On syzygies of algebraic varieties with applications to moduli

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Abstract

In this thesis we study asymptotic syzygies of algebraic varieties and equations of abelian surfaces, with applications to cyclic covers of genus two curves.

First, we show that vanishing of asymptotic p-th syzygies implies p-very ampleness for line bundles on arbitrary projective schemes. For smooth surfaces we prove that the converse holds, when p is small, by studying the Bridgeland-King-Reid-Haiman correspondence for the Hilbert scheme of points. This extends previous results of Ein-Lazarsfeld and Ein-Lazarsfeld-Yang. As an application of our results, we show how to use syzygies to bound the irrationality of a variety.

Furthermore, we confirm a conjecture of Gross and Popescu about abelian surfaces whose ideal is generated by quadrics and cubics. In addition, we use projective normality of abelian surfaces to study the Prym map associated to cyclic covers of genus two curves. We show that the differential of the map is generically injective as soon as the degree of the cover is at least seven, extending a previous result of Lange and Ortega. Moreover, we show that the differentials fails to be injective precisely at bielliptic covers.

Zusammenfassung

Diese Dissertation beschäftigt sich mit asymptotischen Syzygien und Gleichungen Abelscher Varietäten, sowie mit deren Anwendung auf zyklische Überdeckungen von Kurven von Geschlecht zwei.

Was asymptotischen Syzygien angeht, zeigen wir für beliebige Geradenbündel auf projektiven Schemata: Wenn die asymptotischen Syzygien von Grad p eines Geradenbündels verschwinden, dann ist das Geradenbündel p-sehr ampel. Darüber hinaus verwenden wir die Bridgeland-King-Reid-Haiman Korrespondenz, um zu zeigen, dass dieses Ergebnis auch umgekehrt wahr ist, wenn es um eine glatte Fläche und kleine p geht. Dies dehnt Ergebnisse von Ein-Lazarsfeld und Ein-Lazarsfeld-Yang aus. Wir verwenden unsere Ergebnisse, um zu untersuchen, wie Syzygien verwendet werden können, um den Grad der Irrationalität einer Varietät zu begrenzen.

Ferner, beweisen wir eine Vermutung von Gross and Popescu über Abelsche Flächen, deren Ideal durch Quadriken und Kubiken erzeugt wird. Außerdem verwenden wir die projektive Normalität einer Abelschen Fläche, um die Prym Abbildung, die mit zyklischen Überdeckungen von Geschlecht zwei Kurven assoziert ist, zu untersuchen. Wir zeigen, dass das Differential der Abbildung generisch injektiv ist, wenn der Grad der Überdeckung mindestens sieben ist. Wir dehnen damit Ergebnisse von Lange und Ortega aus. Abschließend zeigen wir, dass das Differential genau für bielliptische Überdeckungen nicht injectiv ist.

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Introduction

This thesis focuses on syzygies of projective varieties. This is a classical topic at the intersection of algebraic geometry and commutative algebra, and it studies the algebraic relations between the equations defining a variety in projective space. Counting the number of these relations and their degrees gives a collection of very fine numerical invariants which encode many of the properties of the variety.

Indeed, syzygies were originally introduced by Hilbert in order to compute Hilbert functions. In particular, we can read off them information such as the dimension and the degree of the variety. However, what is most interesting is that they encode subtler geometric properties, which go beyond the Hilbert function.

An example which illustrates nicely these properties is that of four points in the projective plane. There are three possible geometric configurations: no three of them are collinear, exactly three of them are collinear and all four are collinear. The last one is not really a configuration on the plane, and we can restrict our attention to the first two. We collect below the Hilbert functions of the coordinate rings and the syzygies of the defining ideals.

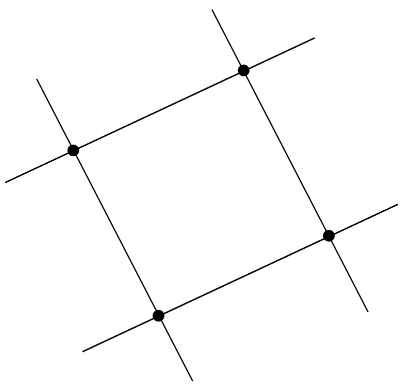

Figure 1: No three collinear points.

n	0	1	2	3	…
$H(n)$	1	3	4	4	…

	0	1
2	2	–
3	–	1

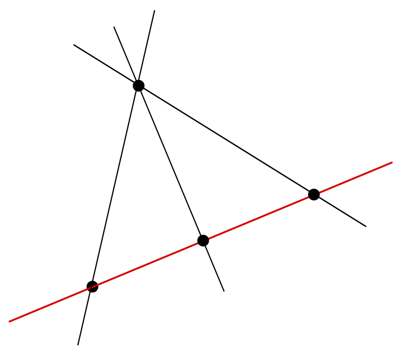

Figure 2: Three collinear points.

n	0	1	2	3	…
$H(n)$	1	3	4	4	…

	0	1
2	2	**1**
3	**1**	1

For each configuration, the table on the left gives the values of the Hilbert function and the other table represents the syzygies: for example, in the first case we see that the homogeneous ideal is generated by two quadrics, with a quadratic relation between them. We see immediately that the Hilbert function cannot distinguish between the two configurations, however the syzygies do. For example, we see that in the second case we need an extra cubic to generate the ideal, and moreover there is a linear relation between the quadrics.

In this spirit, the theory of syzygies studies relations between the algebra of the equations and the geometry of the corresponding variety: nice surveys on the topic are in [Eis05],[AN10],[SE11],[EL16],[Far17].

The main results of the thesis are about detecting special secant spaces to algebraic varieties through syzygies, in a way similar to the above example.

Another minor topic is that of homogeneous ideals of abelian surfaces: we study those generated by quadrics and cubics. Moreover, we apply syzygies of abelian surfaces to study moduli spaces of Prym varieties.

We now describe the results of the thesis in more detail. A word about the general setting and notation: we always work over the complex numbers and if V is a vector space, we denote by $\mathbb{P}(V)$ the projective space of one dimensional quotients of V. If X is a projective scheme and L a line bundle on it, we write $L \gg 0$ if this line bundle has the form $L = P \otimes A^{\otimes d}$, where P is an arbitrary line bundle, A is an ample line bundle and $d \gg 0$.

Asymptotic syzygies and higher order embeddings

We first introduce some notation for syzygies. Let X be a smooth complex projective variety and L an ample and globally generated line bundle: this gives a map $\phi_L\colon X \to \mathbb{P}(H^0(X,L))$ and we can regard the symmetric algebra $S = \operatorname{Sym}^\bullet H^0(X,L)$ as the ring of coordinates of $\mathbb{P}(H^0(X,L))$. For any line bundle B on X we can form a finitely generated graded S-module

$$\Gamma_X(B,L) \overset{\text{def}}{:=} \bigoplus_{q\in\mathbb{Z}} H^0(X, B\otimes L^{\otimes q})$$

and then take its *minimal free resolution*. It is a canonical exact complex of graded S-modules

$$0 \longrightarrow F_s \longrightarrow F_{s-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \Gamma_X(B,L) \longrightarrow 0$$

where the F_i are free graded S-modules of finite rank: F_0 represents the generators of $\Gamma_X(B,L)$, F_1 represents the relations among these generators, F_2 represents the relations among the relations and so on. Taking into account the various degrees, we have a decomposition

$$F_p = \bigoplus_{q\in\mathbb{Z}} K_{p,q}(X,B,L) \otimes_{\mathbb{C}} S(-p-q)$$

for some vector spaces $K_{p,q}(X, B, L)$, called *syzygy groups* or *Koszul cohomology groups*.

A famous open problem in the field of syzygies was the *Gonality Conjecture* of Green and Lazarsfeld [GL86]. It asserts that one can read the gonality of a smooth curve C off the syzygies $K_{h^0(C,L)-2-p,1}(C, \mathcal{O}_C, L)$, for $L \gg 0$.

This conjecture was confirmed for curves on Hirzebruch surfaces [Apr02] and on certain toric surfaces [Kaw08]. Most importantly it was proven for general curves by Aprodu and Voisin [AV03] and Aprodu [Apr04]. However, the conjecture for an arbitrary curve was left open, until Ein and Lazarsfeld recently gave a surprisingly quick proof [EL15], drawing on Voisin's interpretation of Koszul cohomology through the Hilbert scheme [Voi02]. More precisely, Ein and Lazarsfeld's result is a complete characterization of the vanishing of the asymptotic $K_{p,1}(C, B, L)$ in terms of *p-very ampleness*. If B is a line bundle on a smooth projective curve C, we say that B is *p-very ample* if for every effective divisor $\xi \subseteq C$ of degree $p+1$, the evaluation map

$$\mathrm{ev}_\xi \colon H^0(C, B) \longrightarrow H^0(C, B \otimes \mathcal{O}_\xi)$$

is surjective. Hence, 0-very ampleness is the same as global generation, 1-very ampleness is the usual notion of very ampleness, and for $p \geq 2$ a line bundle B is p-very ample if and only if the image of C under $\phi_B \colon C \hookrightarrow \mathbb{P}(H^0(C, B))$ has no $(p+1)$-secant $(p-1)$-planes.

Ein and Lazarsfeld proved the following [EL15, Theorem B]: let C be a smooth curve and B a line bundle. Then

$$K_{p,1}(C, B, L) = 0 \quad \text{for } L \gg 0 \qquad \text{if and only if} \qquad B \text{ is } p\text{-very ample.}$$

In particular, this implies the Gonality Conjecture: indeed, the syzygy group $K_{h^0(C,L)-p-2,1}(C, \mathcal{O}_C, L)$ is dual to $K_{p,1}(C, \omega_C, L)$ and Riemann-Roch shows that a curve C has gonality at least $p+2$ if and only if ω_C is p-very ample.

It is then natural to wonder about an extension of the result for curves in higher dimensions and this was explicitly asked by Ein and Lazarsfeld in [EL16, Problem 4.12] and by Ein, Lazarsfeld and Yang in [ELY16, Remark 2.2]. However, it is not a priori obvious how to generalize the statement, because the concept of p-very ampleness on curves can be extended to higher dimensions in at least three different ways, introduced by Beltrametti, Francia and Sommese in [BFS89].

The first one is by taking essentially the same definition: a line bundle B on a projective scheme X is *p-very ample* if for every finite subscheme $\xi \subseteq X$ of length $p+1$, the evaluation map

$$\mathrm{ev}_\xi \colon H^0(X, B) \to H^0(X, B \otimes \mathcal{O}_\xi)$$

is surjective. If instead we require that the evaluation map ev_ξ is surjective only for curvilinear schemes, the line bundle B is said to be *p-spanned*. Recall that a finite subscheme $\xi \subseteq X$ is *curvilinear* if it is locally contained in a smooth curve, or, more precisely, if $\dim T_P\xi \leq 1$ for all $P \in \xi$. The third extension is the stronger concept of jet very ampleness: a line bundle B on a projective scheme X is called *p-jet very ample* if for every zero cycle $\zeta = a_1P_1 + \cdots + a_rP_r$ of degree $p+1$ the evaluation map

$$\mathrm{ev}_\zeta \colon H^0(X,B) \to H^0(X, B \otimes \mathcal{O}_X/\mathfrak{m}_\zeta), \qquad \mathfrak{m}_\zeta \stackrel{\text{def}}{:=} \mathfrak{m}_{P_1}^{a_1} \ldots \mathfrak{m}_{P_r}^{a_r}$$

is surjective.

It is straightforward to show that p-jet very ampleness implies p-very ampleness, which in turn implies p-spannedness. Moreover, these three concepts coincide on smooth curves, but this is not true anymore in higher dimensions: for arbitrary varieties, they coincide only when $p = 0$ or 1, and they correspond to the usual notions of global generation and very ampleness. Instead, jet very ampleness is stronger than very ampleness as soon as $p \geq 2$.

The question is how these notions of higher order embeddings relate to the asymptotic vanishing of syzygies. This was addressed by Ein, Lazarsfeld and Yang in [ELY16]. They prove in [ELY16, Theorem B] that if X is a smooth projective variety and $K_{p,1}(X,B,L) = 0$ for $L \gg 0$, then the evaluation map $\mathrm{ev}_\xi \colon H^0(X,B) \to H^0(X, B \otimes \mathcal{O}_\xi)$ is surjective for all finite subschemes $\xi \subseteq X$ consisting of $p+1$ distinct points. For the converse, they prove in [ELY16, Theorem A], that if B is p-jet very ample, then $K_{p,1}(X,B,L) = 0$ for $L \gg 0$. In particular, it follows that there is a perfect analog of the result for curves in higher dimensions and $p = 0,1$. However, it is not clear from this whether the statement should generalize to higher p, since in the range $p = 0,1$ spannedness, very ampleness and jet very ampleness coincide.

Our first main theorem is that one implication of the case of curves generalizes in any dimension with p-very ampleness, even for singular varieties. Indeed, the result holds for an arbitrary projective scheme so that it strengthens considerably [ELY16, Theorem B]. Moreover, we can also give an effective result in the case of p-spanned line bundles.

Theorem A. *Let X be a projective scheme and B a line bundle on X.*

If $K_{p,1}(X,B,L) = 0$ *for* $L \gg 0$ *then* *B is p-very ample.*

Moreover, suppose that X is smooth and irreducible of dimension n and let L be a line bundle of the form

$$L = \omega_X \otimes A^{\otimes d} \otimes P^{\otimes (n-1)} \otimes N, \qquad d \geq (n-1)(p+1) + p + 3,$$

where A is a very ample line bundle, P a globally generated line bundle such that $P \otimes B^\vee$ is nef and N a nef line bundle such that $N \otimes B$ is nef. For such a line bundle, it holds that

if $K_{p,1}(X,B,L) = 0$ *then* *B is p-spanned.*

Our second main theorem is that on smooth surfaces we have a perfect analog of the situation for curves, at least when p is small. In particular, this extends the results of [EL15; ELY16].

Theorem B. *Let X be a smooth and irreducible projective surface, B a line bundle and $0 \leq p \leq 3$ an integer. Then*

$$K_{p,1}(X, B, L) = 0 \quad \text{for } L \gg 0 \qquad \text{if and only if} \qquad B \text{ is } p\text{-very ample.}$$

As an application of these results, we generalize part of the Gonality Conjecture to higher dimensions. More precisely, we show how to use syzygies to bound some measures of irrationality discussed recently by Bastianelli, De Poi, Ein, Lazarsfeld and Ullery [Bas+17b]. If X is an irreducible projective variety, the *covering gonality* of X is the minimal gonality of a curve C passing through a general point of X. Instead, the *degree of irrationality* of X is the minimal degree of a dominant rational map $f\colon X \dashrightarrow \mathbb{P}^{\dim X}$. Our result is the following.

Corollary C. *Let X be a smooth and irreducible projective variety of dimension n and suppose that $K_{h^0(X,L)-1-n-p,n}(X, \mathcal{O}_X, L)$ vanishes for $L \gg 0$. Then the covering gonality and the degree of irrationality of X are at least $p+2$.*

In addition, we show that it is enough to check the syzygy vanishing of Corollary C for a single line bundle L in the explicit form of Theorem A. Since syzygies are explicitly computable, this gives in principle an effective way to bound the irrationality of a variety, using for example a computer algebra program.

We briefly describe our strategy. We prove Theorem A by essentially reducing to the case of finite subschemes of projective space. Corollary C follows from Theorem A by adapting some arguments of Bastianelli et al. [Bas+17b].

For Theorem B instead, we work on the Hilbert scheme of points of a smooth surface. The key point is given by some cohomological vanishings for tautological bundles on the Hilbert scheme. To prove these vanishings, we interpret them in the light of the Bridgeland-King-Reid-Haiman correspondence for $X^{[n]}$, introduced by Haiman [Hai02] and further developed by Scala [Sca09] and Krug [Kru14; Kru16]. This correspondence describes the derived category of the Hilbert scheme in terms of the equivariant derived category of the cartesian product. We remark that Yang has already used this correspondence to study Koszul cohomology in [Yan14]. With these tools, we are able to verify the desired vanishing statements for p at most 3, proving Theorem B. We actually believe that these vanishings should hold for every value of p, but they become quickly very hard to check. We include some comments about a possible strategy to attack the problem and we argue that this is essentially a combinatorial statement on the ring $\mathbb{C}[x, y]$.

Equations of abelian surfaces and Prym varieties

A minor subject of this thesis is the study of equations of abelian surfaces. Abelian varieties are projective varieties that have at the same time the structure of an algebraic group and they are ubiquitous in algebraic geometry. They have a complete and explicit description as complex varieties, however this description is inherently transcendental and it is often hard to translate into explicit equations. Hence, there has been much work in trying to understand the qualitative structure of their equations and syzygies: some examples are [Mum66],[Par00],[GP98].

In particular, in their paper [GP98] Gross and Popescu proved that if (A, L) is a general polarized abelian surface of type $(1, d)$ with $d \geq 10$, then its homogeneous ideal in the embedding $A \hookrightarrow \mathbb{P}(H^0(A, L))$ is generated by quadrics. At the end of the same paper, they conjectured that if L is any very ample line bundle of type $(1, d)$ with $d \geq 9$, then the homogeneous ideal of A in the embedding $A \hookrightarrow \mathbb{P}(H^0(A, L))$ is generated by quadrics and cubics.

This result was already proven for $d = 7$ by Manolache and Schreyer in [MS01, Corollary 2.2], where they show that the ideal is generated by cubics and compute the whole minimal free resolution. The case $d = 8$ was proven by Gross and Popescu [GP01, Theorem 6.13] for a general abelian surface and the cases $d \geq 23$ were recently proved by Küronya and Lozovanu [KL15, Theorem 1.3].

In this thesis, we give a complete proof of Gross and Popescu's conjecture, extending it to other types of polarizations.

Theorem D. *Let $A \hookrightarrow \mathbb{P}(H^0(A, L))$ be an abelian surface embedded by a complete linear system not of type $(1, 5)$, $(1, 6)$ or $(2, 4)$. Then its homogeneous ideal is generated by quadrics and cubics.*

Furthermore, we use syzygies of abelian surfaces to study Prym varieties associated to cyclic covers of genus two curves. Recall that, if $f : C \to D$ is finite cover of smooth projective curves, we define the associated *Prym variety* as the kernel of the induced norm map:

$$\operatorname{Prym}(C \to D) \stackrel{\text{def}}{:=} \operatorname{Ker}\left[\operatorname{Nm}(f) \colon \operatorname{Pic}^0(C) \longrightarrow \operatorname{Pic}^0(D)\right].$$

Prym varieties are a classic subject of algebraic geometry, and they have been intensely studied, especially in the case of étale double covers: we refer to [BL04, Chapter 12], [Bea89] and [Far12] for an overview of the topic.

In recent years, Lange and Ortega [Ort03],[LO10],[LO16],[LO18] have studied Prym varieties associated to cyclic étale covers of genus two curves. More precisely, let $C \to D$ be a cyclic étale cover of degree d of a genus two curve. Then the corresponding Prym variety has a natural polarization, obtained as the restriction of the natural principal polarization on the Jacobian $\operatorname{Pic}^0(C)$. It turns out that the type δ of the polarization depends only on the degree d of the cover, hence we get a

Prym map

$$\mathrm{Pr}\colon \mathcal{R}_{2,d} \longrightarrow \mathcal{A}_{\delta}, \qquad [C \to D] \mapsto [\mathrm{Prym}(C \to D)]$$

from the moduli space of cyclic covers of degree d of a genus two curve, to the moduli space of abelian varieties with a polarization of type δ.

In particular, Lange an Ortega proved in [LO10] that the differential of the Prym map for $d = 7$ is injective at a general point, so that the map is generically finite onto its image. Here, we use syzygies of abelian surfaces to extend this result to $d \geq 7$ and we moreover characterize the covers where the differential of the Prym map is not injective.

Theorem E. *The differential of the Prym map* $\mathrm{Pr}\colon \mathcal{R}_{2,d} \longrightarrow \mathcal{A}_{\mathcal{D}}$ *is injective at a cyclic cover in* $\mathcal{R}_{2,d}$ *if and only if* $d \geq 7$ *and the cover is not bielliptic. In particular, the Prym map is generically finite onto its image for* $d \geq 7$.

The key ingredient for the proofs of both Theorem D and Theorem E is a result of Koizumi [Koi76], Ohbuchi [Ohb93], Lazarsfeld [Laz90] and Fuentes García [Gar04] that gives a full classification of projective normality for polarized abelian surfaces. Having this, it is straightforward to obtain Theorem D. For Theorem E, we first describe a construction that associates to a cyclic cover $[C \to D] \in \mathcal{R}_{2,d}$ a polarized abelian surface of type $(1, d)$. Then the theorem is a consequence of the projective normality of this abelian surface.

Structure of the thesis

We briefly describe the structure of the thesis. In Chapter 1 we give some general background on syzygies and on the Hilbert scheme of points on a smooth surface. In particular, we introduce equivariant derived categories and the Bridgeland-King-Reid-Haiman correspondence for the derived category of the Hilbert scheme. Chapter 2 is devoted to the proof of Theorem A, Theorem B and Corollary C. We also discuss in more detail higher order embeddings and measures of irrationality for algebraic varieties. Finally, in Chapter 3 we review some facts about abelian varieties and Prym varieties, we present the result about projective normality of abelian surfaces and we use it to prove Theorem D and Theorem E.

References

The thesis is based on the two papers:

- D. Agostini, *A note on homogeneous ideals of polarized abelian surfaces*, Bulletin of the London Mathematical Society, vol. 49 (2017), 220-225.
- D. Agostini, *Asymptotic syzygies and higher order embeddings*, arXiv:1706.03508.

Chapter 1

Background

In this chapter we give some background on syzygies and on the Hilbert scheme of points on a smooth surface. In particular, we discuss equivariant derived categories and we present the Bridgeland-King-Reid-Haiman correspondence for the Hilbert scheme of points.

1.1 Minimal free resolutions and syzygies

Let V be a complex vector space of finite dimension $r+1$ and let $S = \operatorname{Sym}^{\bullet} V$ be its symmetric algebra, endowed with the standard grading. We can look at S as a coordinate-free version of a polynomial ring in $r+1$ variables.

Let also M be a finitely generated graded S-module. We can regard a choice of a finite set of generators for M as an exact complex

$$F_0 \longrightarrow M \longrightarrow 0$$

where F_0 is a free S-module of finite rank. The kernel of the map $F_0 \longrightarrow M$ consists of the relations, or *syzygies*, between the generators, and it is finitely generated itself because S is noetherian. Hence, choosing a finite set of generators of the kernel, we can extend the previous complex to another exact complex

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where F_1 is again free of finite rank. Of course, we can continue, and we obtain a *free resolution* of M, which is an exact complex

$$\dots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where each F_i is free of finite rank. This algorithm works for every finitely generated module over a noetherian ring, but in the case of a polynomial ring, a fundamental result of Hilbert tells us that we can actually obtain a *finite* free resolution.

Theorem 1.1.1 (Hilbert Syzygy Theorem). *Every finitely generated graded S-module M admits a free resolution of length at most $r+1$. This is an exact complex of graded S-modules*

$$0 \longrightarrow F_\ell \longrightarrow F_{\ell-1} \longrightarrow \dots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where the modules F_p are free of finite rank and $\ell \leq r+1$.

Proof. See [Eis05, Theorem 1.1]. □

Moreover, if at each step of the resolution we choose a minimal set of generators, the complex that we obtain is fundamentally unique and it is a canonical object associated to the module.

Theorem 1.1.2. *Up to isomorphism, there is a unique free resolution of M*

$$0 \longrightarrow F_\ell \longrightarrow F_{\ell-1} \longrightarrow \dots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where the modules F_p have minimal rank.

Proof. See [Eis04, Theorem 20.2]. □

Definition 1.1.3 (Minimal free resolution). The finite free resolution of a finitely generated module M, where the free modules are of minimal rank, is called the *minimal free resolution* of M.

As an example, we consider the configurations of four points in the plane that we have seen in the Introduction.

Example 1.1.4 (Four points in the plane). Consider the projective plane $\mathbb{P}^2$ with coordinates x_0, x_1, x_2. The symmetric algebra $S = \operatorname{Sym}^\bullet H^0(\mathbb{P}^2, \mathcal{O}(1))$ coincides with the polynomial ring $\mathbb{C}[x_0, x_1, x_2]$.

Take the set of points in general position $Z_1 = \{[1,0,0],[0,1,0],[0,0,1],[1,1,1]\}$ and let $I_{Z_1} \subseteq S$ be its homogeneous ideal. The set Z_1 is the complete intersection of the two quadrics $q_1 = x_0(x_1 - x_2), q_2 = x_1(x_2 - x_0)$, and the only relation between q_1 and q_2 is the trivial one: $x_1(x_2 - x_0)q_1 - x_0(x_1 - x_2)q_2 = 0$. Hence, the minimal free resolution of I_{Z_1} is

$$0 \longrightarrow S(-4) \xrightarrow{\begin{pmatrix} x_1(x_2 - x_0) \\ -x_0(x_1 - x_2) \end{pmatrix}} S(-2)^{\oplus 2} \xrightarrow{\begin{pmatrix} x_0(x_1 - x_2) & x_1(x_2 - x_0) \end{pmatrix}} I_{Z_1} \longrightarrow 0.$$

Consider instead the set of points $Z_2 = \{[1,0,0],[0,1,0],[0,0,1],[0,1,1]\}$. These are not in general position, because three of the points are collinear. In this case, the ideal I_{Z_2} is minimally generated by two quadrics $f_1 = x_0x_1, f_2 = x_0x_2$ and one cubic $g = x_1x_2(x_1 - x_2)$. There is a linear relation between the quadrics: $x_2 f_1 - x_1 f_2 = 0$. There is also another relation that involves all three generators:

$x_2^2 f_1 - x_1^2 f_2 + x_0 g = 0$. There is no further relation, so that the minimal free resolution of I_{Z_2} is

$$0 \longrightarrow S(-3) \oplus S(-4) \xrightarrow{\begin{pmatrix} x_2 & x_2^2 \\ -x_1 & -x_1^2 \\ 0 & x_0 \end{pmatrix}} S(-2)^{\oplus 2} \oplus S(-3) \xrightarrow{\begin{pmatrix} x_0x_1 & x_0x_2 & x_1x_2(x_1-x_2) \end{pmatrix}} I_{Z_2} \longrightarrow 0$$

Let us continue with the general discussion. Let M be a finitely generated graded S-module and consider its minimal free resolution:

$$0 \longrightarrow F_\ell \longrightarrow F_{\ell-1} \longrightarrow \ldots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

Since the F_p are free modules, we can write them as

$$F_p = \bigoplus_{q \in \mathbb{Z}} K_{p,q}(M;V) \otimes_{\mathbb{C}} S(-p-q)$$

for certain vector spaces $K_{p,q}(M;V)$ that count the multiplicity of the part of degree $p+q$ in F_p. Following [EL12], we will denote $K_{p,q}(M;V)$ as the group of *p-th syzygies of weight* q of M with respect to V. Since the minimal free resolution is unique, the dimensions of these spaces give a collection of numerical invariants of M.

Definition 1.1.5 (Graded Betti numbers and Betti table)**.** Let M be a finitely generated graded S-module. The numbers

$$b_{p,q}(M) :\stackrel{\text{def}}{=} \dim_{\mathbb{C}} K_{p,q}(M;V)$$

are called the *graded Betti numbers* of M with respect to V. They are usually collected in the *Betti table* of M:

	0	1	2	...	p
$\vdots$					
q	$b_{0,q}$	$b_{1,q}$	$b_{2,q}$	...	$b_{p,q}$
$q+1$	$b_{0,q+1}$	$b_{1,q+1}$	$b_{2,q+1}$	...	$b_{p,q+1}$
$\vdots$					

Example 1.1.6 (Four points in the plane - II)**.** Let us consider again Example 1.1.4 of four points in the plane. When the four points are in general position, we see that the Betti table of the the homogeneous ideal is

	0	1	2
2	2	−	−
3	−	1	−

whereas when there are three collinear points the Betti table is

	0	1	2
2	2	1	–
3	1	1	–

The graded Betti numbers are very fine numerical invariants of the module M. For example, they can be used to compute the Hilbert function of M:

Corollary 1.1.7. *Let M be a finitely generated and graded S-module and let $b_{p,q} = b_{p,q}(M)$ be its graded Betti numbers. The alternate sums $B_j = \sum_{i=0}^{\infty}(-1)^p b_{i,j-i}$ determine the Hilbert function of M via the formula*

$$H_M(n) = \dim_{\mathbb{C}} M_n = \sum_j B_j \binom{r+n-j}{r}$$

Proof. See [Eis05, Corollary 1.2]. □

Remark 1.1.8. Using the formula of Corollary 1.1.7, one can show that the Hilbert function is eventually polynomial [Eis05, Corollary 1.3], and indeed this is the reason why Hilbert proved his Syzygy Theorem.

