π_2 at a general point is equal to

$$q(g, d-g, d, k) + (d-g-r)(r+1-k) - \dim \operatorname{Gr}(d-g-r, d-g+1) = q(g, r, d, k).$$

Hence we conclude that $\dim Q_k(C, \mathbb{P}^r) = q(g, r, d, k)$.

On the other hand, if $q(g, r, d, k) < 0$ then π_2 is not surjective and thus for a general element $\Lambda \in \operatorname{Gr}(d-g-r, d-g+1)$, the variety $Q_k(C, L)[\Lambda]$ and hence $Q_k(C, \mathbb{P}^r)$ is empty. □

3.3 The class of $\overline{\mathfrak{Quad}}^k_{g,n}$

To define the locus $\mathfrak{Quad}^k_{g,n}$, we fix integers g, n, k such that

$$q(g, g-n-1, 2g-2-n, k) = -1.$$

We let

$$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

be the map that forgets the last marked point and $\mathscr{L}$ be the cotangent line bundle on $\overline{\mathcal{M}}_{g,n+1}$. We define the sheaves

$$\mathcal{E} := \pi_* \mathscr{L}\Big(-\sum_{j=1}^{n} \delta_{0:\{j,n+1\}} \Big),$$

and

$$\mathcal{F} := \pi_*\mathscr{L}^{\otimes 2}\Big(-2\cdot\sum_{j=1}^{n}\delta_{0:\{j,n+1\}}\Big),$$

and consider the natural multiplication map

$$\phi : \mathrm{Sym}^2\mathcal{E} \to \mathcal{F}. \tag{3.3}$$

Using the notation of Theorem 1.6.5, we define

$$\mathfrak{Quad}^k_{g,n} := \Sigma^k_{e,f}(\phi)\cap\mathcal{M}_{g,n},$$

where $e = g-n$ and $f = 3g-3-2n$. We use the formula in Theorem 1.6.5 and obtain the following result:

Theorem 3.3.1. *The coefficients of the class* $\left[\overline{\mathfrak{Quad}}^k_{g,n}\right]$ *satisfy the following relations:*

$$\alpha^k_{g,n} = \prod_{t=0}^{g-n-k-1}\frac{\binom{g-n+t}{g-n-k-t}}{\binom{2t+1}{t}},\quad a = \frac{7g-9n+6}{g-n},\quad c = \frac{g+n-6}{g-n},\quad b_{irr} = 1,$$

and $b_{i:s}\geq 1$ *for* $0\leq i\leq g$ *and* $0\leq s\leq n$.

Proof. The proof follows the same line of arguments as in the proof of Theorem 2.2.1. Note that the evaluation map

$$\pi_*\mathscr{L} \xrightarrow{ev} \pi_*\left(\mathscr{L}|_{\sum_{j=1}^{n}\delta_{0:\{j,n+1\}}}\right)$$

fails to be surjective over the boundary divisor $\Delta_{i:S}$ when $i < s$ or $g-i < n-s$, where $s = |S|$. Therefore we break our analysis into two parts. First, we let $\widetilde{\mathcal{M}}_{g,n}$ be the partial compactification defined as the union of $\mathcal{M}_{g,n}$ with the boundary divisors $\Delta_{i:S}$, such that $s\leq i$ and $n-s\leq g-i$. We will deal with the case where $i<s$ or $g-i<n-s$ later.

The first Chern classes of $\mathcal{E}$ and $\mathcal{F}$ over $\widetilde{\mathcal{M}}_{g,n}$ were computed in the proof of Theorem 2.2.1 as

$$c_1(\mathcal{E}) = \lambda - \sum_{j=1}^{n}\psi_j,$$

and

$$c_1(\mathcal{F}) = 13\cdot\lambda - 5\cdot\sum_{j=1}^{n}\psi_j - \delta.$$

Applying Theorem 1.6.5 yields the following formula for the class of $\Sigma^k_{e,f}(\phi)$:

$$[\Sigma^k_{e,f}(\phi)] = A^k_{g-n}\cdot\left(\frac{7g-9n+6}{g-n}\cdot\lambda + \frac{g+n-6}{g-n}\cdot\sum_{j=1}^{n}\psi_j - \delta\right). \tag{3.4}$$

This way we obtained the coefficients a and c. To conclude that $b_{irr} = 1$, one also needs to show that $\Delta_{irr} \not\subset \Sigma^k_{e,f}(\phi)$. The arguments in Lemma 3.2.1 and Proposition 3.2.4 can be repeated verbatim for the canonical image of an irreducible nodal curve to show that Theorem 3.1.1 holds true also for general elements of Δ_{irr}. We skip these details.

From the expression (3.4), we can read off the bound $b_{i:s} \geq 1$ whenever we have that $s \leq i$ and $n - s \leq g - i$. To obtain a bound for the remaining boundary coefficients, we introduce the twist

$$\mathscr{L}' := \mathscr{L}\left(\sum_{\substack{0\leq i\leq g\\ i<s\\ |S|=s}} (i - s - 1)\cdot \delta_{i:S\cup\{n+1\}}\right),$$

and define the sheaves

$$\mathcal{E}' := \pi_*\mathscr{L}'\Big(-\sum_{j=1}^{n} \delta_{0:\{j,n+1\}}\Big),$$

and

$$\mathcal{F}' := \pi_*\mathscr{L}'^{\otimes 2}\Big(-2\cdot\sum_{j=1}^{n} \delta_{0:\{j,n+1\}}\Big).$$

