Overconvergent Fréchet Algebras in Rigid Analysis

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Abstract

We fix a complete field k with respect to a non-Archimedean absolute value $\|\,.\,\|$.

In Chapter 1, we build the overconvergent function algebra $U_{n,\varphi}$ to be a subalgebra of the Tate algebra T_n by putting a growth condition on the coefficients of the power series using a decreasing function φ on $\mathbb{Z}_{\geq 0}$ into $\mathbb{R}_{>0}$ which we call a filter function (satisfying certain conditions). Namely,

$$U_{n,\varphi} := \Big\{ \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u : a_u \in k \text{ and } \lim_{|u|\to\infty} \|a_u\|^{\varphi(u)} = 0 \Big\} \subset T_n$$

where X denotes the tuple $(X_1, \ldots, X_n)$ of n indeterminates and where the norm on $\mathbb{Z}^n_{\geq 0}$ is given by the sum of all coordinates. We endow $U_{n,\varphi}$ with the φ-norm given by

$$\Big| \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u \Big|_{\varphi} := \max_{u\in\mathbb{Z}^n_{\geq 0}} \big\{ \|a_u\|^{\varphi(u)} \big\}.$$

With this setting we prove the following result: $U_{n,\varphi}$ is a Noetherian, Jacobson, unique factorization domain and it is complete with respect to φ-norm, moreover every ideal of $U_{n,\varphi}$ is closed with respect to the induced topology.

In Chapter 2, we define the category of NMK-algebras as the category of all quotients of $U_{n,\varphi}$ (where φ is fixed and n varies). Working in the larger category of Fréchet spaces, we establish Noether normalization and investigate the morphisms between NMK-algebras. Finally, we show that the category of NMK-algebras is closed under completed tensor products.

We investigate certain geometric aspects of the algebra $U_{n,\varphi}$ in Chapter 3, such as the properties of maximal ideals and regularity of $U_{n,\varphi}$. Further, we show that for each $U_{n,\varphi}$ the associated algebraic de Rham complex is exact in positive degrees.

Zusammenfassung

Wir fixieren einen Körper k, der bezüglich eines nicht-archimedischen Absolutbetrags $\| \, . \, \|$ vollständig ist.

In Kapitel 1 konstruieren wir eine Algebra $U_{n,\varphi}$ bestehend aus überkonvergenten Funktionen. Sie ist eine Unteralgebra der Tate-Algebra T_n, wobei mittels einer sogenannten *Filterfunktion* $\varphi\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ eine zusätzliche Wachstumsbedingung an die Koeffizienten der Potenzreihen in $U_{n,\varphi}$ gestellt wird. Explizit definieren wir

$$U_{n,\varphi} := \Big\{ \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u \ : \ a_u \in k \text{ and } \lim_{|u| \to \infty} \|a_u\|^{\varphi(u)} \Big\} \subset T_n,$$

wobei X das Tupel $(X_1, \ldots, X_n)$ in n freien Variablen bezeichnet und die Norm auf $\mathbb{Z}^n_{\geq 0}$ durch die Summe der Koordinaten gegeben ist. Die Algebra $U_{n,\varphi}$ ist mit einer natürlichen φ-Norm ausgestattet vermöge

$$\Big| \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u \Big|_{\varphi} := \max_{u \in \mathbb{Z}^n_{\geq 0}} \big\{ \|a_u\|^{\varphi(u)} \big\}.$$

In diesem Kontext beweisen wir das folgende Resultat: $U_{n,\varphi}$ ist ein Noetherscher, Jacobsonscher, faktorieller Integritätsbereich, der bezüglich der φ-Norm vollständig ist, und jedes Ideal in $U_{n,\varphi}$ ist abgeschlossen in der induzierten Topologie.

In Kapitel 2 definieren wir die Kategorie der NMK-Algebren als die Kategorie der Quotienten der $U_{n,\varphi}$ (wobei φ fixiert und n variabel ist). Indem wir in der größeren Kategorie der Fréchet-Räume arbeiten, beweisen wir die Noethernormalisierung und untersuchen die Morphismen zwischen NMK-Algebren. Schließlich zeigen wir, dass die Kategorie der NMK-Algebren abgeschlossen ist unter vervollständigten Tensorprodukten.

In Kapitel 3 untersuchen wir geometrische Aspekte der Algebren $U_{n,\varphi}$ nämlich Eigenschaften der maximalen Ideale und die Regularität von $U_{n,\varphi}$. Abschließend zeigen wir, dass für jedes $U_{n,\varphi}$ der assoziierte algebraische de Rham-Komplex exakt in positiven Graden ist.

Contents

Introduction

Let k be a field and let $\|\,.\,\|\colon k \to \mathbb{R}_{\geq 0}$ be a non-trivial, non-Archimedean absolute value such that k is complete with respect to $\|\,.\,\|$. The unit disc $k^{\circ} := \{x \in k : \|x\| \leq 1\}$ is a local ring, with maximal ideal $k^{\circ\circ} := \{x \in k : \|x\| < 1\}$. We denote by $\widetilde{k} := k^{\circ}/k^{\circ\circ}$ the residue field of k.

The non-Archimedean absolute value has the property that for all $x, y \in k$ with $\|x\| \neq \|y\|$, we have $\|x + y\| = \max\{\|x\|, \|y\|\}$ and as a consequence of this we have that "all triangles are isosceles" and the open ball $B(x, r)$ centered at x and of radius r is at the same time closed. Since these balls form a basis for the topology, the field k becomes totally disconnected. Doing classical analysis on such a totally disconnected space is not reasonable. For instance, if $p(X)$ and $q(X)$ are two polynomials in $k[X]$, then the function defined piecewisely:

$$\phi(x) = \begin{cases} p(x) & \text{if } x \in B(0,1) \\ q(x) & \text{otherwise} \end{cases}$$

is a continous function. Moreover, the function ϕ is "analytic" in the sense that it is given by a convergent power series in a neighborhood of any given point.

There are a few attemps for solutions to this dilemma and Tate's approach is one of them. It basically involves putting a coarser topology on k, such that the naive definition of "being analytic" does not include the functions as in the example above.

We begin this introduction by summarizing the results concerning the basic theory of Tate's approach.

Definition 0.1. The (free) *Tate algebra in n indeterminates over k* is defined as follows:

$$T_n(k)\colon = \Big\{ \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u \in k[[X_1, X_2, \ldots, X_n]] : a_u \in k \text{ and } \|a_u\| \to 0 \text{ as } |u| \to \infty \Big\}$$

where the multi-index $u = (u_1, u_2, \ldots, u_n) \in \mathbb{Z}^n_{\geq 0}$ with $|u| = u_1 + u_2 + \ldots + u_n$ and $X = (X_1, X_2, \ldots, X_n)$ the n-tuple of variables with $X^u = X_1^{u_1} X_2^{u_2} \ldots X_n^{u_n}$.

Elements of T_n can be thought of as the convergent power series on the polydisc $(k^{\circ})^n$. Roughly speaking, the analytic functions considered in this setting are the elements of a quotient of some Tate algebra.

We will now present a theorem summarizing the basic ring theoretic properties of T_n. The following results play an important role in being able to work with quotients of Tate algebras. Next theorem is an immediate result of the paper [28] and it implies that the algebra T_n shares some common properties with the usual polynomial ring $k[X_1, \ldots, X_n]$.

Theorem 0.2. *(i) The Gauss-norm $\left|\sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u\right| = \max_{u \in \mathbb{Z}^n_{\geq 0}} \{\|a_u\|\}$ on T_n is a multiplicative k-algebra norm and T_n is complete with respect to the corresponding natural metric.*

(ii) *The domain T_n is Noetherian, regular and a unique factorization domain. For every maximal ideal $\mathfrak{m}$ of T_n, the local ring $(T_n)_{\mathfrak{m}}$ has dimension n and the residue class field $T_n/\mathfrak{m}$ has finite degree over k.*

(iii) *The ring T_n is Jacobson: Every prime ideal $\mathfrak{p}$ of T_n is the intersection of the maximal ideals containing it. In particular, if I is an ideal of T_n then an element of T_n/I is nilpotent if and only if it lies in every maximal ideal of T_n/I.*

(iv) *Every ideal in T_n is closed with respect to the Gauss-norm.*

The quotients of the Tate algebras, the so-called *affinoid algebras*, form a category analogous to the category of finite type algebras over a field. In particular, they satisfy a version of the "Noether Normalization Lemma" [26]:

Theorem 0.3. *Let A be a k-affinoid algebra defined as: $A := T_n/I$ where I is an ideal of T_n. Then there is an injective finite morphism $T_d \to A$ for some $d \geq 0$.*

On the other hand, affinoid algebras are Banach spaces and their analytic theory turns out to be not very complicated. For example, all morphisms between affinoid algebras are continuous, which implies that all complete norms are equivalent. What we are basically going to do is to try to somehow generalize the ideas of the results done for the Tate algebra.

So, in this sense non-Archimedean geometry depends on the Tate algebras and their quotients (by ideals). That means, the ideals should not be too complicated to work with. For instance, they should at least be finitely generated, that is why the Noetherian property is crucial. If the ideal is infinitely generated it might not be possible to work with the quotient. The problem is that the Noetherian property is not preserved when we pass to subrings. Therefore, it would always be interesting to find a Noetherian subring of the Tate algebra and try to develop a theory accordingly. For example in the paper [8] a subalgebra is defined as:

$$T_n(\rho,\lambda) = \Big\{ \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u \in k[[X_1,\dots,X_n]] : \|a_u\|\rho_1^{\lambda^{-1}(u_1)}\dots\|a_u\|\rho_n^{\lambda^{-1}(u_n)} \to 0 \text{ as } |u|\to\infty \Big\}$$

for some type of growth function λ. This subring is shown to be Noetherian, and one can try to build a theory of non-Archimedean geometry based on the subring $T_n(\rho,\lambda)$.

It is a fact that the Tate algebras are the building blocks for rigid geometry, but unfortunately they have a severe defect: the de Rham cohomology of the Tate algebras is non-trivial. This nontriviality of de Rham cohomology of T_n, and hence the poly unit disk, violates the intuition of the unit disk being contractible. At this point we must note that being contractible should not be considered in the usual sense because the topology induced by the non-Archimedean absolute value is too complicated to picture. One of the principle motivations of studying such subalgebras is the construction of rigid (de Rham) cohomology given in [2].

For example, at $n = 1$ the de Rham cohomology is given by the cohomology of the complex $\partial\colon T_1 \to T_1$. We have

$$T_1 = \Big\{ \sum_{n=0}^{\infty} a_n X^n \in k[[X]] : \|a_n\| \to 0 \text{ as } n\to\infty \Big\}$$

and we proceed with the following lemma:

Lemma 0.4. *Suppose that the characteristic of k is 0 and the non-Archimedean norm $\| \, . \, \|$ on k is non-trivial. Then, the formal derivation map $\partial : T_1 \to T_1$ where*

$$\sum_{n \geq 0} a_n X^n \mapsto \sum_{n \geq 1} n a_n X^{n-1}$$

is not a surjective map.

Proof. Since k has characteristic 0, it contains a copy of $\mathbb{Q}$ and since the absolute value $\| \, . \, \|$ on k is non-Archimedean, by Ostrowski's Theorem [17], the norm on $\mathbb{Q}$ must be equivalent to a p-adic norm for some p prime.

Now, consider the power series $\sum_{m \geq 0} p^m X^{p^m - 1}$. Since

$$\lim_{m \to \infty} \|p^m\| = \lim_{m \to \infty} \frac{1}{(p^r)^m} = 0$$

(where r is a positive real number) the power series $\sum_{m \geq 0} p^m X^{p^m - 1}$ is an element of T_1. But it is not an element of ∂T_1, because the formal integral $\sum_{m \geq 1} X^{p^m}$, because of having constant non-zero coefficients, does not belong to T_1. Hence, the formal derivation map is not surjective. $\square$

Thus, we deduce that the de Rham cohomology, at least for $n = 1$, is not trivial. On the other hand, we have another subalgebra of T_n, given by:

$$T_n(\rho) = \Big\{ \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u \in k[[X]] : \|a_u\| \rho^{|u|} \to 0 \text{ as } |u| \to \infty \Big\}$$

where $k[[X]] := k[[X_1, X_2, \ldots, X_n]]$ and $\rho \in \mathbb{R}_{>1}$. We define the so-called Washnitzer Algebra:

$$W_n = \bigcup_{\rho > 1} T_n(\rho)$$

i.e.

$$W_n = \Big\{ \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u \in k[[X]] : \text{ there exists } \rho > 1 \text{ such that } \|a_u\| \rho^{|u|} \to 0 \text{ as } |u| \to \infty \Big\}$$

As shown in the paper [12], the algebras W_n have trivial de Rham cohomology (where $\operatorname{char} k = 0$) and a theory of non-Archimedean geometry which is similar to the theory for T_n, has been built by Elmar grosse-Klönne.

But, now the problem is that the algebras W_n are no longer complete for any useful k-Banach norm and the question is:

Question 0.5. *Is there a complete Noetherian subalgebra of T_n (possibly containing W_n) with trivial de Rham cohomology on which we can build a theory of non-Archimedean geometry?*

To get a subalgebra of T_n, we must put a condition on the coefficients. A first attempt would be trying to multiply the norm of the coefficients with a suitable real number, which is done to construct the W_n algebras. A second attempt would be trying to put a power function on the norm of the coefficients. We also would like to construct a subalgebra of T_n with trivial de Rham cohomology.

To get some intuition and for simplicity suppose $n = 1$. We put a condition on the coefficients of a power series of T_n given by a power function φ. Define:

$$U_\varphi = \Big\{ \sum_{n \in \mathbb{Z}_{\geq 0}} a_n X^n : a_n \in k \text{ and } \lim_{n \to \infty} \|a_n\|^{\varphi(n)} = 0 \Big\}$$

for a certain type of function φ on $\mathbb{Z}_{\geq 0}$. So, the elements of U_φ are the power series in T_n such that the coefficients tend to zero faster than a certain rate depending on the function $\varphi : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$.

And, we would like to show that such a candidate U_φ has trivial de Rham cohomology. Note that, if the formal derivation

$$\partial : U_\varphi \to U_\varphi$$

given by

$$\sum_{n=0}^{\infty} a_n X^n \mapsto \sum_{n=1}^{\infty} n a_n X^{n-1}$$

is surjective then the de Rham cohomology is trivial. So, we want to prove the surjectivity of the formal derivation map. For this, we first need to put some condition on the function φ. The next Lemma gives such a condition.

Lemma 0.6. *Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a function. Then $\big(n^{f(n)}\big)_{n \in \mathbb{Z}_{\geq 0}}$ is bounded if and only if*

$$f(n) \leq \frac{C}{\log n}$$

for all large enough n and for some constant $C \in \mathbb{R}_{>0}$.

Proof. Suppose that $f(n) \leq \frac{C}{\log n}$ for some constant $C \in \mathbb{R}_{>0}$. Let M be a real number such that $M > e^C$. Suppose also for the sake of a contradiction that $n^{f(n)}$ is not bounded. Then there exists $m \in \mathbb{N}$ such that $m^{f(m)} > M$, so that we have

$$f(m) \log m > \log M \quad \text{i.e.} \quad \frac{\log M}{\log m} < f(m)$$

Thus, by the assumption, we have

$$\frac{\log M}{\log m} < f(m) \leq \frac{C}{\log m}$$

and it implies the contradiction that $M < e^C$.

For the converse, if $n^{f(n)}$ is bounded by the real number M, i.e. if $n^{f(n)} < M$ then we have $f(n) \log n < \log M$ and hence $f(n) < \frac{\log M}{\log n}$. We can simply take $C = \log M$. □

So, now we have the idea of setting

$$\varphi(n) := \frac{1}{\log n}.$$

Then we have:

Lemma 0.7. *Suppose* $\operatorname{char} k = 0$. *The derivation map on* U_φ *is a surjection. Define* $\partial : U_\varphi \to U_\varphi$ *such that*

$$\sum_{n=0}^{\infty} a_n X^n \mapsto \sum_{n=1}^{\infty} n a_n X^{n-1}$$

Then ∂ *is a surjective map.*

Proof. We need to show that the formal integration of a series in U_φ is also an element of U_φ, i.e. for $\sum_{n=0}^{\infty} a_n X^n \in U_\varphi$ we also have $\sum_{n=1}^{\infty} \frac{a_{n-1}}{n} X^n \in U_\varphi$. Note that for $n \in \mathbb{N}$ we always have the inequality $\|\frac{1}{n}\| \leq n^r$ for some $r \in \mathbb{R}_{>0}$ and hence by the preceeding lemma (Lemma 0.6), $\|\frac{1}{n}\|^{r\varphi(n)}$ is bounded, say by M. Now for large enough values of n,

$$\left\|\frac{a_{n-1}}{n}\right\|^{\varphi(n)} = \|a_{n-1}\|^{\varphi(n)} \left\|\frac{1}{n}\right\|^{\varphi(n)} \leq \|a_{n-1}\|^{\varphi(n)} M \to 0$$

□

Therefore, we conclude that the de Rham cohomology of U_φ is trivial.

So, it turns out that if we define:

$$U_{n,q}(k) = \left\{ \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u : a_u \in k \text{ and } \|a_u\|^{\frac{1}{\log_q |u|}} \to 0 \text{ as} |u| \to \infty \right\}$$

where $q > 1$ any real number, then $U_{n,q}$ becomes a subalgebra of T_n (as will be shown in the next chapter) containing the W_n algebras and the de Rham cohomology of this subalgebra becomes trivial (see Section 3.3 for details). But, to build a reasonable geometry on $U_{n,q}(k)$ we must also have the Noetherian property.

In this text, the function $\frac{1}{\log_q |u|}$ will be generalized to a function which we call φ (a "*filter function*") on the set $\mathbb{Z}_{\geq 0}$ and a theory of non-Archimedean geometry will be built for this particular subalgebra of T_n. The filter function φ must mainly satisfy the following two conditions:

(i) φ is decreasing on $\mathbb{Z}_{\geq 0}$.

(ii) $\frac{\varphi(2n)}{\varphi(n)} > \gamma > 0$ for large enough values of $n \in \mathbb{Z}_{\geq 0}$ and for some $\gamma \in \mathbb{R}_{>0}$.

The necessity of these conditions is explained in Appendix B with counter examples. Using this filter function, for each $n \in \mathbb{Z}_{\geq 0}$ we define:

$$U_{n,\varphi} = \left\{ \sum_{\substack{u \in \mathbb{Z}^n_{\geq 0} \\ u=(u_1,\ldots,u_n)}} a_u X_1^{u_1} \ldots X_n^{u_n} : a_u \in k \text{ and } \lim_{|u| \to \infty} \|a_u\|^{\varphi(|u|)} = 0 \right\}$$

where $X_1, \ldots, X_n$ denote the indeterminates and the norm on $\mathbb{Z}_{\geq 0}$ is given by the sum of all coordinates, i.e. $|u| = u_1 + u_2 + \ldots + u_n$.

Influenced by the Gauss norm on T_n, we put a norm on $U_{n,\varphi}$ using the filter function φ. For $f = \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u \in U_{n,\varphi}$, we define:

$$|f|_\varphi = \max_{u \in \mathbb{Z}^n_{\geq 0}} \left\{ \|a_u\|^{\varphi(|u|)} \right\}$$

where X^u denotes $X_1^{u_1} \ldots X_n^{u_n}$ with $u = (u_1, \ldots, u_n)$.

With this setting we first prove the following result:

Theorem 0.8. *$U_{n,\varphi}$ with the φ-norm forms a normed group structure. $U_{n,\varphi}$ is a subalgebra of T_n and complete with respect to the φ-norm.*

Unfortunately, we do not have a normed ring structure on $U_{n,\varphi}$ (with the φ-norm). Instead, we have a weak version of submultiplicativity on the unit disk:

$$U_{n,\varphi}^{\circ} = \{f \in U_{n,\varphi} : |f|_{\varphi} \leq 1\}.$$

For $f \in U_{n,\varphi}$, by multiplying f by a suitable constant (namely, the inverse of the coefficient with the maximum k-norm) we can pull the element f to the unit disk $U_{n,\varphi}^{\circ}$, i.e. for every $f \in U_{n,\varphi}$ there exists $c \in k$ such that $cf \in U_{n,\varphi}^{\circ}$.

This allows us to establish a "weak-submultiplicative" norm on $U_{n,\varphi}^{\circ}$:

Proposition 0.9. *For $f, g \in U_{n,\varphi}^{\circ}$, we have:*

$$|fg|_{\varphi} \leq |f|_{\varphi}^{\gamma} |g|_{\varphi}^{\gamma}$$

where $\gamma \in \mathbb{R}_{>0}$ is a constant depending on the filter function φ.

This weak-submultiplicativity property of the φ-norm is enough for us to prove Weierstrass Division Theorem and Weierstrass Preparation Theorem for $U_{n,\varphi}$. Then we follow the classical method of Rückert to prove the following result:

Theorem 0.10. *The algebra $U_{n,\varphi}$ is Noetherian, factorial, Jacobson and regular.*

After establishing the Noetherian property for $U_{n,\varphi}$, we build a category on the quotient algebras of $U_{n,\varphi}$ similar to the category of "affinoid algebras". To build such a category we first establish the following result:

Proposition 0.11. *Each ideal of $U_{n,\varphi}$ is closed with respect to the φ-norm.*

This closedness property of ideals allows us to define the natural residue norm on $U_{n,\varphi}/I$ where I is any ideal of $U_{n,\varphi}$:

$$|\bar{f}|_{\varphi} = \inf_{a \in I} \{|f - a|_{\varphi}\}.$$

At this point, we note that for technial reasons we fix the filter function φ and suppose that the set $\|k \setminus \{0\}\|$ is discrete in $\mathbb{R}_{>0}$ (for example when $k = \mathbb{Q}_p$).

We define the category of "nmk algebras" as the collection of all quotient algebras of U_n within the larger category of "k-Fréchet spaces", i.e. more formally:

Definition 0.12. An algebra $\mathcal{N}$ is called an *nmk algebra* if there is a continuous epimorphism $\alpha : U_n \to \mathcal{N}$ for some $n \in \mathbb{Z}_{\geq 0}$.

Next, we establish the following results for the category of nmk algebras:

Theorem 0.13. *All morphisms between nmk algebras are continuous, in particular all norms are equivalent.*

And the "Noether Normalization Lemma":

Theorem 0.14. *Let $\mathcal{N}$ be an nmk algebra. For every finite morphism $\alpha : U_n \to \mathcal{N}$ there is a morphism $\tau : U_d \to U_n$ with $d \leq n$ such that $\alpha \circ \tau : U_d \to \mathcal{N}$ is injective and finite.*

Then we build a completed tensor product theory for the category of nmk algebras. (We note that completed tensor product is the completion of the ordinary tensor product with respect to the induced topology.) And we prove the following:

Theorem 0.15. *The category of nmk algebras is closed under completed tensor products. Moreover we have:* $U_n \hat{\otimes} U_m = U_{n+m}$.

In the final chapter, we investigate certain geometric properties of the algebra U_n (such as maximal ideals) and prove that the associated algebraic de Rham complex for U_n is exact in positive degrees. For this, we add another condition on the filter function φ, namely for $u \in \mathbb{Z}_{\geq 0}^n$ we suppose $\varphi(|u|) \leq \frac{c}{\log |u|}$ for some $c \in \mathbb{R}_{>0}$ (see Appendix B for details). We prove:

Theorem 0.16. $\mathrm{H}^*_{dR}(U_n/k)$ *is trivial.*

In the next part, we start with basic algebraic properties and the consequences of these properties for $U_{n,\varphi}$.

Chapter 1

The Algebra

In this chapter, we will present a substructure of the Tate algebra that will be endowed with a slightly different norm but nevertheless still look very similar to the Tate algebra in an algebraic sense. These type of algebras are called "overconvergent function algebras" in the literature. Then later on, we will establish some basic properties concerning the algebraic structure of these subalgebras. These properties will be helpful to build the geometric structure of the algebra.

1.1 Overconvergent Function Algebras

Definition 1.1. Let $\varphi : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ be any decreasing function $\big($i.e. if $n \leq m$ in $\mathbb{Z}_{\geq 0}$ then we have $\varphi(m) \leq \varphi(n)\big)$ such that for all $n \in \mathbb{Z}_{\geq 0}$ we have

$$\frac{\varphi(2n)}{\varphi(n)} \geq \gamma_\varphi > 0$$

for some $\gamma_\varphi \in \mathbb{R}_{>0}$ depending on φ but independent of the value of n and with

$$\varphi(0) = 1.$$

We extend the function φ to the set $\mathbb{Z}^n_{\geq 0}$ by setting:

$$\varphi(u) := \varphi(|u|) = \varphi\big(u_1 + u_2 + \ldots + u_n\big)$$

where $u \in \mathbb{Z}^n_{\geq 0}$ with $u = (u_1, u_2, \ldots, u_n)$ and the absolute value on $\mathbb{Z}^n_{\geq 0}$ is given by $|u| := u_1 + u_2 + \ldots u_n$. When we write $\varphi : \mathbb{Z}^n_{\geq 0} \to \mathbb{R}_{>0}$ we actually mean $\varphi : \mathbb{Z}^n_{\geq 0} \to \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ and for simplicity instead of $\varphi(|u|)$ we will write just $\varphi(u)$. We will call such a function φ, a *filter function*[1].

Now, using a filter function φ, for all $n \in \mathbb{Z}_{\geq 0}$ we define:

Definition 1.2.

$$U_{n,\varphi} = \Big\{ \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u \in k[[X]] : a_u \in k \text{ and } \|a_u\|^{\varphi(u)} \to 0 \text{ as } |u| \to \infty \Big\}$$

where the symbol X denotes the n-tuple of indeterminates $(X_1, X_2, \ldots, X_n)$. And for $n = 0$ we define $U_{0,\varphi} = k$.

[1]In the sense that the function φ is used to filter out some power series of T_n.

Remark 1.3. *We need to give an example of such a function φ to justify what we are going to do next. For any $q > 1$, the function*

$$\varphi(u) = \frac{1}{\log_q(|u|)} = \frac{\log q}{\log(|u|)}$$

is an example of such a function. For simplicity, throughout the entire text the function φ can be taken to be the function

$$\frac{1}{\log(|u|)}$$

with $\varphi(\bar{0}) := \varphi(0, 0, \ldots, 0) = \varphi(u) = 1$ for all $|u| = 1$ and in this case γ_φ can be taken to be $\frac{1}{2}$.

From now on, for simplicity we will just use the symbol γ instead of γ_φ if the function φ is obvious.

The first thing we will do is to show that $U_{n,\varphi}$ is a actually a subalgebra of T_n. Before proving it, let us fix the function φ and the natural number n.

Again for simplicity we will write U_n instead of $U_{n,\varphi}$.

Then, our first result is:

Proposition 1.4. *The set U_n is a subalgebra of T_n.*

Proof. It is easy to see that $k \subseteq U_n \subseteq T_n$. So, we only need to prove that U_n is closed under addition and multiplication, then it will be a subring of T_n and the desired result will follow.

Let $\sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u$ and $\sum_{u \in \mathbb{Z}^n_{\geq 0}} b_u X^u$ be any two power series in U_n.

- Their sum: $\sum_{u \in \mathbb{Z}^n_{\geq 0}} (a_u + b_u) X^u$ has coefficients $a_u + b_u$ for all $u \in \mathbb{Z}^n_{\geq 0}$. By the strong triangle inequality satisfied by k, we have:

  $$(\|a_u + b_u\|)^{\varphi(u)} \leq (\max\{\|a_u\|, \|b_u\|\})^{\varphi(u)} = \max\{\|a_u\|^{\varphi(u)}, \|b_u\|^{\varphi(u)}\} \to 0 \text{ as } |u| \to \infty$$

  which implies that U_n is closed under addition.

- Their product:

  $$\Big(\sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u\Big)\Big(\sum_{v \in \mathbb{Z}^n_{\geq 0}} b_v X^v\Big) = \sum_{\delta \in \mathbb{Z}^n_{\geq 0}} \Big(\sum_{\delta = u+v} a_u b_v\Big) X^\delta$$

  has the coefficients $\sum_{\delta = u+v} a_u b_v$ for $\delta \in \mathbb{Z}^n_{\geq 0}$, where the addition on $\mathbb{Z}^n_{\geq 0}$ is being done componentwise. We need to show that

  $$\lim_{|\delta| \to \infty} \Big\| \sum_{\delta = u+v} a_u b_v \Big\|^{\varphi(\delta)} = 0$$

  Let $\varepsilon < 1$ be a positive real number.

  By the assumption on φ, there exists $\gamma \in \mathbb{R}_{>0}$ such that for all $u \in \mathbb{Z}^n_{\geq 0}$ we have:

  $$\frac{\varphi(u+u)}{\varphi(u)} = \frac{\varphi(2u)}{\varphi(u)} \geq \gamma > 0.$$

Note that, the power series $\sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u$ and $\sum_{v\in\mathbb{Z}^n_{\geq 0}} b_v X^v$ are already elements of U_n. So:

$$\text{There exists } N\in\mathbb{Z} \text{ such that for all } |u|\geq N \text{ we have } \|a_u\|^{\varphi(u)} < \varepsilon^{\frac{1}{\gamma}}$$

and similarly:

$$\text{For all } |v|\geq N \text{ we have } \|b_v\|^{\varphi(v)} < \varepsilon^{\frac{1}{\gamma}}$$

Then, for all $|u|, |v| \geq N$, we have:

$$\|a_u\| < \varepsilon^{\frac{1}{\gamma\,\varphi(u)}} \text{ and } \|b_v\| < \varepsilon^{\frac{1}{\gamma\,\varphi(v)}}$$

Note that, since $\lim_{|u|\to\infty}\|a_u\| = \lim_{|u|\to\infty}\|b_u\| = 0$ the sets of values $\left(\|b_u\|\right)_{u\in\mathbb{Z}^n_{\geq 0}}$ and $\left(\|a_u\|\right)_{u\in\mathbb{Z}^n_{\geq 0}}$ are bounded, say for all $u\in\mathbb{Z}^n_{\geq 0}$: $\|b_u\|\leq M_1$ and $\|a_u\|\leq M_2$, for some $M_1, M_2\in\mathbb{N}$ and put $M=\max\{M_1, M_2, 1\}$.

Now, set $|\delta| > 2N$. Then we have:

$$\begin{aligned}
\Big\|\sum_{u+v=\delta} a_u b_v\Big\|^{\varphi(\delta)} &= \Big\|\Big(\sum_{\substack{|u|\geq N\\|v|<N\\u+v=\delta}} a_u b_v\Big) + \Big(\sum_{\substack{|v|\geq N\\|u|<N\\u+v=\delta}} a_u b_v\Big) + \Big(\sum_{\substack{|u|\geq N\\|v|\geq N\\u+v=\delta}} a_u b_v\Big)\Big\|^{\varphi(\delta)}\\
&\leq \left(\max\left\{\Big\|\sum_{\substack{|u|\geq N\\|v|<N\\u+v=\delta}} a_u b_v\Big\|\,,\,\Big\|\sum_{\substack{|u|\geq N\\|v|<N\\u+v=\delta}} a_u b_v\Big\|\,,\,\Big\|\sum_{\substack{|u|\geq N\\|v|\geq N\\u+v=\delta}} a_u b_v\Big\|\right\}\right)^{\varphi(\delta)}\\
&\leq \left(\max\left\{\max_{\substack{|u|\geq N\\|v|<N\\u+v=\delta}}\{\|a_u\|\|b_v\|\}\,,\,\max_{\substack{|v|\geq N\\|u|<N\\u+v=\delta}}\{\|a_u\|\|b_v\|\}\,,\,\max_{\substack{|u|\geq N\\|v|\geq N\\u+v=\delta}}\{\|a_u\|\|b_v\|\}\right\}\right)^{\varphi(\delta)}\\
&\leq \left(\max\left\{\max_{\substack{|u|\geq N\\|v|<N\\u+v=\delta}}\{\varepsilon^{\frac{1}{\gamma\varphi(u)}}\|b_v\|\}, \max_{\substack{|v|\geq N\\|u|<N\\u+v=\delta}}\{\|a_u\|\varepsilon^{\frac{1}{\gamma\varphi(v)}}\}, \max_{\substack{|u|\geq N\\|v|\geq N\\u+v=\delta}}\{\varepsilon^{\frac{1}{\gamma\varphi(u)}}\varepsilon^{\frac{1}{\gamma\varphi(v)}}\}\right\}\right)^{\varphi(\delta)}\\
&\leq \left(\max\left\{M_1\max_{\substack{|u|\geq N\\|v|<N\\u+v=\delta}}\{\varepsilon^{\frac{1}{\gamma\varphi(u)}}\}\,,\,M_2\max_{\substack{|v|\geq N\\|u|<N\\u+v=\delta}}\{\varepsilon^{\frac{1}{\gamma\varphi(v)}}\}\,,\,\max_{\substack{|u|\geq N\\|v|\geq N\\u+v=\delta}}\{\varepsilon^{\frac{1}{\gamma\varphi(u)}}\varepsilon^{\frac{1}{\gamma\varphi(v)}}\}\right\}\right)^{\varphi(\delta)}\\
&= \max\left\{M_1^{\varphi(\delta)}\max_{\substack{|u|\geq N\\|v|<N\\u+v=\delta}}\{\varepsilon^{\frac{\varphi(\delta)}{\gamma\varphi(u)}}\}, M_2^{\varphi(\delta)}\max_{\substack{|v|\geq N\\|u|<N\\u+v=\delta}}\{\varepsilon^{\frac{\varphi(\delta)}{\gamma\varphi(v)}}\}, \max_{\substack{|u|\geq N\\|v|\geq N\\u+v=\delta}}\{\varepsilon^{\frac{\varphi(\delta)}{\gamma\varphi(u)}}\varepsilon^{\frac{\varphi(\delta)}{\gamma\varphi(v)}}\}\right\}\\
&\leq \max\left\{M_1^{\varphi(\delta)}\max_{|u|\leq\delta}\left\{\varepsilon^{\frac{\varphi(u+u)}{\gamma\varphi(u)}}\right\}, M_2^{\varphi(\delta)}\max_{|v|\leq\delta}\left\{\varepsilon^{\frac{\varphi(v+v)}{\gamma\varphi(v)}}\right\}, \max_{\substack{|u|\leq|\delta|\\|v|\leq|\delta|}}\left\{\varepsilon^{\frac{\varphi(u+u)}{\gamma\varphi(u)}}\varepsilon^{\frac{\varphi(v+v)}{\gamma\varphi(v)}}\right\}\right\}\\
&\leq \max\left\{M_1^{\varphi(\delta)}\max_{|u|\leq\delta}\left\{\left(\varepsilon^{\frac{1}{\gamma}}\right)^{\gamma}\right\}, M_2^{\varphi(\delta)}\max_{|v|\leq\delta}\left\{\left(\varepsilon^{\frac{1}{\gamma}}\right)^{\gamma}\right\}, \max_{\substack{|u|\leq|\delta|\\|v|\leq|\delta|}}\left\{\left(\varepsilon^{\frac{1}{\gamma}}\right)^{\gamma}\left(\varepsilon^{\frac{1}{\gamma}}\right)^{\gamma}\right\}\right\}\\
&= \max\left\{\varepsilon\, M_1^{\varphi(\delta)}, \varepsilon\, M_2^{\varphi(\delta)}, \varepsilon^2\right\}\\
&\leq \max\left\{\varepsilon\, M^{\varphi(\delta)}, \varepsilon^2\right\}
\end{aligned}$$

$$= \varepsilon\, M^{\varphi(\delta)}$$

Thus for all $0 < \varepsilon < 1$, we have:

$$\lim_{|\delta|\to\infty} \Big\| \sum_{u+v=\delta} a_u b_v \Big\|^{\varphi(\delta)} \leq \lim_{|\delta|\to\infty} \varepsilon\, M^{\varphi(\delta)} = \varepsilon$$

i.e.

$$\lim_{|\delta|\to\infty} \Big\| \sum_{u+v=\delta} a_u b_v \Big\|^{\varphi(\delta)} = 0$$

so that

$$\Big(\sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u \Big)\Big(\sum_{v\in\mathbb{Z}^n_{\geq 0}} b_v X^v \Big) = \sum_{\delta\in\mathbb{Z}^n_{\geq 0}} \Big(\sum_{\delta=u+v} a_u b_v \Big) X^\delta \in U_n$$

Hence, U_n is a subalgebra of T_n. □

In his paper [12], a similar theory for algebras W_n was built by Elmar Grosse-Klönne, this text also aims to imitate and improve his work. It turns out that if we set $\varphi(u) = \frac{1}{\log_q(|u|)}$ for any $q > 1$ real number, we see that $W_n \subseteq U_n$. But the current conditions of the function φ are not enough in general for the inclusion $W_n \subseteq U_n$. In Appendix A, a new condition is introduced and a criterion for this inclusion is given.