Since the graded Betti numbers can compute the Hilbert function of a module, they can also compute various data, such as the dimension or the degree of the module. However, as remarked in the Introduction, the graded Betti numbers can detect more subtle properties of the module M.

Example 1.1.9 (Four points in the plane - III)**.** Let us consider again four points in the projective plane as in Examples 1.1.4 and 1.1.6. As we have noted before, we can use the graded Betti numbers to compute the Hilbert function of I_{Z_1} and I_{Z_2}. Equivalently, we can compute the Hilbert function of the respective coordinate rings, and in both cases we get the same result, namely

$$H(0) = 1, \qquad H(1) = 3, \qquad H(n) = 4 \qquad \text{for all } n \geq 4.$$

Hence, the two ideals I_{Z_1} and I_{Z_2} have the same Hilbert function. However, they have different geometries and this is detected by the Betti numbers.

1.1.1 Koszul cohomology

Koszul cohomology gives a useful way to compute the syzygies in the minimal free resolution of a module. The key observation is the following: let M be a finitely generated graded S-module with a free resolution

$$0 \longrightarrow F_\ell \xrightarrow{\phi_\ell} F_{\ell-1} \longrightarrow \ldots \longrightarrow F_2 \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M \longrightarrow 0.$$

By [Eis05, Theorem 1.6] the resolution is minimal if and only if the image of ϕ_p is contained in S_+F_{p-1}, where $S_+ \subseteq S$ is the homogeneous maximal ideal. Indeed, this property is usually taken as the definition of a minimal resolution [AN10],[Eis05]. We can also rephrase this by saying that, if we tensor the resolution by the S-module $\mathbb{C} = S/S_+$, the maps $\phi_p \otimes_S \mathbb{C}$ are zero. Hence, by the properties of the Tor functor, we get that

$$F_p \cong \mathrm{Tor}_p^S(M, \mathbb{C}), \qquad K_{p,q}(M;V) \cong \mathrm{Tor}_p^S(M,\mathbb{C})_{p+q}.$$

Since the Tor functor is symmetric, we can compute $\mathrm{Tor}_p^S(M,\mathbb{C})$ also by taking a resolution of $\mathbb{C}$ and tensoring it by M. The minimal free resolution of $\mathbb{C}$ is well-known:

Definition 1.1.10 (Koszul complex)**.** The Koszul complex is the complex

$$0 \longrightarrow \wedge^{r+1} V \otimes_{\mathbb{C}} S(-r-1) \longrightarrow \ldots \longrightarrow \wedge^2 V \otimes_{\mathbb{C}} S(-2) \longrightarrow V \otimes_{\mathbb{C}} S(-1) \longrightarrow S \longrightarrow \mathbb{C} \longrightarrow 0$$

where the maps are given by

$$\begin{aligned} d_p \colon \wedge^p V \otimes_{\mathbb{C}} S(-p) &\longrightarrow \wedge^{p-1} V \otimes S(-p+1), \\ v_1 \wedge \cdots \wedge v_p \otimes f &\mapsto \sum_{i=1}^{p} (-1)^i v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_p \otimes v_i \cdot f \end{aligned}$$

Theorem 1.1.11. *The Koszul complex is the minimal free resolution of the S-module $\mathbb{C} = S/S_+$.*

Proof. See [Eis05, Example 2.6] or [AN10, Corollary 1.6]. □

If we tensor the Koszul complex by the module M, we see that the syzygy groups $K_{p,q}(M;V)$ can be computed as the middle cohomology of the Koszul-type complex

$$\wedge^{p+1} V \otimes_{\mathbb{C}} M_{q-1} \xrightarrow{d_{p+1,q-1}} \wedge^p V \otimes_{\mathbb{C}} M_q \xrightarrow{d_{p,q}} \wedge^{p-1} V \otimes_{\mathbb{C}} M_{q+1} \tag{1.1.1}$$

where the differentials are given by

$$\begin{aligned} d_{p,q} \colon \wedge^p V \otimes_{\mathbb{C}} M_q &\longrightarrow \wedge^{p-1} V \otimes_{\mathbb{C}} M_{q+1} \\ v_1 \wedge \cdots \wedge v_p \otimes m &\mapsto \sum_{i=1}^{p} (-1)^i v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_p \otimes v_i \cdot m. \end{aligned}$$

For this reason, we also call the groups $K_{p,q}(M;V)$, the *Koszul cohomology groups* of M with respect to V.

An immediate consequence of interpreting the Koszul cohomology groups using the Tor functor is the long exact sequence in Koszul cohomology:

Lemma 1.1.12. *Let*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be a short exact sequence of finitely generated graded S-modules. Then, we have a long exact sequence in Koszul cohomology

$$\dots \longrightarrow K_{p,q}(M_1;V) \longrightarrow K_{p,q}(M_2;V) \longrightarrow K_{p,q}(M_3;V) \longrightarrow K_{p-1,q+1}(M_1;V) \longrightarrow \dots$$

Proof. This is just a translation of the usual long exact sequence for the functor Tor. See [Gre84, Corollary (1.d.4)] or [AN10, Lemma 1.22] for more details. □

Syzygies with respect to different rings

Sometimes it is useful to compute the syzygies of a module with respect to two different polynomial rings. More precisely, suppose that we have a short exact sequence of vector spaces

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

and a finitely generated graded $\operatorname{Sym}^\bullet W$-module M. Then M is also finitely generated as a $\operatorname{Sym}^\bullet V$-module, and we can compare the Koszul cohomologies computed with respect to W and V.

Lemma 1.1.13. *In the above situation there is an induced map*

$$K_{p,q}(M;V) \longrightarrow K_{p,q}(M;W).$$

Proof. Let $f\colon V \to W$ be the surjective map of above. We then have other surjective maps $\wedge^p f\colon \wedge^p V \to \wedge^p W$, which fit in a commutative diagram

$$\begin{array}{ccccc} \wedge^{p+1}V \otimes_{\mathbb{C}} M_{q-1} & \longrightarrow & \wedge^{p}V \otimes_{\mathbb{C}} M_{q} & \longrightarrow & \wedge^{p-1}V \otimes_{\mathbb{C}} M_{q+1} \\ \downarrow{\scriptstyle \wedge^{p+1}f\otimes \mathrm{id}} & & \downarrow{\scriptstyle \wedge^{p}f\otimes \mathrm{id}} & & \downarrow{\scriptstyle \wedge^{p-1}f\otimes \mathrm{id}} \\ \wedge^{p+1}W \otimes_{\mathbb{C}} M_{q-1} & \longrightarrow & \wedge^{p}W \otimes_{\mathbb{C}} M_{q} & \longrightarrow & \wedge^{p-1}W \otimes_{\mathbb{C}} M_{q+1}. \end{array}$$

This diagram is a morphism between the Koszul complexes of M with respect to V and W. Hence, we get an induced map in Koszul cohomology $K_{p,q}(M;V) \to K_{p,q}(M;W)$. □

Actually, the previous map is surjective. This is a consequence of the following more general result, which is well-known and whose proof can be found for example in [AKL17, Lemma 2.1]. We include a proof also here for completeness.

Lemma 1.1.14. *In the above situation, we have a non-canonical decomposition*

$$K_{p,q}(M;V) \cong \bigoplus_{i=0}^{p} \wedge^{p-i} U \otimes K_{i,q}(M;W)$$

which respects the natural map $K_{p,q}(M;V) \to K_{p,q}(M;W)$ of Lemma 1.1.13. In particular, this map is surjective.

Proof. Fix a splitting $V \cong U \oplus W$. Then $\wedge^p V = \bigoplus_{i=0}^{p} \wedge^{p-i} U \otimes \wedge^i W$ and the Koszul complex behaves well with respect to this splitting. Indeed, since $U \subseteq \mathrm{Ann}(M)$, we see that for all $u_1, \ldots, u_{p-1} \in U$, $w_1, \ldots, w_i \in W$ and $m \in M_q$ we have

$$\begin{aligned}
&d(u_1 \wedge \cdots \wedge u_{p-i} \wedge w_1 \wedge \cdots \wedge w_i \otimes m) \\
&\quad = \sum_{j=0}^{p-i} (-1)^j u_1 \wedge \cdots \wedge \widehat{u_j} \wedge \cdots \wedge u_{p-i} \wedge w_1 \wedge \cdots \wedge w_i \otimes u_i \cdot m \\
&\quad + u_1 \wedge \cdots \wedge u_{p-i} \wedge \left(\sum_{j=0}^{i} (-1)^{p-i+j} \wedge w_1 \wedge \cdots \wedge \widehat{w_k} \wedge \cdots \wedge w_i \otimes w_k \cdot m \right) \\
&\quad = (-1)^{p-i} u_1 \wedge \cdots \wedge u_{p-i} \wedge \left(\sum_{j=0}^{i} (-1)^j \wedge w_1 \wedge \cdots \wedge \widehat{w_k} \wedge \cdots \wedge w_i \otimes w_k \cdot m \right).
\end{aligned}$$

Thus, the Koszul complex of M with respect to V, as well as the Koszul cohomology, split. From the same computations, we also see that the induced projection $K_{p,q}(M;V) \cong \bigoplus_{i=0}^{p} \wedge^{p-i} U \otimes K_{i,q}(M;W) \to K_{p,q}(M;W)$ coincides with the map of Lemma 1.1.13, which is in particular surjective. □

1.1.2 Koszul cohomology in geometry

The language of Koszul cohomology was introduced in the field of syzygies by Green [Gre84]. It is essentially a reformulation of the usual terminology of the Tor functor, but it turns out that it is particularly well suited to geometric situations. We present here some definitions and results about Koszul cohomology in a geometric context.

Let X be a projective scheme, L an ample and globally generated line bundle on X and $V \subseteq H^0(X,L)$ a base-point-free subspace. Then V defines a morphism $\phi_V \colon X \longrightarrow \mathbb{P}(V)$, which is finite [Laz04, Corollary 1.2.15], and we can look at the symmetric algebra $S = \mathrm{Sym}^\bullet V$ as the ring of homogeneous coordinates of the projective space $\mathbb{P}(V)$. For any coherent sheaf $\mathcal{F}$ on X. We can form the associated group of sections

$$\Gamma_X(\mathcal{F},L) \stackrel{\text{def}}{:=} \bigoplus_{q\in\mathbb{Z}} H^0(X, \mathcal{F} \otimes L^{\otimes q})$$

which has a natural structure of graded $\mathrm{Sym}^\bullet V$-module. Hence, if $\Gamma_X(\mathcal{F}, L)$ is finitely generated, we can consider its minimal free resolution with respect to V and define the Koszul cohomology groups

$$K_{p,q}(X, \mathcal{F}, L; V) \stackrel{\text{def}}{:=} K_{p,q}(\Gamma_X(\mathcal{F}, L), V).$$

Moreover, when $V = H^0(X, L)$ or $\mathcal{F} = \mathcal{O}_X$ we define for simplicity

$$\begin{aligned} K_{p,q}(X, \mathcal{F}, L) &\stackrel{\text{def}}{:=} K_{p,q}(X, \mathcal{F}, L; H^0(X, L)), \\ K_{p,q}(X, L; V) &\stackrel{\text{def}}{:=} K_{p,q}(X, \mathcal{O}_X, L; V), \\ K_{p,q}(X, L) &\stackrel{\text{def}}{:=} K_{p,q}(X, L; H^0(X, L)). \end{aligned}$$

By (1.1.1), the group $K_{p,q}(X, \mathcal{F}, L; V)$ can also be computed as the middle cohomology of the Koszul-type complex

$$\wedge^{p+1} V \otimes H^0(X, \mathcal{F} \otimes L^{q-1}) \to \wedge^p V \otimes H^0(X, \mathcal{F} \otimes L^q) \to \wedge^{p-1} V \otimes H^0(X, \mathcal{F} \otimes L^{q+1}). \tag{1.1.2}$$

Remark 1.1.15. In the above situation, consider the pushforward $\phi_{V,*}\mathcal{F}$: this is a coherent sheaf on $\mathbb{P}(V)$ and there is a canonical isomorphism of $\mathrm{Sym}^\bullet V$-modules

$$\begin{aligned} \Gamma_{\mathbb{P}(V)}(\phi_{V,*}\mathcal{F}, \mathcal{O}(1)) &= \bigoplus_q H^0(\mathbb{P}(V), \phi_{V,*}\mathcal{F}(q)) \\ &\cong \bigoplus_q H^0(X, \mathcal{F} \otimes L^q) = \Gamma_X(\mathcal{F}, L). \end{aligned}$$

This yields canonical isomorphisms $K_{p,q}(X, \mathcal{F}, L; V) \cong K_{p,q}(\mathbb{P}(V), \phi_{V,*}\mathcal{F}, \mathcal{O}(1))$ so that all the theory of Koszul cohomology can be reduced to the case of coherent sheaves on projective spaces. However, it is often useful to use the more general language, in order to exploit properties of the variety X.

We briefly discuss how the geometric version of Koszul cohomology compares with the usual syzygies of homogeneous ideals.

Example 1.1.16 (Projective normality and homogeneous ideals)**.** Let X be a projective scheme, L an ample and globally generated line bundle on X and $\phi_L \colon X \longrightarrow \mathbb{P}^r$ the map induced by the complete linear system $H^0(X, L)$. We have an exact sequence of sheaves on $\mathbb{P}^r$

$$0 \longrightarrow \mathcal{I}_{X,\mathbb{P}^r} \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

where we identify $\mathcal{O}_X$ with $\phi_{L,*}\mathcal{O}_X$. Assume that $H^0(X, L^q) = 0$ for all $q < 0$: for example this is true as soon as X is integral of positive dimension. Twisting the previous exact sequence by $\mathcal{O}_{\mathbb{P}^r}(q)$ and taking global sections we get an exact sequence

$$0 \longrightarrow I_X \longrightarrow S \longrightarrow \Gamma_X(\mathcal{O}_X, L),$$

where $S = \operatorname{Sym}^\bullet V$ and $I_X \subseteq S$ is the homogeneous ideal of the image of X in $\mathbb{P}^r$. Hence, if the map $S \to \Gamma_X(\mathcal{O}_X, L)$ is surjective, the module $\Gamma_X(\mathcal{O}_X, L)$ coincides with the ring of homogeneous coordinates S/I_X. The condition that the map $S \to \Gamma_X(\mathcal{O}_X, L)$ is surjective is usually called *projective normality* and it can be phrased in terms of Koszul cohomology by saying that $K_{0,q}(X, L) = 0$ for all $q \geq 2$. If the map $\phi_L \colon X \to \mathbb{P}^r$ is projectively normal, then it is automatically an embedding [Mum70, page 38]. Moreover, in this case we can compute the syzygies of the ideal I_X from the Koszul cohomology on X:

$$K_{p,q}(I_X; H^0(X, L)) \cong K_{p+1,q}(X, L).$$

To conclude this part, we comment briefly on the assumption that the module of sections is finitely generated.

Remark 1.1.17. We have defined the syzygy groups $K_{p,q}(X, \mathcal{F}, L; V)$ under the assumption that $\Gamma_X(\mathcal{F}, L)$ is finitely generated as a $\operatorname{Sym}^\bullet V$ -module. It turns out that finite generation is equivalent to the fact that the sheaf $\mathcal{F}$ has no associated closed points. This is a well known fact [Eis05, page 67], but we give a proof in Lemma 1.1.18 since we were unable to find a reference.

However, the finite generation is not needed to define the Koszul cohomology groups. Indeed, for any coherent sheaf $\mathcal{F}$, we can always define $K_{p,q}(X, \mathcal{F}, L; V)$ as the middle cohomology of the Koszul complex (1.1.2). Moreover, we see that this is consistent with the previous definitions: for each $q \in \mathbb{Z}$, the truncated $\operatorname{Sym}^\bullet V$-module

$$\Gamma_X(\mathcal{F}, L)_{\geq q-1} \overset{\text{def}}{:=} \bigoplus_{h \geq q-1} H^0(X, \mathcal{F} \otimes L^h)$$

is finitely generated [Eis05, p. 67] and it follows from the Koszul complex (1.1.1) that

$$K_{p,q}(X, \mathcal{F}, L) = K_{p,q}(\Gamma_X(\mathcal{F}, L)_{\geq q-1}, H^0(X, L)).$$

We thank Fabio Tonini for a discussion regarding the next lemma.

Lemma 1.1.18. *Let X be a projective scheme, L an ample and globally generated line bundle, $V \subseteq H^0(X, L)$ a base-point-free subspace and $\mathcal{F}$ a coherent sheaf on X. Then $\Gamma_X(\mathcal{F}, L)$ is finitely generated as a* $\operatorname{Sym}^\bullet V$*-module if and only if $\mathcal{F}$ has no associated closed points.*

Proof. First, we observe that since the truncations $\Gamma_X(\mathcal{F}, L)_{\geq q}$ are finitely generated for every $q \in \mathbb{Z}$, the module $\Gamma_X(\mathcal{F}, L)$ is finitely generated if and only if we have the vanishing $H^0(X, \mathcal{F} \otimes L^q) = 0$ for $q \ll 0$. We want to prove that this happens if and only if $\mathcal{F}$ has no associated closed points. We can also assume that $X = \mathbb{P}^r$ and $L = \mathcal{O}_{\mathbb{P}^r}(1)$. Indeed, consider the map $\phi_L \colon X \to \mathbb{P}(H^0(X, L)) = \mathbb{P}^r$ and the coherent sheaf $\phi_{L,*}\mathcal{F}$ on $\mathbb{P}^r$: then $H^0(X, \mathcal{F} \otimes L^q) \cong H^0(\mathbb{P}^r, \phi_{L,*}\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(q))$, and $\mathcal{F}$ has a closed associated point if and only if $\phi_{L,*}\mathcal{F}$ has a closed associated point.

Assume first that $\mathcal{F}$ has an associated closed point P. Then there is an inclusion

$$\mathcal{O}_P \hookrightarrow \mathcal{F}(n)$$

for a certain $n \in \mathbb{Z}$, and twisting by $\mathcal{O}_{\mathbb{P}^r}(q)$ and taking global sections, we see that $H^0(\mathbb{P}^r, \mathcal{F}(q+n)) \supseteq H^0(\mathbb{P}^r, \mathcal{O}_P(q)) \neq 0$ for all $q \in \mathbb{Z}$.

Conversely, suppose that $H^0(\mathbb{P}^r, \mathcal{F}(q)) \neq 0$ for infinitely many $q < 0$. Then we want to show that $\mathcal{F}$ has an associated closed point. We proceed by induction on the dimension r of the projective space. If $r = 0$, then $\mathbb{P}^0$ is a single point so that every nonzero coherent sheaf has an associated closed point. Now let $r > 0$ and suppose that the statement holds for $r-1$. Assume for the moment that the multiplication map

$$m\colon H^0(\mathbb{P}^r, \mathcal{O}(1)) \otimes H^0(\mathbb{P}^r, \mathcal{F}(q-1)) \longrightarrow H^0(\mathbb{P}^r, \mathcal{F}(q)) \qquad (1.1.3)$$

is injective on each factor for all $q \ll 0$. In particular, $h^0(\mathbb{P}^r, \mathcal{F}(q)) \leq h^0(\mathbb{P}^r, \mathcal{F}(q-1))$ for all $q \ll 0$, and since a descending sequence of non-negative integers must stabilize, we get that $h^0(\mathbb{P}^r, \mathcal{F}(q)) = h^0(\mathbb{P}^r, \mathcal{F}(q-1))$ for all $q \ll 0$. However, since the multiplication map m is injective on each factor, the Hopf Lemma [ACGH, page 108] tells us that

$$\begin{aligned} h^0(\mathbb{P}^r, \mathcal{F}(q)) &\geq h^0(\mathbb{P}^r, \mathcal{O}(1)) + h^0(\mathbb{P}^r, \mathcal{F}(q-1)) - 1 \\ &= h^0(\mathbb{P}^r, \mathcal{F}(q-1)) + r > h^0(\mathbb{P}^r, \mathcal{F}(q-1)) \end{aligned}$$

which is a contradiction.

Hence, we want to reduce to the case of (1.1.3) being injective on each factor: first we show that we can assume that for each nonzero $\ell \in H^0(\mathbb{P}^r, \mathcal{O}(1))$, the multiplication map

$$\cdot \ell\colon H^0(\mathbb{P}^r, \mathcal{F}(q-1)) \to H^0(\mathbb{P}^r, \mathcal{F}(q)) \qquad (1.1.4)$$

is injective for $q \ll 0$. If this is not the case, consider the hyperplane $H = \{\ell = 0\}$ and the induced exact sequence of sheaves on $\mathbb{P}^r$:

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F}(-1) \xrightarrow{\cdot \ell} \mathcal{F} \longrightarrow \mathcal{F}_{|H} \longrightarrow 0$$

Observe that the sheaf $\mathcal{G}$ is supported on H, and the multiplication by ℓ fails to be injective for $q \ll 0$ precisely when $H^0(\mathbb{P}^r, \mathcal{G}(q)) \neq 0$ for infinitely many $q < 0$. But in this case the induction hypothesis implies that $\mathcal{G}$ has a closed associated point, which is then also an associated point of $\mathcal{F}$ and we are done.

So, we can assume that for each linear form ℓ the map (1.1.4) is injective for all $q \ll 0$. In particular, we can choose a $\ell_0 \in H^0(\mathbb{P}^r, \mathcal{O}(1))$ such that the maps

$\cdot\ell_0\colon H^0(\mathbb{P}^r, \mathcal{F}(q-1)) \to H^0(\mathbb{P}^r, \mathcal{F}(q))$ are injective for all $q \ll 0$. Consequently, the dimensions $h^0(\mathbb{P}^r, \mathcal{F}(q))$ for $q \ll 0$ form a descending sequence of non-negative integers, which must stabilize. Observe that they stabilize to a positive integer, since infinitely many of them are non-zero by hypothesis. Hence, there exists a $q_0 \in \mathbb{Z}$ such that the multiplication maps $\cdot\ell_0\colon H^0(\mathbb{P}^r, \mathcal{F}(q-1)) \to H^0(\mathbb{P}^r, \mathcal{F}(q))$ are actually isomorphisms for all $q \leq q_0$. We will denote the inverse map by $\cdot\frac{1}{\ell_0}\colon H^0(\mathbb{P}^r, \mathcal{F}(q)) \longrightarrow H^0(\mathbb{P}^r, \mathcal{F}(q-1))$. Now, suppose that there is another nonzero linear form $\ell \in H^0(\mathbb{P}^r, \mathcal{O}(1))$ and a nonzero section $\sigma \in H^0(\mathbb{P}^r, \mathcal{F}(\overline{q}))$, with $\overline{q} < q_0 - 1$, such that $\ell \cdot \sigma = 0$. Then it is easy to see that $\ell \cdot \frac{\sigma}{\ell_0} = 0$ in $H^0(\mathbb{P}^r, \mathcal{F}(\overline{q}+1))$: indeed, multiplication by ℓ_0 is injective on $H^0(\mathbb{P}^r, \mathcal{F}(\overline{q}+1))$ and $\ell_0 \cdot \ell \cdot \frac{\sigma}{\ell_0} = \ell \cdot \sigma = 0$. In the same way, one sees that for all $m \geq 1$ the element $\frac{\sigma}{\ell_0^m} \in H^0(\mathbb{P}^r, \mathcal{F}(\overline{q}-m))$ is nonzero and $\ell \cdot \frac{\sigma}{\ell_0^m} = 0$. So, the multiplication by ℓ map (1.1.4) is not injective for infinitely many $q < 0$, but this contradicts our assumptions. Hence, it must be that the map (1.1.4) is injective for all $q < \overline{q} - 1$ and for each linear form ℓ, but this implies that the map (1.1.3) is injective on each factor for $q \ll 0$, which concludes the proof. □

Remark 1.1.19. In the proof of Lemma 1.1.18, we have used Hopf's Lemma, which holds only on an algebraically closed field. However, the statement is true over any field k, and it reduces to the algebraically closed case. Indeed, if $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^r_k$ and if $\mathcal{F}_{\overline{k}}$ is the corresponding sheaf on $\mathbb{P}^r_{\overline{k}}$, we have that $H^0(\mathbb{P}^r_k, \mathcal{F}_k) = 0$ if and only if $H^0(\mathbb{P}^r_{\overline{k}}, \mathcal{F}_{\overline{k}}) = 0$ [Sta18, Lemma 29.5.2], and moreover $\mathcal{F}_k$ has a closed associated point if and only if $\mathcal{F}_{\overline{k}}$ has a closed associated point [EGAIV.2, Proposition 3.3.6].

Kernel bundles

In geometric situations there are many tools that help us compute Koszul cohomology. Some of the most powerful ones are kernel bundles.

Definition 1.1.20 (Kernel bundle)**.** Let X be a projective scheme, L an ample and globally generated line bundle on X and $V \subseteq H^0(X, L)$ a base-point-free subspace. Then we have an exact sequence

$$0 \to M_V \to V \otimes_{\mathbb{C}} \mathcal{O}_X \to L \to 0 \tag{1.1.5}$$

which defines a vector bundle M_V, called the kernel bundle of L with respect to V. In particular, we set $M_L \stackrel{\text{def}}{:=} M_{H^0(X,L)}$.

Remark 1.1.21. By construction, we see that M_V is a vector bundle of rank $r = \dim V - 1$ and of determinant $\wedge^r M_V \cong L^\vee$.

By a well-known result of Lazarsfeld, the above exact sequence can be used to compute Koszul cohomology:

Proposition 1.1.22 (Lazarsfeld)**.** *With the above notation, let $\mathcal{F}$ be a coherent sheaf on X. Then*

$$K_{p,q}(X,\mathcal{F},L;V) \cong \mathrm{Coker}\left[\wedge^{p+1}V\otimes H^0(X,\mathcal{F}\otimes L^{q-1}) \to H^0(X,\wedge^p M_V\otimes\mathcal{F}\otimes L^q)\right]$$
$$= \mathrm{Ker}\left[H^1(X,\wedge^{p+1}M_V\otimes L^{q-1}\otimes\mathcal{F}) \to \wedge^{p+1}V\otimes H^1(X,L^{q-1}\otimes\mathcal{F})\right].$$

Proof. See e.g. [AN10, Remark 2.6]. □

Assuming some cohomological vanishings, we obtain a bit more from Proposition 1.1.22:

Lemma 1.1.23. *With the above notations, fix $h>0$ and suppose that*

$$H^i(X,\mathcal{F}\otimes L^{q-i}) = 0 \qquad \text{for all } i=1,\dots,h.$$
$$H^i(X,\mathcal{F}\otimes L^{q-i-1}) = 0 \qquad \text{for all } i=1,\dots,h-1.$$

Then

$$K_{p,q}(X,\mathcal{F},L;V) \cong H^h(X,\wedge^{p+h}M_V\otimes\mathcal{F}\otimes L^{q-h}).$$

Proof. We proceed by induction on h. If $h=1$, the statement follows immediately from Proposition 1.1.22. If instead $h>1$, taking exterior powers in the exact sequence (1.1.5) and tensoring by $\mathcal{F}\otimes L^{(q-h)}$ we get an exact sequence

$$0\to\wedge^{p+h}M_V\otimes L^{q-h}\otimes\mathcal{F}\to\wedge^{p+h}V\otimes L^{q-h}\otimes\mathcal{F}\to\wedge^{p+h-1}M_V\otimes L^{q-h+1}\otimes\mathcal{F}\to 0.$$

By hypothesis we have $H^{h-1}(X,L^{q-h}\otimes\mathcal{F}) = H^h(X,L^{q-h}\otimes\mathcal{F}) = 0$. Hence, the long exact sequence in cohomology yields $H^h(X,\wedge^{p+h}M_V\otimes L^{q-h}\otimes\mathcal{F}) \cong H^{h-1}(X,\wedge^{p+h-1}M_V\otimes L^{q-h+1}\otimes\mathcal{F})$. Moreover, we already have the isomorphism $H^{h-1}(X,\wedge^{p+h-1}M_V\otimes L^{q-h+1}\otimes\mathcal{F}) \cong K_{p,q}(X,\mathcal{F},L;V)$ thanks to the induction hypothesis. □

A remark on duality for Koszul cohomology

Serre's duality on a smooth variety translates into duality for Koszul cohomology. The following formulation of the Duality Theorem is due to Green:

Theorem 1.1.24 (Green's Duality Theorem)**.** *Let X be a smooth and irreducible projective variety of dimension n, L an ample and globally generated line bundle on X, $V\subseteq H^0(X,L)$ a base-point-free subspace and E a vector bundle on X. Suppose that*

$$H^i(X,E\otimes L^{q-i}) = H^i(X,E\otimes L^{q-1-i}) = 0, \qquad \text{for all } i=1,\dots,n-1.$$

Then, there is an isomorphism

$$K_{p,q}(X,E,L;V)^\vee \cong K_{\dim V-1-n-p,\,n+1-q}(X,\omega_X\otimes E^\vee,L;V).$$

Proof. See [Gre84, Theorem 2.c.6] or [AN10, Remark 2.25]. □

For later use, we prove a small variant of this result.