It can easily be checked that the ranks of $\mathcal{E}'$ and $\mathcal{F}'$ stay constant away from loci of codimension 2 and thus we have a morphism of vector bundles

$$\phi' : \mathrm{Sym}^2\mathcal{E}' \to \mathcal{F}'. \tag{3.5}$$

extending ϕ. Between the classes $[\Sigma^k_{e,f}(\phi')]$ and $\left[\overline{\mathfrak{Quad}}^k_{g,n}\right]$, we have the relation

$$[\Sigma^k_{e,f}(\phi')] = \left[\overline{\mathfrak{Quad}}^k_{g,n}\right] + \sum d_{i:s}\cdot\delta_{i:S}, \tag{3.6}$$

where $d_{i:s} \geq 0$. Letting $\tilde{b}_{i:s}$ denote the coefficient of $\delta_{i:S}$ in the class $[\Sigma^k_{e,f}(\phi')]$, we obtain the equality $\tilde{b}_{i:s} = b_{i:s} - d_{i:s}$. Using the intersection numbers computed in Lemma 2.2.2 and applying Theorem 1.6.5 once again, we obtain that

$$\tilde{b}_{i:s} = \frac{\big(-i^2(g-2n+3) + i(2g-2sn+6s-3n+3) + s((g-3)s+n-3)\big)}{g-n}, \tag{3.7}$$

for all i, s such that $i < s$. It is easy to see that this quantity is always greater than or equal to 1. □

We believe that the coefficients $d_{0:s}$ in equation (3.6) are equal to zero and therefore $b_{0:s} = \tilde{b}_{0:s}$, but we could prove this only for $k = 4$.

Theorem 3.3.2. *For $k = 4$ and $S \subseteq \{1, \ldots, n\}$ the general element of $\Delta_{0:S}$ does not lie in $\Sigma^k_{e,f}(\phi')$ and we have that*

$$b_{0:s} = \frac{s(gs - 3s + n - 3)}{g - n}.$$

Proof. Let $[X, p_1, \ldots p_n] \in \Delta_{0:S}$ be the curve obtained by gluing the general pointed curves

$$[C, p_0, \{p_j \mid j \in S^c\}] \in \mathcal{M}_{g, S^c \cup \{0\}},$$

and

$$[R, q_0, \{q_j \mid j \in S\}] \in \mathcal{M}_{0, S \cup \{0\}},$$

at the points with label 0 (i.e. $p_0 \sim q_0$). The fiber of the vector bundle map (3.5) over this moduli point can be identified with the map

$$\mathrm{Sym}^2 H^0\bigg(K_C\Big(-\sum_{j \in S^c} p_j - s \cdot p_0\Big)\bigg) \to H^0\bigg(K_C^{\otimes 2}\Big(-2 \cdot \sum_{j \in S^c} p_j - 2s \cdot p_0\Big)\bigg).$$

Therefore, we need to show that there are no rank 4 quadrics containing the image of C under the map given by the linear series

$$\bigg|K_C\Big(-\sum_{j \in S^c} p_j - s \cdot p_0\Big)\bigg|.$$

It is clearly sufficient to specialize to the case where $p_j = p_0$ for all $j \in S^c$ and check this claim for the linear series $|K_C(-n \cdot p_0)|$. First note that for $k = 4$ the numerical condition

$$q(g, g - n - 1, 2g - 2 - n, k) = -1$$

implies that $g = 2n + 3$. Let A be a g^1_a on the curve C such that the pencil pair $\big(A, K_C(-n \cdot p_0) \otimes A^\vee\big)$ corresponds to a rank 4 quadric containing C. By Brill-Noether theory, we have that $a \geq n + 3$. Since $|K_C \otimes A^\vee|$ is a g^{g-a}_{2g-2-a}, the condition that

$$h^0\big(K_C(-n \cdot p_0) \otimes A^\vee\big) \geq 2$$

imposes non trivial conditions on the ramification type of the linear series $|K_C \otimes A^\vee|$ at the point p_0. Precisely, we have the following inequality for the ramification sequence of $|K_C \otimes A^\vee|$:

$$\alpha^{K_C \otimes A^\vee}(p_0) \geq (0, \ldots, 0, a - n - 2, a - n - 2).$$

However, the adjusted Brill-Noether number for such a linear series is equal to

$$\rho(g, g - a, 2g - 2 - a) - 2(a - n - 2) = 2a - 2n - 5 - 2(a - n - 2) = -1,$$

and by Proposition 1.2 in [EH87b], there is no such linear series on a general pointed curve $[C, p_0] \in \mathcal{M}_{g,1}$. Contradiction. $\square$

3.4 A new divisor on $\overline{\mathcal{M}}_{15,8}$

In this section we discuss the proof of Theorem 3.1.3. We note that loci that are defined via the existence of a degenerate pencil of quadrics and expected to be of codimension one, exist in $\mathcal{M}_{g,n}$ for all g, n satisfying

$$\big(g(t), n(t)\big) = \left(\frac{t^2+7t+12}{2}, \frac{t^2+5t+2}{2}\right) \text{ for } t \in \mathbb{N}. \tag{3.8}$$

However, we do not know a method to show that they are all indeed divisors. Therefore, we restricted our attention to the particular case of $t = 2$, which was (due to the numerology) the only case with a potential corollary in birational classification of $\overline{\mathcal{M}}_{g,n}$.

