The arithmetic volume of $\overline{\mathcal{A}}_2$

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Abstract:

Let $\overline{\mathcal{A}}_2$ be the toroidally compactified moduli stack of principally polarized complex abelian surfaces, and let $\mathcal{M}_k(\Gamma_2)$ be the line bundle of Siegel modular forms on $\overline{\mathcal{A}}_2$, equipped with the Petersson metric $\| \cdot \|_{\mathrm{Pet}}$. Viewing $\overline{\mathcal{A}}_2$ as the complex fibre of an arithmetic variety $\overline{\mathscr{A}_2}$ over $\operatorname{Spec}\mathbb{Z}$, and $\mathcal{M}_k(\Gamma_2)$ as the complex line bundle induced by a line bundle on $\overline{\mathscr{A}_2}$, we can ask for the arithmetic degree of this line bundle. We will state a formula for the arithmetic degree $\widehat{\deg}(\mathcal{M}_k(\Gamma_2), \| \cdot \|_{\mathrm{Pet}})$ in terms of special values of the logarithmic derivative of the Riemann ζ-function.

The arithmetic degree consists of a contribution from intersection over $\operatorname{Spec}\mathbb{Z}$, and from an integral of Green forms over the complex fibre. The computation of the summand of the arithmetic degree coming from the complex fibre $\overline{\mathcal{A}}_2$ will be approached by making a specific choice of sections of $\mathcal{M}_k(\Gamma_2)$, whose behaviour is well-known or can be worked out by their representation via ϑ-functions. With an induction argument, we will trace back the integral over the $*$-product of the corresponding Green forms to a sum of integrals over particular cycles on $\overline{\mathcal{A}}_2$ coming from the successive intersection of the divisors of these sections, as well as some boundary terms in the form of integrals around the toroidal boundary. We will prove that the boundary terms vanish, using Minkowski theory and a specific choice of the partition of unity that appears in arithmetic intersection theory for logarithmically singular metrics. The integrals over the special cycles will be traced back to results of Kudla and an application of a modular version of Jensen's formula.

Zusammenfassung:

Es sei $\overline{\mathcal{A}}_2$ der toroidal kompaktifizierte Modulraum prinzipal polarisierter komplexer abelscher Flächen, und $\mathcal{M}_k(\Gamma_2)$ das Geradenbündel Siegel'scher Modulformen auf $\overline{\mathcal{A}}_2$, versehen mit der Petersson-Metrik $\|\cdot\|_{\mathrm{Pet}}$. Betrachtet man $\overline{\mathcal{A}}_2$ als komplexe Faser einer arithmetischen Varietät $\overline{\mathscr{A}_2}$ über $\operatorname{Spec}\mathbb{Z}$, und $\mathcal{M}_k(\Gamma_2)$ als das von einem Geradenbündel auf $\overline{\mathscr{A}_2}$ induzierte Geradenbündel, so kann man die Frage nach dem arithmetischen Grad dieses Geradenbündels stellen. Wir stellen nachfolgend den Grad $\widehat{\deg}(\mathcal{M}_k(\Gamma_2), \|\cdot\|_{\mathrm{Pet}})$ als Ausdruck in speziellen Werten der logarithmischen Ableitung der Riemann'schen ζ-Funktion dar.

Der arithmetische Grad setzt sich aus einem Beitrag vom Schnitt über den endlichen Fasern und einem Integral von Green'schen Formen über die komplexe Faser zusammen. Die Berechnung des von der komplexen Faser $\overline{\mathcal{A}}_2$ induzierten Anteils am arithmetischen Grad erfolgt durch eine spezifische Wahl von Schnitten von $\mathcal{M}_k(\Gamma_2)$, deren Eigenschaften bekannt oder durch ihre Darstellung als Polynome in ϑ-Funktionen ableitbar sind. Mittels eines induktiven Arguments werden wir das Integral über das $*$-Produkt der zugehörigen Green'schen Formen auf eine Summe von Integralen über spezielle Zykel zu-rückführen, die beim sukzessiven Schneiden der zu den Schnitten gehörigen Divisoren auftauchen. Bei diesem Prozess entstehen Randterme in Form von Integralen um den toroidalen Rand. Wir werden zeigen, dass diese verschwinden, indem wir Minkowski-Theorie anwenden und eine bestimmte Wahl der Teilung der Eins treffen, die in der arithmetischen Schnitttheorie für logarithmisch singuläre Metriken auftaucht. Die Integrale über die speziellen Zykel berechnen wir durch Zurückführen auf ein Resultat von Kudla sowie auf eine modulare Version der Jensen-Formel.

Contents

Introduction

Abelian varieties, i.e., varieties with the additional structure of an algebraic group, arise naturally as Jacobians, Albanese varieties, and Prym varieties and form an important research tool in geometry and number theory, but also in adjacent fields to mathematics, such as physics and cryptography. The moduli stacks $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g and their corresponding universal families $\mathcal{X}_g$ are main examples of Shimura varieties and mixed Shimura varieties, respectively, and their algebraic properties are in the focus of researchers such as Siegel, Igusa, and Mumford. In 1943, Siegel [43] inductively computed their geometric volume, i.e., the degree of the associated Hodge bundle ω_g, to equal

$$\mathrm{vol}(\mathcal{A}_g) = \deg(\omega_g) = (g-1)!\pi^{-g}\zeta(2g)\,\mathrm{vol}(\mathcal{A}_{g-1}).$$

The appearing of the ζ-function arises from the interpretation of the volume as a density connected with the solutions of a quadratic matrix equation. Then, the Euler formula for ζ emerges from the product formula linking the p-adic to the real case.
A construction of a model for the moduli stacks $\mathcal{A}_g$ over $\mathrm{Spec}(\mathbb{Z})$ was given by Mumford et al. in [37]. Later, Faltings and Chai [14] were able to construct a model for toroidal compactifications $\overline{\mathcal{A}}_g$ over $\mathrm{Spec}(\mathbb{Z})$.
In order to measure arithmetic complexity on these objects over $\mathrm{Spec}(\mathbb{Z})$, hence, to have a notion of height of a cycle and arithmetic degree of a vector bundle, one needs an intersection theory at hand. An arithmetic intersection theory, i.e., an intersection theory for varieties over $\mathrm{Spec}(\mathbb{Z})$, or, more generally, over the spectra of arithmetic rings A, was first approached by Arakelov [2] (and later extended by Deligne [11]) in the case of relative dimension 1. By „compactifying“ the variety over A by a complex fibre induced by the complex embeddings of A, they were able to define a good notion of intersection, considering on the complex fibre the additional datum of Green functions induced by holomorphic vector bundles with smooth Hermitian metrics. Soon after, this approach was generalized by Gillet and Soulé for arbitrary relative dimension. As the scope of applications of this theory is limited to vector bundles with smooth metrics, it cannot be applied to the case of the metrized Hodge bundle $\overline{\omega}_g$ on $\overline{\mathcal{A}}_g$, as the L^2-metric acquires logarithmic singularities at the toroidal boundary. This fact was one of the motivations for Burgos, Kramer, and Kühn [9] to come up with a more flexible concept of arithmetic Chow groups that can be applied to the situation of mildly singular metrics. Laying the foundation for this generalized intersection theory, Kühn [33] computed the arithmetic height, or arithmetic volume, of $\overline{\mathcal{A}}_1$, which is by definition the arithmetic degree of the metrized Hodge bundle $\overline{\omega}_1$ on $\overline{\mathcal{A}}_1$, to equal

$$\widehat{\mathrm{vol}}(\overline{\mathcal{A}}_1) = \widehat{\deg}(\overline{\omega}_1) = \zeta(-1)\left(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}\right).$$

Further results included the case of Hilbert modular surface by Bruinier, Burgos and Kühn [6], a result connecting integrals over Borcherds forms with special values of Eisenstein series by Kudla [31] that is a main ingredient for the result of this thesis, and an inductive approach for Shimura varieties of orthogonal type by Hörmann [26], which is also applicable to the problem tackled here. More general conjectures about arithmetic intersection numbers by Kramer, Maillot–Roessler [34], and Kudla [32] are stating connections between Fourier coefficients of certain modular forms and classes of algebraic cycles in the arithmetic Chow groups. In particular, the arithmetic degree of the bundle of modular forms is conjectured to be a rational linear combination of logarithmic derivatives of the ζ-function evaluated at negative odd integers. In the thesis at hand, we obtain a formula for the arithmetic height, or arithmetic volume, of $\overline{\mathcal{A}}_2$, by computing the degree of the metrized Hodge bundle on $\overline{\mathcal{A}}_2$ to equal

$$\widehat{\mathrm{vol}}(\overline{\mathcal{A}}_2) = \widehat{\deg}(\overline{\omega}_2) = \zeta(-3)\zeta(-1)\left(2\frac{\zeta'(-3)}{\zeta(-3)} + 2\frac{\zeta'(-1)}{\zeta(-1)} + \frac{17}{6}\right) + c,$$

with a constant c coming from the intersection number at the finite places 2 and 3. We are taking an explicit approach, identifying the Hodge bundle with the bundle of modular forms, choosing certain well-investigated sections of this bundle and tracing the value of the arithmetic volume back to the results mentioned above by giving a recursive formula of integrals over cycles. The thesis is structured as follows:

In the first part of Chapter 1, we introduce principally polarized abelian varieties of dimension g over $\mathbb{C}$, following [35]. Their isomorphism classes are parametrized by the quotient $\mathcal{A}_g := \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$ of the Siegel upper half plane by the symplectic group. Passing to the category of orbifolds, we obtain a universal family $\mathcal{X}_g \longrightarrow \mathcal{A}_g$, with $\mathcal{X}_g = \left(\Gamma_g \ltimes \mathbb{Z}^{2g}\right) \backslash (\mathbb{H}_g \times \mathbb{C}^g)$. The k-th power of the Hodge bundle ω_g associated to the universal family on $\mathcal{A}_g$ can be identified with the line bundle $\mathcal{M}_k(\mathrm{Sp}_{2g}(\mathbb{Z}))$ of Siegel modular forms of weight k. The L^2-metric on ω_g translates to the Petersson metric $\|\cdot\|_{\mathrm{Pet}}$ of modular forms.
In the second part of Chapter 1, we review compactifications of $\mathcal{A}_g$. After observing that the naturally arising Baily–Borel-compactification is highly singular at the boundary for $g > 1$, we will proceed to the theory of toroidal compactifications $\overline{\mathcal{A}}_g$. They are constructed in a way such that the boundary $\partial\mathcal{A}_2$ is a normal crossing divisor on $\overline{\mathcal{A}}_g$. We will describe the process of toroidal compactification in detail, and in particular give an explicit description of the boundary divisor $\partial\mathcal{A}_2$ as a compactification of the universal family $\mathcal{X}_{g-1}$, which will be of use for the computations in Chapter 3.

Turning to the arithmetic side in Chapter 2, we first introduce the notion of an arithmetic variety $\mathscr{X} \longrightarrow \mathrm{Spec}(A)$ over an arithmetic ring A with its additional complex fibre X_∞. We then review the classical arithmetic intersection theory by Gillet and Soulé [19], where elements in the p-th Chow groups are p-codimensional cycles equipped with the additional datum of a Green current on the complex fibre, uniquely (up to smooth currents) defined by a differential equation. The intersection of two such cycles then corresponds to the $*$-product of their Green currents. As in geometric intersection theory, the group $\widehat{\mathrm{Pic}}(\mathscr{X})$ of isometry classes of line bundles equipped with smooth Hermitian metrics can be identified with the first arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathscr{X})$ via the first Chern form $\widehat{\mathrm{c}}_1$. This allows the definition of the arithmetic degree of a smoothly metrized vector bundle as its self intersection number. Unfortunately, in many important examples, the smoothness of the metric is not given. In particular, as previously mentioned, the Petersson metric is logarithmically singular at the boundary $\partial\mathcal{A}_2$, and the arithmetic degree of the Hodge bundle on $\overline{\mathcal{A}}_2$, and, hence, the arithmetic volume of $\overline{\mathcal{A}}_2$, is not defined by this theory. Therefore, we will, as announced, turn to a generalized approach to arithmetic intersection theory, established by Burgos, Kramer, and Kühn [9]. Their approach uses the fact that the defining differential equation of a Green current translates to a cohomological condition in real Deligne–Beilinson-cohomology. This cohomology is computed by a complex $\mathcal{D}_{\log}$. By varying the complex, one can generalize the notion of arithmetic Chow groups, such that it is applicable to logarithmically singular Hermitian metrics on line bundles. Due to a theorem of Burgos [7], there is an explicit expression for the generalized $*$-product of Green forms in this situation that involves a partition of unity adapted to the corresponding cycles. The first Chern form and the notion of arithmetic degree can be defined similarly as in the classical case.

In Chapter 3, we will proceed with the computation of the arithmetic volume of $\overline{\mathcal{A}}_2$, i.e., the arithmetic degree of the line bundle of modular forms. As established in Chapter 2, this number consists of two parts, the intersection of cycles over the integers and an integral over the $*$-product of Green currents over the complex fibre. To tackle the latter, we will first make an explicit choice of sections of the bundle by considering four modular forms E_4, E_6, χ_{10}, and χ_{12} of degree 2, whose divisors are shown to intersect successively properly. Carefully applying a result of Bruinier, Burgos, and Kühn [6], we obtain a recursive formula for the integral over the $*$-product of the four corresponding Green forms g_4, g_6, g_{10}, g_{12} of the shape

$$\int\limits_{\overline{\mathcal{A}}_2} g_{10} * g_6 * g_4 * g_{12} = (A) + (B) + (C) + (D) + (E),$$

where (A) is the integral of g_{10} over $\mathcal{A}_2$, (B) is a limit term involving two integrals around the boundaries $\partial B_\varepsilon(\partial\mathcal{A}_2)$ and $\partial B_\varepsilon(\partial\mathcal{H})$ of tubular ε-neighbourhoods of the boundary divisors $\partial\mathcal{A}_2$ of $\mathcal{A}_2$ and $\partial\mathcal{H}$ of the Humbert surface $\mathcal{H}$, and (C), (D), (E) are integrals over the cycles coming from the successive intersection of the divisors of the chosen modular forms.

The value of (A) is easily computed as a corollary to a result of Kudla about integrals of Borcherd forms over Shimura varieties. The integrands of the boundary integrals in (B) can be expressed in terms of Green forms and partitions of unity adapted to the corresponding divisors. We will write them out explicitely and then show growth conditions that will guarantee the vanishing of the integrals for ε approaching 0. For the integral over $\partial B_\varepsilon(\partial \mathcal{A}_2)$, this is achieved by taking two simplifying steps. First, we enlarge the domain of integration by considering a suitable fundamental domain for the action of $\mathrm{Sp}_4(\mathbb{Z})$ on $\mathbb{H}_2$ via Minkowski theory. By that, we obtain bounds for the local coordinates that will turn out to be sufficient for proving the vanishing of the integral. Second, we choose the partition of unity appearing in the definition of $*$-product carefully, in a way that it becomes constant in a range U, and such that outside U, the local coordinates are bounded. Then, the integrand simplifies on U, and the corresponding forms can be explicitely computed and bounded. We obtain that the integral over U behaves as $\log\log(\varepsilon)/\log(\varepsilon)$ for small ε, and therefore vanishes for ε approaching 0. Outside U, the estimates have to be coarser, due to the non-constant partition of unity. But as here the local coordinates are bounded, the vanishing of the integral still follows with a lemma about the growth of the derivative of ϑ-functions near $\partial \mathcal{A}_2$. In summary, we obtain:

Theorem 3.3.17. The integral

$$\frac{1}{(4\pi i)^2} \int\limits_{\partial B_\varepsilon(\partial \mathcal{A}_2)} [(g_6 * g_4 * g_{12}) \wedge \mathrm{d}^c g_{10} - g_{10} \wedge \mathrm{d}^c (g_6 * g_4 * g_{12})]$$

converges absolutely, and its value tends to 0 for ε approaching 0.

The vanishing of the boundary integral around $\partial B_\varepsilon(\partial \mathcal{H})$ follows in a similar way.

The sum of the integrals (C), (D) and (E) is just the arithmetic height, or arithmetic volume, of the Humbert surface $\mathcal{H}$, and can be traced back to results for $\mathcal{A}_1$. As $\mathcal{H}$ is the symmetric product of two $\mathcal{A}_1$'s, the integrals over the cycles $\overline{\mathcal{H}}$,$\{i\} \times_{\mathcal{H}} \overline{\mathcal{A}}_1$, and $\{i\} \times_{\mathcal{H}} \{\omega\}$ coming from the successive intersection of divisors can be computed applying a modular version of Jensen's formula that provides the integral of Green forms associated to Eisenstein series over $\mathcal{A}_1$. We obtain the equality

$$\widehat{\mathrm{vol}}(\mathcal{H}) = \zeta^2(-1)\left(2\frac{\zeta'(-1)}{\zeta(-1)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2}\right) + \tilde{c},$$

with the constant $\tilde{c}$ coming from the intersection number at the finite places 2 and 3. We note that our result is in compliance with a degeneration of a result of Bruinier and Kühn [6] about the height of Hilbert modular surfaces.

Combining the above results, we finally obtain the main theorem about the arithmetic volume of $\overline{\mathcal{A}}_2$, up to contributions from the places 2 and 3.

Theorem 3.5.4. The arithmetic self intersection number, i.e., the arithmetic degree of the line bundle $\mathcal{M}_k(\Gamma_2)$ of modular forms of weight k on $\overline{\mathcal{A}}_2$, equipped with the Petersson metric, is given as

$$\widehat{\deg}(\mathcal{M}_k(\Gamma_2), \|\cdot\|_{\mathrm{Pet}}) = k^4\zeta(-3)\zeta(-1)\left(2\frac{\zeta'(-3)}{\zeta(-3)} + 2\frac{\zeta'(-1)}{\zeta(-1)} + \frac{17}{6}\right) + c,$$

with c a constant of the form $c_2 \log 2 + c_3 \log 3$ ($c_2, c_3 \in \mathbb{Q}$).

This value is in compliance with what is implied in the literature, e.g., in [5], and strengthens the conjectures about the connection between the geometric and arithmetic volume previously mentioned.

Chapter 1

The moduli space of principally polarized abelian varieties and its compactifications

Abelian varieties, i.e., varieties with the additional structure of an abelian group, appear naturally in algebraic geometry as Jacobians or Albanese varieties. Complex abelian varieties can be identified with complex tori that allow an embedding into a complex projective space. Through this viewpoint, one obtains an interpretation of the moduli space of (principally polarized) abelian varieties as a locally symmetric space. A smooth, geometrically meaningful compactification of this moduli space is provided by the theory of toroidal compactifications.

1.1 Abelian varieties over $\mathbb{C}$

In the following, we will give an overview over complex abelian varieties as algebraic tori and introduce the notion of a polarization. For the basic results, we will follow [4] and [35].

Definition 1.1.1. Let k be any field. An *abelian variety over k* is an irreducible projective algebraic variety X whose points inherit the structure of an abelian group such that the group operation $m\colon X \times X \longrightarrow X$ and the inverse map $\operatorname{inv}\colon X \longrightarrow X$ are both morphisms of varieties.

Proposition 1.1.2. *The group operation of an abelian variety X is commutative.*

Proof. The proposition is a corollary of the rigidity lemma for complete varieties: Let X, Y, Z be varieties with X complete. Let $f\colon X \times Y \longrightarrow Z$ be a morphism such that for some $y_0 \in Y$, $z_0 \in Z$, the identity $f(x, y_0) = z_0$ holds for all $x \in X$. Denote by $p_2\colon X \times Y \longrightarrow Y$ the projection to Y. Then, there is a morphism $g\colon Y \longrightarrow Z$ such that $f = g \circ p_2$, so the point $f(x, y) \in Z$

only depends on the y-coordinate. An easy proof of the rigidity lemma can be found in [35], Chapter II.

Let now X be an abelian variety with identity element e. We apply the rigidity lemma to the map $f\colon X \times X \longrightarrow X$, given by $f(x,y) := (xy)^{-1}x^{-1}y^{-1}$. As $f(x,e) = e$ for all $x \in X$, it follows that f is constant in x, and, choosing $x = e$, one concludes $f(x,y) = e$. Hence, A is commutative. $\square$

From now on, we consider the case $k = \mathbb{C}$. In particular, an abelian variety over $\mathbb{C}$ is a connected compact Lie group. The following theorem states that a connected compact complex Lie group is necessarily a torus. We recall that a g-dimensional complex torus is the quotient V/Λ of a g-dimensional complex vector space V by a lattice $\Lambda \subseteq V$, i.e., a discrete subgroup of V of rank $2g$.

Theorem 1.1.3. *Any connected compact complex Lie group X is a torus, i.e., of the form V/Λ, where V is a complex vector space and Λ a lattice in V.*

Proof. With a similar argument as above, we can show that X is commutative. Let $V := T_e(X)$ be the tangent space to X at the identity element $e \in X$. For every $v \in V$, there is a unique holomorphic homomorphism $\phi_v\colon \mathbb{C} \longrightarrow X$ such that $\mathrm{d}\phi_v$ sends the unit tangent vector $\left(\frac{\partial}{\partial t}\right)_0$ to $\mathbb{C}$ at 0 to $v \in V$. The exponential map $\exp\colon V \longrightarrow X$, defined by $\exp(v) = \phi_v(1)$, is also a homomorphism, as X is commutative. In a neighbourhood of $0 \in V$, the map exp is a homeomorphism. Hence, its image $\exp(V)$ is an open subgroup of the topological group X, and therefore also closed. As X is connected, the exponential map is surjective. Its kernel Λ has to be discrete, as exp is locally injective. Because X is compact, Λ has to be of rank $2g$, hence a lattice, and $X = V/\Lambda$ as claimed. $\square$

In the following we will establish the conditions under which a complex torus is an abelian variety. Chow's theorem states that a closed analytic subspace of a projective space is an algebraic variety. Therefore, the question whether a torus X is a projective variety breaks down to the existence of an ample line bundle on X, as we will see below. Line bundles on X are, up to isomorphism, parametrized by the Picard group $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$ of X. Below we will find a description of $\mathrm{Pic}(X)$ that gives us a better approach to the ampleness of line bundles. It will be motivated by the long exact sequence in cohomology

$$\cdots \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow \cdots, \tag{1.1.1}$$

arising from the short exact exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times \longrightarrow 1.$$

Lemma 1.1.4. *For a complex torus $X = V/\Lambda$, there is a canonical isomorphism*

$$H^n(X,\mathbb{Z}) \cong \mathrm{Alt}^n(\Lambda,\mathbb{Z})$$

from the n-th cohomology with coefficients in $\mathbb{Z}$ to the integer-valued alternating n-forms on Λ.

Proof. By the universal coefficient theorem, there is a natural isomorphism $H^1(X,\mathbb{Z}) \cong \operatorname{Hom}(H_1(X,\mathbb{Z}),\mathbb{Z})$, and $H_1(X,\mathbb{Z}) = \pi_1(X) = \Lambda$. So for $n = 1$, we obtain the identity $H^1(X,\mathbb{Z}) \cong \operatorname{Hom}(\Lambda,\mathbb{Z})$. As a real manifold, a torus X equals the product $S^1 \times \cdots \times S^1$ of $2g$ circles. Using the Künneth formula inductively, it follows that the canonical map $\bigwedge^n H^1(X,\mathbb{Z}) \longrightarrow H^n(X,\mathbb{Z})$ induced by the cup product is an isomorphism. In conclusion, we obtain the identification $\bigwedge^n H^1(X,\mathbb{Z}) \cong \bigwedge^n \operatorname{Hom}(\Lambda,\mathbb{Z}) = \operatorname{Alt}^n(\Lambda,\mathbb{Z})$, and the lemma follows. □

Definition 1.1.5. The image $c_1(L)$ of an element $L \in \operatorname{Pic}(X)$ under the map $c_1\colon \operatorname{Pic}(X) \longrightarrow H^2(X,\mathbb{Z})$ in (1.1.1) is called *the first Chern class of L*. By the above lemma we can identify $c_1(L)$ with an alternating 2-form. Note that in the sequel we will identify a line bundle L with its isomorphism class in $\operatorname{Pic}(X)$ if needed.

We will now establish a condition for alternating forms being Chern classes of line bundles and thereby find a useful description of line bundles of a complex torus X.

Proposition 1.1.6. *Let $X = V/\Lambda$ be a complex torus. An alternating form $E\colon V \times V \longrightarrow \mathbb{R}$ represents the first Chern class of a line bundle L on X if and only if $E(\Lambda,\Lambda) \subseteq \mathbb{Z}$ and $E(iv,iw) = E(v,w)$ for all $v,w \in V$.*

Proof. The proof can be found in [4], Chapter 2, Proposition 1.6. □

Lemma 1.1.7. *There is a 1-1-correspondence between the set of real-valued alternating forms E on V satisfying $E(iv,iw) = E(v,w)$ for all $v,w \in V$ and the set of Hermitian forms H on V. Therefore, the alternating form $E \in \operatorname{Alt}^2(\Lambda,\mathbb{Z})$ representing the first Chern class of a line bundle corresponds to a Hermitian form H on V with the property* $\operatorname{Im}(H)(\Lambda,\Lambda) \subseteq \mathbb{Z}$.

Proof. Given an alternating form E satisfying the above condition, the form H defined by $H(v,w) := E(iv,w) + iE(v,w)$ for all $v,w \in V$ is Hermitian, since

$$H(v,w) = E(iv,w) + iE(v,w) = E(iw,v) - iE(w,v) = \overline{H(w,v)}.$$

Conversely, given a Hermitian form H, the form $E := \operatorname{Im}(H)$ is alternating and $E(iv,iw) = \operatorname{Im}(H(iv,iw)) = \operatorname{Im}(H(v,w)) = E(v,w)$. □

Definition 1.1.8. Let $X = V/\Lambda$ be a complex torus and $H\colon V \times V \longrightarrow \mathbb{C}$ be a Hermitian form whose imaginary part $E := \operatorname{Im}(H)$ fulfills $E(\Lambda,\Lambda) \subseteq \mathbb{Z}$. A *semicharacter α for H* is a map $\alpha\colon \Lambda \longrightarrow S^1 \subseteq \mathbb{C}$ satisfying

$$\alpha(\lambda_1 + \lambda_2) = e^{i\pi E(\lambda_1,\lambda_2)}\alpha(\lambda_1)\alpha(\lambda_2) \quad (\lambda_1,\lambda_2 \in \Lambda)\,.$$

Definition 1.1.9. Let $X = V/\Lambda$ be a complex torus. We denote by $\mathcal{P}(\Lambda)$ the group of tuples (H,α) with H a Hermitian form on V satisfying $\operatorname{Im}(H)(\Lambda,\Lambda) \subseteq \mathbb{Z}$ and α a semicharacter for H.

Proposition 1.1.10. *Let $X = V/\Lambda$ be a complex torus. There is a homomorphism of groups $L\colon \mathcal{P}(\Lambda) \longrightarrow \mathrm{Pic}(X)$ given in the following way: A tuple (H, α) defines an action of Λ on $\mathbb{C} \times V$ by*

$$\lambda(z, v) = (\alpha(\lambda)e^{\pi H(v,\lambda)+\frac{1}{2}\pi H(\lambda,\lambda)}z, v + \lambda) \quad (\lambda \in \Lambda,\, (z, v) \in \mathbb{C} \times V)\,.$$

The image $L(H, \alpha)$ of (H, α) is defined as the quotient $(\mathbb{C}\times V)/\Lambda$ of this action.

Proof. Due to the multiplicative property of α, we see that

$$L(H_1, \alpha_1) \otimes L(H_2, \alpha_2) = L(H_1 + H_2, \alpha_1\alpha_2).$$

The action of Λ on $\mathbb{C} \times V$ induced by (H, α) is free and properly discontinuous. Hence, the quotient is a complex manifold. Considering the projection $L(H, \alpha) \longrightarrow X$ induced by $\mathbb{C} \times V \longrightarrow V$, one easily checks that $L(H, \alpha)$ is a holomorphic line bundle. □

Theorem 1.1.11 (Appell–Humbert theorem)**.** *Any line bundle on the complex torus X is isomorphic to a line bundle of the form $L(H, \alpha)$, for uniquely determined (H, α) as above.*

Proof. By Proposition 1.1.6 and Lemma 1.1.7, the image of c_1 corresponds to a Hermitian form H on V with $\mathrm{Im}(H)(\Lambda \times \Lambda) \subseteq \mathbb{Z}$. Using the exact sequence (1.1.1), one can show that that $\ker(c_1) = \mathrm{Pic}^0(X)$ is isomorphic to the group $\mathrm{Hom}(\Lambda, S^1)$ of characters on Λ. The group $\mathrm{Hom}(\Lambda, S^1)$ can be embedded into $\mathcal{P}(\Lambda)$ by the map $\alpha \mapsto (0, \alpha)$. As the sequence

$$1 \longrightarrow \mathrm{Hom}(\Lambda, S^1) \longrightarrow \mathcal{P}(\Lambda) \longrightarrow \mathrm{im}(c_1) \longrightarrow 0$$

is exact, the theorem follows. The full proof of the theorem can be found in [4], Chapter 2.2. □

Proposition 1.1.12. *Let H be a positive definite Hermitian form, and $E =$ $\mathrm{Im}(H)$. We can express E as a matrix using any $\mathbb{Z}$-basis of Λ. Then, the dimension of the space of global sections of the corresponding line bundle $L(H, \alpha)$ is given by*

$$\dim H^0(X, L(H, \alpha)) = \sqrt{\det E}.$$

Proof. The proof can be found in [35], Chapter I. □

Theorem 1.1.13 (Lefschetz theorem)**.** *Let $X = V/\Lambda$ be a complex torus, and (H, α) as above. Then, the line bundle $L(H, \alpha)$ is ample if and only if H is positive definite. Hence, X is a projective variety if and only if V admits a positive definite Hermitian form H such that $E = \mathrm{Im}(H)$ is integral on $\Lambda \times \Lambda$.*

Proof. We give a sketch of the proof that, for a positive definite H and corresponding α, the third power $L(H, \alpha)^{\otimes 3} = L(3H, \alpha^3)$ gives an embedding of X into a projective space. Proposition 1.1.12 provides a non-zero section ϑ of $L(H, \alpha)$. By verifying the transformation properties, we see that

$\vartheta(v-a)\cdot\vartheta(v-b)\cdot\vartheta(v+a+b)$ is a section of $L(3H,\alpha^3)$ for any $a,b\in V$. For each $v_0\in V$, we can find $a,b\in V$ such that this section doesn't vanish at v_0. Thus, a basis $\{\vartheta_0,\dots,\vartheta_d\}$ of $H^0(X,L(H,\alpha)^{\otimes 3})$ provides a well-defined holomorphic map

$$\Theta\colon X\longrightarrow\mathbb{P}^d.$$

Using Proposition 1.1.12 again, one can show that Θ is in fact an embedding. The complete proof can be found in [35], Chapter I. On the other hand, if H is degenerate, one can show that any morphism $X\longrightarrow\mathbb{P}^n$ has to factor through a proper quotient torus of X. If H is negative definite on a subspace of V of positive dimension, one shows that $L(H,\alpha)$ has no non-zero sections (see [35], Chapter I). $\square$

Definition 1.1.14. A *polarization* on a complex torus $X=V/\Lambda$ is a positive definite Hermitian form H on V satisfying $\mathrm{Im}(H)(\Lambda,\Lambda)\subseteq\mathbb{Z}$, or, equivalently, an ample line bundle L on X. The tuple (X,H) is called a *polarized abelian variety.* The polarization is called *principal* if $\mathrm{Im}(H)\colon\Lambda\times\Lambda\longrightarrow\mathbb{Z}$ is unimodular or, equivalently, if $\dim H^0(X,L)=1$. The tuple (X,H) is then called a *principally polarized abelian variety.*

1.2 The moduli space of principally polarized abelian varieties over $\mathbb{C}$

In the following section, we will establish a parametrization of the isomorphism classes of principally polarized abelian varieties, applying the results of the previous section.

Definition 1.2.1. Let $X=V/\Lambda$ be a complex abelian variety with V a g-dimensional complex vector space and Λ a rank $2g$ lattice in V. Let $\mathcal{B}=\{b_1,\dots,b_g\}$ be a $\mathbb{C}$-basis of V, and $\mathcal{C}=\{\lambda_1,\dots,\lambda_{2g}\}$ be a $\mathbb{Z}$-basis of Λ. Define the *period matrix* $\Omega=(\omega_{j,k})_{j=1,\dots,g;k=1,\dots,2g}\in\mathrm{Mat}_{g,2g}(\mathbb{C})$ by the expansion $\lambda_k=\sum_{j=1}^{g}\omega_{j,k}b_j$. The choice of bases $\mathcal{B}$, $\mathcal{C}$ then induces an isomorphism

$$V/\Lambda\cong\mathbb{C}^g/\Omega\mathbb{Z}^{2g}.$$

Let now $X=V/\Lambda$ be an abelian variety with principal polarization H, and let $\mathrm{Im}(H)$ be the corresponding alternating form. As $\mathrm{Im}(H)\colon\Lambda\times\Lambda\longrightarrow\mathbb{Z}$ is unimodular, by the elementary divisor theorem (see [17]) one can choose a symplectic $\mathbb{Z}$-basis $\mathcal{C}=\{\lambda_1,\dots,\lambda_{2g}\}$ of Λ in a way that the matrix J of $\mathrm{Im}(H)$ with respect to Λ has the form

$$J=\begin{pmatrix}0_g & \mathbb{1}_g\\ -\mathbb{1}_g & 0_g\end{pmatrix},$$

where $\mathbb{1}_g\in\mathrm{Mat}_g(\mathbb{C})$ denotes the identity matrix and $0_g\in\mathrm{Mat}_g(\mathbb{C})$ the zero matrix. Constructing the symplectic basis $\mathcal{C}$ inductively, one can assure that

$\{\lambda_{g+1}, \ldots, \lambda_{2g}\}$ is linearly independent over $\mathbb{C}$. So the $\mathbb{C}$-basis of V can be chosen as $\mathcal{B} = \{\lambda_{g+1}, \ldots, \lambda_{2g}\}$. Hence, the period matrix Ω is of the form $\Omega = (\tau, \mathbb{1}_g)$ with $\tau \in \mathrm{Mat}_g(\mathbb{C})$. The lattice in $\mathbb{C}^g$ corresponding to Λ is then $\Omega\mathbb{Z}^{2g} = \tau\mathbb{Z}^g + \mathbb{Z}^g$.

Definition 1.2.2. We will denote the complex torus $\mathbb{C}^g/(\tau\mathbb{Z}^g + \mathbb{Z}^g)$ with $\tau \in \mathrm{Mat}_g(\mathbb{C})$ by X_τ. Note that, by the considerations above, each principally polarized abelian variety is isomorphic to some X_τ.

The following theorem will specify the shape of the matrix τ.

Theorem 1.2.3 (Riemann relations)**.** *Let $X = V/\Lambda$ be a principally polarized abelian variety with polarization given by the Hermitian form H. Denote by $\mathcal{B} = \{b_1, \ldots, b_g\}$ a $\mathbb{C}$-basis of V and by $\mathcal{C} = \{\lambda_1, \ldots, \lambda_{2g}\}$ a $\mathbb{Z}$-basis of Λ, and let Ω be the corresponding period matrix. By Lemma 1.1.7, the form $H \colon V \times V \longrightarrow \mathbb{C}$ corresponds to an alternating form E by $H(v, w) = E(iv, w) + iE(v, w)$. By Theorem 1.1.13, E takes integral values on $\Lambda \times \Lambda$. Denote by A the matrix of E with respect to the $\mathbb{Z}$-basis $\mathcal{C}$ of Λ. Then, we have the following relations:*

(i) H is Hermitian if and only if $\Omega A^{-1}\Omega^t = 0$.

(ii) H is positive definite if and only if $i\Omega A^{-1}\overline{\Omega}^t$ is positive definite.

Proof. See [4], Chapter 4, Theorem 2.1. □

We will now apply Theorem 1.2.3 to the situation $A = J$ and $\Omega = (\tau, \mathbb{1}_g)$ as above. The fact that H is Hermitian translates into the equation $\Omega J^{-1}\Omega^t = 0$. As one easily sees that

$$\Omega J^{-1}\Omega^t = \tau - \tau^t,$$

the matrix τ has to be symmetric. Furthermore, noting that

$$i\Omega J^{-1}\overline{\Omega}^{-t} = i(\overline{\tau} - \tau) = 2\,\mathrm{Im}(\tau),$$

the positive definiteness of $i\Omega J^{-1}\overline{\Omega}^{-t}$ is equivalent to $\mathrm{Im}(\tau)$ being positive definite. Therefore, we can make the following definition.

Definition 1.2.4. The *Siegel upper half-space* $\mathbb{H}_g$ is the set

$$\mathbb{H}_g := \left\{\tau = x + iy \in \mathrm{Sym}_g(\mathbb{C}) \mid x, y \in \mathrm{Sym}_g(\mathbb{R}),\, y > 0\right\},$$

where the notation $y > 0$ denotes that the matrix $y = \mathrm{Im}(\tau)$ is positive definite. As an open submanifold of $\mathrm{Sym}_g(\mathbb{C})$, it has dimension $g(g + 1)/2$.

The computations above lead to the following observation:

Proposition 1.2.5. *Any principally polarized abelian variety is isomorphic to a torus X_τ with $\tau \in \mathbb{H}_g$. On the other hand, any torus of the form $X_\tau = \mathbb{C}^g/(\tau\mathbb{Z}^g + \mathbb{Z}^g)$ with $\tau \in \mathbb{H}_g$ possesses a principal polarization. It is given by the Hermitian form H whose imaginary part $\mathrm{Im}(H)$ has the matrix $\mathrm{Im}(\tau)^{-1}$ with respect to the standard basis of $\mathbb{C}^g$.*

Proof. The first part of the proposition is already proven. To prove the second part, note that the columns of the matrix $(\tau, \mathbb{1}_g)$ form a basis of the lattice $\mathbb{Z}^g + \tau\mathbb{Z}^g$. With respect to this basis, the matrix of the Hermitian form restricted to the lattice is given by $(\tau, \mathbb{1}_g)^t \operatorname{Im}(\tau)^{-1}(\overline{\tau}, \mathbb{1}_g)$, and the imaginary part of this matrix can be computed to be the matrix J. Hence, $E = \operatorname{Im}(H)$ is unimodular. □

By the previous proposition, the Siegel upper half-space $\mathbb{H}_g$ parametrizes complex tori with principal polarization. In the following we will investigate isomorphisms between two such tori.

Definition 1.2.6. An isomorphism $\phi\colon (X, H) \longrightarrow (X', H')$ between complex polarized abelian varieties is a group isomorphism $X \longrightarrow X'$ that preserves the polarization, i.e., $\phi^* H' = H$.

Let now X_τ and $X_{\tau'}$ be two complex tori with $\tau, \tau' \in \mathbb{H}_g$. A group isomorphism between them is a bijective linear map $\phi\colon \mathbb{C}^g \longrightarrow \mathbb{C}^g$ that induces an isomorphism between the lattices $\tau\mathbb{Z}^g + \mathbb{Z}^g$ and $\tau'\mathbb{Z}^g + \mathbb{Z}^g$. Let $\Phi \in \mathrm{GL}_g(\mathbb{C})$ be the matrix of ϕ with respect to the standard basis of $\mathbb{C}^g$, and let M be the matrix of ϕ with respect to the bases of the lattices given by the columns of the period matrices $(\tau, \mathbb{1}_g)$ and $(\tau', \mathbb{1}_g)$. As ϕ is a bijection on the lattices, the matrix M lies in $\mathrm{GL}_{2n}(\mathbb{Z})$. With the above choice of bases, the matrices Φ and M are linked by the identity

$$\Phi(\tau, \mathbb{1}_g) = (\tau', \mathbb{1}_g)M.$$

Decomposing M as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \quad (A, B, C, D \in \mathrm{Mat}_g(\mathbb{Z}))\,,$$

the above relation translates to

$$\Phi\tau = \tau' A^t + B^t = (A\tau' + B)^t \quad \text{and} \quad \Phi = \tau' C^t + D^t = (C\tau' + D)^t. \quad (1.2.1)$$

Since the matrix Φ is invertible, we can express τ in terms of τ' and $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)^t$ as

$$\tau = \tau^t = (\tau' A^t + B^t)^t \Phi^{-t} = (A\tau' + B)(C\tau' + D)^{-1}.$$

Moreover, the fact that ϕ preserves the Hermitian form is equivalent to the condition

$$M^t J M = J.$$

We can now give the following definitions.

Definition 1.2.7. Let $g > 0$ be a natural number. The *symplectic group* $\mathrm{Sp}_{2g}(\mathbb{R})$ is the subgroup of $\mathrm{GL}_{2g}(\mathbb{R})$ defined via

$$\mathrm{Sp}_{2g}(\mathbb{R}) := \left\{ M \in \mathrm{Mat}_{2g}(\mathbb{R}) \mid M^t J M = J \right\}.$$

By restricting the matrix entries to lie in $\mathbb{Z}$ and $\mathbb{Q}$, one defines $\mathrm{Sp}_{2g}(\mathbb{Z})$ and $\mathrm{Sp}_{2g}(\mathbb{Q})$, respectively, in the same way. The condition $M^tJM = J$ implies $J^tMJ = M^{-1}$, hence, symplectic matrices are in fact invertible.

Definition 1.2.8. The action of the symplectic group on the Siegel upper half-space $\mathbb{H}_g$ is given by the prescription

$$M\tau := (A\tau + B)(C\tau + D)^{-1} \quad \left(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R}),\, \tau \in \mathbb{H}_g \right).$$

With the decomposition $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, one can reformulate the condition $M^tJM = J$ into the three conditions

$$A^tC = C^tA,\ B^tD = D^tB,\ \text{and}\ A^tD - C^tB = \mathbb{1}_g.$$

Furthermore, we have seen in (1.2.1) that the isomorphism between two abelian varieties $X_\tau \longrightarrow X_{M\tau}$ induced by M is just

$$Z \mapsto (C\tau + D)^{-t}Z \tag{1.2.2}$$

With the above observations, we have established the following proposition.

