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Mathematical formalization and diagrammatic reasoning: the case study of the braid group between 1925 and 1950

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The standard historical narrative regarding formalism during the twentieth century indicates the 1920s as a highpoint in the mathematical formalization project. This was marked by Hilbert's statement that the sign stood at the beginning of pure mathematics ['Neubegründung der Mathematik. Erste Mitteilung', *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 1 (1922), 157–177]. If one takes the braid group as a case study of research whose official goal was to symbolically formalize braids and weaving patterns, a reconsideration of this strict definition of formalism is nevertheless required. For example, does it reflect what actually occurred in practice in the mathematical research of this period? As this article shows, the research on the braid group between 1926 and 1950, led among others by Artin, Burau, Fröhlich and Bohnenblust, was characterized by a variety of practices and reasoning techniques. These were not only symbolic and deductive, but also diagrammatic and visual. Against the historical narrative of formalism as based on a well-defined chain of graphic signs that has freedom of interpretation, this article presents how these different ways of reasoning—which were not only sign based—functioned together within the research of the braid group; it will be shown how they are simultaneously necessary and complementary for each other.

Introduction

> The history of mathematics, lacking the guidance of philosophy, has become *blind*, while the philosophy of mathematics, turning its back on the most intriguing phenomena in the history of mathematics, has become *empty*. (Lakatos 1976, 2)

In 1922 David Hilbert published his now well-known article 'The New Grounding of Mathematics'. It is in this article that one of the cornerstones of formalism can be found: 'The solid philosophical attitude that I think is required for the grounding of pure mathematics—as well as for all scientific thought, understanding, and communication—is this: *In the beginning was the sign* [*Am Anfang ist das Zeichen*]' (Hilbert 1922, 163).[1]

Although Hilbert also had favourable views on the explorative role of illustration, visualization and the *Anschauung* in mathematics, as can be seen from his 1920/21 series of lectures *Anschauliche Geometrie* (see Corry 2006), to which I will return shortly below, it is clear that the above citation was taken as a cue when considering the contemporary characterizations of formalism and formalization in mathematics,

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†Excellence Cluster "Image Knowledge Gestaltung".
[1]Translation taken from: (Ewald 1996, 1122).

placing the sign at its centre. Let us for example look at the three criteria which Sybille Krämer in her 1988 book *Symbolische Maschinen* gives to the formalization of mathematical arguments: (1) scribality: using well-defined scribal, graphic signs; (2) the ability to be schematized: i.e. the formalized process can be repeated, it does not have to be repeated every time anew for various specific cases or examples; (3) freedom of interpretation: there is no significance to what the signs themselves mean (Krämer 1988, 1–3). The last criterion may be seen as indicating that the correctness or falsity of an expression within a formal language can and should be decided without reference to any interpretation of that expression, be that an external reference to objects in the world, to two-dimensional diagrams drawn on a paper or three-dimensional material mathematical models. Obviously, it is exactly this third criterion that excludes any visual or diagrammatic reasoning that may be employed during the proving process.[2]

A less strict definition is given by Øystein Linnebo, in his 2017 book *Philosophy of Mathematics*: 'Formalism is the view that mathematics has no need for semantic notions, or at least none that cannot be reduced to syntactic ones' (Linnebo 2017, 39). Once again, this point of view excludes other types of reasoning, visual ones at least, since by definition, they are presented as what cannot be reduced to 'linguistic expressions abstracted from any meaning they may have' (Linnebo 2017, 38).

The question that arises here is indeed whether these theoretical descriptions correspond in practice to how processes of formalization occurred in mathematics during the first half of the twentieth century. Although I do not intend by any means to take these two books as representative of the state of the art regarding formalization in mathematics, or to hint that they do not take the history of mathematics into account (which is not true), I claim that a closer historical investigation of the processes of mathematical formalization is needed. That said, this must take Lakatos's above-cited motto into consideration.

In this paper I will examine the development of braid theory from 1925 to 1950, a mathematical domain whose early development has hardly been investigated historically. Although Moritz Epple already examined the 1926 paper of Emil Artin 'Theorie der Zöpfe' (Epple 1997, 184–188; Epple 1999, 314–320), the research that followed Artin's work has hardly been analyzed within the history or the philosophy of mathematics, and especially not within the current research on mathematical practices. Taking braid theory as a case study, I aim to answer several questions: what role did visual, non-symbolized, non-formalized arguments and reasoning play during the development of braid theory? How did they function together with formalized theory? And what conclusions can be drawn from this case study regarding the limitations of mathematical formalization?

The paper will first survey the development of braid theory from Artin's 1926 paper up to his 1950 paper 'The theory of braids' published in *Scientific American*. Although the survey does not aim to be entirely comprehensive, it does give an overview of how braid theory was formalized—emphasizing the role diagrams and visual arguments played within the process. The second part of the paper will then turn to a more philosophical discussion, attempting to answer the above questions concerning the limits of formalization.

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[2]This point of view, at least regarding diagrams in mathematics, in refined by Krämer in: (Krämer 2016).