From now on, we will fix the function φ and the algebra U_n:

Corollary 1.5. *Let*

$$\varphi : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$$

be a function with the following properties:

(i) $\varphi(0) = 1$

(ii) For $n, m \in \mathbb{Z}_{\geq 0}$ *with* $n \leq m$ *we have:*

$$\varphi(m) \leq \varphi(n)$$

(iii) For all $n \in \mathbb{Z}_{\geq 0}$ *we have:*

$$\frac{\varphi(n+n)}{\varphi(n)} = \frac{\varphi(2n)}{\varphi(n)} \geq \gamma > 0$$

for some $\gamma \in \mathbb{R}_{>0}$.

If we extend the function to:

$$\varphi : \mathbb{Z}^n_{\geq 0} \to \mathbb{R}_{>0} \text{ by } \varphi(u) := \varphi(|u|)$$

then we have:

$$U_{n,\varphi} := \Big\{ \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u \in k[[X]] : a_u \in k \text{ and } \|a_u\|^{\varphi(u)} \to 0 \text{ as } |u| \to \infty \Big\}$$

is a subalgebra of T_n *(possibly containing* W_n*, see Appendix A for details). Here* X *denotes the tuple of* n*-variables* $(X_1, X_2, \ldots, X_n)$ *and the norm on* $\mathbb{Z}^n_{\geq 0}$ *is given by* $|u| = u_1 + u_2 + \ldots + u_n$ *whenever* $u = (u_1, u_2, \ldots, u_n)$.

To sum up:

Remark 1.6. *We define:*

$$U_0 := k$$
$$U_n := k\langle X_1, X_2, \ldots, X_n\rangle^{\varphi} := \Big\{ \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u : a_u \in k \text{ and } \|a_u\|^{\varphi(u)} \to 0 \text{ as } |u| \to \infty \Big\}$$

Whenever we have $m \leq n$, we view U_m as a subalgebra of U_n by using the same function φ restricted to $\mathbb{Z}^m_{\geq 0}$, i.e.

$$U_m := \Big\{ \sum_{v \in \mathbb{Z}^m_{\geq 0}} a_v X^v : a_v \in k \text{ and } \|a_v\|^{(\varphi\restriction_{\mathbb{Z}^m_{\geq 0}})(v)} \to 0 \text{ as } |v| \to \infty \Big\} \subseteq U_n$$

The restriction of φ to the first m coordinates of $\mathbb{Z}^n_{\geq 0}$:

$$(\varphi\restriction_{\mathbb{Z}^m_{\geq 0}}) : \mathbb{Z}^m_{\geq 0} \to \mathbb{R}_{>0}$$

is given by

$$(\varphi\restriction_{\mathbb{Z}^m_{\geq 0}})\big(u_1, u_2, \ldots, u_m\big) := \varphi\big(u_1, \ldots, u_m, 0, \ldots, 0\big)$$

In other words:

$$\varphi\big(u_1, u_2, \ldots, u_n\big) := \varphi(u_1 + u_2 + \ldots + u_m + u_{m+1} + \ldots + u_n)$$

and

$$(\varphi\restriction_{\mathbb{Z}^m_{\geq 0}})(u_1, u_2, \ldots, u_m) := \varphi(u_1 + u_2 + \ldots + u_m)$$

1.2 Basic Properties and Units

In this section we will establish some basic algebraic properties of the algebra U_n. We will first endow U_n with a reasonable norm imitated from the Gauss norm of T_n and then we investigate ring theoretic and topological properties of U_n which will show that the algebra is complete with respect to the norm we will define depending on the function φ. In the last part of the section, we will classify all units of U_n.

First, we recall the definitions of "norm" and "ultrametric" on an abelian group.

Definition 1.7. Let G be an abelian group. A function $|\,.\,| : G \to \mathbb{R}_{\geq 0}$ is said to be an *ultrametric* if it satisfies the following properties:

(i) $|0| = 0$

(ii) $|x - y| \leq \max\{|x|, |y|\}$ for all $x, y \in G$.

Of course, the second condition plays a crucial role for non-Archimedean function theory. Now, the definition of a normed group is:

Definition 1.8. A pair $(G, |\,.\,|)$ consisting of an abelian group G and an ultrametric $|\,.\,| : G \to \mathbb{R}_{\geq 0}$ is called a semi-normed group. The pair $(G, |\,.\,|)$ is called a normed group if $\ker |\,.\,| = \big\{x \in G : |x| = 0\big\} = \{0\}$, in other words:

$$|x| = 0 \text{ if and only if } x = 0.$$

Now, we will define an ultrametric on U_n (inspired from the *Gauss norm* on T_n) and deduce that U_n with this ultrametric gives a normed group structure.

Definition 1.9. For $f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u \in U_n$, define *the φ-norm of f* to be:

$$\Big|\sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u\Big|_{\varphi} = \max\left\{\|a_u\|^{\varphi(u)} : u \in \mathbb{Z}^n_{\geq 0}\right\}$$

Then with this φ-norm, we get a normed group structure on U_n:

Lemma 1.10. *The function* $|\,.\,|_{\varphi} : U_n \to \mathbb{R}_{\geq 0}$ *given by*

$$\sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u \mapsto \max\{\|a_u\|^{\varphi(u)} : u \in \mathbb{Z}^n_{\geq 0}\}$$

defines an ultrametric with trivial kernel on U_n*.*

Proof. We need to check that the function $|\,.\,|_{\varphi}$ satisfies the two properties given in the Definition 1.7. Let $f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u$ and $g = \sum_{u\in\mathbb{Z}^n_{\geq 0}} b_u X^u$ be two elements of U_n. Then:

(i) $|0|_{\varphi} = \Big|\sum_{u\in\mathbb{Z}^n_{\geq 0}} 0X^u\Big|_{\varphi} = \max_{u\in\mathbb{Z}^n_{\geq 0}}\{0^{\varphi(u)}\} = 0$

(ii) Note that since the norm $\|\,.\,\|$ on k is a non-Archimedean norm, it satisfies the ultrametric property which is given by $\|x - y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in k$. Using this fact, we have:

$$\begin{aligned}
|f-g|_{\varphi} &= \max_{u\in\mathbb{Z}^n_{\geq 0}}\{\|a_u - b_u\|^{\varphi(u)}\} \\
&\leq \max_{u\in\mathbb{Z}^n_{\geq 0}}\{(\max\{\|a_u\|, \|b_u\|\})^{\varphi(u)}\} \\
&= \max_{u\in\mathbb{Z}^n_{\geq 0}}\{\max\{\|a_u\|^{\varphi(u)}, \|b_u\|^{\varphi(u)}\}\} \\
&\leq \max_{u\in\mathbb{Z}^n_{\geq 0}}\{\max\{|f|_{\varphi}, |g|_{\varphi}\}\} \\
&= \max\{|f|_{\varphi}, |g|_{\varphi}\}
\end{aligned}$$

For the trivial kernel claim, suppose an element $f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u$ has a non-zero coefficient a_u for some $u \in \mathbb{Z}^n_{\geq 0}$. Then we have

$$0 < \|a_u\|^{\varphi(u)} \leq \max_{u\in\mathbb{Z}^n_{\geq 0}} \|a_u\|^{\varphi(u)} = |f|_{\varphi}$$

Thus, only the element $0 \in U_n$ can have 0-norm, so that the kernel of the ultrametric $|\,.\,|_{\varphi}$ is trivial. □

Hence, the pair $(U_n, |\,.\,|_{\varphi})$ is a normed group.

Remark 1.11. *At this point, it would be good to fix the notations of three different norms we have introduced so far:*

The non-Archimedean norm on k *is* $\|\,.\,\| : k \to \mathbb{R}_{\geq 0}$

The Gauss norm on T_n *is* $|\,.\,| : T_n \to \mathbb{R}_{\geq 0}$ *and*

The φ*-norm on* U_n *is* $|\,.\,|_{\varphi} : U_n \to \mathbb{R}_{\geq 0}$

By the following theorem, we see that the φ-norm on U_n extends the norm $\|\,.\,\|$ on k:

Theorem 1.12. *The norm $|\,.\,|_\varphi$ on U_n extends the norm $\|\,.\,\|$ on k and moreover the polynomial ring $k[X] = k[X_1, X_2, \ldots, X_n]$ is a dense subset of U_n.*

Proof. For $a \in k \subset U_n$ we have;

$$|a|_\varphi = |aX_1^0 X_2^0 \ldots X_n^0|_\varphi = \|a\|^{\varphi(0,0,\ldots,0)} = \|a\|$$

so that the φ-norm extends the norm of k to the algebra U_n.

Now, if $f = \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u$ is a power series in U_n, by considering the initial segments of this power series we can approach f:

$$\sum_{\substack{u \in \mathbb{Z}^n_{\geq 0} \\ |u| < |v|}} a_u X^u \mapsto f \text{ as } |v| \to \infty$$

Each initial segment of f is a polynomial in $k[X_1, \ldots, X_n]$, thus it follows that the polynomials are dense in U_n. □

Actually, the algebra U_n with the φ-norm is a complete non-Archimedean metric space. Before we prove it, we will give a result which we will use in the proof of completeness:

Remark 1.13. *Let $f \in U_n$ be such that $|f|_\varphi \leq 1$ then $|f| \leq |f|_\varphi \leq 1$*

Proof. Note that $|f|_\varphi \leq 1$ implies that for all $u \in \mathbb{Z}^n_{\geq 0}$ we have $\|a_u\|^{\varphi(u)} \leq 1$ and thus we have $\|a_u\| \leq \|a_u\|^{\varphi(u)} \leq 1$, so that:

$$|f| = \max_{u \in \mathbb{Z}^n_{\geq 0}} \|a_u\| \leq \max_{u \in \mathbb{Z}^n_{\geq 0}} \|a_u\|^{\varphi(u)} = |f|_\varphi \leq 1$$

Note also that, for $\|a_u\| \leq 1$ we have $\|a_u\|^{\varphi(u)} \leq 1$ i.e. the same argument also holds for the following statement:

$$\text{For all } f \in U_n \text{ with } |f| \leq 1 \text{ we have } |f| \leq |f|_\varphi \leq 1$$

□

Now, the completeness of the algebra U_n:

Theorem 1.14. *The space U_n equipped with the norm $|\,.\,|_\varphi$ is complete.*

Proof. Let $(f_i)_{i\in\mathbb{N}} = \left(\sum_{u \in \mathbb{Z}^n_{\geq 0}} a_{i,u} X^u\right)_{i\in\mathbb{N}}$ be a Cauchy sequence in U_n. Then, by definition,

$$|f_{i+1} - f_i|_\varphi \to 0 \text{ as } i \to \infty.$$

For fixed $0 < \varepsilon < 1$ any positive real number and for fixed $u \in \mathbb{Z}^n_{\geq 0}$, using Remark 1.13 for all i large enough we get:

$$\begin{aligned} \varepsilon > |f_{i+1} - f_i|_\varphi \geq |f_{i+1} - f_i| & \\ &= \max_{u \in \mathbb{Z}^n_{\geq 0}} \left\{ \|a_{i+1,u} - a_{i,u}\| \right\} \\ &\geq \|a_{i+1,u} - a_{i,u}\| \end{aligned}$$

so that the sequence $\left(a_{i,u}\right)_{i\in\mathbb{N}}$ is also a Cauchy sequence in k. Since k is complete it converges for each $u \in \mathbb{Z}^n_{\geq 0}$, say to a_u.

Now, set $f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u$. We must prove that $f \in U_n$ and $\lim_{i\to\infty} f_i = f$, i.e. we must prove that:

$$\|a_u\|^{\varphi(u)} \to 0 \text{ as } |u| \to \infty$$

and

$$|f - f_i|_\varphi \to 0 \text{ as } i \to \infty$$

Since $(f_i)_{i\in\mathbb{N}}$ is Cauchy, we may find a subsequence $(f_{i_k})_{k\in\mathbb{N}}$ of $(f_i)_{i\in\mathbb{N}}$ with the property that $|f_{i_k} - f_{i_l}|_\varphi \leq \frac{1}{i_l}$ for all $i_k > i_l$ and for $l = 1, 2, 3, \ldots$. Note that, if a subsequence of a Cauchy sequence converges, then the Cauchy sequence itself converges, therefore by replacing $(f_i)_{i\in\mathbb{N}}$ by $(f_{i_k})_{k\in\mathbb{N}}$, we may without loss of generality assume that

$$|f_j - f_i|_\varphi < \frac{1}{i} \text{ for all } j > i \text{ and for } i = 1, 2, 3, \ldots$$

Then for fixed $u \in \mathbb{Z}^n_{\geq 0}$:

$$\begin{aligned}
\|a_{j,u} - a_{i,u}\|^{\varphi(u)} &\leq \max_{v\in\mathbb{Z}^n_{\geq 0}} \|a_{j,v} - a_{i,v}\|^{\varphi(v)} \\
&= |f_j - f_i|_\varphi \\
&\leq \frac{1}{i}
\end{aligned}$$

Note that, for fixed $u \in \mathbb{Z}^n_{\geq 0}$, the function $\phi_u : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $\phi_u(x) = x^{\varphi(u)}$ is continuous and since the non-Archimedean norm $\| \,.\, \|$ on k is also continuous, their composition $(\phi_u \circ \| \,.\, \|) : k \to \mathbb{R}_{\geq 0}$ is also continuous. Therefore we have:

$$\begin{aligned}
\lim_{j\to\infty} \|a_{j,u} - a_{i,u}\|^{\varphi(u)} &= \| \lim_{j\to\infty} (a_{j,u} - a_{i,u})\|^{\varphi(u)} \\
&= \|a_u - a_{i,u}\|^{\varphi(u)} \\
&\leq \frac{1}{i}
\end{aligned}$$

Since $f_i \in U_n$ for each $i = 1, 2, 3, \ldots$ for $|u|$ large enough:

$$\|a_{i,u}\|^{\varphi(u)} \leq \frac{1}{i}$$

Hence we conclude:

$$\begin{aligned}
\|a_u\|^{\varphi(u)} &= \|a_u - a_{i,u} + a_{i,u}\|^{\varphi(u)} \\
&\leq \Big(\max\left\{\|a_u - a_{i,u}\|, \|a_{i,u}\|\right\}\Big)^{\varphi(u)} \\
&= \max\left\{\|a_u - a_{i,u}\|^{\varphi(u)}, \|a_{i,u}\|^{\varphi(u)}\right\} \leq \frac{1}{i}
\end{aligned}$$

Therefore $\|a_u\|^{\varphi(u)} \to 0$ as $|u| \to \infty$ which implies $f \in U_n$.

Furthermore:

$$|f - f_i|_\varphi = \max_{u\in\mathbb{Z}^n_{\geq 0}} \|a_u - a_{i,u}\|^{\varphi(u)} \leq \frac{1}{i}$$

so that

$$|f - f_i|_\varphi \to 0 \text{ as } i \to \infty$$

□

We continue with further results:

Proposition 1.15. *The following are some basic results on the algebraic structure of U_n.*

1. *Let $f, g \in U_n$ be such that $|f|_\varphi \geq 1$ and $|g|_\varphi > 1$ then $|fg|_\varphi > 1$.*

2. *For all $f, g \in U_n$ with $|f|_\varphi, |g|_\varphi \leq 1$ we have $|fg|_\varphi \leq 1$, so that the unit disk $U_n^\circ := \{f \in U_n : |f|_\varphi \leq 1\}$ of U_n is closed under multiplication.*

3. *Let $f, g \in U_n$ be such that $|f|_\varphi = 1$ and $|g|_\varphi = 1$ then $|fg|_\varphi = 1$.*

4. *For all $f = \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u \in U_n$, we have: $|f| = 1$ if and only if $|f|_\varphi = 1$.*

5. *The φ-norm on U_n is not a k-space norm i.e. there exist $f \in U_n$ and $a \in k$ such that $|af|_\varphi \neq \|a\||f|_\varphi$. In fact, we have a slightly different kind of module norm. For all $f \in U_n$ we have:*

$$\begin{cases} \|a\||f|_\varphi \leq |af|_\varphi \leq |f|_\varphi & \text{if } a \in k^\circ \\ |af|_\varphi \leq \|a\||f|_\varphi & \text{if } a \in k \setminus k^\circ \end{cases}$$

   In other words:

$$|af|_\varphi \leq \max\{1, \|a\|\}.|f|_\varphi$$

6. *For every $f \in U_n$ with $|f|_\varphi < 1$ we have $\lim_{n \to \infty} |f^n|_\varphi = 0$*

7. *Let $f = \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u \in U_{n,\varphi}$ where the filter function φ satisfies the property that $\varphi(u) \leq \frac{1}{\log(|u|)}$ for all $u \in \mathbb{Z}^n_{\geq 0}$. Then for each $k \in \mathbb{N}$, $\|a_u\| < \frac{1}{|u|^k}$ for all $|u|$ large enough, which implies that the coefficients of an element of U_n must tend to zero faster than the polynomial rate.*

Proof.

1. Set $f = \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u$ and $g = \sum_{u \in \mathbb{Z}^n_{\geq 0}} b_u X^u$. Then there exist a_u such that $\|a_u\|^{\varphi(u)} \geq 1$ which implies $\|a_u\| \geq 1$ and b_v such that $\|b_v\|^{\varphi(v)} > 1$ which also implies $\|b_v\| > 1$.
   Recall that, the *lexicographical ordering* on $\mathbb{Z}^n$ is defined by;

$$(a_1, a_2, \ldots, a_n) \leq (b_1, b_2, \ldots, b_n)$$

   if and only if there exists m with $1 \leq m \leq n$ such that $a_m < b_m$ and for all $k < m$, $a_k = b_k$.
   Let u_0 be an index which is minimal with respect to the lexicographical ordering such that $\|a_{u_0}\|$ is maximal in $\{\|a_u\| : u \in \mathbb{Z}^n_{\geq 0}\}$ and similarly let v_0 be an index which is minimal with respect to the lexicographical ordering such that $\|b_{v_0}\|$ is maximal in $\{\|b_u\| : u \in \mathbb{Z}^n_{\geq 0}\}$. Then by the maximality of $\|a_{u_0} b_{v_0}\|$, we have:

$$\Big\| \sum_{u+v=u_0+v_0} a_u b_v \Big\| = \max_{u+v=u_0+v_0} \|a_u b_v\| = \|a_{u_0} b_{v_0}\| > 1$$

Therefore:

$$\begin{aligned}|fg|_\varphi &= \max_{\delta\in\mathbb{Z}^n_{\geq 0}} \Big\| \sum_{u+v=\delta} a_u b_v \Big\|^{\varphi(\delta)} \\ &\geq \Big\| \sum_{u+v=u_0+v_0} a_u b_v \Big\|^{\varphi(u_0+v_0)} \\ &= \|a_{u_0} b_{v_0}\|^{\varphi(u_0+v_0)} > 1\end{aligned}$$

2. Let $f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u$ and $g = \sum_{u\in\mathbb{Z}^n_{\geq 0}} b_u X^u$ be any two elements of U_n such that $|f|_\varphi \leq 1$ and $|g|_\varphi \leq 1$. Then we have:

$$\max_{u\in\mathbb{Z}^n_{\geq 0}} \left\{\|a_u\|^{\varphi(u)}\right\} \leq 1 \quad \text{and} \quad \max_{u\in\mathbb{Z}^n_{\geq 0}} \left\{\|b_u\|^{\varphi(u)}\right\} \leq 1$$

which imply that

$$\max_{u\in\mathbb{Z}^n_{\geq 0}} \{\|a_u\|\} \leq 1 \quad \text{and} \quad \max_{u\in\mathbb{Z}^n_{\geq 0}} \{\|b_u\|\} \leq 1$$

So we also have:

$$\begin{aligned}|fg|_\varphi = \max_{\delta\in\mathbb{Z}^n_{\geq 0}} \left\{ \Big\| \sum_{\delta=u+v} a_u b_v \Big\|^{\varphi(\delta)} \right\} &\leq \max_{\delta=u+v} \left\{ \|a_u\|^{\varphi(\delta)} \|b_v\|^{\varphi(\delta)} \right\} \\ &\leq \max_{\delta=u+v} \left\{ 1^{\varphi(\delta)} 1^{\varphi(\delta)} \right\} \leq 1\end{aligned}$$

The same argument and similar calculations also prove the following statements:

- For all $f, g \in U_n$ with $|f|_\varphi, |g|_\varphi < 1$ we have $|fg|_\varphi < 1$.
- For all $f, g \in U_n$ with $|f|_\varphi \leq 1$ and $|g|_\varphi < 1$ we have $|fg|_\varphi < 1$.

3. Let $f := \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u$ and $g := \sum_{u\in\mathbb{Z}^n_{\geq 0}} b_u X^u$ be such that $|f|_\varphi = 1$ and $|g|_\varphi = 1$. Then, there exists $u_0 \in \mathbb{Z}^n_{\geq 0}$ such that $\|a_{u_0}\| = 1$ and for all $u \in \mathbb{Z}^n_{\geq 0}$ we have $\|a_u\| \leq 1$ and similarly there exists $v_0 \in \mathbb{Z}^n_{\geq 0}$ such that $\|b_{v_0}\| = 1$ and for all $u \in \mathbb{Z}^n_{\geq 0}$ we have $\|b_u\| \leq 1$.

$$fg = \Big(\sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u \Big)\Big(\sum_{u\in\mathbb{Z}^n_{\geq 0}} b_u X^u \Big) = \sum_{\delta\in\mathbb{Z}^n_{\geq 0}} \Big(\sum_{u+v=\delta} a_u b_v \Big) X^\delta$$

Since the term $a_{u_0} b_{v_0} X^{u_0+v_0}$ is one of the terms of the product fg, we have

$$|fg|_\varphi \geq \|a_{u_0} b_{v_0}\|^{\varphi(u_0+v_0)} = 1$$

Now, we only need to show that for all $\delta \in \mathbb{Z}^n_{\geq 0}$ we have $\big\| \sum_{u+v=\delta} a_u b_v \big\| \leq 1$.

$$\Big\| \sum_{u+v=\delta} a_u b_v \Big\| \leq \max_{u+v=\delta} \|a_u b_v\| \leq \max_{u\in\mathbb{Z}^n_{\geq 0}} \|a_u\| \max_{u\in\mathbb{Z}^n_{\geq 0}} \|b_u\| \leq 1$$

Hence we deduce that $|fg|_\varphi = 1$.

As a corollary we can also deduce that for all $f, g \in U_n$ with $|f|_\varphi \leq 1$ and $|g|_\varphi \leq 1$ we have, $|fg|_\varphi \leq 1$.

4. Suppose $|f| = \max_{u\in\mathbb{Z}^n_{\geq 0}}\left\{\|a_u\|\right\} = 1$, then there exists $u_0 \in \mathbb{Z}^n_{\geq 0}$ such that $\|a_{u_0}\| = 1$ and for all $u \in \mathbb{Z}^n_{\geq 0}$ we have $\|a_u\| \leq 1$. It implies that for all $u \in \mathbb{Z}^n_{\geq 0}$ we have $\|a_u\|^{\varphi(u)} \leq 1$ and $\|a_{u_0}\|^{\varphi(u_0)} = 1$ and thus $|f|_\varphi = 1$.
   Conversely, if $|f|_\varphi = 1$ then there exists $u_0 \in \mathbb{Z}^n_{\geq 0}$ such that $\|a_{u_0}\|^{\varphi(u_0)} = 1$ which implies $\|a_{u_0}\| = 1$ and for all $u \in \mathbb{Z}^n_{\geq 0}$ we have $\|a_u\|^{\varphi(u)} \leq 1$ which implies $\|a_u\| \leq 1$. Hence we have $|f| = \max_{u\in\mathbb{Z}^n_{\geq 0}}\left\{\|a_u\|\right\} = 1$.
5. Note that $U_1 \subset U_n$ for each $n \geq 1$, so it is enough to give a counter example for $n = 1$. Take $k = \mathbb{Q}_p$. Set $f = 1+X^m$ for some $m \in \mathbb{N}\setminus\{1\}$ and $a = p$. Then $|f|_\varphi = 1$ and

$$|af|_\varphi = |p + pX^m|_\varphi = \left(\frac{1}{p}\right)^{\varphi(m)} \neq \|p\||f|_\varphi = \frac{1}{p}$$

   Now, for the general case set $f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u$. Let $a \in k$ be arbitrary. There are two cases:
   (i) If $a \in k^\circ$ then $\|a\| \leq 1$ thus $1 \geq \|a_u\|^{\varphi(u)} \geq \|a_u\|$ for all $u \in \mathbb{Z}^n_{\geq 0}$ and we have:

$$\begin{aligned}\|a\|.|f|_\varphi &= \|a\| \max_{u\in\mathbb{Z}^n_{\geq 0}} \|a_u\|^{\varphi(u)} \\ &\leq \max_{u\in\mathbb{Z}^n_{\geq 0}} \|a\|^{\varphi(u)}\|a_u\|^{\varphi(u)} = |af|_\varphi = \max_{u\in\mathbb{Z}^n_{\geq 0}} \|aa_u\|^{\varphi(u)} \\ &\leq \max_{u\in\mathbb{Z}^n_{\geq 0}} \|a_u\|^{\varphi(u)} = |f|_\varphi\end{aligned}$$

   (ii) If $a \in k \setminus k^\circ$ then $\|a\| > 1$ thus $\|a_u\| \geq \|a_u\|^{\varphi(u)} > 1$ for all $u \in \mathbb{Z}^n_{\geq 0}$ and we have:

$$\begin{aligned}|af|_\varphi &\leq \max_{u\in\mathbb{Z}^n_{\geq 0}} \|aa_u\|^{\varphi(u)} \\ &= \max_{u\in\mathbb{Z}^n_{\geq 0}} \|a\|^{\varphi(u)}\|a_u\|^{\varphi(u)} \leq \max_{u\in\mathbb{Z}^n_{\geq 0}} \|a\|\|a_u\|^{\varphi(u)} \\ &= \|a\| \max_{u\in\mathbb{Z}^n_{\geq 0}} |a_u\|^{\varphi(u)} = \|a\|.|f|_\varphi\end{aligned}$$

6. Let $f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u$ be any power series in U_n with $|f|_\varphi < 1$, which implies that for all $u \in \mathbb{Z}^n_{\geq 0}$ we have $\|a_u\| < 1$.
   Let $\varepsilon > 0$ be any positive real number. We need to show that $|f^n|_\varphi < \varepsilon$ for all n large enough. Note that,

$$f^n = \sum_{u\in\mathbb{Z}^n_{\geq 0}} \left(\sum_{u_1+\ldots+u_n=u} a_{u_1}a_{u_2}\ldots a_{u_n}\right) X^u$$

   Define $b_u := \sum_{u_1+\ldots+u_n=u} a_{u_1}a_{u_2}\ldots a_{u_n}$, so we need to show that for all $u \in \mathbb{Z}^n_{\geq 0}$ we have $\|b_u\|^{\varphi(u)} < \varepsilon$.
   For any $u \in \mathbb{Z}^n_{\geq 0}$, let a_{u_0} be such that $\|a_{u_0}\| = \max_{|v|<|u|}\left\{\|a_v\|\right\}$. We know that $\|a_u\| < 1$ for any $u \in \mathbb{Z}^n_{\geq 0}$ so that $\|a_{u_0}\|^{\varphi(u)} < 1$, thus there

exists $M \in \mathbb{N}$ such that for all $n > M$ we have $\left(\|a_{u_0}\|^{\varphi(u)}\right)^n < \varepsilon$ i.e. $\|a_{u_0}\| < \varepsilon^{\frac{1}{n\varphi(u)}}$.

Then, for all $n > M$ we have:

$$\begin{aligned}
\|b_u\|^{\varphi(u)} &= \Big\| \sum_{u_1+\ldots+u_n=u} a_{u_1}a_{u_2}\ldots a_{u_n} \Big\|^{\varphi(u)} \\
&\leq \left(\max_{u_1+\ldots+u_n=u} \left\{ \|a_{u_1}a_{u_2}\ldots a_{u_n}\| \right\} \right)^{\varphi(u)} \\
&\leq \left(\max_{u_1+\ldots+u_n=u} \left\{ \|a_{u_0}\|\|a_{u_0}\|\ldots\|a_{u_0}\| \right\} \right)^{\varphi(u)} \\
&= \|a_{u_0}\|^{n\varphi(u)} \\
&< \left(\varepsilon^{\frac{1}{n\varphi(u)}}\right)^{n\varphi(u)} = \varepsilon
\end{aligned}$$

Hence, we deduce that $\lim_{n\to\infty} |f^n|_\varphi = 0$.

7. Since f is an element of U_n, we have:

$$\lim_{|u|\to\infty} \|a_u\|^{\varphi(u)} = 0$$

i.e. for all $\varepsilon \in \mathbb{R}_{>0}$, by definition of convergence, there exists $v \in \mathbb{Z}^n_{\geq 0}$ such that for all $|u| > |v|$ we have $\|a_u\|^{\varphi(u)} < \varepsilon$.

Now, take $\varepsilon = \frac{1}{e^k}$. Note that $\frac{1}{\varphi(u)} \geq \log|u|$. Then, there exists $v \in \mathbb{Z}^n_{\geq 0}$ such that for all $|u| > |v|$ we have $\|a_u\|^{\varphi(u)} < \frac{1}{e^k}$ which implies that:

$$\|a_u\| < \left(\frac{1}{e^k}\right)^{\frac{1}{\varphi(u)}} \leq \left(\frac{1}{e^k}\right)^{\log(|u|)} = \left(e^{\log(|u|)}\right)^{-k} = \frac{1}{|u|^k}$$

□

As T_n, the algebra U_n is similar to the polynomial ring in many aspects. Next, we will define a reduction map from U_n°, the unit disk of U_n, into the polynomial ring (over the residue field $\widetilde{k}$) with n-variables. This map will be very useful for helping us to go between the power series algebra and the polynomial algebra and simplifying the problems at hand in many cases.

Definition 1.16. Define

$$U_n^\circ := \{f \in U_n : |f|_\varphi \leq 1\}$$

and

$$U_n^{\circ\circ} := \{f \in U_n : |f|_\varphi < 1\}$$

Set

$$\widetilde{U}_n = U_n^\circ / U_n^{\circ\circ}.$$

Lemma 1.17. *U_n° is a subring of U_n and $U_n^{\circ\circ}$ is an ideal of U_n°, so that the quotient $U_n^\circ/U_n^{\circ\circ}$ becomes an algebra isomorphic to $\widetilde{k}[X]$ where $\widetilde{k}$ is the residue field $k^\circ/k^{\circ\circ}$ and as usual X denotes the n-tuple of indeterminates $(X_1, X_2, \ldots, X_n)$.*

Proof. By Proposition 1.15 entry 2, we see that U_n° is a subring of U_n and $U_n^{\circ\circ}$ is closed under multiplication with U_n° so that it becomes an ideal.

Now, by extending the canonical epimorphism $\widetilde{\ } : k^\circ \to \widetilde{k}$ we have a map, which will also be denoted by $\widetilde{\ }$, from U_n° into $\widetilde{k}[X]$ given by:

$$\Big(\sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u\Big)^{\sim} := \sum_{u\in\mathbb{Z}^n_{\geq 0}} \widetilde{a_u} X^u \in \widetilde{k}[X]$$

Note that, since $\lim_{|u|\to\infty} \|a_u\| = 0$ almost all coefficients of the power series $\sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u$ vanish under the map $\widetilde{\ }$ so that the image $\sum_{u\in\mathbb{Z}^n_{\geq 0}} \widetilde{a_u} X^u$ is indeed an element of $\widetilde{k}[X]$.