Proposition 1.2.9. *Let* X_τ *and* $X_{\tau'}$ *be principally polarized complex tori. Then, the existence of an isomorphism* $\phi\colon X_\tau \longrightarrow X_{\tau'}$ *compatible with the polarizations* $\mathrm{Im}(\tau)^{-1}$ *and* $\mathrm{Im}(\tau')^{-1}$ *is equivalent to the existence of a matrix* $M \in \mathrm{Sp}_{2g}(\mathbb{Z})$ *satisfying* $M\tau' = \tau$.

Proof. Given an isomorphism $\phi\colon X_\tau \longrightarrow X_{\tau'}$ compatible with the Hermitian forms, we have already seen that the matrix $M \in \mathrm{Sp}_{2g}(\mathbb{Z})$ of ϕ with respect to the symplectic bases of the lattices, given by the columns of the period matrices $(\tau, \mathbb{1}_g)$ and $(\tau', \mathbb{1}_g)$, satisfies $M^t\tau' = \tau$. On the other hand, given a matrix $M \in \mathrm{Sp}_{2g}(\mathbb{Z})$ satisfying $M\tau' = \tau$, the corresponding isomorphism $X_\tau \longrightarrow X_{\tau'}$, whose matrix with respect to the symplectic bases associated with τ and τ' is M^t, is compatible with the Hermitian forms $\mathrm{Im}(\tau)^{-1}$ and $\mathrm{Im}(\tau')^{-1}$ by definition of $\mathrm{Sp}_{2g}(\mathbb{Z})$. □

In conclusion, we obtain the following result.

Theorem 1.2.10. *There is a 1-1-correspondence between isomorphism classes of complex abelian varieties with a principal polarization and points of the orbit space* $\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$ *with the left action of* $\mathrm{Sp}_g(\mathbb{Z})$ *on* $\mathbb{H}_g$ *as in Definition 1.2.8.*

Proof. The theorem combines the results of Propositions 1.2.5 and 1.2.9. □

Notation 1.2.11. We will abbreviate $\Gamma_g := \mathrm{Sp}_{2g}(\mathbb{Z})$ and denote the orbit space $\Gamma_g\backslash\mathbb{H}_g$ by $\mathcal{A}_g$.

For later use, we will present a description of a fundamental domain of the action of Γ_g on $\mathbb{H}_g$. Let therefore denote $L_{k,g}$ the set of vectors in $\mathbb{Z}^g$ whose last $(g - k + 1)$ entries are relatively prime. Then, a fundamental domain of the action is given by the following theorem due to Siegel.

Theorem 1.2.12. *Let $F_g \subseteq \mathbb{H}_g$ be the subset of all matrices $\tau = x + iy \in \mathbb{H}_g$ that satisfy the conditions*

(i) $|\det(C\tau + D)| \geq 1 \quad \left(M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g\right)$*;*

(ii) $y = (y_{j,k})_{j,k=,1\ldots,g}$ *is Minkowski reduced, i.e.,* $y_{k,k} \leq l^t y l$ $(l \in L_{k,g}$*,* $1 \leq k \leq g)$*, and* $y_{k-1,k} \geq 0$ $(1 < k \leq g)$*;*

(iii) $x = (x_{j,k})_{j,k=1,\ldots,g}$ *satisfies* $|x_{j,k}| \leq \frac{1}{2}$ $(1 \leq j, k \leq g)$*.*

Then, F_g intersects with every Γ_g-orbit in $\mathbb{H}_g$. If τ and τ' are two points of F_g with $\tau' = M\tau$ for some $M \in \Gamma_g$, one deduces either $M = \pm \mathbb{1}_g$ and, hence, $\tau = \tau'$, or τ and τ' lie on the boundary of F_g.

Proof. See, e.g., [1], Chapter 2, Theorem 1.20. □

For $g = 1$, these conditions gives us back the well-known fundamental domain

$$F_1 = \{\tau = x + iy \in \mathbb{H}_1 \mid |\tau| \geq 1, |x| \leq 1/2\}.$$

As the action of Γ_g on $\mathbb{H}_g$ has fixed points, $\mathcal{A}_g$ is not a fine moduli space of abelian varieties, as we will see. To have a universal family available, we sometimes want to work with a cover of $\mathcal{A}_g$, which parametrizes abelian varieties together with an extra structure.

Definition 1.2.13. A *level N-structure* on a principally polarized abelian variety $X = (V/\Lambda, H)$ is a choice of a symplectic basis $\{f_j, g_k \mid 1 \leq j, k \leq g\}$ for the N-torsion points

$$X[N] := \{x \in X \mid Nx = e\} \cong \frac{1}{N}\Lambda \cong (\mathbb{Z}^g/N\mathbb{Z}^g)^2,$$

of X, hence, a basis $\{f_j, g_k \mid 1 \leq j, k \leq g\}$ for $X[N]$ as a $\mathbb{Z}/N\mathbb{Z}$ module such that there exists a symplectic basis $\{\lambda_j, \mu_k \mid 1 \leq j, k \leq g\}$ of Λ with

$$f_j \equiv \frac{\lambda_j}{N} \bmod \Lambda \text{ and } g_k \equiv \frac{\mu_k}{N} \bmod \Lambda$$

for all $j, k \in \{1, \ldots, g\}$.

Definition 1.2.14. A symplectic isomorphism between two principally polarized abelian varieties $X = (V/\Lambda, H)$ and $X' = (V'/\Lambda', H')$ with level N-structure, given by symplectic bases $\{\lambda_j, \mu_k \mid 1 \leq j, k \leq g\}$ of Λ and $\{\lambda'_j, \mu'_k \mid 1 \leq j, k \leq g\}$ of Λ', respectively, is an isomorphism $\phi\colon X \longrightarrow X'$ satisfying

$$\phi\left(\frac{\lambda_j}{N}\right) \equiv \frac{\lambda'_j}{N} \bmod \Lambda' \text{ and } \phi\left(\frac{\mu_j}{N}\right) \equiv \frac{\mu'_j}{N} \bmod \Lambda' \quad (1 \leq j \leq g)\,.$$

Analogous to Proposition 1.2.9, it is possible to identify isomorphisms between principally polarized abelian varieties preserving the level N-structure with a subgroup of the symplectic group.

Definition 1.2.15. The *principal congruence subgroup* $\Gamma_g(N)$ of Γ_g is defined as the kernel of the map

$$\Gamma_g = \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})$$

given by reduction of the matrix entries modulo N. In other words,

$$\Gamma_g(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \,\middle|\, A \equiv D \equiv \mathbb{1}_g \mod N,\, B \equiv C \equiv 0_g \mod N \right\}.$$

Proposition 1.2.16. *Isomorphism classes of principally polarized abelian varieties with level N-structure are in 1-1-correspondence with points of the space*

$$\mathcal{A}_g(N) := \Gamma(N)\backslash\mathbb{H}_g.$$

Proof. See [4], Chapter 8, Theorem 3.1. □

The spaces $\mathcal{A}_g$ and $\mathcal{A}_g(N)$ are quasi-projective varieties, as shown in [37] or [3]. Over $\mathcal{A}_g(N)$, $N \geq 3$, there exists a universal family of abelian varieties, which can be constructed as

$$\mathcal{X}_g(N) := \left(\Gamma_g(N) \ltimes \mathbb{Z}^{2g}\right) \backslash \left(\mathbb{H}_g \times \mathbb{C}^g\right).$$

Here, $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g(N)$ acts on $(\tau, Z) \in \mathbb{H}_1 \times \mathbb{C}^g$ by

$$M(\tau, Z) = (M\tau, (C\tau + D)^{-t}Z),$$

since we have already seen that the isomorphism between two abelian varieties $X_\tau \longrightarrow X_{M\tau}$ induced by M is given by $Z \mapsto (C\tau + D)^{-t}Z$. An element $(\lambda, \mu) \in \mathbb{Z}^{2g}$ acts on $Z \in \mathbb{C}^g$ via $Z \mapsto Z + \tau\lambda + \mu$. As all the stabilizers $\Gamma_g(N)_\tau$ of points $\tau \in \mathbb{H}_g$ are trivial for $N \geq 3$, the fibre over a point $\tau \in \mathcal{A}_g(N)$ of the map $\pi_g(N) \colon \mathcal{X}_g(N) \longrightarrow \mathcal{A}_g(N)$, induced by the projection $\mathbb{H}_g \times \mathbb{C}^g \longrightarrow \mathbb{H}_g$, is just X_τ. So the construction yields a universal family.

For $N = 1$, and, hence, $\mathcal{A}_g(1) = \mathcal{A}_g$, this is false. Consider the quotient $\mathcal{X}_g := \left(\Gamma_g \ltimes \mathbb{Z}^{2g}\right) \backslash \left(\mathbb{H}_g \times \mathbb{C}^g\right)$ and the projection $\pi_g \colon \mathcal{X}_g \longrightarrow \mathcal{A}_g$. The element $-\mathbb{1}_{2g}$ fixes all $\tau \in \mathbb{H}_g$ and induces a nontrivial action $Z \mapsto -Z$ on the fibre. Anyway, there are also points with larger stabilizers. For $i\mathbb{1}_g \in \mathbb{H}_g$, we find for instance

$$\Gamma_{g,i\mathbb{1}_g} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g \,\middle|\, C = -B, D = A \right\}.$$

For a classification of the fixed points of the action of Γ_2 on $\mathbb{H}_2$, see [16]. The fibres of the projection $\pi_g \colon \mathcal{X}_g \longrightarrow \mathcal{A}_g$ are therefore not abelian varieties, but quotients of principally polarized abelian varieties by their automorphism group. Nonetheless, if we work with orbifolds (or Deligne–Mumford stacks), we can view $\mathcal{A}_g$ as the orbifold-quotient $\mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})\backslash\backslash\mathcal{A}_g(N)$. Then, the map

$$\pi_g \colon \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})\backslash\backslash\mathcal{X}_g(N) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}/N\mathbb{Z})\backslash\backslash\mathcal{A}_g(N)$$

induced by the projection $\mathbb{H}_g \times \mathbb{C}^g \longrightarrow \mathbb{H}_g$ yields a universal family in the category of orbifolds.

1.3 The Hodge bundle on $\mathcal{A}_g$ and Siegel modular forms over $\mathbb{C}$

In the following section, we will introduce the Hodge bundle on $\mathcal{A}_g$, constructed by considering relative differential forms on the universal family. The Hodge bundle is naturally equipped with a metric. In order to be able to handle it for actual computations, we will give an equivalent description of this line bundle as the line bundle of modular forms and analyze its global sections.

Definition 1.3.1. The *Hodge bundle* ω_g is the determinant of the vector bundle of rank g on $\mathcal{A}_g$ whose fibre over a point $\tau \in \mathcal{A}_g$ is the space $H^{1,0}(X_\tau, \mathbb{C})$ of holomorphic 1-forms on the abelian variety X_τ corresponding to τ. Hence, if $\pi_g \colon \mathcal{X}_g \longrightarrow \mathcal{A}_g$ denotes the universal family and $e_g \colon \mathcal{A}_g \longrightarrow \mathcal{X}_g$ the zero section, the equality

$$\omega_g = \det\left(e_g^* \Omega^1_{\mathcal{X}_g/\mathcal{A}_g}\right)$$

holds. There is a natural metric, the L^2-metric, on ω_g, defined by

$$\|\alpha(\tau)\|^2_{L^2} := \left(\frac{i}{2}\right)^g \int\limits_{X_\tau} \alpha \wedge \overline{\alpha}$$

for a global section α of ω_g.

A global section of ω_g gives rise to a top-degree differential form $f(\tau)\mathrm{d}Z_1 \wedge \ldots \wedge \mathrm{d}Z_g$ on X_τ. Recall that the isomorphism between X_τ and $X_{M\tau}$ is given by $Z \mapsto (C\tau + D)^{-t}Z$, and thus the form has to fulfill the transformation property

$$\begin{aligned} f(\tau)\mathrm{d}Z_1 \wedge \ldots \wedge \mathrm{d}Z_g &= f(M\tau)\mathrm{d}\big((C\tau+D)^{-t}Z\big)_1 \wedge \ldots \wedge \mathrm{d}\big((C\tau+D)^{-t}Z\big)_g \\ &= f(M\tau)\det(C\tau+D)^{-1}\mathrm{d}Z_1 \wedge \ldots \wedge \mathrm{d}Z_g. \end{aligned}$$

This observation motivates the following definition.

Definition 1.3.2. A holomorphic function $f \colon \mathbb{H}_g \longrightarrow \mathbb{C}$ is called *Siegel modular form of degree g and weight $k \in \mathbb{N}$ for Γ_g* if the following conditions are satisfied:

(i) $f(M\tau) = \det(C\tau + D)^k f(\tau) \quad \big(\tau \in \mathbb{H}_g,\, M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g\big)$,

(ii) f is bounded on all subsets of the form $\{\tau = x + iy \in \mathbb{H}_g \mid y - y_0 > 0\} \subseteq \mathbb{H}_g$ with $y_0 \in \mathrm{Sym}_g(\mathbb{R})$ positive definite.

Condition (ii) implies that f is holomorphic at the boundary of $\mathbb{H}_g$, which will be defined in the following sections. For $g > 1$ this condition follows automatically from the holomorphicity of f and condition (i) by the Koecher principle, see, e.g., [15], Chapter I, Theorem 3.5. Siegel modular forms of degree g and weight k obviously form a vector space, which we will denote by

$M_k(\Gamma_g)$. Due to their transformation properties with respect to Γ_g, one can see that they are global sections of a line bundle $\mathcal{M}_k(\Gamma_g)$ on $\mathcal{A}_g$. With the above observation, one obtains the identification

$$H^0\left(\mathcal{A}_g, \omega_g^{\otimes k}\right) \cong M_k(\Gamma_g).$$

The L^2-metric on the Hodge bundle induces the so-called Petersson metric on the line bundle $\mathcal{M}_k(\Gamma_g)$.

Definition 1.3.3. The Petersson metric on $\mathcal{M}_k(\Gamma_g)$ is defined by

$$\|f(\tau)\|^2_{\mathrm{Pet}} := |f(\tau)|^2\big((4\pi)^g \det(\mathrm{Im}(\tau))\big)^k,$$

for f a global section of $\mathcal{M}_k(\Gamma_g)$.

Integrating over a fundamental domain of the action of $\tau\mathbb{Z}^g + \mathbb{Z}^g$ on $\mathbb{C}^g$, one finds

$$\left(\frac{i}{2}\right)^g \int\limits_{X_\tau} \mathrm{d}Z_1 \wedge \mathrm{d}Z_g \wedge \mathrm{d}\overline{Z}_1 \wedge \ldots \wedge \mathrm{d}\overline{Z}_g = \det(\mathrm{Im}(\tau)),$$

and so one deduces easily that, up to scaling, the Petersson metric is in fact the metric on $\mathcal{M}_k(\Gamma_g)$ induced by the L^2-metric on $\omega_g^{\otimes k}$. Hence, we have an isometry of line bundles

$$(\omega_g^{\otimes k}, \|\cdot\|^k_{L^2}) \cong (\mathcal{M}_k(\Gamma_g), \|\cdot\|_{\mathrm{Pet}})\,.$$

In the subsequent computations in Chapter 3, we will work with the line bundle of modular forms equipped with the Petersson metric instead of the Hodge bundle with the L^2-metric. We will therefore investigate Siegel modular forms further and give examples how to construct them. The main reference in the following will be [15].

As in the case $g = 1$, due to its transformation property $f(\tau) = f(\tau + S)$, $S \in \mathrm{Mat}_g(\mathbb{Z})$, a Siegel modular form f possesses a Fourier expansion as

$$f(\tau) = \sum_{\substack{T \in \mathrm{Sym}_g(\mathbb{Z}) \\ T \text{ even}}} a(T) e^{\pi i \cdot \mathrm{tr}(T\tau)},$$

where the sum runs over all symmetric integral matrices whose diagonal entries are even. From the boundedness condition (ii), one deduces that the Fourier coefficients $a(T)$ are only non-zero if T is positive semidefinite. In analogy to the 1-dimensional case, one can now define the subspace of cusp forms.

Definition 1.3.4. The Φ-*operator* is a linear operator $M_k(\Gamma_g) \longrightarrow M_k(\Gamma_{g-1})$, sending a modular form $f \in M_k(\Gamma_g)$ to $\Phi(f) \in M_k(\Gamma_{g-1})$, where $\Phi(f)(\tau)$ ($\tau \in \mathbb{H}_{g-1}$) is defined as the limit

$$\phi(f)(\tau) := \lim_{t \to \infty} f\begin{pmatrix} it & 0 \\ 0 & \tau \end{pmatrix}.$$

Definition 1.3.5. A Siegel modular form f is a *cusp form* if it vanishes at the boundary, i.e., if $f \in \ker(\Phi)$, or, equivalently, if its Fourier expansion satisfies

$$a(T) \neq 0 \Rightarrow T > 0.$$

The following definition gives building blocks for modular forms, the ϑ-series, which are in general not modular forms for Γ_g themselves, but will be used to disassemble some important modular forms in Chapter 3 and investigate their behaviour on certain divisors.

Definition 1.3.6. The ϑ-series for a vector $(a, b) \in (\mathbb{Z}/2\mathbb{Z})^g \times (\mathbb{Z}/2\mathbb{Z})^g$ is given by the equality

$$\vartheta_{a,b}(\tau) := \sum_{n \in \mathbb{Z}^g} e^{2\pi i \left(\frac{1}{2}\left(n+\frac{a}{2}\right)^t \tau \left(n+\frac{a}{2}\right) + \left(n+\frac{a}{2}\right)^t \frac{b}{2} \right)},$$

with $\tau \in \mathbb{H}_g$. It defines a holomorphic function on $\mathbb{H}_g$. A ϑ-series $\vartheta_{a,b}$ is called even if $a^t b$ is even. There are 3 even ϑ-series for degree $g = 1$ and 10 even ϑ-series for degree $g = 2$.

The transformation behaviour of ϑ-series under the action of $M \in \Gamma_g$, see, e.g., [15], establishes a close connection to modular forms, leading to the following theorems.

Theorem 1.3.7. *Any symmetric homogenous polynomial in the 8th powers of the even ϑ-series $\vartheta_{a,b}^8$, with $(a, b) \in (\mathbb{Z}/2\mathbb{Z})^g \times (\mathbb{Z}/2\mathbb{Z})^g$, is a Siegel modular form for Γ_g.*

Proof. See [15], Theorem 3.2. ☐

Theorem 1.3.8. *The product of powers of the even ϑ-series on $\mathbb{H}_g$*

$$\prod_{a^t b \text{ even}} \vartheta_{a,b}^{k_g}(\tau)$$

is a non-zero Siegel modular form of degree g for $k_1 = 8$, $k_2 = 2$, and $k_g = 1$ ($g \geq 3$).

Proof. The proof for $g = 2$ can be found in [22]. The case $g \geq 3$ is proven in [28], Lemma 10. ☐

The modular forms in Theorem 1.3.8 are in fact cusp forms, as one shows by applying the Φ-operator. For $g = 1$, we obtain, after scaling, the well-known modular discriminant

$$\Delta(\tau) = \frac{1}{2^8}(\vartheta_{00}(\tau)\vartheta_{01}(\tau)\vartheta_{10}(\tau))^8 = \exp(2\pi i\tau) \prod_{n \in \mathbb{Z}} (1 - \exp(2\pi i n\tau))^{24}$$

of weight 12. For $g = 2$ and $\tau = \left(\begin{smallmatrix} \tau_1 & \tau_{12}^t \\ \tau_{12} & \tau_2 \end{smallmatrix}\right) \in \mathbb{H}_2$, we obtain the weight 10 cusp form

$$\begin{aligned} \chi_{10}(\tau) &= \frac{1}{2^{12}} \prod_{a^t b \text{ even}} \vartheta_{a,b}^2(\tau) \\ &= \exp\big(2\pi i(\tau_1 + \tau_{12} + \tau_2)\big) \prod_{\substack{n,l,m \in \mathbb{Z} \\ (n,l,m) > 0}} \Big(1 - \exp\big(2\pi i(n\tau_1 + l\tau_{12} + m\tau_2)\big)\Big)^{2f(nm,l)}, \end{aligned}$$

where the exponents of the product expansion are given by the Fourier coefficients $f(nm, l)$ of a weak Jacobi form, see [22], Chapter 4.
Another important class of modular forms is given by holomorphic Eisenstein series, which are defined in the following way.

Definition 1.3.9. Consider the stabilizer of the 0-dimensional boundary component $B_0 = \lim_{t \to \infty} (ti\mathbb{1}_g)$ of $\mathbb{H}_g$, given by

$$\Gamma_{g,B_0} := \left\{ M = \begin{pmatrix} A & B \\ 0 & A^{-t} \end{pmatrix} \in \mathrm{Mat}_{2g}(\mathbb{Z}) \,\middle|\, A \in \mathrm{GL}_g(\mathbb{Z}), A^t B = B^t A \right\},$$

which we will introduce again later. For $k \in \mathbb{N}$ even, the *holomorphic Siegel Eisenstein series of weight k* is defined by

$$E_k(\tau) := \sum_{M \in \Gamma_{g,B_0} \backslash \Gamma_g} \det(C\tau + D)^{-k},$$

where $M \in \Gamma_{g,B_0} \backslash \Gamma_g$ denotes that the sum runs over a full system of representatives of the left cosets of Γ_{g,B_0} in Γ_g.

The functions E_k are in fact modular forms, see [15], Chapter I, Remark 5.3.

1.4 The Baily–Borel compactification

A natural compactification of $\mathcal{A}_g$ is obtained by identifying $\mathbb{H}_g$ with a bounded complex domain D_g such that the action of Γ_g extends to the closure of D_g. In the following section, we will introduce this so-called Satake or Baily–Borel compactification and give a description of the arising boundary components and some corresponding subgroups.

Notation 1.4.1. We will recall some of the notation from section 1.2. The Siegel upper half-space is given by

$$\mathbb{H}_g := \left\{ \tau = x + iy \in \mathrm{Sym}_g(\mathbb{C}) \mid x, y \in \mathrm{Sym}_g(\mathbb{R}),\, y > 0 \right\}.$$

Denoting

$$J = \begin{pmatrix} 0 & \mathbb{1}_g \\ -\mathbb{1}_g & 0 \end{pmatrix},$$

the symplectic group is defined as

$$\begin{aligned}\mathrm{Sp}_g(\mathbb{R}) &= \{M \in \mathrm{GL}_{2g}(\mathbb{R}) \,|\, M^t J M = J\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,\middle|\, A\,, B\,, C\,, D \in \mathrm{Mat}_g(\mathbb{R}) : \begin{matrix} AB^t = BA^t,\ CD^t = DC^t, \\ AD^t - BC^t = \mathbb{1}_g \end{matrix} \right\}.\end{aligned}$$

Furthermore, we denote

$$G_g = \mathrm{Sp}_g(\mathbb{R}) \text{ and } \Gamma_g = \mathrm{Sp}_g(\mathbb{Z}).$$

The group G_g acts on $\mathbb{H}_g$ by

$$M\tau = (A\tau + B)(C\tau + D)^{-1} \quad \left(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_g,\ \tau \in \mathbb{H}_g \right).$$

We would like to construct a "good" compactification of the quotient space $\mathcal{A}_g = \Gamma_g \backslash \mathbb{H}_g$. Therefore, it will be necessary to add suitable boundary components. To have a natural notion of boundary components of $\mathbb{H}_g$ at hand, we will first map $\mathbb{H}_g$ to a bounded domain in the following way.

Definition 1.4.2. Consider the bounded subset of the space of complex symmetric matrices

$$D_g = \{Z \in \mathrm{Sym}_g(\mathbb{C}) \,|\, \mathbb{1}_g - Z\overline{Z} > 0\}.$$

It is isomorphic to $\mathbb{H}_g$ via the Cayley transformation

$$c_g \colon \mathbb{H}_g \longrightarrow D_g,$$

given by the assignment $\tau \mapsto Z = (\tau - i\mathbb{1}_g)(\tau + i\mathbb{1}_g)^{-1}$. Its inverse is

$$c_g^{-1} \colon D_g \longrightarrow \mathbb{H}_g,$$

given by the assignment $Z \mapsto \tau = i(Z+\mathbb{1}_g)(-Z+\mathbb{1}_g)^{-1}$. A matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in G_g acts on D_g via the action induced by c_g. The image MZ of $Z \in D_g$ is given by

$$\big((A-iC)(Z+\mathbb{1}_g)+(iB+D)(Z-\mathbb{1}_g)\big)\big((A+iC)(Z+\mathbb{1}_g)+(iB-D)(Z-\mathbb{1}_g)\big)^{-1}.$$

As D_g is bounded in $\mathrm{Sym}_g(\mathbb{C})$, we can define boundary components as follows.

Definition 1.4.3. Denote by

$$\overline{D}_g = \left\{Z \in \mathrm{Sym}_g(\mathbb{C}) \,\middle|\, \mathbb{1}_g - Z\overline{Z} \geq 0\right\}$$

the topological closure of D_g in $\mathrm{Sym}_g(\mathbb{C})$. A boundary component of D_g is defined as an equivalence class of points in $\overline{D}_g$ that can be connected by holomorphic paths. Formally, two points $Z_1, Z_2 \in \overline{D}_g$ are equivalent if there exist holomorphic maps

$$\xi_j \colon D_1 \longrightarrow \overline{D}_g \quad (j = 1, \ldots, n)$$

such that $\xi_1(0) = Z_1$, $\xi_n(0) = Z_2$, and $\xi_j(D_1) \cap \xi_{j+1}(D_1) \neq \emptyset$ $(j = 1, \ldots, n-1)$.

Remark 1.4.4. The action of G_g on D_g extends to the topological closure $\overline{D}_g$. This can be seen by a short computation, see [38], Proposition 4.3.

The next proposition gives a more handy definition of the above boundary components.

Proposition 1.4.5. *The boundary components of* $\overline{D}_g$ *are in 1-1-correspondence to subspaces* V *of* $\mathbb{R}^{2g}$ *of dimension* $0 \leq g' \leq g$.

Proof. We assign to each $Z \in \overline{D}_g$ a real vector space in the following way: Let W_Z be the subspace of $\mathbb{C}^{2g}$ spanned by the columns of $\left(\begin{smallmatrix} Z+\mathbb{1}_g \\ i(Z-\mathbb{1}_g) \end{smallmatrix}\right)$. Set $V_Z := W_Z \cap \overline{W}_Z$. We define the boundary component corresponding to a subspace $V \subseteq \mathbb{R}^{2g}$ as

$$B(V) := \left\{ Z \in \overline{D}_g \,|\, V_Z = V \right\}.$$

This is a boundary component in the above sense and all boundary components can be obtained this way, see [38], Proposition 4.4. The action of an element $M \in G_g$ on a boundary component $B(V)$ translates to $MB(V) = B(MV)$, where M acts on V by matrix multiplication. □

Definition 1.4.6. Let B be a boundary component of D_g. We denote the *stabilizer of* B *in* Γ_g by

$$\Gamma_{g,B} := \{ M \in \Gamma_g \mid MB = B \}.$$

By the Cayley transformation, we obtain the corresponding boundary components of $\mathbb{H}_g$. Similarly to the case $g = 1$, we only need to consider certain boundary components to compactify the quotient space $\Gamma_g \backslash \mathbb{H}_g$.

Definition 1.4.7. A boundary component $B = B(V)$ is called *rational* if the corresponding subspace $V \subseteq \mathbb{R}^{2g}$ possesses a rational basis, i.e., a basis whose elements lie in $\mathbb{Q}^{2g}$.

Definition 1.4.8. For each $1 \leq g' \leq g$, a particular rational boundary component of D_g is given by

$$B_{g'} := \left\{ \begin{pmatrix} \mathbb{1}_{g-g'} & 0 \\ 0 & Z' \end{pmatrix} \,\middle|\, Z' \in D_{g'} \right\} \cong D_{g'}.$$

It corresponds to the real subspace $V \subseteq \mathbb{R}^{2g}$ spanned by the columns of the matrix $\left(\begin{smallmatrix} \mathbb{1}_{g-g'} \\ 0_{g+g',g-g'} \end{smallmatrix}\right)$. We will refer to it as a *standard boundary component.*

By the Cayley transformation, $B_{g'}$ corresponds to a boundary component of $\mathbb{H}_g$, which we will also denote by $B_{g'}$ and which is given by

$$B_{g'} = \left\{ \lim_{t \to \infty} \begin{pmatrix} it\mathbb{1}_{g-g'} & 0 \\ 0 & \tau_2 \end{pmatrix} \,\middle|\, \tau_2 \in \mathbb{H}_{g'} \right\} \cong \mathbb{H}_{g'}.$$

In the following, let D be either of the spaces D_g or $\mathbb{H}_g$. Denote by

$$\overline{D} = D \sqcup \bigsqcup_{V \subseteq \mathbb{R}^{2g}} B(V) \text{ and } D^* = D \sqcup \bigsqcup_{\substack{V \subseteq \mathbb{R}^{2g}, \\ V \text{ rational}}} B(V)$$

the closure of D and the rational closure of D in $\operatorname{Sym}_g(\mathbb{C})$, respectively.

The following results hold true for any subgroup $\Gamma \subseteq \Gamma_g$ of finite index, in particular they hold for $\Gamma_g(N)$. They elaborate why it is sufficient to only consider rational boundary components for a minimal compactification of $\Gamma\backslash D$.

Proposition 1.4.9. *There exists a fundamental domain F_g for Γ in D such that the topological closure $\overline{F}_g$ is contained in the union of finitely many rational boundary components. Furthermore, $\overline{F}_g$ meets every equivalence class of rational boundary components under the action of Γ.*

Proof. See the results 4.3, 4.6, and 4.7 in [3]. □

With the preceding proposition, the following definition makes sense.

Definition 1.4.10. The *Satake-* or *Baily–Borel compactification* $(\Gamma\backslash D)^*$ of $\Gamma\backslash D$ is the quotient

$$(\Gamma\backslash D)^* := \Gamma\backslash D^*,$$

equipped with the quotient topology.

The space $(\Gamma\backslash D)^*$ is an actual compactification of $\Gamma\backslash D$ with the boundary explicitely given, due to the following theorem of Baily–Borel.

Theorem 1.4.11. *The quotient $(\Gamma\backslash D)^* = \Gamma\backslash D^*$, endowed with the quotient topology, is a compact Hausdorff space, and $\Gamma\backslash D$ is an open, everywhere dense subset. Furthermore, $(\Gamma\backslash D)^*$ is the finite union of subspaces $\Gamma_{B_\alpha}\backslash B_\alpha$, where the B_α are representatives of equivalence classes of rational boundary components modulo Γ.*

Proof. See [3], Corollary 4.11. □

To further investigate the boundary of $\Gamma\backslash D$ in the Baily–Borel compactification, it is necessary to study the subspaces $\Gamma_{B_\alpha}\backslash B_\alpha$ that form the boundary as stated in the theorem. For the case $\Gamma = \Gamma_g$, we have the following result, which states that Γ_g acts transitively on the boundary components.

Remark 1.4.12. Let B be a rational boundary component of D. Then, there exist $0 \leq g' \leq g$ and $M \in \Gamma_g$ with $MB = B_{g'}$, where $B_{g'}$ denotes the standard boundary component from Definition 1.4.8.

Proof. A proof is given in [38], Remark 4.16. □

By the remark, we only have to consider the case $B_\alpha = B_{g'}$ for some $0 \leq g' \leq g$ to fully describe the boundary of $\Gamma_g \backslash D$. The stabilizer $\Gamma_{g,B_{g'}}$ can be determined using the correspondence between boundary components of D_g and subspaces of $\mathbb{R}^{2g}$ described in Proposition 1.4.5. This is for example done in [38], Proposition 4.8, yielding the following result.

Proposition 1.4.13. *The stabilizer $G_{g,B_{g'}} \subseteq \mathrm{Sp}_{2g}(\mathbb{R})$ of a standard boundary component $B_{g'}$ $(0 \leq g' \leq g)$ consists of all matrices M of the form*

$$M = \begin{pmatrix} U & \lambda' & \kappa & \mu' \\ 0 & A & \mu & B \\ 0 & 0 & U^{-t} & 0 \\ 0 & C & -\lambda & D \end{pmatrix} \in \mathrm{Sp}_{2g}(\mathbb{R})$$

with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{g'}(\mathbb{R}),\ U \in \mathrm{GL}_{g-g'}(\mathbb{R})$$

and

$$\lambda, \mu \in \mathrm{Mat}_{g',g-g'}(\mathbb{R}),\ \lambda', \mu' \in \mathrm{Mat}_{g-g',g'}(\mathbb{R}),\ \kappa \in \mathrm{Mat}_{g-g'}(\mathbb{R})$$

such that the conditions

$$(\lambda', \mu') = U(\lambda^t, \mu^t)\begin{pmatrix} A & B \\ C & D \end{pmatrix},\ U^{-1}\kappa - \lambda^t\mu = (U^{-1}\kappa - \lambda^t\mu)^t$$

hold true. The stabilizer $\Gamma_{g,B_{g'}}$ is therefore the group $\Gamma_g \cap G_{g,B_{g'}}$. □

Computing the action of an element $M \in \Gamma_{g,B_{g'}}$ of the stabilizer on $Z \in B_{g'} \subseteq D_g^*$, one obtains

$$MZ = M\begin{pmatrix} \mathbb{1}_{g-g'} & 0 \\ 0 & Z' \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{g-g'} & 0 \\ 0 & \begin{pmatrix} A & B \\ C & D \end{pmatrix} Z' \end{pmatrix},$$

with $Z' \in D_{g'}$. Hence, the subspaces $\Gamma_{g,B_{g'}} \backslash B_{g'}$ in Theorem 1.4.11 are isomorphic to $\Gamma_{g'} \backslash D_{g'}$, and the shape of the Baily–Borel compactification can be specified.

Theorem 1.4.14. *As a set, the Baily–Borel compactification $\mathcal{A}_g^* = \Gamma_g \backslash \mathbb{H}_g^*$ of $\mathcal{A}_g$ equals*

$$\mathcal{A}_g^* = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_0.$$ □

In this decomposition of $\mathcal{A}_g^*$, a point $\tau \in \mathcal{A}_{g'}$ $(0 \leq g' \leq g)$ can be recovered as a limit

$$\lim_{t_1,\dots,t_{g-g'} \to \infty} \begin{pmatrix} it_1 & 0 & \dots & 0 & 0 \\ 0 & it_2 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & it_{g-g'} & 0 \\ 0 & \dots & \dots & 0 & \tau \end{pmatrix}.$$

Remark 1.4.15. As seen above, the codimension of the Baily–Borel boundary $\mathcal{A}_g^* \setminus \mathcal{A}_g$ equals $\dim(\mathcal{A}_g) - \dim(\mathcal{A}_{g-1}) = g$. For $g \geq 2$, the compactification is therefore highly singular.

1.5 Toroidal compactifications of $\mathcal{A}_g$

The Baily–Borel compactification for $\mathcal{A}_g$ ($g \geq 2$), produces a boundary of high codimension. To attempt a smooth compactification, one has to proceed more carefully, taking the quotient by a subgroup of Γ_g first and compactifying certain tori that arise in the process. This yields a boundary of codimension 1 for all $g \geq 1$, but cannot be obtained in a canonical way for $g \neq 1$. We will first define the subgroups necessary for the process, and subsequently describe the steps leading to a toroidal compactification. For the explicit description of the spaces involved, we will restrict to $D = \mathbb{H}_g$.

Definition 1.5.1. Recall the description of the stabilizer $G_{g,B_{g'}}$ of a standard boundary component $B_{g'}$ in Proposition 1.4.13. The groups $G_{g'}$ and $\mathrm{GL}_{g-g'}(\mathbb{R})$ embed naturally into $G_{g,B_{g'}}$ and we define

$$G_h(B_{g'}) := \left\{ \begin{pmatrix} \mathbb{1}_{g-g'} & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & \mathbb{1}_{g-g'} & 0 \\ 0 & C & 0 & D \end{pmatrix} \;\middle|\; \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_{g'} \right\} \cong G_{g'},$$

$$G_l(B_{g'}) := \left\{ \begin{pmatrix} U & 0 & 0 & 0 \\ 0 & \mathbb{1}_{g'} & 0 & 0 \\ 0 & 0 & U^{-t} & 0 \\ 0 & 0 & 0 & \mathbb{1}_{g'} \end{pmatrix} \;\middle|\; U \in \mathrm{GL}_{g-g'}(\mathbb{R}) \right\} \cong \mathrm{GL}_{g-g'}(\mathbb{R}).$$

We will furthermore consider the unipotent radical $\mathcal{W}_{g'}$ of $G_{g,B_{g'}}$, the center $\mathcal{U}_{g'}$ of $\mathcal{W}_{g'}$, and a self-dual open cone $\Omega_{g'}$ in $\mathcal{U}_{g'}$, which have the explicit form

$$\mathcal{W}_{g'} = \left\{ \begin{pmatrix} \mathbb{1}_{g-g'} & \lambda^t & \kappa & \mu^t \\ 0 & \mathbb{1}_{g'} & \mu & 0 \\ 0 & 0 & \mathbb{1}_{g-g'} & 0 \\ 0 & 0 & -\lambda & \mathbb{1}_{g'} \end{pmatrix} \;\middle|\; \begin{array}{c} \lambda, \mu \in \mathrm{Mat}_{g',g-g'}(\mathbb{R}), \\ \kappa \in \mathrm{Mat}_{g-g'}(\mathbb{R}), \\ \kappa - \lambda^t \mu = (\kappa - \lambda^t \mu)^t \end{array} \right\} \subseteq G_{g,B_{g'}},$$

$$\mathcal{U}_{g'} = \left\{ [\kappa] := \begin{pmatrix} \mathbb{1}_{g-g'} & 0 & \kappa & 0 \\ 0 & \mathbb{1}_{g'} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{g-g'} & 0 \\ 0 & 0 & 0 & \mathbb{1}_{g'} \end{pmatrix} \;\middle|\; \kappa \in \mathrm{Sym}_{g-g'}(\mathbb{R}) \right\} \subseteq \mathcal{W}_{g'},$$

$$\Omega_{g'} = \{[\kappa] \mid \kappa > 0\}.$$

Remark 1.5.2. The group $\mathcal{W}_{g'}$ is isomorphic to $\mathbb{R}^{g'(g-g')} \times \mathbb{R}^{g'(g-g')} \times \mathbb{R}^{(g-g')^2}$ with group structure

$$(\lambda, \mu, \kappa) * (\lambda', \mu', \kappa') = (\lambda + \lambda', \mu + \mu', \kappa + \kappa' + \lambda^t \mu' - \mu^t \lambda'),$$

which is a Heisenberg group structure with underlying symplectic form $\langle(\lambda,\mu),(\lambda',\mu')\rangle = 2(\lambda^t\mu' - \mu^t\lambda')$ on $\mathbb{R}^{g'(g-g')} \times \mathbb{R}^{g'(g-g')}$.

Definition 1.5.3. The stabilizer $G_{g,B_{g'}}$ can now be decomposed as

$$G_{g,B_{g'}} = (G_h(B_{g'}) \times G_l(B_{g'})) \ltimes \mathcal{W}_{g'}. \tag{1.5.1}$$

The maps from $G_{g,B_{g'}}$ to $G_h(B_{g'})$ and $G_l(B_{g'})$ will be denoted by p_h and p_l, respectively.

The cocernel $\mathcal{V}_{g'}$ of the embedding

$$0 \longrightarrow \mathcal{U}_{g'} \longrightarrow \mathcal{W}_{g'} \longrightarrow \mathcal{V}_{g'} \longrightarrow 0$$

is a vector group of the form

$$\mathcal{V}_{g'} = \left\{ \begin{pmatrix} \mathbb{1}_{g-g'} & \lambda^t & 0 & \mu^t \\ 0 & \mathbb{1}_{g'} & \mu & 0 \\ 0 & 0 & \mathbb{1}_{g-g'} & 0 \\ 0 & 0 & -\lambda & \mathbb{1}_{g'} \end{pmatrix} \;\middle|\; \lambda,\mu \in \mathrm{Mat}_{g',g-g'}(\mathbb{R}) \right\}.$$

Remark 1.5.4. The action of $G_{g,B_{g'}}$ on an element $\begin{pmatrix} \tau_1 & \tau_{12}^t \\ \tau_{12} & \tau_2 \end{pmatrix} \in \mathbb{H}_g$ with $\tau_1 \in \mathbb{H}_{g-g'}$, $\tau_2 \in \mathbb{H}_{g'}$ is explicitely given by

$$\begin{pmatrix} U & \lambda' & \kappa & \mu' \\ 0 & A & \mu & B \\ 0 & 0 & U^{-t} & 0 \\ 0 & C & -\lambda & D \end{pmatrix} \begin{pmatrix} \tau_1 & \tau_{12}^t \\ \tau_{12} & \tau_2 \end{pmatrix} = \begin{pmatrix} \tau_1' & \tau_{12}'^t \\ \tau_{12}' & \tau_2' \end{pmatrix}$$

with

$$\begin{aligned} \tau_1' &= U\tau_1U^t + \lambda'\tau_{12}U^t + \kappa U^t - (U\tau_{12}^t + \lambda'\tau_2 + \mu')(C\tau_2 + D)^{-1}(C\tau_{12} - \lambda)U^t, \\ \tau_2' &= (A\tau_2 + B)(C\tau_2 + D)^{-1}, \\ \tau_{12}' &= (A\tau_{12} + \mu)U^t - (A\tau_2 + B)(C\tau_2 + D)^{-1}(C\tau_{12} - \lambda)U^t. \end{aligned}$$

Definition 1.5.5. For the toroidal compactification, we will make use of the structure of the spaces

$$\begin{aligned} D(B_{g'}) := \mathcal{U}_{g',\mathbb{C}} \cdot \mathbb{H}_g &= \left\{ \begin{pmatrix} \tau_1 & \tau_{12}^t \\ \tau_{12} & \tau_2 \end{pmatrix} \in \mathrm{Sym}_g(\mathbb{C}) \;\middle|\; \tau_2 \in \mathbb{H}_{g'} \right\} \\ &\cong \mathbb{H}_{g'} \times \mathrm{Mat}_{g',g-g'}(\mathbb{C}) \times \mathrm{Sym}_{g-g'}(\mathbb{C}) \\ &\cong B_{g'} \times \mathcal{V}_{g'} \times \mathcal{U}_{g',\mathbb{C}}, \end{aligned}$$

$$\begin{aligned} D(B_{g'})' := \mathcal{U}_{g',\mathbb{C}} \backslash D(B_{g'}) &\cong \mathbb{H}_{g'} \times \mathrm{Mat}_{g',g-g'}(\mathbb{C}) \\ &\cong B_{g'} \times \mathcal{V}_{g'}. \end{aligned}$$

We obtain a commutative diagram

$$\begin{tikzcd}
D(B_{g'}) \arrow[r, "\cong"] \arrow[d, "\pi'_{g'}"'] \arrow[dd, bend right=60, "\pi_{g'}"'] & B_{g'} \times \mathcal{V}_{g'} \times \mathcal{U}_{g',\mathbb{C}} \arrow[d] \\
D(B_{g'})' \arrow[r, "\cong"] \arrow[d] & B_{g'} \times \mathcal{V}_{g'} \arrow[d] \\
B_{g'} \arrow[r] & B_{g'},
\end{tikzcd}$$

where the horizontal map on top is given by the assignment

$$\begin{pmatrix} \tau_1 & \tau_{12}^t \\ \tau_{12} & \tau_2 \end{pmatrix} \longmapsto \left(\tau_2, \begin{pmatrix} \mathbb{1}_{g-g'} & \lambda^t & 0 & \mu^t \\ 0 & \mathbb{1}_{g'} & \mu & 0 \\ 0 & 0 & \mathbb{1}_{g-g'} & 0 \\ 0 & 0 & -\lambda & \mathbb{1}_{g'} \end{pmatrix}, [\tau_1] \right),$$

with $\tau_{12} = \mu + i\lambda$.