Formalization of braid theory from 1925 to 1950

1925–6: the beginning of braid theory: Artin's 'Theorie der Zöpfe'

It is a rare case in mathematics that one can identify the exact moment when a mathematical theory was developed for the first time. Officially, one may indeed say that Emil Artin's paper '*Theorie der Zöpfe*', written in 1925 and published in 1926, was such a case. As Moritz Epple notes, however, Artin, though officially the sole author of the paper, worked on the subject with Otto Schreier. Mathematical research on braids, moreover, can already be discovered in the 1890s with Adolf Hurwitz; he investigated braids resulting from a motion of branch points on a Riemann surface.[3]

What is unique in Artin's treatment, however, is not only his attempt to systematically research braids as a mathematical object independent of other mathematical objects (such as the Riemann surfaces), but also his attempt to achieve a complete algebraic formalization of the set of braids. It is nevertheless important to note that the focus of Artin's research during these years was not braid theory. He himself spoke of his research on braid theory already in 1926 as 'going astray topologically' [*auf topologischen Abwege*] (Frei and Roquette 2008, 95).[4] But when Artin did work on braids, his aim was explicitly 'to arithmetize' (Artin 1926, 50), i.e. to present, with the tools of group theory, braids symbolically as well as the relations in the braid group and their deformations. In that sense, Artin's paper can be considered the first attempt to formalize braids. This was done officially with the demarcation between the geometrical-topological and algebraic arguments and inferences. Leo Corry describes Artin's image of mathematics as a structural one,[5] and this already reflected Artin's explicit motivation regarding the mathematical investigation of braids. As we will see below, however, for some of the propositions Artin does turn to topological arguments. This already indicates an intertwining of two types of reasoning, those of visual as well as symbolic argumentation.

In several places Artin indeed uses topological considerations; he often talks about the 'geometrical meaning' [*die geometrische Bedeutung*] of his arguments (Artin 1926, 50, 54, 55, 62, 63), and I will return later to the meaning of this term. The term 'strings' [*Fäden*] that he often uses, might be considered technically and as not referring to real strings. That said, as we shall see when examining Artin's paper from 1950, he did indeed think in this direction. As he himself points out, moreover, some of the algebraic arguments he presented can (also) be seen by looking at his diagrams. I will now have a closer look at his paper.

Even if already Artin's explicit aim was to find the *algebraic* structure of the braid group, at the beginning of the paper braids are called 'simpler topological objects'[6] [*topologischer Gebilde*] (Artin 1926, 47) in comparison to knots and

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[3]Epple (1999, 55–80) notes that Gauss also investigated braids in an unpublished note around 1825.

[4]This expression is to be found in a letter from Artin to Helmut Hasse, written on 10 February 1926. All the translations from German to English were done by the author (M.F.) unless stated otherwise.

[5]See (Corry 2004, 60–61), and also (Benis-Sinaceur 1984; Sinaceur 1991, 145–254). Corry mentions: "Van der Waerden's *Moderne Algebra* – the paradigmatic embodiment of the *structural image of algebra* was written under Noether's pervasive and decisive influence, but not only under hers. The contemporary works of *Emil Artin*, *Otto Schreier*, John von Neumann, and others, provided a direct source of inspiration for some of its chapters." (Corry 2004, 220, my emphasis). See also (Epple 1997, 196).

[6]Artin repeats his statement that braids are a "topological object" twice (1926, 47); The expression "topologische Gebilde" appears also at p. 55.

links. Moreover, it is essential to note that during the years 1924 and 1925 Artin did not consider topology (or topological or visual arguments) inferior or as something that strayed beyond his real algebraic interests.[7] In a diary entry from 2 May 1924 Helmut Hasse wrote a 'comment on topology', which was explained to him by Artin sometime in March 1924 (Lemmermeyer and Roquette 2012, 208). Although braids 'give rise to a group' (Artin 1926, 47), he notes that 'schematically, a braid can [...] be represented by a drawing as shown in Figure 2' (Artin 1926, 48). This 'Figure 2' is presented in Figure 1(a) below. One can argue that this figure is only an illustration of two braids concatenated together. Yet in order to facilitate an understanding of what the composition of braids looks like in the braid group, Artin immediately introduces a restriction after this description—one that is completely visual. As he notes, a braid should not turn in reverse: 'In Figure 1 [see Figure 1(b)] a weaving [*Geflecht*] is drawn as an example, one that we do not consider a braid [*Zopf*]' (Artin 1926, 48). Such a restriction is given no further algebraic interpretation.

The crucial point to note here is that while Artin symbolically proves that the braid group is indeed a group, the proof itself is followed by diagrams. And not only that, as for some of the constructions (for example, the inverse braid) Artin notes that 'the geometrical meaning [...] can be recognized immediately' (Artin 1926, 50).[8] He then defines the braid group of n-strings as generated by $n-1$ generators, denoted by σ_i, for $i = 1, \ldots, n-1$, when σ_i is the braid for which the ith string passes once *above* the $(i + 1)^{\text{th}}$ string, whereas the other strings are straight lines (see Figure 1(c)). He then proves that the following relations between the generators hold:

$$\sigma_i \sigma_k = \sigma_k \sigma_i \text{ for } k \neq i, i + 1 \text{ and } \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \ \sigma_{i+1} \text{ for } i = 1, 2, \ldots, n-2.$$

The proof of these relations is entirely visual. Thus, for example, Artin explicates that one can '*extract* from the [following] figures' the relation $\sigma_{i+1}^{\pm 1}\sigma_i = \sigma_i^{\mp 1}\sigma_{i+1}\sigma_i^{\pm 1}\sigma_{i+1}^{\pm 1}$, from which he induces the relation $\sigma_i \ \sigma_{i+1} \ \sigma_i = \sigma_{i+1} \ \sigma_i \ \sigma_{i+1}$. The proof relies on the diagrams depicting (see Figure 2) what happens when one 'shifts' a crossing σ_i from one side of the crossing $\sigma_{i + 1}$ to the other.