The kernel of this map are the power series in U_n° with norm strictly smaller than 1, which implies the fact that $\ker \widetilde{\ } = U_n^{\circ\circ}$. Hence, we have an isomorphism between $U_n^\circ/U_n^{\circ\circ}$ and $\widetilde{k}[X]$. ◻

Remark 1.18. *The map $\widetilde{\ } : U_n^\circ \to \widetilde{k}[X]$ is a surjective ring homomorphism and this map plays an important role simplifying the problems on the power series ring to the problems in the polynomial ring (over the residue field) of n-variables.*

Recall the real number $\gamma \in \mathbb{R}_{>0}$, given in Definition 1.1, is a positive number such that for all $u \in \mathbb{Z}^n_{\geq 0}$ we have

$$\frac{\varphi(2u)}{\varphi(u)} \geq \gamma.$$

It turns out that when we change the norm, even just a bit, we lose many good properties such as the valuation property of the Gauss norm on T_n. To develop a reasonable theory we need a version of submultiplicativity condition on U_n. As we have already seen in Proposition 1.15, entry 5, the φ-norm on U_n does not admit a submultiplicativity condition, but nevertheless if we weaken the condition we get a close notion, which will be called *"weak-submultiplicativity"*. At this point we give it as a definition:

Definition 1.19. The φ-norm is called γ-*submultiplicative* on the subset $X \subseteq U_n$ if it satisfies the condition:

$$|fg|_\varphi \leq |f|_\varphi^\gamma |g|_\varphi^\gamma$$

for all $f, g \in X \subseteq U_n$.

Now, using the positive real number γ, we prove an important result which will be crucial in the future:

Proposition 1.20. *The φ-norm is γ-submultiplicative on U_n°.*

Proof. Put $f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u$ and $g = \sum_{u\in\mathbb{Z}^n_{\geq 0}} b_u X^u$, so that

$$fg = \sum_{\delta\in\mathbb{Z}^n_{\geq 0}} \Big(\sum_{\delta=u+v} a_u b_v\Big)$$

where $\|a_u\| \leq 1$ and $\|b_u\| \leq 1$ for all $u \in \mathbb{Z}^n_{\geq 0}$. Then:

$$|fg|_\varphi = \max_{\delta\in\mathbb{Z}^n_{\geq 0}} \Big\{ \Big\| \sum_{\delta=u+v} a_u b_v \Big\|^{\varphi(\delta)} \Big\}$$

$$
\begin{aligned}
&\le \max_{\delta\in\mathbb{Z}^n_{\ge 0}} \left\{ \left(\max_{\delta=u+v} \{\|a_u b_v\|\} \right)^{\varphi(u+v)} \right\} \\
&= \max_{\delta\in\mathbb{Z}^n_{\ge 0}} \left\{ \max_{\delta=u+v} \left\{ \|a_u\|^{\varphi(u+v)} \|b_v\|^{\varphi(u+v)} \right\} \right\} \\
&\le \max_{\delta\in\mathbb{Z}^n_{\ge 0}} \left\{ \max_{\delta=u+v} \left\{ \|a_u\|^{\varphi(u+v)} \right\} . \max_{\delta=u+v} \left\{ \|b_v\|^{\varphi(u+v)} \right\} \right\} \\
&\le \max_{\delta\in\mathbb{Z}^n_{\ge 0}} \left\{ \max_{|u|\le|\delta|} \left\{ \|a_u\|^{\varphi(u+u)} \right\} . \max_{|v|\le|\delta|} \left\{ \|b_v\|^{\varphi(v+v)} \right\} \right\} \\
&\le \max_{\delta\in\mathbb{Z}^n_{\ge 0}} \left\{ \max_{|u|\le|\delta|} \left\{ \|a_u\|^{\gamma\varphi(u)} \right\} . \max_{|v|\le|\delta|} \left\{ \|b_v\|^{\gamma\varphi(v)} \right\} \right\} \\
&= \max_{\delta\in\mathbb{Z}^n_{\ge 0}} \left\{ \left(\max_{|u|\le|\delta|} \{\|a_u\|^{\varphi(u)}\} \right)^{\gamma} \left(\max_{|v|\le|\delta|} \{\|b_v\|^{\varphi(v)}\} \right)^{\gamma} \right\} \\
&\le \max_{\delta\in\mathbb{Z}^n_{\ge 0}} \left\{ \left(\max_{u\in\mathbb{Z}^n_{\ge 0}} \{\|a_u\|^{\varphi(u)}\} \right)^{\gamma} \left(\max_{v\in\mathbb{Z}^n_{\ge 0}} \{\|b_v\|^{\varphi(v)}\} \right)^{\gamma} \right\} \\
&= \max_{\delta\in\mathbb{Z}^n_{\ge 0}} \left\{ (|f|_\varphi)^{\gamma} (|g|_\varphi)^{\gamma} \right\} \\
&= |f|^{\gamma}_{\varphi} |g|^{\gamma}_{\varphi}
\end{aligned}
$$

□

Now, we go on with an intermediate result:

Proposition 1.21. *The following maps are continuous:*

(i) The identity map $(U_n, |\,.\,|_\varphi) \to (T_n, |\,.\,|)$, and in particular the identity map $(U_n^\circ, |\,.\,|_\varphi) \to (T_n^\circ, |\,.\,|)$,

(ii) If $U_n \subseteq W_n$ then the identity map $(U_n, |\,.\,|_\varphi) \to (W_n, |\,.\,|)$ where the norm $|\,.\,|$ on W_n is the Gauss norm induced from T_n.

Proof. These results are consequences of the fact given in Remark 1.13: For all $|f|_\varphi \le 1$ we have $|f| \le |f|_\varphi \le 1$.

Let $0 < \varepsilon < 1$ be any positive real number. We need to find $\delta \in \mathbb{R}_{>0}$ such that the inequality $|f - g|_\varphi < \delta$ implies that $|f - g| < \varepsilon$.

In both cases, taking $\delta = \varepsilon$ is enough to show the result, i.e.

$$|f - g| \le |f - g|_\varphi < \delta = \varepsilon$$

□

Next, we will classify invertible elements of U_n, this classification plays an important role in the theory and the classification is very similar to the classification of invertible elements of T_n.

Lemma 1.22. *Let $f = \sum_{u\in\mathbb{Z}_{\ge 0}} a_u X^u \in U_n$ be a nonzero power series. A constant multiple of f has norm 1.*

Proof. Since $f \in U_n \subseteq T_n$ we have that the norms of the coefficients of f converge to 0 which implies that the maximum of the norms of the coefficients exist. Let a_v be a coefficient of f with maximum norm. Note that f is non-zero, so $a_v \neq 0$. Then the element:

$$a_v{}^{-1} f = a_v{}^{-1} \sum_{u \in \mathbb{Z}_{\geq 0}} a_u X^u = \sum_{u \in \mathbb{Z}_{\geq 0}} \frac{a_u}{a_v} X^u$$

has φ-norm 1.

Note that for all $a, b \in k$ with $b \neq 0$, we have: $\left\| \frac{a}{b} \right\| = \frac{\|a\|}{\|b\|}$.

So by the maximality of $\|a_v\|$ we deduce that:

$$\left\| \frac{a_u}{a_v} \right\| = \frac{\|a_u\|}{\|a_v\|} \leq 1$$

for all $u \in \mathbb{Z}^n_{\geq 0}$ and obviously $\left\| \frac{a_v}{a_v} \right\| = 1$ which becomes a coefficient with maximum norm. Hence

$$|a_v{}^{-1} f|_\varphi = 1.$$

□

Remark 1.23. *Let $f_i = \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_{i,u} X^u$ be any elements of U_n for $i = 1, 2, \ldots, k$. Put*

$$c := \max_{\substack{u \in \mathbb{Z}^n_{\geq 0} \\ i=1,2,\ldots,k}} \{\|a_{i,u}\|\}.$$

Since $\left\| \frac{a_{i,u}}{c} \right\| \leq 1$, for each $i = 1, 2, \ldots, k$,

$$c^{-1} f_i = \sum_{u \in \mathbb{Z}^n_{\geq 0}} \frac{a_{i,u}}{c} X^u$$

has norm at most 1 (and at least one of them has norm equal to 1).

As mentioned in the Remark 1.18, using the surjective ring homomorphism

$$\sim : U_n^\circ \to \widetilde{k}[X]$$

given by

$$f = \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u \mapsto \widetilde{f} = \sum_{u \in \mathbb{Z}^n_{\geq 0}} \widetilde{a}_u X^u$$

we reduce from power series to polynomials. So, we are able to simplify some problems at hand, as the following results:

Proposition 1.24. *Let $f = \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u \in U_n$ with $|f|_\varphi = 1$. Then $\widetilde{f}$ is a constant polynomial if and only if $\|f(\bar{0})\| = 1$ and $|f - f(\bar{0})|_\varphi < 1$ where $\bar{0} = (0, 0, \ldots, 0) \in \mathbb{Z}^n_{\geq 0}$.*

Proof. If $\widetilde{f} = \sum_{u \in \mathbb{Z}^n_{\geq 0}} \widetilde{a}_u X^u \in \widetilde{k}[X_1, \ldots, X_n]$ is constant then $\widetilde{f} = \widetilde{a}_{\bar{0}} = \widetilde{f(\bar{0})}$ and all the other coefficients of f vanish in the quotient $\widetilde{k}$, i.e. $\|a_u\| < 1$ for all $u \neq \bar{0}$ and since $|f|_\varphi = 1$ we have $\|f(\bar{0})\| = 1$.

Conversely, if $|f - f(\bar{0})|_\varphi < 1$ then $\|a_u\| < 1$ for all $u \neq \bar{0}$ which means $\widetilde{a}_u = 0 \in \widetilde{k}$, hence $\widetilde{f} = \widetilde{a}_{\bar{0}}$ which is a non-zero constant polynomial in $\widetilde{k}[X_1, \ldots, X_n]$.

□

We can actually characterize all the units in U_n° by the following proposition:

Proposition 1.25. *For any $f = \sum_{u \in \mathbb{Z}_{\geq 0}^n} a_u X^u \in U_n$ with $|f|_\varphi \leq 1$, f is a unit in U_n° if and only if $\widetilde{f}$ is a non-zero constant polynomial in $\widetilde{k}[X_1, \ldots, X_n]$.*

Proof. If $f \in U_n^\times$ then there exists $g \in U_n^\times$ such that $fg = 1$. Since the map $U_n^\circ \to \widetilde{k}[X_1, \ldots, X_n]$ is a ring homomorphism we have $\widetilde{f}\widetilde{g} = 1$ which means $\widetilde{f} \in (\widetilde{k}[X_1, \ldots, X_n])^\times = (\widetilde{k})^\times$ i.e. $\widetilde{f}$ is a constant polynomial.

Conversely, suppose $\widetilde{f}$ is a non-zero constant polynomial in $\widetilde{k}[X_1, \ldots, X_n]$ with $|f|_\varphi \leq 1$. By Proposition 1.24, we know that then $\|f(\bar{0})\| = 1$ and $|f - f(\bar{0})|_\varphi < 1$.

We have:

$$f = f(\bar{0}) + \sum_{u \neq \bar{0}} a_u X^u$$

Then by multiplying both sides with $f(\bar{0})^{-1}$, we get:

$$\begin{aligned} f \,.\, f(\bar{0})^{-1} &= \Big(f(\bar{0}) + \sum_{u \neq \bar{0}} a_u X^u\Big) f(\bar{0})^{-1} \\ &= 1 + \sum_{u \neq \bar{0}} a_u f(\bar{0})^{-1} X^u \end{aligned}$$

Define

$$g := \sum_{u \neq \bar{0}} a_u f(\bar{0})^{-1} X^u$$

For any $u \neq \bar{0}$, we have:

$$\|a_u f(\bar{0})^{-1}\| = \|a_u\| \|f(\bar{0})^{-1}\| = \frac{\|a_u\|}{\|f(\bar{0})\|} = \|a_u\|$$

because $\|f(\bar{0})\| = 1$. Note that by the above characterization of f, $\|a_u\| < 1$ for all $u \neq \bar{0}$ i.e. $|g|_\varphi < 1$.

Hence we have

$$f \,.\, f(\bar{0})^{-1} = 1 + g$$

where $g \in U_n$ with $|g|_\varphi < 1$. Now consider the series $\sum_{i=0}^\infty (-1)^i g^i$. This is the formal inverse of the power series $1 + g$. We claim that $\sum_{i=0}^\infty (-1)^i g^i \in U_n$. To show it, we will show that this sum is convergent, i.e. the sequence of the partial sums of this series is convergent. So, it is enough to show that the sequence of partial sums is a Cauchy sequence in U_n, then by the completeness of U_n, we will deduce that $1 + g$ in invertible in U_n, and then the desired result will follow.

Let $\varepsilon > 0$ be any positive real number.

Define $g_k = \sum_{i=0}^k (-1)^i g^i \in U_n$ for $k = 1, 2, \ldots$ Then:

$$\begin{aligned} |g_{k-1} - g_k|_\varphi &= \Big|\sum_{i=0}^{k-1} (-1)^i g^i - \sum_{i=0}^{k} (-1)^i g^i\Big|_\varphi \\ &= \big|-(-1)^k g^k\big|_\varphi = |g^k|_\varphi \end{aligned}$$

Note that since $|g|_\varphi < 1$ by Proposition 1.15 entry 6 we have $\lim_{k \to \infty} |g^k|_\varphi = 0$. Therefore, we have:

$$\lim_{k \to \infty} |g_{k-1} - g_k|_\varphi \leq \lim_{k \to \infty} |g^k|_\varphi = 0$$

Hence, the sequence of the partial sums of the series $\sum_{i=0}^{\infty}(-1)^i g^i$:

$$(g_k)_{k\in\mathbb{N}}$$

is a Cauchy sequence in U_n, hence by the completeness of U_n, it has a limit, i.e. $\sum_{i=0}^{\infty}(-1)^i g^i$ converges, say

$$\sum_{i=0}^{\infty}(-1)^i g^i = h \in U_n$$

Then we have:

$$\begin{aligned} f\,.\,f(\bar{0})^{-1} &= 1+g \quad (\text{ multiply both sides by } h)\\ \big(f\,.\,f(\bar{0})^{-1}\big)h &= (1+g)h = 1 \end{aligned}$$

which means f invertible in U_n.

□

We can merge the two results above into a theorem and it is followed by an easy corollary:

Theorem 1.26. *Let $f \in U_n^\circ$ be any power series. Then the following statements are equivalent:*

(i) f is a unit in U_n, i.e. $f \in U_n^\times$

(ii) $\widetilde{f}$ is a non-zero constant polynomial in $\widetilde{k}[X_1,\dots,X_n]$

(iii) $\|f(\bar{0})\| = 1$ and $|f - f(\bar{0})|_\varphi < 1$

Corollary 1.27. *Every unit in U_n° has φ-norm 1.*

Proof. Let $f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u \in U_n$ be a unit. Then by Theorem 1.26, we have $|f - f(0)|_\varphi < 1$ which means $\|a_u\|^{\varphi(u)} < 1$ for all $u \neq \bar{0}$ and $\|f(\bar{0})\| = 1$ which means $\|a_{\bar{0}}\|^{\varphi(\bar{0})} = 1$. Therefore $|f|_\varphi = 1$. □

Corollary 1.28. *$f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u$ is a unit in U_n if and only if $\|a_u\| < \|a_{\bar{0}}\|$ for all $u \neq \bar{0}$.*

Proof. ($\Longleftarrow$) Suppose $\|a_u\| < \|a_{\bar{0}}\| \neq 0$ for all $u \neq \bar{0}$. Then by multiplying f by $a_{\bar{0}}^{-1}$ we get

$$g := a_{\bar{0}}^{-1} f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_{\bar{0}}^{-1} a_u X^u$$

So, $\|g(\bar{0})\| = \|a_{\bar{0}}^{-1} a_{\bar{0}}\| = 1$ and for all $u \neq \bar{0}$ we have

$$\|a_{\bar{0}}^{-1} a_u\| = \|a_{\bar{0}}^{-1}\|\|a_u\| = \frac{\|a_u\|}{\|a_{\bar{0}}^{-1}\|} < 1$$

so that $g \in U_n^\circ$ and by Theorem 1.26 $g = a_{\bar{0}}^{-1} f$ is invertible in U_n°, hence invertible in U_n which implies $f = a_{\bar{0}} g$ is also invertible in U_n.

($\Longrightarrow$) Conversely, suppose f is a unit in U_n. Then there exists $g \in U_n$ such that $fg = 1$.

We know that there exists c_1 and c_2 in k such that both $c_1 f$ and $c_2 g$ are in U_n°, then we have:

$$(c_1 f)(c_2 g) = c_1 c_2$$

in U_n° which implies that $c_1 f$ is invertible in U_n°. Then by Theorem 1.26,

$$\|c_1 a_{\bar{0}}\| = 1 \text{ and } \|c_1 a_u\| < 1$$

for all $u \neq \bar{0}$, i.e.

$$\|a_{\bar{0}}\| = \|c_1^{-1}\| \text{ and } \|a_u\| < \frac{1}{\|c_1\|} = \|a_{\bar{0}}\|.$$

□

By the proof of Proposition 1.24, above we can deduce the following result:

Remark 1.29. *The set $U_n^\times$ of unit elements in U_n is an open set.*

Proof. Let $f = \sum_{u \in \mathbb{Z}_{\geq 0}^n} a_u X^u \in U_n^\times$ be a unit. Then by Corollary 1.28,

$$\|a_u\| < \|a_{\bar{0}}\| \neq 0$$

for all $u \neq \bar{0}$.

Set $\varepsilon = \min\{\|a_{\bar{0}}\|, 1\}$.

Consider the open ball:

$$B := B(f, \varepsilon) = \left\{g \in U_n : |g - f|_\varphi < \varepsilon\right\}$$

centered at f and of radius ε. We claim that $B \subset U_n^\times$.

Let $g = \sum_{u \in \mathbb{Z}_{\geq 0}^n} b_u X^u \in B$ be an arbitrary element. It is enough to show that g is invertible, i.e. by Corollary 1.28, it is enough to show that $\|b_{\bar{0}}\| \neq 0$ and $\|b_u\| < \|b_{\bar{0}}\|$ for all $u \neq \bar{0}$.

Since $g \in B$, by definition we have $|f - g|_\varphi < \varepsilon$, i.e.

$$|f - g|_\varphi = \Big| \sum_{u \in \mathbb{Z}_{\geq 0}^n} (a_u - b_u) X^u \Big|_\varphi = \max_{u \in \mathbb{Z}_{\geq 0}^n} \left\{ \|a_u - b_u\|^{\varphi(u)} \right\} < \varepsilon$$

Then:

$$\|a_{\bar{0}} - b_{\bar{0}}\| < \varepsilon.$$

Note that, if $\|a_{\bar{0}}\| \neq \|b_{\bar{0}}\|$ then

$$\|a_{\bar{0}} - b_{\bar{0}}\| = \max\left\{\|a_{\bar{0}}\|, \|b_{\bar{0}}\|\right\} < \varepsilon \leq \|a_{\bar{0}}\|$$

hence, $\max\{\|a_{\bar{0}}\|, \|b_{\bar{0}}\|\} = \|b_{\bar{0}}\| < \|a_{\bar{0}}\|$ which is a contradiction. Therefore we must have

$$\|a_{\bar{0}}\| = \|b_{\bar{0}}\|.$$

In particular we have $\|b_{\bar{0}}\| \neq 0$.

Now, we only need to show that $\|b_u\| < \|b_{\bar{0}}\|$ for all $u \neq \bar{0}$. Suppose for a contradiction that $\|b_u\| \geq \|b_{\bar{0}}\| = \|a_{\bar{0}}\|$ for some $u \neq \bar{0}$. Then,

$$\|a_u\| < \|a_{\bar{0}}\| = \|b_{\bar{0}}\| \leq \|b_u\| \text{ i.e. } \|a_u\| \neq \|b_u\|$$

So that,

$$\|a_u - b_u\| = \max\left\{\|a_u\|, \|b_u\|\right\} = \|b_u\| \geq \|a_{\bar{0}}\|$$

Hence

$$|f - g|_\varphi \geq \left(\|a_u - b_u\|\right)^{\varphi(u)} = \|b_u\|^{\varphi(u)} \geq \|a_{\bar{0}}\|^{\varphi(u)} \geq \varepsilon$$

A contradiction. Thus we must have:

$$\|b_u\| < \|b_{\bar{0}}\| \text{ for all } u \neq \bar{0}$$

i.e. g is a unit in U_n. □

With the characterization above, we get the following two technical lemmas:

Lemma 1.30. *For each $f \in U_n$ with $|f|_\varphi = 1$, there is an element $c \in k$ with $\|c\| = 1$ such that $c + f$ is not a unit in U_n.*

Proof. Note that since $|f|_\varphi = 1$, we have $\|f(\bar{0})\| \leq 1$. So, we only need to consider the two cases: $\|f(\bar{0})\| = 1$ and $\|f(\bar{0})\| < 1$:

- If $\|f(\bar{0})\| < 1$, then since $|f|_\varphi = 1$ we must have $|f - f(\bar{0})|_\varphi = 1$. Define $g := 1 + f$, then $|g|_\varphi = 1$ and $|g - g(\bar{0})|_\varphi = 1$. Hence, by Theorem 1.26, g is not a unit U_n and we take $c = 1$.
- If $\|f(\bar{0})\| = 1$, define $g = f - f(\bar{0})$. Then $g(\bar{0}) = 0$ and hence, again by Theorem 1.26, g is not a unit and we take $c = -f(\bar{0})$, which has norm 1.

□

Proposition 1.31. $\displaystyle\bigcap_{\mathfrak{m} \in \mathrm{Sp}(U_n)} \mathfrak{m} = (0)$ *where* $\mathrm{Sp}(U_n)$ *is the set of all maximal ideals of* U_n.

Proof. Suppose there is a non-zero element f contained in all maximal ideals of U_n. We may suppose that $|f| = 1$, otherwise using Lemma 1.22, we may multiply f by a suitable non-zero constant, since we work in an ideal all multiples of f also lie in the ideal. Using the Lemma 1.30, choose $c \in k$ with $\|c\| = 1$ such that $c + f$ is not a unit in U_n. So, since $c + f$ is not a unit, there exists a maximal ideal $\mathfrak{m}$ of U_n that contains $c + f$. By assumption, f is also contained in $\mathfrak{m}$ which implies that $c + f - f = c \in k^\times$ is in $\mathfrak{m}$ and this contradicts the maximality of $\mathfrak{m}$. □

1.3 Weierstrass Theory and Applications

Weierstrass Theory is basically a group of results which can give us further information on U_n. The techniques of Weierstrass Theory will be used later on to apply Rückert's methods and these methods will directly imply useful properties, such as the Noetherian property and the Jacobson property.

In this section we will prove an important result; the so-called Weierstrass Preparation Theorem and we will see some applications of Rückert Theory. Before we prove it, we begin with the definition of being "X_n-distinguished" and later on we will introduce a "division algorithm" on U_n. But before we give the definition we first remark that:

Remark 1.32. *If* $f = \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u \in U_n$ *then we can reformulate the power series expansion of* f *by:*

$$f = \sum_{v=0}^{\infty} f_v X_n^v$$

where $f_v \in k[[X_1, X_2, \ldots, X_{n-1}]]$. *In this case for* $v = 0, 1, 2, \ldots$ *we have* $f_v \in U_{n-1}$.

Note that, since $f \in U_n$ *as* $|u| \to \infty$ *we have* $\|a_u\|^{\varphi(u)} \to 0$.

For each $v = 0, 1, 2, \ldots$*:*

$$f_v = \sum_{u_v \in \mathbb{Z}^{n-1}_{\geq 0}} a_{u_v} X^{u_v}$$

where X *denotes the* $(n-1)$*-tuple of indeterminates* $(X_1, X_2, \ldots, X_{n-1})$ *so that*

$$\|a_{u_v}\|^{\varphi(u_v)} \to 0$$

as $|u_v| \to \infty$.

Now, we give the definition of X_n-distinguished power series.

Definition 1.33. A power series

$$g = \sum_{v=0}^{\infty} g_v(X_1, \ldots, X_{n-1}) X_n^v \in U_n$$

is called X_n-distinguished of degree s if

(i) g_s is a unit in U_{n-1}

(ii) $|g_s|_\varphi = |g|_\varphi$

(iii) $|g_s|_\varphi > |g_v|_\varphi$ for all $v > s$.

Recall that a polynomial is called *unitary* if its highest coefficient is a unit. The next remark is an easy conclusion on the X_n-distinguished power series of norm 1.

Lemma 1.34. *A power series* $g \in U_n$ *with* $|g|_\varphi = 1$ *is* X_n*-distinguished of degree* s *if and only if* $\widetilde{g}$ *is a unitary polynomial of degree* s *in the polynomial ring* $\widetilde{k}[X_1 \ldots, X_{n-1}][X_n]$.

Proof. ($\Longleftarrow$) Suppose $\widetilde{g}$ is a unitary polynomial of degree s in $\widetilde{k}[X_1, \ldots, X_{n-1}][X_n]$. Set

$$\widetilde{g} = \widetilde{g}_0 + \widetilde{g}_1 X_n + \widetilde{g}_2 X_n^2 + \ldots + \widetilde{g}_s X_n^s$$

where $\widetilde{g}_i \in \widetilde{k}[X_1, \ldots, X_{n-1}]$ for $i = 1, 2, \ldots, s$ and $\widetilde{g}_v = 0$ for all $v > s$.

By assumption, $\widetilde{g}_s$ is a unit in $\widetilde{k}[X_1, \ldots, X_{n-1}]$ thus $\widetilde{g}_s \in \widetilde{k}^\times$. Then by Theorem 1.26, g_s is a unit in U_{n-1} so that the first condition in Definition 1.33 is satisfied.

Set $g_s = \sum_{u_s \in \mathbb{Z}^{n-1}_{\geq 0}} g_{u_s} X^{u_s} \in U_{n-1}$. Then since $\widetilde{g}_s \in \widetilde{k}^\times$, Theorem 1.26 implies that $\|g_s(\bar{0})\| = 1$ and $|g_s - g_s(\bar{0})|_\varphi < 1$ which means that $|g_{u_s}|_\varphi < 1$ for all $u_s \neq \bar{0}$. Then:

$$|g_s|_\varphi = \max_{u \in \mathbb{Z}^n_{\geq 0}} \|g_{u_s}\|^{\varphi(u_s)} = \|g(\bar{0})\|^1 = 1.$$

Hence $|g_s|_\varphi = |g|_\varphi = 1$.

For all $v > s$, since $0 = \widetilde{g}_v \in \widetilde{k}[X_1, \ldots, X_{n-1}]$ if we set $g_v = \sum_{u_v \in \mathbb{Z}_{\geq 0}^{n-1}} g_{u_v} X^{u_v}$ then it follows that $\widetilde{g}_{u_v} = 0 \in \widetilde{k}$, i.e. $\|g_{u_v}\| < 1$ for all $u_v \in \mathbb{Z}_{\geq 0}^{n-1}$ hence $|g_v|_\varphi < 1$ for all $v > s$. So that we have

$$1 = |g_s|_\varphi > |g_v|_\varphi \text{ for all } v > s.$$

($\Longrightarrow$) Suppose

$$g = \sum_{v=0}^{\infty} g_v(X_1, \ldots, X_{n-1}) X_n^v$$

is X_n-distinguished of degree s and with norm 1. Then by Definition 1.33, g_s is a unit in U_{n-1}, $|g_s|_\varphi = |g|_\varphi = 1$ and $1 = |g_s|_\varphi > |g_v|_\varphi$ for all $v > s$.

Since $|g_v|_\varphi < 1$, if we set $g_v = \sum_{u_v \in \mathbb{Z}_{\geq 0}^{n-1}} g_{u_v} X^{u_v}$, we see that $\|g_{u_v}\| < 1$ for all $u_v \in \mathbb{Z}_{\geq 0}^{n-1}$, hence $0 = \widetilde{g}_{u_v} \in \widetilde{k}$ for all $u_v \in \mathbb{Z}_{\geq 0}^{n-1}$. Hence, $0 = \widetilde{g} \in \widetilde{k}[X_1, \ldots, X_{n-1}]$. This implies that $\widetilde{g}$ is a polynomial of degree s in $\widetilde{k}[X_1, \ldots, X_{n-1}][X_n]$, say $\widetilde{g} = \widetilde{g}_0 + \widetilde{g}_1 X_n + \ldots + \widetilde{g}_s X_n^s$.

Now, we only need to show that $\widetilde{g}_s$ is a unit in $\widetilde{k}[X_1, \ldots, X_{n-1}]$, i.e. $\widetilde{g}_s \in \widetilde{k}^\times$. Set $g_s = \sum_{u_s \in \mathbb{Z}_{\geq 0}^{n-1}} g_{u_s} X^{u_s}$. We know that g_s is a unit in U_{n-1} then by Theorem 1.26 we have $\|g_s(\bar{0})\| = 1$ and $|g - g_s(\bar{0})|_\varphi < 1$, which means that $\|g_{u_s}\| < 1$ for all $u_s \neq 0$. Then $\widetilde{g}_{u_s} = 0 \in \widetilde{k}$, i.e.

$$\widetilde{g}_s = \sum_{u_s \in \mathbb{Z}_{\geq 0}^{n-1}} \widetilde{g}_{u_s} X^{u_s} = \widetilde{g(\bar{0})} \in \widetilde{k}^\times.$$

□

Now, we give an easy corollary:

Corollary 1.35. *Every unit in U_n° is X_n-distinguished of degree* 0.

Proof. Let $g = \sum_{v=0}^{\infty} g_v X_n^v = \sum_{u \in \mathbb{Z}_{\geq 0}^n} a_u X^u$ be a unit in U_n. Recall that, by Theorem 1.26, $\|a_{\bar{0}}\| = 1$ and $\|a_u\| < 1$ for all $u \neq \bar{0}$. We need to prove the three conditions given in the Definition 1.33:

(i) g_0 is a unit in U_{n-1}:

We see that $g_0 = \sum_{\substack{u=(u_1,u_2,\ldots,u_n) \in \mathbb{Z}_{\geq 0}^n \\ u_n = 0}} a_u X^u$. So that $\|g_0(\bar{0})\| = \|a_{\bar{0}}\| = 1$ and

$$|g_0 - g_0(\bar{0})|_\varphi = \max_{\substack{u=(u_1,u_2,\ldots,u_n) \in \mathbb{Z}_{\geq 0}^n \\ u \neq \bar{0} \text{ and } u_n = 0}} \{\|a_u\|^{\varphi(u)}\} < 1$$

by the assumption, which implies that g_0 is a unit in U_{n-1} again by Theorem 1.26.

(ii) $|g_0|_\varphi = |g|$:

We already proved that g_0 is a unit in U_{n-1}, so by Corollary 1.27, since both g and g_0 are units we have:

$$|g_0|_\varphi = 1 = |g|_\varphi$$

(iii) $|g_0|_\varphi > |g_v|_\varphi$ for all $v > 0$:

Note that since $\|a_u\| < 1$ for all $u \neq \bar{0}$ it follows that $\|g_v\|_\varphi < 1 = |g_0|_\varphi$ for all $v > 0$.

□

Recall the usual "Euclidean Division Algorithm" over a ring R (see [20] Chapter IV for details):

Theorem 1.36. *Let R be a ring. Let $f, g \in R[X]$ be polynomials and suppose g is a unitary polynomial, that is the leading coefficient of g is a unit in R. Then, there exist unique q and r in $R[X]$ such that*

$$f = gq + r$$

where $\deg r = 0$ or $\deg r < \deg g$.

We would like to carry out a "division algorithm" on U_n by a distinguished element g of norm 1. We will basically use the Euclid's Division in $U_{n-1}[X_n]$ and then pull back the results to U_n. Let us first state the so-called Weierstrass Division Theorem which we shortly call WDT:

Theorem 1.37. (WDT: Weierstrass Division Theorem)

Let $g \in U_n$ be X_n-distinguished of degree s. Then, for each $f \in U_n$, there exist uniquely determined elements $q \in U_n$ and $r \in U_{n-1}[X_n]$ with $\deg r < s$ such that

$$f = qg + r$$

If, in addition, f and g are polynomials in $U_{n-1}[X_n]$ and if $\deg g = s$, then q is also a polynomial in $U_{n-1}[X_n]$.

Proof. Note that, if the theorem holds for an X_n-distinguished g of degree s, then it also holds for a scalar multiple of g:

$$f = (c^{-1}q)(cg) + r$$

So, without loss of generality, using Lemma 1.22 we may suppose that $|g|_\varphi = 1$.

Now, we prove the existence of the representation.

Let

$$g = \sum_{v=0}^{\infty} g_v X_n^v$$

be X_n-distinguished of degree s with $g_v \in U_{n-1}$ for $v = 0, 1, 2, \ldots$ and

$$|g_v|_\varphi < |g_s|_\varphi = |g|_\varphi = 1$$

for all $v > s$.

Set

$$g' = \sum_{v=0}^{s} g_v X_n^v \text{ and } g'' = \sum_{v=s+1}^{\infty} g_v X_n^v.$$

Set

$$\varepsilon = |g - g'|_\varphi^\gamma = |g''|_\varphi^\gamma < 1$$

where γ is the positive real number depending on φ-norm, mentioned in Corollary 1.5.

We first prove an intermediate result:

Claim: For any $f \in U_n$ there exist q, f_1 in U_n and $r \in U_{n-1}[X_n]$ with $\deg r < s$ such that $f = qg + r + f_1$ with

$$|f|_\varphi \geq |q|_\varphi, \ |f|_\varphi \geq |r|_\varphi \text{ and } |f_1|_\varphi \leq \varepsilon |f|_\varphi^\gamma$$

Proof. We may approximate f by polynomials in $U_{n-1}[X_n]$. If $f = \sum_{v=0}^{\infty} f_v X_n^v$ then we may write

$$f = \sum_{v=0}^{k} f_v X_n^v + \sum_{v=k+1}^{\infty} f_v X_n^v$$

where

$$f' := \sum_{v=0}^{k} f_v X_n^v \in U_{n-1}[X_n] \text{ and } f'' = \sum_{v=k+1}^{\infty} f_v X_n^v \in U_n$$

such that the norm $|f''|_\varphi$ can be chosen arbitrarily small.