Proof of Theorem 3.1.3. We first show that

$$\overline{\mathfrak{D}}_{15,8} \neq \overline{\mathcal{M}}_{15,8}.$$

Clearly, it is sufficient to find a smooth curve in $C \subseteq \mathbb{P}^6$ of genus 15 and degree 20 such that the pencil $I_2\big(C, \mathcal{O}_C(2)\big)$ is non degenerate. To this end, we pick 15 general points on $\mathbb{P}^2$ and consider the blown up surface $X := \mathrm{Bl}_{15}(\mathbb{P}^2)$. Using Macaulay, we show that the linear system

$$H = 7h - 2(E_1 + \cdots + E_7) - E_8 - \cdots - E_{15}$$

embeds X to $\mathbb{P}^6$, where h is the class of a line in $\mathbb{P}^2$ and E_i is the exceptional divisor corresponding to the i^{th} point. We also check that $\dim I_2\big(X, \mathcal{O}_X(2)\big) = 2$ and that the pencil is non degenerate. Next, we let C to be a general element of the linear system

$$|10h - 3(E_1 + E_2 + E_3) - 2(E_4 + \cdots + E_{15})|\,.$$

Again using Macaulay we check that this linear system is base point free, hence by Bertini's theorem we obtain that C is smooth. It is easy to see that the genus of C is 15 and $C \cdot H = 20$. We also check that

$$H^0\big(\mathcal{O}_X(2H - C)\big) = 0,$$

which implies that the restriction map

$$\rho : H^0\big(\mathcal{O}_X(2)\big) \to H^0\big(\mathcal{O}_C(2)\big)$$

is injective. Therefore, the map

$$H^0\big(\mathcal{O}_{\mathbb{P}^6}(2)\big) \to H^0\big(\mathcal{O}_C(2)\big)$$

factors through ρ and it follows that

$$I_2\big(X, \mathcal{O}_X(2)\big) = I_2\big(C, \mathcal{O}_C(2)\big).$$

To compute the class of $\overline{\mathfrak{D}}_{15,8}$, we let

$$\mathcal{E} := \pi_* \mathscr{L}\Big(- \sum_{j=1}^{8} \delta_{0:\{j,9\}}\Big),$$

and

$$\mathcal{F} := \pi_* \mathscr{L}^{\otimes 2}\Big(- 2 \cdot \sum_{j=1}^{8} \delta_{0:\{j,9\}}\Big),$$

and consider the morphism of vector bundles

$$\phi : \mathrm{Sym}^2 \mathcal{E} \to \mathcal{F} \tag{3.9}$$

over the partial compactification $\widetilde{\mathcal{M}}_{g,n}$, which was defined in the proof of Theorem 3.3.1. Using Theorem 1.6.6, we compute that

$$[\mathfrak{D}\mathfrak{p}(\phi)] = 6 \cdot \Big(39 \cdot \lambda + 17 \cdot \sum_{j=1}^{8} \psi_j - 7 \cdot \delta\Big).$$

This way we obtain that the boundary coefficients of $\big[\overline{\mathfrak{D}}_{15,8}\big]$ are at least -7 (for the boundary divisors that are in $\widetilde{\mathcal{M}}_{g,n}$). To bound the remaining coefficients we use the extension (3.5) of the vector bundle map ϕ. Analogous to the proof of Theorem 3.3.1, we write

$$[\mathfrak{D}\mathfrak{p}(\phi')] = 6 \cdot \Big(39 \cdot \lambda + 17 \cdot \sum_{j=1}^{8} \psi_j - 7 \cdot \delta_{irr} - \sum_{i,s \geq 0} \tilde{b}_{i:s} \cdot \sum_{|S|=s} \delta_{i:S}\Big), \tag{3.10}$$

and compute that

$$\tilde{b}_{i:s} = -2i^2 + i(9 - 10s) + s(12s + 5)$$

for $i < s$. Clearly, $b_{i:s} \geq \tilde{b}_{i:s} \geq 7$. □

3.5 Kodaira dimension of $\overline{\mathcal{M}}_{15,9}$

We conclude this chapter with the proof of Corollary 3.1.4.

Proof. We consider the map

$$\pi_j : \overline{\mathcal{M}}_{15,9} \to \overline{\mathcal{M}}_{15,8}$$

that forgets the point labeled by j. Pulling back the divisor $\overline{\mathfrak{D}}_{15,8}$ via π_j for every $j \in \{1, \ldots, 9\}$ and taking their average, we obtain the effective class

$$\mathfrak{D}_{15,9} = 351 \cdot \lambda + 136 \cdot \sum_{j=1}^{9} \psi_j - b_{irr} \cdot \delta_{irr} - \sum_{i,s \geq 0} b_{i:s} \cdot \sum_{|S|=s} \delta_{i:S},$$

where $b_{irr}, b_{i:s} \geq 63$ for all $i, s \geq 0$.