Proposition 1.1.25. *Let X be a smooth variety of dimension n, L an ample and globally generated line bundle, $V \subseteq H^0(X, L)$ a base-point-free subspace and E a vector bundle such that*

$$H^i(X, E \otimes L^{q-i-1}) = 0 \qquad \text{for all } i = 1, \ldots, n-1,$$
$$H^i(X, E \otimes L^{q-i}) = 0 \qquad \text{for all } i = 2, \ldots, n-1.$$

Then,

$$\dim K_{p,q}(X, E, L; V) \leq \dim K_{\dim V-1-n-p, n+1-q}(X, \omega_X \otimes E^\vee, L; V).$$

Proof. By Proposition 1.1.22, we know that

$$K_{p,q}(X, E, L; V) \subseteq H^1(X, \wedge^{p+1} M_V \otimes L^{q-1} \otimes E).$$

Using Serre's duality, we get

$$\begin{aligned} H^1(X, \wedge^{p+1} M_V \otimes L^{q-1} \otimes E)^\vee &\cong H^{n-1}(X, \wedge^{p+1} M_V^\vee \otimes L^{1-q} \otimes \omega_X \otimes E^\vee) \\ &\cong H^{n-1}(X, \wedge^{r-p-1} M_V \otimes L^{2-q} \otimes \omega_X \otimes E^\vee) \end{aligned}$$

where in the last isomorphism we have used that M_V is a vector bundle of rank $r = \dim V - 1$ and determinant $\wedge^r M_L \cong L^\vee$ (see Remark 1.1.21). To conclude, we will show that

$$H^{n-1}(X, \wedge^{r-p-1} M_V \otimes L^{2-q} \otimes \omega_X \otimes E^\vee) \cong K_{r-n-p, n+1-q}(X, \omega_X \otimes E^\vee, L).$$

This will follow if we can apply Lemma 1.1.23 with $h = n-1$ to the group $K_{r-n-p,n+1-q}(X, \omega_X \otimes E^\vee, L; V)$. The conditions of Lemma 1.1.23 are that

$$H^j(X, \omega_X \otimes E^\vee \otimes L^{n+1-q-j}) = 0 \qquad \text{for all } j = 1, \ldots, n-1,$$
$$H^j(X, \omega_X \otimes E^\vee \otimes L^{n-q-j}) = 0 \qquad \text{for all } j = 1, \ldots, n-2.$$

By Serre duality these are the same as

$$H^{n-j}(X, E \otimes L^{q-(n-j)-1}) = 0 \qquad \text{for all } j = 1, \ldots, n-1$$
$$H^{n-j}(X, E \otimes L^{q-(n-j)}) = 0 \qquad \text{for all } j = 1, \ldots, n-2$$

and setting $i = n-j$, these are precisely the vanishings in our assumptions. So, the conditions of Lemma 1.1.23 are verified and we can conclude. □

Remark 1.1.26. Observe that with the additional vanishing $H^1(X, E \otimes L^{q-1}) = 0$, the two Koszul cohomology groups in Proposition 1.1.25 would be dual to each other thanks to Green's Duality Theorem 1.1.24. Indeed, with this additional vanishing, Proposition 1.1.22 shows that $K_{p,q}(X, E, L; V) \cong H^1(X, \wedge^{p+1} M_V \otimes L^{q-1} \otimes E)$ and then the proof of Proposition 1.1.25 shows that $H^1(X, \wedge^{p+1} M_V \otimes L^{q-1} \otimes E)^\vee \cong K_{\dim V - 1 - n - p, n+1-q}(X, \omega_X \otimes E^\vee, L)$.

1.2 Hilbert schemes of points on surfaces

Another powerful technique for studying geometric syzygies is Voisin's interpretation of Koszul cohomology in terms of the Hilbert scheme of points. This was introduced by Voisin in her breakthrough proof of the general Green's conjecture [Voi02; Voi05] and it was recently used by Ein and Lazarsfeld to prove the Gonality conjecture [EL15].

Here we give some background on the Hilbert scheme of points, and its relation with syzygies. We focus on the case of surfaces and we discuss in particular the derived category of the Hilbert scheme.

1.2.1 The Hilbert scheme of points

Let X be a smooth, irreducible, quasiprojective surface and $n > 0$ a positive integer: we will denote by $X^{[n]}$ the Hilbert scheme of points of X and by $X^{(n)}$ the symmetric product of X. The Hilbert scheme $X^{[n]}$ parametrizes finite subschemes $\xi \subseteq X$ of length n, whereas $X^{(n)}$ parametrizes zero cycles of length n on X. Since X is quasiprojective, both $X^{[n]}$ and $X^{(n)}$ are quasiprojective as well, and they are projective if X is [Göt94, Theorem 1.1.2].

The symmetric product can be obtained as the quotient $X^{(n)} = X^n/\mathfrak{S}_n$, where the symmetric group $\mathfrak{S}_n$ acts naturally on X^n by

$$\sigma \cdot (P_1, \dots, P_n) = (P_{\sigma^{-1}(1)}, \dots, P_{\sigma^{-1}(n)})$$

and we denote by

$$\pi \colon X^n \to X^{(n)}$$

the projection. There is also a canonical Hilbert-Chow morphism [Göt94, Theorem 1.1.7]

$$\mu \colon X^{[n]} \to X^{(n)}, \qquad \xi \mapsto \sum_{P \in X} \ell(\mathcal{O}_{\xi,P}) \cdot P$$

that maps a subscheme to its weighted support. By construction, the Hilbert scheme comes equipped with a universal family $\Xi^{[n]}$, that can be described as

$$\Xi^{[n]} = \{(P,\xi) \in X \times X^{[n]} \mid P \in \xi\}, \qquad p_X \colon \Xi^{[n]} \to X, \qquad p_{X^{[n]}} \colon \Xi^{[n]} \to X^{[n]}.$$

The map $p_{X^{[n]}}$ is finite, flat and of degree n: the fiber of $p_{X^{[n]}}$ over $\xi \in X^{[n]}$ is precisely the subscheme $\xi \subseteq X$.

The same construction of the Hilbert scheme and of the symmetric product can be carried out for every quasiprojective scheme, however, when X is a smooth surface the situation is especially nice, thanks to the following result of Fogarty [Fog68].

Theorem 1.2.1 (Fogarty)**.** *The Hilbert scheme $X^{[n]}$ is a smooth and irreducible variety of dimension $2n$ and the Hilbert-Chow morphism $\mu \colon X^{[n]} \to X^{(n)}$ is a resolution of singularities.*

Proof. See [Fog68] or [Fan+05, Theorem 7.2.3] □

Remark 1.2.2. In fact, it turns out that more is true [Bea83]: the symmetric product $X^{(n)}$ is Gorenstein, and the Hilbert-Chow morphism $\mu \colon X^{[n]} \to X^{(n)}$ is a crepant resolution of singularities. This means that $\mu^* \omega_{X^{(n)}} \cong \omega_{X^{[n]}}$.

Remark 1.2.3. When X is a smooth curve, the Hilbert scheme $X^{[n]}$ is smooth and irreducible of dimension n. Moreover, the Hilbert-Chow morphism $\mu \colon X^{[n]} \to X^{(n)}$ is an isomorphism. For a smooth and irreducible variety X of arbitrary dimension it is still true that $X^{[n]}$ is smooth and irreducible, when $n \leq 3$ [Fan+05, Remark 7.2.5]. However, when X has dimension at least three and $n \geq 4$, the Hilbert scheme $X^{[n]}$ is in general singular [Fan+05, Remark 7.2.6] and reducible [Iar72a], [Fan+05, Example 7.2.8].

The fibers of the Hilbert-Chow morphism are also well understood thanks to a result of Iarrobino and Briançon: to state it, let $O \in \mathbb{A}^2_{\mathbb{C}}$ denote the origin and consider the *punctual Hilbert scheme*

$$H_{O,m} \subseteq (\mathbb{A}^2_{\mathbb{C}})^{[m]}, \qquad H_{O,m} = \{\xi \mid \xi \text{ is supported at } O\}.$$

Iarrobino [Iar72b] proved that $H_{O,m}$ has dimension $m-1$ and Briançon [Bri77] showed that $H_{O,m}$ is irreducible.

It is then straightforward to describe the fibers of the Hilbert-Chow morphism:

Proposition 1.2.4 (Iarrobino, Briançon)**.** *The fiber of $\mu \colon X^{[n]} \to X^{(n)}$ over a cycle $\zeta = \lambda_1 P_1 + \cdots + \lambda_r P_r$, with the points P_i pairwise distinct, is*

$$\mu^{-1}(\zeta) \cong H_{O,\lambda_1} \times \cdots \times H_{O,\lambda_r}.$$

In particular, it is irreducible of dimension $n - r$.

Curvilinear subschemes

To work with syzygies, we will be interested in a particular subset of the Hilbert scheme, the subset of *curvilinear subschemes*. We consider here the general case of curvilinear subschemes on smooth variety of arbitrary dimension, since we will need some of the results later.

Definition 1.2.5 (Curvilinear subschemes)**.** Let X be a quasiprojective variety. A finite subscheme $\xi \subseteq X$ is called *curvilinear* if the tangent space of ξ at each point is at most one dimensional:

$$\dim_{\mathbb{C}} T_P\xi \leq 1 \qquad \text{for all } P \in \xi.$$

We denote the set of curvilinear subschemes of length n as $X^{[n]}_{curv} \subseteq X^{[n]}$.

Example 1.2.6. We give here some examples of curvilinear and non-curvilinear finite schemes of small length.

- Length 1: a finite scheme of length 1 consists of a single point. In particular it is curvilinear.
- Length 2: a finite scheme of length 2 consists of two distinct points or it is of the form $\operatorname{Spec}\mathbb{C}[X]/(X^2)$, so that it is a point together with a tangent direction. In both cases, the schemes are curvilinear.
- Length 3: a finite scheme of length 3 consists of three distinct points, of two distinct points together with a tangent vector, or it is supported at a single point. The schemes supported at a single point are of two possible forms:

  $$\operatorname{Spec}\mathbb{C}[X]/(X^3), \qquad \operatorname{Spec}\mathbb{C}[X,Y]/(X,Y)^2.$$

  The first one is curvilinear and the second is not.

Remark 1.2.7. As suggested from the previous examples, the reason for the name curvilinear comes from the fact that a curvilinear subscheme $\xi \subseteq X$ is locally contained in a smooth curve inside X. More precisely, suppose that X is a smooth quasiprojective variety of dimension n and let ξ be a curvilinear subscheme of length k supported at a point $P \in X$. Then, there are local analytic coordinates $(x_1, \dots, x_n)$ on X centered at P such that the ideal I_ξ of ξ in $\mathbb{C}\{x_1, \dots, x_n\}$ is given by

$$I_\xi = (x_1^k, x_2, \dots, x_n).$$

Proof. Take any set of local analytic coordinates $z_1, \dots, z_n$ centered at P. In the corresponding ring $R = \mathbb{C}\{z_1, \dots, z_n\}$, let $\mathfrak{m} \subseteq R$ be the maximal ideal and $I \subseteq R$ be the ideal corresponding to the scheme ξ. By definition of ξ being curvilinear, the cotangent space has dimension at most one: $\dim_{\mathbb{C}}(\mathfrak{m}/(\mathfrak{m}^2 + I)) \leq 1$. Let

$\bar{z}_1, \ldots, \bar{z}_n$ be the images of the local coordinates in $(\mathfrak{m}/(\mathfrak{m}^2 + I))$. Up to a change of coordinates in R we can suppose that for $i = 2, \ldots, d$ there are $\lambda_i \in \mathbb{C}$ such that $\bar{z}_i = \lambda_i \bar{z}_1$. This is the same as saying that there are $f_i \in \mathfrak{m}^2$ such that $z_i - \lambda_i z_1 - f_i \in I$ for all $i = 2, \ldots, n$. Now set $x_1 = z_1$ and $x_i = z_i - \lambda_i z_1 - f_i$ for all $i = 2, \ldots, n$. Since the linear terms of the x_i are linearly independent, the analytic inverse function theorem shows that $(x_1, \ldots, x_n)$ is another set of local coordinates. For this set of coordinates, we see that $I \supseteq (x_2, \ldots, x_n)$ and in particular, $R/I \cong \mathbb{C}\{x_1\}/I \cap \mathbb{C}\{x_1\}$. However, since R/I has length k, it must be that $I \cap \mathbb{C}\{x_1\} = (x_1^k)$. Hence, it follows immediately that $I = (x_1^k, x_2, \ldots, x_n)$.

In this proof, we have worked in the analytic category for simplicity. For an algebraic treatment see for example [Iar72b, Example page 822] and [Göt94, Remark 2.1.7].

□

Remark 1.2.8. For any smooth quasiprojective variety, the set $X^{[n]}_{curv}$ is an open subset of the Hilbert scheme $X^{[n]}$ [Göt94, Remark 2.1.8]. Indeed, it is easy to see from the explicit description of Remark 1.2.7 that a small perturbation of a curvilinear subscheme is again curvilinear.

Moreover, if X is a smooth and irreducible surface, then the set $X^{[n]}_{curv} \subseteq X^{[n]}$ of curvilinear subschemes is a large open subset. Indeed, its complement has codimension 4 [BFS89, Remark 3.5].

1.2.2 Tautological bundles

Now we suppose again that X is a smooth and irreducible quasiprojective surface. Given a vector bundle on X, there is a canonical way to obtain a bundle on the Hilbert scheme of points. These are the so-called tautological bundles and they have been intensely studied [Leh99],[EGL01],[Dan04], [MO08], [Voi17],[MOP17]. In our case, we are interested in them because of their relation with syzygies.

To define them, recall that over $X^{[n]}$ we have the universal family $\Xi^{[n]}$ together with the maps

$$p_{X^{[n]}} \colon \Xi^{[n]} \longrightarrow X^{[n]}, \qquad p_X \colon \Xi^{[n]} \longrightarrow X.$$

Definition 1.2.9 (Tautological bundle)**.** Let E be a vector bundle on X. We define the corresponding *tautological bundle* on $X^{[n]}$ as

$$E^{[n]} :\stackrel{\text{def}}{=} p_{X^{[n]},*} p_X^*(E).$$

Remark 1.2.10. Since the map $p_{X^{[n]}} \colon \Xi^{[n]} \to X^{[n]}$ is finite and flat of degree n, the sheaf $E^{[n]}$ is a vector bundle of rank $n \cdot rk(E)$ on $X^{[n]}$. By construction, the fiber of $E^{[n]}$ over a point $\xi \in X^{[n]}$ is identified with $H^0(X, E \otimes \mathcal{O}_\xi)$.

In particular, for every vector bundle E on X we obtain a line bundle on $X^{[n]}$ by considering the determinant $\det E^{[n]}$. If we start with the structure sheaf $\mathcal{O}_X$, we obtain a special line bundle on $X^{[n]}$.

Definition 1.2.11. We define a line bundle on $X^{[n]}$ by

$$\mathcal{O}(-\delta_n) :\overset{\text{def}}{=} \det \mathcal{O}_X^{[n]}.$$

Remark 1.2.12. A geometrical interpretation of this line bundle is that the class $2\delta_n$ represents the locus of non-reduced subschemes in $X^{[n]}$, which is the exceptional divisor of the Hilbert-Chow morphism $\mu\colon X^{[n]} \to X^{(n)}$ [Leh99, Lemma 3.7].

There is also another construction of line bundles on the Hilbert scheme. Let L be any line bundle on X, then the line bundle $L^{\boxtimes n} = \bigotimes_{i=1}^n pr_i^* L$ has a $\mathfrak{S}_n$-linearization, so that we can take the sheaf of invariants $L^{(n)} :\overset{\text{def}}{=} \pi_*^{\mathfrak{S}_n}(L^{\boxtimes n})$, which is a coherent sheaf on $X^{(n)}$ (for more details on this construction see Example 1.4.8). In fact, it was proven by Fogarty [Fog73], that $L^{(n)}$ is a line bundle on $X^{(n)}$: this gives a line bundle on $X^{[n]}$ by taking $\mu^* L^{(n)}$. Moreover, he also proved that $\pi^* L^{(n)} \cong L^{\boxtimes n}$ and that the induced map

$$\operatorname{Pic}(X) \to \operatorname{Pic}(X^{(n)}), \qquad L \mapsto L^{(n)} \tag{1.2.1}$$

is a homomorphism of groups.

Remark 1.2.13. In these terms, we can describe easily the canonical bundles of $X^{(n)}$ and $X^{[n]}$. Indeed, the canonical line bundle on $X^{(n)}$ is $\omega_X^{(n)}$ [Bea83] and since the Hilbert-Chow morphism is a crepant resolution, the canonical bundle on $X^{[n]}$ is $\mu^*\omega_X^{(n)}$.

The determinant of a tautological bundle is well-known:

Lemma 1.2.14. *Let L be a line bundle on X. Then*

$$\det L^{[n]} \cong \mu^* L^{(n)} \otimes \mathcal{O}(-\delta_n).$$

Proof. A proof can be found for example in [Leh99]. □

Remark 1.2.15. All the definitions and results in this section are the same if X is a smooth curve. The only one that changes is the description of the canonical bundle: if C is a smooth curve, then the canonical bundle of $C^{[n]}$ is [Mat65] $\omega_{C^{[n]}} \cong \det \omega_C^{[n]} = \mu^*\omega_C^{(n)} \otimes \mathcal{O}(-\delta_n)$, instead of $\mu^*\omega_C^{(n)}$.

1.3 Hilbert scheme and syzygies

The fundamental connection between Hilbert schemes and syzygies was established by Voisin in [Voi02]. To state it, let X be a smooth and irreducible surface, $p \geq 0$ an integer and consider the universal family $\Xi^{[p+1]} \subseteq X \times X^{[p+1]}$ Let $U = X^{[p+1]}_{curv}$ be the open subset of curvilinear scheme: we denote by $\Xi^{[p+1]}_U$ the corresponding universal family, more precisely $\Xi^{[p+1]}_U \stackrel{\text{def}}{:=} \Xi^{[p+1]} \cap (X \times U)$. For a line bundle L on X, we also denote by $L^{[p+1]}_U$ the restriction of the tautological bundle $L^{[p+1]}$ to U. Then Voisin's result is the following:

Theorem 1.3.1 (Voisin). *Let X be a smooth projective surface, B a line bundle and L an ample and globally generated line bundle. Then $K_{p,1}(X, B, L)$ is identified with the cokernel of the restriction map*

$$H^0\left(X \times U, B \boxtimes \det L^{[p+1]}_U\right) \to H^0\left(\Xi^{[p+1]}_U, \left(B \boxtimes \det L^{[p+1]}_U\right)_{|\Xi^{[p+1]}_U}\right).$$

Proof. See [Voi02, Lemma 1] or [AN10, Corollary 5.5, Remark 5.6]. □

Remark 1.3.2. Since $K_{p,q}(X, B, L) = K_{p,1}(X, B \otimes L^{\otimes(q-1)}, L)$, the previous theorem gives a representation of every Koszul cohomology group.

Remark 1.3.3. Voisin's result actually holds for any smooth projective variety X: see again [Voi02, Lemma 1] or [AN10, Corollary 5.5, Remark 5.6].

1.4 The Bridgeland-King-Reid-Haiman correspondence

We collect here some facts about the derived category of the Hilbert scheme of points on a smooth surface X. The fundamental result is due to Haiman [Hai01; Hai02]: using the Bridgeland-King-Reid correspondence [BKR01], he proved that the derived category of $X^{[n]}$ is equivalent to the derived category of equivariant sheaves on the cartesian product X^n. This description is very concrete and it allows to translate geometric problems on the Hilbert scheme to combinatorial problems on the cartesian product. In particular, this has been used by Scala [Sca09; Sca15] and Krug [Kru14; Kru16] to study properties of tautological bundles.

1.4.1 Equivariant sheaves and the Bridgeland-King-Reid correspondence

We give a quick introduction to equivariant sheaves and their derived category. For more details on this section we refer to [BKR01]. We consider a quasiprojective variety X with an action of a finite group G. It is then natural to consider the sheaves for which the action extends.

Definition 1.4.1 (Equivariant sheaves)**.** A *G-equivariant* or *G-sheaf* on X is a quasicoherent sheaf $\mathcal{F}$ on X, together with morphisms $\lambda_g \colon \mathcal{F} \longrightarrow g^*\mathcal{F}$, for all $g \in G$, such that $\lambda_{\mathrm{id}} = \mathrm{id}$ and $\lambda_{hg} = g^*(\lambda_h) \circ \lambda_g$, for every $g, h \in G$. Sometimes the morphisms λ_g are called a *G-linearization* of $\mathcal{F}$.

There is also a natural notion of morphism between two equivariant sheaves:

Definition 1.4.2 (Homomorphism of equivariant sheaves)**.** Let $(\mathcal{F}, \lambda^{\mathcal{F}})$ and $(\mathcal{G}, \lambda^{\mathcal{G}})$ be two equivariant sheaves. A *homomorphism of equivariant sheaves* between them is a homomorphism of $\mathcal{O}_X$-sheaves $\phi \colon \mathcal{F} \to \mathcal{G}$ such that, for every $g \in G$, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \mathcal{G} \\ \downarrow{\scriptstyle \lambda_g^{\mathcal{F}}} & & \downarrow{\scriptstyle \lambda_g^{\mathcal{G}}} \\ g^*\mathcal{F} & \xrightarrow{g^*\phi} & g^*\mathcal{G}. \end{array}$$

Remark 1.4.3. We can give an equivalent description as follows: if $\mathcal{F}$ and $\mathcal{G}$ are two equivariant sheaves, then the group G acts on $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$ by $g \cdot \phi = (\lambda_g^{\mathcal{G}})^{-1} \circ g^*\phi \circ \lambda_g^{\mathcal{F}}$. Then, by definition, the set of fixed points $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})^G$ coincides with the set of morphisms of equivariant sheaves.

Equivariant sheaves, together with the above definition of morphisms, form a category.

Definition 1.4.4 (The category of equivariant sheaves)**.** In the above setting, we denote by $\mathrm{QCoh}_G(X)$ the *category of G-equivariant sheaves on X*.

The category of equivariant sheaves is abelian and it is endowed with the usual functors for quasicoherent sheaves, for example $\mathcal{H}om$ and $\otimes$. If $f \colon X \to Y$ is a G-equivariant map of schemes, we have the usual functors f_*, f^* and they satisfy all the usual properties. Moreover, there are some functors that are exclusive to equivariant sheaves.

Remark 1.4.5. Let $\rho \colon G \to GL(V)$ be a linear representation of G. Let also $\mathcal{F}$ be an equivariant sheaf on X. We can then form another equivariant sheaf $\mathcal{F} \otimes \rho$, whose underlying sheaf is $\mathcal{F} \otimes_{\mathbb{C}} V$, and with linearizations

$$\lambda_g \colon \mathcal{F} \otimes_{\mathbb{C}} V \longrightarrow g^*\mathcal{F} \otimes_{\mathbb{C}} V, \qquad f \otimes v \mapsto \lambda_g(f) \otimes \rho(g)v.$$

This operation defines a functor $\otimes \rho \colon \mathrm{QCoh}_G(X) \to \mathrm{QCoh}_G(X)$, which is moreover exact.

Remark 1.4.6 (Trivial actions and quotients)**.** Suppose that the group G acts trivially on X. An equivariant sheaf on X is simply given by a quasicoherent sheaf $\mathcal{F}$, together with an action of G on $\mathcal{F}$. In this case, each equivariant sheaf on X

splits naturally as $\mathcal{F} \cong \bigoplus_\rho \mathcal{F}_\rho \otimes \rho$, where the sum runs over all the irreducible representations of G and the $\mathcal{F}_\rho$ are usual quasicoherent sheaves with the trivial action of G. In particular, if we consider the trivial representation, we obtain the invariant part $\mathcal{F}^G$ and we get an exact functor

$$(-)^G\colon \operatorname{QCoh}_G(X) \to \operatorname{QCoh}(X), \qquad \mathcal{F} \mapsto \mathcal{F}^G.$$

In particular, we can consider the quotient $\pi\colon X \to X/G$. This is an equivariant morphism, with the trivial action of G on X/G. Hence, by the above discussion we get a functor

$$\pi_*^G\colon \operatorname{QCoh}_G(X) \longrightarrow \operatorname{QCoh}(X/G), \qquad \mathcal{F} \mapsto \pi_*^G(\mathcal{F}) := (\pi_*(\mathcal{F}))^G.$$

Since π is finite, the functor π_* is exact. Since the functor $(-)^G$ is also exact, it follows that the composition π_*^G is exact as well.

Conversely, if $\mathcal{G}$ is a quasicoherent sheaf on X/G, we can consider it as a G-sheaf with the trivial action of G and then we obtain a G-sheaf on X by $\pi^*\mathcal{G}$. This gives a functor

$$\pi^*\colon \operatorname{QCoh}(X/G) \longrightarrow \operatorname{QCoh}_G(X), \qquad \mathcal{G} \mapsto \pi^*\mathcal{G}$$

and for every $\mathcal{G} \in \operatorname{QCoh}(X/G)$ we have natural isomorphisms

$$\pi_*^G\pi^*\mathcal{G} \cong \mathcal{G}.$$

Remark 1.4.7 (Free actions)**.** Suppose that the group G acts freely on the quasiprojective variety X. The two functors

$$\pi_*^G\colon \operatorname{QCoh}_G(X) \longrightarrow \operatorname{QCoh}(X/G), \qquad \pi^*\colon \operatorname{QCoh}(X/G) \longrightarrow \operatorname{QCoh}_G(X)$$

are then inverse of each other and they induce equivalences of categories [Mum74, Proposition 2, p.70].

The main example that we need is that of the symmetric product.

Example 1.4.8 (The symmetric product)**.** Let X be a quasiprojective variety. The symmetric group $\mathfrak{S}_n$ acts naturally on the cartesian product X^n by

$$\sigma \cdot (P_1, \dots, P_n) = (P_{\sigma^{-1}(1)}, \dots, P_{\sigma^{-1}(n)}).$$

For every quasicoherent sheaf $\mathcal{F}$ on X, the sheaf $\mathcal{F}^{\boxtimes n} = pr_1^*\mathcal{F} \otimes \dots \otimes pr_n^*\mathcal{F}$ has a natural $\mathfrak{S}_n$-linearization given by

$$\lambda_\sigma\colon \mathcal{F}^{\boxtimes n} \longrightarrow \mathcal{F}^{\boxtimes n}, \qquad f_1 \otimes \dots \otimes f_n \mapsto f_{\sigma^{-1}(1)} \otimes \dots \otimes f_{\sigma^{-1}(n)}.$$

The quotient $\pi\colon X^n \longrightarrow X^n/\mathfrak{S}_n$ is by definition the symmetric product $X^{(n)}$. Hence, by the previous discussion, from each quasicoherent sheaf $\mathcal{F}$ on X we obtain a quasicoherent sheaf on $X^{(n)}$, by taking

$$\mathcal{F}^{(n)} :\stackrel{\text{def}}{=} \pi_*^{\mathfrak{S}_n}(\mathcal{F}^{\boxtimes n}).$$

Equivariant derived categories

It is natural to consider the derived category associated to equivariant sheaves. For more details, we refer to [BKR01].

Definition 1.4.9 (Equivariant derived category)**.** Let X be a quasiprojective variety with an action of a finite group G. The bounded *equivariant derived category* $D^b_G(X)$ is subcategory of the unbounded derived category of $\mathrm{QCoh}_G(X)$ consisting of complexes with bounded and coherent cohomology.

The equivariant derived category enjoys all the properties of the usual derived category. Among these, we want to spell out a consequence of Grothendieck-Verdier duality for Gorenstein varieties, that we will use later.