Note that, since g is X_n-distinguished of degree s, by Remark 1.33, we have $g_s \in k^\times$ a unit, so that g' is a unitary polynomial of degree s in $U_{n-1}[X_n]$, so that we can perform the usual Euclidean Division Algorithm (given in Theorem 1.36) in the ring $U_{n-1}[X_n]$ to get:

$$f' = qg' + r$$

which implies:

$$f - f'' = q(g - g'') + r$$

i.e.

$$f = (qg + r) + (-qg'' + f'').$$

By multiplying f by a suitable scalar $c \in k$, we may suppose that

$$\max\{|cq|_\varphi, |cr|_\varphi\} = 1.$$

Note that, multiplying f by a scalar does not change the division algorithm. Now we have:

$$cf = ((cq)g + (cr)) + (-cqg'' + cf'')$$

Set

$$f_1 := -cqg'' + cf''$$

Now we want to show that

$$|cf|_\varphi \geq |cq|_\varphi, \ |cf|_\varphi \geq |cr|_\varphi \text{ and } |f_1|_\varphi \leq \varepsilon |cf|_\varphi^\gamma$$

Since $\max\{|cq|_\varphi, |cr|_\varphi\} = 1$, to show $|cf|_\varphi \geq |cq|_\varphi$ and $|cf|_\varphi \geq |cr|_\varphi$, it is enough to show that $|cf|_\varphi \geq 1$. Assume not, i.e. assume for a contradiction that $|cf|_\varphi < 1$. Note that, since $|g''|_\varphi < 1$, by Proposition 1.20, we have

$$|-cqg''|_\varphi \leq |-cq|_\varphi^\gamma |g''|_\varphi^\gamma < 1$$

and since $|f''|_\varphi$ can be chosen arbitrarily small we also have $|-cqg'' + cf''|_\varphi < 1$. Hence, under the reduction map $\widetilde{\ }$ we have:

$$0 = \widetilde{cf} = \widetilde{cq}\widetilde{g} + \widetilde{cr} + \widetilde{f_1} = \widetilde{cq}\widetilde{g} + \widetilde{cr}$$

in $\widetilde{k}[X_1, X_2, \ldots, X_n]$. Since $\deg \widetilde{g} = s > \deg r \geq \deg \widetilde{r}$ it follows that $\widetilde{cq} = \widetilde{cr} = 0$ and this is a contradiction to the assumption that $\max\{|cq|_\varphi, |cr|_\varphi\} = 1$. Thus, we conclude that

$$|cf|_\varphi \geq |cq|_\varphi \text{ and } |cf|_\varphi \geq |cr|_\varphi.$$

Now, we only need to show that $|f_1|_\varphi \leq \varepsilon |cf|_\varphi^\gamma$:

$$\begin{aligned}|f_1|_\varphi &= |-cqg'' + cf''|_\varphi \\ &\leq \max\big\{|-cqg''|_\varphi, |cf''|_\varphi\big\} = |cqg''|_\varphi \\ &\leq |cq|_\varphi^\gamma |g''|_\varphi^\gamma \\ &= |cq|_\varphi^\gamma \varepsilon \leq \varepsilon |cf|_\varphi^\gamma\end{aligned}$$

■

Subclaim: Put $q = \sum q_u X^u$ and $r = \sum r_u X^u$. If $|cf|_\varphi \geq 1 \geq |cq|_\varphi$ (or $|cf|_\varphi \geq 1 \geq |cr|_\varphi$ respectively) where $\left\|\frac{1}{c}\right\| \geq \|q_u\|, \|r_u\|$ for all $u \in \mathbb{Z}_{\geq 0}^n$ then $|f|_\varphi \geq |q|_\varphi$ (or $|f|_\varphi \geq |r|_\varphi$ respectively)

Proof. Note that, we have $1 \geq \|cq_u\|$ and $1 \geq \|cr_u\|$ for all $u \in \mathbb{Z}_{\geq 0}^n$, so that $1 \geq \|cq_u\|^{\varphi(u)}$ and $1 \geq \|cr_u\|^{\varphi(u)}$.

$$\begin{aligned}|f|_\varphi = \max_{u \in \mathbb{Z}_{\geq 0}^n} \left\{\|f_u\|^{\varphi(u)}\right\} &\geq \max_{u \in \mathbb{Z}_{\geq 0}^n} \left\{\|f_u\|^{\varphi(u)} \|cq_u\|^{\varphi(u)}\right\} \\ &= \max_{u \in \mathbb{Z}_{\geq 0}^n} \left\{\|cf_u\|^{\varphi(u)} \|q_u\|^{\varphi(u)}\right\} \\ &\geq \max_{u \in \mathbb{Z}_{\geq 0}^n} \left\{\|q_u\|^{\varphi(u)}\right\} = |q|_\varphi\end{aligned}$$

A similar calculation also holds for r. ■

Now, we prove the existence of Weierstrass Division Algorithm:

Define, $f_0 := (q_0 g + r_0) + f_1$ and proceeding inductively, using the Claim above, for $i = 1, 2, 3, \ldots$ define:

$$f_i = q_i g + r_i + f_{i+1}$$

where

$$|q_i|_\varphi \leq \varepsilon^i |f|_\varphi^\gamma \text{ and } |r_i|_\varphi \leq \varepsilon^i |f|_\varphi^\gamma \text{ and } |f_{i+1}|_\varphi \leq \varepsilon |f|_\varphi^\gamma$$

Hence:

$$\begin{aligned}f &= (q_0 g + r_0) + (q_1 g + r_1) + \ldots + (q_i g + r_i) + \ldots \\ &= \Big(\sum_{i=0}^\infty q_i\Big) g + \Big(\sum_{i=0}^\infty r_i\Big)\end{aligned}$$

Note that, since

$$\lim_{i\to\infty} |q_i|_\varphi = \lim_{i\to\infty} |r_i|_\varphi \leq \lim_{i\to\infty} \varepsilon^i |f|_\varphi^\gamma = 0$$

the sequences $\left(\sum_{i=0}^n q_i\right)_{n\in\mathbb{N}}$ and $\left(\sum_{i=0}^n r_i\right)_{n\in\mathbb{N}}$ are Cauchy sequences in U_n and since U_n is complete they converge. Define:

$$q := \sum_{i=0}^\infty q_i \text{ and } r := \sum_{i=0}^\infty r_i$$

where $q \in U_n$ and $r \in U_{n-1}[X_n]$.

Now, we prove the uniqueness of such a representation. To prove the uniqueness we will use the Gauss norm $|\,.\,|$ on T_n. Suppose

$$f = q_1 g + r_1 \text{ and } f = q_2 g + r_2$$

for some $q_1, q_2 \in U_n \subset T_n$, $r_1, r_2 \in U_{n-1}[X_n] \subset T_{n-1}[X_n]$ and $g \in U_n \subset T_n$ an X_n-distinguished power series of degree s with $|g|_\varphi = |g| = 1$ where $\deg r_1 < s$ and $\deg r_2 < s$.

Note that, the Gauss norm on T_n is a valuation. By multiplying the equation by suitable scalar, without loss of generality, we may suppose that

$$\max\{|q_1|, |r_1|\} = 1$$

Hence, in this case we see that $|f| \leq 1$. Moreover, we have $|f| = 1$, to see this suppose otherwise, i.e. suppose $|f| < 1$. Then, we would have:

$$0 = \widetilde{f} = \widetilde{q}_1 \widetilde{g} + \widetilde{r}_1$$

and since $\deg \widetilde{g} = s > \deg r_1 \geq \deg \widetilde{r}_1$, this would imply that $\widetilde{q}_1 = \widetilde{r}_1 = 0$, i.e. $|q_1| < 1$ and $|r_1| < 1$, but this is a contradiction to the assumption that $\max\{|q_1|, |r_1|\} = 1$. Note that, a similar calculation also holds for the equation $f = q_2 g + r_2$.

Hence we deduce that whenever we have such a representation $f = qg + r$ where $|g| = 1$, we must have

$$|f| = \max\{|q|, |r|\}.$$

Now, since $f = q_1 g + r_1$ and $f = q_2 g + r_2$ we have

$$0 = (q_1 - q_2)g + (r_1 - r_2)$$

hence

$$|0| = 0 = \max\left\{|q_1 - q_2|, |r_1 - r_2|\right\},$$

i.e.

$$q_1 - q_2 = 0 \text{ and } r_1 - r_2 = 0.$$

Now, only the last part of the statement remains to be shown. It is easy to deduce this result. If $g \in U_{n-1}[X_n]$ is a polynomial of degree s then by Remark 1.33 we deduce that g is actually a unitary polynomial so that the usual Euclid Division can be applied in $U_{n-1}[X_n]$ i.e. for every $f \in U_{n-1}[X_n]$ there exists q and r in $U_{n-1}[X_n]$ with $\deg r < s$ such that $f = qg + r$ and since we have already proven the uniqueness of such a representation, the result follows. □

As a corollary we can deduce the following important theorem, so-called the Weierstrass Preparation Theorem, which we shortly call WPT:

Theorem 1.38. (WPT: Weierstrass Preparation Theorem)

Let $g \in U_n$ be X_n-distinguished of degree s. Then there is a unique monic polynomial $\omega \in U_{n-1}[X_n]$ of degree s and a unique unit $e \in U_n$ such that

$$g = e\omega$$

where ω is also X_n-distinguished of degree s with $|\omega|_\varphi = 1$.

Proof. We apply the Weierstrass Division Algorithm (Theorem 1.37) for the monomial X_n^s. So, there exist a unique $e' \in U_n$ and a unique $r' \in U_{n-1}[X_n]$ with $\deg r' < s$ such that

$$X_n^s = e'g + r'.$$

Now, let $c \in k$ be such that

$$\max\left\{|ce'g|_\varphi, |cr'|_\varphi\right\} = 1.$$

then we have

$$cX_n^s = ce'g + cr'$$

and

$$|cX_n^s|_\varphi = |ce'g + cr'|_\varphi \leq \max\left\{|ce'g|_\varphi, |cr'|_\varphi\right\} = 1$$

We must have that $|cX_n^s|_\varphi = 1$. Suppose not, i.e. for a contradiction suppose that

$$|cX_n^s|_\varphi = \|c\|^{\varphi((0,0,\ldots,s))} < 1$$

So that, when we pass the quotient $\widetilde{k}[X_1, \ldots, X_n]$ we get:

$$0 = \widetilde{ce'g} + \widetilde{cr'}$$

but this is a contradiction to the assumption that $\max\left\{|ce'g|_\varphi, |cr'|_\varphi\right\} = 1$ so we deduce that $\|c\|^{\varphi((0,0,\ldots,s))} = 1$, i.e. $\|c\| = 1$.

Note that, if $\|c\| = 1$ then

$$|cf|_\varphi = \max_{u \in \mathbb{Z}^n_{\geq 0}} \left\{\|ca_u\|^{\varphi(u)}\right\} = \max_{u \in \mathbb{Z}^n_{\geq 0}} \left\{\|a_u\|^{\varphi(u)}\right\} = |f|_\varphi$$

for all $f = \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u \in U_n$.

Then we have:

$$|r'|_\varphi = |cr'|_\varphi \leq \max\left\{|ce'g|_\varphi, |cr'|_\varphi\right\} = 1 = |X_n^s|_\varphi = |cX_n^s|_\varphi$$

Now, define $\omega := X_n^s - r'$.

Then, since $r' \in U_{n-1}[X_n]$ of degree $< s$ and $|r'|_\varphi \leq 1$ we see that ω is X_n-distinguished of degree s with $|\omega|_\varphi = 1$ where

$$\omega = e'g.$$

Multiplying the above equation by a suitable constant $c \in k$ we get $\omega = (c^{-1}e')(cg)$ and replacing e' by $c^{-1}e'$ and replacing g by cg we may suppose that $|g|_\varphi = 1$.

Then, we must also have that $|e'|_\varphi \leq 1$. Because otherwise, i.e. if $|e'|_\varphi > 1$ then by Proposition 1.15, entry 1 we must have $|\omega|_\varphi = |e'g|_\varphi > 1$ and this is not possible.

Hence, we can apply the reduction map $\widetilde{\ }$ from U_n° onto $\widetilde{k}[X_1, \ldots, X_n]$ to the equality: $\omega = e'g$ to get:

$$\widetilde{\omega} = \widetilde{e'}\widetilde{g}.$$

Now, note that both ω and g are X_n-distinguished of degree s, hence by Lemma 1.34 both $\widetilde{\omega}$ and $\widetilde{g}$ are unitary polynomials of degree s in $\widetilde{k}[X_1, \ldots, X_{n-1}][X_n]$. Hence the equality $\widetilde{\omega} = \widetilde{e'}\widetilde{g}$ implies that $\deg \widetilde{e'} = 0$ which means $\widetilde{e'}$ is a unit in $\widetilde{k}[X_1, \ldots, X_{n-1}][X_n]$, i.e. $\widetilde{e'}$ is a non-zero constant polynomial in $\widetilde{k}[X_1, \ldots, X_n]$, so that by Proposition 1.25, e' is a unit in U_n. And this proves the existence part of the assertion.

For the uniqueness part, let $\omega \in U_{n-1}[X_n]$ be a monic polynomial of degree s and let e be a unit in U_n such that $g = e\omega$. Then defining $r := X_n^s - \omega$ we get

$$X_n^s = e^{-1}g + (X_n^s - \omega)$$

Since the series g is given, this equality together with the uniqueness of the Weierstrass Division Algorithm prove that the unit e and the monic polynomial ω must be unique. $\square$

Now, we give an easy corollary:

Corollary 1.39. *$U_1 = k\langle X_1\rangle^{\varphi}$ is a Euclidean Domain and in particular a Principal Ideal Domain.*

Proof. Let $g = \sum_{k=1}^{\infty} g_k X_1^k \in U_1$ be any power series. Then we claim that $g' := \frac{g}{|g|} := \sum_{k=1}^{\infty} \frac{g_k}{|g|} X_1^k := \sum_{k=1}^{\infty} g'_k X_1^k$ is X_1-distinguished of degree

$$s = \max_{i \in \mathbb{Z}_{\geq 0}} \{i : \|g_i\| = |g| = \max_{j \in \mathbb{Z}_{\geq 0}} \{\|g_j\|\}\}.$$

Note that $\frac{g}{|g|}$ has Gauss-norm 1, so that since the norms of the coefficients tend to zero, there exists a maximal natural number s such that $\|g_s\| = |g|$.

We need to check the three conditions given in the Definition 1.33:

(i) $g'_s = \frac{g_s}{|g|}$ has Gauss-norm 1, hence it is non-zero so that $g'_s \in k^{\times}$ is a unit.

(ii) $|g'_s|_{\varphi} = \|g'_s\| = \left\|\frac{g_s}{|g|}\right\| = \|1\| = \left|\frac{g}{|g|}\right|_{\varphi} = |g'|_{\varphi}$

(iii) $|g'_s|_{\varphi} = \|g'_s\| = \left\|\frac{g_s}{|g|}\right\| = \|1\| > \left\|\frac{g_v}{|g|}\right\| = \left|\frac{g_v}{|g|}\right|_{\varphi} = |g'_v|_{\varphi}$ for all $v > s$ by the maximality of the natural number s.

Therefore the power series $\frac{g}{|g|}$ is X_1-distinguished.

Now, consider a "degree" function $d : U_1 \to \mathbb{N} \cup \{0\}$ given by

$$g = \sum_{i=0}^{\infty} g_i X_1^i \mapsto d(g) = \max_{s \in \mathbb{Z}_{\geq 0}} \{s : \|g_s\| = |g|\}.$$

Let f and $g \neq 0$ be any two elements of U_1. Then $\frac{g}{|g|}$ is X_1-distinguished of degree $d(g)$. Thus by WDT (Theorem 1.37) there exist $q \in U_1$ and $r \in U_0[X_1]$ with $d(r) \leq \deg r < d(g)$ such that

$$f = q\left(\frac{g}{|g|}\right) + r$$

so that we have

$$f = \left(\frac{q}{|g|}\right)g + r \text{ where } d(r) < d(g)$$

Hence the function d is a Euclidean function on U_1, which means that U_1 is a Euclidean Domain, and in particular a Principal Ideal Domain. $\square$

The polynomials appearing in the Weierstrass Preparation Theorem play an important role in the theory. So we will introduce a special concept for such polynomials:

Definition 1.40. A Weierstrass polynomial (in X_n) is a monic polynomial $\omega \in U_{n-1}[X_n]$ with $|\omega|_\varphi = 1$.

An immediate consequence follows:

Lemma 1.41. *Let ω_1 and ω_2 be monic polynomials in $U_{n-1}[X_n]$. If $\omega_1\omega_2$ is a Weierstrass polynomial, then ω_1 and ω_2 are Weierstrass polynomials.*

Proof. We only need to prove that $|\omega_1|_\varphi = 1$ and $|\omega_2|_\varphi = 1$. Since ω_1 and ω_2 are monic we see that $|\omega_1|_\varphi \geq 1$ and $|\omega_2|_\varphi \geq 1$. Assume without loss of generality that $|\omega_1| > 1$. Then, by Proposition 1.15 entry 1, we deduce that $|\omega_1\omega_2|_\varphi > 1$, which is a contradiction to the assumption given in the statement. Therefore, we must have:

$$|\omega_1|_\varphi = |\omega_2|_\varphi = 1.$$

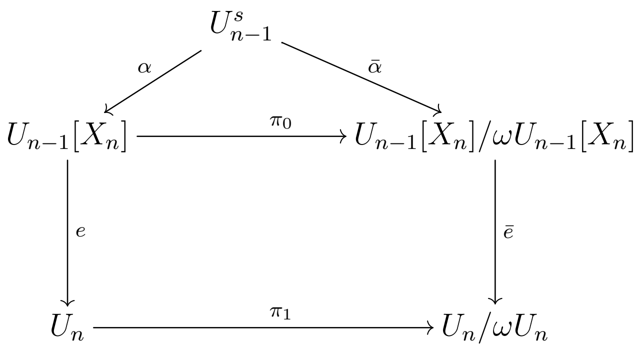
□

The importance of the concept of Weierstrass polynomials can be realized by the fact that, for every X_n-distinguished power series $g \in U_n$, there is a Weierstrass polynomial ω with $\omega U_n = gU_n$. Moreover, we have the following result:

Proposition 1.42. *Let ω be a Weierstrass polynomial of degree s in X_n. Then $U_n/\omega U_n$ is a finite free U_{n-1}-module, moreover:*

$$U_{n-1}[X_n]/\omega U_{n-1}[X_n] \simeq U_n/\omega U_n.$$

Proof. We consider the following commutative diagram of U_{n-1}-module homomorphisms:



where π_0 and π_1 are the canonical residue epimorphisms, e is the natural injection, and $\bar{e}$ is the injection induced by e.

Note that, here U^s_{n-1} is the s-fold normed direct sum of copies of U_{n-1}, i.e.

$$U^s_{n-1} = \bigoplus_{i=1}^{s} U_{n-1}$$

and the map α is given by

$$\alpha(f_1, f_2, \ldots, f_s) = \sum_{v=1}^{s} f_v X_n^v \in U_{n-1}[X_n]$$

for any $(f_1, f_2, \ldots, f_s) \in U^s_{n-1}$. The map $\bar{\alpha}$ is the map induced by α, more precisely:

$$\bar{\alpha} = \pi_0 \circ \alpha.$$

Note also that each map on the diagram is a k-module homomorphism.

We want to show that both $\bar{\alpha}$ and $\bar{e}$ are bijective, then the desired result would follow.

It is enough to show that both

$$\pi_1 \circ e \circ \alpha = \bar{e} \circ \bar{\alpha}$$

and

$$\bar{\alpha} = \pi_0 \circ \alpha$$

are bijective. Then these two results also imply the bijectivity of $\bar{e}$.

Claim: The map $\pi_1 \circ e \circ \alpha$ is bijective.

Proof. Suppose $\bar{f} \in U_n/\omega U_n$. Note that ω is X_n-distinguished of degree s with φ-norm 1, then by the existence part of Weierstrass Division Algorithm there exist unique $h \in U_n$ and $r \in U_{n-1}[X_n]$ such that

$$f = h\omega + r \text{ with } \deg r < s$$

Then $r = f - h\omega$. Set $r := \sum_{v=0}^{k} r_v X^v$ where $r_v \in U_{n-1}$ for each $v = 0, 1, \ldots, k$ and $k < s$. Then putting

$$r' := (r_0, r_1, \ldots, r_k, 0, \ldots, 0) \in U_{n-1}^s$$

we see that r' maps to r under the map α:

$$r' \mapsto \alpha(r') = \sum_{v=0}^{k} r_v X^v = r.$$

Hence:

$$\begin{aligned}(\pi_1 \circ e \circ \alpha)(r') &= (\pi_1 \circ e)(r) = \pi_1(r) \\ &= \pi_1(f - h\omega) = \pi_1(f) - \pi_1(\omega h) \\ &= \pi_1(f) = \bar{f}\end{aligned}$$

This calculation shows that the map $\pi_1 \circ e \circ \alpha$ is surjective, but since for each $f \in U_n$ the values h and r are uniquely given by Weierstrass Division Theorem, we deduce that the map $\pi_1 \circ e \circ \alpha$ is also injective. ∎

Now, we only need to show the bijectivity of $\bar{\alpha}$:

Claim: The map $\bar{\alpha} = \pi_0 \circ \alpha$ is bijective.

Proof. This proof is similar to the proof of previous claim. For any

$$\bar{f} \in U_{n-1}[X_n]/\omega U_{n-1}[X_n],$$

perform the Weierstrass Division Algorithm in $U_{n-1}[X_n]$ for f to get:

$$f = \omega h + r$$

where $h \in U_{n-1}[X_n]$ and $r = \sum_{v=0}^{k} r_v X_n^v$ with $k < s$. Then similarly,

$$r' := (r_0, r_2, \ldots, r_k, 0, \ldots, 0) \mapsto \bar{f}$$

And since h and r are uniquely determined by the Weierstrass Division Theorem, we have that $\bar{\alpha} = \pi_0 \circ \alpha$ is also bijective. ∎

Note that

$$\pi_1 \circ e \circ \alpha = \bar{e} \circ \bar{\alpha}$$

and since both $\pi_1 \circ e \circ \alpha$ and $\bar{\alpha}$ are bijective maps we deduce that $\bar{e}$ is also a bijection. Since all the maps are k-module homomorphism we have:

$$U_{n-1}^s \simeq U_{n-1}[X_n]/\omega U_{n-1}[X_n] \simeq U_n/\omega U_n$$

as k-modules and this, in particular, shows that $U_n/\omega U_n$ is a finite free U_{n-1}-module. □

The preceding results show that, Weierstrass polynomials are useful in reducing problems to similar problems in a lower dimension. But, we need to make sure that there are "enough" of these Weierstrass polynomials. Here "enough" means that every nonzero $f \in U_n$ can be transformed by a suitable automorphism σ into an X_n-distinguished series $\sigma(f)$ which is associated to some Weierstrass polynomial. Before we prove this result, we will establish a lemma first:

Lemma 1.43. *If $g \in U_n$ is X_n-distinguished of degree s then every constant multiple cg, of g where $c \in k^\times$, is also X_n-distinguished of degree s.*

Proof. Set

$$g = \sum_{u \in \mathbb{Z}^n_{\geq 0}} g_u X^u = \sum_{v=0}^{\infty} g_v(X_1, \dots, X_n) X_n^v$$

Then since g is X_n-distinguished of degree s we know that g_s is a unit, $|g|_\varphi = |g_s|_\varphi$ and $|g_s|_\varphi > |g_v|_\varphi$ for all $v > s$.

Let $c \in k^\times$ be a scalar.

We deduce that cg_s is also a unit in $U_{n-1}[X_n]$ because g_s is. So, we only need to show that $|cg_s|_\varphi > |cg_v|_\varphi$ for all $v > s$:

- Suppose for a contradiction that there exists $v > s$ such that

  $$|cg_s|_\varphi \leq |cg_v|_\varphi.$$

  Set

  $$g_s = \sum_{s_w \in \mathbb{Z}^{n-1}_{\geq 0}} g_{s_w} X^{s_w} \text{ and } g_v = \sum_{v_w \in \mathbb{Z}^{n-1}_{\geq 0}} g_{v_w} X^{v_w}$$

  where X denotes the set of $(n-1)$-tuple of variables $(X_1, X_2, \dots, X^{n-1})$. Suppose

  $$|cg_v|_\varphi = \max_{v_w \in \mathbb{Z}^{n-1}_{\geq 0}} \|cg_{v_w}\|^{\varphi(v_w)} = \|cg_w\|^{\varphi(w)}$$

  for some $w \in \mathbb{Z}^{n-1}_{\geq 0}$

□

Recall that the *lexicographical ordering* on $\mathbb{Z}^n$ is defined by;

$$(a_1, a_2, \dots, a_n) < (b_1, b_2, \dots, b_n)$$

if and only if there exists m with $1 \leq m \leq n$ such that $a_m < b_m$ and $a_k = b_k$ for all $k < m$.

Now, we prove that every nonzero power series in U_n can be transformed into an X_n-distinguished power series by a k-algebra automorphism:

Theorem 1.44. *For every nonzero $f \in U_n$, there is a k-algebra automorphism σ of U_n such that $\sigma(f)$ is X_n-distinguished.*

Proof. Let $f = \sum_{\mu \in \mathbb{Z}^n_{\geq 0}} a_\mu X^\mu \in U_n$ be non-zero. We may suppose that $|f|_\varphi = 1$. Otherwise, multiply f by a suitable non-zero constant $c \in k$ to get $|cf|_\varphi = 1$. Note that multiplying by a non-zero constant is also an automorphism of U_n. So, if there is an automorphism σ of U_n where $\sigma(cf)$ is X_n-distinguished then the automorphism $c\sigma$ maps f to an X_n-distinguished power series.

Then, there exists $u \in \mathbb{Z}^n_{\geq 0}$ such that $\|a_u\|^{\varphi(u)} = 1$, i.e. $\|a_u\| = 1$. Let $m = (m_1, \ldots, m_n)$ be the maximal (with respect to lexicographical ordering) n-tuple such that $\|a_m\| = 1$.

Let t be a natural number such that

$$t \geq \max_{1 \leq i \leq n} \mu_i$$

among all indices $\mu = (\mu_1, \ldots, \mu_n)$ with $\|a_\mu\| = 1$.

The automorphism we will use to prove the assertion will be defined by the following rules:

$$\begin{cases} \sigma(X_i) := X_i + X_n^{c_i} & \text{for } i = 1, 2, \ldots, n-1 \\ \sigma(X_n) := X_n \end{cases}$$

where the numbers c_i for $i = 1, 2, \ldots, n-1$ are defined recursively by:

$$\begin{cases} c_n := 1 \quad \text{and} \\ c_{n-j} := 1 + t\sum_{d=0}^{j-1} c_{n-d} & \text{for } j = 1, 2, \ldots, n-1 \end{cases}$$

Note that, this indeed defines a k-algebra automorphism of U_n. More generally, we have the following: Given any $c_1, c_2, \ldots, c_{n-1} \in \mathbb{N}$, define $\phi : U_n \to U_n$ by

$$\begin{cases} \phi(X_i) := X_i + X_n^{c_i} & \textit{for } i = 1, 2, \ldots, n-1 \\ \phi(X_n) := X_n \end{cases}$$

Then ϕ is an automorphism of U_n. To see this, define $\psi : U_n \to U_n$ by

$$\begin{cases} \psi(X_i) := X_i - X_n^{c_i} & \textit{for } i = 1, 2, \ldots, n-1 \\ \psi(X_n) := X_n \end{cases}$$

Then ϕ and ψ are inverses to each other, so we deduce that ϕ is an automorphism of U_n.

Now, we claim that $\sigma(f)$ is X_n-distinguished of degree

$$s := \sum_{i=1}^{n} c_i m_i.$$

To prove this, we will show that $\widetilde{\sigma(f)}$ is a unitary polynomial of degree s in

$$(\widetilde{k}[X_1, \ldots, X_{n-1}])[X_n],$$

and then Lemma 1.34 implies the claim.

First we observe that, for all $\mu = (\mu_1, \ldots, \mu_n)$ with $\|a_\mu\| = 1$ and $\mu \neq m$ we have $\sum_{i=1}^{n} c_i\mu_i < s$. Namely, there is an index p with $1 \leq p \leq n$, such that $\mu_1 = m_1, \ldots, \mu_{p-1} = m_{p-1}$ and $\mu_p < m_p$ by the maximality of the index m. Then,

$$\sum_{i=1}^{n} c_i\mu_i \leq \sum_{i=1}^{p-1} c_i m_i + c_p(m_p - 1) + \sum_{i=p+1}^{n} c_i t = \sum_{i=1}^{p} c_i m_i - 1 < \sum_{i=1}^{n} c_i m_i = s.$$

Now we compute $\widetilde{\sigma(f)}$:

$$\begin{aligned}
\widetilde{\sigma(f)} &= \sum_{\mu} \widetilde{a}_\mu (X_1 + X_n^{c_1})^{\mu_1} \ldots (X_{n-1} + X_n^{c_{n-1}})^{\mu_{n-1}} X_n^{\mu_n} \\
&= \sum_{\substack{\mu \\ \widetilde{a}_\mu \neq 0}} \widetilde{a}_\mu \sum_{\substack{\lambda_1, \ldots, \lambda_{n-1} \\ 0 \leq \lambda_i \leq \mu_i}} \binom{\mu_1}{\lambda_1} \ldots \binom{\mu_{n-1}}{\lambda_{n-1}} X_1^{\mu_1 - \lambda_1} \ldots X_{n-1}^{\mu_{n-1} - \lambda_{n-1}} X_n^{c_1\lambda_1 + \ldots + c_{n-1}\lambda_{n-1} + \mu_n} \\
&= \sum_{i=0}^{s} p_i X_n^i
\end{aligned}$$

where p_i's are suitable elements of $\widetilde{k}[X_1, \ldots, X_{n-1}]$. Thus, by construction we see that $\widetilde{\sigma(f)}$ is a polynomial in X_n of degree s.

Furthermore, $c_1\lambda_1 + \ldots + c_{n-1}\lambda_{n-1} + \mu_n = s$ if and only if $\mu_n = m_n$ and $\mu_i = \lambda_i = m_i$ for $i = 1, 2, \ldots, n-1$. So, the monomial X_n^s appears in the above double sum representation of $\widetilde{\sigma(f)}$ and in this case $p_s = \widetilde{a}_m$ with $\|a_m\| \neq 0$, i.e. $\widetilde{a}_m \in \widetilde{k}^\times$, which means that the highest coefficient of $\widetilde{\sigma(f)}$ is invertible and thus $\widetilde{\sigma(f)}$ is a unitary polynomial of degree s in $(\widetilde{k}[X_1, \ldots, X_{n-1}])[X_n]$. Hence, by Lemma 1.34, $\sigma(f)$ is X_n-distinguished.

□

Now, we will give some applications of the theorems we have given so far. The concepts and results we will introduce now in this part help us to establish some results about the ring structure of the algebra U_n. The results will not be proven, they will just be stated because they are part of a general theory, called Rückert Theory. Detailed proofs and explanations can be found in [26] Section 5.2.5.

Definition 1.45. Let I be a commutative ring with unity. An overring I' of $I[X]$ is called *Rückert* over I if there is a family W of monic polynomials in $I[X]$ such that the following three axioms are satisfied:

(i) If the product of two monic polynomials lies in W, then so do the factors.

(ii) For all $\omega \in W$, there is an isomorphism of I-algebras $I'/\omega I' \simeq I[X]/\omega I[X]$. In particular, the canonical map $I \to I'/\omega I'$ is finite.

(iii) For all $f \in I' \setminus \{0\}$, there is an automorphism σ of I' and a unit e of I' such that $e.\sigma(f) \in W$

The results we have proven in the preceding section imply:

Corollary 1.46. *If we take W to be the set of all Weierstrass polynomials in U_n then U_n is Rückert over U_{n-1}.*

In many aspects, a Rückert ring I' behaves as $I[X]$ does. In particular, some ring theoretic properties of I are inherited by I', as the following three propositions show.

Proposition 1.47. *A Rückert overring I' of a Noetherian ring I is Noetherian.*

Recall that, a ring I is said to be a *Jacobson ring* if for every ideal $\mathfrak{a} \subset I$, the radical $\operatorname{rad} \mathfrak{a}$ equals the Jacobson radical $j(\mathfrak{a})$, which is the intersection of all the maximal ideals of I containing $\mathfrak{a}$. Obviously, any field is a Jacobson ring. But a local ring I is not Jacobson unless $I/\operatorname{rad} I$ is a field. So, one cannot expect that every ring I' which is Rückert over a Jacobson ring I is itself a Jacobson ring ($I := k$ and $I' := k[[X]]$ provide a counter-example), but we have the following close result:

Proposition 1.48. *Let I be a Jacobson ring, and let I' be a Rückert overring of I. Then* $\operatorname{rad} \mathfrak{a} = j(\mathfrak{a})$ *for any non-zero ideal $\mathfrak{a} \subset I'$.*

Recall that a factorial ring is an integral domain I such that each non-unit $f \in I \setminus \{0\}$ can be written as a finite product of prime elements in I and any such product decomposition of f is unique up to units and reordering.

Proposition 1.49. *Every integral domain I', which is Rückert over a factorial ring I, is factorial itself.*

Now, we deduce a crucial result using the Rückert Theory:

Theorem 1.50. *The ring U_n is Noetherian and factorial.*

Proof. As we already remarked: Theorem 1.38, Lemma 1.41, Proposition 1.42, Theorem 1.44 ensure that U_n is Rückert over U_{n-1}, provided W to be taken the set of all Weierstrass polynomials in U_n. We have $U_0 = k$, a Noetherian and factorial ring. Thus by induction on n, we deduce by Proposition 1.47 and Proposition 1.49 that U_n is a Noetherian and factorial ring. □

Corollary 1.51. *U_n is normal.*

Proof. This is an immediate consequence of the fact that any factorial ring R is normal, i.e. integrally closed in its field of fractions. □

Finally, we have:

Theorem 1.52. *U_n is a Jacobson ring.*

Proof. By Proposition 1.31 we know that $j(U_n)$, the intersection of all maximal ideals of U_n, is trivial. Therefore, we conclude from Proposition 1.48 that U_n is a Jacobson ring if U_{n-1} is. Since $U_0 = k$ is a Jacobson ring, as being a field, we deduce the assertion by induction on n. □

1.4 Ideals

In this section, we will investigate ideals of U_n. It is another important property we seek that all ideals of U_n must be closed, so that we will be able to define a natural norm on the quotients (by ideals) of U_n . Before we prove this result, we remind a definition and a theorem from classical functional analysis. Recall that:

Definition 1.53. A *k-Fréchet space* is a locally convex topological vector space X whose topology is defined by a translation invariant metric d such that (X, d) is complete.

Remark 1.54. *We note here that U_n is a k-Fréchet space. For $f, g, h \in U_n$, we have:*

$$|f - g|_\varphi = |(f + h) - (g + h)|_\varphi$$

which shows that the φ-norm is a translation invariant metric on U_n and U_n is complete with respect to this metric. The fact that U_n is a locally convex topological space will be proven in Section 2.1. We need to use the Open Mapping Theorem (which will be stated next) now for the purpose of proving that all ideals of U_n are closed.