On $\overline{\mathcal{M}}_{15,9}$ we also have the pullback of the Brill-Noether divisor from $\overline{\mathcal{M}}_{15}$, whose class is given by the following formula.

$$\left[\overline{\mathcal{M}}^1_{15,8}\right] = 54 \cdot \lambda - 8 \cdot \delta_{irr} - \dots$$

Using these classes we can express $K_{\overline{\mathcal{M}}_{15,9}}$ as follows:

$$K_{\overline{\mathcal{M}}_{15,9}} = \frac{25}{297} \cdot \sum_{j=1}^{9} \psi_j + \frac{2}{297} \cdot \mathfrak{D}_{15,9} + \frac{13}{66} \cdot \left[\overline{\mathcal{M}}^1_{15,8}\right] + E,$$

where E is an effective divisor supported on the boundary of $\overline{\mathcal{M}}_{15,9}$. Since the class $\sum_{j=1}^{9} \psi_j$ is *big*, the result follows. □

Chapter 4

Miscellaneous

4.1 Maximal rank divisors of higher degree

One possible direction to continue our quest for divisor classes on moduli spaces is to investigate the loci determined by the failure of maximal rank conjecture for higher degree hypersurfaces. To this end, we let g, r, d, m be integers such that

$$\rho(g,r,d) = 0 \quad \text{and} \quad \binom{r+m}{m} = md - g + 1, \tag{4.1}$$

and denote by $\mathcal{G}^r_d$ the moduli space of curves of genus g together with a g^r_d on it. The locus of points $[(C, L)] \in \mathcal{G}^r_d$, where the map

$$\mathrm{Sym}^m H^0(C, L) \to H^0(C, L^{\otimes m})$$

fails to be an isomorphism is expected to be of codimension one. If that is indeed the case, by considering the image of this locus under the generically finite forgetful map

$$\sigma : \mathcal{G}^r_d \to \mathcal{M}_g,$$

we obtain a divisor in $\mathcal{M}_g$.

As pointed out in the introduction the divisors arising from the $m = 2$ case of this setup was considered by Farkas [Far09]. The solutions of the equations (4.1) for $m > 2$ and low genus are as follows:

i) $g = 8, r = 3, d = 9, m = 3$,

ii) $g = 28, r = 3, d = 24, m = 8$,

iii) $g = 35, r = 4, d = 32, m = 5$.

(The smallest genus solution following these 3 solutions is $g = 224, r = 7$, $d = 203, m = 5$.) Since the maximal rank conjecture holds in $\mathbb{P}^3$ [BE87a], the first item gives a divisor in $\mathcal{M}_8$, which however, can easily be seen to be equal to the Brill-Noether divisor $\mathcal{M}^2_{8,7}$. Indeed, if C is a general curve of genus 8 with a g^2_7 on it then the g^2_7 maps C to a plane curve C_0 of degree 7 with 7 nodes as singularities [Har86]. As is classically known, a smooth cubic surface is isomorphic to the blow up of the plane at 6 general points. Thus blowing up 6 nodes of C_0, we obtain that the proper transform $\widetilde{C}_0$ lies on a cubic surface $S \subseteq \mathbb{P}^3$ and has the class

$$7\ell - 2e_1 - \cdots - 2e_6 \in \mathrm{Pic}(S),$$

where ℓ denotes the class of the pullback of a line in $\mathbb{P}^2$ and e_i are the exceptional divisors. Since $3\ell - e_1 - \cdots - e_6$ is the class of a hyperplane section of S, we compute that

$$\deg(\widetilde{C}_0) = (7\ell - 2e_1 - \cdots - 2e_6) \cdot (3\ell - e_1 - \cdots - e_6) = 9.$$

Conversely, it follows from Mukai's result [Muk93] (Lemma 3.8 and Proposition 4.2 to be precise) that a curve of degree 9 and genus 8 lying on a cubic surface in $\mathbb{P}^3$ admits a g^2_7.

The other two solutions of the equations (4.1) give new divisor classes in $\mathcal{M}_g$. We summarize our results in the following theorem.

Theorem 4.1.1. *Let $\widetilde{\mathcal{M}}_g$ be the partial compactification $\mathcal{M}_g \cup \Delta_0 \cup \Delta_1$. Then the loci defined as*

$$\mathfrak{D}_{28} := \{[C] \in \mathcal{M}_{28} \mid \exists L \in W^3_{24}(C) \text{ such that } I_8(C, L) \neq 0\},$$

and

$$\mathfrak{D}_{35} := \{[C] \in \mathcal{M}_{35} \mid \exists L \in W^4_{32}(C) \text{ such that } I_5(C, L) \neq 0\}$$

are both divisors and the class of their closures in $\widetilde{\mathcal{M}}_g$ are as follows:

$$\left[\overline{\mathfrak{D}}_{28}\right] = N(28,3,24) \cdot \left(\frac{41633}{39} \cdot \lambda - \frac{19376}{117} \cdot \delta_0 - \frac{11957}{13} \cdot \delta_1\right),$$

and

$$\left[\overline{\mathfrak{D}}_{35}\right] = N(35,4,32) \cdot \left(\frac{10415}{17} \cdot \lambda - \frac{1640}{17} \cdot \delta_0 - 545 \cdot \delta_1\right),$$

where we denote by $N(g, r, d)$ the number of g^r_d's on a general curve of genus g in the case $\rho(g, r, d) = 0$.

Proof. It follows from [BE87a] and [BE09] that these loci are indeed divisors. To compute their classes we employ the methods developed in Section 2 of [Far09]. We will show only the computation of the class $\left[\overline{\mathfrak{D}}_{28}\right]$, the other one being analogous.

We let $\mathcal{G}^3_{24}$ be the space parametrizing curves of genus 28 together with a g^3_{24} on them. Using limit linear series, the forgetful map

$$\sigma : \mathcal{G}^3_{24} \to \mathcal{M}_{28}$$

can be extended to a proper, generically finite map

$$\widetilde{\sigma} : \widetilde{\mathcal{G}}^3_{24} \to \widetilde{\mathcal{M}}^0_{28},$$

where we denote by $\widetilde{\mathcal{M}}^0_{28}$ the partial compactification $\mathcal{M}_{28} \cup \Delta_0^0 \cup \Delta_1^0$. The boundary component Δ_0^0 is equal to the locus of one nodal irreducible curves $C' := C/q \sim y$ with $[C, q] \in \mathcal{M}_{27,1}$ being Brill-Noether general and $y \in C$ together with their degenerations when the points q and y come together, whereas Δ_1^0 is the locus of one nodal curves $C \cup_y E$ with C and $[E, y] \in \mathcal{M}_{1,1}$ being Brill-Noether general.