Although the rest of the paper turns out to be more and more algebraic—for example, in section 4 he algebraically proves that the braid group can be presented as generated by two elements a and σ—Artin does not reject non-formalized methods of reasoning. He refers continually to his diagrams as a means of persuading the reader of the validity of the formalized arguments.[9] In addition, after proving algebraically that in the braid group a certain element commutes with all other elements in the group (this element is presented in Figure 3 for the braid group with four strings), Artin then comments that 'we realize now *anschaulich* that a^n [this braid] commutes with any other braid' (Artin 1926, 54). As I will show later in this article, the terms *anschaulich* as well as 'geometric meaning' were associated with concrete objects, in contrast to abstract formalization.

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[7]In addition to the paper on braids, Artin published another paper on topology called "Zur Isotopie zweidimensionaler Flächen in R_4" (1925).
[8]Compare also Artin's treatment in 1950 (see below).
[9](Artin 1926, 62): "[...] one can easily convince oneself with the help of a sketch that one actually gets it [a relation] in such a way." Or (p. 64): "The condition [...] is *anschaulich* obvious", referring to a figure.
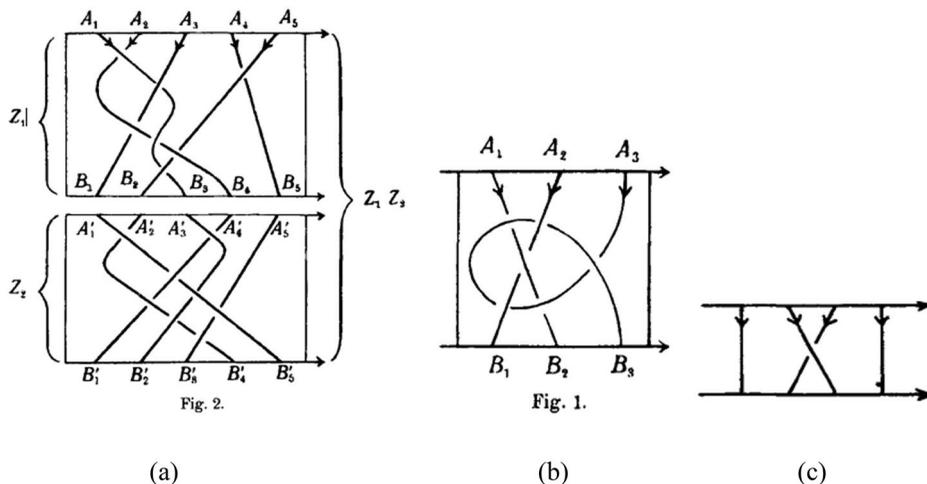
(a)   (b)   (c)

Figure 1.   (a) Artin's drawing of a concatenation of two braids (Artin 1926, 48). (b) A forbidden braid (ibid.) (c) A drawing of the generator σ_i (ibid., 49)

Figure 2.   Inducing the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ (Artin 1926, 51)

Other examples are also discernable. As Epple notes, the word problem[10] was solved in the braid group using extensive topological and geometrical considerations —although the problem was formulated algebraic-symbolically (Epple 1999, 319). Indeed, Artin notes that this problem is solved with the 'concepts of topology' (Artin 1926, 57). Before surveying later papers dealing with braid theory, I will name only two additional examples. While discussing the connection between knots, links and braids, Artin cannot help but praise the proof of James W. Alexander (1923) that every knot or link can be presented as a closure of a braid. Artin states that he 'refers […] to Alexander's work […] [since] [t]he proof there leaves nothing to

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[10]Explicitly, given two braids *A* and *B*, one should determine whether *A* = *B* in the braid group (or equivalently, whether *A* can be deformed to *B* or vice versa).
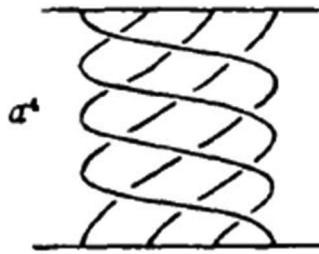
Figure 3.   A drawing of the braid a^4, which commutes with all other braids in the braid group of four strings (Artin 1926, 54)

be desired in simplicity and transparency' (Artin 1926, 56). However, it should also be noted that Alexander uses purely diagrammatic-visual reasoning, as De Toffoli and Giardino have shown (2016), and does not even think of proving his theorem by algebraic means.

Continuing this line of thought, Artin also notes the following: Assuming one cuts a closed braid Z at one place, and then

> one cuts the closed braid at another place, in such a way that first after the part Y the former concatenation place is to be found, then one obtains the braid $Z'' = YZY^{-1}$, since at the back the part Y is missing. (Artin 1926, 55)

No algebraic argument is given, only an implicit demand to visualize a cutting in another place, followed by Figure 4.