The preceeding Definition and the following Theorem are taken from the book [7]. This theorem is the Open Mapping Theorem between k-Fréchet spaces:

Theorem 1.55. *Let V and W be k-Fréchet spaces. Then every surjective continuous linear map $f : V \to W$ is open, that is for every open subset U of V, the image $f(U)$ is open in W.*

Now, we can prove our next result:

Theorem 1.56. *All ideals of U_n are closed.*

Proof. Let $I \subset U_n$ be a proper ideal.

By Remark 1.29, $U_n^\times$, the set of all invertible elements of U_n, is an open set in U_n, hence the closure $\bar{I}$ must also a proper ideal of U_n.

By Theorem 1.50, we know that U_n is Noetherian. So, we can represent

$$\bar{I} = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle$$

(the ideal generated by $\alpha_1, \alpha_2, \ldots, \alpha_k$) for some $k \in \mathbb{N}$.

Note that, since $\langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle = \langle c\alpha_1, c\alpha_2, \ldots, c\alpha_k \rangle$ for any $c \in k^\times$, by multiplying each α_i by a suitable constant we may assume that

$$\max_{i=1}^{k} \left\{ |\alpha_i|_\varphi \right\} = 1.$$

Consider the surjective linear map:

$$\phi : U_n^k \to \bar{I}$$

defined by

$$\phi(f_1, f_2, \ldots, f_k) = \sum_{i=1}^{k} f_i \alpha_i.$$

We claim that ϕ is a continuous function:

Proof. For $f = (f_1, \ldots, f_k)$ and $g = (g_1, \ldots, g_k)$ in U_n^k, set

$$d(f, g) = \max_{i=1}^{k} |f_i - g_i|_\varphi.$$

Then since U_n is a complete metric space, U_n^k is also a complete metric space.

Let $0 < \varepsilon < 1$ be any positive real number. We need to show that there exists $\delta \in \mathbb{R}_{>0}$ such that $d(f,g) < \delta$ implies $d(\phi(f), \phi(g)) < \varepsilon$.

Set $\delta = \varepsilon^{\frac{1}{\gamma}}$ where γ is the constant positive real number depending on the function φ, as it is mentioned in Corollary 1.5.

Suppose $d(f,g) = \max_{i=1}^{k} |f_i - g_i|_\varphi < \delta$. Then:

$$\begin{aligned} d(\phi(f), \phi(g)) &= |\phi(f) - \phi(g)|_\varphi = |\phi(f-g)|_\varphi \\ &= \Big| \sum_{i=1}^{k} (f_i - g_i)\alpha_i \Big|_\varphi \\ &\leq \max_{i=1}^{k} \big|(f_i - g_i)\alpha_i\big|_\varphi \\ &\leq \max_{i=1}^{k} |f_i - g_i|_\varphi^\gamma |\alpha_i|_\varphi^\gamma \\ &\leq \max_{i=1}^{k} \Big(|f_i - g_i|_\varphi\Big)^\gamma < \delta^\gamma = \varepsilon \end{aligned}$$

Hence ϕ is continous. ∎

Recall that, $U_n^{\circ\circ} := \{f \in U_n : |f|_\varphi < 1\}$ is an open subset of U_n, hence $\left(U_n^{\circ\circ}\right)^k$ is an open subset of U_n^k. Note that both U_n^k and $\bar{I}$ are k-Fréchet spaces (Definition 1.53). Then:

$$\phi\Big(\big(U_n^{\circ\circ}\big)^k\Big) = \sum_{i=1}^{k} U_n^{\circ\circ} \alpha_i$$

is a neighborhood of 0 in $\bar{I}$ by Open Mapping Theorem (1.55).

Then we claim that

$$\bar{I} = I + \sum_{i=1}^{k} U_n^{\circ\circ} \alpha_i$$

Proof. It is clear that $I + \sum_{i=1}^{k} U_n^{\circ\circ}\alpha_i \subseteq \bar{I}$. Conversely, since I is dense in its closure $\bar{I}$ and $0 \in I$, for every $y \in \bar{I}$, the sum $y + \sum_{i=1}^{k} U_n^{\circ\circ}\alpha_i$ must intersect with I, i.e. there exists x in I such that $x \in I \cap \big(y + \sum_{i=1}^{k} U_n^{\circ\circ}\alpha_i\big)$. Hence, $y \in x + \sum_{i=1}^{k} U_n^{\circ\circ}\alpha_i \subseteq I + \sum_{i=1}^{k} U_n^{\circ\circ}\alpha_i$. ∎

So, for $i = 1, \ldots, k$ using the equality above we have:

$$\alpha_i = f_i + \sum_{j=1}^{k} a_{i,j}\alpha_j \text{ where } f_i \in I \text{ and } a_{i,j} \in U_n^{\circ\circ}$$

i.e. we have the following equations:

$$\begin{gathered} (a_{1,1}\alpha_1 + a_{1,2}\alpha_2 + \ldots + a_{1,k}\alpha_k) + f_1 = \alpha_1 \\ (a_{2,1}\alpha_1 + a_{2,2}\alpha_2 + \ldots + a_{2,k}\alpha_k) + f_2 = \alpha_2 \\ \vdots \\ (a_{k,1}\alpha_1 + a_{k,2}\alpha_2 + \ldots + a_{k,k}\alpha_k) + f_k = \alpha_k \end{gathered}$$

$$\text{Set } M = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,k} \\ a_{2,1} & a_{2,2} & \dots & a_{2,k} \\ \vdots & \vdots & & \vdots \\ a_{k,1} & a_{k,2} & \dots & a_{k,k} \end{pmatrix}$$

Then we have

$$M\vec{\alpha} + \vec{f} = \vec{\alpha}$$

i.e.

$$(\mathrm{id} - M)\vec{\alpha} = \vec{f}$$

It is not difficult to see that the matrix $(\mathrm{id} - M)$ is an invertible matrix:

By definition:

$$\det(\mathrm{id} - M) = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} b_{i,\sigma(i)}$$

where $\mathrm{Sym}(n)$ is the symmetric group i.e. the set of all bijections of the set $\{1, 2, \dots, n\}$, the function sgn is the sign function depending on the number of the tranpositions of given bijection in $\mathrm{Sym}(n)$ and b_{ij}'s are the entries of the matrix $\mathrm{id} - M$.

If $\sigma \in \mathrm{Sym}(n)$ is not the identity bijection then the product $\prod_{i=1}^{n} b_{i,\sigma(i)}$ is of the form:

$$\prod_{i=1}^{n} b_{i,\sigma(i)} = (1 - a_{i_1,i_1}) \dots (1 - a_{i_k,i_k}) a_{l_1} \dots a_{l_m}$$

where $i_1, \dots, i_k, l_1, \dots, l_m \in \{1, 2, \dots, n\}$ with at least one non-zero a_{l_j} for some j. So that:

$$\prod_{i=1}^{n} b_{i,\sigma(i)} = \sum_{i=1}^{m} a_{t_{i_1},s_{i_1}} \dots a_{t_{i_p},s_{i_p}}$$

which has norm:

$$\left| \prod_{i=1}^{n} b_{i,\sigma(i)} \right|_{\varphi} = \left| \sum_{i=1}^{m} a_{t_i,s_i} \right|_{\varphi} \leq \max_{i=1}^{m} |a_{t_{i_1},s_{i_1}} \dots a_{t_{i_p},s_{i_p}}|_{\varphi} < 1$$

Only when $\sigma = \mathrm{id}$ we have

$$\prod_{i=1}^{n} b_{i,\sigma(i)} = (1 - a_{1,1})(1 - a_{2,2}) \dots (1 - a_{k,k}) = 1 + \sum_{i=1}^{m} a_{i_1,i_1} \dots a_{i_m,i_m}$$

Hence, the determinant $\det(\mathrm{id} - M)$ is of the form $1 + c$ for some $c \in U_n$ with $|c|_{\varphi} < 1$ (by the choice of entries). Therefore, By Theorem 1.26 we have $\det(\mathrm{id} - M) = 1 + c$ is invertible in U_n.

Then we have the equality:

$$\vec{\alpha} = (\mathrm{id} - M)^{-1} \vec{f}$$

so that the generating elements $\alpha_1, \alpha_2, \dots, \alpha_k$ of $\bar{I}$ can be written as a finite sum of the elements in I, therefore $\bar{I} \subseteq I$.

$\square$

For the last part of the chapter we will give the definition of a "strictly closed ideal" and show that at least for a special case of the base field k, the ideals of the algebra U_n are strictly closed.

Definition 1.57. An ideal I of U_n is said to be a *strictly closed ideal* if for any $f \in U_n$ there exists $g \in I$ such that the residue norm on U_n/I defined by

$$|\bar{f}|_I = \inf_{a \in I} |f - a|_\varphi$$

is achieved at g, i.e.

$$|f - g|_\varphi = \inf_{a \in I} |f - a|_\varphi.$$

Then our next result would be:

Proposition 1.58. *Suppose that $\|k \setminus \{0\}\|$ is discrete in $\mathbb{R}_{>0}$ then each ideal of U_n is strictly closed.*

Proof. Let I be an ideal of U_n. Let f be in U_n. Note that if f is an element of the ideal I, then the assertion is obvious, take $g = f$. So, we may suppose that $f \notin I$. Then since I is closed in U_n, we deduce that

$$d(f, I) = |f, I|_\varphi = \inf_{a \in I} |f - a|_\varphi > 0.$$

We need to show that there exists $g \in U_n$ such that

$$|f - g|_\varphi = \inf_{a \in I} |f - a|_\varphi.$$

Note that if we take $a = 0 \in I$ then we see that

$$\inf_{a \in I} |f - a|_\varphi \leq |f|_\varphi.$$

Then we have the inequalities:

$$0 < |f, I|_\varphi = \inf_{a \in I} |f - a|_\varphi \leq |f|_\varphi$$

Since, by the hyphothesis, $\|k \setminus \{0\}\|$ is discrete in $\mathbb{R}_{>0}$, there are only finitely many values of $|f - a|_\varphi$ for $a \in I$ on the interval $(|f, I|_\varphi, |f|_\varphi)$, thus the infimum (in fact the minimum) must be achieved for some $g \in I$. □

Chapter 2

Categorical Properties

In this chapter we study a more general type of U_n algebras, namely the *nmk algebras*. These algebras are basically the quotients of the algebras U_n, by closed ideals. We will establish an important result, the so-called "Noether Normalization Lemma", it states that each nmk algebra $\mathcal{N}$ is actually a finite U_d-module for some $d \geq 0$ and for some injective map $U_d \hookrightarrow \mathcal{N}$. This result makes it possible to reduce certain problems on nmk algebras to the problems on algebras of convergent power series.

For our purposes (for instance to use Proposition 1.58) we will assume that the set $\|k \setminus \{0\}\|$ is discrete in $\mathbb{R}_{>0}$ (for instance when $k = \mathbb{Q}_p$), so that we make sure of the fact that each ideal of U_n is strictly closed in U_n.

We also fix the filter function φ from now on.

We start with the concept of "k-Fréchet Spaces." These spaces are in a sense generalizations of "Banach Spaces." We will restrict ourselves and work in the category of k-Fréchet spaces.

2.1 Pseudo Fréchet Algebras

In this section, we will give the definitions and basic implications and we will prove that the algebra U_n and its quotients (by closed ideals) are all k-Fréchet spaces. The following concepts and remarks were taken from the book "Nonarchimedean Functional Analysis" by Peter Schneider [27].

We will prove that U_n is a k-Fréchet space. In order to do this we first introduce the concepts of a *lattice* and *locally convex topology.*

Definition 2.1. A *lattice* $\mathcal{L}$ in a k-vector space V is a k°-submodule which satisfies the condition that for any vector $v \in V$ there is a nonzero scalar $a \in k^\times$ such that $av \in \mathcal{L}$.

Remark 2.2. *Let $(\mathcal{L}_j)_{j\in J}$ be a nonempty family of lattices in the k-vector space V such that we have:*

(i) For any $j \in J$ and any $a \in k^\times$ there exists a $k \in J$ such that $\mathcal{L}_k \subseteq a\mathcal{L}_j$

and

(ii) For any two $i, j \in J$ there exists a $k \in J$ such that $\mathcal{L}_k \subseteq \mathcal{L}_i \cap \mathcal{L}_j$.

The second condition implies that the intersection of any two "convex" subsets $v + \mathcal{L}_i$ and $v' + \mathcal{L}_j$ either is empty or contains a convex subset of the form $w + \mathcal{L}_k$. This means that *the convex subsets $v+\mathcal{L}_j$ for $v \in V$ and $j \in J$ form the basis of a topology on V which will be called the locally convex topology on V defined by the family $(\mathcal{L}_j)_{j\in J}$. For any vector $v \in V$ the convex subsets $v+\mathcal{L}_j$ (for $j \in J$) form a fundamental system of open and closed neighborhoods of v in this topology.*

Definition 2.3. A locally convex k-vector space is a k-vector space equipped with a locally convex topology.

And we prove that the algebra U_n is a k-Fréchet space:

Proposition 2.4. *U_n is a k-Fréchet space.*

Proof. The metric on U_n is translation invariant and by Theorem 1.14 we know that U_n is complete. Recalling Definition 1.53, we see that we only need to prove that U_n is locally convex. For this we need to establish the locally convex topology structure. We claim that the subsets:

$$\mathcal{L}_r = \{f \in U_n : |f|_\varphi \leq r\}$$

for each $r \in \mathbb{R}_{>0}$ form a locally convex topology structure.

First, we will show that each such $\mathcal{L}_r$ is a k°-submodule. We only need to show that for each fixed $r \in \mathbb{R}_{>0}$ the set $\mathcal{L}_r$ is closed under addition and scalar multiplication:

- Using the ultrametric property given in Lemma 1.10, for any $f, g \in \mathcal{L}_r$:

  $$|f+g|_\varphi \leq \max\left\{|f|_\varphi, |g|_\varphi\right\} \leq r$$

  Hence $\mathcal{L}_r$ is a subgroup.

- If $f \in \mathcal{L}_r$ and $a \in k^\circ$ then by Proposition 1.15 entry 5 we have:

  $$|af|_\varphi \leq |f|_\varphi \leq r$$

  Hence $af \in \mathcal{L}_r$, i.e. $\mathcal{L}_r$ is closed under multiplication with k°.

Thus we conclude that $\mathcal{L}_r$ is a k°-submodule for each $r \in \mathbb{R}_{>0}$.

Now, we will show that the family $(\mathcal{L}_r)_{r\in\mathbb{R}_{>0}}$ of k°-submodules satisfies the conditions given in Remark 2.2:

(i) For any $r \in \mathbb{R}_{>0}$ and any $a \in k^\times$ we must find $s \in \mathbb{R}_{>0}$ such that $\mathcal{L}_s \subseteq a\mathcal{L}_r$. Let $r \in \mathbb{R}_{>0}$ and $a \in k^\times$ be arbitrarily chosen. Set $s = \min\{r, \|a\|r\}$. Then:

$$\mathcal{L}_s \subseteq a\mathcal{L}_r \iff \text{If } |f|_\varphi \leq s \text{ then } f = ag \text{ where } |g|_\varphi \leq r$$

There are two cases:

  - If $\|1/a\| < 1$ then $|f/a|_\varphi \leq |f|_\varphi \leq s \leq r$.
  - If $\|1/a\| \geq 1$ then $|f/a|_\varphi \leq \|1/a\||f|_\varphi \leq \|1/a\|s \leq \|1/a\|\|a\|r = r$.

(ii) For any $r, s \in \mathbb{R}_{>0}$ with $r \leq s$ we have $\mathcal{L}_r \cap \mathcal{L}_s = \mathcal{L}_r$. Thus for any $t \leq s$ we have $\mathcal{L}_t \subseteq \mathcal{L}_r \cap \mathcal{L}_s$.

Hence we deduce that the family $(\mathcal{L}_r)_{r\in\mathbb{R}_{>0}}$ forms a locally convex topology structure on U_n, so that U_n is a locally convex space. □

The following result is Proposition 8.3 in [27]:

Proposition 2.5. *Let V be a k-Fréchet space and let $U \subseteq V$ be a closed vector subspace, then V/U with the quotient topology is a k-Fréchet space as well.*

Thus we deduce the following result:

Corollary 2.6. *All quotient algebras (by closed ideals) of U_n are k-Fréchet spaces, as well.*

Showing that the algebra U_n is a k-Fréchet space justifies working in the category of k-Fréchet spaces. To establish results on tensor and completed tensor products of U_n-algebras (in the third section) we need a slightly more general setting. Recall the definition of an normed R-module, where R is a unitary ring:

Definition 2.7. Let R be a ring with a norm $|\,.\,|$ on it. A *normed R-module M* is an R-module together with the map $M \to \mathbb{R}_{\geq 0}$, denoted by $|\,.\,|$ again, such that for all $x, y \in M$ and $a \in R$ we have:

(i) $|x| = 0$ if and only if $x = 0$

(ii) $|x + y| \leq \max\{|x|, |y|\}$

(iii) $|ax| \leq |a|\,|x|$

Unfortunately, we do not have a normed k-module structure with the algebra U_n. So, we need to establish something close and we define the notion of *pseudo-normed k-space*. In Proposition 1.15 entry 5, we mentioned about a type of norm which can also be related to module norms. Now, we properly define this concept:

Definition 2.8. A k-vector space V is called *pseudo-normed k-space* if there is a function $|\,.\,| : V \to \mathbb{R}_{\geq 0}$ with the following properties for all $x, y \in V$ and $a \in k$:

(i) $|x| = 0$ if and only if $x = 0$

(ii) $|x + y| \leq \max\{|x|, |y|\}$

(iii) $|ax| \leq \max\{1, |a|\}\,|x|$

The map $|\,.\,| : V \to \mathbb{R}_{\geq 0}$ is called pseudo-seminorm if only the conditions (ii) and (iii) are satisfied and (i) possibly not.

Remark 2.9. *Note that by Lemma 1.7 and Proposition 1.15 entry 5, we see that $U_{n,\varphi}$ for each n and for any filter function φ, is a pseudo-normed k-space. Furthermore, if I is an ideal of $U_{n,\varphi}$ then the residue norm on $U_{n,\varphi}/I$ is given by*

$$|\bar{f}|_I = \inf\{|f + b|_\varphi : b \in I\}$$

for all $\bar{f} \in U_{n,\varphi}/I$. So, it is clear that the residue norm satisfies the conditions (i) and (ii) of Definition 2.8 and moreover for $a \in k^\circ \setminus \{0\}$ we have:

$$\begin{aligned}|\overline{af}|_I &= \inf\left\{|af + b|_\varphi : b \in I\right\}\\ &= \inf\left\{|a(f + b/a)|_\varphi : b \in I\right\}\\ &\leq \inf\left\{|(f + b/a)|_\varphi : b \in I\right\}\end{aligned}$$

$$= |\bar{f}|_I$$

and similarly for $a \in k \setminus k^\circ$ we have:

$$\begin{aligned}
|\overline{af}|_I &= \inf\left\{|af+b|_\varphi : b \in I\right\} \\
&= \inf\left\{|a(f+b/a)|_\varphi : b \in I\right\} \\
&\leq \inf\left\{\|a\|\,|(f+b/a)|_\varphi : b \in I\right\} \\
&= \|a\| \inf\left\{|(f+b/a)|_\varphi : b \in I\right\} \\
&= \|a\||\bar{f}|_I
\end{aligned}$$

Hence we conclude that all k-algebra quotients of the k-algebra $U_{n,\varphi}$, i.e. all nmk algebras are pseudo-normed k-spaces.

The concept of k-Fréchet spaces is constructed in the most general way. In general k-Fréchet spaces are not algebras and they do not possess a norm structure. In our case of U_n-algebra we have both of these structures. So, the usual sense of a k-Fréchet space is extended and to overcome this obstacle we will introduce a specific concept, the so-called "pseudo-normed k-Fréchet algebra":

Definition 2.10. A k-algebra A is called *pseudo-normed k-Fréchet algebra* (or shortly pseudo Fréchet algebra when the base field k is obvious), if it possesses a pseudo-norm given in Definition 2.8 and complete with respect to it. We require that the multiplication on pseudo-normed k-Fréchet algebras is continuous.

Remark 2.11. *It is clear that the algebras U_n and its quotients are all pseudo Fréchet algebras. From now on, we will refer U_n-algebras and their quotients as pseudo Fréchet algebras, regarding objects in the category of all k-Freéchet spaces.*

In the next section we will pass to the quotients of algebras U_n by closed ideals and we will investigate some categorical properties of these quotients.

2.2 NMK Algebras

The quotient algebras of U_n are somehow special type of spaces, analogous to "affinoid algebras" mentioned in Chapter 6 of [26]. We will build the category of all quotients of U_n's within the category of k-Fréchet spaces. Now, we give the definition of an NMK[1] algebra.

Definition 2.12. A pseudo Fréchet algebra $\mathcal{N}$ is called an *nmk algebra* if there exists a continuous epimorphism

$$\alpha : U_{n,\varphi} \to \mathcal{N}$$

for some $n \in \mathbb{N}$. We will denote the category of all nmk algebras by $\mathcal{U}$.

Remark 2.13. *By the Open Mapping Theorem (1.55), the map α above is open, hence $\mathcal{N}$ is isomorphic (topologically, as all maps are continous which we will prove later on) to the residue algebra $U_{n,\varphi}/\ker\alpha$. In particular, the residue norm given by*

$$|\bar{f}|_\alpha := |f, \ker\alpha|_\varphi := \inf\{|g|_\varphi : g \in \bar{f}\}$$

[1]NMK: Nesin Matematik Köyü, an institute of mathematics in Izmir, Turkey.

induces the given complete pseudo Fréchet algebra topology on $\mathcal{N}$.

The norm $|\,.\,|_\alpha$ totally depends on the continuous epimorphism α, however all such norms $|\,.\,|_\alpha$ are equivalent, as we will show later on (Proposition 2.22).

On the other hand, we have the canonical residue norm induced from the φ-norm on U_n:

Remark 2.14. *Note that each residue algebra U_n/I of U_n by a (closed) ideal $I \subset U_n$ becomes a complete k-algebra if one defines the residue norm of the residue class $\bar{f}$ of an element $f \in U_n$ by*

$$|\bar{f}|_\varphi := d(f, I) = |f, I|_\varphi := \inf\left\{|f - a|_\varphi : a \in I\right\}$$

So we deduce that each nmk algebra is complete with respect to the residue norm.

Before we prove the fact that all norms $|\,.\,|_\alpha$ on an nmk algebra are equivalent, we need to establish some results first. We start with an analogue of Lemma 1.22 for nmk algebras:

Lemma 2.15. *Let $\mathcal{N}$ be an nmk algebra with the residue norm $|\,.\,|_\varphi$ given in Remark 2.14, then each nonzero vector can be normed to length 1 by multiplication by a scalar.*

Proof. This result immediately follows from Lemmas 1.22 and 1.58. For each $f \in \mathcal{N} \setminus \{0\}$ there exists $g \in I$ such that $|\bar{f}|_\varphi = |f - g|_\varphi$ and each element of U_n can be normed to length 1 by multiplication by a scalar, and so the desired result follows. □

Now, some immediate categorical consequence of the definition of nmk algebras:

Proposition 2.16. *Let $\mathcal{N}$ be an nmk algebra. Then $\mathcal{N}$ is a Noetherian Jacobson ring. Each ideal $I \subset \mathcal{N}$ is closed, and each quotient $\mathcal{N}/I$ (provided with the residue norm) is also an nmk algebra.*

Proof. Let $\alpha : U_n \to \mathcal{N}$ be a continuous epimorphism. Then $\mathcal{N} \simeq U_n/\ker\alpha$ is also a Noetherian Jacobson ring, because U_n is such a ring (see Theorem 1.52 and Theorem 1.50). And since the composition of the maps

$$U_n \xrightarrow{\alpha} \mathcal{N} \xrightarrow{\pi} \mathcal{N}/I$$

is surjective and continuous, where π is the canonical contractive (hence continuous) projection map, we have that $\mathcal{N}/I$ is also an nmk algebra. □

Next, we prove an important result, the so-called "Noether Normalization Lemma":

Theorem 2.17. *(***NNL***)*

1. *For every nmk algebra $\mathcal{N} \neq 0$, there is an injective finite morphism $U_d \to \mathcal{N}$ for some $d \geq 0$.*

2. *For every finite morphism $\alpha : U_n \to \mathcal{N}$, there is a morphism $\tau : U_d \to U_n$ with $d \leq n$ such that $\alpha \circ \tau : U_d \to \mathcal{N}$ is injective finite.*

Proof. We will first show that the first statement is a consequence of the second one. Suppose I is an ideal such that $\mathcal{N} \simeq U_n/I$. Take α to be the natural surjection $\alpha : U_n \to U_n/I \simeq \mathcal{N}$. Then there exists $\tau : U_d \to U_n$, for some $d \leq n$, such that

$$\alpha \circ \tau : U_d \to U_n \to U_n/I \simeq \mathcal{N}$$

is finite and injective.

We will prove the second statement by induction on n.

When $n = 0$, $U_n = U_d = U_0 = k$ is a field and α is trivially injective (as being a finite morphism from $U_0 = k$ to $\mathcal{N}$).

Now, suppose that the assertion holds for $n-1$.

If α is injective, i.e. if $\ker \alpha = 0$, then taking $\tau : U_d \to U_n$ as the natural injection:

$$\tau : k\langle X_1, \ldots, X_d\rangle^{\varphi} \hookrightarrow k\langle X_1, \ldots, X_d, \ldots, X_n\rangle^{\varphi},$$

we trivially get an injection from U_d into $\mathcal{N}$. So, we may without loss of generality assume that $\ker \alpha \neq 0$, i.e. there exists a nonzero $f \in \ker \alpha$. Then by Theorem 1.44, there exists an automorphism σ of U_n such that $\sigma(f)$ is X_n-distinguished and by Theorem 1.38 (WPT), there exists a unique $\omega \in W$ (the set of all Weierstrass polynomials) and a unique unit $e \in U_n^{\times}$ such that

$$\sigma(f) = \omega e$$

i.e.

$$\omega = \sigma(f)e^{-1}$$

i.e.

$$\omega \in W \cap \sigma(\ker \alpha).$$

Replacing the morphism α by $\alpha \circ \sigma^{-1}$ we deduce that

$$\alpha \circ \sigma^{-1}(\omega) = \alpha(f\sigma^{-1}(e^{-1})) = \alpha(\sigma^{-1}(e^{-1}))\alpha(f) = 0$$

Thus we may, without loss of generality, suppose that there exists a Weierstrass polynomial

$$\omega \in \ker \alpha.$$

Then we have the induced morphism:

$$\bar{\alpha} : U_n/\omega U_n \to \mathcal{N}$$

given by

$$g + \omega U_n \mapsto \alpha(g).$$

It is finite (because α is finite).

Note that, by Proposition 1.42, the natural map

$$\phi : U_{n-1} \to U_n/\omega U_n$$

is finite, hence the composition map:

$$\bar{\alpha} \circ \phi : U_{n-1} \to \mathcal{N}$$

is finite (because it is the composition of two finite maps). Then by induction hyphothesis there is a morphism

$$\tau : U_d \to U_{n-1}$$

such that

$$(\bar{\alpha} \circ \phi) \circ \tau : U_d \to \mathcal{N}$$

is injective and this is the desired result. □

Before we go on, we remind the reader of a lemma from classical commutative algebra:

Lemma 2.18. *Let A and B be two integral domains such that there is a finite injective morphism $A \hookrightarrow B$. Then:*

(i) If A is field then so is B.

(ii) If B is a field then so is A.

Proof. (i) Suppose A is a field. Let $x \in B \setminus \{0\}$. Then x is integral over A. If

$$a_0 + a_1 + \ldots + a_{n-1}x^{n-1} + x^n = 0 \text{ with } a_0 \neq 0$$

is an equation satisfied by x then $-a_0^{-1}(x^{n-1}+a_{n-1}x^{n-2}+\ldots+a_1) \in B$ is the inverse of x.

(ii) Suppose B is a field. Let $x \in A \setminus \{0\}$ be arbitrary. Then x^{-1} is integral over A and with a similar calculation of (i), we can deduce that x has an inverse in A. □

Next, we give a result on the maximal ideals of U_n and it will be followed by an easy corollary:

Proposition 2.19. *Let $\mathfrak{m} \subset U_n$ be a maximal ideal. Then $U_n/\mathfrak{m}$ is a finite extension of k.*

Proof. By Noether Normalization Lemma (Theorem 2.17), there exists a finite injective morphism $U_d \to U_n/\mathfrak{m}$ for some $d \leq n$.

Then by Lemma 2.18, since $U_n/\mathfrak{m}$ is a field, we must have that U_d is also a field which implies that $d = 0$ and $U_d = U_0 = k$. Therefore, $U_n/\mathfrak{m}$ is finitely generated over k, and so $U_n/\mathfrak{m}$ is a finite extension of k. □

Corollary 2.20. *Let $\phi : \mathcal{N} \to \mathcal{M}$ be a morphism of nmk algebras and let $\mathfrak{m} \subset \mathcal{M}$ be a maximal ideal. Then $\phi^{-1}(\mathfrak{m})$ is a maximal ideal of $\mathcal{N}$.*

Proof. By Proposition 2.19, for every maximal ideal I of U_n, the quotient U_n/I is a finite extension of k, this fact implies that $\mathcal{M}/\mathfrak{m}$ is also a finite extension of k, i.e. $\mathcal{M}/\mathfrak{m}$ is a finite dimensional k-vector space.

Since $\mathfrak{m}$ is maximal (therefore prime) we have that $\phi^{-1}(\mathfrak{m})$ is a prime ideal of $\mathcal{N}$. Then the morphism

$$\mathcal{N}/\phi^{-1}(\mathfrak{m}) \to \mathcal{M}/\mathfrak{m}$$

induced by the morphism ϕ is an injective morphism between two domains, so that $\mathcal{N}/\phi^{-1}(\mathfrak{m})$ can be seen as a subspace of $\mathcal{M}/\mathfrak{m}$ and it implies that $\mathcal{N}/\phi^{-1}(\mathfrak{m})$ is also a finite dimensional k-vector space. Since $\mathcal{M}/\mathfrak{m}$ is a field, by Lemma 2.18, $\mathcal{N}/\phi^{-1}(\mathfrak{m})$ is also a field, thus $\phi^{-1}(\mathfrak{m})$ is a maximal ideal. □

Now, we finally prove the equivalence of norms on the quotients of U_n, before it we remind the Closed Graph Theorem from functional analysis (Proposition 8.5 of [27]):

Theorem 2.21. *Let $\alpha : \mathbb{F}_1 \to \mathbb{F}_2$ be a linear map between two k-Fréchet spaces. If the graph of α is closed, then α is continuous.*

Note that this theorem applies to all nmk algebras as they are pseudo Fréchet algebras built in the category of k-Fréchet spaces. We end this section with the following crucial result.

Proposition 2.22. *Let $(\mathcal{N}, \|\,.\,\|_1)$ and $(\mathcal{M}, \|\,.\,\|_2)$ be two nmk algebras over k provided with the pseudo-norms which make them pseudo Fréchet algebras. Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a homomorphism of k-algebras. Then α is continuous with respect to the given norms. In particular, all norms on an nmk algebra $\mathcal{K}$ which make $\mathcal{K}$ into a pseudo Fréchet algebra are equivalent.*

Proof. By the Closed Graph Theorem (2.21), we must prove that the graph of $\alpha : \mathcal{N} \to \mathcal{M}$ is closed. Or equivalently, we must prove that if $(x_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{N}$ with

$$\lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} \alpha(x_n) = y$$

then $y = 0$.

Let I be an ideal of $\mathcal{M}$ of finite codimension, i.e. $\mathcal{M}/I$ has finite dimension over k (for example if I is maximal).

Consider the k-algebra homomorphism:

$$\beta : \mathcal{N} \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\pi} \mathcal{M}/I$$

i.e. $\beta = \pi \circ \alpha$ where π is the natural projection map.

Let J be the kernel of β, i.e. $J = \ker \beta \subset \mathcal{N}$. Then J is an ideal of $\mathcal{N}$ of finite codimension, because

$$\mathcal{N}/J \simeq \mathcal{M}/I$$

and $\mathcal{M}/I$ has finite dimension over k.

We can factor β as:

$$\beta : \mathcal{N} \to \mathcal{N}/J \to \mathcal{M}/I.$$

We put on $\mathcal{N}/J$ and $\mathcal{M}/I$ the induced residue norms. Note that both I and J are closed, so the residue seminorm is a norm.

Any linear map between finite dimensional normed k-vector spaces is continuous and therefore β is also continuous, since the map $\mathcal{N} \to \mathcal{N}/J$ is contractive, hence continuous.

This implies that the image of y in $\mathcal{M}/I$ is zero for every ideal $I \subset \mathcal{M}$ of finite codimension.

We will show that for an nmk algebra $\mathcal{K}$, the intersection $\bigcap I$ of all ideals of finite codimension is (0):

> By Proposition 2.19, any maximal ideal $\mathfrak{m} \subset \mathcal{K}$ has finite codimension. Then, it implies that $\mathfrak{m}^n$ has also finite codimension for all $n \geq 1$.
>
> Consider an element $y \in \mathcal{K}$ which lies in $\mathfrak{m}^n$ for every maximal ideal $\mathfrak{m}$ of $\mathcal{K}$ and for all $n \geq 1$. Then
>
> $$J := \{a \in \mathcal{K} : ay = 0\}$$
>
> is an ideal of $\mathcal{K}$.
>
> If $y \neq 0$, then $J \neq \mathcal{K}$ and J lies in a maximal ideal $\mathfrak{m}$ of $\mathcal{K}$.
>
> The image z of y in the localized ring $\mathcal{K}_\mathfrak{m}$ is not zero since $J \subset \mathfrak{m}$ and moreover:
>
> $$z \in \mathfrak{m}^n \mathcal{K}_\mathfrak{m} = (\mathfrak{m}\mathcal{K}_\mathfrak{m})^n$$

for every $n \geq 1$. However, $\mathcal{K}_{\mathfrak{m}}$ is a Noetherian local ring with maximal ideal $\mathfrak{m}\mathcal{K}_{\mathfrak{m}}$ and the intersection of all powers of this maximal ideal is well-known to be (0).

This contradiction shows that $y = 0$.