Remark 1.5.6. By Remark 1.5.4, the submatrix τ_2 corresponding to $B_{g'}$ is only affected by the factor $G_h(B_{g'})$ of decomposition (1.5.1) under the action of $G_{g,B_{g'}}$. The subgroup acts on $\mathbb{H}_{g'}$ by the usual action of $G_{g'}$. Hence, the vertical map $\pi_{g'}$ in the diagram above is equivariant under the induced actions of $G_{g,B_{g'}}$ on $D(B_{g'})$ and $B_{g'}$ in the sense that we obtain the equivariant diagram

$$\begin{tikzcd}
D(B_{g'}) \arrow[r, "\pi_{g'}"] & B_{g'} \\
G_{g,B_{g'}} \arrow[u, bend left] \arrow[r, two heads, "p_h"] & G_h(B_{g'}) \cong G_{g'}, \arrow[u, bend left]
\end{tikzcd}$$

where we recall that p_h denotes the projection to the factor $G_h(B_{g'})$ of $G_{g,B_{g'}}$.

In order to compactify $\mathcal{A}_g = \Gamma_g \backslash \mathbb{H}_g$ toroidally, we will embed $\mathbb{H}_g$ into $D(B_{g'})$ and make use of the product structure of $D(B_{g'}) = B_{g'} \times \mathcal{V}_{g'} \times \mathcal{U}_{g',\mathbb{C}}$. For that purpose, we consider the map $\Phi_{g'} \colon D(B_{g'}) \longrightarrow \mathcal{U}_{g'}$, given by the assignment

$$\Phi_{g'} \begin{pmatrix} \tau_1 & \tau_{12}^t \\ \tau_{12} & \tau_2 \end{pmatrix} = \operatorname{Im}(\tau_1) - \operatorname{Im}(\tau_{12}) \operatorname{Im}(\tau_2)^{-1} \operatorname{Im}(\tau_{12})^t.$$

One shows easily that a matrix $\operatorname{Im}(\tau) \in \operatorname{Sym}_g(\mathbb{R})$ is positive definite if and only if the matrix $\operatorname{Im}(\tau_1) - \operatorname{Im}(\tau_{12}) \operatorname{Im}(\tau_2)^{-1} \operatorname{Im}(\tau_{12})^t \in \operatorname{Sym}_{g-g'}(\mathbb{R})$ is positive definite. Therefore, $\mathbb{H}_g$ is the preimage $\Phi_{g'}^{-1}(\Omega_{g'})$ of the open cone $\Omega_{g'}$ under $\Phi_{g'}$. Again by Remark 1.5.4, the action of an element of $G_l(B_{g'})$ corresponding to a matrix $U \in \operatorname{GL}_{g-g'}(\mathbb{R})$ on an element $\tau \in D(B_{g'})$ translates to the action

$U\Phi_{g'}(\tau)U^t$ on its image in $\mathcal{U}_{g'}$. Therefore, the diagram

$$\begin{tikzcd}
D(B_{g'}) \arrow[r, "\Phi_{g'}"] & \mathcal{U}_{g'} \\
\mathbb{H}_g \arrow[u, hook] \arrow[r] & \Omega_{g'} \arrow[u, hook] \\
G_{g,B_{g'}} \arrow[u, bend left] \arrow[r, two heads, "p_l"] & G_l(B_{g'}) \cong \mathrm{Aut}(\mathcal{U}_{g'}, \Omega_{g'}) \arrow[u, bend left]
\end{tikzcd}$$

commutes.

The previous considerations verify that the map

$$(\pi'_{g'}, \Phi_{g'}) \colon D(B_{g'})/\mathcal{U}_{g'} \overset{\cong}{\longrightarrow} D(B_{g'})' \times \mathcal{U}_{g'}$$

is a $G_{g,B_{g'}}$-equivariant isomorphism.

In the course of toroidally compactifying $\mathcal{A}_g$, we will make use of certain well-behaved subgroups of $\Gamma_g = \mathrm{Sp}_g(\mathbb{Z})$, corresponding to the subgroups of $G_{g'}$ previously defined.

Definition 1.5.7. Let $\Gamma_{g,B_{g'}} = \Gamma_g \cap G_{g,B_{g'}}$ be the stabilizer of the boundary component $B_{g'}$ in Γ_g. Elements of $\Gamma_{g,B_{g'}}$ are explicitely given as matrices

$$M = \begin{pmatrix} U & \lambda' & \kappa & \mu' \\ 0 & A & \mu & B \\ 0 & 0 & U^{-t} & 0 \\ 0 & C & -\lambda & D \end{pmatrix}$$

with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{g'},\ U \in \mathrm{GL}_{g-g'}(\mathbb{Z}),\ \lambda, \lambda', \mu, \mu' \in \mathrm{Mat}_{g',g-g'}(\mathbb{Z}),\ \kappa \in \mathrm{Mat}_{g-g'}(\mathbb{Z}),$$

satisfying the conditions

$$(\lambda', \mu') = U(\lambda^t, \mu^t) \begin{pmatrix} A & B \\ C & D \end{pmatrix},\ U^{-1}\kappa - \lambda^t\mu = (U^{-1}\kappa - \lambda^t\mu)^t.$$

Recall the map $p_l \colon G_{g,B_{g'}} \longrightarrow G_l(B_{g'})$ projecting to the factor $G_l(B_{g'})$ of the decomposition (1.5.1) of $G_{g,B_{g'}}$. Its image $\overline{\Gamma}_{g'} := p_l(\Gamma_{g,B_{g'}})$ is given by

$$\overline{\Gamma}_{g'} = \left\{ \begin{pmatrix} U & 0 & 0 & 0 \\ 0 & \mathbb{1}_{g'} & 0 & 0 \\ 0 & 0 & U^{-t} & 0 \\ 0 & 0 & 0 & \mathbb{1}_{g'} \end{pmatrix} \,\middle|\, U \in \mathrm{GL}_{g-g'}(\mathbb{Z}) \right\} \cong \mathrm{GL}_{g-g'}(\mathbb{Z}).$$

Furthermore, we will define the subgroups in $\Gamma_{g,B_{g'}}$ corresponding to the subspaces $\mathcal{W}_{g'}$, $\mathcal{U}_{g'}$, and $\mathcal{V}_{g'}$ of $G_{g,B_{g'}}$ as

$$\begin{aligned}
W_{g'} &:= \Gamma_g \cap \mathcal{W}_{g'} = \left\{ \begin{pmatrix} \mathbb{1}_{g-g'} & \lambda^t & \kappa & \mu^t \\ 0 & \mathbb{1}_{g'} & \mu & 0 \\ 0 & 0 & \mathbb{1}_{g-g'} & 0 \\ 0 & 0 & -\lambda & \mathbb{1}_{g'} \end{pmatrix} \,\middle|\, \begin{matrix} \lambda, \mu \in \mathrm{Mat}_{g',g-g'}(\mathbb{Z}), \\ \kappa \in \mathrm{Mat}_{g-g'}(\mathbb{Z}), \\ \kappa - \lambda^t\mu = (\kappa - \lambda^t\mu)^t \end{matrix} \right\}, \\
U_{g'} &:= \Gamma_g \cap \mathcal{U}_{g'} = \left\{ \begin{pmatrix} \mathbb{1}_{g-g'} & 0 & \kappa & 0 \\ 0 & \mathbb{1}_{g'} & 0 & 0 \\ 0 & 0 & \mathbb{1}_{g-g'} & 0 \\ 0 & 0 & 0 & \mathbb{1}_{g'} \end{pmatrix} \,\middle|\, \kappa \in \mathrm{Sym}_{g-g'}(\mathbb{Z}) \right\}, \\
V_{g'} &:= W_{g'}/U_{g'}.
\end{aligned}$$

Remark 1.5.8. For $g' = g-1$ and, hence, $\overline{\Gamma}_{g-1} \cong \mathrm{GL}_1(\mathbb{Z}) = \{\pm 1\}$ acting trivially on B_{g-1}, the semi-direct product structure of the group $\Gamma_{g,B_{g-1}}/\overline{\Gamma}_{g-1} \cong \Gamma_{g-1} \ltimes (\mathbb{Z}^{2(g-1)} \times \mathbb{Z})$ reduces to

$$\begin{aligned}
&\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}; \begin{pmatrix} \lambda \\ \mu \end{pmatrix}, \kappa \right) * \left(\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}; \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix}, \kappa' \right) = \\
&\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}; \begin{pmatrix} \lambda \\ \mu \end{pmatrix} + \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-t} \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix}, \kappa + \kappa' + \begin{pmatrix} \lambda \\ \mu \end{pmatrix}^t \begin{pmatrix} A & B \\ C & D \end{pmatrix} J \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} \right).
\end{aligned}$$

This structure is, up to the last factor $\mathbb{Z}$, isomorphic to the structure of the semi-direct product $\Gamma_{g-1} \ltimes \mathbb{Z}^{2(g-1)}$ employed in Section 1.2 to define the universal family $\mathcal{X}_{g-1}$ over $\mathcal{A}_{g-1}$.

Notation 1.5.9. As all the boundary components are equivalent, their stabilizers are conjugate. Hence, the groups and spaces above can be easily defined for any arbitrary rational boundary component B_α. We will denote them by replacing g' by α, for example $\overline{\Gamma}_\alpha$ instead of $\overline{\Gamma}_{g'}$.

In the following, we will give an outline of the procedure of constructing a toroidal compactification. This procedure is carried out in three steps. First we will form the partial quotients of the spaces $D(B_\alpha)$ defined above by the corresponding subgroups U_α. In the quotient objects, there will be tori appearing. These tori will be compactified in the second step. In the third and last step, we will glue the arising objects to a toroidal compactification of $\Gamma_g \backslash \mathbb{H}_g$.

To construct torus embeddings that will compactify the occurring tori partially, we need for every boundary component B_α a cone decomposition for Ω_α, respecting the action of $\overline{\Gamma}_\alpha$ on $D(B_\alpha)$ that restricts to an action by conjugation on Ω_α via the map $\Phi_\alpha \colon D(B_\alpha) \longrightarrow \mathcal{U}_\alpha$. Additionally, the family of cone decompositions parametrized by the B_α has to be compatible with the action of Γ_g on the set of boundary components. Therefore, we make the following definitions.

Definition 1.5.10. Let $\Sigma_\alpha = \{\sigma_\mu \subseteq \overline{\Omega}_\alpha\}$ be a collection of convex rational polyhedral cones in the closure $\overline{\Omega}_\alpha$ of Ω_α in $\mathcal{U}_\alpha$. The collection Σ_α is called a $\overline{\Gamma}_\alpha$-*admissible cone decomposition of* Ω_α if the following requirements are fulfilled:

(i) All faces of a cone $\sigma_\mu \in \Sigma_\alpha$ are again contained in Σ_α.

(ii) The intersection of two cones $\sigma_\mu, \sigma_\nu \in \Sigma_\alpha$ is a face of both σ_μ and σ_ν.

(iii) For any $M \in \overline{\Gamma}_\alpha$ and $\sigma_\mu \in \Sigma_\alpha$, the conjugate cone $M\sigma_\mu M^t$ lies in Σ_α.

(iv) The quotient of the set Σ_α by $\overline{\Gamma}_\alpha$ has only finitely many elements.

(v) The space Ω_α is contained in the union of all $\sigma_\mu \in \Sigma_\alpha$.

Definition 1.5.11. A family $\{\Sigma_\alpha\}_{B_\alpha}$ of cone decompositions of the cones Ω_α is called Γ_g-admissible if it satisfies the two following two conditions:

(vi) If $M \in \Gamma_g$ maps B_α to B_β, the map $\mathcal{U}_\alpha \longrightarrow \mathcal{U}_\beta$, given by conjugation with M, maps Σ_α to Σ_β.

(vii) For $B_\alpha \subseteq \overline{B}_\beta$ contained in the boundary of B_β, the decomposition Σ_β is obtained as $\Sigma_\alpha \cap \mathcal{U}_\beta$.

Given a Γ_g-admissible family $\{\Sigma_\alpha\}_{B_\alpha}$ of $\overline{\Gamma}_\alpha$-admissible cone decompositions of Ω_α, the toroidal compactification can be carried out in three steps.

In the first step, we will take the quotient of $D(B_\alpha)$ by the subgroup $U_\alpha \subseteq \Gamma_{g,B_\alpha}$ to produce tori. Recall the decompositions

$$D(B_\alpha) = \mathcal{U}_{\alpha,\mathbb{C}} \cdot D \cong B_\alpha \times \mathcal{V}_\alpha \times \mathcal{U}_{\alpha,\mathbb{C}}$$

and $D(B_\alpha)' = \mathcal{U}_{\alpha,\mathbb{C}} \backslash D(B_\alpha) \cong B_\alpha \times \mathcal{V}_\alpha$, as well as the quotient map $\pi'_\alpha \colon D(B_\alpha) \longrightarrow D(B_\alpha)'$ and the map $\Phi_\alpha \colon D(B_\alpha) \longrightarrow \mathcal{U}_\alpha$, sending a matrix to its imaginary part and then projecting onto $\mathrm{Im}(\mathcal{U}_{\alpha,\mathbb{C}}) \cong \mathcal{U}_\alpha$ in a way that $\Phi_\alpha^{-1}(\Omega_\alpha) = \mathbb{H}_g$. Both maps descend to the quotient by U_α, as taking the quotient only affects the real part of a matrix in $\mathcal{U}_{\alpha,\mathbb{C}}$. Hence, we obtain the commutative diagram

$$\begin{tikzcd}
\Omega_\alpha \arrow[r, hook] & \mathcal{U}_\alpha & \\
U_\alpha\backslash D \arrow[u] \arrow[r, hook] \arrow[d] & U_\alpha\backslash D(B_\alpha) \arrow[u, "\Phi_\alpha"'] \arrow[r, "\cong"] \arrow[d, "\pi'_\alpha"'] & B_\alpha \times \mathcal{V}_\alpha \times \mathcal{U}_{\alpha,\mathbb{C}}/U_\alpha \arrow[ul, "\mathrm{Im}\circ\mathrm{pr}_3"'] \arrow[d, "(\mathrm{pr}_1{,}\mathrm{pr}_2)"'] \\
D(B_\alpha)' \arrow[r, equal] & D(B_\alpha)' \arrow[r, "\cong"] & B_\alpha \times \mathcal{V}_\alpha
\end{tikzcd}$$

with $T_\alpha := \mathcal{U}_{\alpha,\mathbb{C}}/U_\alpha$ an algebraic torus of dimension $n := (g-g')(g-g'+1)/2$. Thus, the map

$$\pi'_\alpha \colon U_\alpha\backslash D(B_\alpha) \longrightarrow D(B_\alpha)'$$

is a fibre bundle with toric fibre T_α.

In the second step, the spaces $U_\alpha\backslash D$ will be partially compactified using torus embeddings associated to Σ_α.

Definition 1.5.12. Let $\Sigma_\alpha = \{\sigma_\mu\}$ be a $\overline{\Gamma}_\alpha$-admissible cone decomposition of Ω_α. Then, the associated torus embedding X_{Σ_α} of T_α is obtained as follows: The dual cone $\sigma^\vee$ of a cone σ in $\mathcal{U}_\alpha$ is defined to be

$$\sigma^\vee = \{r \in \mathcal{U}_\alpha^\vee \mid \langle r, a\rangle \geq 0,\, a \in \sigma\}.$$

Considering the subsemigroup $\sigma^\vee \cap U_\alpha^\vee$ of $U_\alpha^\vee$, we obtain a torus embedding

$$T_\alpha \longrightarrow X_\sigma := \operatorname{Spec}(\mathbb{C}[\sigma^\vee \cap U_\alpha^\vee]).$$

When σ_μ is a face of σ_ν, and, therefore, $\sigma_\nu^\vee$ is a face of $\sigma_\mu^\vee$, the inclusion

$$\mathbb{C}[\sigma_\nu^\vee \cap U_\alpha^\vee] \subseteq \mathbb{C}[\sigma_\mu^\vee \cap U_\alpha^\vee]$$

holds. Thus, we obtain morphisms

$$X_{\sigma_\mu} \longrightarrow X_{\sigma_\nu},$$

compatible with the embeddings $T_\alpha \longrightarrow X_{\sigma_\mu}$. Via these maps, the torus embeddings X_{σ_μ} ($\sigma_\mu \in \Sigma_\alpha$) glue together to a torus embedding

$$X_{\Sigma_\alpha} = \bigcup_{\sigma_\mu \in \Sigma_\alpha} X_{\sigma_\mu},$$

with $X_{\sigma_\mu} \subseteq X_{\sigma_\nu}$ for σ_μ a face of σ_ν. The embeddings

$$T_\alpha = \operatorname{Spec}(\mathbb{C}[T_1, T_1^{-1}, \ldots, T_n, T_n^{-1}]) \longrightarrow X_{\sigma_\nu}$$

also glue to an action of T_α on X_{Σ_α}.

The fibre bundle $\pi'_\alpha\colon U_\alpha\backslash D(B_\alpha) \longrightarrow D(B_\alpha)'$ with fibre T_α can now be enlarged to a fibre bundle

$$(U_\alpha\backslash D(B_\alpha))_{\Sigma_\alpha} := U_\alpha\backslash D(B_\alpha) \times^{T_\alpha} X_{\Sigma_\alpha}$$

over $D(B_\alpha)'$ with fibre X_{Σ_α}.

Proposition 1.5.13. *The fibre bundle $(U_\alpha\backslash D(B_\alpha))_{\Sigma_\alpha}$ possesses a fibrewise T_α-orbit decomposition*

$$(U_\alpha\backslash D(B_\alpha))_{\Sigma_\alpha} = \coprod_{\sigma_\mu \in \Sigma_\alpha} O(\mu)$$

with the following properties.

(i) *Each orbit $O(\mu)$ is an algebraic torus bundle over $D(B_\alpha)'$.*

(ii) The cone σ_μ is a face of σ_ν if and only if $O(\nu)$ is contained in the closure of $O(\mu)$.

(iii) The dimensions of σ_μ and $O(\mu)$ are linked by the formula

$$\dim(\sigma_\mu) + \dim(O(\mu)) = \dim(D).$$

(iv) The cone $\sigma_\mu = \{0\}$ corresponds to the orbit $O(\mu) = U_\alpha \backslash D(B_\alpha)$.

Proof. See [38], Proposition 6.12. □

Definition 1.5.14. Denote by $(U_\alpha \backslash D)_{\Sigma_\alpha}$ the interior of the closure of $U_\alpha \backslash D$ in $(U_\alpha \backslash D(B_\alpha))_{\Sigma_\alpha}$. Let

$$\overline{\pi}_\alpha \colon (U_\alpha \backslash D)_{\Sigma_\alpha} \longrightarrow D(B_\alpha)' \longrightarrow B_\alpha$$

be the composition of the bundle map with the projection to the boundary component B_α. For $\sigma_\mu \cap \Omega_\alpha \neq \emptyset$, the orbit $O(\mu)$ is contained in $(U_\alpha \backslash D)_{\Sigma_\alpha}$. We may therefore define

$$O(B_\alpha) := \coprod_{\sigma_\mu \cap \Omega_\alpha \neq \emptyset} O(\mu).$$

The collection of orbits $O(B_\alpha)$ can be interpreted as the set of points essentially added with respect to B_α. This is specified in the following proposition, comparing the partial compactification to the Baily–Borel compactification.

Proposition 1.5.15. *The map $U_\alpha \backslash D \longrightarrow \Gamma_g \backslash D$ extends to a holomorphic map $p_\alpha \colon (U_\alpha \backslash D)_{\Sigma_\alpha} \longrightarrow (\Gamma_g \backslash D)^*$, fitting into a commutative diagramm*

$$\begin{tikzcd}
(U_\alpha \backslash D)_{\Sigma_\alpha} \arrow[r, "p_\alpha"] & (\Gamma_g \backslash D)^* \\
O(B_\alpha) \arrow[r] \arrow[u, hook] & \Gamma_{g,B_\alpha} \backslash B_\alpha, \arrow[u, hook]
\end{tikzcd}$$

where the lower map is induced by $\overline{\pi}_\alpha \colon (U_\alpha \backslash D)_{\Sigma_\alpha} \longrightarrow B_\alpha$, and we have $p_\alpha^{-1}(\Gamma_{g,B_\alpha} \backslash B_\alpha) = O(B_\alpha)$.

Proof. See [38], Facts 7.9. □

Example 1.5.16. Consider the case $D = \mathbb{H}_g$, and let B_α be the standard boundary component B_{g-1}. The groups $\mathcal{U}_{g-1}$ and U_{g-1} equal $\mathbb{R}$ and $\mathbb{Z}$, respectively. The arising torus is $T_{g-1} = \mathcal{U}_{g-1,\mathbb{C}}/U_{g-1} = \mathbb{C}/\mathbb{Z}$, and the open cone equals $\Omega_{g-1} = \mathbb{R}_{\geq 0}$. The only possible admissible cone decomposition is given by

$$\Sigma_{g-1} = \{\sigma_0 = \{0\}, \sigma_1 = \mathbb{R}_{\geq 0}\}.$$

As $\sigma_0^\vee = \mathbb{R}$ and $\sigma_1^\vee = \mathbb{R}_{\geq 0}$ hold, the corresponding torus embeddings are

$$X_{\sigma_0} = \operatorname{Spec}(\mathbb{C}[T, T^{-1}]) = \mathbb{C}^\times \text{ and } X_{\sigma_1} = \operatorname{Spec}(\mathbb{C}[T]) = \mathbb{C}.$$

The torus $T_{g-1} = \mathbb{C}/\mathbb{Z}$ is identified with $X_{\sigma_0} = \mathbb{C}^\times$ via the exponential map $z \mapsto \exp(2\pi i z)$. The glueing of X_{σ_0} and X_{σ_1} partially compactifies T_{g-1} by partially compactifying the imaginary axis of $\mathbb{C}/\mathbb{Z}$ by a point $i\infty$, mapped to $0 \in \mathbb{C}$ under the exponential map. We obtain a fibre bundle

$$\begin{aligned}(U_{g-1}\backslash D(B_{g-1})_{\Sigma_{g-1}} &= \big(D(B_{g-1})' \times T_{g-1}\big)_{\Sigma_{g-1}} \\ &= D(B_{g-1})' \times \big(\mathbb{C}^\times \cup \{i\infty\}\big)\,.\end{aligned}$$

The interior of the closure $(U_{g-1}\backslash \mathbb{H}_g)_{\Sigma_{g-1}}$ of $U_{g-1}\backslash \mathbb{H}_g$ in this product is contained in the subspace

$$D(B_{g-1})' \times (\mathbb{H}_1/\mathbb{Z} \cup \{i\infty\})$$

with orbit decomposition

$$\begin{aligned}O_{\sigma_0} &= O(B_g) = (U_{g-1}\backslash \mathbb{H}_g)_{\Sigma_{g-1}} \cap \big(D(B_{g-1})' \times \mathbb{H}_1/\mathbb{Z}\big)\,, \\ O_{\sigma_1} &= O(B_{g-1}) = (U_{g-1}\backslash \mathbb{H}_g)_{\Sigma_{g-1}} \cap \big(D(B_{g-1})' \times \{i\infty\}\big)\,.\end{aligned}$$

In the last step, the spaces $(U_\alpha\backslash D)_{\Sigma_\alpha}$ are glued to a toroidal compactification of $\Gamma_g\backslash D$. The validity of this process arises from the following proposition.

Proposition 1.5.17. *The group Γ_α/U_α acts properly discontinuously on the space $(U_\alpha\backslash D)_{\Sigma_\alpha}$. The quotient space $(\Gamma_\alpha/U_\alpha)\backslash(U_\alpha\backslash D)_{\Sigma_\alpha}$ has a canonical quotient structure of a normal analytic space, and the image of $O(B_\alpha)$ is a closed analytic set in it.*

Proof. The proposition is a consequence of the $\overline{\Gamma}_\alpha$-admissibility of Σ_α, see [38], Proposition 7.11.1 and Theorem 7.12. ☐

There are two basic relations between boundary components B_α and B_β. If B_β is equal to MB_α for some $M \in \Gamma_g$, by the Γ_g-admissibility of the family of cone decompositions, the action of M on D induces an isomorphism

$$\begin{tikzcd}(U_\alpha\backslash D)_{\Sigma_\alpha} \arrow[r, "M", "\cong"'] & (U_\beta\backslash D)_{\Sigma_\beta} \\ U_\alpha\backslash D \arrow[u, hook] \arrow[r, "\cong"] & U_\beta\backslash D. \arrow[u, hook]\end{tikzcd}$$

If B_α lies in the closure of B_β, the condition $\Sigma_\beta = \Sigma_\alpha \cap \mathcal{U}_\beta$ induces an étale map

$$\pi_{\alpha,\beta}\colon \qquad (U_\beta\backslash D)_{\Sigma_\beta} \longrightarrow (U_\alpha\backslash D)_{\Sigma_\alpha}.$$

Definition 1.5.18. We define an equivalence relation on the set

$$(\Gamma_g\backslash D)^+ := \coprod_{B_\alpha} (U_\alpha\backslash D)_{\Sigma_\alpha}$$

as follows: Two points $x_\alpha \in (U_\alpha\backslash D)_{\Sigma_\alpha}$ and $x_\beta \in (U_\beta\backslash D)_{\Sigma_\beta}$ are equivalent if the following two conditions hold:

(i) There exists an element $M \in \Gamma_g$ and a boundary component B_γ such that B_α and MB_β both lie in the closure of this component.

(ii) There is a point $x_\gamma \in (U_\gamma \backslash D)_{\Sigma_\gamma}$ that is mapped to x_α under the map $\pi_{\alpha,\gamma}$, and to Mx_β under the map $\pi_{\beta',\gamma}$, where $B_{\beta'} := MB_\beta$.

A toroidal compactification is defined to be the quotient

$$\overline{(\Gamma_g \backslash D)}^{\text{tor}} := (\Gamma_g \backslash D)^+ / \sim$$

of $(\Gamma_g \backslash D)^+$ by this equivalence relation. Note that depends on the choice of the cone decomposition.

In the case of taking the quotient by the full group Γ_g, there are basically three relations between boundary components that can occur:
Case 1: Let $B_\alpha = MB_{g'}$ $(M \in \Gamma_g \setminus \Gamma_{g,B_{g'}})$. Here, one can apply (ii) from Definition 1.5.18 with $B_\gamma = B_{g'}$. The induced map $M\colon (U_{g'} \backslash D)_{\Sigma_{g'}} \longrightarrow (U_\alpha \backslash D)_{\Sigma_\alpha}$ is an isomorphism. As all rational boundary components can be mapped to a standard boundary component $B_{g'}$ by some $M \in \Gamma_g$, all spaces $(U_\alpha \backslash D)_{\Sigma_\alpha}$ corresponding to boundary components B_α of the same dimension are glued together.
Case 2: Let $B_\alpha = B_\beta = B_{g'}$. As any $M \in \Gamma_{g,B_{g'}}$ gives an automorphism of $B_{g'}$, all points x_α, $x_\beta \in (U_{g'} \backslash D)_{\Sigma_{g'}}$ with $x_\beta = Mx_\alpha$ $(M \in \Gamma_{g,B_{g'}})$ are glued together. We obtain the quotient $\Gamma_{g,B_{g'}} \backslash (U_{g'} \backslash D)_{\Sigma_{g'}}$.
Case 3: Let $B_{g''}$ be in the closure of $B_{g'}$ $(0 \leq g'' < g')$. The map $\pi_{g',g''}$ induced by $M = \mathbb{1}_{2g}$ glues $(U_{g'} \backslash D)_{\Sigma_{g'}}$ into $(U_{g''} \backslash D)_{\Sigma_{g''}}$, producing boundary components.

Example 1.5.19. We will examine the case $D = \mathbb{H}_g$, $B_{g-1} \subseteq \overline{B}_g$, more closely. Note that $B_g = \mathbb{H}_g$. As the group $\mathcal{U}_g = \{\mathbb{1}_{2g}\}$ is trivial and the torus T_g is just a point, we obtain

$$(U_g \backslash \mathbb{H}_g)_{\Sigma_g} = U_g \backslash \mathbb{H}_g = \mathbb{H}_g.$$

As already seen in Example 1.5.16, the partial compactification $(U_{g-1} \backslash \mathbb{H}_g)_{\Sigma_{g-1}}$ is contained in the product $D(B_{g-1})' \times (\mathbb{H}_1/\mathbb{Z} \cup \{i\infty\})$, and, by construction, it has nonempty intersection with any neighbourhood of a point of $D(B_{g-1})' \times \{i\infty\}$. As in *Case 3*, the map $\pi_{g,g-1}$ now glues $B_g = \mathbb{H}_g$ to $(U_{g-1} \backslash \mathbb{H}_g)_{\Sigma_{g-1}}$, and $\mathbb{H}_g$ is partially compactified with a boundary

$$D(B_{g-1})' \times \{i\infty\}.$$

As described in *Case 2*, we now have to consider the action of the groups $\Gamma_{g,B_{g'}}$ on the spaces $(U_{g'} \backslash \mathbb{H}_g)_{\Sigma_{g'}}$ $(g' = g-1,\, g)$. As $\Gamma_{g,B_{g-1}}$ is a subgroup of Γ_g, we only have to consider the action of Γ_g on the interior, and the quotient

$$\Gamma_g \backslash B_g = \Gamma_g \backslash \mathbb{H}_g = \mathcal{A}_g$$

arises away from the boundary. The stabilizer $\Gamma_{g,B_{g-1}}$ acts on the boundary $D(B_{g-1})' \times \{i\infty\}$, and we obtain a boundary component

$$B := \Gamma_{g,B_{g-1}} \backslash D(B_{g-1})' \times \{i\infty\}.$$

We have already established that $D(B_{g-1})'$ is isomorphic to $\mathbb{C}^g \times \mathbb{H}_{g-1}$, and that $\Gamma_{g,B_{g-1}}$ has the structure of a semi-direct product $W_{g-1} \rtimes \Gamma_{g-1}$ with W_{g-1} a lattice isomorphic to $\mathbb{Z}^{2g}$. Hence, the boundary component

$$B \cong (\mathbb{Z}^{2g} \rtimes \Gamma_{g-1}) \backslash (\mathbb{C}^g \times \mathbb{H}_{g-1})$$

is isomorphic to the universal family $\mathcal{X}_{g-1}$ over $\mathcal{A}_{g-1}$, and, therefore, the boundary is of codimension 1.

To conclude, we will state some results about the beneficial properties of toroidal compactifications.

Remark 1.5.20. The toroidal compactification can be defined analogously for any subgroup $\Gamma \subseteq \Gamma_g$ of finite index.

Example 1.5.21. All toroidal compactifications of $\mathcal{A}_1(N)$ coincide with the Baily–Borel compactification.

Proposition 1.5.22. *Let $\Gamma \subseteq \Gamma_g$ be a subgroup of finite index. The holomorphic maps*

$$(\Gamma_\alpha / U_\alpha) \backslash (U_\alpha \backslash D)_{\Sigma_\alpha} \longrightarrow (\Gamma \backslash D)^*$$

glue to a holomorphic map

$$\overline{(\Gamma \backslash D)}^{\text{tor}} \longrightarrow (\Gamma \backslash D)^*$$

to the Baily–Borel compactification that is the identity on the interior. The compactification $\overline{(\Gamma \backslash D)}^{\text{tor}}$ possesses a decomposition into orbits

$$\overline{(\Gamma \backslash D)}^{\text{tor}} = \coprod_{B_\alpha \ mod \ \Gamma} \overline{O}(B_\alpha),$$

where $\overline{O}(B_\alpha) = (\Gamma_\alpha / U_\alpha) \backslash O(B_\alpha)$. By the dimension formula for orbits, one notes that the toroidal boundary is of codimension 1.

Proof. See [38], Observations 7.15. □

Theorem 1.5.23. *A toroidal compactification has at most finite quotient singularities. For any neat arithmetic subgroup $\Gamma \subseteq \Gamma_g$, there is a nonsingular and projective toroidal compactification of $\Gamma \backslash D$.*

Proof. See [38], Theorem 7.23. □

Remark 1.5.24. In the situation of Theorem 1.5.23, the toroidal compactification is in fact the (normalization of) the blow-up of the Baily-Borel compactification at the ideal sheaf corresponding to so-called Fourier-Jacobi series. This interpretation can be applied to construct models of $\overline{\mathcal{A}}_g^{\text{tor}}$ over the integers, see, e.g., [10].

By passing from Γ_g to a congruence subgroup, we can therefore always obtain smoothness at the boundary, We can consider the Hodge bundle to be defined on the toroidal compactification by the following theorem.

Theorem 1.5.25. *The line bundle* $\omega_g^{\otimes k}$ *on* $\mathcal{A}_g$ *extends uniquely to a line bundle* $\overline{\omega}_g^{\otimes k}$ *on* $\overline{\mathcal{A}}_g^{\mathrm{tor}}$*, with*

$$H^0(\mathcal{A}_g, \omega_g^{\otimes k}) \cong H^0(\overline{\mathcal{A}}_g^{\mathrm{tor}}, \overline{\omega}_g^{\otimes k}).$$

The L^2*-metric on* $\omega_g^{\otimes k}$ *extends to a metric on* $\overline{\omega}_g^{\otimes k}$ *that is good in the sense of Mumford [36]. The same holds for the line bundle* $\mathcal{M}_k(\Gamma_g)$ *equipped with the Petersson metric.*

Proof. See [36], Main Theorem 3.1. $\square$

Remark 1.5.26. From now on, we will just write $\overline{\mathcal{A}}_g$ instead of $\overline{\mathcal{A}}_g^{\mathrm{tor}}$ for a suitable toroidal compactification of $\mathcal{A}_g$.

Chapter 2

Arithmetic intersection theory

An intersection theory for varieties of relative dimension 1 over the ring of integers of a number field was first developed by Arakelov [2] in 1974. In 1990, it was extended to arbitrary relative dimension by Gillet and Soulé [19]. Generalizing the PhD-thesis [33], Burgos, Kramer, and Kühn [9] developed a more abstract concept of arithmetic Chow group in 2007, allowing intersection theory to be generalized to line bundles with log-singular metrics on arithmetic varieties.

2.1 Arithmetic Chow groups

In the following section, we will introduce arithmetic varieties, hence, varieties over the spectrum of an arithmetic ring, such as the ring of integers of a number field. The complex embeddings of the arithmetic ring give rise to a corresponding complex variety. We will see how the concept of algebraic cycles can be adapted to this situation by adding to a cycle the information of a Green current of the corresponding complex cycle. This will lead to the introduction of arithmetic Chow groups.

Definition 2.1.1. An *arithmetic ring* is a triple (A, Σ, F_∞) consisting of an excellent regular Noetherian integral domain A, a finite nonempty set Σ of monomorphisms $\sigma\colon A \longrightarrow \mathbb{C}$, and an antilinear involution of $\mathbb{C}$-algebras $F_\infty\colon \mathbb{C}^\Sigma \longrightarrow \mathbb{C}^\Sigma$ fixing the image of A in $\mathbb{C}^\Sigma$ under the map $\delta\colon A \longrightarrow \mathbb{C}^\Sigma$ induced by the family Σ, hence making the diagram

$$\begin{tikzcd}
 & \mathbb{C}^\Sigma \arrow[dd, "F_\infty"] \\
A \arrow[ur, "\delta"] \arrow[dr, "\delta"'] & \\
 & \mathbb{C}^\Sigma
\end{tikzcd}$$

commutative.

Example 2.1.2. Let $\sigma\colon \mathbb{Z} \longrightarrow \mathbb{C}$ denote the canonical embedding. Then $\mathbb{Z}$ together with $\Sigma = \{\sigma\}$ and F_∞ the complex conjugation is an arithmetic ring. More generally, every subring A of $\mathbb{C}$ whose field of fractions is a number field F can be endowed with the structure of an arithmetic ring. The data Σ and F_∞ are naturally given by the set of embeddings $\Sigma = \{\sigma\colon A \longrightarrow \mathbb{C}\}$ and F_∞ acting on $\mathbb{C}^\Sigma = \prod_{\sigma\in\Sigma} \mathbb{C}_\sigma$ by $F_\infty\left((z_\sigma)_{\sigma\in\Sigma}\right) = (\overline{z}_{\overline{\sigma}})_{\sigma\in\Sigma}$.

Definition 2.1.3. Let (A, Σ, F_∞) be an arithmetic ring. An *arithmetic variety over* A is a regular scheme $\pi\colon \mathscr{X} \longrightarrow \operatorname{Spec}(A)$, which is flat and projective. If F is the field of fractions of A, we write $\mathscr{X}_F$ for the generic fibre of $\mathscr{X}$. For $s \in \operatorname{Spec}(A)$, we denote the fibre $\pi^{-1}(s)$ over s by $\mathscr{X}_s$, and for $\sigma \in \Sigma$ we write $X_\sigma := \mathscr{X} \otimes_\sigma \mathbb{C}$ and $X_\Sigma := \coprod_{\sigma\in\Sigma} X_\sigma = \mathscr{X} \otimes_A \mathbb{C}^\Sigma$. We denote by X_∞ the analytic space

$$X_\infty = X_\Sigma(\mathbb{C}) = \coprod_{\sigma\in\Sigma} X_\sigma(\mathbb{C})$$

and call it the *fibre at infinity* or *complex fibre*. Note that X_∞ is a compact complex manifold. The antilinear involution F_∞ on $\mathbb{C}^\Sigma$ induces an antilinear involution F_∞ on X_Σ. This involution gives X_Σ the structure of a real variety (X_Σ, F_∞), which we denote by $X_\mathbb{R}$.

Notation 2.1.4. To ease notation, we will subsequently simply write X instead of X_∞ for the fibre at infinity of an arithmetic variety $\mathscr{X}$. Analogously, we will write Y, Z instead of Y_∞, Z_∞ for the complex subvarieties of X corresponding to the subschemes $\mathscr{Y}, \mathscr{Z} \subseteq \mathscr{X}$, respectively.

Example 2.1.5. Let $\mathcal{A}_1$ be the moduli space of 1-dimensional principally polarized abelian varieties – hence, elliptic curves – as defined in Chapter 1. Any toroidal compactification $\overline{\mathcal{A}}_1$ of $\mathcal{A}_1$ coincides with the Baily–Borel compactification. The space $\overline{\mathcal{A}}_1$ is isomorphic to $\mathbb{P}^1_\mathbb{C}$ via the j-invariant. Hence, $\overline{\mathcal{A}}_1$ is the fibre at infinity for the arithmetic surface $\mathbb{P}^1_\mathbb{Z}$ over $\mathbb{Z}$.

Example 2.1.6. The toroidally compactified moduli stack $\overline{\mathcal{A}}_g$ $(g \geq 2)$ is the fibre at infinity for an arithmetic variety over $\operatorname{Spec}(\mathbb{Z})$, constructed by Faltings and Chai [14]. The Hodge bundle on $\overline{\mathcal{A}}_g$ is the complex bundle associated to the line bundle of invariant g-differentials on the universal semi-abelian scheme also constructed in [14]. The spaces $\overline{\mathcal{A}}_g(N)$ $(N \geq 3)$ are aswell fibres at infinity of arithmetic varieties, over $\mathbb{Z}[1/N, \zeta_N]$ instead of $\mathbb{Z}$, where ζ_N denotes a primitive N-th root of unity.

Definition 2.1.7. A *metrized line bundle* on an arithmetic variety $\mathscr{X}$ is a line bundle $\mathscr{L}$ on $\mathscr{X}$ together with a Hermitian metric $\|\cdot\|_\sigma$ on each complex line bundle $L_\sigma := \mathscr{L} \otimes_\sigma \mathbb{C}$ on X_σ such that the induced metric $\|\cdot\|$ on $L := \mathscr{L} \otimes \mathbb{C}^\Sigma$ is invariant under the action of F_∞. We define the set of isometry classes of metrized line bundles as

$$\widehat{\operatorname{Pic}}(\mathscr{X}) := \{(\mathscr{L}, \|\cdot\|)\}/\sim$$

and call it the *arithmetic Picard group* of $\mathscr{X}$. The class of the metrized line bundle $(\mathscr{L}, \|\cdot\|)$ in $\widehat{\operatorname{Pic}}(\mathscr{X})$ will be denoted by $[\mathscr{L}, \|\cdot\|]$.

To define a suitable intersection theory on $\mathscr{X}$, we will first examine Hermitian metrics and Green currents on complex manifolds X.