The 1930s: Burau and Fröhlich

Following Artin's paper and the growing interest in knot theory, numerous mathematicians were drawn to braid theory. As the set of braids was not previously considered together with the action of their concatenation, presenting the group structure on the set of braids was entirely new. In this subsection I will briefly review the works of two other mathematicians who further researched braids in the 1930s: Werner Burau and W. Fröhlich.
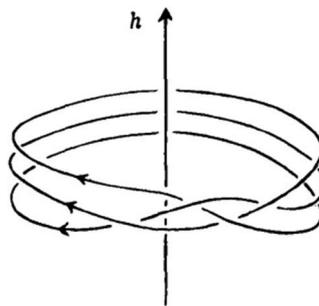

Figure 4.   Artin's drawing for illustrating a cut of a closed braid (Artin 1926, 55)

Burau published several papers on braid and knot theory (1932, 1935a, 1935b), and as Epple notes (1999, 366), he mainly investigated the connections between emerging braid theory and the known invariants of knot theory. During the first half of the 1930s, however, Burau published several results which were important for braid theory itself, but did not necessarily have any connection to knot theory.

In 1932 Burau published his paper 'On braid invariants' [*Über Zopfinvarianten*], whose main aim was to present generators and relations of several subgroups of the braid group, among them the pure braid group. The braid invariants, which Burau looks for, can be extracted 'in an immediate [*unmittelbar*] geometric' fashion (Burau 1932, 117). To recall, the pure braid group is defined as the kernel of the epimorphism from the braid group to the permutation group, sending every braid to its induced permutation. Considered explicitly, the pure braid group contains all the braids which are mapped to the identity permutation. Burau notes that after adding the relations $(\sigma_i)^2 = 1$, for $i = 1, \ldots, n-1$ to the braid group, one can associate a *word* in the braid group to each permutation where the exponent of each of the generators σ_i is 1 and the word has the shortest length: that is, the word 'reproduces the representation of permutation as a product of the least possible number of transpositions' (Burau 1932, 118). The vocabulary Burau uses is symbolic-formal, and points towards the same algebraic formalization at which Artin aimed. Already at the beginning of the paper, however, he notes that Artin introduced braids from the field of topology (Burau 1932, 117). In one of the first theorems presented regarding these shortest words, moreover, several of the steps in the proof are diagrammatic, and cannot be reduced to algebraic-symbolic steps.

Theorem 2 indicates the following: 'In the braid to which one of our shortest words is assigned, two threads may not cross more than once' (Burau 1932, 117). Burau assumes by contradiction that in this word there are two strings, which do cross each other (at least) twice, calling these intersection points *A* and *B* (see Figure 5). He then looks at the area between these two points and divides the other strings into classes, whose definition is visual: (1) strings which end left from *A* and hence start right from *B*, and (2) strings which end left from *B* and hence start right from *A*. Noting that the exponent of each of the generators σ_i is 1 (according to the definition, see above), he concludes that each string from class (2) passes below the strings from class (1). He then notes a crucial visual step: 'Apparently this piece of
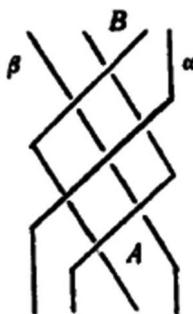

Fig. 1.

Figure 5.   A drawing of Burau, depicting the points *A* and *B* and the two classes of strings, denoted by α and β (Burau 1932, 118)

surface [on which the braid is] can be pulled together [*zusammenziehen*] so far that the points A and B immediately follow each other' (Burau 1932, 117). What is obvious here is visually proved or at least calls for a proof via the imagination of the required action. The proof then follows algebraic-symbolic steps (using the relations of the braid group and the symbolic definition of the shortest word), which concludes it.

Similar visual steps appear in other proofs, mainly sliding threads from one side to another.[11] Moreover, when presenting the relations between the generators of the pure braid group, Burau notes that they stem from 'simple geometric reasons' (Burau 1932, 117),[12] 'whose geometric meaning is easily apparent [*ersichtlich*]' (p. 122). Discussing the algebraic relations between certain elements in the braid group and the generators of the pure group, Burau notes that these relations can be seen 'geometrically' (p. 123). In contrast to Artin, however, the adjective *anschaulich* no longer appears. Obviously the 'geometric immediateness', to which Burau refers at the beginning of his paper, points the way to how *Anschauung* operates. Additionally, the adjective *ersichtlich* does appear, whose meaning is 'evident' (or 'apparent'), and comes from the same family of words *Sicht*, *sehen*: sight, seeing.

The emphasis on geometric meaning and visual reasoning disappears, however, in Burau's now famous 1935 paper, which introduces the Burau representation of the braid group (Burau 1935b).[13] The way the presentation is described is formalized completely in algebraic terms, and there is no discussion as to what it means geometrically —if there is any meaning at all—to associate each generator of the braid group to a matrix.[14] Nevertheless, this algebraic shift does not imply that diagrammatic or visual reasoning was completely abandoned when investigating braids in the 1930s.