Hence the graph of α is closed, which implies by the Closed Graph Theorem that α is continuous. In particular, all norms on an nmk algebra $\mathcal{K}$ are equivalent. □

2.3 Tensors

In this section, we will focus on the tensor products and completed tensor products of nmk algebras. Our aim is to prove that the category $\mathcal{U}$ of all nmk algebras is closed under taking completed tensor products. We start with a reminder of ordinary tensor products:

Remark 2.23. *Let $(A, |\,.\,|_1)$ and $(B, |\,.\,|_2)$ be any two algebras over k, with the ultrametrics $|\,.\,|_1$ and $|\,.\,|_2$. Considering the ordinary tensor product $A \otimes_k B$, we define a function:*

$$|\,.\,| : A \otimes_k B \to \mathbb{R}_{\geq 0}$$

such that for any $g \in A \otimes_k B$,

$$|g| := \inf \left(\max |x_i|_1 |y_i|_2 \right)$$

where the infimum runs over all possible representations of g such as

$$g = \sum_{i=1}^{r} x_i \otimes y_i \text{ with } x_i \in A \text{ and } y_i \in B$$

It can easily be verified that the function $|\,.\,|$ is an ultrametric on the additive group of $A \otimes_k B$. Hence, we deduce that the pair $\big(A \otimes_k B, |\,.\,|\big)$ is a semi-normed group.

The reason we do not go on with the ordinary tensor product is that this structure is not complete and not a normed group in general, but if we take the completion, the resulting space fits our demands. According to Proposition 1.1.7/5 of [26], each semi-normed group admits a completion. And this is how we get the "completed tensor product":

Remark 2.24. *Let A and B be as given above. We construct the completion of $A \otimes_k B$ (as a semi-normed group) to get the complete canonical k-algebra $A \hat{\otimes}_k B$. The resulting k-algebra is called completed tensor product of A and B and it is now a normed group.*

As the ordinary tensor product does, the completed tensor product also satisfies a similar "universal property." We will specifically construct the completed tensor products of nmk algebras and prove that they indeed have the desired universal property. Before we do that, first we give a definition:

Definition 2.25. A k-linear map $\phi : V \to W$ between two pseudo Fréchet algebras is called bounded if there exists a real constant $c > 0$ such that $|\phi(x)| \leq c|x|$ for all $x \in V$. In this case c is referred to as a bound for ϕ.

Now, we prove that boundedness implies continuity:

Lemma 2.26. *Bounded k-linear maps between pseudo Fréchet algebras are continuous.*

Proof. Let $\phi : V \to W$ be a bounded k-linear map between two pseudo Fréchet algebras. Let $\varepsilon > 0$ be arbitrary. We want to find a $\delta > 0$ such that whenever we have $|x - y| \leq \delta$ it will imply that $|\phi(x) - \phi(y)| = |\phi(x - y)| \leq \varepsilon$. Since ϕ is bounded $|\phi(x - y)| \leq c|x - y|$ for some $c > 0$. Hence taking $\delta = \varepsilon/c$ implies the continuity of ϕ. □

Now, we will focus on tensor products and their related bilinear maps. First we give a definition:

Definition 2.27. Let M, N and K be nmk algebras. A k-bilinear map $\phi : M \times N \to K$ is called *bounded* if there exists a real constant $c > 0$ such that $|\phi(x, y)| \leq c|x||y|$ for all $x \in M$ and $y \in N$. Again, c is called a bound for ϕ. A k-linear or k-bilinear map bounded by 1 is called *contractive.*

Now, we give the formal definition of completed tensor products of nmk algebras:

Definition 2.28. Let $\mathcal{M}$ and $\mathcal{N}$ be two nmk algebras. Let $\mathcal{M} \otimes_k \mathcal{N}$ denote the regular tensor product of $\mathcal{M}$ and $\mathcal{N}$, with the seminorm:

$$|g| = \inf \left\{ \max |x_i||y_i| \right\}$$

where $g \in \mathcal{M} \otimes_k \mathcal{N}$ and the infimum is taken over all possible representations:

$$g = \sum_{i=1}^{r} x_i \otimes y_i \text{ with } x_i \in \mathcal{M} \text{ and } y_i \in \mathcal{N} \text{ and } r \in \mathbb{N}.$$

We define $\mathcal{M}\hat{\otimes}_k\mathcal{N}$ as the topological completion of $\mathcal{M} \otimes_k \mathcal{N}$ and we extend the norm to the completion. Then $\mathcal{M}\hat{\otimes}_k\mathcal{N}$ is called the *completed tensor product* of $\mathcal{M}$ and $\mathcal{N}$, it is uniquely determined up to isomorphism.

The attached canonical contractive k-linear map (introduced in Proposition 2.29):

$$\tau : \mathcal{M} \times \mathcal{N} \to \mathcal{M} \otimes_k \mathcal{N} \to \mathcal{M}\hat{\otimes}_k\mathcal{N}$$

given by

$$(x, y) \mapsto x \otimes y \mapsto x\hat{\otimes}y$$

determines the elements of the completed tensor product. In other words we define

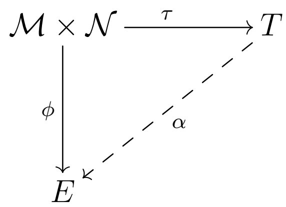
$$x\hat{\otimes}y := \tau(x, y).$$

Note the closure of the image of τ in $\mathcal{N} \to \mathcal{M}\hat{\otimes}_k\mathcal{N}$ is dense and it satisfies the universal property of completed tensor products given in Proposition 2.29 (just as $\mathcal{M}\hat{\otimes}_k\mathcal{N}$ itself does).

Next, we prove the existence of completed tensor products in the context of nmk algebras.

Proposition 2.29. *Let $\mathcal{M}$ and $\mathcal{N}$ be nmk algebras. There exists a contractive k-linear map $\tau : \mathcal{M} \times \mathcal{N} \to T$ into a complete pseudo Fréchet algebra T such that the following universal property holds:*

Given any k-bilinear map $\phi : \mathcal{M} \times \mathcal{N} \to E$ bounded by some $c > 0$ into a pseudo Fréchet algebra, there exists a unique k-linear map $\alpha : T \to E$ bounded by c as well, such that the following diagram commutes:



Proof. To construct the map α, view the ordinary tensor product $\mathcal{M}\otimes_k\mathcal{N}$ as a pseudo-seminormed k-algebra using the seminorm $|\,.\,|:\mathcal{M}\otimes_k\mathcal{N}\to\mathbb{R}_{\geq 0}$ given by:

$$|z| := \inf\left(\max_{i=1}^{r}|x_i||y_i|\right) \text{ for } z\in\mathcal{M}\otimes_k\mathcal{N}$$

where the infimum runs over all possible representations $z=\sum_{i=1}^r x_i\otimes y_i$ for $x_i\in\mathcal{M}$ and $y_i\in\mathcal{N}$.

Thus, we can define the separated completion $T=\mathcal{M}\hat{\otimes}_k\mathcal{N}$. It is in fact now a pseudo Fréchet algebra, since it is complete. For elements $x\in\mathcal{M}$ and $y\in\mathcal{N}$ we write $x\hat{\otimes}y$ for the element in $\mathcal{M}\hat{\otimes}_k\mathcal{N}$ that is induced by the tensor $x\otimes y\in\mathcal{M}\otimes_k\mathcal{N}$. Then the map

$$\tau:\mathcal{M}\times\mathcal{N}\to\mathcal{M}\hat{\otimes}_k\mathcal{N}$$

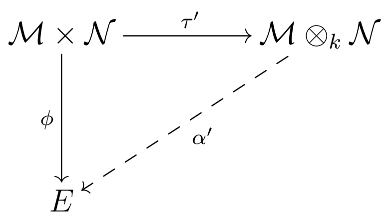
given by

$$(x,y)\mapsto x\hat{\otimes}y$$

is a k-bilinear and contractive map.

The k-algebra $\mathcal{M}\hat{\otimes}_k\mathcal{N}$ together with its extended norm (from the ordinary tensor product norm) is called the *completed tensor product* of $\mathcal{M}$ and $\mathcal{N}$ over the field k.

Now, we will show that the k-bilinear map τ satisfies the universal property of the assertion. So, let $\phi:\mathcal{M}\times\mathcal{N}\to E$ be a bounded k-linear map into a pseudo Fréchet algebra E and let $c>0$ be a bound for ϕ. We have the canonical k-bilinear map τ' : $\mathcal{M}\times\mathcal{N}\to\mathcal{M}\otimes_k\mathcal{N}$ sending the pair (x,y) to the tensor $x\otimes y$, then by the universal property of the ordinary tensor product there exists a unique k-linear map $\alpha':\mathcal{M}\otimes_k\mathcal{N}$ which makes the following diagram commutative:



Then consider some element

$$z=\sum_{i=1}^{r}x_i\otimes y_i\in\mathcal{M}\otimes_k\mathcal{N}$$

where $x_i\in\mathcal{M}$ and $y_i\in\mathcal{N}$. Since (by diagram commutativity)

$$\alpha'(z)=\sum_{i=1}^{r}\phi(x_i,y_i)$$

we get:

$$\left|\alpha'(z)\right|\leq\max_{1\leq i\leq r}\left|\phi(x_i,y_i)\right|\leq\max_{1\leq i\leq r}\left(c|x_i||y_i|\right)=c\max_{1\leq i\leq r}\left(|x_i||y_i|\right)$$

Taking the infimum over all possible representations of z as a sum of tensors $\sum_{i=1}^{r} x_i \otimes y_i$ yields that

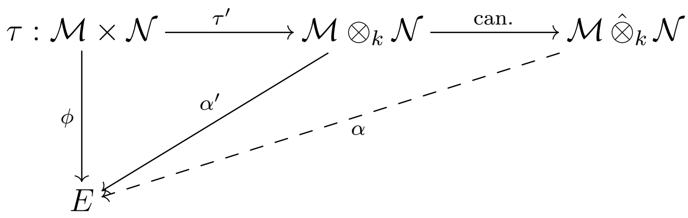
$$\left|\alpha'(z)\right| \leq c|z|$$

and thus we see that α' is also bounded by c.

Now, since E is complete α' gives rise to a k-linear map:

$$\alpha : \mathcal{M} \hat{\otimes}_k \mathcal{N} \to E$$

that is bounded by c, as well. Furthermore, we can enlarge the above commutative diagram to obtain the following commutative diagram:



It only remains to show that α is uniquely determined by the relation $\phi = \alpha \circ \tau$. However, this is clear since α is unique on the image $\tau(\mathcal{M} \times \mathcal{N})$ which generates a dense k-subspace in $\mathcal{M} \hat{\otimes}_k \mathcal{N}$. □

Remark 2.30. *We note that here the universal property of completed tensor products is defined using the bounded maps not continuous maps. It is unclear that all continuous maps between nmk algebras are bounded, but we do have the property that all bounded maps are continuous. So, we remark that the notion of completed tensor product we are using is slightly different than the usual completed tensor products.*

Next, we prove another universal property satisfied by completed tensor products:

Proposition 2.31. *Let $\mathcal{M}$ and $\mathcal{N}$ be two nmk algebras. Then the contractive k-algebra homomorphisms*

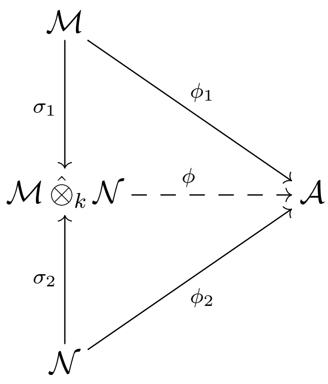
$$\sigma_1 : \mathcal{M} \to \mathcal{M} \hat{\otimes}_k \mathcal{N}, \quad x \mapsto x \hat{\otimes} 1$$

and

$$\sigma_2 : \mathcal{N} \to \mathcal{M} \hat{\otimes}_k \mathcal{N}, \quad y \mapsto 1 \hat{\otimes} y$$

for $x \in \mathcal{M}$ and $y \in \mathcal{N}$, admit the following universal property of amalgamated sums:

Let $\phi_1 : \mathcal{M} \to \mathcal{A}$ and $\phi_2 : \mathcal{N} \to \mathcal{A}$ be two homomorphisms of pseudo Fréchet algebras that are bounded by constants $c_1 > 0$ and $c_2 > 0$ and suppose that $\mathcal{A}$ is complete. Then there is a unique k-algebra homomorphism $\phi : \mathcal{M} \hat{\otimes}_k \mathcal{N} \to \mathcal{A}$ bounded by $c_1 c_2$, which makes the following diagram commutative.



Proof. Consider the homomorphisms of pseudo Fréchet algebras $\phi_1 : \mathcal{M} \to \mathcal{A}$ as well as $\phi_2 : \mathcal{N} \to \mathcal{A}$ where $\mathcal{A}$ is complete and assume that ϕ_1 and ϕ_2 are bounded by the constants $c_1 > 0$ and $c_2 > 0$ respectively. Then the map

$$\mathcal{M} \times \mathcal{N} \to \mathcal{A}, \quad (a_1, a_2) \mapsto \phi_1(a_1)\phi_2(a_2)$$

is a k-bilinear map which is bounded by c_1c_2. Thus the universal property of the completed tensor products in Proposition 2.29 gives rise to a k-linear map

$$\phi : \mathcal{M} \hat{\otimes}_k \mathcal{N} \to \mathcal{A}$$

given by

$$a_1 \hat{\otimes} a_2 \mapsto \phi_1(a_1)\phi_2(a_2)$$

which is also bounded by c_1c_2.

Furthermore for all $a_1, a_1' \in \mathcal{M}$ and $a_2, a_2' \in \mathcal{N}$, the map ϕ satisfies:

$$\begin{aligned} \phi\big((a_1\hat{\otimes}a_2)(a_1'\hat{\otimes}a_2')\big) &= \phi\big(a_1a_1'\hat{\otimes}a_2a_2'\big) \\ &= \phi_1\big(a_1a_1'\big)\phi_2\big(a_2a_2'\big) \\ &= \phi_1(a_1)\,\phi_1(a_1')\,\phi_2(a_2)\,\phi_2(a_2') \\ &= \phi\big(a_1\hat{\otimes}a_2\big)\phi\big(a_1'\hat{\otimes}a_2'\big) \end{aligned}$$

This shows that ϕ is multiplicative on the image of $\mathcal{M} \otimes \mathcal{N}$ in $\mathcal{M} \hat{\otimes}_k \mathcal{N}$ and hence by continuity on $\mathcal{M} \hat{\otimes}_k \mathcal{N}$ itself.

And since for $a_1 \in \mathcal{M}$ and $a_2 \in \mathcal{N}$ we have

$$\phi(a_1\hat{\otimes}a_2) = \phi\big((a_1\hat{\otimes}1)(1\hat{\otimes}a_2)\big) = \phi_1(a_1)\phi_2(a_2)$$

again by continuity we see that ϕ is unique on $\mathcal{M} \hat{\otimes}_k \mathcal{N}$.

□

Next we prove an important consequence of this proposition:

Proposition 2.32. *Let φ be any fixed filter function $\varphi : \mathbb{Z}_{\geq 0} \to \mathbb{R}$. As usual we define:*

$$U_{n+m} := U_{n+m,\varphi} = \Big\{ \sum_{u+v \in \mathbb{Z}_{\geq 0}^{n+m}} a_{u,v}X^uY^v : \|a_{u,v}\|^{\varphi(|u|+|v|)} \to 0 \text{ as } |u| + |v| \to \infty \Big\}$$

where $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_m)$ and the addition on $\mathbb{Z}_{\geq 0}^{n+m}$ is componentwise. Using φ we define

$$U_n := U_{n,\varphi} = \Big\{ \sum_{u \in \mathbb{Z}_{\geq 0}^{n}} a_uX^u : \|a_u\|^{\varphi(|u|)} \to 0 \text{ as } |u| \to \infty \Big\}$$

and $U_m := U_{m,\varphi}$ is defined similarly.

Then, for all $n, m \in \mathbb{N}$:

$$U_n \hat{\otimes}_k U_m = U_{n+m}$$

Proof. Consider the canonical maps:

$$\sigma_1 : U_n \hookrightarrow U_{n+m} \text{ and } \sigma_2 : U_m \hookrightarrow U_{n+m}$$

given by

$$\sigma_1 : (X_1, X_2, \dots, X_n) \mapsto (X_1, X_2, \dots, X_n) \in U_{n+m}$$

and

$$\sigma_2 : (X_1, X_2, \dots, X_m) \mapsto (X_{n+1}, X_{n+2}, \dots, X_{n+m}) \in U_{n+m}$$

We will show that these maps together with the algebra U_{n+m} satisfy the universal property mentioned in Proposition 2.31, so that we can conclude the fact that $U_{n+m} = U_n \hat{\otimes}_k U_m$.

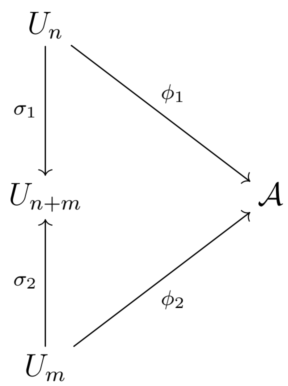
First we note that the maps σ_1 and σ_2 are contractive maps, i.e. they are bounded by 1.

Now, suppose there exist two morphisms of pseudo Fréchet algebras:

$$\phi_1 : U_n \to \mathcal{A} \text{ and } \phi_2 : U_m \to \mathcal{A}$$

into a (complete) pseudo Fréchet algebra $\mathcal{A}$. And suppose that the maps ϕ_1 and ϕ_2 are bounded by the constants c_1 and c_2, respectively.

So, we have the following diagram:



If we can show that there exists a unique k-algebra morphism $\phi : U_{n+m} \to \mathcal{A}$ which is bounded by c_1c_2 making the above diagram commutative, then by Proposition 2.31 the space U_{n+m} satisfies the universal property of $U_n \hat{\otimes}_k U_m$ and hence we are done.

Let

$$f := \sum_{(u,v) \in \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^m_{\geq 0}} a_{(u,v)} X^u Y^v$$

be an arbitrary element of U_{n+m} where X denotes the n-tuple of indeterminates $(X_1, \dots, X_n)$ and Y denotes the m-tuple of indeterminates $(X_{n+1}, \dots, X_{n+m})$.

Define $\phi : U_{n+m} \to \mathcal{A}$ by

$$\phi(f) := \sum_{(u,v) \in \mathbb{Z}^n_{\geq 0} \times \mathbb{Z}^m_{\geq 0}} a_{(u,v)} \phi_1\big(X^u\big) \phi_2\big(Y^v\big)$$

Note that since

$$\big|\phi_1\big(a_{(u,v)} X^u\big)\big| \leq c_1 \big|a_{(u,v)} X^u\big|_\varphi \to 0 \text{ as } |u + v| \to \infty$$

and

$$\big|\phi_2(Y^v)\big| \leq c_2 \big|Y^v\big|_\varphi = c_2$$

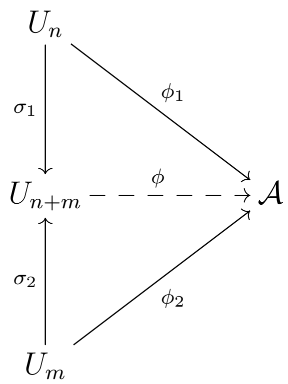
we have:

$$a_{(u,v)} \phi_1\big(X^u\big) \phi_2\big(Y^v\big) \to 0 \text{ as } |u + v| \to \infty$$

Hence the sums of the type $\sum_{(u,v)} a_{(u,v)}\phi_1\big(X^u\big)\phi_2\big(Y^v\big)$ are converging sums in $\mathcal{A}$, i.e. the map ϕ is well-defined and it is bounded by c_1c_2 as shown by the following inequalities:

$$\begin{aligned}
|\phi(f)| &= \Big|\sum_{(u,v)} a_{(u,v)}\phi_1\big(X^u\big)\phi_2\big(Y^v\big)\Big| \\
&\le \max_{(u,v)}\Big\{\big|a_{(u,v)}\phi_1\big(X^u\big)\phi_2\big(Y^v\big)\big|\Big\} \\
&\le \max_{(u,v)}\Big\{c_1c_2\|a_{(u,v)}\|^{\varphi(u,v)}\Big\} \\
&= c_1c_2\max_{(u,v)}\Big\{\|a_{(u,v)}\|^{\varphi(u,v)}\Big\} \\
&= c_1c_2|f|_\varphi
\end{aligned}$$

So since ϕ is bounded, by Lemma 2.26 we deduce that ϕ is a continuous morphism of pseudo Fréchet algebras. Moreover, ϕ is the unique bounded morphism given in Proposition 2.31 making the following diagram commutative:



□

Now as the final result in this chapter, we will prove that the category $\mathcal{U}$ of all nmk algebras is closed under completed tensor products:

Proposition 2.33. *Let $\mathcal{N}$ and $\mathcal{M}$ be two nmk algebras. Then $\mathcal{N}\,\hat{\otimes}_k\,\mathcal{M}$ is also an nmk algebra.*

Proof. We choose two (continuous) surjections of k-algebras:

$$\alpha_1 : U_n \to \mathcal{N}$$

and

$$\alpha_2 : U_m \to \mathcal{M}.$$

By using Corollary 2.32, we have the canonical morphism of k-algebras:

$$\alpha : U_{n+m} = U_n\,\hat{\otimes}_k\,U_m \to \mathcal{N}\,\hat{\otimes}_k\,\mathcal{M}$$

is surjective and we claim that its kernel is generated by $\ker\alpha_1$ and $\ker\alpha_2$, thus this gives rise to an isomorphism of k-algebras:

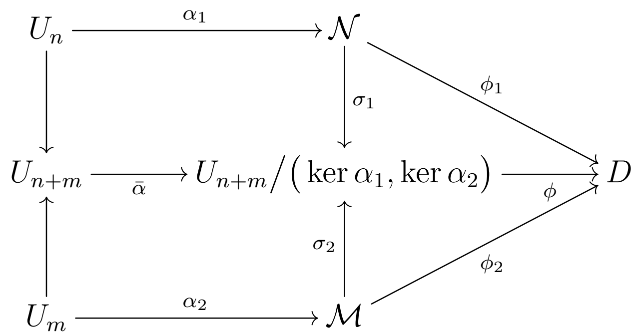
$$U_{n+m}/\big(\ker\alpha_1, \ker\alpha_2\big) \to^{\simeq} \mathcal{N}\,\hat{\otimes}_k\,\mathcal{M}.$$

So, to prove the claimed isomorphism we need to show that

$$U_{n+m}/\big(\ker\alpha_1, \ker\alpha_2\big)$$

satisfies the universal property of completed tensor products.

Consider the commutative diagram of k-algebras:



where σ_1 and σ_2 are induced by the inclusions $U_n \hookrightarrow U_{n+m}$ and $U_m \hookrightarrow U_{n+m}$ and $\bar{\alpha}$ is the canonical projection.

If D is a complete pseudo Fréchet algebra and $\phi_1 : \mathcal{N} \to D$ and $\phi_2 : \mathcal{M} \to D$ are homomorphisms that are bounded by the constants c_1 and c_2 respectively, then by interpreting $U_{n+m} = U_n \hat{\otimes}_k U_m$ we see that there exists a canonical homomorphism of k-algebras $U_{n+m} \to D$ that is bounded by c_1c_2 which factors through the quotient

$$U_{n+m}/\big(\ker\alpha_1, \ker\alpha_2\big)$$

via the unique homomorphism of k-algebras

$$\phi : U_{n+m}/\big(\ker\alpha_1, \ker\alpha_2\big) \to D$$

which makes the diagram commutative.

Now, consider $U_{n+m}/\big(\ker\alpha_1, \ker\alpha_2\big)$ as an nmk algebra equipped with the residue norm. Note that the maps σ_1 and σ_2 are contractive maps and note further that ϕ is bounded by c_1c_2 and since $\bar{\alpha}$ is a contractive map the composition $\phi \circ \bar{\alpha}$ is also bounded by c_1c_2.

Hence, we conclude that $U_{n+m}/\big(\ker\alpha_1, \ker\alpha_2\big)$ along with the contractions σ_1 and σ_2 satisfies the universal property completed tensor product $\mathcal{N} \hat{\otimes}_k \mathcal{M}$.

□

Chapter 3

Geometric Properties

3.1 Maximal Ideals

In this section we will give some results on the maximal ideals of the algebra U_n.

Lemma 3.1. *Let $\mathfrak{m} \subset U_n$ be a maximal ideal. Let K be a finite extension of k that we endow with the multiplicative norm induced from k. Let $\phi : U_n/\mathfrak{m} \to K$ be a morphism of k-algebras. Then ϕ is continuous and for all $f \in U_n^\circ$ we have:*

$$\|\phi(\bar{f})\| \leq |f|_\varphi^\gamma$$

Proof. The continuity of ϕ follows easily, because by Proposition 2.19 $U_n/\mathfrak{m}$ is a finite extension of k and so that it is also a k-vector space of finite dimension and ϕ is k-linear.

Then there exists $C > 0$ such that for all $\bar{f} \in U_n/\mathfrak{m}$:

$$\|\phi(\bar{f})\| \leq C|\bar{f}|_\mathfrak{m}$$

Applying this inequality to f^n for $n = 1, 2, 3, \ldots$ we get:

$$\|\phi(\bar{f}^n)\| \leq C|\bar{f}^n|_\mathfrak{m}.$$

Note that the norm $\| \, . \, \|$ on K is multiplicative as being the induced norm from k, and the map ϕ is a k-algebra morphism, so we have:

$$\|\phi(\bar{f}^n)\| = \|\phi(\bar{f})^n\| = \|\phi(\bar{f})\|^n$$

It is easy to see that for each $f \in U_n$:

$$\inf_{a \in \mathfrak{m}} \left\{|f - a|_\varphi\right\} = |\bar{f}|_\mathfrak{m} \leq |f|_\varphi.$$

Then using Proposition 1.20, we deduce that for all $f \in U_n^\circ$:

$$\|\phi(\bar{f})\|^n = \|\phi(\bar{f}^n)\| \leq C|\bar{f}^n|_\mathfrak{m} \leq C|f^n|_\varphi \leq C\big(|f|_\varphi^\gamma\big)^n$$

so that

$$\|\phi(\bar{f})\| \leq C^{\frac{1}{n}}|f|_\varphi^\gamma$$

Letting n tend to infinity we deduce that:

$$\|\phi(\bar{f})\| \leq |f|_\varphi^\gamma$$

for all $f \in U_n^\circ$. $\square$

Now, using this lemma we will establish a surjection from B (the unit polydisc defined below) to the set of maximal ideals of U_n. From now on, by $\mathrm{Sp}(A)$ we will denote the set of all maximal ideals of some algebra A. Then our next result is:

Proposition 3.2. *Define*

$$B^n := \{(a_1, a_2, \dots, a_n) \in \bar{k}^n : \|a_i\| \le 1 \text{ for } i = 1, 2, \dots, n\}$$

as the unit polydisk where $\bar{k}$ denotes the algebraic closure of k. For $a := (a_1, a_2, \dots, a_n) \in B^n$ define

$$\tau(a) := \{f \in U_n : f(a) = 0\}$$

Then τ defines a surjective map between B^n and $\mathrm{Sp}(U_n)$.

Proof. For any $a := (a_1, a_2, \dots, a_n) \in B^n$, consider "the evaluation map at a"

$$e_a : U_n \to k(a_1, a_2, \dots, a_n)$$

given by

$$e_a(f) := f(a)$$

It is easy to see that this map is surjective. So it induces an isomorphism:

$$U_n/\tau(a) \simeq k(a_1, a_2, \dots, a_n)$$

Hence we deduce that $\tau(a)$ is a maximal ideal of U_n for each $a \in B^n$.

Now, let $\mathfrak{m} \in \mathrm{Sp}(U_n)$ be any maximal ideal of U_n. Consider $U_n/\mathfrak{m}$ as a normed k-vector space with the residue norm. Then by Proposition 2.19 $U_n/\mathfrak{m}$ is a finite dimensional k-vector space and therefore there exists an embedding

$$i : U_n/\mathfrak{m} \hookrightarrow \bar{k}$$

By Lemma 3.1 the embedding i is continuous. Set

$$a_i := i(\bar{X}_i)$$

Note that, X_i has norm 1 for each $i = 1, 2, \dots, n$ i.e. $X_i \in U_n^\circ$. Then again by Lemma 3.1 we have:

$$\|i(\bar{X}_i)\| \le |X_i|_\varphi^\gamma = 1$$

Therefore

$$a := (a_1, a_2, \dots, a_n) := (i(\bar{X}_1), i(\bar{X}_2), \dots, i(\bar{X}_n)) \in B^n$$

On the other hand, the canonical map $\phi : U_n \to U_n/\mathfrak{m}$ is continuous (in fact it is uniformly continuous), so that the maps

$$i \circ e_a : U_n \to \bar{k}$$

and

$$i \circ \phi : U_n \to \bar{k}$$

are also continuous maps, and they coincide on $(X_1, X_2, \dots, X_n)$ so that they must be equal and we conclude that $\tau(a) = \mathfrak{m}$. □

Proposition 3.3. *Let $\mathfrak{m} \in \mathrm{Sp}(U_n)$ be a maximal ideal. Set $\mathfrak{m}' := \mathfrak{m} \cap k[X]$. Then:*

(i) $\mathfrak{m}'$ *is a maximal ideal in* $k[X]$ *and* $k[X]/\mathfrak{m}' \simeq U_n/\mathfrak{m}$

(ii) $\mathfrak{m} = \mathfrak{m}'U_n$

where as usual $k[X] = k[X_1, X_2, \ldots, X_n]$ *is the polynomial ring with* n *variables.*

Proof. Using Proposition 3.2, write $\mathfrak{m} = \tau(a)$ for some $a \in B^n$. Define:

$$\phi_a : k[X] \to k(a) \simeq U_n/\mathfrak{m}$$

given by

$$\phi_a(p(X)) = p(a).$$

Then

$$\ker \phi_a = \{p(X) \in k[X] : p(a) = 0\} = \tau(a) \cap k[X] = \mathfrak{m} \cap k[X] = \mathfrak{m}'$$

Hence ϕ_a induces an isomorphism:

$$\bar{\phi_a} : k[X]/\mathfrak{m}' \simeq U_n/\mathfrak{m}$$

and this shows that $\mathfrak{m}'$ is maximal in $k[X]$.

We have the following commutative diagram:

$$\begin{array}{ccc} k[X]/\mathfrak{m}' & \xrightarrow{\ i\ } & U_n/\mathfrak{m}'U_n \\ \downarrow{\scriptstyle \bar{\phi_a}} & \swarrow{\scriptstyle \pi} & \\ U_n/\mathfrak{m} & & \end{array}$$

Note that $\bar{\phi_a}$ is a bijective map, so the map i must be injective and the map π must be surjective.

Since i is injective, the image of the finite dimensional (over k) field $k[X]/\mathfrak{m}'$ under the map i in $U_n/\mathfrak{m}'U_n$ is also a field of finite dimension over k, i.e. if we put $A := i(k[X]/\mathfrak{m}')$ then A is field of finite dimension over k. Therefore it is complete and it implies that A is closed in $U_n/\mathfrak{m}'U_n$.

Moreover, A is dense in $U_n/\mathfrak{m}'U_n$ because $k[X]$ is dense in U_n. As being a closed dense subset we must have $A = U_n/\mathfrak{m}'U_n$ so that i is also a surjection. This implies that the map π is also an injective map, i.e. both of them are isomorphisms. Then we have $\mathfrak{m} = \mathfrak{m}'U_n$. □

Corollary 3.4. *Let* $\mathfrak{m} \in \mathrm{Sp}(U_n)$ *be a maximal ideal. Then there are polynomials* $p_i \in k[X_1, X_2, \ldots, X_i]$ *which are monic in* X_i *for* $i = 1, 2, \ldots, n$ *such that the ideals* $\mathfrak{m}' := \mathfrak{m} \cap k[X_1, \ldots, X_n]$ *and* $\mathfrak{m}$ *are both generated by* $p_1, p_2, \ldots, p_n$*.*

Proof. Let $a = (a_1, a_2, \ldots, a_n) \in B^n(\bar{k})$ be such that $\mathfrak{m} = \tau(a)$ where τ is the function from $B^n(\bar{k})$ to $\mathrm{Sp}(U_n)$ given in the Proposition 3.3.

For $i = 1, 2, \ldots, n$, let $\bar{p_i} \in k(a_1, a_2, \ldots, a_{i-1})[X_i]$ denote the minimal polynomial of a_i over $k(a_1, \ldots, a_{i-1})$. Let $p_i \in k[X_1, X_2, \ldots, X_i]$ be a monic polynomial in X_i such that p_i represents $\bar{p_i}$ under the canonical projection $k[X_1, \ldots, X_i] \to k(a_1, \ldots, a_{i-1})[X_i]$. Note that all inverse images of $\bar{p_i}$, under this canonical morphism, vanish on $a = (a_1, \ldots, a_n)$, because if $p_i \in k[X_1, \ldots, X_i] \subseteq k[X_1, \ldots, X_n]$, then $p_i(a_1, \ldots, a_i, \ldots, a_n) = 0$ by the choice of $\bar{p_i}$. So, since $\mathfrak{m} = \tau(a) = \{f \in U_n : f(a) = 0\}$ for each $i = 1, 2, \ldots, n$ then

$$p_i \in \mathfrak{m}' = \mathfrak{m} \cap k[X_1, \ldots, X_n].$$

We have $k[X_1]/p_1 = k(a_1)$ and by induction it is easy to see that

$$k[X_1, \dots, X_n]/(p_1, \dots, p_n) = k(a_1, \dots, a_n)$$

Thus, $p_1, \dots, p_n$ generate the maximal ideal $\mathfrak{m}'$ in $k[X_1, \dots, X_n]$ and by Proposition 3.3 we conclude that $p_1, \dots, p_n$ also generates $\mathfrak{m} \subset U_n$. $\square$

Remark 3.5. *For T_n the Tate algebra we also have the surjective map $B^n \to \mathrm{Sp}(T_n)$ given by $a \mapsto \tau(a) = \{f \in T_n : f(a) = 0\}$ See [26] Section 7.1 for details.*

Proposition 3.6. *The map $\mathrm{Sp}(T_n) \to \mathrm{Sp}(U_n)$ given by $\mathfrak{m} \mapsto \mathfrak{m} \cap U_n$ is bijective.*

Proof. First, we show that $\mathfrak{m} \cap U_n$ is a maximal ideal of U_n for each $\mathfrak{m} \in \mathrm{Sp}(T_n)$. Choose $a \in B^n$ and write $\mathfrak{m} = \tau(a)$, then $T_n/\mathfrak{m} \simeq k(a)$. Then the morphism $U_n \to T_n/\mathfrak{m}$ defined by $f \mapsto f(a)$ has kernel $\{f \in U_n : f(a) = 0\} = \{f \in T_n : f(a) = 0\} \cap U_n = \mathfrak{m} \cap U_n$, hence we have the induced isomorphism:

$$U_n/\mathfrak{m} \cap U_n \to T_n/\mathfrak{m}$$

and since $T_n/\mathfrak{m}$ is a field is shows that $\mathfrak{m} \cap U_n$ is a maximal ideal of U_n.