Let $f\colon X \to Y$ a morphism of Gorenstein varieties and let ω_X and ω_Y be the corresponding dualizing line bundles. We can define the functor

$$f^!\colon D^b(Y) \longrightarrow D^b(X), \qquad \mathcal{G} \mapsto R\mathcal{H}om_X(Lf^*R\mathcal{H}om_Y(G,\omega_Y),\omega_X).$$

Grothendieck-Verdier duality is the following result:

Theorem 1.4.10 (Grothendieck-Verdier duality)**.** *Let $f\colon X \to Y$ be a morphism as above. Then, for every $\mathcal{F} \in D^b(X)$ and $\mathcal{G} \in D^b(Y)$ we have natural isomorphisms*

$$Rf_*R\mathcal{H}om_X(\mathcal{F}, f^!\mathcal{G}) \cong R\mathcal{H}om_Y(Rf_*\mathcal{F},\mathcal{G}).$$

As a corollary, we obtain the following result in the equivariant derived category.

Corollary 1.4.11. *Let X be a Gorenstein variety with an action of a finite group G such that the quotient X/G is Gorenstein as well. Consider the projection morphism $\pi\colon X \to X/G$. For every $\mathcal{F} \in D^b_G(X)$ and for every $\mathcal{G} \in D^b(X/G)$ we have natural isomorphisms*

$$\pi_*^G R\mathcal{H}om_X(\mathcal{F}, f^!\mathcal{G}) \cong R\mathcal{H}om_{X/G}(\pi_*^G\mathcal{F},\mathcal{G}).$$

Proof. The usual Grothendieck-Verdier duality of Theorem 1.4.10 gives

$$\pi_* R\mathcal{H}om_X(\mathcal{F}, f^!\mathcal{G}) \cong R\mathcal{H}om_{X/G}(\pi_*\mathcal{F},\mathcal{G}). \tag{1.4.1}$$

Observe that we do not need to take the derived functor $R\pi_*$, because π is finite, and therefore π_* is exact. We know by Remark 1.4.6 that $\pi_*\mathcal{F}$ splits as $\pi_*\mathcal{F} = \bigoplus_\rho (\pi_*\mathcal{F})_\rho \otimes \rho$, where ρ runs along the irreducible representations of G and $(\pi_*\mathcal{F})_\rho$ are sheaves with the trivial action of G. Hence, the right hand side in (1.4.1) splits accordingly and taking G-invariants on both sides we conclude. □

1.4.2 The Bridgeland-King-Reid correspondence

We recall quickly the ideas of the Bridgeland-King-Reid correspondence that was applied by Haiman on the Hilbert scheme of points. Again, we refer to [BKR01] for details.

Let M be a smooth and irreducible quasiprojective variety with an action of a finite group G. Nakamura introduced in [Nak01] the *G-Hilbert scheme* $G\text{-Hilb}\, M$, which parametrizes *G-clusters* on M: these are G-invariant finite subschemes $\xi \subseteq M$ such that $\mathcal{O}_\xi$ is isomorphic to the standard representation $\mathbb{C}[G]$ of G. In particular, every free orbit of G gives such a subscheme: we denote by $\operatorname{Hilb}_G M$ the irreducible component of $G\text{-Hilb}\, M$ containing the free orbits. We also have a G-Hilbert-Chow morphism $\tau\colon \operatorname{Hilb}_G M \to M/G$ that sends each subscheme to the corresponding orbit. If we denote by $\mathcal{Z} \subseteq M \times \operatorname{Hilb}_G M$ the universal family of G-clusters over $\operatorname{Hilb}_G M$, we get a commutative diagram.

$$\begin{array}{ccc} \mathcal{Z} & \xrightarrow{\quad p \quad} & M \\ \big\downarrow q & & \big\downarrow \pi \\ \operatorname{Hilb}_G M & \xrightarrow{\quad \tau \quad} & M/G. \end{array}$$

This diagram is G-equivariant, if we consider the trivial action on $\operatorname{Hilb}_G M$ and M/G: in particular, if $\mathcal{F}$ is any quasicoherent sheaf on $\operatorname{Hilb}_G M$, we can consider it as a G-sheaf with the trivial action of G, and then we obtain a G-sheaf on M by taking $p_*(q^*\mathcal{F})$. This way, we get a functor

$$\Phi\colon D^b(\operatorname{Hilb}_G M) \longrightarrow D^b_G(M), \qquad \Phi :\stackrel{\text{def}}{=} Rp_* \circ q^*.$$

On the reverse direction, we have a functor

$$\Psi\colon D^b_G(M) \longrightarrow D^b(\operatorname{Hilb}_G M), \qquad \Psi :\stackrel{\text{def}}{=} q^G_* \circ Lp^*.$$

Observe that in the definition of Φ and Ψ, we do not need to derive q^* and q_* since q is finite and flat. The result is that these two functors give equivalences of categories if certain conditions are satisfied.

Theorem 1.4.12 (Bridgeland-King-Reid)**.** *With the above notation, assume that M/G is Gorenstein and that*

$$\dim(\operatorname{Hilb}_G M \times_{M/G} \operatorname{Hilb}_G M) \leq \dim M + 1.$$

Then the G-Hilbert-Chow morphism is a crepant resolution of singularities and the functors

$$\begin{aligned} \Phi\colon D^b(\operatorname{Hilb}_G M) \longrightarrow D^b_G(M), \qquad & \Phi = Rp_* \circ q^* \\ \Psi\colon D^b_G(M) \longrightarrow D^b(\operatorname{Hilb}_G M), \qquad & \Psi = q^G_* \circ Lp^* \end{aligned}$$

are equivalences of categories.

Proof. In [BKR01, Theorem 1.1] the authors prove that the G-Hilbert-Chow morphism is a crepant resolution and that the functor Φ is an equivalence of categories. The fact that Ψ is also an equivalence of categories is an easy consequence and we present a proof here following Krug [Kru16, Proposition 2.9]. We can look at Φ as the equivariant Fourier-Mukai transform associated to the object $\mathcal{O}_{\mathcal{Z}} \in D^b(\mathrm{Hilb}_G M \times M)$. Since this is an equivalence of categories, if follows that the equivariant Fourier-Mukai transform in the reverse direction is also an equivalence of categories, and this is no other than the functor Ψ. □

1.4.3 The derived category of the Hilbert scheme of points

Finally, we present here the results of Haiman, Scala and Krug. For more complete references about this part, one can look at [Hai01; Hai02; Sca09; Kru16].

In [Hai01] Haiman defines the *isospectral Hilbert scheme* as the reduced fiber product

$$B_n = (X^{[n]} \times_{X^{(n)}} X^n)_{red}.$$

This is the set

$$B_n = \left\{ (\xi, (P_1, \dots, P_n)) \in X^{[n]} \times X^n \mid \mu(\xi) = P_1 + \cdots + P_n \right\} \subseteq X^{[n]} \times X^n$$

and it fits into a commutative diagram

$$\begin{array}{ccc} B_n & \xrightarrow{p} & X^n \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \pi} \\ X^{[n]} & \xrightarrow{\mu} & X^{(n)}. \end{array}$$

Haiman's main result is the following:

Theorem 1.4.13 (Haiman)**.** *With the same notations as in Theorem 1.4.12, there is an identification* $X^{[n]} \cong \mathrm{Hilb}_{\mathfrak{S}_n} X^n$*. Under this identification, the Hilbert-Chow morphism* μ *coincides with the G-Hilbert-Chow morphism* τ *and the isospectral Hilbert scheme* B_n *coincides with the universal family* $\mathcal{Z}$*.*

Proof. See [Hai01, Theorem 5.1]. □

Now it is straightforward to apply Theorem 1.4.12.

Corollary 1.4.14 (Haiman)**.** *The two functors*

$$\Phi = Rp_* \circ q^*, \qquad \Psi = q_*^{\mathfrak{S}_n} \circ Lp^*$$

give equivalences of categories

$$\Phi\colon D^b(X^{[n]}) \to D^b_{\mathfrak{S}_n}(X^n), \qquad \Psi\colon D^b_{\mathfrak{S}_n}(X^n) \to D^b(X^{[n]}).$$

Proof. To show that Φ is an equivalence of categories, we need to check that the conditions of Theorem 1.4.12 are satisfied. We know that $X^{(n)}$ is Gorenstein from Remark 1.2.2. Then, we need to check the condition on the dimension: what we want is that

$$\dim(X^{[n]} \times_{X^{(n)}} X^{[n]}) \leq 2n+1.$$

However, we have in Proposition 1.2.4 an explicit description of the fibers of $\mu\colon X^{[n]} \to X^{(n)}$. Using this, one can compute that $\dim(X^{[n]} \times_{X^{(n)}} X^{[n]}) = 2n$, so that the condition is verified. □

Remark 1.4.15. Haiman proved the above results in the case of $X = \mathbb{A}^2_{\mathbb{C}}$. The case of a smooth quasiprojective surface is essentially the same, as it was pointed out by Scala [Sca09, Section 1.5] and Krug [Kru16, Proposition 2.9].

An important part of Corollary 1.4.14 is that the equivalences Φ and Ψ are explicitly computable. In particular Scala was able to compute the image under Φ of the tautological bundles $E^{[n]}$. More precisely, consider the space $X \times X^n = \{(P_0, \dots, P_n) \mid P_i \in X\}$ with the two projections

$$\begin{aligned} pr_0\colon X \times X^n &\to X\ , \quad (P_0, \dots, P_n) \mapsto P_0 \\ pr_{[1,n]}\colon X \times X^n &\to X^n, \quad (P_0, \dots, P_n) \mapsto (P_1, \dots, P_n) \end{aligned} \tag{1.4.2}$$

and the subscheme

$$D_n \subseteq X \times X^n \qquad D_n = \Delta_{01} \cup \Delta_{02} \cup \dots \cup \Delta_{0n} \tag{1.4.3}$$

where Δ_{ij} denotes the partial diagonal $\Delta_{ij} = \{(P_0, \dots, P_n) \mid P_i = P_j\}$. Scala showed the following in [Sca09, Theorem 2.2.2]:

Theorem 1.4.16 (Scala)**.** *Let E be a vector bundle on X and let $E^{[n]}$ be the corresponding tautological bundle on $X^{[n]}$. Then $\Phi(E^{[n]}) \cong Rpr_{[1,n],*}(pr_0^* E \otimes \mathcal{O}_{D_n})$. Moreover, $\Phi(E^{[n]})$ is a sheaf on X^n and it has an exact resolution*

$$0 \longrightarrow \Phi(E^{[n]}) \longrightarrow \mathsf{C}^0_E \longrightarrow \mathsf{C}^1_E \longrightarrow \dots \longrightarrow \mathsf{C}^n_E \longrightarrow 0$$

for a certain explicit complex $\mathsf{C}^\bullet_E$.

By saying that $\Phi(E^{[n]})$ is a sheaf we mean that, as a complex in the derived category $D^b(X^{[n]})$, it has nontrivial cohomology only in degree zero.

Remark 1.4.17. The first term of the complex $\mathsf{C}_E^\bullet$ is

$$\mathsf{C}_E^0 :\stackrel{\text{def}}{=} \bigoplus_{i=1}^n pr_i^* E$$

and, in particular, it is locally free. For the other terms, we are not going to give an explicit description, since we will not use it later. However we will need the following key property proven by Krug in [Kru14, Proof of Lemma 3.3].

Theorem 1.4.18 (Krug). *Let E be a vector bundle on X. Then for all $i \geq 0$ we have*

$$\mathcal{E}xt^j_{X^n}(\mathsf{C}_E^i, \mathcal{O}_{X^n}) = 0 \qquad \textit{if } j \neq 2i.$$

To conclude this section, we have the following result of Krug, who noted that it is worth considering both equivalences Φ and Ψ at the same time.

Proposition 1.4.19 (Krug). *For any $F \in D^b(X^{[n]})$ and $G \in D^b_{\mathfrak{S}_n}(X^n)$ there is an isomorphism in $D^b(X^{(n)})$*

$$R\mu_*(F \overset{L}{\otimes} \Psi(G)) \cong \pi_*^{\mathfrak{S}_n}(\Phi(F) \overset{L}{\otimes} G)$$

which is functorial in F and G. In particular we have isomorphisms of functors

$$R\mu_* \circ \Psi \cong \pi_*^{\mathfrak{S}_n}, \qquad \pi_*^{\mathfrak{S}_n} \circ \Phi \cong R\mu_*.$$

Proof. For the first part see [Kru16, Proposition 5.1]. The second part is a consequence of the first, together with the observation that $\Psi(\mathcal{O}_{X^n}) \cong \mathcal{O}_{X^{[n]}}$ [Kru16, Remark 3.10] and $\Phi(\mathcal{O}_{X^{[n]}}) \cong \mathcal{O}_{X^n}$ [Sca09, Proposition 1.3.3]. □

Chapter 2

Asymptotic syzygies and higher order embeddings

In this chapter we prove Theorem A, Theorem B and Corollary C about asymptotic syzygies and higher order embeddings. We recall them here for convenience.

Theorem A. *Let X be a projective scheme and B a line bundle on X. Then*

$$\text{if} \quad K_{p,1}(X,B,L) = 0 \quad \text{for } L \gg 0 \qquad \text{then} \qquad B \text{ is } p\text{-very ample.}$$

Moreover, suppose that X is smooth and irreducible of dimension n and let L be a line bundle of the form

$$L = \omega_X \otimes A^{\otimes d} \otimes P^{\otimes(n-1)} \otimes N, \qquad d \geq (n-1)(p+1) + p + 3,$$

where A is a very ample line bundle, P a globally generated line bundle, such that $P \otimes B^{\vee}$ is nef and N a nef line bundle, such that $N \otimes B$ is nef. For such a line bundle, it holds that

$$\text{if} \quad K_{p,1}(X,B,L) = 0 \qquad \text{then} \qquad B \text{ is } p\text{-spanned.}$$

Theorem B. *Let X be a smooth and irreducible projective surface, B a line bundle and $0 \leq p \leq 3$ an integer. Then*

$$K_{p,1}(X,B,L) = 0 \quad \text{for } L \gg 0 \qquad \text{if and only if} \qquad B \text{ is } p\text{-very ample.}$$

Corollary C. *Let X be a smooth and irreducible projective variety of dimension n and suppose that $K_{h^0(X,L)-1-n-p,n}(X, \mathcal{O}_X, L)$ vanishes for $L \gg 0$. Then the covering gonality and the degree of irrationality of X are at least $p+2$.*

We begin in Section 2.1, by discussing in more detail the notions of higher order embeddings which we have seen in the Introduction. Afterwards, Section 2.2 is devoted to the proof of Theorem A: the strategy is essentially a reduction to finite subschemes of projective spaces. In Section 2.3, we describe some quantitative measures of irrationality, and we prove Corollary C as an application of Theorem A.

The last two Sections 2.4 and 2.5 contain the proof of Theorem B. We follow the strategy of Ein and Lazarsfeld for curves, working on the Hilbert scheme of points. The additional difficulty for a surface X is that the Hilbert scheme of points $X^{[n]}$ does not coincide with the symmetric product $X^{(n)}$. We proceed to study the Hilbert-Chow morphism $\mu\colon X^{[n]} \to X^{(n)}$ more closely and we get in Proposition 2.4.4 a characterization of the asymptotic vanishing of $K_{p,1}(X,B,L)$ purely in terms of B. We then show in Proposition 2.5.1 that a p-very ample line bundle B satisfies this criterion, assuming some cohomological vanishings about tautological bundles on the Hilbert scheme.

To prove these vanishings, we interpret them in the light of the Bridgeland-King-Reid-Haiman correspondence for $X^{[n]}$. Using this correspondence, we are able to verify the desired vanishing statements for p at most 3, proving Theorem B. At the end of the chapter, we discuss some open problems. In particular, we include some comments about a possible strategy to extend Theorem B to higher p and we argue that this is essentially a combinatorial problem on the ring $\mathbb{C}[x,y]$.

2.1 Higher order embeddings

First, we would like to present in more detail the notions of higher order embeddings that we have seen in the Introduction of the thesis. These notions were introduced by Beltrametti, Francia and Sommese in [BFS89] and they have been subjects of considerable attention: for a sample of the work done on this topic one can look at [BS90],[BS93],[BS97],[Knu01].

The first notion is that of p-very ampleness.

Definition 2.1.1 (Higher very ampleness)**.** Let $p \geq 0$ be an integer. A line bundle B on a projective scheme X is called *p-very ample*, if for every finite subscheme $\xi \subseteq X$ of length $p+1$ the evaluation map

$$\mathrm{ev}_\xi\colon H^0(X,B) \longrightarrow H^0(X, B\otimes \mathcal{O}_\xi)$$

is surjective.

Remark 2.1.2. From the above definition, we see that a line bundle B is 0-very ample if and only if it is globally generated, and that it is 1-very ample if and only if it is very ample in the usual sense [Har77, Remark II.7.8]. For higher p, we can interpret failure of p-very ampleness via the existence of special secant varieties: if B is very ample, then it is p-very ample for $p \geq 2$ if and only if under the embedding

$$\phi_B\colon X \hookrightarrow \mathbb{P}(H^0(X,B))$$

there is no $(p-1)$-space that is $(p+1)$-secant to X. For example, B is 2-very ample if and only if it is very ample and the variety $\phi_B(X) \subseteq \mathbb{P}(H^0(X,B))$ has no trisecant lines.

A weaker version of p-very ampleness is given by p-spannedness. This condition essentially means that there are no special secant varieties to smooth curves contained in the variety. The precise definition is the following:

Definition 2.1.3 (Higher spannedness)**.** Let $p \geq 0$ be an integer. A line bundle B on a projective scheme X is called *p-spanned* if for every finite and curvilinear subscheme $\xi \subseteq X$ of length $p+1$ the evaluation map

$$\mathrm{ev}_\xi \colon H^0(X,B) \longrightarrow H^0(X, B\otimes \mathcal{O}_\xi)$$

is surjective.

Remark 2.1.4. It is obvious from the definition that p-very ampleness implies p-spannedness, and it is also clear that the reverse implication holds on a smooth projective curve. On an arbitrary projective scheme, the two notions are the same when $p \leq 2$. It is expected that the two notions diverge on smooth surfaces as soon as $p \geq 3$, but we do not know of any explicit example. Interestingly, for some surfaces such as abelian surfaces [Ter98] and K3 surfaces [Knu01] the two notions coincide.

Proof. We show here that p-spannedness implies p-very ampleness when $p \leq 2$. For $p = 0, 1$ this is true because by Example 1.2.6 every subscheme of length 1 or 2 is curvilinear. Example 1.2.6 shows also that for $p = 2$, it is enough to consider a subscheme $\xi \subseteq X$ isomorphic to $\operatorname{Spec} \mathbb{C}[X,Y]/(X,Y)^2$. In particular, such a scheme is supported at a single point $P \in X$ and, if $\mathfrak{m}_P$ is the ideal sheaf of the point, we have that $\xi \subseteq \operatorname{Spec} \mathcal{O}_\xi/\mathfrak{m}_P^2$. Hence, the map ev_ξ factors as

$$H^0(X,B) \longrightarrow H^0(X, B\otimes \mathcal{O}_X/\mathfrak{m}_P^2) \longrightarrow H^0(X, B\otimes\mathcal{O}_\xi)$$

and it is enough to prove that the first map is surjective. However, this map is surjective, because B is very ample [Har77, Proposition II.7.3]. □

Another notion of higher order embedding, stronger than p-very ampleness, is given by p-jet very ampleness.

Definition 2.1.5 (Jet very ampleness)**.** Let $p \geq 0$ be an integer. A line bundle B on a projective scheme X is called *p-jet very ample* if for every zero cycle $\zeta = a_1P_1 + \cdots + a_rP_r$ of degree $p+1$, the evaluation map

$$\mathrm{ev}_\zeta \colon H^0(X,B) \longrightarrow H^0(X, B\otimes\mathcal{O}_X/\mathfrak{m}_\zeta), \qquad \mathfrak{m}_\zeta := \mathfrak{m}_{P_1}^{a_1}\dots\mathfrak{m}_{P_r}^{a_r} \tag{2.1.1}$$

is surjective.

Remark 2.1.6. If a line bundle is p-jet very ample, then it is also p-very ample and a fortiori it is p-spanned. Moreover, these three notions coincide on a smooth curve. For a general projective scheme, p-jet very ampleness coincides with p-very ampleness and p-spannedness for $p = 0, 1$, and it is strictly stronger than p-very ampleness for $p \geq 2$.

Proof. We first show that a p-jet very ample line bundle is p-very ample. Proceeding as in the proof of Remark 2.1.4, it is enough to show that, if $\xi \subseteq X$ is a scheme of length $p+1$ supported at a point P, then $\xi \subseteq \operatorname{Spec} \mathcal{O}_X/\mathfrak{m}_P^{p+1}$. To do this, let $\mathfrak{m}_P\mathcal{O}_\xi$ be the ideal corresponding to $\mathfrak{m}_P$ in $\mathcal{O}_\xi$: then we have a filtration of length $p+2$

$$\mathcal{O}_\xi \supseteq \mathfrak{m}_P\mathcal{O}_\xi \supseteq \mathfrak{m}_P^2\mathcal{O}_\xi \supseteq \cdots \supseteq \mathfrak{m}_P^{p+1}\mathcal{O}_\xi \supseteq 0$$

and since $\mathcal{O}_\xi$ has length $p+1$, not all of these inclusions are proper. It follows that $\mathfrak{m}_P^i\mathcal{O}_\xi = \mathfrak{m}_P^{i+1}\mathcal{O}_\xi$, for a certain $1 \leq i \leq p+1$, and Nakayama's Lemma implies that $\mathfrak{m}_P^{p+1}\mathcal{O}_\xi = 0$. But this is saying exactly that $\mathcal{O}_\xi \subseteq \operatorname{Spec} \mathcal{O}_X/\mathfrak{m}_P^{p+1}$.

Next, it is clear that p-jet very ampleness is the same as p-very ampleness and p-spannedness on a smooth curve. The proof of Remark 2.1.4 shows that these notions coincide on an arbitrary scheme for $p = 0, 1$.

To conclude, we prove that p-jet very ampleness is in general strictly stronger than p-very ampleness for $p \geq 2$. To do this, let X be a K3 surface and L an ample line bundle that is not globally generated: [BRS00] shows that L^{p+2} is not p-jet very ample for $p \geq 2$, whereas the criterion of [Knu01, Theorem 1.1] proves that L^{p+2} is p-very ample. □

2.1.1 Higher order embeddings via Hilbert schemes

The various concepts of higher order embeddings have natural interpretations in terms of tautological bundles on the Hilbert scheme. We restrict here to the case of a smooth quasiprojective surface X.

We fix an integer $p \geq 0$ and we consider the universal family $\Xi^{[p+1]}$ over the Hilbert scheme $X^{[p+1]}$, together with the projections

$$p_{X^{[p+1]}} \colon \Xi^{[p+1]} \longrightarrow X^{[p+1]}, \qquad p_X \colon \Xi^{[p+1]} \longrightarrow X.$$

For any line bundle B on X we have the usual evaluation map $H^0(X, B) \otimes \mathcal{O}_X \to B$. Pulling back this map to $\Xi^{[p+1]}$ along p_X and pushing forward to $X^{[p+1]}$ via $p_{X^{[n]}}$, we obtain another evaluation map

$$\mathrm{ev}_B \colon H^0(X, B) \otimes_{\mathbb{C}} \mathcal{O}_{X^{[n]}} \to B^{[n]}. \tag{2.1.2}$$

By construction, the fiber of the map over each point $\xi \in X^{[p+1]}$ is precisely the map

$$\mathrm{ev}_\xi \colon H^0(X,B) \longrightarrow H^0(X, B \otimes \mathcal{O}_\xi)$$

that evaluates the sections of B on the subscheme $\xi \subseteq X$. Hence, the line bundle B is p-very ample on X if and only if the evaluation map ev_B is surjective on $X^{[p+1]}$. Moreover, B is p-spanned if and only if the map ev_B is surjective on the open subset of curvilinear subschemes $X^{[p+1]}_{curv} \subseteq X^{[p+1]}$.

Remark 2.1.7. In fact, the map $\mathrm{ev}_B \colon H^0(X,B) \otimes_{\mathbb{C}} \mathcal{O}_{X^{[p+1]}} \to B^{[p+1]}$ induces an isomorphism $H^0(X,B) \cong H^0(X^{[p+1]}, B^{[p+1]})$ [Kru16, Corollary 4.2]. Thus, we can say that B is p-very ample if and only if $B^{[p+1]}$ is globally generated. In this case, we observe that we have an induced morphism from the Hilbert scheme into a Grassmannian variety:

$$X^{[p+1]} \longrightarrow G(p+1, H^0(X,B)), \qquad \xi \mapsto [H^0(X,B) \longrightarrow H^0(X, B \otimes \mathcal{O}_\xi)].$$

Moreover, since the complement of the open subset $X^{[p+1]}_{curv} \subseteq X^{[p+1]}$ has codimension at least two (see Remark 1.2.8), we see that $H^0(X^{[p+1]}, B^{[p+1]}_{curv}) \cong H^0(X^{[p+1]}, B^{[p+1]}) \cong H^0(X,B)$. So, we can say that B is p-spanned if and only if the restriction $B^{[p+1]}_{curv}$ is globally generated.

There is also a connection between tautological bundles and jet very ampleness, which is stated already in [ELY16] in a different language. Consider again a line bundle B on X: in [ELY16, Lemma 1.5], the authors construct a coherent sheaf $\mathcal{E}_{p+1,B}$ on X^{p+1} such that the fiber over a point $(P_1, \dots, P_{p+1}) \in X^{p+1}$ is given by

$$\mathcal{E}_{p+1,B_{|(P_1,\dots,P_{p+1})}} = H^0(X, B \otimes \mathcal{O}_X/\mathfrak{m}_\zeta), \qquad \zeta = P_1 + \cdots + P_{p+1}.$$

Moreover, they construct an evaluation map

$$\mathrm{ev}_{\mathcal{E}_B} \colon H^0(X,B) \otimes \mathcal{O}_{X^{p+1}} \to \mathcal{E}_{p+1,B} \tag{2.1.3}$$

which on fibers coincides with the jet evaluation of (2.1.1), so that B is p-jet very ample if and only if $\mathrm{ev}_{\mathcal{E}_B}$ is surjective. Looking at the construction of [ELY16], one actually sees that $\mathcal{E}_{p+1,B}$ is obtained as $\mathcal{E}_{p+1,B} \cong pr_{[1,p+1],*}(pr_0^* B \otimes \mathcal{O}_{D_{p+1}})$, using the notation of (1.4.2) and (1.4.3). This is precisely the sheaf that appears in Scala's Theorem 1.4.16, that we can then rephrase as follows.

Corollary 2.1.8 (Scala). *Let B be a line bundle on X and $p \geq 0$ an integer. Then $\Phi(B^{[p+1]}) \cong \mathcal{E}_{p+1,B}$ in $D^b_{\mathfrak{S}_n}(X^{p+1})$ and the evaluation map $\mathrm{ev}_{\mathcal{E}_B}$ (2.1.3) corresponds to the map*

$$H^0(X,B) \otimes_{\mathbb{C}} \Phi(\mathcal{O}_{X^{[p+1]}}) \to \Phi(B^{[p+1]})$$

that we obtain applying the functor Φ to the evaluation map ev_B (2.1.2).

2.2 Asymptotic syzygies and finite subschemes

In this section we prove Theorem A. The basic fact that we employ is the following observation:

Lemma 2.2.1. *Let V be a vector space of dimension $p+1$ and let $N = \bigoplus_{q\geq 0} N_q$ be a graded* $\operatorname{Sym}^\bullet V$*-module such that*

$$(0 :_{N_0} V) \stackrel{def}{:=} \{y \in N_0 \,|\, v\cdot y = 0 \text{ for all } v \in V\} = 0.$$

Then, for any submodule $M \subset N$, such that $M_0 \subsetneq N_0$ and $M_1 = N_1$, we have $K_{p,1}(M;V) \neq 0$.

Proof. We have a short exact sequence of $\operatorname{Sym}^\bullet V$-modules

$$0 \longrightarrow M \longrightarrow N \longrightarrow N/M \longrightarrow 0$$

which induces a long exact sequence in Koszul cohomology (see Lemma 1.1.12):

$$\ldots \longrightarrow K_{p+1,0}(N;V) \longrightarrow K_{p+1,0}(N/M;V) \longrightarrow K_{p,1}(M;V) \longrightarrow \ldots$$

Thanks to our hypotheses on M, the Koszul complex (1.1.1) shows immediately that $K_{p+1,0}(N/M;V) \cong \wedge^{p+1}V \otimes (N/M)_0 \neq 0$. To conclude, it suffices to show that $K_{p+1,0}(N;V) = 0$. The Koszul complex (1.1.1) shows that

$$K_{p+1,0}(N;V) = \operatorname{Ker}\left[d_{p+1,0}\colon\ \wedge^{p+1} V \otimes N_0 \to \wedge^p V \otimes N_1\right].$$

Now fix a basis $x_0, \ldots, x_p$ of V: for every $y \in N_0$ we have

$$d_{p+1,0}(x_0 \wedge \cdots \wedge x_{p+1} \otimes y) = \sum_{i=0}^{p+1} (-1)^i x_0 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_p \otimes x_i \cdot y.$$

Hence, $d_{p+1,0}(x_0 \wedge \cdots \wedge x_{p+1} \otimes y) = 0$ if and only if $x_i \cdot y = 0$ for all i. But by assumption, this implies $y = 0$, which concludes the proof. □

The previous lemma implies the following general non-vanishing result of syzygies.