Remark 2.1.8. A Hermitian metric on a line bundle L on a complex manifold X can also be given in the following way: Let $\{U_j\}$ be a covering of X such that $L|_{U_j}$ is free eover $\mathcal{O}_X|_{U_j}$. Let $\{\phi_j \colon L|_{U_j} \longrightarrow \mathcal{O}_X|_{U_j}\}$ be a family of holomorphic trivializations and denote $\phi_{jk} := \phi_j \phi_k^{-1}$. Fixing a metric on L is then equivalent to giving a family of positive functions $\rho = \{\rho_j \colon U_j \longrightarrow \mathbb{R}_{>0}\}$ such that on $U_j \cap U_k$, we have $\rho_j = |\phi_{jk}|^2 \rho_k$. For s a section of L and $x \in U_j$, the metric corresponding to the family ρ is then defined by

$$\|s(x)\|^2 := (\rho_j(x))^{-1} |\phi_j(s)(x)|^2 \quad (x \in U_j).$$

Notation 2.1.9. Let X be a d-dimensional compact complex manifold. We denote by $\mathrm{A}^{p,q}(X)$ the space of smooth $\mathbb{C}$-valued (p,q)-forms on X, and by $\mathrm{D}_{p,q}(X)$ its dual, the space of Schwartz-continuous (p,q)-currents. We denote by

$$\mathrm{A}^n(X) := \bigoplus_{p+q=n} \mathrm{A}^{p,q}(X)$$

the space of smooth $\mathbb{C}$-valued n-forms on X, and by

$$\mathrm{D}_n(X) := \bigoplus_{p+q=n} \mathrm{D}_{p,q}(X)$$

the space of n-currents on X.

Definition 2.1.10. Let $\|\cdot\|$ be a Hermitian metric on a line bundle L on a compact complex manifold X, and let $\rho = \{\rho_j \colon U_j \longrightarrow \mathbb{R}_{>0}\}$ be the corresponding family as in Remark 2.1.8. We define the *first Chern form* $\mathrm{c}_1(L, \|\cdot\|) \in \mathrm{A}^{1,1}(X)$ to be

$$\mathrm{c}_1(L, \|\cdot\|) := \mathrm{dd}^c \log \rho_j$$

on U_j. As the functions ϕ_{jk} are holomorphic, and, thus, $\mathrm{dd}^c \log |\phi_{j,k}|^2 = 0$, we obtain $\mathrm{dd}^c \log \rho_j = \mathrm{dd}^c \log \rho_k$ on $U_j \cap U_k$. Therefore, the form is globally defined. For a non-zero meromorphic section s of L, we have the equality

$$\mathrm{c}_1(L, \|\cdot\|) = -\mathrm{dd}^c \log \|s\|^2$$

outside of $\mathrm{div}(s)$.

Definition 2.1.11. Let X be a d-dimensional compact complex manifold. We fix the orientation on X that is induced via the chart maps to $\mathbb{C}^d$ by the volume form

$$\left(\frac{i}{2}\right)^d \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \ldots \wedge \mathrm{d}z_d \wedge \mathrm{d}\bar{z}_d = \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}x_d \wedge \mathrm{d}y_d$$

on $\mathbb{C}^d$ with coordinates $z_j = x_j + iy_j$ $(j = 1, \ldots, d)$. With the induced orientation of X, any closed p-dimensional submanifold $\iota \colon M \hookrightarrow X$ defines a current $\delta_M \in \mathrm{D}_{p,p}$ by setting

$$\delta_M(\eta) := \int_M \iota^* \eta$$

for $\eta \in \mathrm{A}^{p,p}(X)$. More generally, any p-dimensional analytic subvariety $\iota\colon Y \hookrightarrow X$ induces a current $\delta_Y \in \mathrm{D}_{p,p}(X)$ by setting

$$\delta_Y(\eta) := \int\limits_{\widetilde{Y}} \pi^*\iota^*\eta$$

for $\eta \in \mathrm{A}^{p,p}(X)$; here, $\pi\colon \widetilde{Y} \longrightarrow Y$ denotes a resolution of singularities of Y. For a p-dimensional analytic cycle, i.e., a finite formal sum $Y = \sum_{j=1}^{m} n_j Y_j$ ($n_j \in \mathbb{Z}$) of p-dimensional closed analytic subvarieties $Y_j \subseteq X$, this definition can be extended to $\delta_Y := \sum_{j=1}^{m} n_j \delta_{Y_j}$.

Definition 2.1.12. Let $T \in \mathrm{D}_n(X)$ be a current and $\eta \in \mathrm{A}^m(X)$ be a smooth form. The assignment $(T\wedge\eta)(\beta) := T(\eta\wedge\beta)$ ($\beta \in A_{n-m}(X)$) defines a product

$$\wedge\colon \mathrm{D}_n(X) \otimes \mathrm{A}^m(X) \longrightarrow \mathrm{D}_{n-m}(X),$$

which is compatible with the bigrading, hence, decomposes into a product

$$\mathrm{D}_{p,q}(X) \otimes \mathrm{A}^{r,s}(X) \longrightarrow \mathrm{D}_{p-r,q-s}(X) \quad (p+q=n, r+s=m).$$

In particular, the assignment $\eta \mapsto \delta_X \wedge \eta =: [\eta]$ ($\eta \in \mathrm{A}^{p,q}(X)$) gives a map

$$\mathrm{A}^{p,q}(X) \longrightarrow \mathrm{D}_{d-p,d-q}(X).$$

More generally, denote by $\mathrm{L}^1_{\mathrm{loc}}\big(X, \Omega_X^{p,q}\big)$ the (p,q)-forms α on X that have locally integrable coefficients in every coordinate patch. The assignment $\alpha \mapsto \delta_X \wedge \alpha$ defines a map

$$\mathrm{L}^1_{\mathrm{loc}}\big(X, \Omega_X^{p,q}\big) \longrightarrow \mathrm{D}_{d-p,d-q}(X).$$

Definition 2.1.13. With the natural topology described, e.g., in [12], the maps $\mathrm{A}^{p,q}(X) \longrightarrow \mathrm{D}_{d-p,d-q}(X)$ are continuous. Their images, the smooth currents, are dense in $\mathrm{D}_{d-p,d-q}(X)$. We will therefore use the notation

$$\mathrm{D}^{p,q}(X) := \mathrm{D}_{d-p,d-q}(X).$$

The exterior derivative $\mathrm{d} = \partial + \overline{\partial}\colon \mathrm{A}^n(X) \longrightarrow \mathrm{A}^{n+1}(X)$ induces a dual homomorphism

$$\mathrm{d}' = \partial' + \overline{\partial'}\colon \mathrm{D}_{n+1}(X) \longrightarrow \mathrm{D}_n(X),$$

by setting $\mathrm{d}'T(\eta) := T(\mathrm{d}\eta)$ for $T \in \mathrm{D}_{n+1}(X)$, $\eta \in \mathrm{A}^n(X)$. Here, ∂ and $\overline{\partial}$ induce the maps $\partial'\colon \mathrm{D}_{p,q}(X) \longrightarrow \mathrm{D}_{p-1,q}(X)$ and $\overline{\partial'}\colon \mathrm{D}_{p,q}(X) \longrightarrow \mathrm{D}_{p,q-1}(X)$, respectively. Using Stokes' theorem, we compute for $\eta \in \mathrm{A}^n(X)$ and $\alpha \in \mathrm{A}^{d-n-1}(X)$

$$[\mathrm{d}\eta](\alpha) = \int\limits_X \mathrm{d}\eta \wedge \alpha = \int\limits_X \mathrm{d}(\eta \wedge \alpha) - \int\limits_X (-1)^n \eta \wedge \mathrm{d}\alpha = (-1)^{n+1}\mathrm{d}'[\eta](\alpha).$$

Therefore, by abuse of notation, we define the exterior derivative

$$\mathrm{d} := (-1)^{n+1}\mathrm{d}'\colon \mathrm{D}^n(X) \longrightarrow \mathrm{D}^{n+1}(X),$$

and analogously the partial derivatives ∂, $\overline{\partial}$. These now commute with the map $\mathrm{A}^n(X) \longrightarrow \mathrm{D}^n(X)$. Note that for non-smooth forms α, we have in general $[\mathrm{d}\alpha] \neq \mathrm{d}[\alpha]$. By Stokes' theorem, we obtain the equality $\mathrm{d}\delta_Y = \delta_{\partial Y}$, where ∂Y denotes the boundary of Y. For both, forms and currents, we will from now on denote

$$\mathrm{d} = \partial + \overline{\partial}, \qquad \mathrm{d}^c = \frac{i}{4\pi}(\overline{\partial} - \partial),$$

and hence

$$\mathrm{dd}^c = \frac{i}{2\pi}\partial\overline{\partial}.$$

Definition 2.1.14. We define the quotients

$$\begin{aligned}\widetilde{\mathrm{A}}^{p,q}(X) &:= \mathrm{A}^{p,q}(X)/(\partial\mathrm{A}^{p-1,q}(X) + \overline{\partial}\mathrm{A}^{p,q-1}(X)),\\ \widetilde{\mathrm{D}}^{p,q}(X) &:= \mathrm{D}^{p,q}(X)/(\partial\mathrm{D}^{p-1,q}(X) + \overline{\partial}\mathrm{D}^{p,q-1}(X)).\end{aligned}$$

By [19], Theorem 1.2.2, any smooth current ω of the form $\omega = \partial u + \overline{\partial} v$, with u, v currents, can be represented as $\omega = \partial\alpha + \overline{\partial}\beta$, with α, β smooth currents. Therefore, the natural map $\widetilde{\mathrm{A}}^{p,q}(X) \longrightarrow \widetilde{\mathrm{D}}^{p,q}(X)$ is an injection. By construction, the homomorphism $\partial\overline{\partial}$ on forms and currents factors through $\widetilde{\mathrm{A}}^{p,q}(X)$ and $\widetilde{\mathrm{D}}^{p,q}(X)$, respectively. The kernel of $\partial\overline{\partial}\colon \widetilde{\mathrm{D}}^{p,q}(X) \longrightarrow \mathrm{D}^{p+1,q+1}(X)$ is contained in $\widetilde{\mathrm{A}}^{p,q}(X)$, as, again by [19], Theorem 1.2.2, a current γ with $\partial\overline{\partial}\gamma$ smooth can be represented in $\widetilde{\mathrm{D}}^{p,q}(X)$ by a current corresponding to a smooth form.

Definition 2.1.15. We define two subspaces of $\mathrm{A}^{p,p}(X)$ and $\mathrm{D}^{p,p}(X)$ by

$$\begin{aligned}\mathrm{A}^{p,p}(X_{\mathbb{R}}) &:= \{\alpha \in \mathrm{A}^{p,p}(X) \mid \alpha \text{ real}, F_\infty^*\alpha = (-1)^p\alpha\},\\ \mathrm{D}^{p,p}(X_{\mathbb{R}}) &:= \{T \in \mathrm{D}^{p,p}(X) \mid T \text{ real}, F_\infty^*T = (-1)^pT\}.\end{aligned}$$

They correspond to the smooth forms and currents on the real variety $X_{\mathbb{R}} = (X, F_\infty)$, respectively.

Definition 2.1.16. Let $Y = \sum_j n_jY_j$ be a p-codimensional analytic cycle on the complex manifold X. A *Green current for* Y is an element $g_Y \in \widetilde{\mathrm{D}}^{p-1,p-1}(X)$ which is the class of a current in $\mathrm{D}^{p-1,p-1}(X_{\mathbb{R}})$, such that there exists a smooth (p,p)-form ω with

$$\mathrm{dd}^c g_Y + \delta_Y = [\omega]$$

in $\mathrm{D}^{p,p}(X)$. Note that $\mathrm{dd}^c g_Y$ is independent of the choice of a representative of g_Y in $\mathrm{D}^{p-1,p-1}(X_{\mathbb{R}})$.
Let $|Y|$ be the support of Y in X. A *Green form for* Y is defined to be a form $g_Y \in \mathrm{L}^1_{\mathrm{loc}}\big(X, \Omega_X^{p-1,p-1}\big)$, smooth on $X \setminus |Y|$, such that the class of $[g_Y]$ in $\widetilde{\mathrm{D}}^{p-1,p-1}(X)$ is a Green current for Y.

Remark 2.1.17. To simplify notation, we denote a representative of a Green current $g_Y \in \widetilde{\mathrm{D}}^{p-1,p-1}(X)$ in $\mathrm{D}^{p-1,p-1}(X)$ again by g_Y. We also denote both Green forms and Green currents for a cycle Y by g_Y. If a Green current comes from a Green form g_Y, we write it as $[g_Y]$. Later, we will see that, in our situation, it is not necessary to distinguish between the two, as all Green currents come from Green forms.

Lemma 2.1.18. *If X is compact and projective, Green currents for cycles Y on X always exist.*

Proof. By [13], Theorem 1.4, the d-, ∂-, and $\overline{\partial}$-cohomology of currents on a complex manifold X is the same as the d-, ∂-, and $\overline{\partial}$-cohomology of smooth forms on X. As $\mathrm{d}\delta_Y = 0$ by Stokes' theorem, there exists a smooth real (p,p)-form ω representing the cohomology class of δ_Y. Therefore, the current $[\omega] - \delta_Y$ is d-exact. By the assumptions, X is Kähler, and the $\partial\overline{\partial}$-Lemma can be applied, see, e.g., [19], Theorem 1.2.1. Hence, there exists a current $g_Y \in \mathrm{D}^{p-1,p-1}$ such that $\frac{i}{2\pi}\partial\overline{\partial}g_Y = \mathrm{dd}^c g_Y = [\omega] - \delta_Y$. ☐

Lemma 2.1.19. *Any two Green currents for a p-codimensional analytic cycle Y on X differ by an element in the image of $\widetilde{\mathrm{A}}^{p-1,p-1}(X)$ in $\widetilde{\mathrm{D}}^{p-1,p-1}(X)$.*

Proof. Let g_1 and g_2 be Green currents for Y, hence, $\mathrm{dd}^c g_j = [\omega_j] - \delta_Y$ $(j = 1, 2)$. Then, $\mathrm{dd}^c(g_1 - g_2) = [\omega_1] - [\omega_2]$ is a smooth current. By [19], Theorem 1.2.2, $g_1 - g_2$ can be written as $[\omega] + \partial\alpha + \overline{\partial}\beta$, with α, β currents and ω a smooth form. Therefore, $g_1 - g_2$ lies in the image of $\widetilde{\mathrm{A}}^{p-1,p-1}(X)$ in $\widetilde{\mathrm{D}}^{p-1,p-1}(X)$. ☐

For a Hermitian line bundle on the compact complex manifold X, Green currents can be obtained in a natural way.

Theorem 2.1.20. *Let L be a holomorphic line bundle on X, and let $\|\cdot\|$ be a smooth Hermitian metric on L. For a non-zero meromorphic section s of L, we will consider the divisor $Y = \mathrm{div}(s)$ corresponding to L. A Green form g_Y for Y is explicitely given by $g_Y = -\log\|s\|^2$.*

Proof. The function $\log\|s\|^2$ is locally integrable on X and hence induces a current on X. By definition of the first Chern class of a metrized line bundle, we have

$$[-\mathrm{dd}^c \log\|s\|^2] = [\mathrm{c}_1(L, \|\cdot\|)]$$

outside Y. By the Poincaré–Lelong formula (see, e.g., [21], p. 388), we obtain the equality

$$\mathrm{dd}^c[-\log\|s\|^2] = [\mathrm{c}_1(L, \|\cdot\|)] - \delta_Y,$$

which proves the claim ☐

We now recall the notion of Chow groups of Noetherian schemes.

Definition 2.1.21. Let $\mathscr{X}$ be a Noetherian scheme. We denote by $\mathrm{Z}^p(\mathscr{X})$ the group of p-codimensional cycles on $\mathscr{X}$, i.e., the free abelian group generated by the p-codimensional integral subschemes of $\mathscr{X}$. For a $(p-1)$-codimensional subscheme $\mathscr{Y}$ with generic point y and f in the function field $k(y)^\times$, we obtain the p-codimensional cycle

$$\operatorname{div}(f) = \sum_{\mathscr{V}} \operatorname{ord}_{\mathscr{V}}(f)\mathscr{V},$$

where the sum runs over all p-codimensional integral subschemes $\mathscr{V} = \overline{\{x\}}$ with $x \in \mathscr{X}^{(p)} \cap \mathscr{Y}$, and $\operatorname{ord}_{\mathscr{V}}(f)$ denotes the order of f in the local ring $\mathcal{O}_{\mathscr{Y},x}$, which is in fact a discrete valuation ring. Let $\mathrm{Rat}^p(\mathscr{X}) \subseteq \mathrm{Z}^p(\mathscr{X})$ be the subgroup generated by all cycles of the form $\operatorname{div}(f)$. The quotient

$$\mathrm{CH}^p(\mathscr{X}) := \mathrm{Z}^p(\mathscr{X})/\mathrm{Rat}^p(\mathscr{X})$$

is called the *p-th Chow group of* $\mathscr{X}$.

Definition 2.1.22. Let $\mathscr{Y}$ be a closed subscheme of $\mathscr{X}$ with support $|\mathscr{Y}|$. We write $\mathrm{Z}^p_{\mathscr{Y}}(\mathscr{X})$ for the group of cycles with support in $|\mathscr{Y}|$, and $\mathrm{Rat}^p_{\mathscr{Y}}(\mathscr{X}) \subseteq \mathrm{Z}^p_{\mathscr{Y}}(\mathscr{X})$ for the subgroup generated by cycles of the form $\operatorname{div}(f)$, where f lies in the function field $k(y)^\times$ for some $y \in \mathscr{X}^{(p-1)} \cap \mathscr{Y}$. The quotient

$$\mathrm{CH}^p_{\mathscr{Y}}(\mathscr{X}) := \mathrm{Z}^p_{\mathscr{Y}}(\mathscr{X})/\mathrm{Rat}^p_{\mathscr{Y}}(\mathscr{X})$$

is called the *p-th Chow group with support in* $\mathscr{Y}$.

In the following, let $\mathscr{X}$ be an arithmetic variety. We are now able to define the arithmetic Chow groups of $\mathscr{X}$.

Definition 2.1.23. Let $\widehat{\mathrm{Z}}^p(\mathscr{X}) \subseteq \mathrm{Z}^p(\mathscr{X}) \oplus \widetilde{\mathrm{D}}^{p-1,p-1}(X_{\mathbb{R}})$ be the subgroup consisting of pairs of the form $(\mathscr{Z}, g_Z)$, where $\mathscr{Z}$ is a p-codimensional cycle on X and g_Z is a Green current for Z. We will define the subgroup $\widehat{\mathrm{Rat}}^p(\mathscr{X})$ extending $\mathrm{Rat}^p(\mathscr{X})$ as follows. For $\iota\colon \mathscr{Y} \hookrightarrow \mathscr{X}$ a $(p-1)$-codimensional integral subscheme, we denote by $\pi\colon \widetilde{Y} \longrightarrow Y$ a proper resolution of singularities of Y. A function $f \in k(\mathscr{Y})^\times$ restricts to a rational function $\widetilde{f}$ on $\widetilde{Y}$. As the function $\log|\widetilde{f}|^2$ is real valued and integrable on $\widetilde{Y}$, it defines a current in $\mathrm{D}^{0,0}(\widetilde{Y}_{\mathbb{R}})$, and we obtain a current $\iota_*\pi_*[\log|\widetilde{f}|^2] \in \mathrm{D}^{p-1,p-1}(X_{\mathbb{R}})$ by pushforward. This current is independent of the choice of the resolution of singularities and will therefore be denoted by $\iota_*[\log|f|^2]$. By abuse of notation, its class in $\widetilde{\mathrm{D}}^{p-1,p-1}(X_{\mathbb{R}})$ will also be denoted by $\iota_*[\log|f|^2]$. By the Poincaré–Lelong lemma, the pair

$$\widehat{\operatorname{div}}(f) := (\operatorname{div}(f), -\iota_*[\log|f|^2])$$

is an element of $\widehat{\mathrm{Z}}^p(\mathscr{X})$. The subgroup of $\widehat{\mathrm{Z}}^p(\mathscr{X})$ generated by all elements of the form $\widehat{\operatorname{div}}(f)$ will be denoted by $\widehat{\mathrm{Rat}}^p(\mathscr{X})$.

Definition 2.1.24. The *p-th arithmetic Chow group* is defined to be the quotient

$$\widehat{\mathrm{CH}}^p(\mathscr{X}) := \widehat{\mathrm{Z}}^p(\mathscr{X})/\widehat{\mathrm{Rat}}^p(\mathscr{X}).$$

The class of a pair $(\mathscr{Z}, g_Z) \in \widehat{\mathrm{Z}}^p(\mathscr{X})$ in $\widehat{\mathrm{CH}}^p(\mathscr{X})$ will be denoted by $[\mathscr{Z}, g_Z]$.

Proposition 2.1.25. *There is an isomorphism*

$$\widehat{\mathrm{c}}_1 \colon \widehat{\mathrm{Pic}}(\mathscr{X}) \longrightarrow \widehat{\mathrm{CH}}^1(\mathscr{X}),$$

given by sending the class $[\mathscr{L}, \|\cdot\|] \in \widehat{\mathrm{Pic}}(\mathscr{X})$ *to the class*

$$\widehat{\mathrm{c}}_1(\mathscr{L}, \|\cdot\|) := [\mathrm{div}(s), [-\log \|s\|^2]] \in \widehat{\mathrm{CH}}^1(\mathscr{X}).$$

Here, s denotes a non-zero rational section of $\mathscr{L}$ as well as the induced section of the complex bundle L on X.

Proof. The map is well-defined, as two sections of $\mathscr{L}$ differ by a rational function on $\mathscr{X}$. Its inverse is given by the assignment $[\mathscr{Z}, g_Z] \mapsto [\mathcal{O}_{\mathscr{X}}(\mathscr{Z}), \|\cdot\|]$, where $\|\cdot\|$ is locally defined as $\|f\| := |f|e^{-g_Z}$ for a section f of $\mathcal{O}_X(Z)$, see [44], III.4, Proposition 1. $\square$

2.2 The arithmetic intersection product

To equip the arithmetic Chow groups with the structure of a ring, it is necessary to define a product structure on Green currents that is compatible with the intersection product of the corresponding algebraic cycles. In order to define this so-called $*$-product, we will first restrict ourselves to the case of currents coming from forms of logarithmic type, certain well-behaved and hence integrable forms that allow the definition of a product structure, and then extend this structure to arbitrary Green currents by representing them by forms of logarithmic type.

Definition 2.2.1. Let X be a smooth projective complex variety, and let Y be a subvariety of X which does not contain any irreducible components of X. A form $\alpha \in \mathrm{A}^n(X \setminus Y)$ is *of logarithmic type* along Y if there exist a projective morphism $\pi \colon \widetilde{X} \longrightarrow X$ (obtained as an embedded resolution of singularities of Y) and a smooth form $\beta \in \mathrm{A}^n\big(\widetilde{X} \setminus \pi^{-1}(Y)\big)$ with the following properties:

(i) The preimage $\pi^{-1}(Y)$ is a normal crossing divisor, and the morphism $\pi \colon \widetilde{X} \setminus \pi^{-1}(Y) \longrightarrow X \setminus Y$ is smooth.

(ii) The form α is the direct image of β under π, and for $z_1 \cdot \ldots \cdot z_k = 0$ a local equation of $\pi^{-1}(Y)$, β is of the form

$$\beta = \sum_{j=1}^{k} \alpha_j \log |z_j|^2 + \gamma,$$

where the α_j are ∂- and $\overline{\partial}$- closed smooth forms and γ is a smooth form.

Remark 2.2.2. If a form α is of logarithmic type along Y, it is locally integrable on X, and therefore induces a current $[\alpha]$ on X which equals $\pi_*[\beta]$.

Example 2.2.3. Let $(L, \|\cdot\|)$ be a metrized line bundle on X. If $Y = \operatorname{div}(s)$ for a non-zero meromorphic section s of L, the Green form $-\log\|s\|^2$ for Y defined in Theorem 2.1.20 is of logarithmic type.

Theorem 2.2.4. *Let X be a smooth projective complex variety and $Y \subseteq X$ a closed irreducible p-codimensional subvariety . Then, there exists a Green form of logarithmic type for Y.*

Proof. See [19], Theorem 1.3.5. □

Remark 2.2.5. Let X, Y be as in Theorem 2.2.4, and let g_Y be any Green current for Y. By the previous theorem, there exists a Green form $\widetilde{g}_Y$ of logarithmic type for Y. By Lemma 2.1.19, the difference $g_Y - [\widetilde{g}_Y]$ is a smooth current, and without loss of generality we can assume $g_Y = [\widetilde{g}_Y]$ in $\widetilde{\mathrm{D}}^{p-1,p-1}(X)$.

Lemma 2.2.6. *Let X be a smooth projective complex variety and $Y \subseteq X$ a closed subvariety of X which does not contain any irreducible component of X. Then, the property of being of logarithmic type along Y is preserved under pull-back and push-forward in the following sense:*

(i) Let $f\colon X' \longrightarrow X$ be a morphism of smooth projective varieties and let α be a form of logarithmic type along Y. If $f^{-1}(Y)$ does not contain any components of X', then $f^*\alpha$ is of logarithmic type along $f^{-1}(Y)$.

(ii) Let $f\colon X \longrightarrow X'$ be a morphism of smooth projective varieties and let α be a form of logarithmic type along Y. If f is smooth outside Y, and $f(Y)$ does not contain any component of X', then $f_*\alpha$ is of logarithmic type along $f(Y)$. Furthermore, the equality $f_*[\alpha] = [f_*\alpha]$ of currents holds.

Proof. See [19], Lemma 1.3.3. □

Definition 2.2.7. Let $Y, Z \subseteq X$ be closed irreducible subvarieties such that $Z \not\subseteq Y$, and let g_Y be a Green form of logarithmic type for Y. Let $\pi\colon \widetilde{Z} \longrightarrow Z$ be a resolution of singularities of Z, and let $q\colon \widetilde{Z} \longrightarrow X$ be its composite with the inclusion $\iota\colon Z \longrightarrow X$. By the previous Lemma 2.2.6, q^*g_Y is of logarithmic type along $q^{-1}(Y)$ in $\widetilde{Z}$, and hence integrable. So by defining

$$g_Y \wedge \delta_Z := q_*[q^*g_Y],$$

we obtain a current on X. Note that for a birational morphism $f\colon X' \longrightarrow X$ and a locally integrable form α on X, the equality of currents $f_*[f^*\alpha] = [\alpha]$ holds. Therefore, the definition of $g_Y \wedge \delta_Z$ does not depend on the choice of the resolution of singularities π.

Definition 2.2.8. Let $Y, Z \subseteq X$ be closed irreducible p- and q-codimensional subvarieties, respectively, such that $Z \not\subseteq Y$. Let g_Y be a Green form of logarithmic type for Y and g_Z be any Green current for Z. We define the *$*$-product of the currents $[g_Y]$ and g_Z* to be

$$[g_Y] * g_Z := g_Y \wedge \delta_Z + [\omega_Y] \wedge g_Z \in \widetilde{\mathrm{D}}^{p+q-1,p+q-1}(X).$$

Definition 2.2.9. Let $\mathscr{X}$ be a regular scheme, denote by $\mathscr{X}^{(p)}$ the set of p-codimensional points of $\mathscr{X}$, and let $\mathscr{Y}$, $\mathscr{Z}$ be p- and q-codimensional cycles on $\mathscr{X}$, respectively. Serre [42] defined the *intersection multiplicity at a generic point x* of $\mathscr{Y} \cap \mathscr{Z}$ to be

$$\chi^x(\mathscr{Y}, \mathscr{Z}) := \sum_{j \geq 0} (-1)^j \ell_{\mathcal{O}_{\mathscr{X},x}} \mathrm{Tor}_j^{\mathcal{O}_{\mathscr{X},x}}(\mathcal{O}_{\mathscr{Y},x}, \mathcal{O}_{\mathscr{Z},x}).$$

If $\mathscr{Y}$ and $\mathscr{Z}$ intersect properly, their intersection product turns out to be given by the formula

$$\mathscr{Y} \cdot \mathscr{Z} = \sum_x \chi^x(\mathscr{Y}, \mathscr{Z}) \overline{\{x\}} \quad (x \in \mathscr{Y} \cap \mathscr{Z} \cap \mathscr{X}^{(p+q)}).$$

Theorem 2.2.10. *Let X be a smooth projective complex variety and let Y, Z be p- and q-codimensional cycles on X, respectively. If Y and Z intersect properly, the equality*

$$\mathrm{dd}^c\left([g_Y] * g_Z\right) + \sum_x \chi^x(X, Y) \delta_{\overline{\{x\}}} = [\omega_Y \wedge \omega_Z] \quad (x \in Y \cap Z \cap X^{(p+q)}),$$

with $\chi^x(X, Y)$ Serre's intersection multiplicity defined above, holds. Hence, $[g_Y] * g_Z$ is a Green current for $Y \cdot Z$.

Proof. See [44], II.3, Theorem 4. □

Definition 2.2.11. Let $Y, Z \subseteq X$ be closed irreducible subvarieties with $Z \not\subseteq Y$, and let g_Y, g_Z be any two Green currents for Y and Z, respectively. We define their $*$-product in the following way: By Remark 2.2.5, we find a Green form $\widetilde{g}_Y$ of logarithmic type for Y such that $g_Y = [\widetilde{g}_Y]$ in $\widetilde{\mathrm{D}}^{p-1,p-1}$. We set

$$g_Y * g_Z := [\widetilde{g}_Y] * g_Z.$$

One shows that this definition does not depend on the choice of $\widetilde{g}_Y$.

Theorem 2.2.12. *The $*$-product is commutative and associative.*

Proof. See [19], Section 2.2. □

Theorem 2.2.13. *Let $\mathscr{X}$ be a regular scheme, and let $\mathscr{Y}$, $\mathscr{Z}$ be closed subschemes of $\mathscr{X}$. Then, there exists a pairing*

$$\mathrm{CH}^p_{\mathscr{Y}}(\mathscr{X})_{\mathbb{Q}} \otimes \mathrm{CH}^q_{\mathscr{Z}}(\mathscr{X})_{\mathbb{Q}} \longrightarrow \mathrm{CH}^{p+q}_{\mathscr{Y} \cap \mathscr{Z}}(\mathscr{X})_{\mathbb{Q}}$$

that makes $\bigoplus_{p,\mathscr{Y}} \mathrm{CH}^p_{\mathscr{Y}}(\mathscr{X})_{\mathbb{Q}}$ *into a commutative ring with unit element* $[\mathscr{X}] \in \mathrm{CH}^0(\mathscr{X})_{\mathbb{Q}}$*. Furthermore, it commutes with the change of supports induced by the inclusion of cycles* $\mathscr{Y}' \subseteq \mathscr{Y}$ *and* $\mathscr{Z}' \subseteq \mathscr{Z}$*, respectively, and for classes* $[\mathscr{Y}_1] \in \mathrm{CH}^p_{\mathscr{Y}}(\mathscr{X})$ *and* $[\mathscr{Z}_1] \in \mathrm{CH}^q_{\mathscr{Z}}(\mathscr{X})$*, with* $\mathscr{Y}_1, \mathscr{Z}_1$ *intersecting properly, it coincides with the expected formula*

$$[\mathscr{Y}_1] \cdot [\mathscr{Z}_1] = \sum_{x \in \mathscr{Y}_1 \cap \mathscr{Z}_1 \cap \mathscr{X}^{(p+q)}} \chi^x(\mathscr{Y}_1, \mathscr{Z}_1)[\overline{\{x\}}],$$

where $\chi^x(\mathscr{Y}, \mathscr{Z})$ *denotes Serre's intersection multiplicity.*

Proof. Let $\mathrm{K}_0^{\mathscr{Y}}(\mathscr{X})$ be the Grothendieck group of the category of bounded complexes of locally free coherent $\mathcal{O}_{\mathscr{X}}$-modules acyclic outside $\mathscr{Y}$. The rational Chow group $\mathrm{CH}^p_{\mathscr{Y}}(\mathscr{X})_{\mathbb{Q}}$ can the be identified with the graded group $\mathrm{Gr}^p_\gamma \mathrm{K}_0^{\mathscr{Y}}(\mathscr{X})_{\mathbb{Q}}$ associated to the filtration on K-theory with supports, see [20]. There is a graded ring structure on $\bigoplus_{\mathscr{Y}} \mathrm{Gr}^*_\gamma \mathrm{K}_0^{\mathscr{Y}}(\mathscr{X})_{\mathbb{Q}}$ induced by the cup product, i.e., the tensor product of complexes on $\mathrm{K}_0^{\mathscr{Y}}(\mathscr{X})_{\mathbb{Q}}$. This gives a product structure on $\bigoplus_{\mathscr{Y}} \mathrm{CH}^p_{\mathscr{Y}}(\mathscr{X})_{\mathbb{Q}}$, fulfilling the required properties. A complete proof can be found in [20], implicitely using a moving lemma for K_1-chains later proven in [24]. □

Theorem 2.2.14. *Let* $\mathscr{X}$ *be an arithmetic variety. There is a pairing*

$$\widehat{\mathrm{CH}}^p(\mathscr{X}) \otimes \widehat{\mathrm{CH}}^q(\mathscr{X}) \longrightarrow \widehat{\mathrm{CH}}^{p+q}(\mathscr{X})_{\mathbb{Q}}$$

that induces a ring structure on

$$\widehat{\mathrm{CH}}^*(\mathscr{X})_{\mathbb{Q}} := \bigoplus_{p \geq 0} \widehat{\mathrm{CH}}^p(\mathscr{X})_{\mathbb{Q}}.$$

For $[\mathscr{Y}] \in \mathrm{CH}^p(\mathscr{X})$*,* $[\mathscr{Z}] \in \mathrm{CH}^q(\mathscr{X})$ *such that* $\mathscr{Y}$ *and* $\mathscr{X}$ *intersect properly in the generic fibre of* $\mathscr{X}$*, it is given by the formula*

$$[\mathscr{Y}, g_Y] \cdot [\mathscr{Z}, g_Z] := [\mathscr{Y} \cdot \mathscr{Z}, g_Y * g_Z].$$

Proof. If the intersection in the generic fibre is not proper, there are moving lemmas for K_1-chains available, see [44], III.2, Theorem 2. □

With the intersection product at hand, we can now define the notion of an arithmetic degree for a metrized line bundle.

Proposition 2.2.15. *Consider the case* $\mathscr{X} = \mathrm{Spec}(\mathcal{O}_F)$*, where* $\mathcal{O}_F$ *is the ring of integers of a number field* F*, and* Σ *is the set of embeddings of* $\mathcal{O}_F$ *into* $\mathbb{C}$*. Then, there is a homomorphism*

$$\widehat{\deg}' : \widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathcal{O}_F)) \longrightarrow \mathbb{R}.$$

The image of the class $[\mathscr{Z}, g_Z] \in \widehat{\mathrm{CH}}^1(\mathrm{Spec}(\mathcal{O}_F))$*, where* $\mathscr{Z} = \sum_{\wp} n_{\wp}[\wp] \in \mathrm{CH}^1(\mathrm{Spec}(\mathcal{O}_F))$*,* $\wp \subseteq \mathcal{O}_F$ *a prime ideal, and* $g_Z = \{g_{Z,\sigma}\}_{\sigma\in\Sigma}$ *with* $g_{Z,\sigma} \in \mathbb{R}$*, is given by the real number*

$$\widehat{\deg}'(\mathscr{Z}, g_Z) := \sum_{\wp} n_{\wp} \log(\#\mathcal{O}_F/\wp) + \frac{1}{2}\int\limits_X g_Z = \sum_{\wp} n_{\wp} \log(\#\mathcal{O}_F/\wp) + \frac{1}{2}\sum_{\sigma\in\Sigma} g_{Z,\sigma}.$$

In the case $\mathcal{O}_F = \mathbb{Z}$*, the map* $\widehat{\deg}'$ *is an isomorphism.*

Proof. See [19], 3.4.3. □

Theorem 2.2.16. *Let* $\mathscr{X}$*,* $\mathscr{Y}$ *be regular equidimensional projective schemes with* $\delta = \dim(\mathscr{X}) - \dim(\mathscr{Y})$*. Let* $f\colon \mathscr{X} \longrightarrow \mathscr{Y}$ *be a proper morphism such that* $f\colon \mathscr{X}_{\mathbb{Q}} \longrightarrow \mathscr{Y}_{\mathbb{Q}}$ *is smooth. Then, there is a push-forward homomorphism*

$$f_*\colon \widehat{\mathrm{CH}}^p(\mathscr{X}) \longrightarrow \widehat{\mathrm{CH}}^{p-\delta}(\mathscr{Y}),$$

which is defined below.

Proof. The push-forward of a class $[\mathscr{Z}, g_Z] \in \widehat{\mathrm{CH}}^p(\mathscr{X})$ is defined as follows. If the cycle $\mathscr{Z}$ is irreducible with generic point z, $f_*\mathscr{Z}$ is given by $[k(z) : k(f(z))]\overline{\{f(z)\}}$ if $\dim(z) = \dim(f(z))$, and 0 otherwise. The push-forward f_*g_Z of the current g_Z is a Green current for f_*Z as it fulfills

$$\mathrm{dd}^c f_*g_Z + \delta_{f_*Z} = [f_*\omega_Z],$$

where the push-forward of the smooth form ω_Z is obtained by integrating along the fibres of f. See [44], III.3, Theorem 3 or [19], Theorem 3.6.1, for details. □

We are now able to define the arithmetic degree of the class of a metrized line bundle $[\mathscr{L}, \|\cdot\|] \in \widehat{\mathrm{Pic}}(\mathscr{X})$.

Definition 2.2.17. Let $[\mathscr{L}, \|\cdot\|] \in \widehat{\mathrm{Pic}}(\mathscr{X})$ be the class of a Hermitian line bundle on the arithmetic variety $\pi\colon \mathscr{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_F)$ of relative dimension d, and let

$$\widehat{\mathrm{c}}_1(\mathscr{L}, \|\cdot\|) = [\mathrm{div}(s), [-\log\|s\|^2]] \in \widehat{\mathrm{CH}}^1(\mathscr{X}),$$

be its first Chern form, with s denoting a non-zero rational section of $\mathscr{L}$ and the induced section of the complex bundle, respectively. Then, the *arithmetic degree* of $[\mathscr{L}, \|\cdot\|]$ is defined as

$$\widehat{\deg}(\mathscr{L}, \|\cdot\|) := \widehat{\deg}'\left(\pi_*\left(\widehat{\mathrm{c}}_1(\mathscr{L}, \|\cdot\|)^{d+1}\right)\right) \in \mathbb{R}.$$

Remark 2.2.18. With the notation of the above definition, assuming that

$$\widehat{\mathrm{c}}_1(\mathscr{L}, \|\cdot\|)^{d+1} = [\mathscr{Z}, g_Z] \in \widehat{\mathrm{CH}}^{d+1}(\mathscr{X}),$$

with $\mathscr{Z} = \sum_{\mathcal{P}\in\mathscr{X}} n_{\mathcal{P}}[\mathcal{P}]$, we have

$$\widehat{\deg}(\mathscr{L}, \|\cdot\|) = \sum_{\mathcal{P}\in\mathscr{X}} n_{\mathcal{P}} \log(\#\mathcal{O}_F/\pi(\mathcal{P})) + \frac{1}{2}\int\limits_X g_Z.$$

2.3 Arithmetic Chow groups associated to Gillet complexes

As the Petersson metric $\| \cdot \|_{\mathrm{Pet}}$ on the line bundle of modular forms $\mathcal{M}_k(\Gamma_g)$ on $\overline{\mathcal{A}}_g$, defined in Chapter 1, Definitions 1.3.2 and 1.3.3, has logarithmic singularities at the boundary divisor for any toroidal compactification $\overline{\mathcal{A}}_g$ of $\mathcal{A}_g$, methods of the arithmetic intersection theory introduced in the previous section cannot be applied immediately. Burgos, Kramer, and Kühn, [9] tackled this problem by choosing a more abstract approach to arithmetic Chow groups. They introduced the additional datum of an auxiliary complex to the definition that allows to adjust the intersection theory to more general situations, in particular to the one of line bundles with log-singular metrics. In the following section, we will introduce these Chow groups.

We will first recall some notions of complexes from [9].

Definition 2.3.1. Let R be a commutative ring and $k \in \mathbb{N}$. A *k-complex* $A = (A^{*,\dots,*}, \mathrm{d}_1, \dots, \mathrm{d}_k)$ is a k-graded R-module A together with k endomorphisms $\mathrm{d}_1, \dots, \mathrm{d}_k$ of multi-degrees $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ respectively, such that the equalities $\mathrm{d}_j^2 = 0$ and $\mathrm{d}_j \mathrm{d}_l = -\mathrm{d}_l \mathrm{d}_j$ $(j, l = 1, \dots, k,\ j \neq k)$ hold. If the second equality is replaced by $\mathrm{d}_j \mathrm{d}_l = \mathrm{d}_l \mathrm{d}_j$, we call A a *k-iterated complex.* The k-complex $\mathcal{C}_k(A)$ associated to a k-iterated complex $A = (A^{n_1,\dots,n_k}, \mathrm{d}_1, \dots, \mathrm{d}_k)$ is given by $\mathcal{C}_k(A) := (A^{n_1,\dots,n_k}, \mathrm{d}_1, (-1)^{n_1}\mathrm{d}_2, \dots, (-1)^{n_1+\dots+n_{k-1}}\mathrm{d}_k)$.

Definition 2.3.2. Let $A = (A^{*,\dots,*}, \mathrm{d}_1, \dots, \mathrm{d}_k)$ be a k-complex. The *associated simple complex* $s(A)$ is defined by

$$s(A)^n := \bigoplus_{n_1+\dots+n_k=n} A^{n_1,\dots,n_k}, \quad \mathrm{d} := \sum_{j=1}^{k} \mathrm{d}_j.$$

For a k-iterated complex A, the associated simple complex is defined as $s(A) := s(\mathcal{C}_k(A))$.