Next, we show the injectivity. Suppose $\mathfrak{m} \cap U_n = \mathfrak{m}' \cap U_n$ for some $\mathfrak{m}$ and $\mathfrak{m}'$ in $\mathrm{Sp}(T_n)$. Since $\mathfrak{m} \cap U_n$ and $\mathfrak{m}' \cap U_n$ are both maximal ideals of U_n, there exists a and b in B^n such that $\mathfrak{m} \cap U_n = \tau(a) = \mathfrak{m}' \cap U_n = \tau(b)$. Therefore we have $\{f \in U_n : f(a) = 0\} = \{f \in U_n : f(b) = 0\}$ and it implies that $a = b$.

Finally, we show that the morphism is surjective, i.e. every maximal ideal of U_n can be written of the form $\mathfrak{m} \cap U_n$ for some maximal ideal of T_n. Let I be a maximal ideal of U_n. Choose $a \in B^n$ such that $\tau(a) = I$. Note that $\tau(a)$ considered in T_n gives a maximal ideal, namely $\{f \in T_n : f(a) = 0\}$. Then $I = \tau(a) = \{f \in U_n : f(a) = 0\} = \{f \in T_n : f(a) = 0\} \cap U_n$. Hence I is of the form $\mathfrak{m} \cap U_n$ for some maximal ideal $\mathfrak{m}$ of T_n. $\square$

Corollary 3.7. *We know that the natural map $\mathrm{Sp}(T_n) \to \mathrm{Sp}(W_n)$ is bijective, (see [12] for details). Hence, by the preceeding Proposition, we deduce that the maps*

$$\mathrm{Sp}(U_n) \to \mathrm{Sp}(W_n) \text{ if } W_n \subseteq U_n$$

and

$$\mathrm{Sp}(W_n) \to \mathrm{Sp}(U_n) \text{ if } U_n \subseteq W_n$$

are also bijective.

3.2 Regularity

In this part, we prove that the algebra U_n is a regular ring. We begin by recalling some facts about dimension theory of rings (which can be found in [1]):

Definition 3.8. Let R be a ring and $\mathfrak{p}$ be a prime ideal. We define the height of $\mathfrak{p}$ by

$$\mathrm{ht}(\mathfrak{p}) := \sup\left\{n : \text{ There exists a chain of prime ideals } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n = \mathfrak{p}\right\}$$

and we define the dimension of R by

$$\dim(R) := \sup\left\{ \mathrm{ht}(\mathfrak{p}) : \mathfrak{p} \subset R \text{ is a prime ideal } \right\}$$

Lemma 3.9. *Let A be a local ring with maximal ideal $\mathfrak{m}$. Then the quotient $\mathfrak{m}/\mathfrak{m}^2$ is a vector space over the residue field $A/\mathfrak{m}$. We have $\dim(A) = \mathrm{ht}(\mathfrak{m})$.*

Lemma 3.10. *Let A be a Noetherian ring then $\dim(A) \leq \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$.*

Lemma 3.11. *We have that $\dim(R) = \sup\left\{ \dim R_{\mathfrak{m}} : \mathfrak{m} \text{ is a maximal ideal of } R\right\}$*

Lemma 3.12. *For any maximal ideal $\mathfrak{m}$ of a ring R, we have*

$$R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}} \simeq R/\mathfrak{m}$$

Definition 3.13.

- The Noetherian local ring A is called *regular* if

$$\dim(A) = \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$$

- A Noetherian ring R is called regular if for all prime ideals $\mathfrak{p} \subset R$, the local ring $R_{\mathfrak{p}}$ is regular.

Now, we prove that the localization of U_n by a maximal ideal is a regular ring.

Proposition 3.14. *For every maximal ideal $\mathfrak{m}$ of U_n, the localization $(U_n)_{\mathfrak{m}}$ is a regular ring of dimension n.*

Proof. Recall that the algebra U_n is Noetherian and hence $(U_n)_{\mathfrak{m}}$ is also Noetherian. Consider the polynomials $p_1, p_2, \ldots, p_n$ given in Corollary 3.4. Since $\mathfrak{m}$ can be generated by n elements, we have

$$\dim_{U_n/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 \leq n.$$

On the other hand, the ideals

$$\mathfrak{m}_i := (p_1, p_2, \ldots, p_i) \text{ for } i = 1, 2, \ldots, n$$

are all prime ideals. Thus $\mathrm{ht}(\mathfrak{m}) \geq n$ and we conclude that

$$n \leq \dim\left(U_n\right)_{\mathfrak{m}} \leq \dim_{U_n/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 \leq n.$$

□

Corollary 3.15. $\dim U_n = n$

Proof. The chain of prime ideals

$$\{0\} \subsetneq (X_1) \subsetneq (X_1, X_2) \subsetneq \ldots \subsetneq (X_1, X_2, \ldots, X_n)$$

implies that $\dim U_n \geq n$. On the other hand combining Lemma 3.11 and Proposition 3.14 we see that $\dim U_n \leq n$. □

3.3 de Rham Cohomology

In this section, we aim to prove that the de Rham cohomology of U_n is trivial. We start with giving the definitions of "derivations" and "universal finite differential module", then we will introduce the concepts "de Rham complex" and the cohomology of this complex, the so-called "de Rham cohomology".

We start with a more general setting. Let A be a k-algebra. Let M be a finitely generated A-module. A *derivation* $D : A \to M$ (over k) is a k-linear map satisfying the usual rule:

$$D(ab) = aD(b) + bD(a)$$

or for simplicity

$$D(ab) = aDb + bDa.$$

Note that $D(1) = 1D(1) + 1D(1) = 2D(1)$ i.e. $D(1) = 0$ and hence $D(k) = 0$.

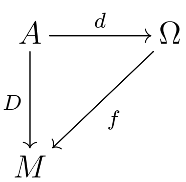
The set of all derivations form an A-module, denoted as $\mathrm{Der}_k(A, M)$. Here we define

$$(aD)(b) = aD(b).$$

By a *universal finite derivation* for A over k, we mean a finite A-module Ω and a derivation

$$d : A \to \Omega$$

such that given any derivation $D : A \to M$ into a finitely generated A-module M, there exists a unique A-homomorphism $f : \Omega \to M$ making the following diagram commutative:



It is a well-known fact that a universal finite derivation (d, Ω) is uniquely determined up to a unique isomorphism, so that we have a functorial isomorphism

$$\mathrm{Der}_k(A, M) \simeq \mathrm{Hom}_A(\Omega, M).$$

Next, we will give the definitions of "de Rham complex" and "de Rham cohomology". To do this we will introduce the concept of "exterior power":

Definition 3.16. Let E be a module over a ring R. Let $T^r(E)$ denote the tensor product of E with itself r-times, i.e. $T^r(E) = E^{\otimes r} = E \otimes \ldots \otimes E$. Let $\mathfrak{a}_r$ be the subspace of the tensor product $T^r(E)$ generated by the elements of the type

$$x_1 \otimes x_2 \otimes \ldots \otimes x_r$$

where $x_i = x_j$ for some $i \neq j$. We define the exterior r-power of E by

$$\bigwedge^r(E) = T^r(E)/\mathfrak{a}_r.$$

Remark 3.17. *We have an r-multilinear canonical map $E^r \to \bigwedge^r(E)$ obtained from the decomposition*

$$E^r \to T^r(E) \to T^r(E)/\mathfrak{a}_r = \bigwedge^r(E).$$

The image of an element $(x_1, \ldots, x_r) \in E^r$ in the above canonical map into $\bigwedge^r(E)$ will be denoted by $x_1 \wedge \ldots \wedge x_r$. It is also the image of $x_1 \otimes \ldots \otimes x_r$ in the factor homomorphism $T^r(E) \to \bigwedge^r(E)$.

Example 3.18. *For $r = 2$, we define*

$$x \wedge y := x \otimes y + \mathfrak{a}_2$$

for $x, y \in E$ where $\mathfrak{a}_2$ is the ideal of $E \otimes E$ generated by the elemets of the form $x \otimes x$.

It follows immediately from the definition that this product is anticommutative on elements of E:

$$\begin{aligned} 0 &= (x + y) \wedge (x + y) \\ &= x \wedge x + x \wedge y + y \wedge x + y \wedge y \\ &= x \wedge y + y \wedge x \end{aligned}$$

i.e. $x \wedge y = -y \wedge x$.

More generally, if σ is a permutation of the set of integers $\{1, 2, \ldots, r\}$ and $x_1, \ldots, x_r$ are elements of E then we have:

$$x_{\sigma(1)} \wedge x_{\sigma(2)} \wedge \ldots \wedge x_{\sigma(r)} = \operatorname{sgn}(\sigma) x_1 \wedge x_2 \wedge \ldots \wedge x_r$$

where sgn is the signature of the permutation σ.

In particular, if $x_i = x_j$ for some $i \neq j$ then we have $x_1 \wedge x_2 \wedge \ldots \wedge x_r = 0$.

The next theorem summarizes what we have done. It is taken from the book [20] (Chapter XIX, Proposition 1.1).

Theorem 3.19. *Let E be a free module of rank n over a commutative ring R. If $r > n$ then $\bigwedge^r(E) = 0$. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of E over R. If $1 \leq r \leq n$ then $\bigwedge^r(E)$ is free over R and the elements*

$$v_{i_1} \wedge \ldots \wedge v_{i_r}, \ i_1 < \ldots < i_r$$

form a basis of $\bigwedge^r(E)$ over R. We have $\dim_R \bigwedge^r(E) = \binom{n}{r}$.

Next we introduce the de Rham complex and de Rham cohomology with the following theorem taken from [20] (Chapter XIX, Theorem 3.2).

Theorem 3.20. *Let A be a k-algebra. For $i \geq 0$ define*

$$\Omega^i_{A/k} := \bigwedge^i \Omega^1_{A/k}$$

where $\Omega^0_{A/k} = A$ and $\Omega^1_{A/k}$ is the universal finite derivation of A. Then there exists a unique sequence of k-homomorphisms

$$d_i : \Omega^i_{A/k} \to \Omega^{i+1}_{A/k}$$

such that for $\omega \in \Omega^i$ and $\eta \in \Omega^j$ we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^i \omega \wedge d\eta$$

Furthermore $d \circ d = 0$.

Definition 3.21. Recall that a complex of modules is a sequence of homomorphisms

$$\cdots \to E^{i-1} \xrightarrow{d_{i-1}} E^i \xrightarrow{d_i} E^{i+1} \to \cdots$$

such that $d_i \circ d_{i-1} = 0$. One usually omits the subscript on the map d. With this terminology we see that the sequence $\Omega^i_{A/k}$ for $i \geq 0$ given in Theorem 3.20 form a complex, called *de Rham complex*:

$$0 \to A = \Omega^0_{A/k} \xrightarrow{d_0} \Omega^1_{A/k} \xrightarrow{d_1} \Omega^2_{A/k} \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} \Omega^n_{A/k} \xrightarrow{d_n} 0$$

The cohomology of this complex is called the *de Rham cohomology*, denoted by $\mathrm{H}^*_{\mathrm{dR}}(\,.\,)$, i.e.

$$\mathrm{H}^0_{\mathrm{dR}}(A/k) = \ker d_0$$
$$\mathrm{H}^k_{\mathrm{dR}}(A/k) = {}^{\ker d_k}/_{\mathrm{im}\, d_{k-1}} \text{ for } k \geq 1.$$

For clarification, we will explicitely give the maps d_i's given in Definition 3.21:
First of all the map $d_0 : A \to \Omega^1_{A/k}$ is the universal derivation.
The map d_1 is induced by the map

$$a \otimes b \mapsto (ab, adb)$$

where d is the universal derivation $d : A \to \Omega^1_{A/k}$ given by

$$a \mapsto 1 \otimes a - a \otimes 1$$

We define the map

$$d_1 : \Omega^1_{A/k} \to \Omega^2_{A/k} = \bigwedge^2 \Omega^1_{A/k}$$

by

$$adb \mapsto da \wedge db.$$

We use the map d_0 to define the higher powers. To get the k-linear map

$$d_n : \bigwedge^n \Omega^1_{A/k} = \Omega^n_{A/k} \to \Omega^{n+1}_{A/k} = \bigwedge^{n+1} \Omega^1_{A/k}$$

we define

$$d_n : \sum f_{i_1,\ldots,i_n} dX_{i_1} \wedge \ldots \wedge dX_{i_n} \mapsto \sum d_0(f_{i_1,\ldots,i_n}) \wedge dX_{i_1} \wedge \ldots \wedge dX_{i_n}.$$

Now, we can establish the universal finite derivation for the algebra U_n:

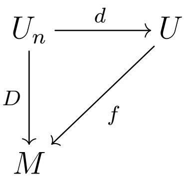
Proposition 3.22. *The universal finite derivation d for the algebra $U_n = k\langle X_1, X_2, \ldots, X_n\rangle^\varphi$ is given by the U_n-module:*

$$\Omega^1_{U_n/k} := U_n dX_1 \oplus U_n dX_2 \oplus \ldots \oplus U_n dX_n$$

and the map d is defined by the sum of the partial derivatives with respect to each X_i:

$$d(f) = df := \sum_{i=1}^{n} \frac{\partial f}{\partial X_i} dX_i.$$

Proof. We claim that given any derivation $D : U_n \to M$ where M is a finitely generated U_n-module, there exists a unique U_n-linear map $f : U \to M$ such that the following diagram commutes.



Note that the map f is U_n linear so we only need to define it on the basis elements $dX_1, dX_2, \dots, dX_n$. Define:

$$f(dX_i) = D(X_i) \text{ for } i = 1, 2, \dots, n.$$

Also define

$$E := D - f \circ d : U_n \to M$$

We will show that the map E is in fact the zero map and this would imply the commutativity of the above diagram.

Firstly, notice that

$$E(X_i) = D(X_i) - (f \circ d)(X_i) = D(X_i) - D(X_i) = 0$$

for $i = 1, 2, \dots, n$. Also notice that:

$$\begin{aligned} E(ab) &= (D - f \circ d)(ab) \\ &= D(ab) - (f \circ d)(ab) \\ &= aDb + bDa - f(adb + bda) \\ &= aDb + bDa - af(db) - bf(da) \\ &= a(Db - f(db)) + b(Da - f(da)) \\ &= a(D - f \circ d)(b) + b(D - f \circ d)(a) \\ &= aEb + bEa \end{aligned}$$

Hence we deduce that E is also a derivation.

Since $E(X_i) = 0$ for $i \in \{1, 2, \dots, n\}$, we deduce that the product rule (of being a derivation) on E implies that $E = 0$ on the polynomial algebra $k[X_1, \dots, X_n]$. To see this, let $\sum_{i=1}^r a_i X_1^{i_1} \dots X_n^{i_r}$ be arbitrary in $k[X_1, \dots, X_n]$. Then:

$$\begin{aligned} E(f) &= E\Big(\sum_{i=1}^r a_i X_1^{i_1} \dots X_n^{i_r}\Big) \\ &= \sum_{i=1}^r a_i E(X_1^{i_1} \dots X_n^{i_r}) = 0 \end{aligned}$$

Let $\mathfrak{m}$ be the ideal in U_n generated by $X_1, \dots, X_n$, i.e. $\mathfrak{m} = \langle X_1, \dots, X_n \rangle$. Note that $U_n/\mathfrak{m} = k$, therefor $\mathfrak{m}$ is maximal. We will prove that the image of U_n under the map E is contained in the submodule $\mathfrak{m}^r M$ for all $r \geq 1$. For this let $f \in U_n$ be arbitrary and $r \in \mathbb{N}$.

Write:

$$f = \sum_{\substack{u=(u_1,\dots,u_n) \\ u \in \mathbb{Z}^n_{\geq 0}}} a_u X_1^{u_1} \dots X_n^{u_n}$$

$$
\begin{aligned}
&= \sum_{\substack{u=(u_1,\dots,u_n)\\ u_i\le r}} a_u X_1^{u_1}\dots X_n^{u_n} + \sum_{\substack{u=(u_1,\dots,u_n)\\ u_i> r}} a_u X_1^{u_1}\dots X_n^{u_n}\\
&= \sum_{\substack{u=(u_1,\dots,u_n)\\ u_i\le r \text{ for all } i}} a_u X_1^{u_1}\dots X_n^{u_n} + X_1^r\dots X_n^r \sum_{\substack{u=(u_1,\dots,u_n)\\ u_i> r \text{ for all } i}} a_u X_1^{u_1-r}\dots X_n^{u_n-r}\\
&= g + X_1^r\dots X_n^r h
\end{aligned}
$$

where

$$
g := \sum_{\substack{u=(u_1,\dots,u_n)\\ u_i\le r \text{ for all } i}} a_u X_1^{u_1}\dots X_n^{u_n} \text{ and } h := \sum_{\substack{u=(u_1,\dots,u_n)\\ u_i> r \text{ for all } i}} a_u X_1^{u_1-r}\dots X_n^{u_n-r}.
$$

Note that g is a polynomial in $k[X_1,\dots,X_n]$ and $h\in U_n$. Now, since $E=0$ on the polynomial algebra $k[X_1,\dots,X_n]$ we have:

$$
\begin{aligned}
E(f) &= E(g+X_1^r\dots X_n^r h)\\
&= E(g)+E(X_1^r\dots X_n^r h)\\
&= 0 + X_1^r\dots X_n^r E(h) + E(X_1^r\dots X_n^r)h\\
&= X_1^r\dots X_n^r E(h) + 0h\\
&= (X_1\dots X_n)^r E(h)\in \mathfrak{m}^r M
\end{aligned}
$$

Hence $E(U_n)\subseteq \mathfrak{m}^r M$ for all $r\ge 1$.

Now, since $\mathfrak{m}$ is maximal we have the following descending chain:

$$
\mathfrak{m} \supsetneq \mathfrak{m}^2 \supsetneq \dots \supsetneq \mathfrak{m}^r \supsetneq \dots
$$

Thus we get the following descending chain on the finitely generated (in other words Noetherian) module M:

$$
\mathfrak{m}M \supsetneq \mathfrak{m}^2 M \supsetneq \dots \supsetneq \mathfrak{m}^r M \supsetneq \dots
$$

Since M is Noetherian, the intersection

$$
\bigcap_{r\ge 1} \mathfrak{m}^r M = 0.
$$

And since $E(U_n)\subseteq \mathfrak{m}^r M$ for all $r\ge 1$, we have:

$$
E(U_n)\subseteq \bigcap_{r\ge 1} \mathfrak{m}^r M = 0
$$

which implies that $E=0$.

□

Corollary 3.23. *Let $\mathcal{N}$ be an nmk algebra. Suppose $\mathcal{N}\simeq U_n/I$ for some n and $I\trianglelefteq U_n$ with $I=\langle f_1,\dots,f_r\rangle$. Then the universal finite differential module for $\mathcal{N}$ is given by*

$$
\bigoplus_{i=1}^n U_n dX_i \Big/ \bigoplus_{i=1}^r U_n df_i
$$

where d is the universal finite differential map $d: U_n\to \bigoplus_{i=1}^n U_n dX_i$ given in Theorem 3.22.

Proof. Set
$$\Omega_{\mathcal{N}/k} := \mathcal{N} \otimes_{U_n} \left(\bigoplus_{i=1}^{n} U_n dX_i \Big/ \bigoplus_{i=1}^{r} U_n df_i \right).$$
We have the derivation $D : \mathcal{N} \to \Omega_{\mathcal{N}/k}$, derived from:
$$U_n \xrightarrow{d} \bigoplus_{i=1}^{n} U_n dX_i \to \Omega_{\mathcal{N}/k}.$$
We observe that D maps the ideal $\langle f_1, \ldots, f_n \rangle$ to 0.

The induced derivation D can be shown, in the same way as done for U_n in Theorem 3.22, to have the universal property. □

Next, we remark an easy observation:

Lemma 3.24. *Formal integration in $U_n = k\langle X_1, \ldots, X_n \rangle^{\varphi}$ exists with respect to any X_i.*

Proof. Let
$$f = \sum_{\substack{u \in \mathbb{Z}^n_{\geq 0} \\ u=(u_1,\ldots,u_n)}} a_u X_1^{u_1} \ldots X_n^{u_n} \in U_n$$
Then
$$\int f dX_i = \sum_{\substack{u \in \mathbb{Z}^n_{\geq 0} \\ u=(u_1,\ldots,u_n)}} \frac{a_u}{u_i + 1} X_1^{u_1} \ldots X_i^{u_i+1} \ldots X_n^{u_n}$$
By Lemma 0.6 the values $\|\frac{1}{u_i+1}\|^{\varphi(|u|+1)}$ are bounded so that we get:
$$\lim \left\| \frac{a_u}{u_i + 1} \right\|^{\varphi(|u|+1)} = 0 \text{ as } |u| \to \infty.$$
□

Now, we apply Proposition 3.22 to $U_1 = k\langle X_1 \rangle^{\varphi}$ and confirm what we have already realized in Lemma 0.7:

Corollary 3.25. *$\mathrm{H}^*_{dR}(U_1/k)$ is trivial.*

Proof. To calculate the de Rham cohomology groups we consider the following de Rham complex:
$$0 \to \Omega^0_{U_1/k} \xrightarrow{d_0} \Omega^1_{U_1/k} \xrightarrow{d_1} 0$$
Note that by definition $\Omega^0_{U_1/k} = U_1$ and by Proposition 3.22 we know that $\Omega^1_{U_1/k} = U_1 dX_1$. The map d_0 is the universal derivation of U_1 given in Proposition 3.22:
$$d_0(f) = \frac{\partial f}{\partial X_1} dX_1.$$
And the map d_1 is clearly the 0-map.

By definition:
$$\mathrm{H}^0_{\mathrm{dR}}(U_1/k) = \ker d_0$$
Note that if $f \in \ker d_0$ then
$$\frac{\partial f}{\partial X_1} dX_1 = 0 \text{ i.e. } \frac{\partial f}{\partial X_1} = 0$$

which implies that f is constant in k. Hence $\ker d_0 = k$, that is:

$$\mathrm{H}^0_{\mathrm{dR}}(U_1/k) = k.$$

Again by definition,

$$\mathrm{H}^1_{\mathrm{dR}}(U_1/k) = {}^{\ker d_1}/_{\operatorname{im} d_0}$$

By Lemma 0.7, the map d_0 is a surjection i.e. $\operatorname{im} d_0 = U_1 dX_1$. Hence:

$$\mathrm{H}^1_{\mathrm{dR}}(U_1/k) = {}^{U_1 dX_1}/_{U_1 dX_1} = 0$$

Note also that since the universal differential module $U_1 dX_1$ has dimension 1 over U_1, by Theorem 3.19 we deduce that

$$\mathrm{H}^n_{\mathrm{dR}}(U_1/k) = 0 \text{ for all } n > 1.$$

□

Now, we proceed with the general case:

Theorem 3.26. $\mathrm{H}^*_{dR}(U_n/k)$ *is trivial. More precisely, the following de Rham chain complex has trivial cohomology groups (or in other words it is exact):*

$$0 \to k \hookrightarrow \Omega^0_{U_n/k} \xrightarrow{d_0} \Omega^1_{U_n/k} \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Omega^n_{U_n/k} \xrightarrow{d_n} 0$$

where

$$U_n = k\langle X_1, \ldots, X_n\rangle^{\varphi},$$
$$\Omega^0_{U_n/k} = U_n,$$
$$\Omega^1_{U_n/k} = \bigoplus_{i=1}^{n} U_n dX_i$$

and

$$\Omega^r_{U_n/k} = \bigwedge_{i=1}^{r} \Omega^1_{U_n/k}.$$

The universal finite differential map $d_0 : U_n \to \bigoplus_{i=1}^n U_n dX_i$ *is given by*

$$f \mapsto \sum_{i=1}^{n} \frac{\partial f}{\partial X_i} dX_i$$

and for $i \geq 1$, $d_i : \Omega^i_{U_n/k} \to \Omega^{i+1}_{U_n/k}$ *is given by*

$$\sum f_{i_1 \ldots i_r} dX_{i_1} \wedge \ldots \wedge dX_{i_r} \mapsto \sum d_0(f_{i_1 \ldots i_r}) \wedge dX_{i_1} \wedge \ldots \wedge dX_{i_r}.$$

Proof. By induction on n. If $n = 0$:

$$\mathrm{H}^0_{\mathrm{dR}}(U_n/k) = {}^{\ker d_0}/_{\operatorname{im}(k \hookrightarrow U_n)}$$

For $f \in U_n$ with $d_0(f) = 0$ we have:

$$\sum_{i=1}^{n} \frac{\partial f}{\partial X_i} dX_i = 0$$

and since dX_i's are U_n-linearly independent in $\Omega^1_{U_n/k}$ we get $\frac{\partial f}{\partial X_i} = 0$ for $i = 1, \ldots, n$ which implies that $f \in k$. Hence

$$\ker d_0 / \mathrm{im}(k \hookrightarrow U_n) = k/k = 0$$

This also means that

$$\mathrm{H}^0_{\mathrm{dR}}(U_n/k) = \ker d_0 = k.$$

Now, let $\omega \in \Omega^r_{U_n/k}$ be such that $d_r(\omega) = 0$. Note that

$$\mathrm{H}^r_{\mathrm{dR}}(U_n/k) = \ker d_r / \mathrm{im}\, d_{r-1}$$

To show that this cohomology group is trivial we need to show that $\ker d_r = \mathrm{im}\, d_{r-1}$. By construction of the de Rham chain complex we have $\mathrm{im}\, d_i \subseteq \ker d_{i+1}$ for all i. So, we only need to show that $\ker d_r \subseteq \mathrm{im}\, d_{r-1}$. Let ω be arbitrary in $\ker d_r$. To prove the above inclusion, we must find $\eta \in \Omega^{r-1}_{U_n/k}$ such that $d_{r-1}(\eta) = \omega$.

Write $\omega = \omega' dX_1 + \omega''$ where ω' and ω'' are in $\Omega^r_{U_n/k}$ with no dX_1-term involved. Define:

$$\eta' = \int \omega' dX_1$$

where the integration is taken formally, i.e. the algebraic process of replacing X_1^r term by $\frac{X_1^{r+1}}{r+1}$ whenever it occurs. Then $\eta' \in \Omega^{r-1}_{U_n/k}$ and

$$d_{r-1}(\eta') = \omega' dX_1 + \omega'''$$

where ω''' has no dX_1-term involved.

Now consider:

$$\begin{aligned} \omega - d_{r-1}(\eta') &= (\omega' dX_1 + \omega'') - (\omega' dX_1 + \omega''') \\ &= \omega'' - \omega''' \end{aligned}$$

It does not involve any dX_1-term.

Note that if we can fin $\eta'' \in \Omega^{r-1}_{U_n/k}$ such that

$$d_{r-1}(\eta'') = \omega - d_{r-1}(\eta')$$

then we get

$$d_{r-1}(\eta' + \eta'') = \omega$$

and setting $\eta = \eta' + \eta''$ we reach the desired result. Hence we can replace ω by $\omega - d_{r-1}(\eta')$ and without loss of generality suppose that ω does not contain any dX_1-term.

The we can write:

$$\omega = \sum f_{i_1,\ldots,i_r} dX_{i_1} \wedge \ldots \wedge dX_{i_r}$$

where $i_j > 1$ for $j = 1, \ldots, r$.

Since $d_r(\omega) = 0$ we have

$$\sum d_0(f_{i_1,\ldots,i_r}) \wedge dX_{i_1} \wedge \ldots \wedge dX_{i_r} = 0.$$

Hence:

$$\frac{\partial f_{i_1,\ldots,i_r}}{\partial X_1} = 0$$

for all possible $i_1, \ldots, i_r$. This means that the set of power series $f_{i_1,\ldots,i_r}$ do not contain any X_1-term and therefore we reduce the case of the algebra U_{n-1}, more precisely $k\langle X_2, X_3, \ldots, X_n\rangle^{\varphi}$. And by induction hyphothesis there exists η such that

$$d_{r-1}(\eta) = \omega.$$

□

Next as an example, we will establish the result for the algebra $U_2 = k\langle X_1, X_2\rangle^{\varphi}$ without using the induction procedure:

Example 3.27. *$\mathrm{H}^*_{dR}(U_2/k)$ is trivial.*

Proof. We have the following de Rham complex:

$$0 \to U_2 \xrightarrow{d_0} U := U_2 dX_1 + U_2 dx_2 \xrightarrow{d_1} U_2 dX_1 \wedge dX_2 \xrightarrow{d_2} 0$$

Note that by Theorem 3.19, the space $\Omega^2_{U_2/k}$ has rank 1 over U_2 since the universal differential module U has rank 2 over U_2, and since it is generated by elements of the form $dX_1 \wedge dX_2$, we get $\Omega^2_{U_2/k} = U_2 dX_1 \wedge dX_2$.

Again by Theorem 3.19, we know that $\mathrm{H}^r_{dR}(U_2/k) = 0$ for all $r > 2$.

The map d_0 is the universal derivation on U_2 given by:

$$f \mapsto df = \frac{\partial f}{\partial X_1} dX_1 + \frac{\partial f}{dX_2} dX_2.$$

By definition $\mathrm{H}^0_{dR}(U_2/k) = \ker d_0$, if $f \in \ker d_0$ then we get

$$\frac{\partial f}{\partial X_1} dX_1 + \frac{\partial f}{dX_2} dX_2 = 0$$

and since dX_1 and dX_2 are U_2-linearly independent we deduce that

$$\frac{\partial f}{\partial X_1} = \frac{\partial f}{\partial X_2} = 0$$

which means $f \in k$, i.e. $\ker d_0 = k$ hence $\mathrm{H}^0_{dR}(U_2/k) = k$

Also for the last cohomology group:

$$\mathrm{H}^2_{dR}(U_2/k) = {}^{\ker d_2}\!/_{\mathrm{im}\, d_1}.$$

Here the map d_2 is the 0-map and the map d_1 is is surjective by:

For any $f \in U_2$, set $h := \int f dX_1$. Note that $\frac{\partial h}{\partial X_1} = f$. Then we have:

$$hdX_2 \mapsto \Big(\frac{\partial h}{\partial X_1} dX_1 + \frac{\partial h}{\partial X_2} dX_2\Big) \wedge dX_2 = \frac{\partial h}{\partial X_1} dX_1 \wedge dX_2 = f dX_1 \wedge dX_2$$

so that $d_1(hdX_2) = f dX_1 \wedge dX_2$ which implies that the map d_1 is surjective hence the second cohomology group is

$$\mathrm{H}^2_{dR}(U_2/k) = 0.$$

Next:

$$\mathrm{H}^1_{dR}(U_2/k) = {}^{\ker d_1}\!/_{\mathrm{im}\, d_0}.$$

We will show that $\ker d_1 = \operatorname{im} d_0$ and this will imply that the first cohomology group is also trivial. Recall that by construction $\operatorname{im} d_0 \subseteq \ker d_1$.

For the converse:

$$\operatorname{im} d_0 = \Big\{ \frac{\partial f}{\partial X_1} dX_1 + \frac{\partial f}{dX_2} dX_2 : f \in U_n \Big\}$$

To find $\ker d_1$ suppose $f dX_1 + g dX_2 \in \ker d_1$, then:

$$\begin{aligned}
0 &= d_1(f dX_1 + g dX_2) \\
&= df \wedge dX_1 + dg \wedge dX_2 \\
&= \Big(\frac{\partial f}{\partial X_1} dX_1 + \frac{\partial f}{dX_2} dX_2 \Big) \wedge dX_1 + \Big(\frac{\partial g}{\partial X_1} dX_1 + \frac{\partial g}{dX_2} dX_2 \Big) \wedge dX_2 \\
&= \frac{\partial f}{\partial X_2} dX_2 \wedge dX_1 + \frac{\partial g}{\partial X_1} dX_1 \wedge dX_2 \\
&= \Big(\frac{\partial f}{\partial X_2} - \frac{\partial g}{\partial X_1} \Big) dX_1 \wedge dX_2
\end{aligned}$$

i.e. if $f dX_1 + g dX_2 \in \ker d_1$ we get:

$$\frac{\partial f}{\partial X_2} = \frac{\partial g}{\partial X_1}$$

Set:

$$f = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{(n,m)} X_1^n X_2^m \quad \text{and} \quad g = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_{(n,m)} X_1^n X_2^m$$

Then:

$$\frac{\partial f}{\partial X_2} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} m a_{(n,m)} X_1^n X_2^{m-1} = \frac{\partial g}{\partial X_1} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} n b_{(n,m)} X_1^{n-1} X_2^m.$$

Therefore for all n and m, we have:

$$\frac{a_{(n,m+1)}}{n+1} = \frac{b_{(n+1,m)}}{m+1} \tag{3.1}$$

We need to find $h \in U_2$ such that $d_0(h) = \frac{\partial h}{\partial X_1} + \frac{\partial h}{\partial X_2} = f dX_1 + g dX_2$ i.e.

$$\Big(\frac{\partial h}{\partial X_1} - f \Big) dX_1 = \Big(\frac{\partial h}{\partial X_2} - g \Big) dX_2$$

which means:

$$\frac{\partial h}{\partial X_1} = f \text{ and } \frac{\partial h}{\partial X_2} = g$$

Let h_1 and h_2 be the formal integrations of f and g with respect to X_1 and X_2, respectively:

$$h_1 = \int f dX_1 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_{(n,m)}}{n+1} X_1^{n+1} X_2^m \quad \text{and} \quad h_2 = \int g dX_2 = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{b_{(n,m)}}{m+1} X_1^n X_2^{m+1}$$

Note that $\frac{\partial h_1}{\partial X_1} = f$ and $\frac{\partial h_2}{\partial X_2} = g$.