Lemma 2.2.2. *Let X be a projective scheme, B a line bundle on X and $\xi \subseteq X$ a finite subscheme of length $p+1$, such that the evaluation map*

$$\operatorname{ev}_\xi\colon H^0(X,B) \longrightarrow H^0(X, B\otimes \mathcal{O}_\xi)$$

is not surjective. Let also L be an ample and globally generated line bundle on X such that

1. *$H^1(X, \mathcal{I}_\xi \otimes B \otimes L^{\otimes q}) = 0$ for all $q > 0$.*

2. $K_{p-1,2}(X, \mathcal{I}_\xi \otimes B, L) = 0$.

3. $H^1(X, \mathcal{I}_\xi \otimes L) = 0$.

Then $K_{p,1}(X, B, L) \neq 0$.

Proof. Consider the short exact sequence of sheaves on X:

$$0 \longrightarrow \mathcal{I}_\xi \otimes B \longrightarrow B \longrightarrow B \otimes \mathcal{O}_\xi \longrightarrow 0.$$

Twisting by powers of L and taking global sections, we get an exact sequence of graded $\operatorname{Sym}^\bullet H^0(X, L)$-modules

$$0 \longrightarrow \bigoplus_{q \geq 0} H^0(X, \mathcal{I}_\xi \otimes B \otimes L^{\otimes q}) \longrightarrow \bigoplus_{q \geq 0} H^0(X, B \otimes L^{\otimes q}) \longrightarrow M \longrightarrow 0.$$

Assumption (1) shows that M is a submodule of $\bigoplus_{q \geq 0} H^0(X, B \otimes L^{\otimes q} \otimes \mathcal{O}_\xi)$ such that

$$M_0 \subsetneq H^0(X, B \otimes \mathcal{O}_\xi), \qquad M_q = H^0(X, B \otimes L^{\otimes q} \otimes \mathcal{O}_\xi) \qquad \text{for all } q > 0. \quad (2.2.1)$$

The sequence 2.2 induces an exact sequence in Koszul cohomology as in Lemma 1.1.12:

$$\ldots \longrightarrow K_{p,1}(X, B, L) \longrightarrow K_{p,1}(M; H^0(X, L)) \longrightarrow K_{p-1,2}(X, \mathcal{I}_\xi \otimes B, L) \longrightarrow \ldots$$

and assumption (2) shows that map $K_{p,1}(X, B, L) \to K_{p,1}(M; H^0(X, L))$ is surjective. Hence, it is enough to prove that $K_{p,1}(M; H^0(X, L)) \neq 0$. To do this, observe that the structure of $\operatorname{Sym}^\bullet H^0(X, L)$-module on M is induced by the structure of $\operatorname{Sym}^\bullet H^0(X, L \otimes \mathcal{O}_\xi)$-module. Moreover, assumption (3) shows that the evaluation map $H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_\xi)$ is surjective. Hence, Lemma 1.1.14 gives a surjection

$$K_{p,1}(M; H^0(X, L)) \longrightarrow K_{p,1}(M; H^0(X, L \otimes \mathcal{O}_\xi))$$

and to conclude it is enough to show that $K_{p,1}(M; H^0(X, L \otimes \mathcal{O}_\xi)) \neq 0$. However, this follows immediately from the description of M in (2.2.1) and Lemma 2.2.1. □

To apply the previous lemma, we need a statement for the asymptotic vanishing of high degree syzygies of weight at least two. This is probably already known but we include a proof for completeness.

Lemma 2.2.3. *Let X be a projective scheme, A an ample line bundle and P an arbitrary line bundle on X. For any integer $d > 0$, set $L_d = A^{\otimes d} \otimes P$. Fix a coherent sheaf $\mathcal{F}$ on X and two integers $p \geq 0, q \geq 2$. Then $K_{p,q}(X, \mathcal{F}, L_d) = 0$ for infinitely many d.*

Proof. First suppose that X is smooth. In this case we claim that $K_{p,q}(X, \mathcal{F}, L_d) = 0$ for all $d \gg 0$. If $\mathcal{F}$ is locally free, we have $K_{p,q}(X, \mathcal{F}, L_d) = 0$ for $d \gg 0$, thanks for example to [Yan14, Proof of Theorem 4]. Assume now that $\mathcal{F}$ is an arbitrary coherent sheaf. Since X is smooth, $\mathcal{F}$ has a finite resolution by locally free sheaves [Har77, Exercise III.6.9]: we can choose a resolution with the minimum length ℓ, so that we get an exact complex

$$0 \to E_\ell \longrightarrow E_{\ell-1} \longrightarrow \dots \longrightarrow E_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

where the E_i are locally free. We proceed to prove the lemma by induction on ℓ. If $\ell = 0$, then $\mathcal{F}$ is locally free and we are done. If $\ell > 0$, we can split the resolution into two exact complexes

$$0 \longrightarrow \mathcal{G} \longrightarrow E_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$
$$0 \longrightarrow E_\ell \longrightarrow E_{\ell-1} \longrightarrow \dots \longrightarrow E_1 \longrightarrow \mathcal{G} \longrightarrow 0.$$

Since $d \gg 0$, we get $H^1(X, \mathcal{G} \otimes L_d^{\otimes q}) = 0$ for all $q \geq 1$, so that we obtain a short exact sequence of $\operatorname{Sym}^\bullet H^0(X, L_d)$-graded modules:

$$0 \to \bigoplus_{h \geq 1} H^0(X, \mathcal{G} \otimes L_d^{\otimes h}) \to \bigoplus_{h \geq 1} H^0(X, E_0 \otimes L_d^{\otimes h}) \to \bigoplus_{h \geq 1} H^0(X, \mathcal{F} \otimes L_d^{\otimes h}) \to 0.$$

Since $q \geq 2$, this sequence induces an exact sequence in Koszul cohomology as in Lemma 1.1.12:

$$\dots \longrightarrow K_{p,q}(X, E_0, L_d) \longrightarrow K_{p,q}(X, \mathcal{F}, L_d) \longrightarrow K_{p-1,q+1}(X, \mathcal{G}, L_d) \longrightarrow \dots$$

If $d \gg 0$, we know that $K_{p,q}(X, E_0, L_d) = 0$ because E_0 is locally free. Moreover, $K_{p-1,q+1}(X, \mathcal{G}, L_d) = 0$ by induction hypothesis. Hence, $K_{p,q}(X, \mathcal{F}, L_d) = 0$ as well, and we are done.

Now take an arbitrary projective scheme X. We claim that it is enough to find a closed embedding $j \colon X \hookrightarrow Y$ such that Y is smooth and so that it has two line bundles $\widetilde{A}, \widetilde{P}$, with $\widetilde{A}$ ample, such that $j^*\widetilde{A} \cong A$ and $j^*\widetilde{P} \cong P$. Indeed, in this case set $\widetilde{L}_d = \widetilde{P} \otimes \widetilde{A}^{\otimes d}$. If $d \gg 0$, we can assume that the restriction map

$$H^0(Y, \widetilde{L}_d) \to H^0(X, L_d) \tag{2.2.2}$$

is surjective. Since Y is smooth, what we have already proven gives the vanishing $K_{p,q}(Y, j_*\mathcal{F}, \widetilde{L}_d) = 0$ for $d \gg 0$. However, the structure of $\operatorname{Sym}^\bullet H^0(Y, L_d)$-module on

$$\bigoplus_h H^0(Y, j_*\mathcal{F} \otimes \widetilde{L}_d^{\otimes h}) = \bigoplus_h H^0(X, \mathcal{F} \otimes L_d^{\otimes h})$$

is induced by the structure of $\operatorname{Sym}^\bullet H^0(X, L_d)$-module via the map (2.2.2). Hence, Lemma 1.1.14 on Koszul cohomology with respect to two different rings gives that $K_{p,q}(X, \mathcal{F}, L_d) = 0$ as well.

We just need to find the embedding $j\colon X \hookrightarrow Y$. Observe that in the original statement we can replace P by a translate $P \otimes A^{\otimes h}$, and A by a positive multiple $A^{\otimes m}$. Hence, we choose h, k positive such that both $P \otimes A^{\otimes h}$ and $P^\vee \otimes A^{\otimes k}$ are very ample, and consider the induced closed embedding $\varphi\colon X \hookrightarrow \mathbb{P}^n \times \mathbb{P}^m$. Then we see that $\varphi^*\mathcal{O}(1,0) = P \otimes A^{\otimes h}$, $\varphi^*\mathcal{O}(1,1) = A^{\otimes(h+k)}$. Since $\mathcal{O}(1,1)$ is ample, this completes the proof. □

Using this, we could already give the proof of the first part of Theorem A, but we postpone this until the end of the next section, so that we can also prove the second part.

2.2.1 An effective result for spanned line bundles

In this section, we give a proof of the second part of Theorem A. The idea is to find effective bounds for the conditions of Lemma 2.2.2. The essential reason for restricting to spannedness instead of very ampleness is to have an effective vanishing statement along the lines of Lemma 2.2.3: this is given by the following result of Ein and Lazarsfeld [EL93, Theorem 2].

Theorem 2.2.4 (Ein, Lazarsfeld)**.** *Let X be a smooth projective variety of dimension n, B a line bundle and $p \geq 0$ an integer. Let also L be a line bundle of the form*

$$L \cong \omega_X \otimes A^{\otimes d} \otimes N$$

where A is a very ample line bundle, N is a nef line bundle such that $N \otimes B$ is nef and $d \geq n+1+p$. Then $K_{p,q}(X,B,L) = 0$ for all $q \geq 2$.

Proof. This is a reformulation of [EL93, Theorem 2]. □

With this, we can prove the second part of Theorem A by induction on the dimension of X. For the inductive step, we need the next two lemmas.

Lemma 2.2.5. *Let X be a smooth projective variety, L an ample and globally generated line bundle, B another line bundle and $p \geq 0$ an integer. Let also $D \subseteq X$ be a divisor such that:*

1. $H^1(X, L^{\otimes q} \otimes B \otimes \mathcal{O}_X(-D)) = 0$ *for all* $q \geq 0$.
2. $K_{p-1,2}(X, B \otimes \mathcal{O}_X(-D), L) = 0$.
3. $H^1(X, L \otimes \mathcal{O}_X(-D)) = 0$.

Then the natural maps

$$H^0(X,B) \longrightarrow H^0(D, B_{|D}), \qquad K_{p,1}(X,B,L) \longrightarrow K_{p,1}(D, B_{|D}, L_{|D})$$

are surjective.

Proof. The proof goes along the same lines as that of Lemma 2.2.2, so we provide here just a sketch. Assumption (1) yields a short exact sequence of graded $\mathrm{Sym}^\bullet H^0(X,L)$-modules:

$$0 \to \bigoplus_{q\geq 0} H^0(X, L^{\otimes q} \otimes B \otimes \mathcal{O}_X(-D)) \to \bigoplus_{q\geq 0} H^0(X, B\otimes L^{\otimes q}) \to M \to 0$$

where $M = \bigoplus_{q\geq 0} H^0(D, L_{|D}^{\otimes q} \otimes B_{|D})$. In particular, the map $H^0(X,B) \to H^0(D, B_{|D})$ is surjective. The long exact sequence in Koszul cohomology of Lemma 1.1.12 and assumption (2) show that the natural map

$$K_{p,1}(X,B,L) \to K_{p,1}(M; H^0(X,L))$$

is surjective. Moreover, assumption (3) and Lemma 1.1.14 imply that the natural map

$$K_{p,1}(M; H^0(X,L)) \to K_{p,1}(D, B_D, L_D).$$

is also surjective. In particular, the composite map $K_{p,1}(X,B,L) \to K_{p,1}(X, B_D, L_D)$ is surjective, and this is the map we were looking for. □

Lemma 2.2.6. *Let X be a smooth and irreducible projective variety of dimension at least two. Let $\xi \subseteq X$ be a curvilinear subscheme of length k and H an ample and k-jet very ample line bundle on X. Then there exists a smooth and irreducible divisor $D \in |H|$ such that $\xi \subseteq D$.*

Proof. Consider the linear system $V = H^0(X, H \otimes \mathcal{I}_\xi)$. We will show that a general divisor in $|V|$ is smooth and irreducible. We first show that V has base points only at the points of ξ. If $P \notin \xi$, the subscheme $\xi \cup \{P\}$ has length $k+1$, and since H is in particular k-very ample (see Remark 2.1.6), the evaluation map $\mathrm{ev}_\xi \colon H^0(X,H) \to H^0(X, H \otimes \mathcal{O}_{\xi\cup\{P\}})$ is surjective. Hence, P is not a base point of V. Now, Bertini's theorem [Har77, Remark III.10.9.1] tells us that a general divisor $D \in |V|$ is irreducible and nonsingular away from the support of ξ. We need to check what happens at the points of ξ, and for this we can suppose that ξ is supported at a single point P. Since ξ is curvilinear, we can find by Remark 1.2.7 analytic coordinates $(x_1, \dots, x_n)$ around P, so locally we have $\mathcal{I}_\xi = (x_1, \dots, x_{n-1}, x_n^k)$. Moreover, as H is k-jet very ample, the map $H^0(X,H) \to H^0(X, H \otimes \mathcal{O}_X/\mathfrak{m}_P^k)$ is surjective. Hence, the power series expansion of a general section $\sigma \in V$ around P has a nonzero coefficient for x_1, so that σ defines a divisor which is nonsingular at P. □

Now we can give the proof of the second part of Theorem A. The first case is that of curves.

Proposition 2.2.7. *Let C be a smooth, projective and irreducible curve of genus g, and B a* *line bundle which is not p-very ample. Let also L be a line bundle such that*

$$\deg L \geq 2g + p + 1, \qquad \deg(L \otimes B) \geq 2g + p + 1.$$

Then $K_{p,1}(C, B, L) \neq 0$.

Proof. Observe that L is ample and globally generated by Riemann-Roch. Suppose first that $h^0(C, B) \geq p + 1$. Let $\xi \subseteq C$ be an effective divisor of degree $p + 1$ such that the evaluation map $\mathrm{ev}_\xi \colon H^0(C, B) \to H^0(C, B \otimes \mathcal{O}_\xi)$ is not surjective. We show that L satisfies the conditions of Lemma 2.2.2. Since $\deg L \geq 2g + p$ and B is effective, it is easy to see that conditions (1) and (3) hold. To check condition (2), we need to show that $K_{p,1}(C, B \otimes \mathcal{O}_C(-\xi), L) = 0$. By Proposition 1.1.22, it is enough to show that $H^1(C, \wedge^p M_L \otimes L \otimes B \otimes \mathcal{O}_C(-\xi)) = 0$. Since $\deg L \geq 2g + p$, a result of Green [Gre84, Theorem (4.a.1)] implies that $H^1(C, \wedge^p M_L \otimes L) = 0$. Hence, if we can prove that $B \otimes \mathcal{O}_C(-\xi)$ is effective, it follows that $H^1(C, \wedge^p M_L \otimes L \otimes B \otimes \mathcal{O}_C(-\xi)) = 0$ as well. To check that $B \otimes \mathcal{O}_C(-\xi)$ is effective, observe that $h^0(C, B) \geq p + 1$ by assumption, and moreover the evaluation map ev_ξ is not surjective, so that $h^0(C, B \otimes \mathcal{O}_C(-\xi)) > h^0(C, B) - p - 1 \geq 0$, and we are done.

Assume now that $h^0(C, B) \leq p$. By Proposition 1.1.5, the syzygy group $K_{p,1}(C, B, L)$ is the cokernel of the map

$$\wedge^{p+1} H^0(C, L) \otimes H^0(C, B) \to H^0(C, \wedge^p M_L \otimes L \otimes B).$$

Thus, to prove what we want it is enough to show that

$$\dim_{\mathbb{C}} \wedge^{p+1} H^0(C, L) \otimes H^0(C, B) < \dim_{\mathbb{C}} H^0(C, \wedge^p M_L \otimes L \otimes B). \tag{2.2.3}$$

To do this, set $d = \deg L$ and $b = \deg B$. We can estimate the dimension of $H^0(C, \wedge^p M_L \otimes L \otimes B)$ via the Euler characteristic, which is easy to compute with Riemann-Roch:

$$h^0(C, \wedge^p M_L \otimes L \otimes B) \geq \chi(C, \wedge^p M_L \otimes L \otimes B) = \binom{d-g}{p} \left(-p \cdot \frac{d}{d-g} + d + b \right).$$

Now, suppose that $0 < h^0(C, B) \leq p$: in particular $b \geq 0$. We can just bound the left hand side of (2.2.3) by $\binom{d+1-g}{p+1} p$ and then a computation shows that (2.2.3) holds, thanks to $d \geq 2g + p + 1$ and $b \geq 0$.

The last case is when $h^0(C, B) = 0$. To prove (2.2.3), it is enough to show that $\chi(C, \wedge^p M_L \otimes L \otimes B) > 0$. This can be checked by a computation, using the assumption that $d + b \geq 2g + p + 1$. □

Remark 2.2.8. Going through the proof of Proposition 2.2.7 more carefully, it is not hard to show that the assumption on L can be weakened to $\deg L \geq 2g + p$, at least when C has genus $g \geq 2$. In this case, setting $B = \omega_C$, Proposition 2.2.7 and Lemma

2.3.8 imply that if C has gonality k, then

$$K_{k-1,1}(C,\omega_C,L) \cong K_{h^0(L)-k-1,1}(C,\mathcal{O}_C,L) \neq 0$$

for every line bundle L of degree $\deg L \geq 2g+k-1$. This is a well-known consequence of the Green-Lazarsfeld non-vanishing Theorem [Gre84, Appendix]. Conversely, Farkas and Kemeny proved a vanishing theorem in [FK16, Theorem 0.2]: if C is a general k-gonal curve of genus at least 4, then $K_{h^0(L)-k,1}(C,\mathcal{O}_C,L) = 0$, when $\deg L \geq 2g+k-1$. However, they note in the same paper that this vanishing does not hold for every curve.

We can now give the full proof for the second part of Theorem A: we rewrite the statement below for clarity, and we formulate it as a non-vanishing statement.

Theorem 2.2.9. *Let X be a smooth and irreducible projective variety of dimension n, and B a line bundle on X which is not p-spanned. Then $K_{p,1}(X,B,L) \neq 0$, for every line bundle L of the form*

$$L = \omega_X \otimes A^{\otimes d} \otimes P^{\otimes(n-1)} \otimes N, \qquad d \geq (n-1)(p+1)+p+3,$$

where A is a very ample line bundle, P a globally generated line bundle such that $P \otimes B^{\vee}$ is nef, and N is a nef line bundle such that $N \otimes B$ is nef.

Proof. We observe that any L as in the statement of the theorem is very ample: indeed, Kodaira vanishing shows that $L \otimes A^{-1}$ is 0-regular with respect to A in the sense of Castelnuovo-Mumford [Laz04, Definition 1.8.4]. In particular, it is globally generated [Laz04, Theorem 1.8.5]. Hence $L = (L \otimes A^{-1}) \otimes A$ is very ample.

We proceed to prove the theorem by induction on n. If $n = 1$, set g to be the genus of the curve X: then we see that $\deg L \geq 2g-2+d \geq 2g+p+1$, and the same holds for $\deg(L \otimes B)$. Hence, the conclusion follows from Proposition 2.2.7.

Suppose now that $n \geq 2$ and that the result is true for $n-1$. Fix a finite, curvilinear scheme $\xi \subseteq X$ of length $p+1$ such that the evaluation map

$$\mathrm{ev}_\xi \colon H^0(X,B) \to H^0(X,B \otimes \mathcal{O}_\xi)$$

is not surjective. Consider the line bundle $H = P \otimes A^{\otimes(p+1)}$: since P is globally generated and A is very ample, H is $(p+1)$-jet very ample (see [BS93, Lemma 2.2]). Therefore, Lemma 2.2.6 shows that there is a smooth and irreducible divisor $D \in |H|$ such that $\xi \subseteq D$.

Now, let L be as in the statement of the theorem: we claim that L, B and D satisfy the hypotheses of Lemma 2.2.5. Indeed, we see that

$$L \otimes \mathcal{O}_X(-D) \cong L \otimes H^{-1} \cong \omega_X \otimes A^{\otimes(d-p-1)} \otimes P^{\otimes(n-2)} \otimes N$$

and the assumption on d shows that $A^{\otimes(d-p-1)} \otimes P^{\otimes(n-2)} \otimes N$ is ample, so that $H^1(X, L \otimes \mathcal{O}_X(-D)) = 0$, by Kodaira vanishing. A similar reasoning shows that

$H^1(X, L^{\otimes q} \otimes B \otimes \mathcal{O}_X(-D)) = 0$, for all $q \geq 1$. To check that $H^1(X, B \otimes \mathcal{O}_X(-D)) = 0$, observe that $H^1(X, B \otimes \mathcal{O}_X(-D))^\vee \cong H^{n-1}(X, \omega_X \otimes B^\vee \otimes H)$ and $H \otimes B^{-1} = P \otimes B^{-1} \otimes A^{\otimes(p+1)}$ is clearly ample, so that we can use Kodaira vanishing again, together with the assumption $n \geq 2$.

Finally, Theorem 2.2.4 shows that $K_{p-1,2}(X, B \otimes \mathcal{O}_X(-D), L)$ vanishes: indeed, we can write

$$L \cong \omega_X \otimes A^{\otimes(n+p)} \otimes A^{\otimes(d-n-p)} \otimes P^{\otimes(n-1)} \otimes N$$

and since $d - n - p \geq (n-1)p + 2$ we see that $A^{\otimes(d-n-p)} \otimes P^{\otimes(n-1)} \otimes N$ is nef. Furthermore,

$$A^{\otimes(d-n-p)} \otimes P^{\otimes(n-1)} \otimes N \otimes B \otimes \mathcal{O}_X(-D) \cong A^{\otimes(d-n-2p-1)} \otimes P^{\otimes(n-2)} \otimes B \otimes N$$

and since $d - n - 2p - 1 \geq (n-2)p + 1$, we see again that this is nef. Then the aforementioned Theorem 2.2.4 applies and we get that $K_{p-1,2}(X, B \otimes \mathcal{O}_X(-D), L) = 0$.

Applying Lemma 2.2.5, we obtain that the two natural restriction maps

$$H^0(X, B) \longrightarrow H^0(D, B_{|D}), \qquad K_{p,1}(X, B, L) \longrightarrow K_{p,1}(D, B_{|D}, L_{|D})$$

are surjective. In particular, since $\xi \subseteq D$, we see that B_D is not p-spanned on D. Moreover, the adjunction formula shows that

$$L_{|D} = K_D \otimes A_{|D}^{\otimes(d-(p-1))} \otimes P_{|D}^{\otimes(n-2)} \otimes N_{|D}$$

which clearly satisfies the induction hypothesis for $n-1$. Hence $K_{p,1}(D, B_{|D}, L_{|D}) \neq 0$, and since $K_{p,1}(X, B, L) \to K_{p,1}(D, B_{|D}, L_{|D})$ is surjective, this concludes the proof. □

We can now prove Theorem A.

Proof of Theorem A. We start from the first part. Let X be a projective scheme, and B a line bundle on X. Fix also an ample line bundle A, another line bundle P and set $L_d = P \otimes A^{\otimes d}$ for any integer $d > 0$. Assume that $K_{p,1}(X, B, L_d) = 0$ for $d \gg 0$. We want to show that B is p-very ample. So, we assume that B is not p-very ample and we claim that $K_{p,1}(X, B, L_d) \neq 0$ for infinitely many d.

To prove this claim, let $\xi \subseteq X$ be a finite subscheme of length $p+1$ such that the evaluation map

$$\mathrm{ev}_\xi \colon H^0(X, B) \to H^0(X, B \otimes \mathcal{O}_\xi)$$

is not surjective. Then it is enough to show that the hypotheses in Lemma 2.2.2 are verified for infinitely many d. Hypotheses (1) and (3) hold for all $d \gg 0$ thanks to Serre vanishing. Lemma 2.2.3 implies that hypothesis (2) is satisfied, so we are done.

The second part of Theorem A is exactly Theorem 2.2.9. □

2.3 Measures of irrationality

In this section we prove Corollary C as an application of Theorem A. We first review some quantitative measures of irrationality for algebraic varieties, which have been extensively studied recently: see for example [Bas12],[BCP13],[Bas+17b],[Bas+17a], [GK18],[SU18],[Voi18].

The starting point is the fundamental notion of gonality of a curve.

Definition 2.3.1 (Gonality)**.** Let C be an irreducible and reduced projective curve. The *gonality* of C is the minimal degree of a dominant rational map

$$f\colon C \dashrightarrow \mathbb{P}^1.$$

Remark 2.3.2. If C is a smooth and irreducible curve, then every rational and dominant map $C \dashrightarrow \mathbb{P}^1$ extends to a finite map $C \to \mathbb{P}^1$ of the same degree. Hence, the above definition of gonality agrees with the usual one for smooth curves and extends it to singular curves.

By definition, a curve has gonality one if and only if it is rational, hence we can look at the gonality as a measure of the irrationality of C. In higher dimension, this notion generalizes naturally to the degree of irrationality.

Definition 2.3.3 (Degree of irrationality)**.** Let X be a reduced and irreducible projective variety. The *degree of irrationality* $\operatorname{irr}(X)$ of X is the minimal degree of a dominant rational map

$$f\colon X \dashrightarrow \mathbb{P}^{\dim X}.$$

Another way to generalize the gonality to a higher dimensional variety X is via the covering gonality.

Definition 2.3.4 (Covering gonality)**.** Let X be a reduced and irreducible projective variety. The *covering gonality* $\operatorname{cov.gon}(X)$ of X is the minimal gonality of a reduced and irreducible curve C passing through a general point of X.

Remark 2.3.5. By definition, a variety X has degree of irrationality one if and only if it is rational. Instead, it has covering gonality one if and only if it is uniruled.

Remark 2.3.6. We can see that for any variety X we have

$$\operatorname{irr}(X) \geq \operatorname{cov.gon}(X).$$

Proof. Indeed, let X be a reduced and irreducible projective variety of dimension n, with a dominant rational map $f\colon X \dashrightarrow \mathbb{P}^n$ of degree d. For a general point $P \in X$, let $L \subseteq \mathbb{P}^n$ be a line passing through $f(P)$ and let C be a reduced and irreducible curve in $f^{-1}(L)$ passing through P. Since the point P was chosen generally, we have a rational dominant map $f_{|C}\colon C \dashrightarrow L \cong \mathbb{P}^1$ of degree at most d, hence $\operatorname{gon}(C) \leq d$. □

We are particularly interested in a relation, first stated in [Bas+17b], between the covering gonality and a variation on the concept of p-very ampleness.

Definition 2.3.7 (Birational p-very ampleness)**.** Let X be a projective scheme and $p \geq 0$ an integer. A line bundle B is called *birationally p-very ample* if there is an open and dense subset $U \subseteq X$ such that for every finite subscheme $\xi \subseteq U$ of length $p+1$ the evaluation map

$$\mathrm{ev}_\xi \colon H^0(X, B) \longrightarrow H^0(X, B \otimes \mathcal{O}_\xi)$$

is surjective.

The following basic observation is a simple consequence of Riemann-Roch:

Lemma 2.3.8. *Let C be a smooth and irreducible curve. Then the following are equivalent*

(i) ω_C *is p-very ample.*

(ii) ω_C *is birationally p-very ample.*

(iii) $\mathrm{gon}(C) \geq p+2$.

Proof. The proof of the equivalence between (i) and (ii) is in [Bas+17b, Lemma 1.3], but we provide the full proof here for completeness.