Turning to W. Fröhlich: in his 1936 paper '*Über ein spezielles Transformationsproblem bei einer besonderen Klasse von Zöpfen*', he investigated several sub-groups U_{kn} of the pure group of n strings. These sub-groups consist of all the braids, such that after removing the kth string from them, the obtained braid is the identity braid. Fröhlich indicates that 'with each braid of U_{kn} it is possible to achieve by suitable deformation that all threads with the exception of the k-th run straight and parallel' (Fröhlich 1936, 226). Even if a symbolic-formalized proof to this claim is not provided, there is certainly an implicit demand that the reader will imagine the deformation. Moreover, when describing the same procedure in a different section, Fröhlich comments that this deformation, being the stretching of the kth string, is the 'physical meaning' of the normal form of braids from U_{kn} (Fröhlich 1936, 228). This expression surely resonates with the similar expressions of 'geometrical meaning'[15] made by Artin and Burau.

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[11](Burau 1932, 119): "Between both types of strings, however, A can be pushed to the end of the braid [...]", or (p. 122): "Pushing away of the strings".

[12]The same expression appears on p. 120.

[13]The (unreduced) Burau representation (Burau 1935b, 180) is a homomorphism from the braid group to the group of matrices over $Z[t, t^{-1}]$, given by a map on the generators (the matrix I_k is the $k \times k$ identity matrix and all the other entries are 0):

$$\sigma_i \mapsto \begin{pmatrix} I_{i-1} & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & I_{n-i-1} \end{pmatrix}$$

[14]The adjectives geometric, *anschaulich*, or topological are not mentioned even once.

[15]This expression does not appear at all in Fröhlich's paper.
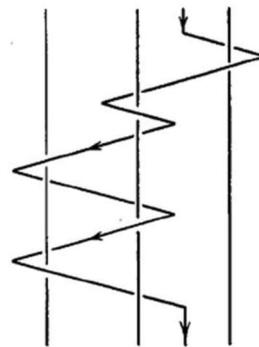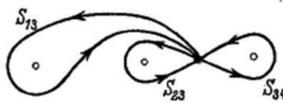
Figure 6. Fröhlich's depiction of finding the fundamental curves of *K* (Fröhlich 1936, 227)

A similar demand to visualize arguments occurs later in the paper where Fröhlich indicates that after projecting a braid from U_{kn} to the normal plane, one obtains from the kth string a closed curve K that twines around the other $n-1$ points, which are the projections of the other $n-1$ strings. He then notes that K, 'as is well known, can be formed into a canonical form by means of (continuous) deformations, which is the power product [*Potenzprodukt*] of $n-1$ fundamental curves' (Fröhlich 1936, 226), where these fundamental curves are the projection of the elements $S_{1k}, \ldots, S_{kn}$, being the generators of the pure braid group. How this deformation is done is only depicted (see Figure 6), however. Moreover, from the figure that Fröhlich draws it is unclear how, from a possible deformation, one obtains algebraically from a braid in U_{kn} the expression of a *Potenzprodukt* composed from the elements S_{ij}.

1947: the algebraization of the braid group: Artin and Bohnenblust

As we have clearly seen, Fröhlich's visual arguments were somewhat vague, and Burau's 1935 paper did not convey any 'geometrical meaning' to his representation. In Fröhlich's papers from 1938 and 1939 on braid theory (1938, 1939), such visual arguments are almost entirely absent.[16] In the late 1940s visual reasoning and diagrammatic arguments were rejected from research in braid theory. This can be noticed in two papers, both published in 1947: the first Artin's 'Theory of Braids', the second Bohnenblust's 'The Algebraical Braid Group'.

Already at the beginning of his 1947 paper on braids, Artin declares that in his 1926 '*Theorie der Zöpfe*',

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[16]In 1939 Fröhlich briefly mentions the "geometric interpretation [*Deutung*]" of several elements in the braid group (Fröhlich 1939, 289).

> Most of the proofs are entirely intuitive. […] It is possible to correct the proofs. The
> difficulties that one encounters if one tries to do so come from the fact that the pro-
> jection of the braid, which is an excellent tool for intuitive investigations, is a very
> clumsy one for rigorous proofs. (Artin 1947a, 101)

Artin's paper presents the braid group and its properties in a much more algebraic way.
Several of the results 'may [even] escape […] intuitive investigation.' A similar rhetoric
appears in Bohnenblust's 'The Algebraical Braid Group'. As Bohnenblust notes, 'the
proofs [in Artin's 1926 paper] […] are partially intuitive.' Bohnenblust also notes that
while 'by its nature the problem [of finding the relations between the generators of the
braid group] is geometrical [… ,] throughout the major part of his [1947] paper Artin
makes extensive use of geometrical considerations', which can be considered as an
implicit critique. This is since Bohnenblust's aim is to consider 'a purely *group-theoretical*
problem' and hence not the geometrical setting. Bohnenblust declares that 'the 'algebrai-
cal braid group' is defined […] [and] with the algebraical approach it is finally shown that
the defining relations […] are defining relations for the geometrical braid group' (Boh-
nenblust 1947, 127). Moreover, he adds at the end of his paper that '[t]he identity
between the geometrical and the algebraical braid group is now evident, since both
are represented by the same group of automorphisms of a free group'; that is, the geo-
metric braid group is presented by another group (the group of automorphisms of a free
group), which can be defined without turning to topological explanations.