Note also that:

$$\frac{\partial h_2}{\partial X_1} = \frac{\partial}{\partial X_1}\Big(\int g dX_2\Big) = \frac{\partial}{\partial X_1}\Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{b_{(n,m)}}{m+1}X_1^n X_2^{m+1}\Big)$$
$$= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{nb_{(n,m)}}{m+1}X_1^{n-1}X_2^{m+1}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(n+1)b_{(n+1,m)}}{m+1}X_1^n X_2^{m+1}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{(n,m+1)}X_1^n X_2^{m+1}\Big(\text{ by } 3.1 : \frac{(n+1)b_{(n+1,m)}}{m+1} = a_{(n,m+1)}\Big)$$
$$= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}a_{(n,m)}X_1^n X_2^m$$
$$= f - \sum_{n=0}^{\infty}a_{(n,0)}X_1^n$$

Similarly:

$$\frac{\partial h_1}{\partial X_2} = \frac{\partial}{\partial X_2}\Big(\int f dX_1\Big) = g - \sum_{m=0}^{\infty}b_{(0,m)}X_2^m.$$

Now, define:

$$h = \frac{1}{2}\Big(h_1 + h_2 + \sum_{n=0}^{\infty}\frac{a_{(n,0)}}{n+1}X_1^{n+1} + \sum_{m=0}^{\infty}\frac{b_{(0,m)}}{m+1}X_2^{m+1}\Big)$$

Then:

$$\frac{\partial h}{\partial X_1} = \frac{1}{2}\Big(\frac{\partial h_1}{\partial X_1} + \frac{\partial h_2}{\partial X_1} + \sum_{n=0}^{\infty}a_{(n,0)}X_1^n + 0\Big)$$
$$= \frac{1}{2}\Big(f + (f - \sum_{n=0}^{\infty}a_{(n,0)}X_1^n) + \sum_{n=0}^{\infty}a_{(n,0)}X_1^n\Big) = \frac{1}{2}(f+f) = f$$

And similarly:

$$\frac{\partial h}{\partial X_2} = g.$$

This shows that $f dX_1 + g dX_2$ lies in $\operatorname{im}(d_0)$ which concludes that $\ker d_1 \subseteq \operatorname{im} d_0$ and therefore

$$\mathrm{H}^1_{dR}(U_2/k) = {}^{\ker d_1}/_{\operatorname{im} d_0} = 0$$

□

We observe that the main argument we are using to prove Theorem 3.26 is that the space we are working on admits integration, i.e. the process of formal integration gives again power series of the same type. We can use this observation conjecture a more general result:

Question 3.28. *Let $\mathcal{N}$ be an nmk algebra. Is it true that $\mathcal{N}$ and $\mathcal{N}\hat{\otimes}U_1$ have the same de Rham cohomology groups?*

Appendix A

Relation to W_n

First we note that we assume that the characteristic of k is 0, then by Ostrowski's Theorem (some reference here) we know that the non-Archimedean absolute value on k is equivalent to the induced p-adic norm for some p prime, i.e. we assume that $\mathbb{Q}_p \subseteq k$.

Recall the definition of *Washnitzer Algebra*:

$$W_n = \left\{ \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_u X^u \in k[[X]] : \text{ there exists } \rho > 1 \text{ such that } \|a_u\| \rho^{|u|} \to 0 \text{ as } |u| \to \infty \right\}$$

where $k[[X]] := k[[X_1, X_2, \dots, X_n]]$ and $\rho \in \mathbb{R}_{>1}$.

In this part we will give a criterion for the inclusion $W_n \subseteq U_n$. The next remark shows that for this purpose we need another condition on the function φ.

Remark 3.29. *Define:*

$$\varphi(u) = \frac{1}{|u|} \text{ for } u \in \mathbb{Z}^n_{\geq 0}.$$

Then $\varphi(u)$ is a decreasing function bounded above by the function $\frac{1}{\log |u|}$ and

$$\frac{\varphi(2u)}{\varphi(u)} = \frac{|u|}{|2u|} = \frac{1}{2} > 0,$$

so that φ satisfies all the conditions given in Definition 1.1 but still W_n is not a subset of U_n.

Proof. We need to give a counter-example. Take $k = \mathbb{Q}_p$. We only need to show it for $n = 1$. We claim that the element $f = \sum_{n=0}^{\infty} p^n X^n$ is an element of W_1, but not of U_1. Take $\rho = \sqrt{p} > 1$, then:

$$\lim_{n\to\infty} \|p^n\| \rho^n = \lim_{n\to\infty} \frac{1}{p^n} \left(\sqrt{p}\right)^n = \lim_{n\to\infty} \left(\frac{\sqrt{p}}{p}\right)^n = \lim_{n\to\infty} \left(\frac{1}{\sqrt{p}}\right)^n = 0$$

Hence $f \in W_1$, on the other hand:

$$\lim_{n\to\infty} \|p^n\|^{\varphi(n)} = \left(\frac{1}{p^n}\right)^{\frac{1}{n}} = \frac{1}{p} \neq 0$$

Hence, $f \notin U_1$. □

So, we need an extra condition on the function φ, the next proposition gives this condition:

Proposition 3.30. *$W_n \subset U_n$ if and only if* $\lim_{|u|\to\infty} |u|\,\varphi(u) = \infty$

Proof. ($\Longrightarrow$) Note that $\mathbb{Q}_p \subseteq k$. Then the series $f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} p^{|u|}X^u$ belongs to W_n. Taking $\rho = \sqrt{p}$ we see that:

$$\lim_{|u|\to\infty} \|p^{|u|}\|\,\rho^{|u|} = \lim_{|u|\to\infty} \left(\frac{1}{\sqrt{p}}\right)^{|u|} = 0.$$

We know that $W_n \subset U_n$, i.e. $f \in U_n$. Then:

$$\lim_{|u|\to\infty} \|p^{|u|}\|^{\varphi(u)} = 0$$

So,

$$\lim_{|u|\to\infty} \|p^{|u|}\|^{\varphi(u)} = \lim_{|u|\to\infty} \left(\frac{1}{p^{|u|}}\right)^{\varphi(u)} = \lim_{|u|\to\infty} \frac{1}{p^{|u|\,\varphi(u)}} = 0 \Longleftrightarrow \lim_{|u|\to\infty} p^{|u|\,\varphi(u)} = \infty$$

which implies that $|u|\varphi(u)$ diverges to infinity as $|u|$ goes to infinity.

($\Longleftarrow$) Suppose $\lim_{|u|\to\infty} |u|\,\varphi(u) = \infty$. Let $f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u$ be any element of W_n. Then there exists $\rho > 1$ such that

$$\lim_{|u|\to\infty} \|a_u\|\rho^{|u|} = 0.$$

We want to show that $\lim_{|u|\to\infty} \|a_u\|^{\varphi(u)} = 0$.

Let $\varepsilon \in \mathbb{R}_{>0}$ be arbitrary.

Since $f \in W_n$, there exists $u_0 \in \mathbb{Z}^n_{\geq 0}$ such that for all $|u| > |u_0|$, we have:

$$\|a_u\|\rho^{|u|} < \varepsilon \quad \text{i.e.} \quad \|a_u\| < \frac{\varepsilon}{\rho^{|u|}}$$

Then for all $|u| > |u_0|$ we have:

$$\lim_{|u|\to\infty} \|a_u\|^{\varphi(u)} \leq \lim_{|u|\to\infty} \left(\frac{\varepsilon}{\rho^{|u|}}\right)^{\varphi(u)} = \lim_{|u|\to\infty} \frac{\varepsilon}{\rho^{|u|\,\varphi(u)}} = 0$$

Hence $W_n \subseteq U_n$. ∎

Finally, we give an example of a U_n algebra which is indeed a structure lying between T_n and W_n:

Example 3.31. *Let $q > 1$ be any real number. Set*

$$\varphi(u) = \frac{1}{\log_q |u|} = \frac{\log q}{\log |u|}$$

for $|u| \neq 0, 1$ with $\varphi(\bar{0}) = 1$ and $\varphi(u) = 1$ for all $|u| = 1$. Then the inclusions $W_n \subseteq U_n \subseteq T_n$ are strict.

Proof. First, we will prove that $U_n \subsetneq T_n$. It is enough to give an example for $n = 1$. We need to find a power series f such that $f \in T_1$ but $f \notin U_1$. Take $k = \mathbb{Q}_p$ and set

$$f := \sum_{n\in\mathbb{N}} p^{\lfloor \log n \rfloor} X^n$$

where $\lfloor . \rfloor : \mathbb{R} \to \mathbb{Z}$ is the floor function on $\mathbb{R}$ given by $\lfloor x \rfloor = \max_{n\in\mathbb{Z}}\{n : n \leq x\}$. We claim that $f \in T_1$ but $f \notin U_1$. Since $\log n \to \infty$ as $n \to \infty$ we deduce that:

$$\lim_{n\to\infty} \left\| p^{\lfloor \log(n) \rfloor} \right\| = \lim_{n\to\infty} \frac{1}{p^{\lfloor \log n \rfloor}} = 0$$

Hence $f \in T_1$. On the other hand:

$$\lim_{n\to\infty} \left\| p^{\lfloor \log n \rfloor} \right\|^{\frac{1}{\log_q n}} = \lim_{n\to\infty} \left(\frac{1}{p^{\lfloor \log n \rfloor}} \right)^{\frac{\log q}{\log n}} = \lim_{n\to\infty} \frac{1}{p^{\lfloor \log n \rfloor \frac{\log q}{\log n}}} \geq \frac{1}{p^{\log q}}$$

Hence $f \notin U_1$.

Now, we will prove the strictness of the inclusion $W_n \subsetneq U_n$. It is similarly enough to give an example for $n = 1$. Set

$$f = \sum_{n\in\mathbb{N}} p^{\lfloor \frac{n}{\log n} \rfloor} X^n$$

We claim that $f \in U_1$ and $f \notin W_1$. First, we show that $f \notin W_1$. Let $\rho > 1$ be any real number. Note that, since the function $\frac{1}{\log n}$ tends to zero, as n goes to infinity, for all n large enough $p^{\frac{1}{\log n}} < \rho$, so that $\frac{\rho}{p^{\frac{1}{\log n}}} > 1$. Then, we have:

$$\lim_{n\to\infty} \|p^{\lfloor \frac{n}{\log n} \rfloor}\| \rho^n = \lim_{n\to\infty} \frac{1}{p^{\lfloor \frac{n}{\log n} \rfloor}} \rho^n \geq \lim_{n\to\infty} \frac{\rho^n}{p^{\frac{n}{\log n}}} = \lim_{n\to\infty} \left(\frac{\rho}{p^{\frac{1}{\log n}}} \right)^n > 1$$

Hence $f \notin W_1$.

Now, we claim that $f \in U_1$:

$$\lim_{n\to\infty} \|p^{\lfloor \frac{n}{\log n} \rfloor}\|^{\frac{1}{\log_q n}} = \lim_{n\to\infty} \left(\frac{1}{p^{\lfloor \frac{n}{\log n} \rfloor}} \right)^{\frac{\log q}{\log n}} \leq \lim_{n\to\infty} \left(\frac{1}{p^{\frac{n}{\log n} - 1}} \right)^{\frac{\log q}{\log n}} = \lim_{n\to\infty} \frac{p^{\frac{\log q}{\log n}}}{p^{\frac{n \log q}{\log^2 n}}} = 0$$

Hence, $f \in U_1$ and we deduce that $W_n \subsetneq U_n$. □

In fact, the algebra U_n might be contained in the algebra W_n. The next result gives a criteria for such an inclusion:

Proposition 3.32. *$U_n \subset W_n$ if and only if the function $|u|\varphi(u)$ is bounded.*

Proof. ($\Longleftarrow$) Suppose

$$|u|\varphi(u) < M$$

for some $M \in \mathbb{N}$. Let $f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u$ be any power series in U_n. We will show that $f \in W_n$, i.e. there exists $\rho > 1$ such that $\lim_{|u|\to\infty} \|a_u\|\rho^{|u|} = 0$.

Since $f \in U_n$, $\lim_{|u|\to\infty} \|a_u\|^{\varphi(u)} = 0$ so that for all $|u|$ large enough, we have: $\|a_u\|^{\varphi(u)} < \frac{1}{p}$. Set $\rho = p^{\frac{1}{M+1}}$. Then for all $|u|$ large enough we have:

$$\|a_u\|\rho^{|u|} = \|a_u\| \left(p^{\frac{1}{M+1}} \right)^{|u|} < \left(\frac{1}{p} \right)^{\frac{1}{\varphi(u)}} p^{\frac{1}{M+1}|u|}$$

$$= \left(\frac{1}{p}\right)^{\frac{1}{\varphi(u)} - \frac{|u|}{M+1}} = \left(\frac{1}{p}\right)^{\frac{1-\frac{1}{M+1}|u|\varphi(u)}{\varphi(u)}} = \left(\left(\frac{1}{p}\right)^{1-\frac{1}{M+1}|u|\varphi(u)}\right)^{\frac{1}{\varphi(u)}}$$

$$< \left(\left(\frac{1}{p}\right)^{1-\frac{M}{M+1}}\right)^{\frac{1}{\varphi(u)}} = \left(\frac{1}{p^{\frac{1}{M+1}}}\right)^{\frac{1}{\varphi(u)}}$$

Hence

$$\lim_{|u|\to\infty} \|a_u\|\rho^{|u|} \leq \lim_{|u|\to\infty} \frac{1}{p^{\frac{1}{(M+1)\varphi(u)}}} = 0$$

which implies that $f \in W_n$.

($\Longrightarrow$) Conversely, suppose $U_n \subset W_n$. Let $f = \sum_{u\in\mathbb{Z}^n_{\geq 0}} a_u X^u \in W_n \setminus U_n$ be any power series. Then there exists $\rho > 1$ such that $\lim_{|u|\to\infty} \|a_u\|\rho^{|u|} = 0$ and $\lim_{|u|\to\infty} \|a_u\|^{\varphi(u)} \neq 0$.

By assumption there exist infinitely many $u \in \mathbb{Z}^n_{\geq 0}$ such that $\|a_u\|^{\varphi(u)} > \varepsilon$ for some $\varepsilon \in \mathbb{R}_{>0}$ and we know that for all $|u|$ large enough $\|a_u\|\rho^{|u|} < \varepsilon$ i.e. $\|a_u\| < \frac{\varepsilon}{\rho^{|u|}}$. Then for infinitely many $u \in \mathbb{Z}^n_{\geq 0}$ we have:

$$0 < \varepsilon < \|a_u\|^{\varphi(u)} < \left(\frac{\varepsilon}{\rho^{|u|}}\right)^{\varphi(u)} = \frac{\varepsilon^{\varphi(u)}}{\rho^{|u|\varphi(u)}} < \frac{1}{\rho^{|u|\varphi(u)}}$$

which implies that $\{|u|\varphi(u)\}_{u\in\mathbb{Z}^n_{\geq 0}}$ is bounded. □

We end this part with a concrete example:

Example 3.33. *Set*

$$\varphi(u) = \frac{1}{|u|}$$

for $|u| \neq 0, 1$ *and* $\varphi(\bar{0}) = \varphi(u) = 1$ *for all* $|u| = 1$. *Then by Proposition 3.32 we deduce that* $U_n \subsetneq W_n$ *and this inclusion is strict.*

Proof. We will give an example of a power series f such that $f \in W_n$ but $f \notin U_n$. It is enough to give the example for $n = 1$.

Set $f = \sum_{n=0}^{\infty} p^n X^n \in T_1$, then we claim that $f \in W_1$ but $f \notin U_1$.

Set $\rho = \sqrt{p} > 1$, then:

$$\|p\|\rho^n = \left(\frac{1}{p^n}\right)\left(\sqrt{p}\right)^n = \left(\frac{\sqrt{p}}{p}\right)^n = \frac{1}{\left(\sqrt{p}\right)^n} \to 0 \text{ as } n \to \infty$$

which implies the fact that $f \in W_1$.

Now, since

$$\|p^n\|^{\varphi(n)} = \left(\frac{1}{p^n}\right)^{\frac{1}{n}} = \frac{1}{p} \neq 0$$

we must have $f \notin U_1$.

□

Appendix B

Filter Functions

In this part firstly, we will investigate the filter functions further and we will establish some further results on the structural behaviour of our algebra. Recall that a *filter function* is a function $\varphi : \mathbb{Z}^n_{\geq 0} \to \mathbb{R}_{>0}$ such that:

(i) φ is decreasing, in the sense that if $|u| \leq |v|$ then $\varphi(v) \leq \varphi(u)$ where $|u| = u_1 + u_2 + \ldots + u_n$ and $|v| = v_1 + v_2 + \ldots + v_n$ if $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$,

(ii) $\varphi\big((0, 0, \ldots, 0)\big) = 1$

(iii) $\dfrac{\varphi(u+u)}{\varphi(u)} = \dfrac{\varphi(2u)}{\varphi(u)} \geq \gamma > 0$ for some $\gamma \in \mathbb{R}_{>0}$ and for all of u.

Later on, we will show that these conditions given above on filter functions are all crucial.

Note also that, if φ is any filter function then for all $r \in \mathbb{R}_{>0}$ if we define the function $\varphi^r : \mathbb{Z}^n_{\geq 0} \to \mathbb{R}_{>0}$ by

$$\varphi^r(u) = \big(\varphi(u)\big)^r$$

then the function φ^r is also a filter function. In this chapter, we will prove that whenever we have a filter function φ, the set $U_{n,\varphi}$ is a Noetherian Jacobson subring of T_n with all ideals closed. Thus for any filter function φ and for any $r \in \mathbb{R}_{>0}$ since the function φ^r is also a filter function, we will also have the same properties for the algebras U_{n,φ^r} for all $r \in \mathbb{R}$.

To add new variables on U_n, we define the set:

$$U_n\langle X_1, \ldots, X_m\rangle^{\varphi} := \Big\{ \sum_{v \in \mathbb{Z}^m_{\geq 0}} f_v X^{v_1}_{n+1} \ldots X^{v_m}_{n+m} | \ f_v := \sum_{u \in \mathbb{Z}^n_{\geq 0}} a_{(u,v)} X^u \in U_n$$
$$\text{and } \|a_{(u,v)}\|^{\varphi(v)} \to 0 \text{ as } |v| \to \infty \text{ for any fixed } u \Big\}$$

Now, we will show that this set contains U_{n+m}.

Proposition 3.34. $U_{n+m} \subseteq U_n\langle X_1, X_2, \ldots, X_m\rangle^{\varphi}$

Proof. Before we start proving, we will remind the definition of the set:

$$U_{n+m} = \Big\{ \sum_{\substack{u=(u_1,u_2,\ldots,u_n) \\ v=(v_1,v_2,\ldots,v_m) \\ u \in \mathbb{Z}^n_{\geq 0}, v \in \mathbb{Z}^m_{\geq 0}}} a_{(u,v)} X^{u_1}_1 \ldots X^{u_n}_n X^{v_1}_{n+1} \ldots X^{v_m}_{n+m} : \|a_{(u,v)}\|^{\varphi(u,v)} \to 0 \text{ as } |u+v| \to \infty \Big\}$$

and the definition of $U_{n,\varphi}\langle X_1, \ldots, X_m\rangle^\varphi$ is given as above.

Now, we prove the inclusion:

Let
$$f = \sum_{\substack{u=(u_1,u_2,\ldots,u_n)\\ v=(v_1,v_2,\ldots,v_m)\\ u\in\mathbb{Z}^n_{\geq 0},\, v\in\mathbb{Z}^m_{\geq 0}}} a_{(u_1,\ldots,u_n,v_1,\ldots,v_m)} X_1^{u_1}\ldots X_n^{u_n} X_{n+1}^{v_1}\ldots X_{n+m}^{v_m} \in U_{n+m}$$

be arbitrary. So, we know that $\|a_{(u,v)}\|^{\varphi(u,v)} = \|a_{(u,v)}\|^{\varphi(u+v)} \to 0$ as $|u+v| \to \infty$.

For any $v = (v_1, \ldots, v_m) \in \mathbb{Z}^m_{\geq 0}$ set
$$f_v := \sum_{\substack{u=(u_1,\ldots,u_n)\\ u\in\mathbb{Z}^n_{\geq 0}}} a_{(u_1,\ldots,u_n,v)} X_1^{u_1}\ldots X_n^{u_n}$$

then
$$f = \sum_{\substack{v=(v_1,\ldots,v_m)\\ v\in\mathbb{Z}^m_{\geq 0}}} f_v X_{n+1}^{v_1}\ldots X_{n+m}^{v_m}$$

We need to prove the following two assertions:

(i) $f_v \in U_n$ for each $v \in \mathbb{Z}^m_{\geq 0}$.

(ii) $\|a_{(u_1,\ldots,u_n,v)}\|^{\varphi(v)} \to 0$ as $|v| \to \infty$.

(i) To show that $f_v = \sum_{\substack{u=(u_1,\ldots,u_n)\\ u\in\mathbb{Z}^n_{\geq 0}}} a_{(u_1,\ldots,u_n,v)} X_1^{u_1}\ldots X_n^{u_n} \in U_{n,\varphi}$, we need to show that $\|a_{(u_1,\ldots,u_n,v)}\|^{\varphi(u_1,\ldots,u_n)} \to 0$ as $|u| \to \infty$.

Note that for $|u| = u_1+u_2+\ldots+u_n$ large enough we have $\|a_{(u_1,\ldots,u_n,v}\| < 1$, thus since φ is a decreasing function < 1 we get:

$$\begin{aligned}\|a_{(u_1,\ldots,u_n,v)}\|^{\varphi(u_1,\ldots,u_n)} &< \|a_{(u_1,\ldots,u_n,v)}\|^{\varphi(u_1,\ldots,u_n,v)}\\ &= \|a_{(u,v)}\|^{\varphi(u,v)} \to 0\end{aligned}$$

So we conclude that $f_v \in U_{n,\varphi}$ for each $v \in \mathbb{Z}^m_{\geq 0}$.

(ii) Now, we will show that $\|a_{(u_1,\ldots,u_n,v)}\|^{\varphi(v)} \to 0$ as $|v| \to \infty$. A similar calculation leads us:

$$\|a_{(u_1,\ldots,u_n,v)}\|^{\varphi(v)} \leq \|a_{(u_1,\ldots,u_n,v)}\|^{\varphi(u_1,\ldots,u_n,v)} = \|a_{(u,v)}\|^{\varphi(u,v)} \to 0$$

for all $|v|$ large enough.

□

Question 3.35. *Is it true that*

$$U_{n+m} = U_n\langle X_1, X_2, \ldots, X_m\rangle^\varphi\,?$$

If not, is there a suitable definition of $U_n\langle X_1, X_2, \ldots, X_m\rangle^\varphi$ *(suitable for the consistency of the theory) for which we have the above equality?*

Next, we will show that the conditions on the function φ are all necessary conditions.

According to the Chapter 1, we have a subalgebra U_n of T_n but the algebra U_n heavily depends on the function φ and there are a few conditions on this function. In this part we explain that these conditions are crucial.

Proposition 3.36. *The conditions on the function φ are all necessary to build a consistent theory.*

Proof. (i) For all $|u|$ large enough, $\varphi(u) \leq \frac{C}{\log |u|}$ for some $C \in \mathbb{R}_{>0}$.

The condition $\varphi(u) \leq \frac{C}{\log |u|}$ is necessary for U_n to have trivial de Rham cohomology (see Section 3.3 for details). The proof of Lemma 0.7 shows us that we need to have this condition. Note that, in any non-Archimedean field k we have the inequality: $\left\|\frac{1}{m}\right\| \leq m$ for all $m \in \mathbb{N}$ and this is a sharp condition, in the sense that taking $k = \mathbb{Q}_p$ and $m = p^k$ for any $k \in \mathbb{N}$ imply the fact that

$$\left\|\frac{1}{m}\right\| = \left\|\frac{1}{p^k}\right\| = p^k = m$$

so that, to prove that the limit of $\left\|\frac{a_{m-1}}{m}\right\|^{\varphi(m)}$ is zero for all $\sum_{m \in \mathbb{Z}_{\geq 0}} a_m X^m$ in U_1 in Lemma 0.7, we must have $\left\|\frac{1}{m}\right\|^{\varphi(m)}$ as a bounded real number and Lemma 0.6 gives the necessary and sufficient condition which happens to be

$$\varphi(m) \leq \frac{C}{\log m} \text{ for some } C \in \mathbb{R}_{>0}.$$

(ii) φ is a decreasing function in the sense that for $|u| \leq |v|$ we have $\varphi(v) \leq \varphi(u)$.

We cannot allow the function φ to make big jumps between the values of $\mathbb{Z}^n_{\geq 0}$. Consider such an example where φ is not a decreasing function. For $u \in \mathbb{Z}^n_{\geq 0}$, define:

$$\varphi(u) = \begin{cases} 1/\log |u| & \text{if } |u| \text{ is odd} \\ 1/|u| & \text{if } |u| \text{ is even} \end{cases}$$

Then, $\varphi(u) \leq \frac{1}{\log |u|}$ and for all $u \in \mathbb{Z}^n_{\geq 0}$, $\frac{\varphi(u+u)}{\varphi(u)} \geq \frac{|u|}{|2u|} = \frac{1}{2} > 0$, so that the other two conditions are satisfied for this particular function φ. But, with this function we do not have a ring structure on U_n. It is enough to show it for U_1. Set $k = \mathbb{Q}_p$ for some p prime and

$$f := \sum_{k=0}^{\infty} p^{2k+1} X^{2k+1} = pX + p^3X^3 + p^5X^5 + \ldots + p^{2k+1}X^{2k+1} + \ldots$$

Then

$$\lim_{k\to\infty} \|p^{2k+1}\|^{\varphi(2k+1)} = \lim_{k\to\infty} \left(\frac{1}{p^{2k+1}}\right)^{\varphi(2k+1)} = \lim_{k\to\infty} \left(\frac{1}{p^{2k+1}}\right)^{\frac{1}{\log|2k+1|}}$$
$$= \lim_{k\to\infty} \frac{1}{p^{\frac{2k+1}{\log(2k+1)}}} = 0$$

Thus, $f \in U_1$. But $f^2 = f.f \notin U_1$:

$$f.f = \left(\sum_{k=0}^{\infty} p^{2k+1} X^{2k+1}\right)\left(\sum_{k=0}^{\infty} p^{2k+1} X^{2k+1}\right) = \sum_{n=0}^{\infty} np^{2n} X^{2n}$$

Then for infinitely many $n \in \mathbb{N}$ (the ones which are relatively prime to p), we have:

$$\|np^{2n}\|^{\varphi(2n)} = \left(\frac{1}{p^{2n}}\right)^{\frac{1}{2n}} = \frac{1}{(p^{2n})^{\frac{1}{2n}}} = \frac{1}{p}$$

thus the limit,

$$\lim_{n\to\infty} \|np^{2n}\|^{\varphi(2n)}$$

does not exist i.e. f^2 is not an element of U_1.

(iii) For all $u \in \mathbb{Z}^n_{\geq 0}$, $\frac{\varphi(2u)}{\varphi(u)} \geq \gamma$ for some $\gamma \in \mathbb{R}_{>0}$

The third condition is that there exists $\gamma > \mathbb{R}_{>0}$ such that for all $u \in \mathbb{Z}^n_{\geq 0}$ we have $\frac{\varphi(u+u)}{\varphi(u)} \geq c > 0$ for some $c \in \mathbb{R}_{>0}$. Suppose for another counter example that:

$$\varphi(u) = \frac{1}{e^{|u|}} \qquad \text{for } u \in \mathbb{Z}^n_{\geq 0}$$

Then φ is a decreasing function bounded by the function $\frac{1}{\log |u|}$, i.e. the other two conditions are satisfied, and we also have:

$$\begin{aligned}\lim_{|u|\to\infty} \frac{\varphi(u+u)}{\varphi(u)} &= \lim_{|u|\to\infty} \frac{1/e^{|u+u|}}{1/e^{|u|}} = \lim_{|u|\to\infty} \frac{e^{|u|}}{e^{|u+u|}} \\ &= \lim_{|u|\to\infty} \frac{e^{|u|}}{e^{|u|+|u|}} = \lim_{|u|\to\infty} \frac{e^{|u|}}{e^{|u|}.e^{|u|}} = \lim_{|u|\to\infty} \frac{1}{e^{|u|}} = 0\end{aligned}$$

We will again give a counter example for U_1. Set $k = \mathbb{Q}_p$ where $p = 3$ and

$$f = \sum_{n=0}^{\infty} p^{p^{2n+1}} X^{2n+1}$$

Then:

$$\begin{aligned}\lim_{n\to\infty} \left\|p^{p^{2n+1}}\right\|^{\varphi(2n+1)} &= \lim_{n\to\infty} \left(\frac{1}{p^{p^{2n+1}}}\right)^{\frac{1}{e^{2n+1}}} = \lim_{n\to\infty} \frac{1}{p^{\frac{p^{2n+1}}{e^{2n+1}}}} \\ &= \lim_{n\to\infty} \frac{1}{p^{(\frac{p}{e})^{2n+1}}} = 0\end{aligned}$$

Thus, f is an element of U_1. But $f^2 = f.f$ is not an element of U_1:

$$\begin{aligned}f^2 = f.f &= \left(\sum_{n=0}^{\infty} p^{p^{2n+1}} X^{2n+1}\right)\left(\sum_{n=0}^{\infty} p^{p^{2n+1}} X^{2n+1}\right) \\ &= \sum_{n=0}^{\infty}\Big(\sum_{\substack{i+j=2n \\ i,j \text{ odd}}} p^{p^i} p^{p^j}\Big) X^{2n}\end{aligned}$$

thus we only need to show that the sequence $\left(\left\|\sum_{\substack{i+j=2n\\ i,j \text{ odd}}} p^{p^i}p^{p^j}\right\|^{\varphi(2n)}\right)_{n\in\mathbb{N}}$ does not converge to 0. Note that it has a subsequence:

$$\left(\left\|\sum_{\substack{i+j=4n+2\\ i,j \text{ odd}}} p^{p^i}p^{p^j}\right\|^{\varphi(4n+2)}\right)_{n\in\mathbb{N}}$$

We have:

$$\begin{aligned}\left\|\sum_{\substack{i+j=4n+2\\ i,j \text{ odd}}} p^{p^i}p^{p^j}\right\|^{\varphi(4n+2)} &= \left\|p^{p^{2n+1}}p^{p^{2n+1}}\right\|^{\frac{1}{e^{4n+2}}}\\ &= \left(\frac{1}{p^{p^{2n+1}}p^{p^{2n+1}}}\right)^{\frac{1}{e^{4n+2}}} = \frac{1}{p^{\frac{2p^{2n+1}}{e^{4n+2}}}}\end{aligned}$$

Since

$$\frac{2p^{2n+1}}{e^{4n+2}} = 2\left(\frac{p}{e^2}\right)^{2n+1} \to 0$$

as n goes to infinity, thus the sequence:

$$\left(\left\|\sum_{\substack{i+j=2n\\ i,j \text{ odd}}} p^{p^i}p^{p^j}\right\|^{\varphi(2n)}\right)_{n\in\mathbb{N}}$$

can not possibly converge to 0, therefore $f^2 = f.f$ is not an element of U_1.

□

Question 3.37. *Is there a criterion for the two functions φ_1 and φ_2 (both satisfying the conditions given in the Definition 1.1) such that*

$$U_{n,\varphi_1} \simeq U_{n,\varphi_2}$$

(as algebras)?

Bibliography

[1] M.F. Atiyah and I.G. Macdonald. *Introduction to Commutative Algebra.* Addison Wesley Publishing Company, 1969.

[2] Pierre Berthelot. Cohomologie rigide et cohomologie rigide à supports propres. *IR-MAR 96-03*, 1996.

[3] S. Bosch and W. Lütkebohmert. Formal and rigid geometry. *Math. Ann.*, 295:291–317, 1993.

[4] Siegfried Bosch. *Lectures on Formal and Rigid Geometry.* Springer, 2014.

[5] N. Bourbaki. *Elements of Mathematics - Topological Vector Spaces Chapters 1-5.* Springer, 1987.

[6] Brian Conrad. Several approaches to non-archimedean geometry. In *University Lecture Series 45.* American Mathematics Society, 2008.

[7] John B. Conway. *A Course in Functional Analysis.* Springer, 1985. Graduate Texts in Mathematics 96.

[8] Damek Davis and Daqing Wan. Factorial and noetherian subrings of power series rings. *Proceedings of the American Mathematical Society*, 139(3):823–834, 2011.

[9] de A. J. Jong and M. van der Put. Etale cohomology of rigid analytic spaces. *Docum. Math.*, 1:1–56, 1996.

[10] Max Flander. *Non-Archimedean Analysis and its Applications in Tropical Geometry.* PhD thesis, The University of Melbourne, 2007.

[11] Jean Fresnel and Marius van der Put. *Rigid Analytic Geometry and Its Applications.* Birhäuser, 2004.

[12] Elmar Große-Klönne. Rigid analytic spaces with overconvergent structure sheaf. *J.eine angew. Math.*, 519:73–95, 2000.

[13] Elmar Große-Klönne. Finiteness of de rham cohomology in rigid analysis. *Duke Mathematical Journal*, 113(1):57–91, 2002.

[14] Elmar Große-Klönne. De rham cohomology of rigid spaces. *Mathematische Zeitschrift*, 247:223–240, 2004.

[15] Robin Hartshorne. On the de rham cohomology of algebraic varieties. *Publications mathmatiques de l'I.H..S*, 45:5–99, 1975.

[16] Robin Hartshorne. *Algebraic Geometry*. Springer, 1997.

[17] Nathan Jacobson. *Basic Algebra II*. W.H.Freeman and Company, 1989.

[18] Erwin Kreyszig. *Introductory Functional Analysis with Applications*. John Wiley and Sons Inc., 1978.

[19] Saunders Mac Lane. *Homology*. Springer, 1995.

[20] Serge Lang. *Algebra*. Springer, 2002.

[21] W. Lütkebohmert. Formal algebraic and rigid analytic geometry. *Math. Ann.*, 286:341–371, 1990.

[22] Jürgen Neukirch. *Algebraic Number Theory*. Springer, 1999.

[23] Marius Van Der Put. De rham cohomology of affinoid spaces. *Compositio Mathematica*, 73:223–239, 1990.

[24] Walter Rudin. *Principles of mathematical analysis*. McGraw-Hill, Inc., 1976.

[25] Walter Rudin. *Functional Analysis*. McGraw-Hill Inc-, 1991.

[26] S.Bosch, U.Güntzer, and R.Remmert. *Non-Archimedean Analysis*. Springer, 1984.

[27] Peter Schneider. *Nonarchimedean Functional Analysis*. Springer, 2002.

[28] John Tate. Rigid analytic spaces. *Inventiones Math. Springer*, 12:257–289, 1971.

Selbstständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbstsändig und nur unter Verwendung der von mir gemäß §7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014, angegebenen Hilfsmittel angefertigt habe.

Berlin, den 12.02.2018 Uğur Doğan