It is straightforward that (i) implies (ii). To see that (ii) implies (iii), let $f \colon C \to \mathbb{P}^1$ be a map of degree $d \leq p+1$ and set $A = f^*\mathcal{O}_{\mathbb{P}^1}(1)$. Then $h^0(C, A) \geq 2$ and by Riemann-Roch, this is the same as $h^0(C, \omega_C \otimes A^{-1}) \geq h^0(C, \omega_C) - (d+1)$. Hence, for any divisor $\xi \in |A|$, the exact sequence

$$0 \longrightarrow H^0(C, \omega_C \otimes A^{-1}) \longrightarrow H^0(C, \omega_C) \xrightarrow{\mathrm{ev}_\xi} H^0(C, \omega_C \otimes \mathcal{O}_\xi)$$

shows that the map $\mathrm{ev}_\xi \colon H^0(C, \omega_C) \longrightarrow H^0(C, \omega_C \otimes \mathcal{O}_\xi)$ is not surjective. Now let $U \subseteq C$ be a non-empty open subset and take a point $P \in \mathbb{P}^1$ such that $P \notin f(C \setminus U)$. Then $\xi = f^*(P)$ is a divisor in U of degree d for which the evaluation map ev_ξ is not surjective. Thus, B is not birationally $(d-1)$-very ample and a fortiori is not birationally p-very ample either.

To conclude, we prove that (iii) implies (i): suppose we have a divisor $\xi \subseteq C$ of degree $p+1$ such that the evaluation map of $H^0(C, \omega_C)$ at ξ is not surjective. Then the exact sequence of before shows that $h^0(C, \omega_C(-\xi)) \geq h^0(C, \omega_C) - (p+1)$, which, by Riemann-Roch, is the same as $h^0(C, \mathcal{O}_C(\xi)) \geq 2$. Hence, $\mathrm{gon}(C) \leq \deg \xi = p+1$. □

In particular, this lemma tells us that if ω_C is birationally p-very ample, then $\mathrm{gon}(C) \geq p+2$. The analogous result for the covering gonality was proven by Bastianelli et al. in [Bas+17b, Theorem 1.10]. We prove here the same statement, but we replace birational very ampleness with spannedness.

Proposition 2.3.9. *Let X be a smooth and irreducible projective variety and $p \geq 0$ an integer. If the canonical bundle ω_X is p-spanned, then the covering gonality of X is at least $p+2$.*

Proof. We follow here the proof of [Bas+17b, Theorem 1.10] with some small modifications.

Set $c = \operatorname{cov.gon}(X)$. Then there is a smooth family $\pi\colon \mathcal{C} \longrightarrow T$ of curves of gonality c, together with a generically finite dominant map $f\colon \mathcal{C} \longrightarrow X$, such that for a general fiber $\mathcal{C}_t = \pi^{-1}(t)$ the induced map $f_t = f_{|\mathcal{C}_t}\colon \mathcal{C}_t \longrightarrow X$ is birational onto its image. We want to use Lemma 2.3.8, so we need to prove that for a general fiber $\mathcal{C}_t$, the canonical bundle $\omega_{\mathcal{C}_t}$ is birationally p-very ample. For a general fiber, we have $\omega_{\mathcal{C}_t} \cong \omega_{\mathcal{C}|\mathcal{C}_t}$ and we know that $\omega_{\mathcal{C}} \cong f^*\omega_X \otimes \mathcal{O}_{\mathcal{C}}(E)$, where E is the ramification divisor of f. The general fiber intersects the ramification divisor properly, so $\omega_{\mathcal{C}_t} \cong f_t^*\omega_X \otimes \mathcal{O}_{\mathcal{C}_t}(E_t)$, where $E_t = \mathcal{C}_t \cap E$. Since $E_t \subseteq \mathcal{C}_t$ is a proper closed subset, it is easy to see that if $f_t^*\omega_X$ is birationally p-very ample, then $f_t^*\omega_X \otimes \mathcal{O}_{\mathcal{C}_t}(E_t)$ is birationally p-very ample as well [Bas+17b, Example 1.2.(i)]. To prove that $f_t^*\omega_X$ is birationally p-very ample, let $U_t \subseteq \mathcal{C}_t$ be an open subset such that $f_{t|U_t}\colon U_t \longrightarrow X$ is an isomorphism onto the image: then every finite subscheme $\xi \subseteq U_t$ of length $p+1$ can be seen as a curvilinear subscheme $f_t(\xi) \subseteq X$ of length $p+1$. Since ω_X is p-spanned, the evaluation map $H^0(X, \omega_X) \longrightarrow H^0(X, \omega_X \otimes \mathcal{O}_{f_t(\xi)})$ is surjective and by pullback along f_t^*, it follows that $H^0(\mathcal{C}_t, f_t^*\omega_X) \longrightarrow H^0(\mathcal{C}_t, f_t^*\omega_X \otimes \mathcal{O}_\xi)$ is surjective as well. Hence, $f_t^*\omega_X$ is birationally p-very ample and this completes the proof. □

2.3.1 Asymptotic syzygies and measures of irrationality

Now we can prove Corollary C. First we present a related result: more precisely we show that vanishing of asymptotic syzygies implies p-very ampleness of the canonical bundle. In particular, this extends [ELY16, Corollary C] of Ein, Lazarsfeld and Yang. We also observe that we do not require the condition $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$, which is present in [ELY16, Corollary C].

Corollary 2.3.10. *Let X be a smooth and irreducible projective variety of dimension n.*

$$\text{If} \quad K_{h^0(X,L)-1-n-p,n}(X, \mathcal{O}_X, L) = 0 \quad \text{for } L \gg 0 \qquad \text{then} \qquad \omega_X \text{ is } p\text{-very ample.}$$

Proof. Since $L \gg 0$, we see that $H^{n-i}(X, L^{\otimes i}) = 0$ for all $i = 1, \ldots, n-1$ and $H^{n-i}(X, L^{\otimes (i-1)}) = 0$ for all $i = 2, \ldots, n-1$. Hence, using Serre's duality and Proposition 1.1.25, we get

$$\dim K_{p,1}(X, \omega_X, L) \leq \dim K_{h^0(X,L)-1-n-p,n}(X, \mathcal{O}_X, L).$$

Thus, $K_{h^0(X,L)-1-n-p,n}(X, \mathcal{O}_X, L) = 0$ implies $K_{p,1}(X, \omega_X, L) = 0$ as well, so that we conclude using Theorem A. □

A similar argument, together with results from [Bas+17b], provides a proof of Corollary C. We actually give here a more precise version, which contains the effective result mentioned in the Introduction.

Corollary 2.3.11. *Let X be a smooth and irreducible projective variety of dimension n. Let L be a line bundle of the form*

$$L = \omega_X \otimes A^{\otimes d} \otimes P^{\otimes (n-1)} \otimes N, \qquad d \geq (n-1)(p+1) + p + 3,$$

where A is a very ample line bundle, P is a globally generated line bundle such that $P \otimes \omega_X^{\vee}$ is nef and N is a nef line bundle such that $N \otimes \omega_X$ is nef. If $K_{h^0(X,L)-1-n-p,n}(X, \mathcal{O}_X, L) = 0$ then the covering gonality and the degree of irrationality of X are at least $p+2$.

Proof. For such a line bundle L, Kodaira Vanishing implies that $H^{n-i}(X, L^{\otimes i}) = 0$ for all $i = 1, \dots, n-1$ and $H^{n-i}(X, L^{\otimes (i-1)}) = 0$ for all $i = 2, \dots, n-1$. Hence, Serre's duality and Proposition 1.1.25 imply

$$\dim K_{p,1}(X, \omega_X, L) \leq \dim K_{h^0(X,L)-1-n-p,n}(X, \mathcal{O}_X, L).$$

Thus, $K_{h^0(X,L)-1-n-p,n}(X, \mathcal{O}_X, L) = 0$ yields $K_{p,1}(X, \omega_X, L) = 0$ as well. Therefore, Theorem A shows that ω_X is p-spanned and then Proposition 2.3.9 implies that the covering gonality of X is at least $p+2$. Since the covering gonality is always smaller or equal than the degree of irrationality (see Remark 2.3.6), this concludes the proof. □

2.4 Asymptotic syzygies and the Hilbert scheme

Now we turn to the case of surfaces, with the aim of proving Theorem B. The first step is given by Voisin's Theorem 1.3.1, which interprets syzygies via the universal family on the Hilbert scheme of points. Ein and Lazarsfeld noticed that one can actually work on the Hilbert scheme itself.

More precisely, let X be a smooth projective surface, B a line bundle and L an ample and globally generated line bundle. For any integer $p \geq 0$, we have the evaluation map $\mathrm{ev}_B \colon H^0(X, B) \otimes \mathcal{O}_{X^{[p+1]}} \to B^{[p+1]}$ and we can twist it by $\det L^{[p+1]}$ to get another map

$$\mathrm{ev}_{B,L} \colon H^0(X, B) \otimes \det L^{[p+1]} \to B^{[p+1]} \otimes \det L^{[p+1]}. \tag{2.4.1}$$

Ein and Lazarsfeld observed that one can compute the Koszul cohomology groups from the map induced on global sections. They proved this in [EL15, Lemma 1.1] for smooth curves and we present here a proof for surfaces.

Lemma 2.4.1 (Voisin, Ein-Lazarsfeld)**.** *Let X be a smooth projective surface, B a line bundle, L be an ample and globally generated line bundle and $p \geq 0$ an integer. Then*

$K_{p,1}(X,B,L)$ is isomorphic to the cokernel of the map

$$H^0(X,B)\otimes H^0(X^{[p+1]},\det L^{[p+1]}) \longrightarrow H^0(X^{[p+1]},B^{[p+1]}\otimes \det L^{[p+1]}). \qquad (2.4.2)$$

In particular, $K_{p,1}(X,B,L)=0$ if and only if the map $\mathrm{ev}_{B,L}$ (2.4.1) *is surjective on global sections.*

Proof. Let $U = X^{[p+1]}_{curv} \subseteq X^{[p+1]}$ be the open subset of curvilinear subschemes. Then we know from Voisin's Theorem 1.3.1 that $K_{p,1}(X,B,L)$ coincides with the cokernel of the restriction map

$$H^0\left(X\times U, B\boxtimes \det L^{[p+1]}_U\right) \to H^0\left(\Xi^{[p+1]}_U, \left(B\boxtimes \det L^{[p+1]}_U\right)_{|\Xi^{[p+1]}_U}\right). \qquad (2.4.3)$$

We want to rewrite (2.4.3). By definition, we see that it is the map induced on global sections by the morphism of sheaves on $X\times U$:

$$pr_X^*B\otimes pr_U^*(\det L^{[p+1]}_U) \to (pr_X^*B\otimes pr_U^*(\det L^{[p+1]}_U))\otimes \mathcal{O}_{\Xi^{[p+1]}_U}. \qquad (2.4.4)$$

Hence, we can look at (2.4.3) also as the map induced on global sections by the pushforward of (2.4.4) along pr_U: by the projection formula we can write this pushforward as

$$pr_{U,*}(pr_X^*B)\otimes \det L^{[p+1]}_U \to pr_{U,*}(pr_X^*B\otimes \mathcal{O}_{\Xi^{[p+1]}_U})\otimes \det L^{[p+1]}_U.$$

Using the definition of tautological bundles together with flat base change along $U\hookrightarrow X^{[p+1]}$, we can rewrite this as

$$H^0(X,B)\otimes_{\mathbb{C}} \det L^{[p+1]}_U \to B^{[p+1]}_U\otimes \det L^{[p+1]}_U \qquad (2.4.5)$$

where the map is the restriction of the evaluation map (2.1.2) to U. Using the fact that $X^{[p+1]}$ is smooth and and that U is a dense open subset whose complement has codimension at least two (see Remark 1.2.8), we see that the map induced by (2.4.5) on global sections is the same as the map (2.4.2) and we conclude. □

Using this lemma, we want to study the asymptotic vanishing of $K_{p,1}(X,B,L)$. The idea is to pushforward the map $\mathrm{ev}_{B,L}$ (2.4.1) to the symmetric product via the Hilbert-Chow morphism $\mu\colon X^{[p+1]}\to X^{(p+1)}$. This allows us to give a characterization of the vanishing of $K_{p,1}(X,B,L)$ purely in terms of B.

We first need an easy lemma, that we prove for completeness.

Lemma 2.4.2. *Let X be a projective scheme and $\phi\colon \mathcal{F}\to\mathcal{G}$ a map of coherent sheaves on X. Then ϕ is surjective if and only if the induced map $\mathcal{F}\otimes L\to\mathcal{G}\otimes L$ is surjective on global sections when $L\gg 0$.*

Proof. There is an exact sequence of sheaves

$$0 \to \operatorname{Ker}\phi \to \mathcal{F} \to \mathcal{G} \to \operatorname{Coker}\phi \to 0$$

and for $L \gg 0$ we have that $H^1(X, \operatorname{Ker}\phi \otimes L) = H^1(X, \operatorname{Im}\phi \otimes L) = 0$ thanks to Serre's vanishing. Hence, on global sections we obtain an exact sequence

$$0 \to H^0(X, \operatorname{Ker}\phi \otimes L) \to H^0(X, \mathcal{F} \otimes L) \to H^0(X, \mathcal{G} \otimes L) \to H^0(X, \operatorname{Coker}\phi \otimes L) \to 0.$$

As $L \gg 0$, the sheaf $\operatorname{Coker}\phi \otimes L$ is globally generated, so that $H^0(X, \operatorname{Coker}\phi \otimes L) = 0$ if and only if $\operatorname{Coker}\phi = 0$. By the previous exact sequence, this is exactly what we had to prove. □

We are going to use this lemma together with the following observation:

Lemma 2.4.3. *Let X be a smooth projective surface, L a line bundle and $p \geq 0$ an integer. If $L \gg 0$ then $L^{(p+1)} \gg 0$ as well.*

Proof. By definition, $L \gg 0$ means that there are an ample line bundle A and an arbitrary line bundle P such that $L = P \otimes A^{\otimes d}$ and $d \gg 0$. Since the map $L \mapsto L^{(p+1)}$ of (1.2.1) is a homomorphism of groups, it follows that $L^{(p+1)} = P^{(p+1)} \otimes (A^{(p+1)})^{\otimes d}$. If we can prove that $A^{(p+1)}$ is ample, we conclude. This is true, because, under the finite map $\pi \colon X^{p+1} \to X^{(p+1)}$, the pullback $\pi^* A^{(p+1)} \cong A^{\boxtimes(p+1)}$ is ample and then the conclusion follows from [Laz04, Lemma 1.2.28]. □

Now we can state our criterion. In what follows, we will denote by $\mathfrak{a}_n$ the alternating representation of $\mathfrak{S}_n$: recall from Remark 1.4.5 that tensoring with $\mathfrak{a}_n$ yields exact functors

$$(-) \otimes \mathfrak{a}_n \colon \operatorname{QCoh}_{\mathfrak{S}_n}(X^n) \longrightarrow \operatorname{QCoh}_{\mathfrak{S}_n}(X^n), \qquad (-) \otimes \mathfrak{a}_n \colon D^b_{\mathfrak{S}_n}(X^n) \longrightarrow D^b_{\mathfrak{S}_n}(X^n).$$

Proposition 2.4.4. *Let X be a smooth projective surface and B a line bundle on X. Then $K_{p,1}(X, B, L) = 0$ for $L \gg 0$ if and only if the induced map of sheaves on $X^{(p+1)}$*

$$H^0(X, B) \otimes_{\mathbb{C}} \mu_*(\mathcal{O}(-\delta_{p+1})) \longrightarrow \mu_*(B^{[p+1]} \otimes \mathcal{O}(-\delta_{p+1})) \tag{2.4.6}$$

is surjective. Moreover, this map is isomorphic to the map

$$H^0(X, B) \otimes \pi_*^{\mathfrak{S}_{p+1}}(\mathcal{O}_{X^n} \otimes \mathfrak{a}_{p+1}) \longrightarrow \pi_*^{\mathfrak{S}_{p+1}}(\mathcal{E}_{p+1,B} \otimes \mathfrak{a}_{p+1}). \tag{2.4.7}$$

Proof. We know from Lemma 2.4.1 that $K_{p,1}(X, B, L) = 0$ if and only if the map

$$H^0(X, B) \otimes_{\mathbb{C}} \det L^{[p+1]} \longrightarrow B^{[p+1]} \otimes \det L^{[p+1]}$$

is surjective on global sections. Taking the pushforward along μ, this is equivalent to saying that

$$H^0(X,B) \otimes_{\mathbb{C}} \mu_*(\det L^{[p+1]}) \longrightarrow \mu_*(B^{[p+1]} \otimes \det L^{[p+1]})$$

is surjective on global sections. However, since $\det L^{[p+1]} \cong \mathcal{O}(-\delta_{p+1}) \otimes \mu^* L^{(p+1)}$ by Lemma 1.2.14, we can rewrite the last map using the projection formula as

$$\Big(H^0(X,B) \otimes_{\mathbb{C}} \mu_*(\mathcal{O}(-\delta_{p+1}))\Big) \otimes L^{(p+1)} \longrightarrow \mu_*(B^{[p+1]} \otimes \mathcal{O}(-\delta_{p+1})) \otimes L^{(p+1)}.$$

Now suppose that $L \gg 0$: then Lemma 2.4.2 and Lemma 2.4.3 show that this map is surjective on global sections if and only if the map (2.4.6) is surjective.

To conclude, we need to show that the maps (2.4.6) and (2.4.7) are isomorphic: to do this we will use the equivalences in Haiman's Theorem 1.4.14. First, Krug has proven in [Kru16, Theorem 1.1] that $\mathcal{O}(-\delta_{p+1}) \cong \Psi(\mathcal{O}_{X^{p+1}} \otimes \mathfrak{a}_{p+1})$, so that we can rewrite (2.4.6) as

$$H^0(X,B) \otimes \mu_*(\Psi(\mathcal{O}_{X^{p+1}} \otimes \mathfrak{a}_{p+1})) \to \mu_*(B^{[p+1]} \otimes \Psi(\mathcal{O}_{X^{p+1}} \otimes \mathfrak{a}_{p+1})).$$

Now, using Lemma 1.4.19, we get functorial isomorphisms in $D^b(X^{(p+1)})$:

$$\mu_*(\Psi(\mathcal{O}_{X^{p+1}} \otimes \mathfrak{a}_{p+1})) \cong \pi_*^{\mathfrak{S}_{p+1}}(\mathcal{O}_{X^{p+1}} \otimes \mathfrak{a}_{p+1}),$$
$$\mu_*(B^{[p+1]} \otimes \Psi(\mathcal{O}_{X^{p+1}} \otimes \mathfrak{a}_{p+1})) \cong \pi_*^{\mathfrak{S}_{p+1}}(\Phi(B^{[p+1]}) \otimes \mathfrak{a}_{p+1}),$$

so that the map (2.4.6) corresponds to

$$H^0(X,B) \otimes \pi_*^{\mathfrak{S}_{p+1}}(\mathcal{O}_{X^{p+1}} \otimes \mathfrak{a}_{p+1}) \to \pi_*^{\mathfrak{S}_{p+1}}(\Phi(B^{[p+1]}) \otimes \mathfrak{a}_{p+1}).$$

Since $\Phi(B^{[p+1]}) \cong \mathcal{E}_{p+1,B}$ by Corollary 2.1.8, we conclude. □

Remark 2.4.5. The characterization of the asymptotic vanishing of $K_{p,1}(X,B,L)$ via the surjectivity of the map (2.4.6) holds also on smooth curves. Since for curves the Hilbert-Chow morphism is an isomorphism, the surjectivity of (2.4.6) is equivalent to the p-very ampleness of B. This is how Ein and Lazarsfeld proved their result for curves [EL15, Theorem B].

To illustrate the criterion of Proposition 2.4.4, we use it to give alternative proofs to Theorems A and B from [ELY16] in the case of surfaces:

Corollary 2.4.6. *[ELY16, Theorem A] Let X be a smooth projective surface and B a p-jet very ample line bundle on X. Then $K_{p,1}(X,B,L) = 0$ for $L \gg 0$.*

Proof. By Proposition 2.4.4, $K_{p,1}(X,B,L) = 0$ for $L \gg 0$ if and only if the map

$$H^0(X,B) \otimes \pi_*^{\mathfrak{S}_{p+1}}(\mathcal{O}_{X^n} \otimes \mathfrak{a}_{p+1}) \to \pi_*^{\mathfrak{S}_{p+1}}(\mathcal{E}_{p+1,B} \otimes \mathfrak{a}_{p+1})$$

is surjective. The assumption that B is p-jet very ample means that the map

$$H^0(X,B) \otimes \mathcal{O}_{X^{p+1}} \to \mathcal{E}_{p+1,B}$$

is surjective. Since both functors of tensoring by $\mathfrak{a}_{p+1}$ and taking pushforward $\pi_*^{\mathfrak{S}_{p+1}}$ are exact, it follows that the first map is surjective as well. □

Corollary 2.4.7. *[ELY16, Theorem B] Let X be a smooth projective surface and B a line bundle on X. If $K_{p,1}(X,B,L) = 0$ for $L \gg 0$, then the evaluation map*

$$\mathrm{ev}_\xi \colon H^0(X,B) \to H^0(X, B \otimes \mathcal{O}_\xi)$$

is surjective for any subscheme $\xi \in X^{[p+1]}$ consisting of $p+1$ distinct points.

Proof. By Proposition 2.4.4, if $K_{p,1}(X,B,L) = 0$ for $L \gg 0$, then the map

$$H^0(X,B) \otimes \mu_*\mathcal{O}(-\delta_{p+1}) \to \mu_*(B^{[p+1]} \otimes \mathcal{O}(-\delta_{p+1}))$$

is surjective. This map restricted to the open subset $V \subseteq X^{(p+1)}$ consisting of reduced cycles is again surjective. Now it is easy to see that $\mu_{|\mu^{-1}(V)} \colon \mu^{-1}(V) \to V$ is an isomorphism, so that the map

$$H^0(X,B) \otimes \mathcal{O}(-\delta_{p+1}) \to B^{[p+1]} \otimes \mathcal{O}(-\delta_{p+1})$$

is surjective on $\mu^{-1}(V)$. Tensoring by $\mathcal{O}(\delta_{p+1})$, we obtain the desired assertion. □

2.5 Higher order embeddings and asymptotic syzygies on surfaces

Using Proposition 2.4.4, we can prove Theorem B from the Introduction. The key conditions are some local cohomological vanishing for tautological bundles. We would like to thank Victor Lozovanu for a discussion about the following proposition.

Proposition 2.5.1. *Let X be a smooth projective surface, $p \geq 0$ an integer and suppose that*

$$R^{i+1}\mu_*\left(\mathrm{Sym}^i\, \mathcal{O}_X^{[p+1]\vee}\right) = 0 \qquad \text{for all } 0 \leq i < p. \tag{2.5.1}$$

Then for any p-very ample line bundle B on X we have that $K_{p,1}(X,B,L) = 0$ for $L \gg 0$.

Proof. Since B is p-very ample, the map of sheaves on $X^{[p+1]}$

$$\mathrm{ev}_B \colon H^0(X,B) \otimes \mathcal{O}_{X^{[p+1]}} \longrightarrow B^{[p+1]}$$

is surjective. Now, we want to use Proposition 2.4.4 and we need to prove that the map of sheaves on $X^{(p+1)}$

$$H^0(X,B) \otimes \mu_* \mathcal{O}_{X^{[n]}}(-\delta_{p+1}) \to \mu_*(B^{[p+1]} \otimes \mathcal{O}(-\delta_{p+1}))$$

is surjective. This map is surjective if and only if it is surjective when tensored by the line bundle $B^{(p+1)}$. Using (1.2.14) and the projection formula, we can rewrite the tensored map as

$$\mu_*(H^0(X,B) \otimes_{\mathbb{C}} \det B^{[p+1]}) \to \mu_*(B^{[p+1]} \otimes \det B^{[p+1]}). \tag{2.5.2}$$

Set $h^0(X,B) = r+1$. Taking the Buchsbaum-Rim complex [Laz04, Theorem B.2.2] associated to the surjective map $H^0(X,B) \otimes \mathcal{O}_{X^{[p+1]}} \to B^{[p+1]}$ and tensoring by $\det B^{[p+1]}$ we get an exact complex of vector bundles

$$0 \to E_{r-p-1} \to \cdots \to E_1 \to E_0 \to H^0(X,B) \otimes \det B^{[p+1]} \to B^{[p+1]} \otimes \det B^{[p+1]} \to 0$$

with

$$E_i = \wedge^{p+2+i} H^0(X,B) \otimes_{\mathbb{C}} \operatorname{Sym}^i (B^{[p+1]})^\vee.$$

Breaking this complex into short exact sequences, we see that if $R^{i+1}\mu_*(E_i) = 0$ for all $0 \le i \le r-p-1$, then the map (2.5.2) is surjective. Since the fibers of the Hilbert-Chow morphism have dimension at most p (see 1.2.4), it is enough to have $R^{i+1}\mu_*(E_i) = 0$ for all $0 \le i < p$. This is the same as

$$R^{i+1}\mu_*(\operatorname{Sym}^i B^{[p+1]^\vee}) = 0 \text{ for all } 0 \le i < p. \tag{2.5.3}$$

Now, Scala shows in [Sca15, Lemma 3.1] that we can find an open cover of X^{p+1} composed of sets V^{p+1}, where $V \subseteq X$ is an open affine subset where B is trivial. Then it follows by the construction of the symmetric product and the Hilbert scheme that we have an open cover of $X^{(p+1)}$ of the form $V^{(p+1)}$ and that over these sets the Hilbert-Chow morphism restricts to $\mu_V \colon V^{[p+1]} \to V^{(p+1)}$. To conclude, it is straightforward to show that

$$\begin{aligned} R^{i+1}\mu_*(\operatorname{Sym}^i B^{[p+1]^\vee})_{|V^{(p+1)}} &\cong R^{i+1}\mu_{V*}(\operatorname{Sym}^i((B_{|V})^{[p+1]})^\vee) \\ \cong R^{i+1}\mu_{V*}(\operatorname{Sym}^i((\mathcal{O}_V^{[p+1]})^\vee) &\cong R^{i+1}\mu_*(\operatorname{Sym}^i \mathcal{O}_X^{[p+1]^\vee})_{|V^{(p+1)}}. \end{aligned}$$

In particular, condition (2.5.3) is equivalent to hypothesis (2.5.1). □

To conclude the proof of Theorem B we need to verify the cohomological vanishings of Proposition 2.5.1. This is done in the next lemma.

Lemma 2.5.2. *Let X be a smooth surface. Then for every $n \geq 1$ we have*

$$R^1\mu_*(\mathcal{O}_{X^{[n]}}) = 0, \qquad R^2\mu_*((\mathcal{O}_X^{[n]})^\vee) = 0, \qquad R^3\mu_*((\mathcal{O}_X^{[n]} \otimes \mathcal{O}_X^{[n]})^\vee) = 0.$$

Proof. For the first vanishing, we have that $R\mu_*\mathcal{O}_{X^{[n]}} = \mathcal{O}_{X^{(n)}}$: indeed, we know from [Sca09, Proposition 1.3.3] that $\Phi(\mathcal{O}_{X^{[n]}}) \cong \mathcal{O}_{X^n}$, and then Proposition 1.4.19, gives that

$$R\mu_*(\mathcal{O}_{X^{[n]}}) \cong \pi_*^{\mathfrak{S}_n}(\Phi(\mathcal{O}_{X^{[n]}})) \cong \pi_*^{\mathfrak{S}_n}(\mathcal{O}_{X^n}) \cong \mathcal{O}_{X^{(n)}}.$$

For the second, we see from [Sca09, Theorem 3.2.1] that

$$R\mu_*\mathcal{O}_X^{[n]} \cong \pi_*^{\mathfrak{S}_n}(\mathsf{C}^0_{\mathcal{O}_X})$$

and then Lemma 1.4.11 shows that

$$R\mu_*((\mathcal{O}_X^{[n]})^\vee) \cong \pi_*^{\mathfrak{S}_n}((\mathsf{C}^0_{\mathcal{O}_X})^\vee).$$

Since $\mathsf{C}^0_{\mathcal{O}_X}$ is locally free and $\pi_*^{\mathfrak{S}_n}$ an exact functor, it follows that $R\mu_*((\mathcal{O}_X^{[n]})^\vee)$ is concentrated in degree zero and in particular $R^2\mu_*((\mathcal{O}_X^{[n]})^\vee) = 0$.