On $\widetilde{\mathcal{G}}^3_{24}$ there are two vector bundles E and F whose fibers can be described as follows. Their fibers over $[C, L] \in \mathcal{G}^3_{24}$ are given as

$$E(C, L) = H^0(C, L) \quad \text{and} \quad F(C, L) = H^0(C, L^{\otimes 8}).$$

If we let $s = [C/q \sim y, L] \in \widetilde{\sigma}^{-1}(\Delta_0^0)$ and $t = [C \cup_y E, L] \in \widetilde{\sigma}^{-1}(\Delta_1^0)$ then

$$E(s) = H^0\big(C, L(-q-y)\big) \oplus \mathbb{C} \cdot u \quad \text{and} \quad F(s) = H^0\big(C, L^{\otimes 8}(-q-y)\big) \oplus \mathbb{C} \cdot u^8,$$

where $u \in H^0(C, L) \setminus H^0\big(C, L(-q-y)\big)$ and

$$E(t) = H^0(C, L_C) \quad \text{and} \quad F(t) = H^0\big(C, L_C^{\otimes 8}(-2y)\big) \oplus \mathbb{C} \cdot u^8,$$

where L_C is the C aspect of the limit linear series L on the nodal curve $C \cup_y E$ and $u \in H^0(C, L_C) \setminus H^0\big(C, L_C(-y)\big)$. There is a natural map of vector bundles

$$\phi : \mathrm{Sym}^8 E \to F,$$

and $\overline{\mathfrak{D}}_{28}$ is the pushforward of the locus, where this map fails to be an isomorphism, via the map $\widetilde{\sigma}$.

In order to compute the class of $\overline{\mathfrak{D}}_{28}$ we introduce three test curves. We let $[C, y] \in \mathcal{M}_{27,1}$ be a general pointed curve and attach to it a pencil of plane cubics at the point y. We denote this curve by R. We let C^0 denote the test curve obtained by identifying y with a moving point q on C. Finally, we glue a genus 1 curve to C at the point y and let this point vary along C and denote the resulting test curve by C^1.

If we denote the class of $\overline{\mathfrak{D}}_{28}$ as

$$\big[\overline{\mathfrak{D}}_{28}\big] = a\lambda - b_0\delta_0 - b_1\delta_1,$$

we have the equations

i) $R \cdot \big[\overline{\mathfrak{D}}_{28}\big] = a - 12b_0 + b_1$,

ii) $C^0 \cdot \big[\overline{\mathfrak{D}}_{28}\big] = 54b_0 - b_1$,

iii) $C^1 \cdot \big[\overline{\mathfrak{D}}_{28}\big] = 52b_1$.

To compute $R \cdot \left[\overline{\mathfrak{D}}_{28}\right]$, we note that the vector bundles E and F only depend on the genus 27 aspect of the limit g^3_{24} and on the point which the pencil of plane cubics is attached to. Since these data do not vary in the test family R, we conclude that the restrictions of the bundles to R are trivial. Therefore, we have that $R \cdot \left[\overline{\mathfrak{D}}_{28}\right] = 0$. Next, we use the formulas in Lemma 2.6, Proposition 2.12 and Lemma 2.13 in [Far09] to compute that

$$C^0 \cdot \left[\overline{\mathfrak{D}}_{28}\right] = 8023 \cdot N(28,3,24) \quad \text{and} \quad C^1 \cdot \left[\overline{\mathfrak{D}}_{28}\right] = 47828 \cdot N(28,3,24).$$

Solving these three equations we obtain the λ, δ_0 and δ_1 coefficients as stated in the theorem. □

Note that the ratio of the λ and δ_0 coefficients of the classes $\left[\overline{\mathfrak{D}}_{28}\right]$ and $\left[\overline{\mathfrak{D}}_{35}\right]$ are approximately equal to 6, 45 and 6, 35 and they are greater than the slopes of the respective Brill-Noether divisors.

4.2 More on the correspondence of pencils and rank 4 quadrics

For nonspecial embeddings (i.e. $r = d - g$) the rank 4 case of Theorem 3.1.1 implies that for a general line bundle $L \in \mathrm{Pic}^d(C)$ on a general curve C, every component of $Q_4(C, L)$ has dimension $2d - 3g - 4$. Under the light of Lemma 1.6.4, we know that each of these irreducible components correspond to a different choice of (d_1, d_2), which are the degrees of the pencils giving rise to the rank 4 quadric in consideration. What one cannot conclude from Theorem 3.1.1 however, is whether $Q_4(C, L)$ has quadrics corresponding to all possible degree types (d_1, d_2). With the following theorem we answer this question in the affirmative.

Theorem 4.2.1. *Let C be a general curve of genus $g \geq 2$ and d_1, d_2 integers satisfying*

i) $2d_1 + 2d_2 - 3g - 4 \geq 0$,

ii) $\rho(g, 1, d_i) \geq 0$ *for* $i = 1, 2$.