Example 2.3.3. Let $f \colon (A^*, \mathrm{d}_A) \longrightarrow (B^*, \mathrm{d}_B)$ be a morphism of complexes. We can define an associated 2-iterated complex $(f^{*,*}, d_1, d_2)$ by setting

$$f^{0,q} := A^q, \quad f^{1,q} := B^q, \quad \mathrm{d}_1 := f, \quad \mathrm{d}_2\big|_{f^{0,*}} := \mathrm{d}_A, \quad \mathrm{d}_2\big|_{f^{1,*}} := \mathrm{d}_B.$$

The corresponding simple complex $s(f)$ is then given by

$$s(f)^n = A^n \oplus B^{n-1}, \quad \mathrm{d}(a, b) = (\mathrm{d}_A a, f(a) - \mathrm{d}_B b).$$

Definition 2.3.4. For $k \in \mathbb{N}$, the *shifted complex* $A[k]$ of a complex (A^*, d_A) is defined by $A[k]^n := A^{n+k}$ with differential $\mathrm{d} := (-1)^k \mathrm{d}_A$.

Definition 2.3.5. The *mapping cone* of a morphism of complexes $f \colon (A^*, \mathrm{d}_A) \longrightarrow (B^*, \mathrm{d}_B)$ is defined as

$$\mathrm{cone}(f)^n := A^{n+1} \oplus B^n, \quad \mathrm{d}(a, b) := (-\mathrm{d}_A a, f(a) + \mathrm{d}_B b).$$

Note that $s(f) = \mathrm{cone}(-f)[-1]$.

Definition 2.3.6. Let $A = (A^*, \mathrm{d}_A)$ be a complex of R-modules. We denote by $\mathrm{Z}^n(A) := \ker(\mathrm{d}_A \colon A^n \longrightarrow A^{n+1})$ the submodule of cocyles of A^n. Furthermore, we set $\widetilde{A}^n := A^n / \operatorname{im} \mathrm{d}_A$ and write $\widetilde{a}$ for the class of $a \in A^n$ in $\widetilde{\mathrm{A}}^n$.

Definition 2.3.7. Let $f \colon (A^*, \mathrm{d}_A) \longrightarrow (B^*, \mathrm{d}_B)$ be a morphism of complexes of R-modules. The *relative cohomology groups associated to* f are defined by setting

$$\mathrm{H}^n(A, B) := \mathrm{H}^n(s(f)).$$

The cocycles of the complex $s(f)$ are the pairs $(a, b) \in s(f)^n$ satisfying the conditions $\mathrm{d}_A a = 0$ and $\mathrm{d}_B b = f(a)$. The subgroup of coboundaries is generated by the elements of the form $(\mathrm{d}_A a, f(a))$ and $(0, \mathrm{d}_B b)$. We denote the class of an element $(a, b) \in \mathrm{Z}^n(s(f))$ in $\mathrm{H}^n(A, B)$ by $[a, b]$.
The *truncated relative cohomology groups associated to* f are defined as

$$\widehat{\mathrm{H}}^n(A, B) := \mathrm{H}^n(\sigma_n A, B) = \left\{ (a, \widetilde{b}) \in \mathrm{Z}^n(A) \oplus \widetilde{B}^{n-1} \mid f(a) = \mathrm{d}_B b \right\},$$

with

$$(\sigma_p A)^n = \begin{cases} A^n, & n \geq p, \\ 0, & n < p, \end{cases}$$

the bête filtration. The class map

$$\mathrm{cl} \colon \widehat{\mathrm{H}}^n(A, B) \longrightarrow \mathrm{H}^n(A, B)$$

is defined via the assignment $(a, \widetilde{b}) \mapsto [a, b]$.

Definition 2.3.8. A *Gillet complex over* $\mathbb{C}$ is a graded complex $\mathcal{G} = (\mathcal{G}^*(*), \mathrm{d})$ of sheaves in the big Zariski site of the category of smooth schemes over $\mathbb{C}$, together with a pairing in the derived category of graded complexes of abelian sheaves that is associative, graded-commutative and has a unit. For each closed immersion $Y \longrightarrow X$ of smooth schemes, the *cohomology groups of* X *with coefficients in* $\mathcal{G}$ *and support* Y are defined to be the relative hypercohomology groups

$$\mathrm{H}^k_Y(X, \mathcal{G}(j)) = \mathbb{H}^k_Y(\overline{X}, \mathcal{G}(j)),$$

where $\overline{X}$ denotes a smooth compactification of X with $\overline{X} \setminus X$ a normal crossing divisor. A *Gillet cohomology* is a Gillet complex together with a homology theory, i.e., a covariant functor from a category of schemes and proper morphisms, containing closed immersions to smooth schemes, to the category of bigraded abelian groups $X \mapsto \bigoplus_{k \geq 0, j \in \mathbb{Z}} \mathrm{H}_k(X, \mathcal{G}(j))$ such that the cohomology and homology theory satisfy the Gillet axioms, see [18], Definition 1.2.

Definition 2.3.9. We define the groups $\mathrm{R}^p_p(X)$ and $\mathrm{R}^p_{p-1}(X)$ to be

$$\mathrm{R}^p_p(X) := \bigoplus_{x \in X^{(p)}} \mathrm{K}_0(k(x)) = \bigoplus_{x \in X^{(p)}} \mathbb{Z},$$
$$\mathrm{R}^{p-1}_p(X) := \bigoplus_{x \in X^{(p-1)}} \mathrm{K}_1(k(x)) = \bigoplus_{x \in X^{(p-1)}} k(x)^\times.$$

The group $\mathrm{R}^p_p(X)$ equals the group $\mathrm{Z}^p(X)$ of p-codimensional cycles on X. An element f in $\mathrm{R}^{p-1}_p(X)$ can be written as $f = \sum_{x \in X^{(p-1)}} f_x$ $(f_x \in k(x)^\times)$.

Remark 2.3.10. The two groups $\mathrm{R}_p^p(X)$ and $\mathrm{R}_p^{p-1}(X)$ come from the Gersten-Quillen spectral sequence in K-theory. The differential $\mathrm{d}\colon \mathrm{R}_p^{p-1}(X) \longrightarrow \mathrm{R}_p^p(X)$ is given by $\mathrm{d}f = \mathrm{div}(f) = \sum_{x\in X^{(p-1)}} \mathrm{div}(f_x)$. Quillen [39] proved the identification

$$\mathrm{CH}^p(X) = \mathrm{R}_p^p(X)/\mathrm{d}\,\mathrm{R}_p^{p-1}(X).$$

Proposition 2.3.11. *There exist well-defined morphisms*

$$\begin{aligned} \mathrm{cl}_{\mathcal{G}}\colon\ \mathrm{R}_p^p(X) &\longrightarrow \bigoplus_{x\in X^{(p)}} \mathrm{H}_x^{2p}(X,\mathcal{G}(p)),\\ \mathrm{cl}_{\mathcal{G}}\colon\ \mathrm{R}_p^{p-1}(X) &\longrightarrow \bigoplus_{x\in X^{(p-1)}} \mathrm{H}_x^{2p-1}(X,\mathcal{G}(p)), \end{aligned}$$

called the Chern characters of $\mathcal{G}$. Here, $\mathrm{H}_x^k(X,\mathcal{G}(p)) := \varinjlim_U \mathrm{H}^k_{\overline{\{x\}}\cap U}(X,\mathcal{G}(p))$, where the limit is taken over all open neighbourhoods U of x in X.

Proof. See [9], Lemma 1.49 and Definition 1.50. □

It will be useful to be able to work with more general complexes, but still have a notion of characteristic classes available that inherit the good properties of characteristic classes in Gillet cohomology. Therefore, we will define a class of complexes whose cohomology theories factor through a Gillet cohomology.

Definition 2.3.12. Let $\mathcal{G}$ be a Gillet complex. A *$\mathcal{G}$-complex over X* is a graded complex $\mathcal{C} = (\mathcal{C}^*(*), d)$ of sheaves of abelian groups over X together with a morphism

$$\mathfrak{c}_{\mathcal{C}}\colon \mathcal{G} \longrightarrow \mathcal{C}$$

in the derived category of graded complexes of sheaves of abelian groups over X such that the sheaves $\mathcal{C}^n(p)$ satisfy the Mayer–Vietoris principle for all n, p, or, equivalently, if they are totally acyclic.

Definition 2.3.13. For $U \subseteq X$ open, we will denote the sections of $\mathcal{C}^*(p)$ over U by $\mathcal{C}^*(U,p)$. The *$\mathcal{C}$-cohomology groups* of U are the cohomology groups of complexes

$$\mathrm{H}^*_{\mathcal{C}}(U,p) := \mathrm{H}^*(\mathcal{C}(U,p)).$$

For a cycle Y on X, $|Y|$ its support, and $U = X\setminus|Y|$, we define the *(truncated) relative $\mathcal{C}$-cohomology groups* to be the groups

$$\begin{aligned} \mathrm{H}^*_{\mathcal{C},Y}(X,p) &:= \mathrm{H}^*(\mathcal{C}(X,p),\mathcal{C}(U,p)),\\ \widehat{\mathrm{H}}^*_{\mathcal{C},Y}(X,p) &:= \widehat{\mathrm{H}}^*(\mathcal{C}(X,p),\mathcal{C}(U,p)), \end{aligned}$$

with $\widehat{\mathrm{H}}$ as in Definition 2.3.7.

As the sheaves $\mathcal{C}^n(p)$ are totally acyclic, and, hence, their hypercohomology coincides with the cohomology of global sections, the map $\mathfrak{c}_{\mathcal{C}}$ induces morphisms

$$\begin{aligned} \mathrm{H}^*(U, \mathcal{G}(p)) &\longrightarrow \mathrm{H}^*_{\mathcal{C}}(U, p), \\ \mathrm{H}^*_Y(X, \mathcal{G}(p)) &\longrightarrow \mathrm{H}^*_{\mathcal{C},Y}(X, p), \end{aligned}$$

which, by abuse of notation, will also be denoted by $\mathfrak{c}_{\mathcal{C}}$. We are now able to define characteristic classes in $\mathcal{C}$-cohomology.

Definition 2.3.14. Let Y be a p-codimensional cycle on X. We define the class $\mathrm{cl}_{\mathcal{C}}(Y) \in \mathrm{H}^{2p}_{\mathcal{C},Y}(X, p)$ to be

$$\mathrm{cl}_{\mathcal{C}}(Y) := \mathfrak{c}_{\mathcal{C}}(\mathrm{cl}_{\mathcal{G}}(Y)),$$

with $\mathrm{cl}_{\mathcal{G}}$ as in Proposition 2.3.11. For $f \in \mathrm{R}^{p-1}_p(X)$, $Y = \mathrm{div}(f)$, $|Y|$ the support of Y, and $U = X \setminus |Y|$, we define $\mathrm{cl}_{\mathcal{C}}(f) \in \mathrm{H}^{2p-1}_{\mathcal{C}}(U, p)$ to be the class

$$\mathrm{cl}_{\mathcal{C}}(f) := \mathfrak{c}_{\mathcal{C}}(\mathrm{cl}_{\mathcal{G}}(f)).$$

We can now generalize the notion of Green currents for a cycle Y in terms of relative truncated cohomology as follows.

Definition 2.3.15. Let Y be a p-codimensional cycle on X with support $|Y|$. A *Green object for* Y is an element $\mathfrak{g}_Y \in \widehat{\mathrm{H}}^{2p}_{\mathcal{C},Y}(X, p)$ such that

$$\mathrm{cl}(\mathfrak{g}_Y) = \mathrm{cl}_{\mathcal{C}}(Y) \in \mathrm{H}^{2p}_{\mathcal{C},Y}(X, p),$$

with cl the class map as in Definition 2.3.7. By the same Definition 2.3.7, $\mathfrak{g}_Y$ is given by a pair $(\omega_Y, \widetilde{g}_Y)$, with $\omega_Y \in \mathrm{Z}\mathcal{C}^{2p}(X, p)$ and $\widetilde{g}_Y \in \widetilde{\mathcal{C}}^{2p-1}(X \setminus |Y|, p)$ satisfying $\omega_Y = \mathrm{d}g_Y$ for g_Y a representative of $\widetilde{g}_Y$ in $\mathcal{C}^{2p-1}(X \setminus |Y|, p)$, and such that $\mathfrak{g}_Y$ represents the class of Y in $\mathrm{H}^{2p}_{\mathcal{C},Y}(X, p)$.

Proposition 2.3.16. *Let $\mathcal{C}$ be a $\mathcal{G}$-complex on X, and let Y be a p-codimensional cycle on X. Then, there exists a Green object $\mathfrak{g}_Y \in \widehat{\mathrm{H}}^{2p}_{\mathcal{C},Y}(X, p)$ for Y.*

Proof. See [9], Proposition 3.23. □

Definition 2.3.17. Let $f \in \mathrm{R}^{p-1}_p(X)$, set $Y = \mathrm{div}(f)$, and let $|Y|$ be the support of Y. Let

$$\mathrm{b}\colon \mathrm{H}^{n-1}_{\mathcal{C}}(X \setminus |Y|, p) \longrightarrow \widehat{\mathrm{H}}^{n}_{\mathcal{C},Y}(X, p)$$

be the map to relative truncated cohomology, sending $[b]$ to $(0, -\widetilde{b})$. We denote by $\mathfrak{g}(f)$ the Green object

$$\mathfrak{g}(f) := \mathrm{b}(\mathrm{cl}_{\mathcal{C}}(f)) \in \widehat{\mathrm{H}}^{2p}_{\mathcal{C},Y}(X, p).$$

Lemma 2.3.18. *The element $\mathfrak{g}(f)$ is a well-defined Green object for the cycle $Y = \mathrm{div}(f)$.*

Proof. The lemma follows from the fact that the characteristic classes $\mathrm{cl}_{\mathcal{C}}$ inherit their properties from the characteristic classes $\mathrm{cl}_{\mathcal{G}}$. Denote by

$$\delta\colon \mathrm{H}^{2p-1}_{\mathcal{C}}(X \setminus |Y|, p) \longrightarrow \mathrm{H}^{2p}_{\mathcal{C},Y}(X, p)$$

the connecting homomorphism from the long exact sequence in relative cohomology. Then, the equality

$$\mathrm{cl}_{\mathcal{C}}(\mathrm{div}(f)) = \delta\,(\mathrm{cl}_{\mathcal{C}}(f)) \in \mathrm{H}^{2p}_{\mathcal{C},Y}(X, p)$$

holds, see [9], Lemma 3.9. By the equality $\delta = \mathrm{cl} \circ b$, we obtain that $\mathfrak{g}(f)$ is a Green object for $Y = \mathrm{div}(f)$. The well-definedness follows from the same lemma. □

Definition 2.3.19. Let Y, Z be closed subsets of X, and set $U = X \setminus Y$, $V = X \setminus Z$. Consider the Mayer–Vietoris sequence

$$0 \longrightarrow \mathcal{C}^n(U \cup V, p) \xrightarrow{i} \mathcal{C}^n(U, p) \oplus \mathcal{C}^n(V, p) \xrightarrow{j} \mathcal{C}^n(U \cap V, p) \longrightarrow 0,$$

where $i(\eta) = (\eta, \eta)$ and $j(\alpha, \eta) = -\alpha + \eta$. The sequence induces the quasi-isomorphism

$$\iota\colon \mathcal{C}^n(U \cup V, p) \longrightarrow s(-j),$$

sending η to $(i(\eta), 0)$.

Proposition 2.3.20. *The quasi-isomorphism ι gives rise to isomorphisms of (truncated) $\mathcal{C}$-cohomology groups*

$$\begin{aligned}\mathrm{H}^n_{\mathcal{C},Y\cap Z}(X, p) &\longrightarrow \mathrm{H}^n(\mathcal{C}(X, p), s(-j)),\\ \widehat{\mathrm{H}}^n_{\mathcal{C},Y\cap Z}(X, p) &\longrightarrow \widehat{\mathrm{H}}^n(\mathcal{C}(X, p), s(-j)),\end{aligned}$$

for all $n, p \in \mathbb{N}$.

Proof. The proposition follows from the fact that ι is a quasi-isomorphism and [9], Corollary 2.63. □

Definition 2.3.21. Let $\mathcal{C}$ be a $\mathcal{G}$-complex. A $\mathcal{G}$-pairing is a pairing of graded complexed $\bullet\colon \mathcal{C} \otimes \mathcal{C} \longrightarrow \mathcal{C}$ that is compatible with the pairing on $\mathcal{G}$ with respect to the maps $\mathfrak{c}_{\mathcal{C}} \otimes \mathfrak{c}_{\mathcal{C}}$ and $\mathfrak{c}_{\mathcal{C}}$.

Theorem 2.3.22. *Let Y, Z be closed subsets of X. A $\mathcal{G}$-pairing $\bullet$ on a $\mathcal{G}$-complex $\mathcal{C}$ on X induces pairings*

$$\bullet\colon \mathrm{H}^n_{\mathcal{C},Y}(X, p) \otimes \mathrm{H}^m_{\mathcal{C},Z}(X, q) \longrightarrow \mathrm{H}^{n+m}_{\mathcal{C},Y\cap Z}(X, p + q)$$

in relative $\mathcal{C}$-cohomology and

$$*\colon \widehat{\mathrm{H}}^n_{\mathcal{C},Y}(X, p) \otimes \widehat{\mathrm{H}}^m_{\mathcal{C},Z}(X, q) \longrightarrow \widehat{\mathrm{H}}^{n+m}_{\mathcal{C},Y\cap Z}(X, p + q)$$

in truncated relative $\mathcal{C}$-cohomology that commute with the class map cl *in the sense that $\bullet \circ (\mathrm{cl} \otimes \mathrm{cl}) = \mathrm{cl} \circ *$.*

Proof. As proven in [9], Theorem 2.47 and Proposition 2.66, there are natural pairings with values in $\mathrm{H}^{n+m}(\mathcal{C}(X,p+q),s(-j))$ and $\widehat{\mathrm{H}}^{n+m}(\mathcal{C}(X,p+q),s(-j))$, respectively. Now one can use the isomorphisms in Proposition 2.3.20 induced by the quasi-isomorphism ι to obtain the statement. $\square$

Theorem 2.3.23. *Let Y, Z be p- and q-codimensional cycles on X with support $|Y|$ and $|Z|$, respectively. Let $\mathfrak{g}_Y \in \widehat{\mathrm{H}}^{2p}_{\mathcal{C},Y}(X,p)$ be a Green object for Y and $\mathfrak{g}_Z \in \widehat{\mathrm{H}}^{2q}_{\mathcal{C},Z}(X,q)$ be a Green object for Z. Then, the following statements hold:*

(i) If Y and Z intersect properly, the $*$-product $\mathfrak{g}_Y * \mathfrak{g}_Z \in \widehat{\mathrm{H}}^{2p+2q}_{\mathcal{C},Y\cap Z}(X,q)$ is a Green object for $Y \cdot Z \in \mathrm{CH}^{p+q}_{Y\cap Z}(X)$.

(ii) Writing $\mathfrak{g}_Y = (\omega_Y, \widetilde{g}_Y)$ and $\mathfrak{g}_Z = (\omega_Z, \widetilde{g}_Z)$, the element $\mathfrak{g}_Y * \mathfrak{g}_Z$ is represented in $\widehat{\mathrm{H}}^{2p+2q}(\mathcal{C}(X,p+q),s(-j))$ by the element

$$(\omega_Y \bullet \omega_Z, ((g_Y \bullet \omega_Z, \omega_Y \bullet g_Z), -g_Y \bullet g_Z)^{\sim})$$

in $\mathrm{Z}^{2p+2q}(\mathcal{C}(X,p+q)) \oplus s(f)^{2p+2q-1}$.

Proof. Statement (i) follows from the compatibility of the products $\bullet$ and $*$ with the product on $\mathcal{G}$ and the good properties of the morphisms $\mathfrak{c}_{\mathcal{C}}$ and $\mathrm{cl}_{\mathcal{G}}$, see [9], Theorems 3.37 and 3.39. The second statement (ii) follows from the explicit descriptions of the morphisms induced by i, j and ι in the cohomology of complexes, given in [9], Proposition 2.20 and Theorem 2.47, or [8], in the considerations before Definition 4.5. $\square$

Remark 2.3.24. To obtain an explicit formula for the $*$-product $\mathfrak{g}_Y * \mathfrak{g}_Z \in \widehat{\mathrm{H}}^{2p+2q}_{\mathcal{C},Y\cap Z}(X,q)$ for particular choices of the complex $\mathcal{C}$, one needs to give an explicit inverse to the isomorphism induced by the quasi-isomorphism ι in Proposition 2.3.20. Later, this will be done for the complex $\mathcal{D}_{\log}$.

Remark 2.3.25. The assumption of proper intersection in Theorem 2.3.23 can be dropped by considering *weak Green objects*, see [9], Definition 3.20 and Theorem 3.37.

Theorem 2.3.26. *If the $\mathcal{G}$-pairing $\bullet$ on $\mathcal{C}$ is pseudo-associative, i.e., if it is associative up to homotopy and satisfies a certain vanishing condition on cocycles (see [9], Definition 3.41), the induced $*$-product in truncated relative cohomology is associative. Similarly, if the pairing $\bullet$ is pseudo-commutative (see [9], Definition 3.48), the $*$-product is commutative.*

Proof. See [9], Theorems 3.42 and 3.49. $\square$

To obtain independence from the support $|Y|$ of a cycle Y in relative cohomology, we make the following definition.

Definition 2.3.27. Let $\mathcal{Z}^p = \mathcal{Z}^p(X)$ be the set of all closed subsets of X of codimension greater or equal than p ordered by inclusion. We will write

$$\begin{aligned} \mathcal{C}(X \setminus \mathcal{Z}^p, p) &:= \varinjlim_{Y \in \mathcal{Z}^p} \mathcal{C}(X \setminus Y, p), \\ \mathrm{H}^*_{\mathcal{C},\mathcal{Z}^p}(X,p) &:= \varinjlim_{Y \in \mathcal{Z}^p} \mathrm{H}^*_{\mathcal{C},Y}(X,p), \\ \widehat{\mathrm{H}}^*_{\mathcal{C},\mathcal{Z}^p}(X,p) &:= \varinjlim_{Y \in \mathcal{Z}^p} \widehat{\mathrm{H}}^*_{\mathcal{C},Y}(X,p). \end{aligned}$$

In order to define Chow groups, we can now consider $\widehat{\mathrm{H}}^*_{\mathcal{C},\mathcal{Z}^p}(X,p)$ to be the common set of definition for Green objects for different p-codimensional cycles on X.

Definition 2.3.28. Let Y be a p-codimensional cycle on X. A *Green object for the class of* Y *in* $\mathrm{H}^{2p}_{\mathcal{C},\mathcal{Z}^p}(X,p)$ is an element $\mathfrak{g}_Y \in \widehat{\mathrm{H}}^{2p}_{\mathcal{C},\mathcal{Z}^p}(X,p)$ such that

$$\mathrm{cl}(\mathfrak{g}_Y) = \mathrm{cl}_{\mathcal{C}}(Y) \in \mathrm{H}^{2p}_{\mathcal{C},\mathcal{Z}^p}(X,p).$$

Definition 2.3.29. Let $\mathscr{X}$ be an arithmetic variety, and let $\mathcal{C}$ be a $\mathcal{G}$-complex on $X_{\mathbb{R}}$. The *group of p-codimensional arithmetic cycles on* $\mathscr{X}$ *with value in* $\mathcal{C}$ is defined as the set

$$\widehat{\mathrm{Z}}^p(\mathscr{X},\mathcal{C}) := \big\{(\mathscr{Y},\mathfrak{g}_Y) \in \mathrm{Z}^p(\mathscr{X}) \oplus \widehat{\mathrm{H}}^{2p}_{\mathcal{C},\mathcal{Z}^p}(X,p) \,\big|\, \mathrm{cl}_{\mathcal{C}}(Y) = \mathrm{cl}(\mathfrak{g}_Y)\big\},$$

together with the natural additive structure. Here $\mathscr{Y}$ denotes a cycle on $\mathscr{X}$, and Y denotes the induced complex cycle on the complex fibre X.

Definition 2.3.30. Let $f \in \mathrm{R}^{p-1}_p(\mathscr{X})$, and set $\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), \mathfrak{g}(f))$. We define the group $\widehat{\mathrm{Rat}}^p(\mathscr{X},\mathcal{C})$ to be

$$\widehat{\mathrm{Rat}}^p(\mathscr{X},\mathcal{C}) := \big\{\widehat{\mathrm{div}}(f) \,\big|\, f \in \mathrm{R}^{p-1}_p(\mathscr{X})\big\} \subseteq \widehat{\mathrm{Z}}^p(X,\mathcal{C}).$$

As shown in Lemma 2.3.18, this is a subgroup of $\widehat{\mathrm{Z}}^p(\mathscr{X},\mathcal{C})$.

Definition 2.3.31. The *p-th* $\mathcal{C}$*-arithmetic Chow group of* $\mathscr{X}$ is defined to be the quotient

$$\widehat{\mathrm{CH}}^p(\mathscr{X},\mathcal{C}) := \widehat{\mathrm{Z}}^p(\mathscr{X},\mathcal{C})\big/\widehat{\mathrm{Rat}}^p(\mathscr{X},\mathcal{C}).$$

Theorem 2.3.32. *Let* $\mathscr{X}$ *be an arithmetic variety, and let* $\mathcal{C}$ *be a* $\mathcal{G}$*-complex on* $X_{\mathbb{R}}$*. A* $\mathcal{G}$*-pairing on* $\mathcal{C}$ *induces a pairing*

$$\widehat{\mathrm{CH}}^p(\mathscr{X},\mathcal{C}) \otimes \widehat{\mathrm{CH}}^q(\mathscr{X},\mathcal{C}) \longrightarrow \widehat{\mathrm{CH}}^{p+q}(\mathscr{X},\mathcal{C})_{\mathbb{Q}}.$$

For cycles $[\mathscr{Y}] \in \mathrm{CH}^p(\mathscr{X})$, $[\mathscr{Z}] \in \mathrm{CH}^q(\mathscr{X})$ *such that* $\mathscr{Y}$ *and* $\mathscr{Z}$ *intersect properly in the generic fibre of* $\mathscr{X}$*, it is given by the formula*

$$[\mathscr{Y},\mathfrak{g}_Y] \cdot [\mathscr{Z},\mathfrak{g}_Z] := [\mathscr{Y} \cdot \mathscr{Z}, \mathfrak{g}_Y * \mathfrak{g}_Z].$$

Proof. See [9], Theorem 4.19. □

2.4 The complex $\mathcal{D}_{\log}$ and arithmetic Chow groups for logarithmically singular metrics

A connection between arithmetic Chow groups and real Deligne–Beilinson cohomology was already drawn by Gillet and Soulé in [19]. In his paper [8], Burgos showed that the complex $\mathcal{D}_{\log}$ defined below computes real Deligne–Beilinson cohomology and, setting $\mathcal{G} = \mathcal{C} = \mathcal{D}_{\log}$, gives back the arithmetic Chow Groups of Gillet and Soulé defined in Section 2.1. Nevertheless, another choice of the complex $\mathcal{C}$ allows to adapt to the situation of line bundles with logarithmically singular metrics.

Definition 2.4.1. Let X be a smooth complex variety, and let $j\colon X \longrightarrow \overline{X}$ be a smooth compactification with $D = \overline{X} \setminus X$ a normal crossing divisor. Set $\mathbb{R}(p) := (2\pi i)^p\mathbb{R} \subseteq \mathbb{C}$. The corresponding constant sheaves on X will also be denoted by $\mathbb{R}(p)$. Let Ω^*_X be the sheaf of holomorphic forms on X and $\Omega^*_{\overline{X}}(\log D)$ be the sheaf of holomorphic forms on $\overline{X}$ with logarithmic singularities along D. Let F^p be the Hodge filtration of $\Omega^*_{\overline{X}}(\log D)$, i.e.,

$$F^p\Omega^*_{\overline{X}}(\log D) := \bigoplus_{p'\geq p} \Omega^{p'}_{\overline{X}}(\log D)\,.$$

The *Deligne–Beilinson complex* $\mathbb{R}(p)_{\mathcal{D}}$ of the pair $(X, \overline{X})$ is defined to be the simple complex $s(u)$ of the map

$$u\colon \mathrm{R}j_*\mathbb{R}(p) \oplus F^p\Omega^*_{\overline{X}}(\log D) \longrightarrow j_*\Omega^*_X,$$

given by the assignment $u(a, f) := -a + f$. The *Deligne–Beilinson cohomology groups* are the hypercohomology groups $\mathbb{H}^*(\overline{X}, \mathbb{R}(p)_{\mathcal{D}})$ of the sheaf $\mathbb{R}(p)_{\mathcal{D}}$. They can be shown to be independent of the choice of smooth compactification.

As remarked above, the Deligne–Beilinson cohomology groups are also computable as the Gillet cohomology groups of a Gillet complex $\mathcal{D}_{\log}$. To construct this complex, we will first introduce Deligne algebras.

Definition 2.4.2. A *Dolbeault complex* $A = (A^*, \mathrm{d}_A)$ is a graded complex of real vector spaces, which is bounded from below, together with a bigrading

$$A^n_{\mathbb{C}} = \bigoplus_{p+q=n} A^{p,q}$$

on $A^*_{\mathbb{C}} = A^* \otimes_{\mathbb{R}} \mathbb{C}$, satisfying the following properties:

(i) The differential d_A can be decomposed as a sum $\mathrm{d}_A = \partial + \overline{\partial}$, with ∂ an operator of type $(1, 0)$ and $\overline{\partial}$ an operator of type $(0, 1)$.

(ii) The symmetry property $\overline{A^{p,q}} = A^{q,p}$ holds.

Notation 2.4.3. Given a Dolbeault complex $A = (A^*, \mathrm{d}_A)$, we have a decreasing filtration of $A^*_{\mathbb{C}}$, the Hodge filtration F, given by

$$F^p A^n := F^p A^n_{\mathbb{C}} := \bigoplus_{p' \geq p} A^{p', n-p'},$$

and its complex conjugate $\overline{F}$, defined by $\overline{F}^p A^n := \overline{F^p A^n_{\mathbb{C}}}$. For $x \in A^*_{\mathbb{C}}$, we denote by $x^{k,k'}$ its component in $A^{k,k'}$ $(k, k' \geq 0)$. We define the operator $F^{k,k'} \colon A^*_{\mathbb{C}} \longrightarrow A^*_{\mathbb{C}}$ by the assignment

$$F^{k,k'}(x) := \sum_{l \geq k, l' \geq k'} x^{l,l'},$$

i.e., $F^{k,k'}$ is the projection of $A^*_{\mathbb{C}}$ onto the subspace $F^k A^* \cap \overline{F}^{k'} A^*$. Denoting $A^n(p) = (2\pi i)^p A^n \subseteq A^n_{\mathbb{C}}$, we define an operator

$$\pi_p \colon A^*_{\mathbb{C}} \longrightarrow A^*(p)$$

by setting $\pi_p(x) := \frac{1}{2}(x + (-1)^p \overline{x})$.

To any Dolbeault complex, we can now associate a complex which is homotopically equivalent to the simple complex $s(u)$ of the map $u \colon A^* \oplus F^p A \longrightarrow A_{\mathbb{C}}$, given by $u(a, f) = -a + f$.

Definition 2.4.4. Let $A = (A^*, \mathrm{d}_A)$ be a Dolbeault complex. The *Deligne complex* $(\mathcal{D}^*(A, *), \mathrm{d}_{\mathcal{D}})$ *associated to* A is the graded complex given by

$$\mathcal{D}^n(A, p) = \begin{cases} A^{n-1}(p-1) \cap F^{n-p, n-p} A^{n-1}_{\mathbb{C}}, & n \leq 2p - 1, \\ A^n(p) \cap F^{p,p} A^n_{\mathbb{C}}, & n \geq 2p, \end{cases}$$

with differential $\mathrm{d}_{\mathcal{D}} x$ $(x \in \mathcal{D}^n(A, p))$ defined by

$$\mathrm{d}_{\mathcal{D}} x = \begin{cases} -F^{n-p+1, n-p+1} \mathrm{d}_A x, & n < 2p - 1, \\ -2\partial\overline{\partial} x, & n = 2p - 1, \\ \mathrm{d}_A x, & n > 2p - 1. \end{cases}$$

We now observe that an algebra structure on a Dolbeault algebra induces an algebra structure on its corresponding Deligne algebra.

Definition 2.4.5. A *Dolbeault algebra* $A = (A^*, \mathrm{d}_A, \wedge)$ is a Dolbeault complex equipped with an associative and graded commutative product

$$\wedge \colon A^* \times A^* \longrightarrow A^*,$$

such that the induced product on $A^*_{\mathbb{C}}$ is compatible with the bigrading, i.e., $A^{p,q} \wedge A^{p'q'} \subseteq A^{p+p', q+q'}$.

Definition 2.4.6. Let A be a Dolbeault algebra as above. The product on A induces a graded commutative product $\bullet\colon \mathcal{D}^n(A,p) \times \mathcal{D}^m(A,q) \longrightarrow \mathcal{D}^{n+m}(A,p+q)$ on the Deligne complex $\mathcal{D}^*(A,*)$, given by

$$x \bullet y = \begin{cases} (-1)^n r_p(x) \wedge y + x \wedge r_q(y), & n < 2p, m < 2q, \\ F^{l-r,l-r}(x \wedge y), & n < 2p, m \geq 2q, l < 2r, \\ F^{r,r}(r_p(x) \wedge y) + 2\pi_r(\partial(x \wedge y)^{r-1,l-r}), & n < 2p, m \geq 2q, l \geq 2r, \\ x \wedge y, & n \geq 2p, m \geq 2q. \end{cases}$$

where $l = n+m$, $r = p+q$, and $r_p(x) = 2\pi_p(F^p \mathrm{d}_A x)$. We call the algebra $(\mathcal{D}^*(A,*), \mathrm{d}_{\mathcal{D}}, \bullet)$ the *Deligne algebra associated to A.*

In the following, we will consider Dolbeault algebras of differential forms.

Definition 2.4.7. Let X be a smooth complex variety, D a normal crossing divisor in X, $U = X \setminus D$ its complement and $j\colon U \longrightarrow X$ the inclusion. We denote by $\mathscr{A}_X^*$ and $\mathscr{A}_U^*$ the sheaf of smooth complex differential forms on X and U, respectively. The *complex $\mathscr{A}_X^*(\log D)$ of differential forms having logarithmic singularities along D* is the $\mathscr{A}_X^*$- subalgebra of $j_*\mathscr{A}_U^*$ locally generated by the sections

$$\log z_j \overline{z}_j, \frac{\mathrm{d}z_j}{z_j}, \frac{\mathrm{d}\overline{z}_j}{\overline{z}_j} \text{ for } j = 1, \ldots, k,$$

where $z_1 \cdot \ldots \cdot z_k = 0$ is a local equation for D.

Notation 2.4.8. We denote by $\mathrm{A}_X^*(\log D)$ the global sections $\Gamma(X, \mathscr{A}_X^*(\log D))$ of the sheaf $\mathscr{A}_X^*(\log D)$, and analogously by $\mathrm{A}_{X,\mathbb{R}}^*(\log D)$ the global sections $\Gamma(X, \mathscr{A}_{X,\mathbb{R}}^*(\log D))$ of $\mathscr{A}_{X,\mathbb{R}}^*(\log D)$. The complex

$$\mathrm{A}_X(\log D) := \big(\mathrm{A}_{X,\mathbb{R}}^*(\log D), \mathrm{d}\big)$$

naturally has the structure of a Dolbeault algebra, with the usual differential d and the product given by the wedge product of forms.

We now provide a variant of the above definition which is independent of the particular divisor D.

Remark 2.4.9. Let X be a smooth complex variety. We let I be the category of smooth compactifications of X, whose elements are tuples $(\overline{X}_\alpha, j_\alpha)$ with $j_\alpha\colon X \longrightarrow \overline{X}_\alpha$, such that $\mathrm{D}_\alpha = \overline{X}_\alpha \setminus X$ is a normal crossing divisor. The morphisms of I are the maps $f\colon \overline{X}_\alpha \longrightarrow \overline{X}_\beta$ that fulfill $f \circ j_\alpha = j_\beta$. The opposite category I° is filtered, and we obtain the following definition.

Definition 2.4.10. For a smooth complex variety X, we define the vector space

$$\mathrm{A}_{\log}^*(X)^\circ := \varinjlim_{\overline{X}_\alpha \in I^\circ} \mathrm{A}_{\overline{X}_\alpha}^*(\log D_\alpha).$$

The vector spaces $\mathrm{A}_{\log}^*(X)^\circ$ form a complex of presheaves in the Zariski topology. We denote by $\mathrm{A}_{\log}^*$ the associated complex of sheaves.

Definition 2.4.11. Let $\mathcal{D}^n_{\log}(p)$ denote the sheaf over the Zariski site of complex regular schemes that assigns the group

$$\mathcal{D}^n_{\log}(X,p) := \mathcal{D}^n_{\log}(X(\mathbb{C}),p) = \mathcal{D}^n(\mathrm{A}_{\log}X(\mathbb{C}),p)$$

to a scheme X. The induced presheafs of graded complexes of real vector spaces on X are in fact totally acyclic sheafs, see [9], Proposition 5.33. They will be denoted by $\mathcal{D}_{\log,X} = \mathcal{D}^*_{\log,X}(*)$.

Theorem 2.4.12. *The complex of sheaves $\mathcal{D}_{\log}$ is a Gillet complex for regular schemes over $\mathbb{C}$ which computes real Deligne–Beilinson cohomology, i.e.,*

$$\mathrm{H}^*(\mathcal{D}_{\log}(X,p)) = \mathbb{H}^*(\overline{X},\mathbb{R}(p)_{\mathcal{D}}).$$

The product $\bullet$ induced by the wedge product of forms makes $\mathcal{D}_{\log}$ into a graded commutative and pseudo-associative algebra.

Proof. See [9], Corollary 5.31 and Theorem 5.34. □

For the concrete complex $\mathcal{D}_{\log}$, an explicit formula for the $*$-product on the $\mathcal{D}_{\log}$-arithmetic Chow groups can be obtained as follows:

Remark 2.4.13. Let Y and Z be p- and q-codimensional cycles on X with support $|Y|$ and $|Z|$, respectively. Let $\mathfrak{g}_Y = (\omega_Y, \widetilde{g}_Y) \in \widehat{\mathrm{H}}^{2p}_{\mathcal{D}_{\log},Y}(X,p)$ and $\mathfrak{g}_Z = (\omega_Z, \widetilde{g}_Z) \in \widehat{\mathrm{H}}^{2q}_{\mathcal{D}_{\log},Z}(X,q)$ be Green objects for Y and Z, respectively. Assume that neither $|Y|$ nor $|Z|$ are contained in the intersection $|Y| \cap |Z|$. Put $U = X \setminus |Y|$ and $V = X \setminus |Z|$. We recall the Mayer–Vietoris sequence from Definition 2.3.19

$$0 \longrightarrow \mathcal{D}^n_{\log}(U \cup V,p) \xrightarrow{i} \mathcal{D}^n_{\log}(U,p) \oplus \mathcal{D}^n_{\log}(V,p) \xrightarrow{j} \mathcal{D}^n_{\log}(U \cap V,p) \longrightarrow 0,$$

with $i(\eta) = (\eta,\eta)$ and $j(\alpha,\eta) = -\alpha+\eta$. It gives rise to the quasi-isomorphism

$$\iota \colon \mathcal{D}^n_{\log}(U \cup V,p) \longrightarrow s(-j)$$

sending η to $(i(\eta),0)$. In [7], Burgos proves that there is a resolution of singularities $\pi \colon \widetilde{X} \longrightarrow X$ of $|Y| \cup |Z|$ factoring through embedded resolutions of $|Y|$, $|Z|$, $|Y| \cap |Z|$, so we can assume that $\pi^{-1}(|Y|)$, $\pi^{-1}(|Z|)$, $\pi^{-1}(|Y| \cap |Z|)$ are normal crossing divisors. Denote by $\widetilde{Y}$ and $\widetilde{Z}$, the normal crossing divisor formed by the components of $\pi^{-1}(|Y|)$ and $\pi^{-1}(|Z|)$, respectively, that are not contained in $\pi^{-1}(|Y| \cap |Z|)$. As $\widetilde{Y}$ and $\widetilde{Z}$ do not meet, we can find two smooth, F_∞-invariant functions σ_{YZ} and σ_{ZY} satisfying $0 \leq \sigma_{YZ}, \sigma_{ZY} \leq 1$ and $\sigma_{YZ} + \sigma_{ZY} = 1$, such that $\sigma_{YZ} = 1$ in a neighbourhood of $\widetilde{Y}$ and $\sigma_{ZY} = 1$ in a neighbourhood of $\widetilde{Z}$. Hence, given $\alpha \in \mathrm{A}^n_{\log}(U \cap V)$, we get $\sigma_{YZ}\alpha \in \mathrm{A}^n_{\log}(U)$ and $\sigma_{ZY}\alpha \in \mathrm{A}^n_{\log}(V)$. Therefore, we have a section of the map j, given by the assignment $\alpha \mapsto (-\sigma_{YZ}\alpha, \sigma_{ZY}\alpha)$.

With the following lemma, we are able to state an explicit formula for the $*$-product.