For the third vanishing we observe that $R\mu_*(\mathcal{O}_X^{[n]} \otimes \mathcal{O}_X^{[n]})$ is concentrated in degree zero [Sca09, Corollary 3.3.1]. Hence, using the first part of Lemma 1.4.11, we get that

$$R^3\mu_*((\mathcal{O}_X^{[n]} \otimes \mathcal{O}_X^{[n]})^\vee) \cong \mathcal{E}xt^3_{X^{(n)}}(\mu_*(\mathcal{O}_X^{[n]} \otimes \mathcal{O}_X^{[n]}), \mathcal{O}_{X^{(n)}}).$$

Now, Scala gives in [Sca09, Theorem 3.5.2] an exact sequence of sheaves on $X^{(n)}$:

$$0 \to \mu_*(\mathcal{O}_X^{[n]} \otimes \mathcal{O}_X^{[n]}) \to \pi_*^{\mathfrak{S}_n}(\mathsf{C}^0_{\mathcal{O}} \otimes \mathsf{C}^0_{\mathcal{O}}) \to \pi_*^{\mathfrak{S}_n}(\mathsf{C}^1_{\mathcal{O}} \otimes \mathsf{C}^0_{\mathcal{O}}) \to 0$$

where the $\mathsf{C}^i_\bullet$ are the sheaves appearing in Theorem 1.4.16. Therefore, it is enough to show that

$$\mathcal{E}xt^3_{X^{(n)}}(\pi_*^{\mathfrak{S}_n}(\mathsf{C}^0_{\mathcal{O}} \otimes \mathsf{C}^0_{\mathcal{O}}), \mathcal{O}_{X^{(n)}}) = 0, \qquad \mathcal{E}xt^4_{X^{[n]}}(\pi_*^{\mathfrak{S}_n}(\mathsf{C}^1_{\mathcal{O}} \otimes \mathsf{C}^0_{\mathcal{O}}), \mathcal{O}_{X^{(n)}}) = 0.$$

For the first one we see, using Lemma 1.4.11, that

$$\mathcal{E}xt^3_{X^{(n)}}(\pi_*^{\mathfrak{S}_n}(\mathsf{C}^0_{\mathcal{O}} \otimes \mathsf{C}^0_{\mathcal{O}}), \mathcal{O}_{X^{(n)}}) \cong \pi_*^{\mathfrak{S}_n}(\mathcal{E}xt_{X^n}(\mathsf{C}^0_{\mathcal{O}} \otimes \mathsf{C}^0_{\mathcal{O}}, \mathcal{O}_{X^n})) = 0$$

where the last vanishing follows from the fact that $\mathsf{C}^0_{\mathcal{O}}$ is locally free. For the second, we use again Lemma 1.4.11 and we get

$$\begin{aligned}\mathcal{E}xt^4_{X^{[n]}}(\pi_*^{\mathfrak{S}_n}(\mathsf{C}^1_{\mathcal{O}}\otimes\mathsf{C}^0_{\mathcal{O}}),\mathcal{O}_{X^{(n)}}) &\cong \pi_*^{\mathfrak{S}_n}\mathcal{E}xt^4_{X^n}(\mathsf{C}^1_{\mathcal{O}}\otimes\mathsf{C}^0_{\mathcal{O}},\mathcal{O}_{X^n})\\ &\cong \pi_*^{\mathfrak{S}_n}\left({\mathsf{C}^0_{\mathcal{O}}}^{\vee}\otimes\mathcal{E}xt^4_{X^n}(\mathsf{C}^1_{\mathcal{O}},\mathcal{O}_{X^n})\right),\end{aligned}$$

where in the last step we have used again the fact that $\mathsf{C}^0_{\mathcal{O}}$ is locally free. To conclude we observe that

$$\mathcal{E}xt^4_{X^n}(\mathsf{C}^1_{\mathcal{O}},\mathcal{O}_{X^n})=0$$

by Theorem 1.4.18. □

It is now straightforward to prove Theorem B:

Proof of Theorem B. Let X be a smooth projective surface and B a line bundle on X. Fix also an integer $0\leq p\leq 3$. If $K_{p,1}(X,B,L)=0$ for $L\gg 0$, Theorem A shows that B is p-very ample. We prove the converse through Proposition 2.5.1. We need to check the vanishings in (2.5.1): the cases $p=0,1,2$ follow immediately from Lemma 2.5.2. For the case $p=3$, we use again Lemma 2.5.2, together with the observation that $\mathrm{Sym}^2((\mathcal{O}_X^{[n]})^{\vee})$ is a direct summand of $(\mathcal{O}_X^{[n]}\otimes\mathcal{O}_X^{[n]})^{\vee}$. □

2.5.1 An algebro-combinatorial approach

It is possible that the statement of Theorem B remains true for any p, but the key part in our proof was in Lemma 2.5.2, where we proved the vanishings

$$R^{k+1}\mu_*\left(\mathcal{H}om_{X^{[n]}}\left(\left(\mathcal{O}_X^{[n]}\right)^{\otimes k},\mathcal{O}_{X^{[n]}}\right)\right)=0 \tag{2.5.4}$$

for $k=0,1,2$. In particular, we were able to do so because Scala [Sca09] gives a relatively simple description of the sheaves $R\mu_*((\mathcal{O}_X^{[n]})^{\otimes k})$, when $k=0,1,2$ is small. However, as k increases, these sheaves become increasingly more complicated and it is not clear whether it is possible to check the vanishings explicitly as we have done in Lemma 2.5.2.

Here we would like to discuss another point of view on the problem and argue that the above statement is essentially combinatorial. We first observe that in the proof of Lemma 2.5.2 we did not use anything about the particular geometry of X. Indeed, we can look at the vanishings (2.5.4) as being basically local statements on X, so that we can restrict to the case of $X=\mathbb{A}^2_{\mathbb{C}}$. The precise statement is the following:

Lemma 2.5.3. *Let X be an arbitrary smooth quasiprojective surface and let $k\geq 0$ be an integer. Then the vanishing 2.5.4 holds if and only if it holds for $X=\mathbb{A}^2_{\mathbb{C}}$.*

Proof. We argue as in [Sca09, p. 8]. The vanishing that we want to prove is local on $X^{(n)}$: consider a cycle $\zeta=a_1P_1+\cdots+a_rP_r\in X^{[n]}$, where the $P_i\in X$ are distinct points and the a_i are positive integers such that $a_1 + \cdots + a_r = n$. For each P_i, we can take a small analytic coordinate chart $V_i \subseteq X$ centered at P_i and isomorphic to a small ball in $\mathbb{C}^2$. We can form the open subset $V = V_1 \cup \cdots \cup V_r$. Then $V^{(n)} \subseteq X^{(n)}$ is an open neighborhood of ζ, and the restriction of the sheaf in 2.5.4 to this open subset is isomorphic to

$$R^{k+1}\mu_{V*}\left(\mathcal{H}om_{V^{[n]}}\left(\left(\mathcal{O}_V^{[n]}\right)^{\otimes k}, \mathcal{O}_{V^{[n]}}\right)\right)$$

where $\mu_V \colon V^{[n]} \longrightarrow V^{(n)}$ is the Hilbert-Chow morphism for V. Since V is isomorphic to a disjoint union of small balls, the vanishing of this sheaf is independent of the original variety X, and this concludes the proof. □

Thanks to this lemma, in order to prove the vanishings (2.5.4), we can restrict to the case $X = \mathbb{A}^2_{\mathbb{C}}$, where we can employ explicit commutative algebra. At first, using Lemma 1.4.11 and Proposition 1.4.19 we see that

$$R^{k+1}\mu_*\left(\mathcal{H}om_{X^{[n]}}\left(\left(\mathcal{O}_X^{[n]}\right)^{\otimes k}, \mathcal{O}_{X^{[n]}}\right)\right) \cong \pi_*^{\mathfrak{S}_n}\left(\mathcal{E}xt^{k+1}_{X^n}\left(\Phi\left(\left(\mathcal{O}_X^{[n]}\right)^{\otimes k}\right), \mathcal{O}_{X^n}\right)\right). \tag{2.5.5}$$

In the case of $X = \mathbb{A}^2_{\mathbb{C}}$, Haiman gave an explicit description of $\Phi((\mathcal{O}_X^{[n]})^{\otimes k})$ for any $k \geq 0$. To state his result, let $S_n = \mathbb{C}[X_1, Y_1, \ldots, X_n, Y_n]$ be the ring of X^n, with the natural action of $\mathfrak{S}_n$, and let $S_n[A_1, B_1, \ldots, A_k, B_k]$ be the ring of $X^n \times X^k$. For every function $f \colon \{1, \ldots, k\} \to \{1, \ldots, n\}$, define a linear subspace $W_f \subseteq X^n \times X^k$ by

$$\begin{aligned} W_f &\stackrel{\text{def}}{:=} \operatorname{Spec} S_n[A_1, B_1, \ldots, A_k, B_k]/I_f, \\ I_f &\stackrel{\text{def}}{:=} \left(A_i - X_{f(i)}, B_i - Y_{f(i)} \mid i = 1, \ldots, k\right). \end{aligned}$$

The union of these subspaces is called Haiman's polygraph.

Definition 2.5.4 (Haiman's Polygraph)**.** The *Haiman's polygraph* is the union

$$Z(n,k) \stackrel{\text{def}}{:=} \bigcup_f W_f.$$

We denote its coordinate ring by

$$R(n,k) \stackrel{\text{def}}{:=} S_n[A_1, B_1, \ldots, A_k, B_k] / \bigcap_f I_f.$$

This coordinate ring is the key to the computation that we are interested in:

Theorem 2.5.5 (Haiman)**.** *For $X = \mathbb{A}^2_{\mathbb{C}}$ there is an isomorphism*

$$\Phi\left(\left(\mathcal{O}_X^{[n]}\right)^{\otimes k}\right) \cong R(n,k)$$

where we look at $R(n,k)$ as a $\mathfrak{S}_n$-sheaf on X^n.

Proof. See [Hai02, Theorem 2.1, Proposition 5.3]. □

As a consequence, we immediately obtain the following result:

Corollary 2.5.6. *Fix an integer $p \geq 0$ and suppose that*

$$\operatorname{Ext}^{k+1}_{S_{p+1}}\left(R(p+1,k), S_{p+1}\right)^{\mathfrak{S}_{p+1}} = 0, \qquad \textit{for all } 0 \leq k < p. \tag{2.5.6}$$

Then for any p-very ample line bundle B on an arbitrary smooth projective surface X we have that

$$K_{p,1}(X,B,L) = 0 \qquad \textit{for } L \gg 0.$$

Proof. Let X be a smooth projective surface. Lemma 2.5.3, the isomorphism (2.5.5) and Theorem 2.5.5 imply that the vanishings (2.5.6) are equivalent to

$$R^{k+1}\mu_*\left(\mathcal{H}om_{X^{[n]}}\left(\left(\mathcal{O}_X^{[p+1]}\right)^{\otimes k}, \mathcal{O}_{X^{[p+1]}}\right)\right) = 0, \qquad \text{for all } 0 \leq k < p.$$

In particular, since $\operatorname{Sym}^k \mathcal{O}_X^{[p+1]}$ is a direct summand of $\left(\mathcal{O}_X^{[p+1]}\right)^{\otimes k}$, it follows that

$$R^{k+1}\mu_*\left(\operatorname{Sym}^k \mathcal{O}_X^{[p+1]\vee}\right) = 0, \qquad \text{for all } 0 \leq k < p,$$

which are exactly the vanishing conditions of Proposition 2.5.1. □

This corollary tells us that we can regard Theorem B as a consequence of an essentially combinatorial statement about the ring $\mathbb{C}[x,y]$. Moreover, this statement is completely explicit and in principle it can be verified by a computer. We wrote a program in Macaulay2 [M2] to check these vanishings, but the problem becomes computationally very expensive as p grows, and we were not able to obtain better results than those already proven before.

2.5.2 Concluding remarks

We include a couple of comments on some possible extensions of our results.

- A topic that we do not discuss at all is how to make the statement of Theorem B effective. Indeed, for a curve C, Ein and Lazarsfeld give in in [EL15, Proposition 2.1] a lower bound on the degree of a line bundle L such that, if B is a p-very ample line bundle, then $K_{p,1}(C, B, L) = 0$. The bound has later been improved by Rathmann [Rat16] for any curve and by Farkas and Kemeny for a general curve and $B = \omega_C$ [FK16]. It is natural to ask for a similar result for surfaces.

- Instead, it is not clear whether one should expect Theorem B to be valid for varieties of dimension greater than two. Our proof used various properties of tautological bundles that may break down in higher dimensions, since the Hilbert scheme may be singular or reducible. A possible strategy could be to follow Voisin's Theorem 1.3.1 and restrict the attention to the component of the Hilbert scheme containing the curvilinear subschemes. About this, see also a comment Ein, Lazarsfeld and Yang in [EL16, Footnote 9]. However, for any smooth and irreducible projective variety X, the Hilbert scheme $X^{[3]}$ is again smooth and irreducible, so that one could expect that Theorem B holds in any dimension for $p = 2$.

Chapter 3

Equations of abelian surfaces and the cyclic Prym map in genus two

In this chapter we prove Theorem D about equations of abelian surfaces and Theorem E on the Prym map for cyclic covers of genus two curves. We recall them here.

Theorem D. *Let $A \hookrightarrow \mathbb{P}(H^0(A,L))$ be an abelian surface embedded by a complete linear system not of type $(1,5)$, $(1,6)$ or $(2,4)$. Then its homogeneous ideal is generated by quadrics and cubics.*

Theorem E. *The differential of the Prym map $Pr\colon \mathcal{R}_{2,d} \longrightarrow \mathcal{A}_{\mathcal{D}}$ is injective at a cyclic cover in $\mathcal{R}_{2,d}$ if and only if $d \geq 7$ and the cover is not bielliptic. In particular, the Prym map is generically finite onto its image for $d \geq 7$.*

We start by reviewing some facts on abelian varieties in Section 3.1. In Section 3.2, we prove Theorem D. The main step in the proof is a classification of projective normality for polarized abelian surfaces, due to Koizumi, Ohbuchi, Lazarsfeld and Fuentes García. We state this result in Theorem 3.2.7 and we also give the proof of the most interesting case, that of a $(1,d)$-polarization. Using this fact and duality for Koszul cohomology, it is straightforward to prove Theorem D. Finally, in Section 3.3 we discuss the Prym map for cyclic covers of genus two curves and we prove Theorem E. In the proof, we present a construction that associates a polarized abelian surface to a cyclic cover of a genus two curve. We then show how to use Theorem 3.2.7 on this abelian surface to conclude.

3.1 General facts on abelian varieties

Here we briefly recall some facts on abelian varieties, that we are going to use throughout this chapter. The first one is about cohomology of line bundles.

Proposition 3.1.1. *Let A be an abelian variety of dimension g.*

1. *For any line bundle L on A, we have that $\chi(X, L) = \frac{(L^g)}{g!}$.*
2. *If L is an ample line bundle on A then $H^i(A, L) = 0$ for all $i > 0$.*
3. *If $L \in \mathrm{Pic}^0(A)$ and $L \not\cong \mathcal{O}_A$, then $H^i(A, L) = 0$ for all i.*
4. *For the trivial bundle we have that $h^i(A, \mathcal{O}_A) = \binom{g}{i}$.*

Proof. See [BL04, Chapter 3]. □

Next, we discuss Heisenberg groups associated to ample line bundles. Let L be an ample line bundle on an abelian variety A: it defines a homomorphism of abelian varieties

$$\varphi_L \colon A \longrightarrow \mathrm{Pic}^0(A), \qquad P \mapsto t_P^* L \otimes L^{-1}$$

and, since L is ample, this is an isogeny [BL04, Proposition 2.4.8], so that the subgroup

$$K(L) \overset{\mathrm{def}}{:=} \mathrm{Ker}\, \varphi_L = \{P \in A \mid t_P^* L \cong L\}$$

is a finite abelian group. This group can be extended to the Heisenberg group of L.

Definition 3.1.2 (Heisenberg group)**.** Let L be an ample line bundle on an abelian variety A. The associated *Heisenberg group* is

$$G(L) \overset{\mathrm{def}}{:=} \{(\alpha, P) \mid \alpha \colon L \longrightarrow t_P^* L \text{ isomorphism } \}.$$

The Heisenberg group $G(L)$ it is a central extension of $K(L)$ by $\mathbb{C}^*$: more precisely, we have an exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow G(L) \longrightarrow K(L) \longrightarrow 0$$

such that $\mathbb{C}^*$ is precisely the center of $G(L)$ [Mum66, Theorem 1, p.293]. In particular, we can define a skew-symmetric bilinear form

$$e^L \colon K(L) \times K(L) \longrightarrow \mathbb{C}^*$$

as follows: for two elements $P, Q \in K(L)$, let $\widetilde{P}, \widetilde{Q} \in G(L)$ be any two elements that lie over them. Then we set $e^L(P, Q) \overset{\mathrm{def}}{:=} [P, Q] = \widetilde{P} \cdot \widetilde{Q} \cdot \widetilde{P}^{-1} \cdot \widetilde{Q}^{-1}$. Since $K(L)$ is abelian, it follows that $e^L(P, Q) \in \mathrm{Ker}\,(G(L) \to K(L)) = \mathbb{C}^*$. It is also easy to see that the definition is independent of the lifts $\widetilde{P}, \widetilde{Q}$. Finally, usual properties of the commutator show that e^L is indeed a skew-symmetric bilinear form.

Moreover, since the center of $G(L)$ is given precisely by $\mathbb{C}^*$, this form is non-degenerate [Mum66, p.293], meaning that it gives an isomorphism

$$K(L) \longrightarrow \mathrm{Hom}(K(L), \mathbb{C}^*), \qquad P \mapsto e(P, -).$$

The canonical form of skew-symmetric matrices [Mum66, p.293] shows that there are e^L - isotropic subgroups $K_1, K_2 \subseteq K(L)$ such that $K(L) \cong K_1 \oplus K_2$ and such that the induced map

$$K_1 \longrightarrow \mathrm{Hom}(K_2, \mathbb{C}^*), \qquad P \mapsto e(P, -)$$

is an isomorphism. It turns out [BL04, Lemma 6.6.5] that the group K_1, and hence also K_2, is of the form

$$K_1 \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_g\mathbb{Z}$$

for g integers d_i such that $d_i | d_{i+1}$. Thus, the group $K(L)$ is isomorphic to

$$K(L) \cong \left(\mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_g\mathbb{Z}\right)^2.$$

By the theory of elementary divisors, such a sequence $(d_1, \ldots, d_g)$ is unique. We summarize this discussion in the following definition

Definition 3.1.3 (Type of a line bundle)**.** Let L be an ample line bundle on an abelian variety A of dimension g. Then there are unique positive integers $(d_1, \ldots, d_g)$, with $d_i | d_{i+1}$, such that

$$K(L) \cong \left(\mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_g\mathbb{Z}\right)^2.$$

The sequence $(d_1, \ldots, d_g)$ is called the *type of the line bundle L*.

We can characterize the global sections of a line bundle in terms of its type

Lemma 3.1.4. *Let L be an ample line bundle of type $(d_1, \ldots, d_g)$ on an abelian variety A of dimension g. Then*

$$h^0(A, L) = d_1 d_2 \ldots d_g.$$

Proof. See [BL04, Corollary 3.2.8]. □

To conclude, we briefly discuss polarizations and their behavior under isogenies. Recall that if L is an ample line bundle on A, the associated polarization is the first Chern class $c_1(L) \in H^2(A, \mathbb{Z})$. Two line bundles L_1, L_2 give the same polarization if and only if they differ by an element of $\mathrm{Pic}^0(A)$: in particular in this case we have $K(L_1) = K(L_2)$ [BL04, Lemma 2.4.7.(a)]. We can define the *type of a polarization* to be the type of any line bundle that induces it. In particular, a polarization of type $(1, 1, \ldots, 1)$ is called a *principal polarization*. Polarized abelian varieties have nice moduli spaces [BL04, Chapter 8].

Definition 3.1.5 (Moduli spaces of polarized abelian varieties)**.** Fix a type $\delta = (d_1, \ldots, d_g)$. We denote by $\mathcal{A}_\delta$ the moduli space of polarized abelian varieties of type δ. This is a quasiprojective variety whose points represent isomorphism classes of polarized abelian varieties of type δ.

To conclude, we prove a result about pullbacks of principal polarizations under isogenies. This is well-known , however we decided to include a proof because we were not able to find a straightforward reference.

Lemma 3.1.6. *Let $F\colon A \longrightarrow B$ be an isogeny of abelian varieties of dimension g, and let M be a principally polarized line bundle on B. Then there is a unique sequence $(d_1, \dots, d_g)$ of positive integers, with $d_i | d_{i+1}$, such that*

$$\operatorname{Ker} F \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_g\mathbb{Z}$$

and the pullback $L = F^*M$ is a line bundle of type $(d_1, \dots, d_g)$.

Proof. From the definition of type, it is enough to prove that $K(L) \cong (\operatorname{Ker} F)^2$ as abstract groups. To do so, first observe that we have a commutative diagram [BL04, Corollary 2.4.6.(d)]:

$$\begin{array}{ccc} A & \xrightarrow{\varphi_L} & \operatorname{Pic}^0(A) \\ \downarrow F & & \uparrow F^* \\ B & \xrightarrow{\varphi_M} & \operatorname{Pic}^0(B) \end{array}$$

Since M induces a principal polarization, φ_M is an isomorphism. Moreover, F^* is an isogeny of the same degree as F [BL04, Proposition 2.4.3]. Hence, if $d = \deg F = |\operatorname{Ker} F|$, we have that $|K(L)| = \deg \varphi_L = d^2$. Consider now the bilinear form $e^L\colon K(L) \times K(L) \to \mathbb{C}^*$. By construction of L, we have that $\operatorname{Ker} F \subseteq K(L)$ and it follows from general properties of isogenies [BL04, Proposition 6.3.3], that $e^L(\operatorname{Ker} F \times \operatorname{Ker} F) = 1$. Since L is non-degenerate, the map

$$K(L) \longrightarrow \operatorname{Hom}(\operatorname{Ker} F, \mathbb{C}^*), \qquad P \mapsto e(P, -)$$

is surjective, so that we can choose a section, whose image is a subgroup $\widetilde{\operatorname{Ker} F} \subseteq K(L)$. In particular, $\widetilde{\operatorname{Ker} F}$ is abstractly isomorphic to $\operatorname{Ker} F$. To conclude, we will show that $K(L) \cong \operatorname{Ker} F \oplus \widetilde{\operatorname{Ker} F}$: we have $|\operatorname{Ker} F \oplus \widetilde{\operatorname{Ker} F}| = d^2 = |K(L)|$, so that it is enough to prove $\operatorname{Ker} F \cap \widetilde{\operatorname{Ker} F} = 0$. Thus, let $P \in \operatorname{Ker} F \cap \widetilde{\operatorname{Ker} F}$: since $\operatorname{Ker} F$ is isotropic, we have $e^L(P, Q) = 1$ for all $Q \in \operatorname{Ker} F$, but by definition of $\widetilde{\operatorname{Ker} F}$ this implies $P = 0$. □

3.2 Equations of abelian surfaces

The key fact that we are going to use is a classification of projective normality for abelian surfaces. Recall from Remark 1.1.16 that an ample line bundle L on a projective variety X is called projectively normal if the multiplication map

$$\operatorname{Sym}^q H^0(X, L) \longrightarrow H^0(X, L^q)$$

is surjective for all $q \geq 2$. In terms of Koszul cohomology, this means that $K_{0,q}(X, L) = 0$ for all $q \geq 2$. We also know from Remark 1.1.16 that an ample and projectively normal line bundle is automatically very ample.

For abelian surfaces, there is a complete classification of projective normality, due to results of Koizumi [Koi76], Ohbuchi [Ohb93], Lazarsfeld [Laz90] and Fuentes García [Gar04].

We are going to state the full result later in Theorem 3.2.7. Now we want to start with the most important case, given by the result of Lazarsfeld and Fuentes García:

Theorem 3.2.1 (Lazarsfeld, Fuentes García)**.** *Let A be an abelian surface and L an ample line bundle on A of type $(1,d)$. Then L is projectively normal if and only if it is very ample and $d \geq 7$.*

This theorem was proven by Lazarsfeld [Laz90] in the cases $d = 7, 9, 11$ and $d \geq 13$. The remaining cases $d = 8, 10, 12$ were solved by Fuentes García [Gar04]. We sketch here Lazarsfeld's proof of the result, in particular because the original preprint [Laz90] is quite hard to find: we would like to thank Robert Lazarsfeld for having made a copy of it available to us.

First we observe the following:

Lemma 3.2.2. *Let A be an abelian surface and L a very ample line bundle on it. Then L is projectively normal if and only if the multiplication map*

$$Sym^2 H^0(A,L) \longrightarrow H^0(A,L^2)$$

is surjective.

Proof. This is proven for an abelian variety of any dimension by Iyer in [Iye99, Proposition 2.1]. In the case of abelian surfaces, we can give a quick proof via Koszul cohomology as follows: suppose that the above multiplication map is surjective, then $K_{0,2}(A,L) = 0$ and by Remark 1.1.16 we just need to prove that $K_{0,q}(A,L) = 0$ for all $q \geq 3$. In this case, we see from Lemma 3.1.1 that $H^1(A, L^{q-1}) = H^1(A, L^{q-2}) = 0$ so that Theorem 1.1.24 implies

$$K_{0,q}(A,L)^{\vee} \cong K_{r-2,3-q}(A,L)$$

where $r = h^0(L) - 1$. Now, we observe that $r - 2 > 0$, since L is very ample, and $3 - q \leq 0$ by hypothesis. If $3 - q < 0$, the Koszul complex (1.1.2) shows immediately that $K_{r-2,3-q}(A,L) = 0$. If instead $3 - q = 0$, the Koszul complex (1.1.2) gives that

$$K_{r-2,0}(A,L) \cong \operatorname{Ker}\left[d\colon \wedge^{r-2} H^0(A,L) \longrightarrow \wedge^{r-3} H^0(A,L) \otimes H^0(A,L)\right]$$

and since $r - 2 > 0$, it is easy to see that this vanishes as well. □

Let us now fix an abelian surface embedded by a complete linear system $A \hookrightarrow \mathbb{P}(H^0(A,L))$ with L of type $(1,d)$ and $d \geq 7$. There is an exact sequence

$$0 \longrightarrow I \longrightarrow \mathrm{Sym}^2 H^0(A,L) \longrightarrow H^0(A,L^2) \longrightarrow U \longrightarrow 0 \qquad (3.2.1)$$

and we want to prove that $U = 0$. We start with the following lemma, which is a clarification of [Gar03, Lemma 2.6].

Lemma 3.2.3. *Let $X \subseteq \mathbb{P}^N$ be a reduced and irreducible surface of degree t, not contained in any hyperplane. Then*

$$h^0(\mathbb{P}^N, \mathcal{I}_{X,\mathbb{P}^N}(2)) \leq \frac{N(N-1)}{2} - \min\{t, 2N-5\}$$

Proof. First recall that $t \geq N-1$ [EH87, Proposition 0]. Choose $H \subseteq \mathbb{P}^N$ to be a general linear subspace of codimension 2. Then, $H \cap X$ consists of t distinct points in linearly general position in H: in particular, they span H, since $t \geq \dim H + 1$.

Now we observe that there is no quadric $Q \subseteq \mathbb{P}^N$ containing both X and H. Indeed, suppose that there is such a Q: then, since X is non-degenerate, it would have rank at least 3, so that its singular locus $\mathrm{Sing}(Q)$ would be a linear subspace of codimension at least 3, which cannot contain $H \cap X$. This shows that $X \cap H \cap (Q \setminus \mathrm{Sing}(Q)) \neq \emptyset$ and since $H \cap (Q \setminus \mathrm{Sing}(Q))$ is a Cartier divisor on $Q \setminus \mathrm{Sing}(Q)$, it follows from Krull's principal ideal theorem that every irreducible component of $X \cap H \cap (Q \setminus \mathrm{Sing}(Q))$ has positive dimension, which gives a contradiction.

This shows that the restriction map

$$H^0(\mathbb{P}^N, \mathcal{I}_{X,\mathbb{P}^N}(2)) \longrightarrow H^0(H, \mathcal{I}_{X\cap H,H}(2))$$

is injective. To conclude, we can just apply Castelnuovo's argument for which $t \geq N-1$ points in linearly general position in $\mathbb{P}^{N-2}$ impose at least $\min\{t, 2N-5\}$ independent conditions on quadrics: see [ACGH, Lemma p.115]. □

In our case, we get the following:

Lemma 3.2.4. *With the notations of* (3.2.1)*, we have* $\dim U \leq 6$.