Then the multiplication map

$$\begin{aligned} \mu_{d_1,d_2} : W^1_{d_1}(C) \times W^1_{d_2}(C) &\to \mathrm{Pic}^{d_1+d_2}(C) \\ (L_1, L_2) &\mapsto L_1 \otimes L_2 \end{aligned}$$

is surjective.

Proof. We let $\mathcal{H}_{g,d-g,d}$ be the unique component of $\mathrm{Hilb}_{g,d-g,d}$ parametrizing curves with general moduli and denote by $\mathcal{Q}$ (for notational convenience) the parameter space $\mathcal{Q}_{g,d-g,d,4}$ defined in Section 3.1.

The variety $\mathcal{Q}$ can easily be seen to be determinantal: On $Q_4(\mathbb{P}^{d-g})$ there is the tautological line bundle, whose fiber over a quadric is the one dimensional space of polynomials defining that quadric. We pull this line bundle back to

$$\mathcal{H}_{g,d-g,d} \times Q_4(\mathbb{P}^{d-g}),$$

and denote the resulting line bundle by $\mathcal{L}$. Moreover, there is a sheaf $\tilde{\mathcal{F}}$ on $\mathcal{H}_{g,d-g,d}$ whose fiber over the moduli point $[C \subseteq \mathbb{P}^{d-g}]$ is naturally isomorphic to $H^0\big(\mathcal{O}_C(2)\big)$. For degree reasons we have that $H^1\big(\mathcal{O}_C(2)\big) = 0$, therefore $\tilde{\mathcal{F}}$ is a vector bundle of rank $2d - g + 1$. We denote the pullback of this vector bundle to $\mathcal{H}_{g,d-g,d} \times Q_4(\mathbb{P}^{d-g})$ as $\mathcal{F}$. There is a natural morphism of vector bundles

$$\mathcal{L} \to \mathcal{F},$$

which restricts a quadratic polynomial to a quadratic section of the curve (i.e. to an element of $H^0\big(\mathcal{O}_C(2)\big)$). The variety $\mathcal{Q}$ is simply the locus where this morphism is identically zero. From this determinantal description, it follows readily that if Z is an irreducible component of $\mathcal{Q}$ then we have the lower bound that

$$\dim Z \geq \dim \mathcal{H}_{g,d-g,d} + 2d - 3g - 4. \tag{4.2}$$

As we remarked above, $\mathcal{Q}$ splits into irreducible components characterized by the degrees of the pencils corresponding to the rank 4 quadric. We denote by $\mathcal{Q}(d_1, d_2)$ the irreducible component, whose general element is a pair

$$\big([C \subseteq \mathbb{P}^{d-g}], Q\big) \in \mathcal{Q},$$

where Q corresponds to a pair of base point free pencils of degrees d_1 and d_2 on C. To prove the theorem it suffices to show that the projection map

$$\mathcal{Q}(d_1, d_2) \to \mathcal{H}_{g,d-g,g}$$

is surjective. We confirm this by applying induction on the genus. Note first that if $d_i \geq g + 1$ then $W^1_{d_i}(C) = \mathrm{Pic}^{d_i}(C)$ and hence the theorem is trivially true. Therefore from now on we assume that $d_1, d_2 \leq g$. The base step $g = 2$ of the induction is clear, since also in this case we have that $W^1_{d_i}(C) = \mathrm{Pic}^{d_i}(C)$ for either $i = 1$ or $i = 2$.

For the inductive step, fix integers g and $d = d_1 + d_2$ satisfying the inequalities in the statement of the theorem and assume that $d_1 \geq d_2$. By induction assumption, for a general curve C of genus $g - 1$, the map $\mu_{d_1 - 1, d_2}$ is surjective. Fix a general line bundle L_0 of degree $d - 1$ on C. By Theorem 3.1.1, we have that

$$\dim Q_4(C, L_0) = 2d - 3g - 3 \geq 1.$$

Our aim now is to find a secant line ℓ of the embedded curve $C \subseteq \mathbb{P}^{d-g}$ and a quadric $Q \in Q_4(C \cup \ell, \mathbb{P}^{d-g})$ such that

a) the variety $Q_4(C \cup \ell, \mathbb{P}^{d-g})$ has the expected dimension $2d - 3g - 4$,

b) the projection map $\mathbb{P}^{d-g} \setminus \mathrm{Sing}(Q) \to \mathbb{P}^3$ maps $C \cup \ell$ to a (d_1, d_2) curve in a smooth quadric in $\mathbb{P}^3$.

This will finish the proof since b) implies that the element

$$\big([C \cup \ell \subseteq \mathbb{P}^{d-g}], Q\big) \in \mathcal{Q}$$

lies on the component $\mathcal{Q}(d_1, d_2)$ and a) together with the lower bound (4.2) implies that the map $\mathcal{Q}(d_1, d_2) \to \mathcal{H}_{g,d-g,d}$ is surjective. To carry out this outlined strategy we let $p_1, p_2 \in C$ be general points and define the variety

$$W^1_{d_2}(C, p_1, p_2) := \{L \in W^1_{d_2}(C) \mid H^0\big(L(-p_1 - p_2)\big) \neq 0\}.$$

We consider the map

$$\varphi : W^1_{d_1-1}(C) \times W^1_{d_2}(C, p_1, p_2) \to \mathrm{Pic}^{d_1+d_2-1}(C).$$

We claim that φ is surjective as well. To see this, note that for a general line bundle $L \in \mathrm{Pic}^{d_1+d_2-1}(C)$ we have that

$$\dim \mu^{-1}_{d_1-1,d_2}(L) \geq 1. \tag{4.3}$$

For every pair $(L_1, L_2) \in \mu^{-1}_{d_1-1,d_2}(L)$, possibly outside a set of high codimension, there is a single effective divisor in $|L_2(-p_1)|$. If p_2, which was chosen to be general, is not contained in any of these divisors as (L_1, L_2) varies in $\mu^{-1}_{d_1-1,d_2}(L)$, it would mean that this family of divisors are supported on a finite subset of C. That in turn would mean that there are finitely many distinct line bundles L_2 and hence $\mu^{-1}_{d_1-1,d_2}(L)$ is a finite set, contradicting with (4.3). Therefore, φ is surjective.