    If we return to Artin's 1947 paper, it is clear that he still relies on topological con-
siderations.[17] Yet as Bohnenblust noted, the emphasis on algebraic formalization is
much stronger than in the 1926 paper. When commenting on the methods used in
this paper, Artin notes that they were 'geometric and can easily be made rigorous by
means of the tools developed in this paper' (Artin 1947a, 115). The construction of
the concatenation of two braids is given a 'formal definition' (p. 103): and indeed in
contrast to the diagrammatic illustration given in the 1926 paper. Moreover, hardly
any diagrams are contained in the 1947 paper, and the diagrams that are there serve
merely as an illustration of what certain braids look like. In contrast to the 1926
paper, however, these diagrams could have been omitted, as they play only an illustra-
tive role. In addition, Artin once more fails to prove what the relations between the
generators of the braid group are; he indeed notes that the methods he used in the
1926 paper were 'geometric' but then refers to Bohnenblust's paper, 'which is essen-
tially algebraic and leads deeper into the theory of the group' (Artin 1947a, 115).
The concluding remarks are an eye-opener regarding the shift that is taking place
with respect to how 'geometrical meaning' should be taken into consideration. Artin
finishes the paper with the following remark:

> The geometric meaning of the normal form of [the braid in the pure braid group] is
> now […] clear. […] Although it has been proved that every braid can be deformed into
> a similar normal form the writer is convinced that any attempt to carry this out on a
> living person would only lead to violent protests and discrimination against math-
> ematics. He would therefore discourage such an experiment. (Artin 1947a, 126)

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[17]When discussing the generators of the "Poincaré group of the punctured plane" (Artin 1947a, 107), for
example, Epple (1999, 319) notes that Artin also uses in (Artin 1947a) topological arguments while
solving the word problem.

1950: Artin's return to visual reasoning?

Two papers that appeared shortly after Artin's and Bohnenblust's 1947 papers praise the algebraic approach to braid theory. As Chow wrote in 1948,

> Artin has recently given a new elegant and completely rigorous treatment of his theory of braids. […] Bohnenblust has derived some of the main results, in particular the completeness of the relations of the braid group, by an algebraical analysis of the abstract group. (Chow 1948, 654)

The conclusion of the paper notes the 'isomorphism' between the 'geometric braid group' and the 'algebraic braid group' (p. 658). The expression 'geometric braid group' appears also in Bohnenblust's paper, and one might say that it became a synonym with the 'intuitive', topological definition of the braid group. Dealing with the 'representations of the group of *n*-braids by transitive permutation groups of *n* letters' (Artin 1947b, 643), Artin's 1947 'Braids and Permutation' does not even hint at the use of visual reasoning or diagrams—the paper is completely algebraic.

In 1950, however, Artin published a paper entitled 'The Theory of Braids', in *Scientific American*. As this paper was not a paper for a mathematical journal, he was less strict in his discussion. Hence one may claim that there is a gap between the public presentation of mathematics to the layperson and the way mathematics was really practised. As I claim, however, Artin in this paper was still thinking of braid theory as a theory that cannot be completely formalized in an algebraic-symbolic manner. Indeed, as Artin admits from the outset, two styles of reasoning exist when dealing with braid theory: topology, 'used in the definition of braids', and 'the theory of groups, used in their treatment' (Artin 1950, 112). Artin starts with the definition of a 'weaving pattern', noting implicitly the material origin of the occupation with braids. A braid is defined as 'a weaving pattern together with the permission to deform it according to [certain] rules' (Artin 1950, 113). The requirements demanded for a set of braids to be a group are presented with diagrams, shown 'intuitively speaking' (p. 114), and only sometimes symbolically. Only *afterwards* are the relations within the braid group shown and written down formally. Thus, to give an example, Artin notes that given a braid A,

> tied to it [in the drawn figure] [is] its exact reflexion […] which we call A^{-1}. The reader can convince himself that the combined braid AA^{-1} can be disentangled if he starts by removing crossings from the middle outwards. (Artin 1950, 115)

Here the reader is convinced by means of a visual argument, and should imagine the material action of the disentanglement, or 'combing', as Artin phrases it elsewhere in the paper. Moreover, while several statements are only symbolically proven (e.g. that the group can be generated only with two elements, as in the 1926 paper), other results—such as how to obtain a unique normal form, which 'describes the braid uniquely'—are described by requiring the reader to imagine the stretching of several threads and by referring to a diagram (Artin 1950, 119, 118).

To conclude, Artin's 1950 approach—in contrast to the approaches that the 'geometric braid group' can be presented in terms of an automorphism group—is that there is a 'translation of our geometric procedure into group theoretic language' (Artin 1950, 119), a translation that does not cancel or marginalize either side of the process. I will discuss this conception of translation in the following section.

Formalization and visual arguments: between translation and hybridization

What stands behind Artin's explicit aim of symbolically formalizing braids? One may claim that the official aim was to formalize, in the sense of Krämer, the mathematical treatment of braids, and to present an abstract, formal structure, to such an extent that the theorems and conclusions drawn from it would be independent of any material or imaginative interpretation. Algebraization was considered by Artin as a way to eliminate the role of any diagrammatic or visual reasoning involved.