Proof. Since L is of type $(1, d)$ the abelian surface A is embedded in $\mathbb{P}^{d-1}$ and it has degree $2d$. Then Lemma 3.2.3 implies

$$\begin{aligned}\dim U &= \dim H^0(A, L^2) - \dim \mathrm{Sym}^2 H^0(A, L) + \dim I \\ &\leq 4d - \binom{d+1}{2} + \frac{d(d-7)}{2} + 6 = 6.\end{aligned}$$

□

Therefore, it is enough to show that if $U \neq 0$, then $\dim U \geq 7$. Lazarsfeld's idea is to use the representation theory of the Heisenberg's group $G(L)$. We recall some of the theory here. A linear representation of $G(L)$, where $\mathbb{C}^*$ acts by the character $\lambda \mapsto \lambda^k$ is called a *representation of weight k*. The space $H^0(A, L)$ has a natural linear action of $G(L)$ given by

$$(\alpha, x) \cdot \sigma = t^*_{-x}(\alpha(\sigma)).$$

Up to isomorphism, this is the unique irreducible representation of $G(L)$ of weight 1 [Mum66, Proposition 3, Theorem 2]. This representation induces other representations of weight 2 on $\mathrm{Sym}^2 H^0(A, L)$ and $H^0(A, L^2)$ such that the multiplication map in (3.2.1) is $G(L)$-equivariant. In particular, I and U can be regarded as $G(L)$-representations of weight 2. The irreducible ones have been classified by Iyer [Iye99, Proposition 3.2].

Proposition 3.2.5 (Iyer)**.** *Let L be an ample line bundle of type $(1, d)$ on an abelian surface A. Then:*

1. *if d is odd, there is, up to isomorphism, a unique irreducible $G(L)$-representation of weight 2. This representation has dimension d.*
2. *if $d = 2m$ is even, then there are, up to isomorphism, four distinct $G(L)$-representations of weight 2. Each irreducible representation has dimension m.*

This discussion proves Theorem 3.2.1 for most cases:

Proof of Theorem 3.2.1. Suppose that d is odd and greater than 7 or even and greater than 14. Assume that the embedding is not projectively normal. Then Lemma 3.2.2 shows that in (3.2.1) we have $U \neq 0$. Since U is a $G(L)$-representation, it must be by Proposition 3.2.5 that $\dim U \geq 7$. This is however impossible, because $\dim U \leq 6$ by Lemma 3.2.4.

This leaves the cases $d = 8, 10, 12$. These were solved by Fuentes García in [Gar04] using the involutions in $G(L)$ coming from the 2-torsion points of $K(L)$, together with geometric results about polarized abelian surfaces of small degree. □

Remark 3.2.6. Theorem 3.2.1 can be proven for $d \geq 9$ also using results of Küronya-Lozovanu and Ito. Indeed, a special case of [KL15, Theorem 1.1] shows that if L is an ample line bundle on an abelian surface A with $(L^2) \geq 20$, then L is projectively normal if and only if there is no elliptic curve $E \subseteq A$, such that $(L \cdot E) \leq 2$. Moreover, as a consequence of [Ito17, Theorem 1.2] this is true even when $(L^2) > 16$.

In particular, if L is a very ample line bundle of type $(1, d)$, with $d \geq 9$, these conditions are satisfied: we see that $(L^2) = 2d \geq 18$ and for any elliptic curve $E \subseteq X$ we have that $L_{|E}$ is very ample, which implies $(L \cdot E) \geq 3$.

Now we can give the full classification of projective normality for abelian surfaces.

Theorem 3.2.7 (Koizumi, Ohbuchi, Lazarsfeld, Fuentes García)**.** *Let L be a very ample line bundle on an abelian surface A. Then L is projectively normal, unless it is of type $(1, 5), (1, 6), (2, 4)$. In these cases, it is never projectively normal.*

Proof. Suppose that L is of type $(d_1, d_1 m)$: if $d_1 \geq 3$, then the result was first proven by Koizumi [Koi76]. Another proof can be found in [BL04, Theorem 7.3.1].

If $d_1 = 2$ and $m \geq 3$ then projective normality follows from a result by Ohbuchi [Ohb93]. Alternatively, we can reason as in Remark 3.2.6. Ohbuchi also shows in [Ohb93, Lemma 6] that, if $m = 2$, then L is not projectively normal. For another proof of this, Barth has shown in [Bar87, Theorem 2.11] that the ideal I_A contains precisely 6 linearly independent quadrics. Hence $\mathrm{Sym}^2 H^0(A, L) \longrightarrow H^0(A, L^2)$ has image of dimension $36 - 6 = 30$, which is less than the dimension of $H^0(A, L^2)$.

If $d_1 = 1$, then this is Theorem 3.2.1. Observe that there cannot be projective normality for L of type $(1,5)$ or $(1,6)$, because in these cases $\mathrm{Sym}^2 H^0(A, L)$ has dimension smaller than $H^0(A, L^2)$.

In all the other cases, it is easy to see that the line bundle cannot be very ample. □

As a consequence of this result, it is easy to prove Theorem D.

Proof of Theorem D. Let A be an abelian surface and L a very ample line bundle not of type $(1,5), (1,6)$ and $(2,4)$. Then we know from Theorem 3.2.7 that L is projectively normal. Hence, thanks to Remark 1.1.16 we see that the homogeneous ideal of A is generated by quadrics and cubics if and only if $K_{1,q}(A, L) = 0$ for all $q \geq 3$. To do this, observe that $H^1(A, L^{q-1}) = H^1(A, L^{q-2}) = 0$, so that Theorem 1.1.24 gives an isomorphism

$$K_{1,q}(A, L)^\vee \cong K_{r-3,3-q}(A, L)$$

where $r = h^0(A, L) - 1$. In particular, we see that $r - 3 > 0$, because there is no abelian surface embedded in $\mathbb{P}^3$: the only smooth surfaces in $\mathbb{P}^3$ with trivial canonical bundle are quartic surfaces, and these are all K3 surfaces. Moreover, $3 - q \leq 0$, so, reasoning as in the proof of Lemma 3.2.2, we see that $K_{r-3,3-q}(A, L) = 0$ and we conclude. □

Remark 3.2.8. We can also consider the exceptional cases. For a very ample line bundle of type $(1,5)$ Manolache has proven [Man88, Theorem 1] that the homogeneous ideal is generated by 3 quintics and 15 sextics. For the case $(1,6)$ Gross and Popescu [GP01, Remark 4.8.(2)] have proven that to generate the ideal of such a surface one needs cubics and quartics. For the case $(2,4)$ Barth [Bar87, Theorem 2.14,Theorem 4.9] gives explicit quadrics which generate the ideal sheaf of the surface: it is then easy (for example with Macaulay2 [M2]) to compute examples where the homogeneous ideal is generated by quadrics and quartics.

3.3 The Prym map for cyclic covers of genus two curves

We introduce now the Prym map for cyclic étale covers of curves, following [LO10].

3.3.1 Cyclic covers of smooth varieties

We collect here some facts about étale cyclic covers of smooth varieties. Let X be a smooth quasiprojective variety, together with a free action of the cyclic group $\mathbb{Z}/d\mathbb{Z}$. Since the action is free, the quotient $Y = X/(\mathbb{Z}/d\mathbb{Z})$ is again a smooth and irreducible projective variety, and the quotient map $f\colon X \longrightarrow Y$ is finite and étale of degree d [Mum74, Theorem p.66]. Such a map $\pi\colon X \longrightarrow Y$ is called a *cyclic étale cover of degree d*.

As in Remark 1.4.6 the pushforward $f_*\mathcal{O}_X$ has a structure of $\mathcal{O}_Y$-algebra that decomposes according to the irreducible representations of $\mathbb{Z}/d\mathbb{Z}$: this decomposition has the form

$$f_*\mathcal{O}_X \cong \bigoplus_{i=0}^{d-1} \eta^{-i}$$

where $\eta \in \operatorname{Pic}(Y)$ is a d-torsion line bundle, meaning that there is an isomorphism $\eta^d \to \mathcal{O}_Y$. We note that the $\mathcal{O}_Y$-algebra structure on $f_*\mathcal{O}_X$ gives us one specific isomorphism $\varphi\colon \eta^d \to \mathcal{O}_Y$.

Conversely, take a d-torsion line bundle η on a smooth quasiprojective variety Y. If we fix an isomorphism $\varphi\colon \eta^d \to \mathcal{O}_Y$, we can endow the sheaf $\bigoplus_{i=0}^{d-1} \eta^{-i}$ with the structure of a $\mathcal{O}_Y$-algebra, and it is easy to see that

$$\operatorname{Spec} \bigoplus_{i=0}^{d-1} \eta^{-i} \longrightarrow Y \tag{3.3.1}$$

is a cyclic étale cover of degree d. Hence, there is a correspondence between cyclic étale covers of degree d over Y and d-torsion line bundles η, together with an isomorphism $\eta^d \to \mathcal{O}_Y$. Moreover, if we choose two different isomorphisms $\eta^d \to \mathcal{O}_Y$, it is easy to see that the corresponding cyclic covers of Y are isomorphic.

We will need later the following lemma.

Lemma 3.3.1. *Suppose that Y is projective and connected and let $f\colon X \to Y$ be an étale cyclic cover given by a d-torsion line bundle η. Then, the kernel of the pullback map $f^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ is precisely the subgroup generated by η.*

Proof. Suppose that L is a line bundle on Y such that $f^*L \cong \mathcal{O}_X$. The projection formula gives that $L \otimes f_*\mathcal{O}_X \cong f_*\mathcal{O}_X$, so that

$$\bigoplus_{i=0}^{d-1} L \otimes \eta^{-i} \cong \bigoplus_{i=0}^{d-1} \eta^{-i}.$$

Then, by the Krull-Schmidt theorem [Ati56, Theorem 1,Theorem 3], it follows that $L \cong \eta^{-i}$, for a certain i. For the converse, we need to prove that $f^*\eta \cong \mathcal{O}_X$. However, the previous reasoning shows that $h^0(X, f^*\eta) \neq 0$ and $h^0(X, f^*\eta^{-1}) \neq 0$: this way we get two injective maps $\mathcal{O}_X \to f^*\eta \to \mathcal{O}_X$, and since the composition is an isomorphism, it follows that both of them are isomorphisms as well. □

3.3.2 Cyclic covers of curves and the Prym map

We can specialize the previous discussion to smooth curves: let D be a smooth and irreducible curve of genus g. Then, by what we have remarked before, isomorphism classes of étale cyclic covers of degree d of D correspond to d-torsion line bundles $\eta \in \operatorname{Pic}^0(D)$.

Remark 3.3.2. Moreover, we observe that a cover $f\colon C \to D$ corresponding to η is connected if and only if η has order precisely d.

Proof. The cover C is connected if and only if $h^0(C, \mathcal{O}_C) = 1$, but $h^0(C, \mathcal{O}_C) = h^0(D, f_*\mathcal{O}_C) = \sum_{i=0}^{d-1} h^0(D, \eta^{-i}) = 1 + \sum_{i=1}^{d-1} h^0(D, \eta^{-i})$, and since $\eta \in \operatorname{Pic}^0(D)$ we see that $h^0(D, \eta^{-i}) \neq 0$ if and only if $\eta^i \cong \mathcal{O}_D$. □

From this discussion, we see that we have a moduli space of cyclic covers as follows:

Definition 3.3.3 (The space $\mathcal{R}_{g,d}$)**.** The *moduli space of cyclic covers of degree d* is the space $\mathcal{R}_{g,d}$ of isomorphism classes $[D, \eta]$, where D is a smooth curve of genus g and $\eta \in \operatorname{Pic}^0(D)$ is a torsion bundle of order d. Such a couple (D, η) is sometimes called also a *level curve* of order d.

Remark 3.3.4. The space $\mathcal{R}_{g,d}$ is irreducible [Ber99] and since each curve has a finite number of torsion line bundles, we see that $\dim \mathcal{R}_{g,d} = \dim \mathcal{M}_g = 3g - 3$.

At this point we recall the construction of the Prym variety associated to a cyclic cover. This was classically studied for double étale covers and then extended to arbitrary covers.

Consider a level curve $[D, \eta] \in \mathcal{R}_{g,d}$ and let $f\colon C \to D$ be a corresponding cyclic cover. Then we have the induced norm homomorphism between the Jacobians

$$\operatorname{Nm}(f)\colon \operatorname{Pic}^0(C) \longrightarrow \operatorname{Pic}^0(D), \qquad \mathcal{O}_C\Big(\sum P_i\Big) \mapsto \mathcal{O}_D\Big(\sum f(P_i)\Big).$$

Using this, we can define the Prym variety as follows.

Definition 3.3.5 (Prym variety)**.** The *Prym variety* associated to $[D, \eta] \in \mathcal{R}_{g,d}$ is the principal connected component of the kernel of the norm map:

$$\Pr(D, \eta) :\stackrel{\text{def}}{=} (\operatorname{Ker} \operatorname{Nm}(f))^0.$$

By construction, the Prym variety is an abelian subvariety of $\operatorname{Pic}^0(C)$. In particular, it has a natural polarization obtained by restricting the canonical principal polarization of $\operatorname{Pic}^0(C)$. The type of this polarization has been computed:

Lemma 3.3.6. *With the above notations, the Prym variety* $\mathrm{Pr}(D,\eta)$ *is an abelian variety of dimension* $(d-1)(g-1)$ *and the natural polarization has type*

$$\delta = (1,1,1,\dots,1,d,d,d,\dots,d)$$

where 1 *is repeated* $(d-2)(g-1)$ *times and* d *is repeated* $g-1$ *times.*

Proof. We can compute the dimension as follows:

$$\begin{aligned}\dim \mathrm{Pr}(D,\eta) &= \dim \mathrm{Pic}^0(C) - \dim \mathrm{Pic}^0(D) \\ &= g(C) - g(D) = d(g-1)+1-g = (d-1)(g-1).\end{aligned}$$

For the type of the polarization, see [BL04, Corollary 12.1.5, Lemma 12.3.1]. □

Let us denote by $\mathcal{A}_\delta$ the moduli space of abelian varieties with a polarization of the type in Lemma 3.3.6. Then the Prym construction gives a map of moduli spaces:

Definition 3.3.7 (The Prym map)**.** The Prym map is the map

$$\mathrm{Pr}_{g,n} \colon \mathcal{R}_{g,n} \longrightarrow \mathcal{A}_\delta, \qquad [D,\eta] \mapsto [\mathrm{Prym(D,\jmath)}].$$

Lange and Ortega have considered in [LO10],[LO16] the differential of the Prym map for cyclic covers and they have proved that it is very often injective. In particular, it follows that the Prym map is generically finite.

Here we want to describe this differential, following [LO10]. Consider again a level curve $[D,\eta] \in \mathcal{R}_{g,d}$ and let $f\colon C \to D$ be a corresponding cyclic cover. Since the cover is étale, we have $f^*\omega_D \cong \omega_C$, and the projection formula together with (3.3.1) gives

$$f_*\omega_C \cong \omega_D \otimes f_*\mathcal{O}_C \cong \bigoplus_{i=0}^{d-1} \omega_D \otimes \eta^{-i}.$$

Taking global sections, we get that

$$H^0(C,\omega_C) = \bigoplus_{i=0}^{d-1} H^0(D,\omega_D\otimes\eta^{-i}) \tag{3.3.2}$$

and this is exactly the decomposition of $H^0(C,\omega_C)$ into $(\mathbb{Z}/d\mathbb{Z})$-representations. We single out the non-trivial representations and we set

$$W \stackrel{\text{def}}{:=} \bigoplus_{i=1}^{d-1} H^0(D,\omega_D\otimes\eta^{-i}) \subseteq H^0(C,\omega_C). \tag{3.3.3}$$

With this, we can state the result about the differential of the Prym map.

Proposition 3.3.8 (Lange-Ortega)**.** *With the above notation, the dual of the differential of the Prym map at $[D,\eta] \in \mathcal{R}_{g,d}$ is the multiplication map*

$$m\colon \operatorname{Sym}^2 W \longrightarrow H^0(C,\omega_C^2).$$

Proof. See [LO10, Proposition 4.1]. □

3.3.3 The differential of the Prym map for genus two curves

Now we show how to associate a polarized abelian surface to a cyclic cover of a genus two curve. We then use this abelian surface to study the differential of the Prym map and to prove Theorem E.

Take a level curve $[D,\eta] \in \mathcal{R}_{2,d}$ and let $B = \operatorname{Pic}^0(D)$ be the Jacobian variety of D. We fix a point $P_0 \in D$, so that we have the corresponding Abel-Jacobi map

$$\alpha\colon D \hookrightarrow B, \qquad P \mapsto \mathcal{O}_D(P - P_0)$$

which realizes D as a divisor on B. Standard properties of the Abel-Jacobi map imply that the line bundle $M = \mathcal{O}_B(D)$ is a principal polarization on B [BL04, Corollary 11.2.3] and also that the pullback map

$$\alpha^*\colon \operatorname{Pic}^0(B) \longrightarrow \operatorname{Pic}^0(D)$$

is an isomorphism [BL04, Lemma 11.3.1]. In particular, the line bundle $\eta_B \stackrel{\text{def}}{:=} (\alpha^*)^{-1}(\eta)$ on B is again a torsion bundle of order d. If we choose an isomorphism $\varphi_B\colon \eta_B^d \to \mathcal{O}_B$, we can pull it back via α to an isomorphism $\varphi \stackrel{\text{def}}{:=} \alpha^*(\varphi_B)\colon \eta^d \to \mathcal{O}_D$. We can take the corresponding cyclic covers,

$$C \stackrel{\text{def}}{:=} \operatorname{Spec} \bigoplus_{i=0}^{d-1} \eta^{-i}, \qquad A \stackrel{\text{def}}{:=} \operatorname{Spec} \bigoplus_{i=0}^{d-1} \eta_B^{-i}$$

and we have the following:

Lemma 3.3.9. *With the above notation, we have a fibered square*

$$\begin{array}{ccc} C & \xrightarrow{\ j\ } & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ D & \xrightarrow{\ \alpha\ } & B \end{array}$$

Moreover, A is an abelian surface, the line bundle $L \stackrel{\text{def}}{:=} F^*M$ is ample of type $(1,d)$ and under the embedding $j\colon C \hookrightarrow A$ the curve C can be considered as a divisor $C \in |L|$.

Proof. By construction we have an isomorphism $\alpha^*\left(\bigoplus_{i=0}^{d-1}\eta_B^{-i}\right) \cong \bigoplus_{i=0}^{d-1}\eta^{-i}$ of sheaves of $\mathcal{O}_D$-algebras. Hence, we get a fibered square as above from the properties of the relative Spec [Sta18, Lemma 26.4.6.(2)]. To see that A is an abelian surface, one observes first that it is connected, since η_B has exactly order d, and then $h^0(A,\mathcal{O}_A) = \sum_{i=0}^{d-1} h^0(B,\eta_B^{-i}) = 1$ by Proposition 3.1.1. Since $F\colon A \longrightarrow B$ is an étale finite map, and A is connected, it follows from the Serre-Lang theorem [Mum74, Theorem IV.18] that A is an abelian surface and that the map F is an isogeny.

To conclude, we need to prove that L is ample and of type $(1,d)$. Since the map F is finite and M is ample, it follows from [Laz04, Proposition 1.2.13] that $L = F^*M$ is ample. For the type, we want to use Lemma 3.1.6: we need to prove that $\operatorname{Ker} F \cong \mathbb{Z}/d\mathbb{Z}$. Thanks to [BL04, Proposition 2.4.3], it is enough to show the same for $\operatorname{Ker} F^*$: however we know from Lemma 3.3.1, that $\operatorname{Ker} F^*$ is precisely the subgroup generated by η, so that $\operatorname{Ker} F^* \cong \mathbb{Z}/d\mathbb{Z}$. □

Recall from (3.3.2) that we have a decomposition $H^0(C,\omega_C) = \bigoplus_{i=0}^{d-1} H^0(D,\omega_D \otimes \eta^{-i})$ into $\mathbb{Z}/d\mathbb{Z}$-representations. We have defined in (3.3.3) the linear system $W = \bigoplus_{i=1}^{d-1} H^0(D,\omega_D \otimes \eta^{-i})$ and now we want to give an interpretation of it in terms of the abelian surface A.

Lemma 3.3.10. *With notations as before, W coincides with the image of the restriction map from $H^0(A,L)$ to $H^0(C,\omega_C)$:*

$$W = \operatorname{Im}\left(H^0(A,L) \longrightarrow H^0(C,\omega_C)\right).$$

Proof. First we observe that the restriction map makes sense, since $C \in |L|$ by Lemma 3.3.9, so that the adjunction formula gives $\omega_C \cong \omega_A \otimes L_{|C} \cong L_{|C}$. Let $\tau \in H^0(B,M)$ be a section such that $D = \{\tau = 0\}$. Using again the adjunction formula, we have an exact sequence of sheaves on B

$$0 \longrightarrow \mathcal{O}_B \xrightarrow{\cdot\tau} M \longrightarrow \omega_D \longrightarrow 0$$

and Lemma 3.3.9 shows that, pulling back via F^*, we get an exact sequence

$$0 \longrightarrow \mathcal{O}_A \xrightarrow{\cdot\sigma} L \longrightarrow \omega_C \longrightarrow 0$$

where $\sigma := F^*(\tau) \in H^0(A,L)$. By construction, we see that this is actually an exact sequence of $(\mathbb{Z}/d\mathbb{Z})$-sheaves and, moreover, if we take the pushforward along F_* and then take the $(\mathbb{Z}/d\mathbb{Z})$-invariant part, we get a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_*(\mathcal{O}_A) & \xrightarrow{F_*(\cdot\sigma)} & F_*L & \longrightarrow & F_*\omega_C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle(-)^{\mathbb{Z}/d\mathbb{Z}}} & & \downarrow{\scriptstyle(-)^{\mathbb{Z}/d\mathbb{Z}}} & & \downarrow{\scriptstyle(-)^{\mathbb{Z}/d\mathbb{Z}}} & & \\
0 & \longrightarrow & \mathcal{O}_B & \xrightarrow{\cdot\tau} & M & \longrightarrow & \omega_D & \longrightarrow & 0
\end{array}$$

Passing to global sections, we get another commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}\sigma & \longrightarrow & H^0(A,L) & \longrightarrow & H^0(C,\omega_C) & \longrightarrow & H^1(A,\mathcal{O}_A) & \longrightarrow & 0 \\ & & \big\downarrow (-)^{\mathbb{Z}/d\mathbb{Z}} & & \big\downarrow (-)^{\mathbb{Z}/d\mathbb{Z}} & & \big\downarrow (-)^{\mathbb{Z}/d\mathbb{Z}} & & \big\downarrow (-)^{\mathbb{Z}/d\mathbb{Z}} & & \\ 0 & \longrightarrow & \mathbb{C}\tau & \longrightarrow & H^0(B,M) & \longrightarrow & H^0(D,\omega_D) & \longrightarrow & H^1(B,\mathcal{O}_B) & \longrightarrow & 0 \end{array} \tag{3.3.4}$$

Since M gives a principal polarization, we have that $h^0(B,M)=1$ by Lemma 3.1.4, so that the map $\mathbb{C}\tau \to H^0(B,M)$ is an isomorphism. Hence, the map $H^0(B,M) \to H^0(D,\omega_D)$ is zero, and since the diagram is commutative, it follows that

$$\operatorname{Im}\left(H^0(A,L) \longrightarrow H^0(C,\omega_C)\right) \subseteq W = \operatorname{Ker}\left((-)^{\mathbb{Z}/d\mathbb{Z}} \colon H^0(C,\omega_C) \longrightarrow H^0(D,\omega_D)\right). \tag{3.3.5}$$

To conclude it is enough to show that the two spaces in (3.3.5) have the same dimension. To prove this, we look again at diagram (3.3.4) and we see that

$$\begin{aligned} \dim_{\mathbb{C}} \operatorname{Im}\left(H^0(A,L) \longrightarrow H^0(C,\omega_C)\right) = h^0(C,\omega_C) - h^1(A,\mathcal{O}_A) = h^0(C,\omega_C) - 2, \\ \dim_{\mathbb{C}} W = h^0(C,\omega_C) - h^0(D,\omega_D) = h^0(C,\omega_C) - 2. \end{aligned}$$

□

With this lemma, we can reinterpret the codifferential of the Prym map in Proposition 3.3.8 as a multiplication map on the abelian surface A:

Lemma 3.3.11. *With the same notations of before, we have that*

$$\operatorname{Sym}^2 W \longrightarrow H^0(C,\omega_C^2) \text{ is surjective}$$

if and only if

$$\operatorname{Sym}^2 H^0(A,L) \longrightarrow H^0(A,L^2) \text{ is surjective.}$$

Proof. We could give a general proof using Koszul cohomology, but since this is a very simple case we follow a more elementary approach. We first observe that in the statement we can replace $\operatorname{Sym}^2 W$ and $\operatorname{Sym}^2 H^0(A,L)$ with $W^{\otimes 2}$ and $H^0(A,L)^{\otimes 2}$ respectively. We take again a section $\sigma \in H^0(A,L)$ such that $C = \{\sigma = 0\}$: then Lemma 3.3.10 gives the exact sequence

$$0 \longrightarrow \mathbb{C}\sigma \longrightarrow H^0(A,L) \longrightarrow W \longrightarrow 0. \tag{3.3.6}$$

Instead, if we take global sections in the exact sequence

$$0 \longrightarrow L \xrightarrow{\cdot\sigma} L^2 \longrightarrow \omega_C^2 \longrightarrow 0$$

and we use that $H^1(A,L) = 0$ (see Proposition 3.1.1), we get an exact sequence

$$0 \longrightarrow H^0(A,L)\sigma \longrightarrow H^0(A,L^2) \longrightarrow H^0(C,\omega_C^2) \longrightarrow 0. \tag{3.3.7}$$

Putting together (3.3.6) and (3.3.7), we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sigma\otimes H^0(A,L)+H^0(A,L)\otimes\sigma & \longrightarrow & H(A,L)^{\otimes 2} & \longrightarrow & W^{\otimes 2} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^0(A,L)\sigma & \longrightarrow & H^0(A,L^2) & \longrightarrow & H^0(C,\omega_C^2) & \longrightarrow & 0 \end{array}$$

Since the map

$$\sigma\otimes H^0(A,L)+H^0(A,L)\otimes\sigma \longrightarrow H^0(A,L)\sigma$$

is clearly surjective, the Snake Lemma proves that

$$\operatorname{Coker}\left(H^0(A,L)^{\otimes 2} \longrightarrow H^0(A,L^2)\right) \cong \operatorname{Coker}\left(W^{\otimes 2} \longrightarrow H^0(C,\omega_C^2)\right)$$

which implies the statement. □

Before concluding, we introduce bielliptic curves and bielliptic covers.

Definition 3.3.12 (Bielliptic curve)**.** A smooth and irreducible projective curve C is called *bielliptic* if it admits a map of degree two $C \to E$ to an elliptic curve E. The map $C \to E$ is called a *bielliptic map* and the corresponding involution $C \to C$ is called a *bielliptic involution*.

Remark 3.3.13. If D is a curve of genus two, a bielliptic map $D \to F$ is ramified at 2 points by Riemann-Hurwitz. Hence, a bielliptic curve of genus two can be chosen by specifying two points on an elliptic curve: since $\dim \mathcal{M}_{1,2} = 2$ and $\dim \mathcal{M}_2 = 3$, this shows that the general curve of genus two is not bielliptic and that the bielliptic locus is a divisor on $\mathcal{M}_2$.

Definition 3.3.14 (Bielliptic cover)**.** A cover $f\colon C \to D$ of smooth and irreducible projective curves is called *bielliptic* if and only if there exist compatible bielliptic quotients of C and D. This means that there is a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{2:1} & E \\ \downarrow f & & \downarrow \\ D & \xrightarrow{2:1} & F \end{array}$$

where E and F are elliptic curves and $C \to E$ and $D \to F$ are maps of degree two.

Remark 3.3.15. Remark 3.3.13 shows that the general cyclic cover $C \to D$ of a genus two curve is not bielliptic.

Bielliptic cover appear in our situation because of the following result of Ramanan:

Theorem 3.3.16 (Ramanan)**.** *In the situation of Lemma 3.3.9, the line bundle L is very ample if and only if $d \geq 5$ and the cover $C \to D$ is not bielliptic*

Proof. See [Ram85, Theorem 3.1]. □

Finally, we give the proof of Theorem E:

Proof of Theorem E. We know from Lemma 3.3.11 that the differential of the Prym map is injective at $[D, \eta]$ if and only the multiplication map $\operatorname{Sym}^2 H^0(A, L) \to H^0(A, L^2)$ is surjective, where (A, L) is the polarized abelian surface which corresponds to (C, η), as in Lemma 3.3.9. By Theorem 3.2.1, this happens if and only if $d \geq 7$ and the line bundle L is very ample. Hence, we conclude thanks to Ramanan's Theorem 3.3.16. □

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Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß §7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, den 17. April 2018 Daniele Agostini