Lemma 2.4.14. *A homotopy inverse of the quasi-isomorphism ι above is given by the morphism of complexes $s(-j) \longrightarrow \mathcal{D}^n_{\log}(U \cup V, p)$, defined via*

$$((a,b),c) \mapsto \sigma_{ZY} a + \sigma_{YZ} b + \sigma_{YZ} \mathrm{d}_{\mathcal{D}} c - \mathrm{d}_{\mathcal{D}}(\sigma_{YZ} c).$$

Proof. As, by the previous remark, there is a section for the map j available, Lemma 2.41 of [9] can be applied, which gives a formula for the homotopy inverse of ι in terms of such a section. □

Theorem 2.4.15. *Let Y, Z and σ_{YZ}, σ_{YZ} be as in Remark 2.4.13. Let $\mathfrak{g}_Y = (\omega_Y, \widetilde{g}_Y) \in \widehat{\mathrm{H}}^{2p}_{\mathcal{D}_{\log},Y}(X,p)$, $\mathfrak{g}_Z = (\omega_Z, \widetilde{g}_Z) \in \widehat{\mathrm{H}}^{2q}_{\mathcal{D}_{\log},Z}(X,q)$ be Green objects for Y and Z, respectively. Then, the element $\mathfrak{g}_Y * \mathfrak{g}_Z \in \widehat{\mathrm{H}}^{2(p+q)}_{\mathcal{D}_{\log},Y\cap Z}(X, p+q)$ can be computed as*

$$\mathfrak{g}_Y * \mathfrak{g}_Z = (\omega_Y \wedge \omega_Z, (-2\sigma_{ZY} g_Y \wedge \partial\bar{\partial} g_Z - 2\partial\bar{\partial}(\sigma_{YZ} g_Y) \wedge g_Z)^{\sim}).$$

Proof. Applying the previous lemma and the explicit descriptions of the product $\bullet$ and the differential $\mathrm{d}_{\mathcal{D}}$ given in Definitions 2.4.6 and 2.4.4 to the element

$$\mathfrak{g}_Y * \mathfrak{g}_Z = (\omega_Y \bullet \omega_Z, ((g_Y \bullet \omega_Z, \omega_Y \bullet g_Z), -g_Y \bullet g_Z)^{\sim})$$

in $\widehat{\mathrm{H}}^{2(p+q)}(\mathcal{D}_{\log}(X), s(-j), p+q)$ gives the stated formula for the $*$-product. □

For the Gillet complex $\mathcal{D}_{\log}$, the cohomological condition $\mathrm{cl}(\mathfrak{g}_Y) = \mathrm{cl}_{\mathcal{D}_{\log}}(Y) \in \mathrm{H}^{2p}_{\mathcal{D}_{\log},Y}(X,p)$ for Green forms can be translated back into the differential equation $[\omega_Y] - \mathrm{dd}^c[g_Y] = \delta_Y$ for Green currents in the sense of Gillet and Soulé, and we obtain the following theorem.

Theorem 2.4.16. *For an arithmetic variety $\mathscr{X}$, there is an isomorphism*

$$\widehat{\mathrm{CH}}^p(\mathscr{X}, \mathcal{D}_{\log}) \longrightarrow \widehat{\mathrm{CH}}^p(\mathscr{X})$$

from the $\mathcal{D}_{\log}$-arithmetic Chow groups to the arithmetic Chow groups of Gillet and Soulé that is compatible with products and pull-backs. It is given by the assignment

$$[\mathscr{Y}, \mathfrak{g}_Y] \mapsto [\mathscr{Y}, 2(2\pi i)^{d-p+1}[g_Y]],$$

where $\mathfrak{g}_Y = (\omega_Y, \widetilde{g}_Y)$, and g_Y is a representative of $\widetilde{g}_Y$ in $\mathcal{D}^{2p-1}_{\log}(X \setminus |Y|, p)$.

Proof. Note that we assumed an arithmetic variety to be projective. For the proofs that $2(2\pi i)^{d-p+1}[g_Y]$ is a Green current in the sense of Gillet and Soulé and that the notions of rational equivalence are compatible, see [7], Theorem 4.4, and [8], Theorem 7.2. The normalization factor $2(2\pi i)^{d-p+1}$ arises from the normalization of the correspondence between forms and currents in [9], in order to make their Dolbeault algebra structures compatible. □

Lemma 2.4.17. *For smooth projective complex varieties X, the $*$-product for the $\mathcal{D}_{\log}$-arithmetic Chow groups coincides with the $*$-product for Green currents defined by Gillet and Soulé. Therefore, the above isomorphism $\widehat{\mathrm{CH}}^p(\mathscr{X}, \mathcal{D}_{\log}) \longrightarrow \widehat{\mathrm{CH}}^p(\mathscr{X})$ induces an isomorphism of Chow rings.*

Proof. The two currents $[-2\sigma_{ZY} g_Y \wedge \partial\overline{\partial} g_Z - 2\partial\overline{\partial}(\sigma_{YZ} g_Y) \wedge g_Z]$ and $g_Y \wedge \delta_Z + [\omega_Y] \wedge [g_Z]$ can be shown to differ only by an element in $\partial \mathrm{D}^{p+q-2,p+q-1}(X) + \overline{\partial}\mathrm{D}^{p+q-1,p+q-2}(X)$, see [7], Lemma 4.17. $\square$

The Petersson metric on the line bundle of modular forms on the toroidally compactified moduli space $\overline{\mathcal{A}}_g$ defined in Chapter 1 has logarithmic singularities along the boundary. Therefore, a Green form $-\log \|s\|^2_{\mathrm{Pet}}$ for the divisor $\mathrm{div}(s)$ of a section has log-log-singularities at the boundary in addition to the log-singularities at $\mathrm{div}(s)$. To have an intersection theory for this situation available, we will modify the $\mathcal{D}_{\log}$-complex $\mathcal{C}$.

Definition 2.4.18. Let X be a d-dimensional smooth complex variety, D a normal crossing divisor on X, $U = X \setminus D$, and $j \colon U \longrightarrow X$ the inclusion. A smooth function f on U is said to have log-*growth along* D if we have

$$|f(z_1, \ldots, z_d)| \prec \prod_{j=1}^{k} (\log |z_j|)^M$$

for any local equation $z_1 \cdot \ldots \cdot z_k$ of D in a coordinate neighbourhood V of D and some $M \in \mathbb{N}$. The *sheaf of differential forms on X with* log-*growth along* D is the subalgebra of $j_* \mathscr{A}^*_U$ that is generated by the functions with log-growth along D as well as the differentials

$$\frac{\mathrm{d}z_j}{z_j}, \frac{\mathrm{d}\overline{z}_j}{\overline{z}_j} \ (j = 1, \ldots, k) \qquad \text{and} \qquad \mathrm{d}z_j, \mathrm{d}\overline{z}_j \ (j = k+1, \ldots, d),$$

for any coordinate neighbourhood V of D. A differential form η with log-growth along D such that $\partial\eta$, $\overline{\partial}\eta$, and $\partial\overline{\partial}\eta$ also have log-growth along D is called a *pre*-log-*form*.

Definition 2.4.19. Let X, D, U be as above. A smooth function f on U is said to have log-log-*growth along* D if we have

$$|f(z_1, \ldots, z_d)| \prec \prod_{j=1}^{k} (\log \log |z_j|)^M$$

for any local equation $z_1 \cdot \ldots \cdot z_k$ of D in a coordinate neighbourhood V of D and some $M \in \mathbb{N}$. A smooth function f on U is said to have *Poincaré growth along* D if it is bounded in a neighbourhood of each point of D. The *sheaf of differential forms on X with* log-log-*growth along* D is the subalgebra of $j_* \mathscr{A}^*_U$ that is generated by the functions with log-log-growth along D as well as the differentials

$$\frac{\mathrm{d}z_j}{z_j \log |z_j|}, \frac{\mathrm{d}\overline{z}_j}{\overline{z}_j \log |z_j|} \ (j = 1, \ldots, k) \qquad \text{and} \qquad \mathrm{d}z_j, \mathrm{d}\overline{z}_j \ (j = k+1, \ldots, d),$$

for any coordinate neighbourhood V of D. The *sheaf of differential forms on X with Poincaré growth along D* is the subalgebra of $j_*\mathscr{A}_U^*$ that is generated by the functions with Poincaré growth along D as well as the above differentials. A differential form η with log-log-growth along D such that $\partial\eta$, $\overline{\partial}\eta$, and $\partial\overline{\partial}\eta$ also have log-log-growth along D is called a *pre*-log-log-*form*. A differential form η with Poincaré growth along D such that $\mathrm{d}\eta$ also has Poincaré growth along D is called *good*.

Definition 2.4.20. Let X be a smooth complex variety, and let B, D be two normal crossing divisors on X that may have common components. We define the *sheaf of mixed forms* $\mathscr{A}_X^*\langle D\langle B\rangle\rangle_{\mathrm{pre}}$ to be the subalgebra of $j_*\mathscr{A}_{X\setminus B\cup D}^*$ generated by the forms that have pre-log-growth along D and pre-log-log-growth along B.

Remark 2.4.21. The sheaf $\mathscr{A}_X^*\langle D\langle B\rangle\rangle_{\mathrm{pre}}$ is naturally a Dolbeault algebra. Therefore, it gives rise to a $\mathcal{D}_{\log}$-complex $\mathcal{D}_{\mathrm{pre}}$, which is constructed analogously to $\mathcal{D}_{\log}$ in Definition 2.4.11. The $*$-product of Green objects in the $\mathcal{D}_{\mathrm{pre}}$-arithmetic Chow groups can be explicitely given as in Theorem 2.4.15.

Example 2.4.22. Denote by B the boundary of a toroidal compactification $\overline{\mathcal{A}}_g$ of $\mathcal{A}_g$ described in Section 5 of Chapter 1. Let s be a non-zero rational section of the line bundle of modular forms $\mathcal{M}_k(\Gamma_g)$ on $\overline{\mathcal{A}}_g$ defined in Chapter 1, Definition 1.3.2, and let $\|\cdot\|_{\mathrm{Pet}}$ be the Petersson metric on $\mathcal{M}_k(\Gamma_g)$ defined in Chapter 1, Definition 1.3.3. Denote by Y the support of $\mathrm{div}(s)$. Let $\pi\colon \widetilde{X} \longrightarrow X$ be an embedded resolution of singularities of Y such that $D := \pi^{-1}(Y)$, $\widetilde{B} := \pi^{-1}(B)$ and $\pi^{-1}(B\cup Y)$ are normal crossing divisors. Then, we have

$$-\log\|s\|_{\mathrm{Pet}}^2 \in \Gamma(\widetilde{X}, \mathscr{A}_{\widetilde{X}}^*\langle D\langle\widetilde{B}\rangle\rangle_{\mathrm{pre}}),$$

and the form $\partial\overline{\partial}\log\|s\|_{\mathrm{Pet}}$ is a pre-log-log-form for $\widetilde{B}$.

Proof. See [9], Proposition 7.15. □

This motivates a generalization of the results about $\mathcal{D}_{\log}$-arithmetic Chow groups above.

Definition 2.4.23. Let X be a smooth complex variety, B a normal crossing divisor on X and $U = X \setminus B$. Let L be a line bundle on X. A smooth Hermitian metric $\|\cdot\|$ on $L|_U$ is said to be *good along B* if for all non-zero rational sections s of L, the functions $\|s\|$, $\|s\|^{-1}$ have log-growth along B and the form $\partial\log\|s\|$ is good.

Proposition 2.4.24. *Let $\mathscr{X}$ be an arithmetic variety, and let B be a normal crossing divisor on the corresponding complex variety X. We set $\underline{\mathscr{X}} := (\mathscr{X}, B)$ and denote by $\widehat{\mathrm{Pic}}(\underline{\mathscr{X}})$ the group of isometry classes of line bundles with Hermitian metrics that are good along B. Then, there is a morphism*

$$\widehat{\mathrm{c}}_1\colon \widehat{\mathrm{Pic}}(\underline{\mathscr{X}}) \longrightarrow \widehat{\mathrm{CH}}^1(\mathscr{X}, \mathcal{D}_{\mathrm{pre}}),$$

sending the class $[\mathscr{L}, \|\cdot\|]$ *to* $[\mathrm{div}(s), (-2\partial\overline{\partial} g_s, \widetilde{g}_s)]$, *with* s *denoting a non-zero rational section of* $\mathscr{L}$ *as well as the induced section of the complex bundle* L, *and* $g_s := -\frac{1}{2}\log\|s\|^2$.

Proof. See [9], Proposition 7.50. □

Remark 2.4.25. The log-singular Petersson metric is in particular good along the boundary of a toroidal compactification $\overline{\mathcal{A}}_g$ of $\mathcal{A}_g$. By the previous proposition, one can assign an arithmetic degree to the Hermitian line bundle $(\mathcal{M}_k(\Gamma_g), \|\cdot\|_{\mathrm{Pet}})$ on $\overline{\mathcal{A}}_g$ in analogy to Definition 2.2.17.

Chapter 3

The arithmetic volume of $\overline{\mathcal{A}}_2$

In the following chapter, we will compute the arithmetic volume of $\overline{\mathcal{A}}_2$ directly, employing the formula for the arithmetic degree established in Chapter 2, by choosing appropriate sections of the line bundle of modular forms whose divisors can be computed explicitely.

3.1 Four Siegel modular forms of genus 2

We will introduce the Siegel modular forms that will subsequently be used to compute the arithmetic volume and show that their divisors intersect properly. First, we recall some notation and results that will be useful for the following computations.

Notation 3.1.1. Denote the coordinates on $\mathbb{H}_2$ by

$$\tau := \begin{pmatrix} \tau_1 & \tau_{12} \\ \tau_{12} & \tau_2 \end{pmatrix} = x + iy = \begin{pmatrix} x_1 & x_{12} \\ x_{12} & x_2 \end{pmatrix} + i \begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix},$$

and let $\pi_2 \colon \mathbb{H}_2 \longrightarrow \mathcal{A}_2$ be the quotient morphism. We recall that the boundary divisor of the toroidal compactification of $\mathcal{A}_2$ is isomorphic to the universal family $\mathcal{X}_1$ over $\mathcal{A}_1$, and, therefore, the boundary is of codimension 1, i.e., we have

$$\partial\mathcal{A}_2 = \overline{\mathcal{A}_2} \setminus \mathcal{A}_2 \cong (\mathbb{Z}^2 \rtimes \Gamma_1)\backslash(\mathbb{C}^2 \times \mathbb{H}_1).$$

Local coordinates in a neighbourhood of $\partial\mathcal{A}_2$ are given by

$$t := \exp(2\pi i \tau_1),\ \tau_2 = x_2 + iy_2,\ \tau_{12} = x_{12} + iy_{12}.$$

We will also consider the divisor on $\overline{\mathcal{A}}_2$ given by the compactification $\overline{\mathcal{H}}$ of the diagonal divisor

$$\begin{aligned} \mathcal{H} &:= \pi_2\left(\left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \,\middle|\, \tau_1, \tau_2 \in \mathbb{H}_1 \right\}\right) \\ &\cong \mathrm{Sym}_2(\mathcal{A}_1) = \mathcal{A}_1 \times \mathcal{A}_1 \big/ \big((\tau_1, \tau_2) \sim (\tau_2, \tau_1)\big), \end{aligned}$$

the Humbert surface of invariant 1. Here, the symmetrization $(\tau_1, \tau_2) \sim (\tau_2, \tau_1)$ arises from the action of the matrix

$$S := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \Gamma_2.$$

Due to the isomorphism $\mathcal{H} \cong \mathrm{Sym}_2(\mathcal{A}_1)$, we will denote a subset of $\mathcal{H}$ that is a product of two subsets S_1, S_2 of $\mathcal{A}_1$ by

$$S_1 \times_{\mathcal{H}} S_2 := \pi_2\left(\left\{ \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} \middle| \tau_1 \in S_1, \tau_2 \in S_2 \right\}\right). \tag{3.1.1}$$

The Siegel modular forms in question will now be constructed by means of ϑ-series.

Definition 3.1.2. The *ϑ-series of a vector* $(a, b) \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2$ is given by the equality

$$\vartheta_{a,b}(\tau) : = \sum_{n \in \mathbb{Z}^2} e^{2\pi i \left(\frac{1}{2} \left(n + \frac{a}{2}\right)^t \tau \left(n + \frac{a}{2}\right) + \left(n + \frac{a}{2}\right)^t \frac{b}{2} \right)}.$$

With the coordinates as in Notation 3.1.1 and $n = (n_1, n_2)^t$, $a = (a_1, a_2)^t$, $b = (b_1, b_2)^t$, this can be rewritten as

$$\vartheta_{a,b}(\tau) = \\ \sum_{n_1, n_2 \in \mathbb{Z}} e^{\pi i \left(\left(n_1 + \frac{a_1}{2}\right)^2 \tau_1 + 2\left(n_1 + \frac{a_1}{2}\right)\left(n_2 + \frac{a_2}{2}\right)\tau_{12} + \left(n_2 + \frac{a_2}{2}\right)^2 \tau_2 + \left(n_1 + \frac{a_1}{2}\right) b_1 + \left(n_2 + \frac{a_2}{2}\right) b_2 \right)}. \tag{3.1.2}$$

We note that $\vartheta_{a,b}$ is non-trivial if and only if $a^t b$ is even; we then call $\vartheta_{a,b}$ *even*. There are 10 even ϑ-series of degree 2.

Definition 3.1.3. The cusp form χ_{10} of weight 10 is given (up to normalization) by the product of the squares of the 10 even ϑ-series

$$\begin{aligned} \chi_{10}(\tau) &:= \frac{1}{2^{12}} \prod_{(a,b) \text{ even}} \vartheta^2_{a,b}(\tau) \\ &= e^{2\pi i(\tau_1 + \tau_{12} + \tau_2)} \prod_{\substack{n,l,m \in \mathbb{Z} \\ (n,l,m) > 0}} \left(1 - e^{2\pi i(n\tau_1 + l\tau_{12} + m\tau_2)}\right)^{2f(nm,l)}. \end{aligned}$$

The exponents $f(nm, l)$ of the product expansion are given by the Fourier coefficients of a weak Jacobi form, see [22], Chapter 4. The notation $(n, m, l) > 0$ means that $m, n \geq 0$, and $l = -1$ for $m = n = 0$. Note that the normalization differs from the convention in the literature by the factor 4, and rather corresponds to the one of X_{10} in [29]. This is done in order to have integer coprime Fourier coefficients, see also [40].

Proposition 3.1.4. *The divisor of* χ_{10} *is given by*

$$\mathrm{div}(\chi_{10}) = \partial\mathcal{A}_2 + \overline{\mathcal{H}}.$$

Proof. With the local coordinate $t = \exp(2\pi i \tau_1)$, we see that the first factor of the product expansion of χ_{10} in Definition 3.1.3 has a simple zero along $\partial\mathcal{A}_2$. It is a well known result that the vanishing locus of χ_{10} on $\mathcal{A}_2$ is $\mathcal{H}$, see, e.g. [45], Chapter IX, Proposition 3.3. As $\mathcal{H}$ is the fixed point locus of the endomorphism

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \Gamma_2$$

of $\mathbb{H}_2$ that sends τ_{12} to $-\tau_{12}$, a local parameter around $\mathcal{H}$ is given by τ_{12}^2. Considering the Fourier expansion of χ_{10} in [27], Theorem 3, or noting that the coefficient in the product expansion of χ_{10} for $m = n = 0$ and $l = -1$ equals $f(0, -1) = 1$, one sees that the vanishing order of χ_{10} along $\mathcal{H}$ also equals 1. □

Definition 3.1.5. We will further consider the following three modular forms of degree 2 given by

$$E_4(\tau) := \frac{1}{4} \sum_{(a,b) \text{ even}} \vartheta_{a,b}^8(\tau),$$

$$E_6(\tau) := \frac{1}{4} \sum \pm \big(\vartheta_{a_1,b_1}(\tau)\vartheta_{a_2,b_2}(\tau)\vartheta_{a_3,b_3}(\tau)\big)^4,$$

$$\chi_{12}(\tau) := \frac{1}{2^{15}} \sum \big(\vartheta_{a_1,b_1}(\tau) \cdot \ldots \cdot \vartheta_{a_6,b_6}(\tau)\big)^4.$$

The second sum runs over all syzygous triples (a_j, b_j) $(j = 1, 2, 3)$, and the third sum runs over the complements of the syzygous quadruples as described in [28], Chapter 4. A triple (m_1, m_2, m_3) is called *syzygous* if $m_1 + m_2 + m_3$ is even. A quadruple (m_1, m_2, m_3, m_4) is called syzygous if any triple (m_j, m_k, m_l) $(1 \le j < k < l \le 4)$ is syzygous. The signs in the second sum arise from a symmetrization process, making it a modular form for Γ_2. For details, see [29]. Note that the normalization of χ_{12} again differs from the convention in the literature by a factor 12, and rather corresponds to the one of X_{12} in [29], in order to have integer coprime Fourier coefficients, see again also [40].

Remark 3.1.6. With the above normalizations, the Fourier coefficients around the cusp $\tau_1 = i\infty$ of the modular forms E_4, E_6, χ_{10}, χ_{12} are all integer and coprime. Therefore, the forms are defined over $\mathrm{Spec}(\mathbb{Z})$ by the q-expansion principle, see, e.g., [30].

Proposition 3.1.7. *The divisors corresponding to the modular forms in Definitions 3.1.3 and 3.1.5 intersect successively properly. In particular, we have* *the following intersections*

$$\begin{aligned}
&\mathrm{div}(E_6)\cdot\mathrm{div}(\chi_{10}) = \partial\mathcal{A}_2\cdot\mathrm{div}(E_6) + \frac{1}{2}\big(\{i\}\times_{\mathcal{H}}\overline{\mathcal{A}}_1\big),\\
&\mathrm{div}(E_4)\cdot\mathrm{div}(E_6)\cdot\mathrm{div}(\chi_{10}) = \frac{1}{6}(\{i\}\times_{\mathcal{H}}\{\omega\}),\\
&\mathrm{div}(\chi_{12})\cdot\mathrm{div}(E_4)\cdot\mathrm{div}(E_6)\cdot\mathrm{div}(\chi_{10}) = \emptyset,
\end{aligned}$$

with $\omega = e^{2\pi i/3}$*, and* $\times_{\mathcal{H}}$ *denoting the product of subsets of* $\mathcal{A}_1$ *defined in (3.1.1).*

Proof. If $\tau_{12} = 0$, i.e., if

$$\tau = \begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix},$$

we see from Definition 3.1.2 that the ϑ-series $\vartheta_{a,b}$ with $a = (a_1, a_2)^t$ and $b = (b_1, b_2)^t$ decomposes as

$$\vartheta_{a,b}(\tau) = \vartheta_{a_1,b_1}(\tau_1)\vartheta_{a_2,b_2}(\tau_2). \tag{3.1.3}$$

Applying this decomposition to the formulas for the modular forms in Definition 3.1.5, their restrictions to $\overline{\mathcal{H}}$ decompose as

$$E_4(\tau) = E_4(\tau_1)E_4(\tau_2), \quad E_6(\tau) = E_6(\tau_1)E_6(\tau_2), \quad \chi_{12}(\tau) = 12\Delta(\tau_1)\Delta(\tau_2), \tag{3.1.4}$$

with the degree 1 modular forms

$$\begin{aligned}
E_4(\tau_1) &= \frac{1}{2}\big(\vartheta_{00}^8(\tau_1) + \vartheta_{01}^8(\tau_1) + \vartheta_{10}^8(\tau_1)\big),\\
E_6(\tau_1) &= \frac{1}{2}\big(\vartheta_{00}^4(\tau_1) + \vartheta_{01}^4(\tau_1)\big)\big(\vartheta_{00}^4(\tau_1) + \vartheta_{10}^4(\tau_1)\big)\big(\vartheta_{01}^4(\tau_1) - \vartheta_{10}^4(\tau_1)\big),\\
\Delta(\tau_1) &= \frac{1}{2^8}\big(\vartheta_{00}(\tau_1)\vartheta_{01}(\tau_1)\vartheta_{10}(\tau_1)\big)^8 = \exp(2\pi i\tau_1)\prod_{n\in\mathbb{Z}}(1 - e^{2\pi i n\tau_1})^{24}.
\end{aligned}$$

Note that with this normalization we have

$$E_4^3(\tau_1) - E_6^2(\tau_1) = 12^3\Delta(\tau_1). \tag{3.1.5}$$

Similarly, one sees that for $\tau = \left(\begin{smallmatrix} i\infty & \tau_{12}\\ \tau_{12} & \tau_2\end{smallmatrix}\right) \in \partial\mathcal{A}_2$, the ϑ-series reduce to

$$\begin{aligned}
&\vartheta_{a,b}(\tau) = \vartheta_{a_2,b_2}(\tau_2), && \text{if } (a_1, b_1) = (0, b_1),\\
&\vartheta_{a,b}(\tau) = 0, && \text{if } (a_1, b_1) = (1, b_1).
\end{aligned}$$

Hence, on $\partial\mathcal{A}_2$, the Eisenstein series restrict to

$$E_4(\tau)\big|_{\partial\mathcal{A}_2} = E_4(\tau_2), \quad E_6(\tau)\big|_{\partial\mathcal{A}_2} = E_6(\tau_2). \tag{3.1.6}$$

Therefore, restricting E_6 to $\overline{\mathcal{H}}$ and $\partial\mathcal{A}_2$, one obtains by means of (3.1.4) and (3.1.6) that

$$\operatorname{div}(E_6)\cdot\operatorname{div}(\chi_{10})=\pi_2\left(\left\{\begin{pmatrix} i\infty & \tau_{12}\\ \tau_{12} & i\end{pmatrix}\,\middle|\,\tau_{12}\in\mathbb{C}\right\}\right)+\frac{1}{2}(\{i\}\times_{\mathcal{H}}\overline{\mathcal{A}_1}),$$

as $E_6(\tau_2)$ has a zero of order $1/2$ for $\tau_2=i$. Restricting E_4 to $\operatorname{div}(E_6)\cdot\operatorname{div}(\chi_{10})$ and noting that $E_4(\tau_2)$ has a zero of order $1/3$ at $\tau_2=\omega$, one obtains by means of (3.1.4) and (3.1.6) that

$$\operatorname{div}(E_4)\cdot\operatorname{div}(E_6)\cdot\operatorname{div}(\chi_{10})=\frac{1}{6}(\{i\}\times_{\mathcal{H}}\{\omega\}).$$

Finally, as χ_{12} vanishes only at the boundary $\partial\mathcal{A}_2$, one sees that

$$\operatorname{div}(\chi_{12})\cdot\operatorname{div}(E_4)\cdot\operatorname{div}(E_6)\cdot\operatorname{div}(\chi_{10})=\emptyset,$$

so the divisors intersect properly. $\square$

Notation 3.1.8. From now on, we will denote the Green form $-\log\|E_4\|_{\mathrm{Pet}}$ corresponding to the modular forms E_4 by g_4, and the corresponding Chern form $4\pi i\mathrm{dd}^c g_4$ by ω_4. Analogously, we define the Green forms g_6, g_{10}, g_{12} and Chern forms ω_6, ω_{10}, ω_{12} corresponding to E_6, χ_{10}, χ_{12}, respectively.

3.2 A formula for the arithmetic volume of $\overline{\mathcal{A}}_2/\mathbb{Z}$

We will give an explicit formula for the arithmetic self intersection number of the Hodge bundle on $\overline{\mathcal{A}}_2$ by applying a proposition of [6], recalling some of the theory of Chapter 2.

Remark 3.2.1. Let $\underline{X}=(X,D)$ be a proper smooth real variety of dimension d with fixed normal crossing divisor D. Let $Y\in \mathrm{Z}^p(X_{\mathbb{R}})$, $Z\in\mathrm{Z}^q(X_{\mathbb{R}})$ be two cycles on X of codimension p, q with support $|Y|$, $|Z|$, respectively. Assume that neither $|Y|$ nor $|Z|$ are contained in the intersection $|Y|\cap|Z|$. In [7], Burgos proves that one finds a resolution $\pi:\widetilde{X}_{\mathbb{R}}\to X_{\mathbb{R}}$ of singularities of $|Y|\cup|Z|$ which factors through embedded resolutions of $|Y|$, $|Z|$, $|Y|\cap|Z|$. Denote by $\widehat{Y}$ the normal crossing divisor formed by the components of $\pi^{-1}(|Y|)$ that are not contained in $\pi^{-1}(|Y|\cap|Z|)$. Analogously, denote by $\widehat{Z}$ the normal crossing divisor formed by the components of $\pi^{-1}(|Z|)$ that are not contained in $\pi^{-1}(|Y|\cap|Z|)$. Hence, $\widehat{Y}$ and $\widehat{Z}$ are closed subsets of $\widetilde{X}$ that do not meet. Therefore, there exist two smooth, F_∞-invariant functions σ_{YZ} and σ_{ZY} satisfying $0\leq\sigma_{YZ},\sigma_{ZY}\leq 1$, $\sigma_{YZ}+\sigma_{ZY}=1$ with $\sigma_{YZ}=1$ in a neighborhood of $\widehat{Y}$ and $\sigma_{ZY}=1$ in a neighborhood of $\widehat{Z}$. $\square$

Theorem 3.2.2. *Let* $\mathfrak{g}_Y=(\omega_Y,\widetilde{g}_Y)\in\widehat{\mathrm{H}}^{2p}_{\mathcal{D}_{\mathrm{pre}},Y}(X,p)$ *and* $\mathfrak{g}_Z=(\omega_Z,\widetilde{g}_Z)\in\widehat{\mathrm{H}}^{2q}_{\mathcal{D}_{\mathrm{pre}},Z}(X,q)$ *be Green objects corresponding to the cycles* Y, Z *as in Remark 3.2.1. Then, their* $*$*-product* $\mathfrak{g}_Y*\mathfrak{g}_Z\in\widehat{\mathrm{H}}^{2(p+q)}_{\mathcal{D}_{\mathrm{pre}},Y\cap Z}(X,p+q)$ *is given* *as*

$$\begin{aligned}\mathfrak{g}_Y * \mathfrak{g}_Z =&\big(\omega_Y \wedge \omega_Z, (-2(\sigma_{ZY} g_Y) \wedge \partial\bar{\partial} g_Z - 2\partial\bar{\partial}(\sigma_{YZ} g_Y) \wedge g_Z)^{\sim}\big)\\ =&\big(\omega_Y \wedge \omega_Z, (4\pi i(\sigma_{ZY} g_Y) \wedge \mathrm{dd}^c g_Z + 4\pi i \mathrm{dd}^c(\sigma_{YZ} g_Y) \wedge g_Z)^{\sim}\big).\end{aligned}$$

Proof. See Chapter 2, Theorem 2.4.15. □

Notation 3.2.3. Let X be a real variety, and let $Y \subseteq X$ be a closed subset. We will denote an ε-neighbourhood of Y by $B_\varepsilon(Y)$.

We will add a proof of the next proposition, as the computations involved will be useful to obtain results in a neighbourhood of the boundary.

Proposition 3.2.4. *Let $\underline{X}$ be as in Remark 3.2.1. Assume that $D = D_1 \cup D_2$, where D_1 and D_2 are normal crossing divisors of X satisfying $D_1 \cap D_2 = \emptyset$. Let Y and Z be cycles of codimension p and q with corresponding Green objects $\mathfrak{g}_Y = (\omega_Y, \widetilde{g}_Y)$ and $\mathfrak{g}_Z = (\omega_Z, \widetilde{g}_Z)$, respectively. Assume that $p+q = d+1$, and that Y and Z intersect properly, i.e., $|Y| \cap |Z| = \emptyset$. Furthermore, assume $|Y| \cap D_2 = \emptyset$ and $|Z| \cap D_1 = \emptyset$. Then, we have the decomposition*

$$\begin{aligned}&\frac{1}{(2\pi i)^d} \int\limits_X g_Y * g_Z\\ &= \lim_{\varepsilon \to 0} \left(\frac{1}{(2\pi i)^d} \int\limits_{X \setminus B_\varepsilon(D)} g_Y \wedge \omega_Z - \frac{2}{(2\pi i)^{d-1}} \int\limits_{\partial B_\varepsilon(D_1)} [g_Z \wedge \mathrm{d}^c g_Y - g_Y \wedge \mathrm{d}^c g_Z] \right)\\ &\quad + \frac{1}{(2\pi i)^{q-1}} \int\limits_{\overline{Y \setminus Y \cap D_1}} g_Z.\end{aligned}$$

Proof. The proof can be found in [6], Theorem 1.14. Let $\widetilde{X}$ be a resolution of singularities of $|Y| \cup |Z|$ and $\{\sigma_{YZ}, \sigma_{ZY}\}$ a partition of unity as in Remark 3.2.1. Write Y' and Z' for the strict transforms of Y and Z in $\widetilde{X}$, respectively, and let Y'' be the strict transform of the closure of $Y \setminus (Y \cap D)$. Assume that $\sigma_{YZ} = 1$ in a neighborhood of D_1 and $\sigma_{YZ} = 0$ in a neighborhood of D_2. By Theorem 3.2.2, we have

$$g_Y * g_Z = 4\pi i\big((\sigma_{ZY} g_Y) \wedge \mathrm{dd}^c g_Z + \mathrm{dd}^c(\sigma_{YZ} g_Y) \wedge g_Z\big).$$

Set

$$X_\varepsilon := \widetilde{X} \setminus \big(B_\varepsilon(D) \cup B_\varepsilon(Y'') \cup B_\varepsilon(Z')\big).$$

Taking into account that

$$(\sigma_{ZY} g_Y) \wedge \mathrm{dd}^c g_Z = g_Y \wedge \mathrm{dd}^c g_Z - (\sigma_{YZ} g_Y) \wedge \mathrm{dd}^c g_Z$$

and that

$$\begin{aligned}\mathrm{dd}^c(\sigma_{YZ} g_Y) \wedge g_Z &= \mathrm{d}\big(\mathrm{d}^c(\sigma_{YZ} g_Y) \wedge g_Z\big) - \mathrm{d}^c(\sigma_{YZ} g_Y) \wedge \mathrm{d} g_Z\\ &= \mathrm{d}\big(\mathrm{d}^c(\sigma_{YZ} g_Y) \wedge g_Z\big) - \mathrm{d}(\sigma_{YZ} g_Y) \wedge \mathrm{d}^c g_Z,\end{aligned}$$

we can rewrite $g_Y * g_Z$ as

$$\begin{aligned} &4\pi i\big(g_Y \wedge \mathrm{dd}^c g_Z + \mathrm{d}\big(\mathrm{d}^c(\sigma_{YZ} g_Y) \wedge g_Z\big) - \mathrm{d}(\sigma_{YZ} g_Y) \wedge \mathrm{d}^c g_Z - (\sigma_{YZ} g_Y) \wedge \mathrm{dd}^c g_Z\big) \\ &= 4\pi i\big(g_Y \wedge \mathrm{dd}^c g_Z + \mathrm{d}\big(\mathrm{d}^c(\sigma_{YZ} g_Y) \wedge g_Z - (\sigma_{YZ} g_Y) \wedge \mathrm{d}^c g_Z\big)\big). \end{aligned} \tag{3.2.1}$$

We obtain

$$\begin{aligned} \int\limits_{X_\varepsilon} g_Y * g_Z &= 4\pi i \int\limits_{X_\varepsilon} [(\sigma_{ZY} g_Y) \wedge \mathrm{dd}^c g_Z + \mathrm{dd}^c(\sigma_{YZ} g_Y) \wedge g_Z] \\ &= \int\limits_{X_\varepsilon} g_Y \wedge (4\pi i)\mathrm{dd}^c g_Z + 4\pi i \int\limits_{X_\varepsilon} \mathrm{d}\big(g_Z \wedge \mathrm{d}^c(\sigma_{YZ} g_Y) - (\sigma_{YZ} g_Y) \wedge \mathrm{d}^c g_Z\big). \end{aligned}$$

Applying Stokes' theorem to the latter integral and observing that for ε sufficiently small, σ_{YZ} is 1 on $B_\varepsilon(D_1)$ and $B_\varepsilon(Y'')$, and 0 on $B_\varepsilon(Z')$ and $B_\varepsilon(D_2)$, we obtain

$$\begin{aligned} &\int\limits_{X_\varepsilon} \mathrm{d}\big(g_Z \wedge \mathrm{d}^c(\sigma_{YZ} g_Y) - (\sigma_{YZ} g_Y) \wedge \mathrm{d}^c g_Z\big) \\ &= - \int\limits_{\partial(B_\varepsilon(D_1) \cup B_\varepsilon(Y''))} [g_Z \wedge \mathrm{d}^c g_Y - g_Y \wedge \mathrm{d}^c g_Z] \\ &= - \int\limits_{\partial B_\varepsilon(D_1)} [g_Z \wedge \mathrm{d}^c g_Y - g_Y \wedge \mathrm{d}^c g_Z] - \int\limits_{\partial(B_\varepsilon(Y'') \backslash B_\varepsilon(Y'') \cap B_\varepsilon(D_1))} [g_Z \wedge \mathrm{d}^c g_Y - g_Y \wedge \mathrm{d}^c g_Z]\,. \end{aligned} \tag{3.2.2}$$

Note that, with the orientation of $\partial B_\varepsilon(Y)$ induced by $X \backslash B_\varepsilon(Y)$, for a p-codimensional cycle $Y = \sum_j n_j Y_j$ and a differential form α the equality

$$- \lim_{\varepsilon \to 0} \int\limits_{\partial B_\varepsilon(Y)} \alpha \wedge \mathrm{d}^c g_Y = \frac{(2\pi i)^{p-1}}{2} \sum_j n_j \int\limits_{Y_j} \alpha$$

holds. Then, the last integral on the right hand side of (3.2.2) becomes

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int\limits_{\partial(B_\varepsilon(Y'') \backslash B_\varepsilon(Y'') \cap B_\varepsilon(D_1))} [g_Z \wedge \mathrm{d}^c g_Y - g_Y \wedge \mathrm{d}^c g_Z] \\ &= \lim_{\varepsilon \to 0} \int\limits_{\partial B_\varepsilon(Y'' \backslash Y'' \cap D_1)} [g_Z \wedge \mathrm{d}^c g_Y - g_Y \wedge \mathrm{d}^c g_Z] \\ &= \frac{(2\pi i)^{p-1}}{2} \int\limits_{\overline{Y'' \backslash Y'' \cap D_1}} g_Z. \end{aligned}$$

Combining the above computations completes the proof. ☐

We now apply Proposition 3.2.4 several times with the concrete choice of Green forms given above.

Definition 3.2.5. Let $\{\sigma_{4,6}, \sigma_{6,4}\}$ denote a partition of unity adapted to the divisors $\operatorname{div}(E_4)$ and $\operatorname{div}(E_6)$ on $\overline{\mathcal{A}}_2$ as in Remark 3.2.1. By abuse of notation, we will denote the restriction of the partition of unity to a subset of $\overline{\mathcal{A}}_2$ again by $\{\sigma_{4,6}, \sigma_{6,4}\}$.

Theorem 3.2.6. *The integral over the $*$-product of Green forms $g_{10}*g_6*g_4*g_{12}$ can be computed as*

$$\frac{1}{(2\pi i)^3}\int\limits_{\overline{\mathcal{A}}_2} g_{10}*g_6*g_4*g_{12} = \frac{1}{(2\pi i)^3}\int\limits_{\overline{\mathcal{A}}_2} g_{10}\wedge\omega_6\wedge\omega_4\wedge\omega_{12}$$

$$-\lim_{\varepsilon\to 0}\left(\frac{2}{(2\pi i)^2}\int\limits_{\partial B_\varepsilon(\partial\mathcal{A}_2)} [(g_6*g_4*g_{12})\wedge \mathrm{d}^c g_{10} - g_{10}\wedge \mathrm{d}^c(g_6*g_4*g_{12})]\right.$$

$$\left. + \frac{2}{2\pi i}\int\limits_{\partial B_\varepsilon(\partial\mathcal{H})} [g_4\wedge \mathrm{d}^c(\sigma_{6,4}g_6)\wedge\omega_{12} - (\sigma_{6,4}g_6)\wedge \mathrm{d}^c g_4\wedge\omega_{12}]\right)$$

$$+\frac{1}{(2\pi i)^2}\int\limits_{\overline{\mathcal{H}}} g_6\wedge\omega_4\wedge\omega_{12} + \frac{1}{4\pi i}\int\limits_{\{i\}\times_{\mathcal{H}}\overline{\mathcal{A}}_1} g_4\wedge\omega_{12} + \frac{1}{6}\int\limits_{\{i\}\times_{\mathcal{H}}\{\omega\}} g_{12},$$

where $\{\sigma_{4,6}, \sigma_{6,4}\}$ denotes a partition of unity adapted to $\operatorname{div}(E_4)$ and $\operatorname{div}(E_6)$ as in Definition 3.2.5.