As this historical survey has shown, one can notice a growing tendency towards formalization within the mathematical treatment of braid theory. This is especially true in 1947–48 when a marginalization of visual reasoning (seen for example with the introduction of Burau's representation) occurred. In taking the role of diagrammatic and visual reasoning into account, however, I aim in this section to show that these types of arguments could not be reduced entirely to symbolic-deductive procedure, at least during the 1920s and 1930s. This will shed light on how processes of formalization operate together with visual reasoning.

To reiterate, recently there has been a call to consider philosophically the practices of mathematics, and especially the role diagrams play within mathematical reasoning. As Mancosu, for example, states: 'the epistemology of mathematics needs to be extended well beyond its present confines to address epistemological issues having to do […] [with] visualization, diagrammatic reasoning, understanding, explanation […]' (2008, 2). Indeed, the role of visualization in mathematical practice has been re-assessed, and sketches, diagrams and three-dimensional material models can be considered as an essential part of the mathematical proof and practice. Furthermore, they are no mere approximation or second-order representation of the abstract object obtained within the symbolic realm. As Netz (1998, 1999) and Manders (2008) have shown, already in Antiquity several geometrical proofs were impossible to understand without accompanying diagrams, which not only provided additional information, but were also used to draw inferences. This has certainly shown that one can speak of diagrammatic reasoning and its inferential and justificatory power, *along with* symbolic-deductive practice and its inferential power.[18]

With this in mind, it is essential to recall that while Artin was influenced by and was also a part of the structural approach to algebra, he was also influenced by the topological tradition, mainly due to his work with Schreier.[19] Hence in the 1926 paper, despite his 'official declaration', diagrams for Artin were not merely heuristic or illustrative. Moreover, algebraic formalization could not be regarded as a way of bypassing diagrammatic or visual reasoning. As Epple notes, in the 1926 paper Artin shows 'that the group-theoretical *game* may be embedded […] into the topological one' (1997, 186), i.e. one can (re)describe certain topological situations in an algebraic way. Hence Artin uses two types of reasoning (which Epple calls 'games'): the first, algebraic ('purely symbolic') and the second, within a topological context. For

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[18]There are numerous studies of such diagrammatic reasoning, also beyond Antiquity, see e.g. (Avigad *et al.* 2009; Leitgeb 2009; De Toffoli and Giardino 2014; Rabouin 2015; Larvor 2012, 2017).

[19]Epple (1997, 184) notes that the topological concepts found in Artin's 1926 paper were not defined "too explicitly". Moreover, Otto Schreier, who moved from Vienna to Hamburg in 1923, brought with him the topological ideas of Wirtinger und Reidemeister to his work with Artin, who was based in Hamburg, and the two worked together on the braid group (Epple 1999, 314).

Epple, arguments using only algebraic reasoning indicate 'their (relatively) poor argumentative context', while 'the use of two mathematical methods […] are (relatively) concrete […]'. Since 'they use the methods of two mathematical games, […] the argumentative context is [thus also] (relatively) rich' (Epple 1997, 187).

I would like to make a small detour and remind the reader here that the diagrams that Artin, Burau and Fröhlich were working with were concrete objects, and to those concrete objects and their deformations several adjectives were assigned: *anschaulich*, 'intuitive', having 'geometrical' or 'physical meaning', or being 'grounded geometrically'. While the adjective *anschaulich* appears only in Artin's 1926 paper (though re-appearing in 1950 in English as 'intuitive'), it is clear that its extent, together with the extent of the other adjectives, is restricted to imaginable objects—with the restriction that these objects can be in theory materially constructed or drawn. Though I do not aim to discuss here the (changing) role of the concept of *Anschauung* in relation to diagrammatic reasoning in the first decades of the twentieth century, the fact that these *anschauliche* objects were very concrete and not abstract can also be seen with Hilbert's public series of lectures *Anschauliche Geometrie*, which took place during the years 1920–21. In the introduction to Hilbert's 1932 book which compiled these lectures, he indicates the following: 'In mathematics […] we come across two tendencies: the tendency towards abstraction […] and the other tendency towards *Anschaulichkeit*, which is rather based on a lively understanding of the objects and their contextual relationships' (Hilbert and Cohn-Vossen 1932, v). While Hilbert states explicitly how new mathematical domains evolved as a result of abstraction, he does not draw an equivalent development with *Anschauung*. Hilbert, for example, does present numerous diagrams as well as photos of material, three-dimensional models during the lectures and in the book,[20] but it seems that these *anschauliche* models were only there to illustrate, to function as a 'lively understanding of the *objects*', i.e. of specific mathematical objects.[21]

Returning to our discussion, and considering the usage of both types of arguments —an abstract, formalized, symbolic one, and a topological, imaginative, *concrete* and diagrammatic one—I claim that during the 1920s and the 1930s, for Artin, Burau and Fröhlich, diagrams and their manipulation (either in practice or via their imagining) were a way to stabilize and make arguments rigorous. This is clearly the case given the fact that several arguments in the course of the proofs could only be shown or imagined. Is it any wonder then that expressions such as 'topological object', 'geometric meaning' and 'physical meaning' often appear? Or that in 1950 Artin emphasizes the materiality as well as material actions lying at the base of braid theory?[22] Due to the necessity of visual reasoning, one can say that the symbolic-formalized arguments were lacking essential inferential steps. Hence they were in need of stabilization,

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[20]There are numerous diagrams (also of three-dimensional surfaces) throughout the book. Photos of three-dimensional objects can be found for example in (Hilbert and Cohn-Vossen, 17) (cardboard model of a movable ellipsoid), (14–15) (string models of hyperboloid) or (193–4) (models from plaster of surfaces and their geodesic lines).