Now, we let $(L_1, L_2) \in \varphi^{-1}(L_0)$ be a general element. By the generality assumption, we have that

$$h^0\big(L_1(-p_1 - p_2)\big) = 0 \text{ and } h^0\big(L_2(-p_1 - p_2)\big) = 1.$$

Geometrically this means that the line $\ell := \overline{p_1, p_2}$ is contained in one and only one of the rulings of the rank 4 quadric Q corresponding to the pencil pair (L_1, L_2). This implies that ℓ does not meet the singular locus of Q and therefore the projection from $\mathrm{Sing}(Q)$ maps C to a (d_1, d_2) curve in the image quadric in $\mathbb{P}^3$. Since p_1, p_2 were chosen to be general, to contain ℓ imposes a nontrivial condition on the quadrics in $Q_4(C, L_0)$. Hence, we obtain that

$$\dim Q_4(C \cup \ell, \mathbb{P}^{d-g}) = 2d - 3g - 4,$$

which finishes the proof. □

It follows from the theorem that in the special case where

$$2d_1 + 2d_2 - 3g - 4 = 0,$$

the map μ_{d_1,d_2} is generically finite. Since we know the degree of the variety $Q_4(C, \mathbb{P}^{d-g})$ by Theorem 1.6.2, we already know the sum of the degrees of the maps μ_{d_1,d_2} for admissible pairs d_1, d_2. It would be interesting to know the degree of these maps individually, though we have no idea how to approach this question.

We can deal with the first few cases of this problem using simple arguments. We close this discussion by exhibiting these small genus cases.

Case i) $g = 2$**:** In this case we have $d_1 + d_2 = 5$. Therefore, the only admissible case is $d_1 = 3$ and $d_2 = 2$. Since $W^1_3(C) = \mathrm{Pic}^3(C)$ and there is a unique g^1_2 on C, the map μ_{d_1,d_2} is an isomorphism.

Case ii) $g = 4$**:** We have that $d_1 + d_2 = 8$. Since the general curve of genus 4 is not hyperelliptic, the possible pairs for d_1, d_2 are $(5, 3)$ and $(4, 4)$. In the case $(d_1, d_2) = (5, 3)$ we again have that $W^1_5(C) = \mathrm{Pic}^5(C)$ and therefore the degree of $\mu_{5,3}$ is equal to the number of g^1_3's on C. Using Theorem 1.3.4, we compute that

$$\deg(\mu_{5,3}) = 2.$$

Moreover since $\deg\left(Q_4(C, \mathbb{P}^4)\right) = 5$, it follows that

$$\deg(\mu_{4,4}) = 3.$$

Case iii) $g = 6$**:** In this case $d_1 + d_2 = 11$ and the possible pairs for d_1, d_2 are $(7, 4), (6, 5)$. As in the previous case, we compute that

$$\deg(\mu_{7,4}) = \#\{g^1_4\text{'s on } C\} = 5,$$

and

$$\deg(\mu_{6,5}) = \deg\left(Q_4(C, \mathbb{P}^5)\right) - 5 = 30.$$

The next case is $g = 8$ and the possible pairs for (d_1, d_2) are $(5, 9), (6, 8), (7, 7)$. Clearly, with the methods we used above we cannot say anything more than that

$$\deg(\mu_{5,9}) = \#\{g^1_5\text{'s on } C\} = 14,$$

and

$$\deg(\mu_{6,8}) + \deg(\mu_{7,7}) = \deg\left(Q_4(C, \mathbb{P}^6)\right) - 14 = 280.$$

4.3 Kodaira dimension of $\overline{\mathcal{M}}_{12,10}$

In this short section we show that the moduli space $\overline{\mathcal{M}}_{12,10}$ is of general type. This result is an immediate consequence of the divisor class computations done in the papers [FV14] and [FV13]. We noticed this overlooked corollary accidentally, while we were exploring the birational geometric consequences of the class $\overline{\mathfrak{Quad}}_{12,6}$.

Theorem 4.3.1. *The moduli space $\overline{\mathcal{M}}_{12,10}$ is of general type.*

Proof. We define the locus of curves

$$\mathfrak{D}_{12} := \{[C] \in \mathcal{M}_{12} \mid \exists L \in W^4_{14}(C) \text{ such that } I_2(C, L) \neq 0\}.$$

In [FV14] it is proven that this locus is a divisor and the class of its closure in $\overline{\mathcal{M}}_{12}$ is equal to

$$\left[\overline{\mathfrak{D}}_{12}\right] = 13245 \cdot \lambda - 1926 \cdot \delta_0 - \dots$$

Next, we define the locus

$$\mathcal{F}_{12,1} := \left\{[C, p_1, \dots, p_{10}] \in \mathcal{M}_{12,10} \mid \exists A \in W^1_{11}(C) \text{ with } h^0\Big(A\big(-\sum_{i=1}^{10} p_i\big)\Big) \neq 0\right\}.$$