Proof. We apply Proposition 3.2.4 first with

$$X = \overline{\mathcal{A}}_2, \quad D_1 = \partial\mathcal{A}_2, \quad D_2 = \emptyset,$$
$$Y = \operatorname{div}(\chi_{10}) = \partial\mathcal{A}_2 + \overline{\mathcal{H}}, \quad Z = \operatorname{div}(E_6)\cdot\operatorname{div}(E_4)\cdot\operatorname{div}(\chi_{12}).$$

Note that $|Z|\cap D_1 = \emptyset$, as D_1 is a component of $\operatorname{div}(\chi_{10})$. We obtain the equality

$$\frac{1}{(2\pi i)^3}\int\limits_{\overline{\mathcal{A}}_2} g_{10}*(g_6*g_4*g_{12}) = \lim_{\varepsilon\to 0}\left(\frac{1}{(2\pi i)^3}\int\limits_{\overline{\mathcal{A}}_2\setminus B_\varepsilon(\partial\mathcal{A}_2)} g_{10}\wedge\omega_6\wedge\omega_4\wedge\omega_{12}\right.$$

$$\left. -\frac{2}{(2\pi i)^2}\int\limits_{\partial B_\varepsilon(\partial\mathcal{A}_2)} [(g_6*g_4*g_{12})\wedge \mathrm{d}^c g_{10} - g_{10}\wedge \mathrm{d}^c(g_6*g_4*g_{12})]\right)$$

$$+\frac{1}{(2\pi i)^2}\int\limits_{\overline{\mathcal{H}}} g_6*g_4*g_{12}.$$

We will see later that the first integral in the decomposition converges for ε approaching 0 and can therefore be taken out of the limit. To comply with the prerequisites of Proposition 3.2.4, we will next consider

$$X = \overline{\mathcal{H}}, \quad D_1 = \emptyset, \quad D_2 = \partial\mathcal{H},$$
$$Y = \operatorname{div}(E_6)\cdot\operatorname{div}(E_4)\cdot\overline{\mathcal{H}} = \frac{1}{6}(\{i\}\times_{\mathcal{H}}\{\omega\}), \quad Z = \operatorname{div}(\chi_{12})\cdot\overline{\mathcal{H}} = \{i\infty\}\times_{\mathcal{H}}\overline{\mathcal{A}}_1.$$

We then obtain

$$\begin{aligned}&\frac{1}{(2\pi i)^2}\int\limits_{\overline{\mathcal{H}}}(g_6 * g_4) * g_{12}\\ &\qquad = \lim_{\varepsilon\to 0}\left(\frac{1}{(2\pi i)^2}\int\limits_{\overline{\mathcal{H}}\setminus B_\varepsilon(\partial\mathcal{H})}(g_6 * g_4)\wedge\omega_{12}\right) + \frac{1}{6}\int\limits_{\{i\}\times_{\mathcal{H}}\{\omega\}} g_{12}.\end{aligned}$$

We rewrite $g_6 * g_4$ as in (3.2.1) in the proof of Proposition 3.2.4 and obtain

$$\begin{aligned}g_6 * g_4 &= 4\pi i\big((\sigma_{4,6}g_6)\mathrm{dd}^c g_4 + \mathrm{dd}^c(\sigma_{6,4}g_6)\wedge g_4\big)\\ &= 4\pi i\big(g_6\wedge\mathrm{dd}^c g_4 + \mathrm{d}\big(\mathrm{d}^c(\sigma_{6,4}g_6)\wedge g_4 - (\sigma_{6,4}g_6)\wedge\mathrm{d}^c g_4\big)\big),\end{aligned}$$

with $\{\sigma_{4,6},\sigma_{6,4}\}$ a partition of unity as in Definition 3.2.5. Note that this expression is singular along $\operatorname{div}(E_6)\cdot\overline{\mathcal{H}} = \frac{1}{2}(\{i\}\times_{\mathcal{H}}\overline{\mathcal{A}}_1)$. We therefore set $\mathcal{H}_\varepsilon := \overline{\mathcal{H}}\setminus B_\varepsilon(\partial\mathcal{H}\cup\{i\}\times_{\mathcal{H}}\overline{\mathcal{A}}_1)$ and obtain

$$\begin{aligned}\frac{1}{(2\pi i)^2}\int\limits_{\mathcal{H}_\varepsilon}(g_6 * g_4)\wedge\omega_{12}) &= \frac{1}{(2\pi i)^2}\int\limits_{\mathcal{H}_\varepsilon} g_6\wedge\omega_4\wedge\omega_{12}\\ &+ \frac{4\pi i}{(2\pi i)^2}\int\limits_{\mathcal{H}_\varepsilon}\mathrm{d}\big(g_4\wedge\mathrm{d}^c(\sigma_{6,4}g_6)\wedge\omega_{12} - (\sigma_{6,4}g_6)\wedge\mathrm{d}^c g_4\wedge\omega_{12}\big).\end{aligned}$$

Applying Stokes' theorem and taking the limit $\varepsilon\to 0$, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{(2\pi i)^2}\int\limits_{\mathcal{H}_\varepsilon}(g_6 * g_4)\wedge\omega_{12} &= \frac{1}{(2\pi i)^2}\int\limits_{\overline{\mathcal{H}}} g_6\wedge\omega_4\wedge\omega_{12} + \frac{1}{4\pi i}\int\limits_{\{i\}\times_{\mathcal{H}}\overline{\mathcal{A}}_1} g_4\wedge\omega_{12}\\ &- \lim_{\varepsilon\to 0}\frac{2}{2\pi i}\int\limits_{\partial B_\varepsilon(\partial\mathcal{H})}\left[g_4\wedge\mathrm{d}^c(\sigma_{6,4}g_6)\wedge\omega_{12} - (\sigma_{6,4}g_6)\wedge\mathrm{d}^c g_4\wedge\omega_{12}\right].\end{aligned}$$

This completes the proof of the theorem. □

3.3 The boundary integrals

We will use local coordinates and an explicit description of the boundary $\partial\mathcal{A}_2$ to show that the boundary integrals, i.e., the integrals along $\partial B_\varepsilon(\partial\mathcal{A}_2)$ and $\partial B_\varepsilon(\partial\mathcal{H})$ appearing in the decomposition stated in Theorem 3.2.6 vanish for ε approaching 0.

In the following, we will first proof the vanishing of the integral along $\partial B_\varepsilon(\partial\mathcal{A}_2)$. For this purpose, we will give an estimate for the integrand

$$(g_6 * g_4 * g_{12}) \wedge \mathrm{d}^c g_{10} - g_{10} \wedge \mathrm{d}^c(g_6 * g_4 * g_{12}),$$

and then bound the domain of integration $\partial B_\varepsilon(\partial\mathcal{A}_2)$. Using the explicit computations in this proof, the vanishing of the integral along $\partial B_\varepsilon(\partial\mathcal{H})$ follows easily by restricting the forms under consideration to $\mathcal{H}$.

Lemma 3.3.1. *On $\partial B_\varepsilon(\partial\mathcal{A}_2)$, we have the equality*

$$\begin{aligned}&\frac{1}{(4\pi i)^2}(g_6 * g_4 * g_{12}) \wedge \mathrm{d}^c g_{10}\\ &\quad = (\sigma_{4,6} g_6) \wedge \mathrm{dd}^c g_4 \wedge \mathrm{dd}^c g_{12} \wedge \mathrm{d}^c g_{10} + \mathrm{dd}^c(\sigma_{6,4} g_6) \wedge g_4 \wedge \mathrm{dd}^c g_{12} \wedge \mathrm{d}^c g_{10}.\end{aligned}$$

Proof. To compute the form $g_6 * g_4 * g_{12}$, we first consider the non-intersecting cycles $\mathrm{div}(E_6) \cdot \mathrm{div}(E_4)$ and $\mathrm{div}(\chi_{12}) = \partial\mathcal{A}_2$. Let now $\{\sigma_{(6,4),12}, \sigma_{12,(6,4)}\}$ be a partition of unity adapted to $\mathrm{div}(E_6)\cdot\mathrm{div}(E_4)$ and $\mathrm{div}(\chi_{12})$ as in Remark 3.2.1. By the definition of the $*$-product in Theorem 3.2.2, we obtain

$$g_6 * g_4 * g_{12} = 4\pi i\big((\sigma_{12,(6,4)}(g_6 * g_4)) \wedge \mathrm{dd}^c g_{12} + \mathrm{dd}^c(\sigma_{(6,4),12}(g_6 * g_4)) \wedge g_{12}\big).$$

In a neighbourhood of $\mathrm{div}(\chi_{12}) = \partial\mathcal{A}_2$, we have $\sigma_{(6,4),12} = 0$ and $\sigma_{12,(6,4)} = 1$. For ε small, we therefore obtain on $\partial B_\varepsilon(\partial\mathcal{A}_2)$ the equality

$$g_6 * g_4 * g_{12} = 4\pi i(g_6 * g_4) \wedge \mathrm{dd}^c g_{12}.$$

Using the definition of the $*$-product to compute $g_6 * g_4$, we then obtain on $\partial B_\varepsilon(\partial\mathcal{A}_2)$ the equality

$$g_6 * g_4 * g_{12} = (4\pi i)^2\big((\sigma_{4,6} g_6) \wedge \mathrm{dd}^c g_4 \wedge \mathrm{dd}^c g_{12} + \mathrm{dd}^c(\sigma_{6,4} g_6) \wedge g_4 \wedge \mathrm{dd}^c g_{12}\big),$$

with $\{\sigma_{4,6}, \sigma_{6,4}\}$ a partition of unity as in Definition 3.2.5. This proves the claim of the lemma. ☐

Lemma 3.3.2. *On $\partial B_\varepsilon(\partial\mathcal{A}_2)$, we have the equality*

$$\begin{aligned}&\frac{1}{(4\pi i)^2} g_{10} \wedge \mathrm{d}^c(g_6 * g_4 * g_{12})\\ &\quad = g_{10} \wedge \mathrm{d}^c(\sigma_{4,6} g_6) \wedge \mathrm{dd}^c g_4 \wedge \mathrm{dd}^c g_{12} + g_{10} \wedge \mathrm{dd}^c(\sigma_{6,4} g_6) \wedge \mathrm{d}^c g_4 \wedge \mathrm{dd}^c g_{12}.\end{aligned}$$

Proof. As shown in the proof of Lemma 3.3.1, on $\partial B_\varepsilon(\partial\mathcal{A}_2)$ the form $g_6 * g_4 * g_{12}$ restricts to

$$g_6 * g_4 * g_{12} = (4\pi i)^2\big((\sigma_{4,6} g_6) \wedge \mathrm{dd}^c g_4 \wedge \mathrm{dd}^c g_{12} + \mathrm{dd}^c(\sigma_{6,4} g_6) \wedge g_4 \wedge \mathrm{dd}^c g_{12}\big).$$

On $\partial B_\varepsilon(\partial\mathcal{A}_2)$, we therefore obtain the equality

$$\begin{aligned}&\mathrm{d}^c(g_6 * g_4 * g_{12})\\ &\quad = (4\pi i)^2\big(\mathrm{d}^c(\sigma_{4,6} g_6) \wedge \mathrm{dd}^c g_4 \wedge \mathrm{dd}^c g_{12} + \mathrm{dd}^c(\sigma_{6,4} g_6) \wedge \mathrm{d}^c g_4 \wedge \mathrm{dd}^c g_{12}\big),\end{aligned}$$

which proves the claim of the lemma. ☐

Remark 3.3.3. For the convenience of the reader, we will state the formulas for the derivations d, d^c, and dd^c on $\mathcal{A}_2$, that will be used frequently below. Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a smooth function. With $\tau_j = x_j + iy_j$ $(j = 1, 2, 12)$, we have

$$\begin{aligned}
\mathrm{d}f &= (\partial + \overline{\partial})f = \sum_{j=1,2,12} \left(\frac{\partial f}{\partial x_j}\mathrm{d}x_j + \frac{\partial f}{\partial y_j}\mathrm{d}y_j \right), \\
\mathrm{d}^c f &= \frac{1}{4\pi i}(\partial - \overline{\partial})f = \frac{1}{4\pi} \sum_{j=1,2,12} \left(\frac{\partial f}{\partial x_j}\mathrm{d}y_j - \frac{\partial f}{\partial y_j}\mathrm{d}x_j \right), \\
\mathrm{dd}^c f &= -\frac{1}{2\pi i}\partial\overline{\partial}f \\
&= \frac{1}{4\pi} \sum_{j,k=1,2,12} \left(\left(\frac{\partial^2 f}{\partial x_j \partial x_k} + \frac{\partial^2 f}{\partial y_j \partial y_k} \right) \mathrm{d}x_j\mathrm{d}y_k + \frac{\partial^2 f}{\partial x_k \partial y_j}\mathrm{d}x_j\mathrm{d}x_k \right. \\
&\qquad\qquad \left. - \frac{\partial^2 f}{\partial x_j \partial y_k}\mathrm{d}y_j\mathrm{d}y_k \right).
\end{aligned}$$

Remark 3.3.4. Recall the local coordinate along the boundary $\partial\mathcal{A}_2$, given by

$$t = \exp(2\pi i \tau_1) = \exp(2\pi i x_1)\exp(-2\pi y_1).$$

The coordinate t can be expressed in polar coordinates as $t = re^{i\theta}$ $(\theta \in [0, 2\pi), r \in \mathbb{R}_{\geq 0})$. Then, the set $\partial B_\varepsilon(\partial\mathcal{A}_2)$ is defined by the condition $r = \varepsilon$, hence, by $y_1 = M$ with

$$M := -\frac{\log \varepsilon}{2\pi}.$$

Note that t is independent of the other local coordinates τ_2, τ_{12}. With the equalities

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}r}{r} + i\mathrm{d}\theta, \ \frac{\mathrm{d}\overline{t}}{\overline{t}} = \frac{\mathrm{d}r}{r} - i\mathrm{d}\theta,$$

and noting that r is constant on $\partial B_\varepsilon(\partial\mathcal{A}_2)$, we obtain the transformations

$$\begin{aligned}
\mathrm{d}x_1 &= \frac{1}{4\pi i}\left(\frac{\mathrm{d}t}{t} - \frac{\mathrm{d}\overline{t}}{\overline{t}} \right) = \frac{1}{2\pi}\mathrm{d}\theta, \\
\mathrm{d}y_1 &= -\frac{1}{4\pi}\left(\frac{\mathrm{d}t}{t} + \frac{\mathrm{d}\overline{t}}{\overline{t}} \right) = -\frac{1}{2\pi r}\mathrm{d}r = 0.
\end{aligned}$$

Furthermore, we recall that, for $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ a smooth function, we can express $\mathrm{d}^c f$ in polar coordinates as

$$\mathrm{d}^c f = \frac{1}{4\pi}\frac{\partial f}{\partial r} r\mathrm{d}\theta + \frac{1}{4\pi}\frac{\partial f}{\partial \theta}\frac{1}{r}\mathrm{d}r.$$

Lemma 3.3.5. *The Chern forms* ω_k *corresponding to the Green forms* g_k *($k = 4, 6, 10, 12$) are given by*

$$\omega_k = -4\pi i \frac{k}{2}\mathrm{dd}^c \log(\det y).$$

Proof. The Green forms g_k corresponding to the Eisenstein series E_k ($k = 4, 6$) are given by

$$\begin{aligned} g_k = -\log \|E_k\| &= -\log\bigl(|E_k|(4\pi)^k(\det y)^{k/2}\bigr) \\ &= -\log |E_k| - k\log(4\pi) - \frac{k}{2}\log(\det y). \end{aligned}$$

As the Eisenstein series E_k are holomorphic, we have $\mathrm{dd}^c \log |E_k| = 0$. The same considerations apply to the Green forms corresponding to the cusp forms χ_{10} and χ_{12}. Hence, we have

$$\omega_k = 4\pi i \mathrm{dd}^c g_k = -4\pi i\frac{k}{2}\mathrm{dd}^c \log(\det y).$$

□

We now compute $\mathrm{dd}^c g_k$ in terms of the local coordinates $t = re^{i\theta}$, $\tau_2 = x_2+iy_2$, $\tau_{12} = x_{12} + iy_{12}$ (see Notation 3.1.1).

Lemma 3.3.6. *Let $M = -\log\varepsilon/2\pi$. On $\partial B_\varepsilon(\partial\mathcal{A}_2)$ the form $4\pi\mathrm{dd}^c\log(\det y)$ restricts to*

$$\begin{aligned} &4\pi\mathrm{dd}^c \log(\det y) \\ &= -\frac{y_{12}^2}{\bigl(My_2 - y_{12}^2\bigr)^2}\frac{1}{2\pi}\mathrm{d}\theta\mathrm{d}y_2 + \frac{2y_2y_{12}}{\bigl(My_2 - y_{12}^2\bigr)^2}\frac{1}{2\pi}\mathrm{d}\theta\mathrm{d}y_{12} - \frac{M^2}{\bigl(My_2 - y_{12}^2\bigr)^2}\mathrm{d}x_2\mathrm{d}y_2 \\ &\quad + \frac{2My_{12}}{\bigl(My_2 - y_{12}^2\bigr)^2}(\mathrm{d}x_2\mathrm{d}y_{12} + \mathrm{d}x_{12}\mathrm{d}y_2) - \frac{2\bigl(My_2 + y_{12}^2\bigr)}{\bigl(My_2 - y_{12}^2\bigr)^2}\mathrm{d}x_{12}\mathrm{d}y_{12}. \end{aligned}$$

Proof. With the above coordinates, the form $4\pi\mathrm{dd}^c\log(\det y)$ decomposes as

$$\begin{aligned} 4\pi\mathrm{dd}^c \log(\det y) &= 4\pi\mathrm{dd}^c \log(y_1y_2 - y_{12}^2) \\ &= -\frac{y_2^2}{(\det y)^2}\mathrm{d}x_1\mathrm{d}y_1 - \frac{y_{12}^2}{(\det y)^2}(\mathrm{d}x_1\mathrm{d}y_2 + \mathrm{d}x_2\mathrm{d}y_1) \\ &\quad + \frac{2y_2y_{12}}{(\det y)^2}(\mathrm{d}x_1\mathrm{d}y_{12} + \mathrm{d}x_{12}\mathrm{d}y_1) - \frac{y_1^2}{(\det y)^2}\mathrm{d}x_2\mathrm{d}y_2 \\ &\quad + \frac{2y_1y_{12}}{(\det y)^2}(\mathrm{d}x_2\mathrm{d}y_{12} + \mathrm{d}x_{12}\mathrm{d}y_2) - \frac{2(y_1y_2 + y_{12}^2)}{(\det y)^2}\mathrm{d}x_{12}\mathrm{d}y_{12}. \end{aligned}$$

Observing that

$$\mathrm{d}x_1 = \frac{1}{2\pi}\mathrm{d}\theta, \quad \mathrm{d}y_1 = 0$$

on $\partial B_\varepsilon(\partial\mathcal{A}_2)$, the claim follows. □

Lemma 3.3.7. *Let $M = -\log\varepsilon/2\pi$. On $\partial B_\varepsilon(\partial\mathcal{A}_2)$, the form $16\pi^2 \mathrm{dd}^c\log(\det y)\wedge \mathrm{dd}^c\log(\det y)$ restricts to*

$$\begin{aligned}
&16\pi^2\mathrm{dd}^c\log(\det y)\wedge\mathrm{dd}^c\log(\det y)\\
&= -\frac{4My_{12}}{\left(My_2-y_{12}^2\right)^3}\frac{1}{2\pi}\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}y_{12} - \frac{4y_{12}^2}{\left(My_2-y_{12}^2\right)^3}\frac{1}{2\pi}\mathrm{d}\theta\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12}\\
&\quad + \frac{4M^2}{\left(My_2-y_{12}^2\right)^3}\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12}.
\end{aligned}$$

Proof. With the formula for $4\pi\mathrm{dd}^c\log(\det y)$ in Lemma 3.3.6, we compute

$$\begin{aligned}
&16\pi^2\mathrm{dd}^c\log(\det y)\wedge\mathrm{dd}^c\log(\det y)\\
&= \frac{4My_{12}^3}{\left(My_2-y_{12}^2\right)^4}\frac{1}{2\pi}\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}y_{12} + \frac{4\left(My_2+y_{12}^2\right)y_{12}^2}{\left(My_2-y_{12}^2\right)^4}\frac{1}{2\pi}\mathrm{d}\theta\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12}\\
&\quad - \frac{4M^2y_2y_{12}}{\left(My_2-y_{12}^2\right)^4}\frac{1}{2\pi}\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}y_{12} - \frac{8My_2y_{12}^2}{\left(My_2-y_{12}^2\right)^4}\frac{1}{2\pi}\mathrm{d}\theta\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12}\\
&\quad + \frac{4M^2\left(My_2+y_{12}^2\right)}{\left(My_2-y_{12}^2\right)^4}\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12} - \frac{8M^2y_{12}^2}{\left(My_2-y_{12}^2\right)^4}\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12},
\end{aligned}$$

and the claim follows. ☐

Lemma 3.3.8. *Let $M = -\log\varepsilon/2\pi$. On $\partial B_\varepsilon(\partial\mathcal{A}_2)$, the forms $\mathrm{d}^c g_k$ corresponding to the Eisenstein series E_k $(k = 4, 6)$ restrict to*

$$\begin{aligned}
4\pi\mathrm{d}^c g_k =& \left(-\varepsilon\frac{1}{|E_k|}\frac{\partial}{\partial r}|E_k| + \frac{k}{4\pi}\frac{y_2}{My_2-y_{12}^2}\right)\mathrm{d}\theta\\
&+ \left(\frac{1}{|E_k|}\frac{\partial}{\partial y_2}|E_k| + \frac{k}{2}\frac{M}{My_2-y_{12}^2}\right)\mathrm{d}x_2\\
&+ \left(\frac{1}{|E_k|}\frac{\partial}{\partial y_{12}}|E_k| - \frac{k}{2}\frac{2y_{12}}{My_2-y_{12}^2}\right)\mathrm{d}x_{12}\\
&- \frac{1}{|E_k|}\frac{\partial}{\partial x_2}|E_k|\mathrm{d}y_2 - \frac{1}{|E_k|}\frac{\partial}{\partial x_{12}}|E_k|\mathrm{d}y_{12}.
\end{aligned}$$

Proof. The Green forms g_k corresponding to the Eisenstein series E_k $(k = 4, 6)$ are given by

$$\begin{aligned}
g_k &= -\log\left(|E_k|(4\pi)^k\left(\frac{\log r}{-2\pi}y_2 - y_{12}^2\right)^{\frac{k}{2}}\right)\\
&= -\log|E_k| - k\log(4\pi) - \frac{k}{2}\log\left(\frac{\log r}{-2\pi}y_2 - y_{12}^2\right).
\end{aligned}$$

On $\partial B_\varepsilon(\partial\mathcal{A}_2)$, we obtain in the above local coordinates

$$\begin{aligned}4\pi \mathrm{d}^c g_k = &-r\frac{1}{|E_k|}\frac{\partial}{\partial r}|E_k|\mathrm{d}\theta - \frac{k}{2}r\frac{\partial}{\partial r}\log\left(\frac{\log r}{-2\pi}y_2 - y_{12}^2\right)\mathrm{d}\theta\\ &+\sum_{j=2,12}\left(-\frac{1}{|E_k|}\frac{\partial}{\partial x_j}|E_k|\mathrm{d}y_j - \frac{k}{2}\frac{\partial}{\partial x_j}\log\left(\frac{\log r}{-2\pi}y_2 - y_{12}^2\right)\mathrm{d}y_j\right.\\ &\left.\qquad+\frac{1}{|E_k|}\frac{\partial}{\partial y_j}|E_k|\mathrm{d}x_j + \frac{k}{2}\frac{\partial}{\partial y_j}\log\left(\frac{\log r}{-2\pi}y_2 - y_{12}^2\right)\mathrm{d}x_j\right).\end{aligned}$$

Computing the partial derivatives and setting $r=\varepsilon$ gives the claimed formula. □

Lemma 3.3.9. *Let $M = -\log\varepsilon/2\pi$. On $\partial B_\varepsilon(\partial\mathcal{A}_2)$, the form $\mathrm{d}^c g_{10}$ corresponding to the cusp form χ_{10} restricts to*

$$\begin{aligned}4\pi \mathrm{d}^c g_{10} =&\left(-\varepsilon\frac{1}{|\chi_{10}|}\frac{\partial}{\partial r}|\chi_{10}| + \frac{5}{2\pi}\frac{y_2}{My_2 - y_{12}^2}\right)\mathrm{d}\theta\\ &+\left(\frac{1}{|\chi_{10}|}\frac{\partial}{\partial y_2}|\chi_{10}| + 5\frac{M}{My_2 - y_{12}^2}\right)\mathrm{d}x_2\\ &+\left(\frac{1}{|\chi_{10}|}\frac{\partial}{\partial y_{12}}|\chi_{10}| - 5\frac{2y_{12}}{My_2 - y_{12}^2}\right)\mathrm{d}x_{12}\\ &-\frac{1}{|\chi_{10}|}\frac{\partial}{\partial x_2}|\chi_{10}|\mathrm{d}y_2 - \frac{1}{|\chi_{10}|}\frac{\partial}{\partial x_{12}}|\chi_{10}|\mathrm{d}y_{12}.\end{aligned}$$

Proof. The proof runs along the same lines as the proof of Lemma 3.3.8. □

To give estimates for the integrand $(g_6 * g_4 * g_{12})\wedge \mathrm{d}^c g_{10} - g_{10}\wedge \mathrm{d}^c(g_6 * g_4 * g_{12})$, we will embed the domain of integration $\partial B_\varepsilon(\partial\mathcal{A}_2)$ into a fundamental domain for $\Gamma_2\backslash\mathbb{H}_2$.

Lemma 3.3.10. *Let $M = -\log\varepsilon/2\pi$. There is a fundamental domain $\mathcal{F}\subseteq\mathbb{H}_2$ for the action of Γ_2 on $\mathbb{H}_2$ such that the preimage of $\partial B_\varepsilon(\partial\mathcal{A}_2)$ under the quotient morphism $\pi_2\colon \mathbb{H}_2 \longrightarrow \mathcal{A}_2$ restricted to $\mathcal{F}$ is contained in the set S_ε given by restricting the local coordinates under consideration (see Notation 3.1.1 and Remark 3.3.4) as follows:*

$$y_1 = M,\ y_2\in[1/2, M],\ y_{12}\in[0, y_2/2],\ \theta\in[0,2\pi),\ x_2, x_{12}\in[-1/2,1/2]. \tag{3.3.1}$$

Proof. Recall from Chapter 1, Theorem 1.2.12, that a fundamental domain $\mathcal{F}_2$ for the action of Γ_2 on $\mathbb{H}_2$ is given by the following Minkowski conditions on $\tau = x + iy \in \mathbb{H}_2$:

(i) For all $\gamma = \left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right)\in\Gamma_2$ the inequality $|\det(C\tau + D)|\geq 1$ holds.

(ii) The matrix y is Minkowski reduced, i.e., for all $l \in \mathbb{Z}^2$ such that the last $g-k+1 = 3-k$ entries are relatively prime, we have $y_k \leq l^t y l$ $(k = 1, 2)$; furthermore, $y_{12} \geq 0$ holds.

(iii) The matrix x satisfies $|x_k| \leq \frac{1}{2}$ $(k = 1, 2, 12)$.

Applying condition (ii) with $k = 1$ and $l = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, one obtains $y_1 \leq y_2$. For $k = 2$ and $l = \left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)$, one obtains

$$y_2 \leq \begin{pmatrix} 1 \\ -1 \end{pmatrix}^t \begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = y_1 + y_2 - 2y_{12},$$

and the condition $2y_{12} \leq y_1$ follows. We immediately deduce the inequality $\det y \leq y_1 y_2 \leq 2 \det y$. Furthermore, applying condition (i) for $C = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, together with condition (iii), one gets $\tau_1 \in \mathcal{F}_1$, where $\mathcal{F}_1$ is the standard fundamental domain for the action of Γ_1 on $\mathbb{H}_1$. Hence, we can assume $y_1 \geq \sqrt{3}/2 > 1/2$. Interchanging the roles of τ_1 and τ_2 by translating $\mathcal{F}_2$ employing the action of the matrix S from Notation 3.1.1 on $\mathbb{H}_2$, we obtain a new fundamental domain $\mathcal{F} = S\mathcal{F}_2$ such that $\partial B_\varepsilon(\partial \mathcal{A}_2)$ lies in the set S_ε as claimed. □

Remark 3.3.11. From now on, we will identify $\partial B_\varepsilon(\partial \mathcal{A}_2)$ with its preimage in the fundamental domain $\mathcal{F}$ introduced in Lemma 3.3.10.

Lemma 3.3.12. *The partition of unity* $\{\sigma_{4,6}, \sigma_{6,4}\}$ *adapted to the divisors* $\mathrm{div}(E_4)$ *and* $\mathrm{div}(E_6)$ *as in Definition 3.2.5 can be chosen in a way such that in a small neighbourhood of* $\partial \mathcal{A}_2$*, and hence on* $\partial B_\varepsilon(\partial \mathcal{A}_2)$ *(for small* $\varepsilon > 0$*), the following properties hold:*

(i) The value of $\sigma_{4,6}$ *and* $\sigma_{6,4}$ *depends only on the value of the coordinate* τ_2*, i.e.,* $\sigma_{4,6}(\tau) = \sigma_{4,6}(\tau_2)$ *and* $\sigma_{6,4}(\tau) = \sigma_{6,4}(\tau_2)$*.*

(ii) We have $\sigma_{4,6}(\tau_2) = \sigma_{6,4}(\tau_2) = 1/2$ *for* $y_2 > 2$*.*

Proof. By (3.1.6), the value of E_4 and E_6 restricted to $\partial \mathcal{A}_2$ only depends on the coordinate τ_2. Hence, we can assume the same for a partition of unity $\{\widetilde{\sigma}_{4,6}(\tau_2), \widetilde{\sigma}_{6,4}(\tau_2)\}$ adapted to the cycles $\mathrm{div}(E_4) \cdot \partial \mathcal{A}_2$ and $\mathrm{div}(E_6) \cdot \partial \mathcal{A}_2$ on $\partial \mathcal{A}_2$. In a small neighbourhood of $\partial \mathcal{A}_2$, the functions $\widetilde{\sigma}_{4,6}(\tau_2)$ and $\widetilde{\sigma}_{6,4}(\tau_2)$ will still be equal to 1 in neighbourhoods of the divisors $\mathrm{div}(E_4)$ and $\mathrm{div}(E_6)$, respectively, and, therefore, satisfy (i). Furthermore, we note that the cycles $\mathrm{div}(E_4) \cdot \partial \mathcal{A}_2$ and $\mathrm{div}(E_6) \cdot \partial \mathcal{A}_2$ are supported in the open set defined by the condition $y_2 < 2$, as $\mathrm{Im}(i), \mathrm{Im}(\omega) < 2$. The same holds true for $\mathrm{div}(E_4)$ and $\mathrm{div}(E_6)$ in a small neighbourhood of $\partial \mathcal{A}_2$. Therefore, we can choose $\widetilde{\sigma}_{4,6}(\tau_2)$ and $\widetilde{\sigma}(\tau_2)$ to equal 1/2 outside this range, so they satisfy (ii). Finally, as the neighbourhood of $\partial \mathcal{A}_2$ where the partition $\{\widetilde{\sigma}_{4,6}(\tau_2), \widetilde{\sigma}_{6,4}(\tau_2)\}$ is defined does not depend on ε, we can interpolate the partition $\{\sigma_{4,6}(\tau_2), \sigma_{6,4}(\tau_2)\}$ from Definition 3.2.5 in this neighbourhood such that (i) and (ii) hold. □

We will now prove the vanishing of the integral along $\partial B_\varepsilon(\partial\mathcal{A}_2)$ by suitably subdividing the integration domain.

Definition 3.3.13. Let $\{\sigma_{4,6}, \sigma_{6,4}\}$ be a partition of unity as in Lemma 3.3.12. Let $N_4 \subseteq \partial B_\varepsilon(\partial\mathcal{A}_2)$ be a neighbourhood of $|\operatorname{div}(E_4)| \cap \partial B_\varepsilon(\partial\mathcal{A}_2)$ such that $\sigma_{4,6} = 1$ on N_4, and let $N_6 \subseteq \partial B_\varepsilon(\partial\mathcal{A}_2)$ be a neighbourhood of $|\operatorname{div}(E_6)| \cap \partial B_\varepsilon(\partial\mathcal{A}_2)$ such that $\sigma_{6,4} = 1$ on N_6. We define an open subset U of $\partial B_\varepsilon(\partial\mathcal{A}_2)$ by setting

$$U := N_4 \cup N_6 \cup (\partial B_\varepsilon(\partial\mathcal{A}_2) \cap \{\tau \in \mathcal{F} \,|\, y_2 > 2\}) \subseteq \partial B_\varepsilon(\partial\mathcal{A}_2).$$

We note that this is a disjoint union.

Proposition 3.3.14. *For $U \subseteq \partial B_\varepsilon(\partial\mathcal{A}_2)$ as in Definition 3.3.13, the integral*

$$\frac{1}{(4\pi i)^2} \int\limits_U [(g_6 * g_4 * g_{12}) \wedge \mathrm{d}^c g_{10} - g_{10} \wedge \mathrm{d}^c(g_6 * g_4 * g_{12})]$$

converges absolutely, and its value tends to 0 for ε approaching 0.

Proof. Applying Lemmas 3.3.1 and 3.3.2, we see that the form

$$\frac{1}{(4\pi i)^2}\big((g_6 * g_4 * g_{12}) \wedge \mathrm{d}^c g_{10} - g_{10} \wedge \mathrm{d}^c(g_6 * g_4 * g_{12})\big)$$

restricts on N_4 to

$$g_6 \wedge \mathrm{dd}^c g_4 \wedge \mathrm{dd}^c g_{12} \wedge \mathrm{d}^c g_{10} - g_{10} \wedge \mathrm{d}^c g_6 \wedge \mathrm{dd}^c g_4 \wedge \mathrm{dd}^c g_{12}, \tag{3.3.2}$$

on N_6 to

$$\mathrm{dd}^c g_6 \wedge g_4 \wedge \mathrm{dd}^c g_{12} \wedge \mathrm{d}^c g_{10} - g_{10} \wedge \mathrm{dd}^c g_6 \wedge \mathrm{d}^c g_4 \wedge \mathrm{dd}^c g_{12}, \tag{3.3.3}$$

and on $\partial B_\varepsilon(\partial\mathcal{A}_2) \cap \{\tau \in \mathcal{F} \,|\, y_2 > 2\}$ to

$$\begin{aligned}\frac{1}{2}\big(&g_6 \wedge \mathrm{dd}^c g_4 \wedge \mathrm{dd}^c g_{12} \wedge \mathrm{d}^c g_{10} - g_{10} \wedge \mathrm{d}^c g_6 \wedge \mathrm{dd}^c g_4 \wedge \mathrm{dd}^c g_{12} \\ &+\mathrm{dd}^c g_6 \wedge g_4 \wedge \mathrm{dd}^c g_{12} \wedge \mathrm{d}^c g_{10} - g_{10} \wedge \mathrm{dd}^c g_6 \wedge \mathrm{d}^c g_4 \wedge \mathrm{dd}^c g_{12}\big),\end{aligned} \tag{3.3.4}$$

as here $\sigma_{4,6} = \sigma_{6,4} = 1/2$ holds. With formulas (3.3.2), (3.3.3), and (3.3.4), we can explicitely express the integrand on U in the form

$$f(\tau)\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12},$$

where $f(\tau)$ is a smooth function depending on ε. We will first bound $f(\tau)$ on N_4, and then deduce that the same bound holds for the whole U. Applying

Lemmas 3.3.7, 3.3.8 and 3.3.9 to compute (3.3.2), and with $M = -\log \varepsilon / 2\pi$, one finds

$$\begin{aligned}
f(\tau) =& \frac{3}{16\pi^3}\left(-\log|E_6| - 6\log(4\pi) - 3\log\left(My_2 - y_{12}^2\right)\right) \times \\
& \left(\frac{4M^2}{\left(My_2 - y_{12}^2\right)^3}\left(-\varepsilon\frac{1}{|\chi_{10}|}\frac{\partial}{\partial r}|\chi_{10}| + \frac{5}{2\pi}\frac{y_2}{My_2 - y_{12}^2}\right)\right. \\
& + \frac{4y_{12}^2}{\left(My_2 - y_{12}^2\right)^3}\frac{1}{2\pi}\left(\frac{1}{|\chi_{10}|}\frac{\partial}{\partial y_2}|\chi_{10}| + 5\frac{M}{My_2 - y_{12}^2}\right) \\
& \left. + \frac{4My_{12}}{\left(My_2 - y_{12}^2\right)^3}\frac{1}{2\pi}\left(\frac{1}{|\chi_{10}|}\frac{\partial}{\partial y_{12}}|\chi_{10}| - 10\frac{y_{12}}{My_2 - y_{12}^2}\right)\right) \\
& - \frac{3}{16\pi^3}\left(-\log|\chi_{10}| - 10\log(4\pi) - 5\log\left(My_2 - y_{12}^2\right)\right) \times \\
& \left(\frac{4M^2}{\left(My_2 - y_{12}^2\right)^3}\left(-\varepsilon\frac{1}{|E_6|}\frac{\partial}{\partial r}|E_6| + \frac{3}{2\pi}\frac{y_2}{My_2 - y_{12}^2}\right)\right. \\
& + \frac{4y_{12}^2}{\left(My_2 - y_{12}^2\right)^3}\frac{1}{2\pi}\left(\frac{1}{|E_6|}\frac{\partial}{\partial y_2}|E_6| + 3\frac{M}{My_2 - y_{12}^2}\right) \\
& \left. + \frac{4My_{12}}{\left(My_2 - y_{12}^2\right)^3}\frac{1}{2\pi}\left(\frac{1}{|E_6|}\frac{\partial}{\partial y_{12}}|E_6| - 6\frac{y_{12}}{My_2 - y_{12}^2}\right)\right). \qquad (3.3.5)
\end{aligned}$$

The modular form χ_{10} has a simple zero along $\partial\mathcal{A}_2$, and a simple zero along $\mathcal{H}$, given by the equality $\tau_{12}^2 = 0$ for the local coordinate τ_{12}^2 around $\mathcal{H}$. Therefore, it decomposes as $\chi_{10} = r\tau_{12}^2\phi_{10}(\tau)$, with $\phi_{10}(\tau)$ a non-zero smooth function in a neighbourhood of $\partial\mathcal{A}_2$. Hence, the terms

$$\varepsilon\frac{1}{|\chi_{10}|}\frac{\partial}{\partial r}|\chi_{10}|, \quad \frac{1}{|\chi_{10}|}\frac{\partial}{\partial y_2}|\chi_{10}|, \quad \frac{y_{12}}{|\chi_{10}|}\frac{\partial}{\partial y_{12}}|\chi_{10}|$$

are bounded from above on $\partial B_\varepsilon(\partial\mathcal{A}_2)$, and the bound is independent of ε. Note that, as we are outside N_6, the term $\log|E_6|$ and its partial derivatives occuring in (3.3.5) are bounded independently of ε as well. Applying these considerations and noting that the term $\log|\chi_{10}|$ is of order M, we can bound the absolute value of $f(\tau)$ on N_4 by

$$\begin{aligned}|f(\tau)| \prec & \frac{\log\left(My_2 - y_{12}^2\right)}{\left(My_2 - y_{12}^2\right)^3} \cdot \Bigg(M^2\left(1 + \frac{y_2}{My_2 - y_{12}^2}\right) \\ & + y_{12}^2\left(1 + \frac{M}{My_2 - y_{12}^2}\right) + M\left(1 + \frac{y_{12}^2}{My_2 - y_{12}^2}\right)\Bigg) \\ & + \frac{M}{\left(My_2 - y_{12}^2\right)^3} \cdot \Bigg(M^2\left(\varepsilon + \frac{y_2}{My_2 - y_{12}^2}\right) \\ & + y_{12}^2\left(1 + \frac{M}{My_2 - y_{12}^2}\right) + My_{12}\left(1 + \frac{y_{12}}{My_2 - y_{12}^2}\right)\Bigg).\end{aligned}$$

With Lemma 3.3.10, one obtains the estimates

$$\frac{1}{2} \leq y_2 \leq M, \quad 0 \leq y_{12} \leq y_2, \quad \frac{1}{My_2} \leq \frac{1}{My_2 - y_{12}^2} \leq \frac{2}{My_2};$$

moreover, $\varepsilon \leq 1/M$ for small $\varepsilon > 0$. Hence, we can further bound $|f(\tau)|$ as

$$\begin{aligned}|f(\tau)| \prec & \frac{\log M}{(My_2)^3} \cdot \left(M^2\left(1 + \frac{y_2}{My_2}\right) + y_2^2\left(1 + \frac{M}{My_2}\right) + M\left(1 + \frac{y_2^2}{My_2}\right)\right) \\ & + \frac{M}{(My_2)^3} \cdot \left(M^2\left(\varepsilon + \frac{y_2}{My_2}\right) + y_2^2\left(1 + \frac{M}{My_2}\right) + My_2\left(1 + \frac{y_2}{My_2}\right)\right). \\ \prec & \log M\Big(\frac{1}{My_2^3} + \frac{1}{M^3y_2} + \frac{1}{M^2y_2^3}\Big) + \frac{1}{My_2^3} + \frac{1}{M^2y_2} + \frac{1}{My_2^2} \\ \prec & \frac{\log M}{My_2^3} + \frac{1}{My_2^2}. \qquad (3.3.6)\end{aligned}$$

By analogous computations, one easily sees that the same bound for $|f(\tau)|$ holds on the whole of U. As, again by Lemma 3.3.10, the domain of integration is contained in the set S_ε given by the restrictions

$$y_1 = M,\ y_2 \in [1/2, M],\ y_{12} \in [0, y_2/2],\ \theta \in [0, 2\pi),\ x_2, x_{12} \in [-1/2, 1/2],$$

the value of the integral of $[(g_6 * g_4 * g_{12}) \wedge \mathrm{d}^c g_{10} - g_{10} \wedge \mathrm{d}^c(g_6 * g_4 * g_{12})]$ over the open set U can be bounded from above by the integral of the estimate for

$|f(\tau)| \mathrm{d}\theta \mathrm{d}x_2 \mathrm{d}y_2 \mathrm{d}x_{12} \mathrm{d}y_{12}$ given in (3.3.6) over the set S_ε. One obtains

$$\begin{aligned} &\left| \frac{1}{(4\pi i)^2} \int\limits_U [(g_6 * g_4 * g_{12}) \wedge \mathrm{d}^c g_{10} - g_{10} \wedge \mathrm{d}^c(g_6 * g_4 * g_{12})] \right| \\ &\quad \prec \int\limits_{S_\varepsilon} \left(\frac{\log M}{M y_2^3} + \frac{1}{M y_2^2} \right) \mathrm{d}\theta \mathrm{d}x_2 \mathrm{d}y_2 \mathrm{d}x_{12} \mathrm{d}y_{12} \\ &\quad \prec \int\limits_{1/2}^{M} \left(\frac{\log M}{M y_2^2} + \frac{1}{M y_2} \right) \mathrm{d}y_2 \prec \frac{\log M}{M}, \end{aligned}$$

as integrating over θ and x_j $(j = 1, 12, 2)$ gives a factor 2π, and integration over y_{12} multiplies the integrand by $y_2/2$. As $\log M/M$ tends to 0 for ε approaching 0, the claim follows. ☐

To prove the vanishing of the boundary integral outside the open set U as in Definition 3.3.13, we first prove the following lemma.

Lemma 3.3.15. *For any ϑ-series $\vartheta_{a,b}$ as in Definition 3.1.2, we have the bound*

$$\left| \frac{\partial}{\partial x_1} \vartheta_{a,b}(\tau) \right| = 2\pi \left| \frac{\partial}{\partial \theta} \vartheta_{a,b}(\tau) \right| \prec \varepsilon^{\frac{1}{16}}$$

for $\tau \in S_\varepsilon$ defined in Lemma 3.3.10.