[21]A similar coupling between *Anschauung* and visualization (*Veranschaulichung*) with external objects is to be seen also in the 1930s in Hans Hahn's lecture "The crisis of the Anschauung" (1933). See (Volkert 1986, 259).

[22]This is despite Artin's vehement rejection of material constructions in 1947. Artin's call to imagine these material actions might be understood therefore as a call for a form of "manipulative imagination" (De Toffoli and Giardino 2014, 829), which might have been more reliable.

which the visual arguments certainly provided. In short, here a stabilization of formalized reasoning is provided by the visual arguments.[23]

Taking such stabilization into account, I claim that a certain *dynamic* exists between the symbolic text and the diagrams—this is the case since the diagrams express claims, which are not expressed and sometimes cannot be expressed symbolically in the text itself. This also means that there is a specific way of engaging the diagrams, and the relations between the diagrams and the symbolic-deductive practice may change. I claim that this dynamic came to be expressed in two forms: translation and hybridization.

First, there is a *translation* (or attempts at translation) from one practice to another, mainly from the diagrammatic to the formal. As Artin notes in 1950, the formal-algebraic braid theory can be thought of as a theoretical way to formalize manual 'weaving', and he explicitly terms this process 'translation' (1950, 119). This means that although a group may be thought of as a geometrical object (for example, when considering the Cayley graph), within the research on braid theory the group was considered as belonging to the opposite pole with respect to geometrical objects and operations.[24] This translation is not a complete one, however, as the deformation and moving of the strings is an essential part of the proofs during the 1920s and the 1930s, which cannot be translated into a symbolic language. This is a continuation of the diagrammatic-visual reasoning of Alexander in 1923, proving that every knot or link can be presented as a closure of a braid. Alexander, Artin, Burau and Fröhlich each note and use the possible deformations of knots and braids, either with a real diagram or as a 'mental model'. Following Brendan Larvor's cue, I suggest that there is in their works a usage of 'internalised diagrammatic practices' (2017, 15). Though Larvor refers to the work of de Toffoli and Giardino on Alexander's 1923 paper on knot theory, it is clear that, following de Toffoli and Giardino, the diagrams during the 1920s and 1930s (but also Artin's 1950 paper) within the research on braid theory were also a form of 'manipulative imagination' (De Toffoli and Giardino 2014, 829). This was not something (easily) translatable into formalized-symbolic arguments.

Secondly, there was *hybridization*. This can be understood as bringing the two types of argumentation together: i.e. the mathematical process of discovery and proof is composed of a hybrid of visual and symbolic-deductive reasoning that functions side by side. This hybridization not only considers diagrams as having 'at the same time diagrammatic and symbolic elements' (De Toffoli and Giardino 2014, 830), but also can be seen in *how* formalization and visualization work together and complement each other. Along with the incompleteness of the translation of arguments, this hybridization shows that formalization is not a 'smooth' practice, without disturbances, that happens in an instant. Moreover, it also points to the limits of formalization as such. Whereas it is clear that diagrammatic reasoning needs formalized steps to show its

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[23]Along similar lines, David Rabouin discusses the stabilization of mathematical knowledge in relation to Proclus's mathematical work: "one [can] accept that mathematical knowledge stabilizes itself in the interplay between these two regimes, which Proclus calls 'discursive reasoning' and 'imagination'." (Rabouin 2015, 134).

[24]Moreover, it is important to note that the usage of the adjective "algebraic" is not so often compared to the usage of "group-theoretic", and the term "algebraic braid group" turned into a conceptual unit used mostly after Bohnenblust's paper.

generality,[25] symbolic-deductive proofs (sometimes) used (imaginative) diagrams. This implies that their dynamic features and the operations carried out on them were an essential, irreplaceable step in the course of the proof. Hence, proofs involving these two types of arguments can be considered hybrid. In that sense, these visual steps were also regarded as sufficiently general—as what can be repeated, following Krämer's characterization.

Returning to the characterizations of formalism presented at the beginning of this article, one has to wonder at least whether the ahistorical description of 'freedom of interpretation' has to be refined or re-defined. As the case study of the braid group has shown, a rejection of formalism or of any other reasoning involved except a symbolic one not only does not stand the test of reality, but also delivers a distorted image of formalism as well as the processes of formalization. Formalization, as I have attempted to show, may and can operate with visual arguments, either by translating them *partially* into symbolic language, or by forming an entangled hybrid between the two practices of reasoning.

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[25]Apart from the obvious fact that "in order to become meaningful, it has to be considered inside a particular context of use, and therefore interpreted in such a context" (De Toffoli and Giardino 2014, 830), where context also means symbolic-formalized one.

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