It is proven in [FV13] that $\mathcal{F}_{12,1}$ is a divisor and its class is given as follows.

$$\left[\overline{\mathcal{F}}_{12,1}\right] = 9 \cdot \sum_{i=1}^{10} \psi_i - \delta_{irr} - \dots$$

Using these two classes we can write the canonical class of $\overline{\mathcal{M}}_{12,10}$ as

$$K_{\overline{\mathcal{M}}_{12,10}} = \frac{59}{4415} \sum_{j=1}^{10} \psi_j + \frac{13}{13245} \left[\overline{\mathfrak{D}}_{12}\right] + \frac{484}{4415} \left[\overline{\mathcal{F}}_{12,1}\right] + E,$$

with E being an effective divisor supported on $\overline{\mathcal{M}}_{12,10} \setminus (\mathcal{M}_{12,10} \cup \Delta_{irr})$. □

Appendix A

Macaulay code

Here we present the Macaulay code we used in the proof of Theorem 3.1.3. We first define the "NondegenerateQuadrics" function, which takes an ideal I as an argument and checks if the variety of singular quadrics containing the variety defined by I is reduced or not. It returns "True" if it is reduced and "False" otherwise.

```
i1 : NondegenerateQuadrics = I -> (
        k = coefficientRing(ring I);

        R = k[D_1..D_(numgens(ring I))];

        g = map(R, ring I, vars R);

        J = g(I);

        use R;

        n = hilbertFunction(2,R) - hilbertFunction(2,J);

        b = submatrix(mingens J, toList(0..(n-1)));

        Jac = jacobian super b;

        quadrics = for i from 0 to n-1 list jacobian
        transpose matrix Jac_i;

        S = QQ[F_0..F_(n-1)];

        f = map(S,R);
```

```
    Q = 0;

    for i from 0 to n-1 do Q = Q + F_i*f(quadrics_i);

    singQuadrics = det(Q);

    return ideal(det(Q)) == radical(ideal(det Q))
  )

o1 = NondegenerateQuadrics

o1 : FunctionClosure
```

Next, we choose 15 points in the plane. For computational convenience we choose these points in groups of 3,4 and 8 points.

```
i2 : k = ZZ/101;

i3 : R = k[X,Y,Z];

i4 : P1 = ideal(X,Y);

o4 : Ideal of R

i5 : P2 = ideal(X,Z);

o5 : Ideal of R

i6 : P3 = ideal(Y,Z);

o6 : Ideal of R

i7 : I3 = intersect(P1,P2,P3);

o7 : Ideal of R

i8 : M4 = random(R^{-2,-2},R^{-4});

                2         1
o8 : Matrix R  <--- R

i9 : I4 = minors(1,M4);

o9 : Ideal of R
```

```
i10 : M8 = random(R^{-4,-2},R^{-6});

                  2           1
o10 : Matrix R   <---- R

i11 : I8 = minors(1,M8);

o11 : Ideal of R
```

As a reality check, we confirm that we have chosen 15 points in total and that they are distinct from each other.

```
i12 : degree intersect(I3,I4,I8) == 15

o12 = true

i13 : intersect(I3,I4,I8) == radical(intersect(I3,I4,I8))

o13 = true
```

We compute a basis of the linear system

$$|7h - 2(E_1 + .. + E_7) - (E_8 + ... + E_{15})|,$$

and check that it has indeed projective dimension 6.

```
i14 : I7 = intersect(I3,I4);

o14 : Ideal of R

i15 : IH = saturate intersect(I7^2,I8);

o15 : Ideal of R

i16 : BH = basis(7,module IH);

o16 : Matrix

i17 : hilbertFunction(7,R)-hilbertFunction(7,IH) == 7

o17 = true
```

We compute a basis of the linear system

$$A := |10h - 3(E_1 + .. + E_3) - 2(E_4 + .. + E_{15})|,$$

and check that the linear system has no base points.

```
i18 : I12 = intersect(I4,I8);

o18 : Ideal of R

i19 : IA = saturate intersect(I3^3,I12^2);

o19 : Ideal of R

i20 : BA = basis(10,module IA);

o20 : Matrix

i21 : baseLocusA = saturate ideal super BA

o21 : Ideal of R

i22 : baseLocusA == IA

o22 = true
```

To show that $H^0\big(\mathcal{O}_X(2H-C)\big)=0$ we note that

$$|2H-C| = |4h-(E_1+E_2+E_3)-2(E_4+E_5+E_6+E_7)|,$$

and compute a basis for the linear system $|2H-C|$ and show that it has no elements.

```
i23 : ID = saturate intersect(I3,I4^2);

o23 : Ideal of R

i24 : basis(4,module ID)

o24 = 0

o24 : Matrix
```

We define the ring of $\mathbb{P}^6$ and the map $f:\mathbb{P}^2 \dashrightarrow \mathbb{P}^6$ induced by the linear system $|7h-2(E_1+..+E_7)-(E_8+...+E_{15})|$.

```
i25 : T = k[W_0..W_6];

i26 : fH = map(R,T,super BH);

o26 : RingMap R <--- T
```

We check that the resulting surface $X\subseteq\mathbb{P}^6$ lies on two linearly independent quadrics.

```
i27 : IH = kernel fH;

o27 : Ideal of T

i28 : hilbertFunction(2,T) - hilbertFunction(2,IH)

o28 = 2
```

Finally we check that this pencil of quadrics is nondegenerate.

```
i29 : NondegenerateQuadrics(IH)

o29 = true
```

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