Proof. On $\partial\mathcal{A}_2$, the coordinate τ_1 is constant, equal to $i\infty$. Hence, the partial derivative $\frac{\partial}{\partial x_1}\vartheta_{a,b}$ vanishes on $\partial\mathcal{A}_2$. To determine its vanishing order, we note that for a matrix $\tau \in S_\varepsilon$ the inequalities $y_{12} \leq y_2/2$ and $y_2 < y_1$ hold by definition of S_ε. We deduce the inequality $(y_1/2)y_2 - y_{12}^2 > 0$. Therefore, the matrix

$$\tau' = \begin{pmatrix} \tau_1 - i\frac{y_1}{2} & \tau_{12} \\ \tau_{12} & \tau_2 \end{pmatrix},$$

obtained from τ by replacing the coordinate y_1 by $y_1/2$, lies in $\mathbb{H}_2$. Letting $n = (n_1, n_2)^t \in \mathbb{Z}^2$ and $a = (a_1, a_2)^t, b = (b_1, b_2)^t \in (\mathbb{Z}/2\mathbb{Z})^2$ as in Definition 3.1.2, we now compute

$$\begin{aligned} \vartheta_{a,b}(\tau) &= \sum_{n \in \mathbb{Z}^2} e^{2\pi i \left(\frac{1}{2}\left(n+\frac{a}{2}\right)^t \tau \left(n+\frac{a}{2}\right) + \left(n+\frac{a}{2}\right)^t \frac{b}{2} \right)} \\ &= \sum_{n \in \mathbb{Z}^2} e^{\pi i \left(\left(n_1+\frac{a_1}{2}\right)^2 \tau_1 + 2\left(n_1+\frac{a_1}{2}\right)\left(n_2+\frac{a_2}{2}\right)\tau_{12} + \left(n_2+\frac{a_2}{2}\right)^2 \tau_2 + \left(n_1+\frac{a_1}{2}\right)b_1 + \left(n_2+\frac{a_2}{2}\right)b_2 \right)} \\ &= \sum_{n \in \mathbb{Z}^2} e^{-\pi\left(n_1+\frac{a_1}{2}\right)^2 \frac{y_1}{2}} e^{2\pi i \left(\frac{1}{2}\left(n+\frac{a}{2}\right)^t \tau' \left(n+\frac{a}{2}\right) + \left(n+\frac{a}{2}\right)^t \frac{b}{2} \right)} \end{aligned}$$

For $n_1 \neq 0$ or $a_1 \neq 0$, we deduce with $y_1 = -\log\varepsilon/2\pi$ that

$$e^{-\pi\left(n_1+\frac{a_1}{2}\right)^2\frac{y_1}{2}} \leq e^{\frac{1}{16}\log\varepsilon} = \varepsilon^{\frac{1}{16}}.$$

For the partial derivative $\frac{\partial}{\partial x_1}\vartheta_{a,b}$, we now obtain

$$\begin{aligned}
&\left|\frac{\partial}{\partial x_1}\vartheta_{a,b}(\tau)\right| \\
&= \left|\sum_{n\in\mathbb{Z}^2} e^{-\pi\left(n_1+\frac{a_1}{2}\right)^2\frac{y_1}{2}}\pi i\left(n_1+\frac{a_1}{2}\right)^2 e^{2\pi i\left(\frac{1}{2}\left(n+\frac{a}{2}\right)^t\tau'\left(n+\frac{a}{2}\right)+\left(n+\frac{a}{2}\right)^t\frac{b}{2}\right)}\right| \\
&\leq \varepsilon^{\frac{1}{16}}\sum_{n\in\mathbb{Z}^2}\left|\pi i\left(n_1+\frac{a_1}{2}\right)^2 e^{2\pi i\left(\frac{1}{2}\left(n+\frac{a}{2}\right)^t\tau'\left(n+\frac{a}{2}\right)+\left(n+\frac{a}{2}\right)^t\frac{b}{2}\right)}\right|,
\end{aligned}$$

and the latter sum converges, as the sum

$$\sum_{n\in\mathbb{Z}^2}\pi i\left(n_1+\frac{a_1}{2}\right)^2 e^{2\pi i\left(\frac{1}{2}\left(n+\frac{a}{2}\right)^t\tau'\left(n+\frac{a}{2}\right)+\left(n+\frac{a}{2}\right)^t\frac{b}{2}\right)} = \frac{\partial}{\partial x_1}\vartheta_{a,b}(\tau')$$

converges absolutely for $\tau' \in \mathbb{H}_2$. This proves the claim. □

Proposition 3.3.16. *For $U \subseteq \partial B_\varepsilon(\partial\mathcal{A}_2)$ as in Definition 3.3.13, the integral*

$$\frac{1}{(4\pi i)^2}\int\limits_{\partial B_\varepsilon(\partial\mathcal{A}_2)\setminus U}\left[(g_6 * g_4 * g_{12})\wedge \mathrm{d}^c g_{10} - g_{10}\wedge\mathrm{d}^c(g_6 * g_4 * g_{12})\right]$$

converges absolutely, and its value tends to 0 for ε approaching 0.

Proof. By Lemmas 3.3.1 and 3.3.2, the integrand has the form

$$\begin{aligned}
&(\sigma_{4,6}g_6)\wedge\mathrm{dd}^c g_4\wedge\mathrm{dd}^c g_{12}\wedge\mathrm{d}^c g_{10} + \mathrm{dd}^c(\sigma_{6,4}g_6)\wedge g_4\wedge\mathrm{dd}^c g_{12}\wedge\mathrm{d}^c g_{10} \\
&- g_{10}\wedge\mathrm{d}^c(\sigma_{4,6}g_6)\wedge\mathrm{dd}^c g_4\wedge\mathrm{dd}^c g_{12} - g_{10}\wedge\mathrm{dd}^c(\sigma_{6,4}g_6)\wedge\mathrm{d}^c g_4\wedge\mathrm{dd}^c g_{12}
\end{aligned} \tag{3.3.7}$$

on $\partial B_\varepsilon(\partial\mathcal{A}_2)\setminus U$, with $\{\sigma_{4,6},\sigma_{6,4}\}$ a partition of unity adapted to $\mathrm{div}(E_4)$ and $\mathrm{div}(E_6)$ as in Lemma 3.3.12. In the following, we will give bounds for the forms occuring in (3.3.7). For forms α,β on $\partial B_\varepsilon(\partial\mathcal{A}_2)$, we will use the notation $\alpha \prec \beta$ if there exists a positive real constant C such that $\int_V \alpha \leq C\int_V \beta$ for all closed subsets $V \subseteq \partial B_\varepsilon(\partial\mathcal{A}_2)$ whenever the integrals are defined. In analogy to Lemma 3.3.8, we compute

$$\begin{aligned}
\mathrm{d}g_6 = &-\frac{1}{|E_6|}\frac{\partial}{\partial\theta}|E_6|\mathrm{d}\theta - \frac{1}{|E_6|}\frac{\partial}{\partial x_2}|E_6|\mathrm{d}x_2 - \frac{1}{|E_6|}\frac{\partial}{\partial x_{12}}|E_6|\mathrm{d}x_{12} \\
&-\left(\frac{1}{|E_6|}\frac{\partial}{\partial y_2}|E_6| + 3\frac{M}{My_2-y_{12}^2}\right)\mathrm{d}y_2 \\
&-\left(\frac{1}{|E_6|}\frac{\partial}{\partial y_{12}}|E_6| - 3\frac{2y_{12}}{My_2-y_{12}^2}\right)\mathrm{d}y_{12}.
\end{aligned}$$

As we can express E_6 in terms of ϑ-series as in Definition 3.1.5, we can apply Lemma 3.3.15 and obtain the bound

$$\left|\frac{\partial}{\partial x_1}E_6\right| = \frac{1}{2\pi}\left|\frac{\partial}{\partial \theta}E_6\right| \prec \varepsilon^{\frac{1}{16}} \prec \frac{1}{M}$$

on S_ε for ε small.

Applying the conditions $1/2 < y_2 < 2$ and $y_{12} \leq y_2$ to the coordinate expansions of $\mathrm{d}^c g_k$ and $\mathrm{dd}^c g_k$ $(k = 4, 6, 12)$ in Lemmas 3.3.6, 3.3.8, and 3.3.9, and noting that $\sigma_{6,4}$ only depends on the local coordinate τ_2, we obtain the estimates

$$\mathrm{d}\sigma_{6,4}, \mathrm{d}^c\sigma_{6,4} \prec \mathrm{d}x_2 + \mathrm{d}y_2,$$

$$\mathrm{dd}^c\sigma_{6,4} \prec \mathrm{d}x_2\mathrm{d}y_2,$$

$$\mathrm{d}^c g_6, \mathrm{d}g_6, \mathrm{d}^c g_4 \prec \frac{1}{M}\mathrm{d}\theta + \mathrm{d}x_2 + \mathrm{d}y_2 + \mathrm{d}x_{12} + \mathrm{d}y_{12},$$

$$\mathrm{d}^c g_{10} \prec \mathrm{d}\theta + \mathrm{d}x_2 + \mathrm{d}y_2 + \mathrm{d}x_{12} + \mathrm{d}y_{12},$$

$$\begin{aligned}&\mathrm{dd}^c g_4, \mathrm{dd}^c g_6, \mathrm{dd}^c g_{12}\\ &\qquad \prec \frac{1}{M^2}(\mathrm{d}\theta\mathrm{d}y_2 + \mathrm{d}\theta\mathrm{d}y_{12}) + \frac{1}{M}(\mathrm{d}x_2\mathrm{d}y_{12} + \mathrm{d}x_{12}\mathrm{d}y_2 + \mathrm{d}x_{12}\mathrm{d}y_{12}) + \mathrm{d}x_2\mathrm{d}y_2\end{aligned}$$

on $\partial B_\varepsilon(\partial\mathcal{A}_2) \setminus U$. Using these estimates to bound the summands in (3.3.7) on $\partial B_\varepsilon(\partial\mathcal{A}_2) \setminus U$, one obtains

$$\begin{aligned}
(\sigma_{4,6}g_6) \wedge \mathrm{dd}^c g_4 \wedge \mathrm{dd}^c g_{12} \wedge \mathrm{d}^c g_{10} &\prec \frac{\log M}{M}\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12},\\
\mathrm{dd}^c\sigma_{6,4} \wedge g_6 \wedge g_4 \wedge \mathrm{dd}^c g_{12} \wedge \mathrm{d}^c g_{10} &\prec \frac{(\log M)^2}{M}\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12},\\
\mathrm{d}^c\sigma_{6,4} \wedge \mathrm{d}g_6 \wedge g_4 \wedge \mathrm{dd}^c g_{12} \wedge \mathrm{d}^c g_{10} &\prec \frac{\log M}{M}\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12},\\
\mathrm{d}\sigma_{6,4} \wedge \mathrm{d}^c g_6 \wedge g_4 \wedge \mathrm{dd}^c g_{12} \wedge \mathrm{d}^c g_{10} &\prec \frac{\log M}{M}\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12},\\
\sigma_{6,4} \wedge \mathrm{dd}^c g_6 \wedge g_4 \wedge \mathrm{dd}^c g_{12} \wedge \mathrm{d}^c g_{10} &\prec \frac{\log M}{M}\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12},\\
g_{10} \wedge \mathrm{d}^c\sigma_{4,6} \wedge g_6 \wedge \mathrm{dd}^c g_4 \wedge \mathrm{dd}^c g_{12} &\prec \frac{\log M}{M^2}\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12},\\
g_{10} \wedge \sigma_{4,6} \wedge \mathrm{d}^c g_6 \wedge \mathrm{dd}^c g_4 \wedge \mathrm{dd}^c g_{12} &\prec \frac{1}{M}\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12},\\
g_{10} \wedge \mathrm{dd}^c\sigma_{6,4} \wedge g_6 \wedge \mathrm{d}^c g_4 \wedge \mathrm{dd}^c g_{12} &\prec \frac{\log M}{M}\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12},\\
g_{10} \wedge \mathrm{d}^c\sigma_{6,4} \wedge \mathrm{d}g_6 \wedge \mathrm{d}^c g_4 \wedge \mathrm{dd}^c g_{12} &\prec \frac{1}{M}\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12},\\
g_{10} \wedge \mathrm{d}\sigma_{6,4} \wedge \mathrm{d}^c g_6 \wedge \mathrm{d}^c g_4 \wedge \mathrm{dd}^c g_{12} &\prec \frac{1}{M}\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12},\\
g_{10} \wedge \sigma_{6,4} \wedge \mathrm{dd}^c g_6 \wedge \mathrm{d}^c g_4 \wedge \mathrm{dd}^c g_{12} &\prec \frac{1}{M}\mathrm{d}\theta\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12}.
\end{aligned}$$

Therefore, their integral over the bounded domain $\partial B_\varepsilon(\partial\mathcal{A}_2) \setminus U$ tends to 0 for ε approaching 0 and, hence, M approaching ∞. □

Theorem 3.3.17. *The integral*

$$\frac{1}{(4\pi i)^2} \int\limits_{\partial B_\varepsilon(\partial\mathcal{A}_2)} [(g_6 * g_4 * g_{12}) \wedge \mathrm{d}^c g_{10} - g_{10} \wedge \mathrm{d}^c(g_6 * g_4 * g_{12})]$$

converges absolutely, and its value tends to 0 for ε approaching 0.

Proof. The Theorem combines the results of Propositions 3.3.14 and 3.3.16. □

We now deduce the vanishing of the boundary integral along $\partial B_\varepsilon(\partial\mathcal{H})$ using similar methods.

Theorem 3.3.18. *Let $\{\sigma_{4,6}, \sigma_{6,4}\}$ be a partition of unity adapted to* $\mathrm{div}(E_4)$ *and* $\mathrm{div}(E_6)$ *as in Lemma 3.3.12. Then, the integral*

$$\frac{4}{2\pi i} \int\limits_{\partial B_\varepsilon(\partial\mathcal{H})} [g_4 \wedge \mathrm{d}^c(\sigma_{6,4} g_6) \wedge \omega_{12} - (\sigma_{6,4} g_6) \wedge \mathrm{d}^c g_4 \wedge \omega_{12}]$$

converges absolutely, and its value tends to 0 for ε approaching 0.

Proof. By Lemma 3.3.6, noting that $\tau_{12} = 0$ and $y_1 = M$ are constant on $\partial B_\varepsilon(\partial\mathcal{H})$, the form ω_{12} reduces to

$$\omega_{12} = 4\pi i \mathrm{d}\mathrm{d}^c g_{12} = 6i \frac{\mathrm{d}x_2 \mathrm{d}y_2}{y_2^2}$$

on $\partial B_\varepsilon(\partial\mathcal{H})$. As $\sigma_{6,4}$ depends only on the coordinate τ_2, the form $\mathrm{d}^c\sigma_{6,4} \wedge \omega_{12}$ vanishes. Applying Lemma 3.3.8 to give estimates for $\mathrm{d}^c g_4$ and $\mathrm{d}^c g_6$, we can bound the integrand by

$$|g_4 \wedge \mathrm{d}^c(\sigma_{6,4} g_6) \wedge \omega_{12} - (\sigma_{6,4} g_6) \wedge \mathrm{d}^c g_4 \wedge \omega_{12}| \prec (|g_4| + |g_6|) \frac{1}{M} \mathrm{d}\theta \frac{\mathrm{d}x_2 \mathrm{d}y_2}{y_2^2}.$$

Furthermore, note that on $\partial B_\varepsilon(\partial\mathcal{H})$, we have the equality $\tau_1 = x_1 + iM$, with $M = -\log\varepsilon/2\pi$. The equality $E_k(\tau) = E_k(\tau_1)E_k(\tau_2)$ on $\mathcal{H}$ $(k = 4, 6)$, see (3.1.4), induces the decomposition of the Green forms $g_4(\tau)$ and $g_6(\tau)$ as

$$g_k(\tau) = g_k(\tau_1) + g_k(\tau_2) \quad (k = 4, 6)$$

on $\partial B_\varepsilon(\partial\mathcal{H})$. As the degree 1 Eisenstein series $E_4(\tau_1)$ and $E_6(\tau_1)$ do not vanish for M approaching ∞, the term

$$g_k(\tau_1) = -\log \|E_k(\tau_1)\|_{\mathrm{Pet}} = -|E_k(\tau_1)| - k/2\log(4\pi) - k/2\log M \quad (k = 4, 6)$$

is of order $\log M$ for $\tau_1 = x_1 + iM$. Therefore, on $\partial B_\varepsilon(\partial\mathcal{H})$, we obtain

$$|g_k(\tau)| \prec \log M + |g_k(\tau_2)| \quad (k = 4, 6).$$

The domain of integration $\partial B_\varepsilon(\partial\mathcal{H}) \subseteq \mathcal{H}$ is of the form $\partial B_\varepsilon(i\infty) \times_{\mathcal{H}} \mathcal{A}_1$. Hence, we can estimate

$$\left| \frac{4}{2\pi i} \int\limits_{\partial B_\varepsilon(\partial\mathcal{H})} [g_4 \wedge \mathrm{d}^c(\sigma_{6,4} g_6) \wedge \omega_{12} - (\sigma_{6,4} g_6) \wedge \mathrm{d}^c g_4 \wedge \omega_{12}] \right|$$
$$\prec \frac{\log M}{M} + \frac{1}{M} \int\limits_0^{2\pi} \int\limits_{\mathcal{A}_1} [|g_4(\tau_2)| + |g_6(\tau_2)|] \, \frac{\mathrm{d}x_2 \mathrm{d}y_2}{y_2^2} \mathrm{d}\theta.$$

As the integrals of $g_4(\tau_2)$ and $g_6(\tau_2)$ over $\mathcal{A}_1$ converge absolutely, the last sum vanishes for ε approaching 0, and, hence, M approaching ∞, and the claim follows. □

3.4 Computation of the lower dimensional integrals

In the following, we will compute the last three integrals appearing in Theorem 3.2.6 by tracing them back to integrals of Green forms corresponding to Eisenstein series on $\mathcal{A}_1$. These can then be evaluated applying the following modular version of Jensen's formula stated by Rohrlich [41].

Lemma 3.4.1. *Let f be a modular form of weight k with $f(i\infty) = 1$. Then, the equality*

$$\int\limits_{\mathcal{A}_1} \log \|f(\tau_1)\|_{\mathrm{Pet}} \frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2}$$
$$= -k \left(\frac{1}{2}\zeta(-1) + \zeta'(-1) \right) - \frac{1}{12} \sum_{\tau_0 \in |\operatorname{div}(f)|} \mathrm{ord}_{\tau_0}(f) \log \|\Delta(\tau_0)\|_{\mathrm{Pet}}$$

holds. Here, $\tau_1 = x_1 + iy_1$ is the coordinate on $\mathcal{A}_1$, and the term $\|\Delta(\tau_0)\|_{\mathrm{Pet}}$ denotes the quantity $(4\pi)^6 |\Delta(\tau_0) y_0^6|$.

Proof. The lemma follows from a Theorem of Rohrlich, see [41]. For the computations leading to the stated version of the formula, see the proof of Theorem 1.6.1 in [33]. □

Proposition 3.4.2. *The integral over the Humbert surface $\overline{\mathcal{H}}$ on the right hand side of Theorem 3.2.6 has the value*

$$\frac{1}{(2\pi i)^2} \int\limits_{\overline{\mathcal{H}}} g_6 \wedge \omega_{12} \wedge \omega_4 = 48 \left(\frac{1}{2}\zeta(-1) + \zeta'(-1) \right) + \frac{1}{3} \log \|\Delta(i)\|_{\mathrm{Pet}}.$$

Proof. Applying Lemma 3.3.5 and Lemma 3.3.6, and noting that $\tau_{12} = 0$ on $\mathcal{H}$, we see that the Chern forms ω_{12} and ω_4 reduce to

$$\omega_{12} = 6 \cdot 4\pi i \left(\frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2} + \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2} \right), \quad \omega_4 = 2 \cdot 4\pi i \left(\frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2} + \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2} \right)$$

on $\mathcal{H}$. For their product $\omega_4 \wedge \omega_{12}$, one obtains

$$\omega_4 \wedge \omega_{12} = 24(4\pi i)^2 \frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2} \wedge \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2}.$$

Recalling the isomorphism $\mathcal{H} \cong \mathrm{Sym}_2(\mathcal{A}_1)$ from Notation 3.1.1, we note that $\overline{\mathcal{A}}_1 \times \overline{\mathcal{A}}_1$ is a double cover of $\overline{\mathcal{H}}$. We obtain

$$\begin{aligned} \frac{1}{(2\pi i)^2} \int\limits_{\overline{\mathcal{H}}} g_6 \wedge \omega_4 \wedge \omega_{12} &= 24 \frac{(4\pi i)^2}{(2\pi i)^2} \int\limits_{\overline{\mathcal{H}}} g_{E_6} \wedge \frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2} \wedge \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2} \\ &= 96 \int\limits_{\overline{\mathcal{H}}} g_6 \wedge \frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2} \wedge \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2} = 48 \int\limits_{\overline{\mathcal{A}}_1 \times \overline{\mathcal{A}}_1} g_6 \wedge \frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2} \wedge \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2}. \end{aligned}$$

With $g_6 = -\log \|E_6\|_{\mathrm{Pet}}$ and the decomposition (3.1.4) of E_6 on $\mathcal{H}$ as $E_6(\tau) = E_6(\tau_1)E_6(\tau_2)$, the latter integral writes as

$$\begin{aligned} &48 \int\limits_{\overline{\mathcal{A}}_1 \times \overline{\mathcal{A}}_1} g_6 \wedge \frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2} \wedge \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2} \\ &= 48 \int\limits_{\overline{\mathcal{A}}_1 \times \overline{\mathcal{A}}_1} -\log \|E_6(\tau)\|_{\mathrm{Pet}} \wedge \frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2} \wedge \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2} \\ &= 48 \int\limits_{\overline{\mathcal{A}}_1} \int\limits_{\overline{\mathcal{A}}_1} \left[-\log \|E_6(\tau_1)\|_{\mathrm{Pet}} - \log \|E_6(\tau_2)\|_{\mathrm{Pet}}\right] \frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2} \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2}. \end{aligned}$$

With respect to the volume form $\frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2}$, the volume of $\mathcal{A}_1$ equals $\frac{1}{12}$. Hence, we can simplify the last integral to

$$\begin{aligned} &48 \int\limits_{\overline{\mathcal{A}}_1} \int\limits_{\overline{\mathcal{A}}_1} \left[-\log \|E_6(\tau_1)\|_{\mathrm{Pet}} - \log \|E_6(\tau_2)\|_{\mathrm{Pet}}\right] \frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2} \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2} \\ &= 48 \left(\frac{1}{12} \int_{\overline{\mathcal{A}}_1} -\log \|E_6(\tau_1)\|_{\mathrm{Pet}} \frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2} + \frac{1}{12} \int\limits_{\overline{\mathcal{A}}_1} -\log \|E_6(\tau_2)\|_{\mathrm{Pet}} \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2} \right) \\ &= -8 \int\limits_{\overline{\mathcal{A}}_1} \log \|E_6(\tau_1)\|_{\mathrm{Pet}} \frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2}. \end{aligned} \tag{3.4.1}$$

Noting that $\mathrm{ord}_i(E_6) = \frac{1}{2}$, Lemma 3.4.1 yields

$$\int\limits_{\mathcal{A}_1} \log \|E_6\|_{\mathrm{Pet}} \frac{\mathrm{d}x \mathrm{d}y}{4\pi y^2} = -6 \left(\frac{1}{2} \zeta(-1) + \zeta'(-1) \right) - \frac{1}{24} \log \|\Delta(i)\|_{\mathrm{Pet}}. \tag{3.4.2}$$

Plugging (3.4.2) into (3.4.1), the proposition follows. □

Proposition 3.4.3. *The integral over $\{i\} \times \overline{\mathcal{A}}_1$ on the right hand side of Theorem 3.2.6 has the value*

$$\frac{1}{4\pi i} \int\limits_{\{i\} \times_{\mathcal{H}} \overline{\mathcal{A}}_1} g_4 \wedge \omega_{12}$$
$$= 24 \left(\frac{1}{2}\zeta(-1) + \zeta'(-1) \right) - \frac{1}{6} \log \|\Delta(i)\|_{\mathrm{Pet}} + \frac{1}{6} \log \|\Delta(\omega)\|_{\mathrm{Pet}} - \log 2 - \frac{1}{2} \log 3.$$

Proof. As in the proof of Proposition 3.4.2, one sees that on $i \times_{\mathcal{H}} \mathcal{A}_1$, the form ω_{12} reduces to

$$\omega_{12} = 6 \cdot 4\pi i \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2}.$$

With $g_4 = -\log \|E_4\|_{\mathrm{Pet}}$ and the decomposition (3.1.4) of E_4 on $\mathcal{H}$ as $E_4(\tau) = E_4(\tau_1)E_4(\tau_2)$, it follows that

$$\begin{aligned} \frac{1}{4\pi i} \int\limits_{\{i\} \times_{\mathcal{H}} \overline{\mathcal{A}}_1} g_4 \wedge \omega_{12} &= -6 \int\limits_{\overline{\mathcal{A}}_1} \left[\log \|E_4(i)\|_{\mathrm{Pet}} + \log \|E_4(\tau_2)\|_{\mathrm{Pet}} \right] \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2} \\ &= -\frac{1}{2} \log \|E_4(i)\|_{\mathrm{Pet}} - 6 \int\limits_{\overline{\mathcal{A}}_1} \log \|E_4(\tau_2)\|_{\mathrm{Pet}} \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2}, \end{aligned} \tag{3.4.3}$$

as the volume of $\mathcal{A}_1$ with respect to the volume form $\frac{\mathrm{d}x_1 \mathrm{d}y_1}{4\pi y_1^2}$ equals $\frac{1}{12}$. Noting that $\mathrm{ord}_\omega(E_4) = \frac{1}{3}$, the integral of $\log \|E_4(\tau_2)\|_{\mathrm{Pet}}$ over $\mathcal{A}_1$ can again be computed with Lemma 3.4.1 and has the value

$$\int\limits_{\mathcal{A}_1} \log \|E_4(\tau_2)\|_{\mathrm{Pet}} \frac{\mathrm{d}x_2 \mathrm{d}y_2}{4\pi y_2^2} = -4 \left(\frac{1}{2}\zeta(-1) + \zeta'(-1) \right) - \frac{1}{36} \log \|\Delta(\omega)\|_{\mathrm{Pet}}.$$

Employing this equality to (3.4.3), we obtain

$$\frac{1}{4\pi i} \int\limits_{\{i\} \times_{\mathcal{H}} \overline{\mathcal{A}}_1} g_4 \wedge \omega_{12}$$
$$= -\frac{1}{2} \log \|E_4(i)\|_{\mathrm{Pet}} + 24 \left(\frac{1}{2}\zeta(-1) + \zeta'(-1) \right) + \frac{1}{6} \log \|\Delta(\omega)\|_{\mathrm{Pet}}. \tag{3.4.4}$$

With the relation $E_4^3(\tau_1) - E_6^2(\tau_1) = 12^3 \Delta(\tau_1)$, see (3.1.5), we deduce

$$\log \|E_4(i)\|_{\mathrm{Pet}} = \frac{1}{3} \log \|\Delta(i)\|_{\mathrm{Pet}} + 2 \log 2 + \log 3, \tag{3.4.5}$$

and, combining (3.4.4) and (3.4.5), the proposition follows. □

Proposition 3.4.4. *The integral over* $\{i\} \times_{\mathcal{H}} \{\omega\}$ *on the right hand side of Theorem 3.2.6 has the value*

$$\frac{1}{6}\int\limits_{\{i\}\times_{\mathcal{H}}\{\omega\}} g_{12} = -\frac{1}{6}\log\|\Delta(i)\|_{\mathrm{Pet}} - \frac{1}{6}\log\|\Delta(\omega)\|_{\mathrm{Pet}} - \frac{1}{3}\log 2 - \frac{1}{6}\log 3.$$

Proof. Applying the decomposition (3.1.4) of χ_{12} on $\mathcal{H}$ as $\chi_{12}(\tau) = 12\Delta(\tau_1)\Delta(\tau_2)$, the proposition follows.

□

3.5 The main result

We can now collect the results of the preceeding sections to compute the arithmetic intersection number, applying the following result of Kudla [31] to compute the value of the first integral on the right hand side of Theorem 3.2.6.

Theorem 3.5.1. *Let* Δ_5 *denote the modular form (with non-trivial quadratic character) given by the product of even* ϑ*-series*

$$\Delta_5(\tau) = \frac{1}{2^6}\prod_{(a,b)\ \mathrm{even}} \vartheta_{a,b}(\tau).$$

Then, the equality

$$\begin{aligned}&-\operatorname{vol}(\mathcal{A}_2)^{-1}\int\limits_{\mathcal{A}_2}\log\bigl(|\Delta_5(\tau)|^2\det(y)^5\bigr)\Omega^3\\ &\quad= 10\left(-\frac{4}{3} - 2\frac{\zeta'(-3)}{\zeta(-3)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2}\log 2 + \log\pi\right) - 7\log 2\end{aligned}$$

holds, where the volume form Ω^3 *is given in coordinates by*

$$\Omega^3 = -\frac{3}{16\pi^3}(\det y)^{-3}\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_{12}\mathrm{d}y_{12}.$$

Idea of proof. The form Δ_5 is a Borcherds form corresponding to a modular form f_5 with values in $\mathbb{C}[M^\vee/M]$ for a certain lattice M which appears in the construction of $\mathcal{A}_2$ as a quotient of a homogeneous domain. The generating series of the Heegner divisors attached to points in $M^\vee/M$ is an Eisenstein series. The above integral can be computed relating it to the order 1 term of the Laurent expansion of its Fourier coefficients. The full proof can be found in [31].

Corollary 3.5.2. *The integral over* $\overline{\mathcal{A}}_2$ *on the right hand side of Theorem 3.2.6 has the value*

$$\begin{aligned}&\frac{1}{(2\pi i)^3}\int\limits_{\overline{\mathcal{A}}_2} g_{10}\wedge\omega_6\wedge\omega_4\wedge\omega_{12}\\ &\quad= 10\cdot 6\cdot 4\cdot 12\,\zeta(-3)\zeta(-1)\left(\frac{4}{3} + 2\frac{\zeta'(-3)}{\zeta(-3)} - \frac{\zeta'(-1)}{\zeta(-1)} + \frac{6}{5}\log 2\right).\end{aligned}$$

Proof. Siegel [43] computed the volume of $\mathcal{A}_2$ with respect to the volume form $(\det y)^{-3}\mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2y_2\mathrm{d}x_{12}\mathrm{d}y_{12}$ to be $2\pi^{-3}\zeta(2)\zeta(4)$. Hence, the volume $\mathrm{vol}(\mathcal{A}_2)$ with respect to the volume form Ω^3 equals

$$\mathrm{vol}(\mathcal{A}_2) = \int\limits_{\mathcal{A}_2} \Omega^3 = \zeta(-1)\zeta(-3).$$

In order to compare the volume forms Ω^3 and $\omega_6 \wedge \omega_4 \wedge \omega_{12}$, we apply Lemmas 3.3.5 and 3.3.6 to express the Chern forms in coordinates and obtain

$$\omega_6 \wedge \omega_4 \wedge \omega_{12} = -36(4\pi i)^3\Omega^3.$$

Furthermore, noting that $\Delta_5^2(\tau) = \frac{1}{2^{12}}\prod_{(a,b)\text{ even}} \vartheta_{a,b}^2(\tau) = \chi_{10}(\tau)$, we have

$$g_{10}(\tau) = -\log\left(|\Delta_5(\tau)|^2(4\pi)^{10}\det(y)^5\right),$$

and we compute with Theorem 3.5.1

$$\begin{aligned}
&\frac{1}{(2\pi i)^3}\int\limits_{\overline{\mathcal{A}}_2} g_{10} \wedge \omega_6 \wedge \omega_4 \wedge \omega_{12} = 288\int\limits_{\overline{\mathcal{A}}_2} \log\left(|\Delta_5(\tau)|^2(4\pi)^{10}\det(y)^5\right)\Omega^3 \\
&= 288\zeta(-3)\zeta(-1)\log\left((4\pi)^{10}\right) + 288\int\limits_{\overline{\mathcal{A}}_2} \log\left(|\Delta_5(\tau)|^2\det(y)^5\right)\Omega^3 \\
&= 288\zeta(-3)\zeta(-1)\,10(\log\pi + 2\log 2) \\
&\quad - 288\zeta(-3)\zeta(-1)\left(10\left(-\frac{4}{3} - 2\frac{\zeta'(-3)}{\zeta(-3)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2}\log 2 + \log\pi\right) - 7\log 2\right) \\
&= 10 \cdot 6 \cdot 4 \cdot 12\,\zeta(-3)\zeta(-1)\left(\frac{4}{3} + 2\frac{\zeta'(-3)}{\zeta(-3)} - \frac{\zeta'(-1)}{\zeta(-1)} + \frac{6}{5}\log 2\right).
\end{aligned}$$

Note that the convergence of the integral claimed in the proof of Theorem 3.2.6 is now confirmed by Kudla's result.

□

Proposition 3.5.3. *The contribution for the arithmetic self intersection number coming from the finite fibres is a rational linear combination of* $\log 2$ *and* $\log 3$.

Proof. The statement can be verified by considering ϑ-embeddings for $\mathcal{A}_2(2)$ into projective space, see, e.g., [27], for the defining equations for its image in $\mathbb{P}^4$. These are only defined over $\mathbb{Z}[\frac{1}{2}]$. The description of the above modular forms as polynomials in ϑ-series, and hence, in projective coordinates, shows that they have empty intersection in the finite fibres, except for $p = 3$. Over $\mathbb{F}_3$, there are 6 points that are common zeroes of the modular forms. They are all equivalent under the action of $\mathrm{Sp}_4(\mathbb{Z}/2\mathbb{Z})$.

□

Theorem 3.5.4. *The arithmetic self intersection number, i.e., the arithmetic degree of the line bundle $\mathcal{M}_k(\Gamma_2)$ of modular forms of weight k on $\overline{\mathcal{A}}_2$, equipped with the Petersson metric, is given as*

$$\begin{aligned}&\widehat{\deg}(\mathcal{M}_k(\Gamma_2), \|\cdot\|_{\mathrm{Pet}})\\ &\qquad = k^4\left(\zeta(-3)\zeta(-1)\left(2\frac{\zeta'(-3)}{\zeta(-3)} + 2\frac{\zeta'(-1)}{\zeta(-1)} + \frac{17}{6}\right) + c_2\log 2 + c_3\log 3\right),\end{aligned}$$

with $c_2, c_3 \in \mathbb{Q}$.

Proof. Summing up the values of the integrals over $\overline{\mathcal{H}}$, $\{i\}\times_{\mathcal{H}}\overline{\mathcal{A}}_1$, and $\{i\}\times_{\mathcal{H}}\{\omega\}$ on the right hand side of Theorem 3.2.6, we obtain

$$\begin{aligned}&\frac{1}{(2\pi i)^2}\int\limits_{\overline{\mathcal{H}}} g_6\wedge\omega_4\wedge\omega_{12} + \frac{1}{4\pi i}\int\limits_{\{i\}\times_{\mathcal{H}}\overline{\mathcal{A}}_1} g_4\wedge\omega_{12} + \frac{1}{6}\int\limits_{\{i\}\times_{\mathcal{H}}\{\omega\}} g_{12}\\
&= \frac{1}{(2\pi i)^2}\int\limits_{\overline{\mathcal{H}}} g_6\wedge\omega_4\wedge\omega_{12} + \frac{1}{2}\frac{1}{2\pi i}\int\limits_{\{i\}\times_{\mathcal{H}}\overline{\mathcal{A}}_1} g_4\wedge\omega_{12} + \frac{1}{6}\int\limits_{\{i\}\times_{\mathcal{H}}\{\omega\}} g_{12}\\
&= 48\Big(\frac{1}{2}\zeta(-1)+\zeta'(-1)\Big) + \frac{1}{3}\log\|\Delta(i)\|_{\mathrm{Pet}}\\
&\quad + 24\Big(\frac{1}{2}\zeta(-1)+\zeta'(-1)\Big) - \frac{1}{6}\log\|\Delta(i)\|_{\mathrm{Pet}} + \frac{1}{6}\log\|\Delta(\omega)\|_{\mathrm{Pet}} - \log 2 - \frac{1}{2}\log 3\\
&\quad - \frac{1}{6}\log\|\Delta(i)\|_{\mathrm{Pet}} - \frac{1}{6}\log\|\Delta(\omega)\|_{\mathrm{Pet}} - \frac{1}{3}\log 2 - \frac{1}{6}\log 3\\
&= 72\Big(\frac{1}{2}\zeta(-1)+\zeta'(-1)\Big) - \frac{4}{3}\log 2 - \frac{2}{3}\log 3\\
&= -6\Big(\frac{1}{2}+\frac{\zeta'(-1)}{\zeta(-1)}\Big) - \frac{4}{3}\log 2 - \frac{2}{3}\log 3.\end{aligned}$$

Hence, applying the decomposition in Theorem 3.2.6 and Corollary 3.5.2, we obtain

$$\begin{aligned}&\frac{1}{(2\pi i)^3}\int\limits_{\overline{\mathcal{A}}_2} g_{10}*g_6*g_4*g_{12}\\
&= 10\cdot 6\cdot 4\cdot 12\,\zeta(-3)\zeta(-1)\left(\frac{4}{3}+2\frac{\zeta'(-3)}{\zeta(-3)} - \frac{\zeta'(-1)}{\zeta(-1)} + \frac{6}{5}\log 2\right)\\
&\qquad\qquad\qquad\qquad - 6\Big(\frac{1}{2}+\frac{\zeta'(-1)}{\zeta(-1)}\Big) - \frac{4}{3}\log 2 - \frac{2}{3}\log 3\\
&= 10\cdot 6\cdot 4\cdot 12\,\zeta(-3)\zeta(-1)\left(2\frac{\zeta'(-3)}{\zeta(-3)} + 2\frac{\zeta'(-1)}{\zeta(-1)} + \frac{17}{6} + \frac{28}{15}\log 2 + \frac{1}{3}\log 3\right),\end{aligned}$$

noting that

$$\zeta(-1) = -\frac{1}{12} \quad\text{and}\quad \zeta(-3) = \frac{1}{120}.$$

As the degree is multilinear in the weights of the chosen modular forms, the result follows. □

Remark 3.5.5. The intermediate result in the proof of Theorem 3.5.4 about the height of the Humbert surface

$$\begin{aligned}
&\frac{1}{(2\pi i)^2}\int\limits_{\overline{\mathcal{H}}} g_6 * g_4 * g_{12} \\
&\quad = \frac{1}{(2\pi i)^2}\int\limits_{\overline{\mathcal{H}}} g_6 \wedge \omega_4 \wedge \omega_{12} + \frac{1}{4\pi i}\int\limits_{\{i\}\times_{\mathcal{H}}\overline{\mathcal{A}}_1} g_4 \wedge \omega_{12} + \frac{1}{6}\int\limits_{\{i\}\times_{\mathcal{H}}\{\omega\}} g_{12} \\
&\quad = -6\left(\frac{1}{2}+\frac{\zeta'(-1)}{\zeta(-1)}\right) - \frac{4}{3}\log 2 - \frac{2}{3}\log 3
\end{aligned}$$

is in fact a degeneration of a result by Bruinier, Burgos, and Kühn. In [6], they consider the situation of a real quadratic field K with discriminant D, and the Hilbert modular group $\Gamma_K := \mathrm{SL}_2(\mathcal{O}_K)$ acting on $(\tau_1, \tau_2) \in \mathbb{H}^2$ via $M(\tau_1, \tau_2) = (M\tau_1, M'\tau_2)$, where M' denotes the conjugate of $M \in \Gamma_K$. The arithmetic self intersection number of the metrized bundle $\overline{\mathcal{M}_k}(\Gamma_K)$ of Hilbert modular forms of weight k together with the Petersson metric is then computed as

$$\overline{\mathcal{M}_k}(\Gamma_K)^3 = -k^3\zeta_K(-1)\left(\frac{\zeta_K'(-1)}{\zeta_K(-1)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2} + \frac{1}{2}\log(D)\right).$$

(In fact, the result is obtained for a congruence subgroup $\Gamma_K(N)$ ($N \geq 3$) instead of Γ_K, but can be lifted to $N = 1$ as explicated in [6].) For the degenerate case $K = \mathbb{Q} \oplus \mathbb{Q}$ and $D = 1$, the Humbert surface $\mathcal{H}$ of invariant 1 is the image of $\Gamma_K \backslash \mathbb{H}^2$ in $\mathcal{A}_2$ under the morphism induced by the diagonal embedding $\mathbb{H}_1^2 \longrightarrow \mathbb{H}_2$. For $K = \mathbb{Q} \oplus \mathbb{Q}$, the function ζ_K splits as $\zeta_K = \zeta^2$. Denoting by I_{fin} the contribution to the intersection number coming from the finite fibres and applying above formula with $\zeta_K = \zeta^2$, one obtains

$$\frac{1}{(2\pi i)^2}\int\limits_{\mathcal{H}} g_6 * g_4 * g_{12} = -6 \cdot 4 \cdot 12\zeta^2(-1)\left(2\frac{\zeta'(-1)}{\zeta(-1)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2}\right) - I_{\mathrm{fin}},$$

confirming the calculations in Section 3.4.

Remark 3.5.6. The conjectured value for the arithmetic degree of the line bundle of modular forms of weight k on $\overline{\mathcal{A}}_2$, see, e.g., [5], is

$$\widehat{\deg}(\mathcal{M}_k(\Gamma_2), \|\cdot\|_{\mathrm{Pet}}) = k^4\zeta(-3)\zeta(-1)\left(2\frac{\zeta'(-3)}{\zeta(-3)} + 2\frac{\zeta'(-1)}{\zeta(-1)} + \frac{17}{6}\right).$$

The conjectured value is therefore supported by Theorem 3.5.4.

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Selbständigkeitserklärung

Hiermit versichere ich, die vorgelegte Dissertation selbständig und nur unter Verwendung der angegebenen Hilfen und Hilfsmittel angefertigt zu haben.

Ich habe mich nicht anderwärts um einen Doktorgrad in dem Promotionsfach beworben und besitze keinen entsprechenden Doktorgrad. Die Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 42 am 11. Juli 2018, habe ich zur Kenntnis genommen.

Berlin, den 24. August 2018 Barbara Jung