Tools for Superstring Amplitudes

Dissertation
zur Erlangung des akademischen Grades

doctor rerum naturalium

(Dr. rer. nat.)

im Fach (Promotionsfach): Physik

Spezialisierung: Theoretische Physik

eingereicht an der

Mathematisch-Naturwissenschaftlichen Fakultät

der Humboldt-Universität zu Berlin

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Tag der mündlichen Prüfung:	9. Oktober 2019

Declaration of independent work

I declare that I have completed the thesis independently using only the aids and tools specified. I have not applied for a doctor's degree in the doctoral subject elsewhere and do not hold a corresponding doctor's degree. I have taken due note of the Faculty of Mathematics and Natural Sciences PhD Regulations, published in the Official Gazette of Humboldt-Universität zu Berlin no. 42/2018 on 11/07/2018.

Abstract

In this thesis, we develop computational tools to calculate tree and one-loop superstring amplitudes. In particular, we provide a recursive method to construct kinematic factors of tree level open superstring amplitudes and present systematic tools to manifest the supersymmetric cancellations in n-boson-two-fermion amplitudes at the one-loop order of the RNS superstring.

For tree level open superstring amplitudes, we present simplified recursions for multiparticle superfields, which can be applied to construct kinematic parts of open superstring amplitudes at tree level. We also discuss the gauge transformations which enforce their Lie symmetries as suggested by the Bern-Carrasco-Johansson duality between color and kinematics. Another gauge transformation due to Harnad and Shnider is shown to streamline the theta-expansion of multiparticle superfields, bypassing the need to use their recursion relations beyond the lowest components. The findings of this work greatly simplify the component extraction from kinematic factors in pure spinor superspace.

We then investigate massless n-point one-loop amplitudes of the open RNS superstring with two external fermions and determine their world-sheet integrands. The contributing correlation functions involving spin-1/2 and spin-3/2 operators from the fermion vertices are evaluated to any multiplicity. Moreover, we introduce techniques to sum these correlators over the spin structures of the world-sheet fermions, such as to manifest all cancellations due to spacetime supersymmetry. These spin-summed correlators can be expressed in terms of doubly-periodic functions known from the mathematics literature on elliptic multiple zeta values. On the boundary of moduli space, our spin-summed correlators specialize to compact representations of fermionic one-loop integrands for ambitwistor strings.

Zusammenfassung

In dieser Arbeit entwickeln wir Rechenwerkzeuge zur Berechnung von Baum– und Einschleifensuperstringamplituden. Insbesondere stellen wir eine rekursive Methode zur Konstruktion kinematischer Faktoren für die Amplitude offener Superstrings auf Baumniveau bereit und präsentieren systematische Werkzeuge, um die supersymmetrischen Auslöschungen in n-Boson-Zwei-Fermion Amplituden auf Einschleifenniveau des RNS Superstrings zu manifestieren.

Für offene Superstringamplituden auf Baumniveau stellen wir vereinfachte Rekursionen für Mehrteilchensuperfelder vor, mit denen kinematische Teile von offenen Superstringamplituden auf Baumniveau konstruiert werden können. Wir diskutieren auch die Eichtransformationen, die ihre Lie-Symmetrien erzwingen, wie dies durch die Bern-Carrasco-Johansson-Dualität zwischen Farbe und Kinematik nahegelegt wird. Eine weitere Eichtransformation aufgrund von Harnad und Shnider soll die Theta-Expansion von Mehrteilchensuperfeldern vereinfachen und die Notwendigkeit umgehen, ihre Rekursionsrelationen über die niedrigsten Komponenten hinaus zu verwenden. Die Ergebnisse dieser Arbeit vereinfachen die Komponentenextraktion aus kinematischen Faktoren im reinen Spinor Superspace erheblich.

Wir untersuchen dann masselose n-Punkt-Ein-Schleifen-Amplituden des offenen RNS Superstrings mit zwei externen Fermionen und bestimmen ihre Weltflächen-Integranden. Die beitragenden Korrelationsfunktionen, an denen Spin-1/2– und Spin-3/2-Operatoren aus den Fermionen-Vertices beteiligt sind, werden zu einer beliebigen Multiplizität ausgewertet. Darüber hinaus führen wir Techniken ein, um diese Korrelatoren über die Spinstrukturen der Weltflächen-Fermionen zu summieren, um alle Auslöschungen aufgrund der Supersymmetrie der Raumzeit zu manifestieren. Diese spinsummierten Korrelatoren können in Form von doppeltperiodischen Funktionen ausgedrückt werden, die aus der mathematischen Literatur über elliptische Multiple-Zeta-Werte bekannt sind. Unsere spinsummierten Korrelatoren an der Grenze des Modulraums sind auf kompakte Darstellungen von fermionischen Ein-Schleifen-Integranden für ambitwistorische Strings spezialisiert.

Acknowledgments

I am grateful to my supervisor Oliver Schlotterer for his great support. Oliver has never hesitated to share his excellent insight into various fields of physics with me, and give practical advice on my doctoral study in general. I would like to thank my secondary supervisors Massimo Taronna and Daniele Oriti, especially on sharing their knowledge on physics outside of string theory. I also appreciate my official supervisor Prof. Hermann Nicolai on accepting me to join the IMPRS program and wish to thank IMPRS coordinators Axel Kleinschmidt and Hadi Godazgar for their administrative support during the program. I should not forget to thank my master thesis supervisor Prof. Dimitri Polyakov since everything can be started due to his incredible support during the application process to the IMPRS program.

My gratitude also goes to my friends and colleagues in the AEI (in absolutely no particular order): Olof Ahlén, Sebastian Bramberger, Marco Finocchiaro, Jan Gerken, Alexander Kegeles, Pan Kessel, Isha Kotecha, Lars Kreutzer, and Olaf Krüger. I particularly have enjoyed discussions with Jan about physics and politics, and Alex, Isha and Marco always have provided conversations on topics in quantum gravity about which I have been happy to talk.

Finally, I wish to express my love and gratitude to my family and my friends in Korea. My wife Hyerin always grabbed me out from the swamp of the daydream into which I often fell. My parents and my brother invariably have been a reliable anchor for my life. My friends in Korea were always ready to go for a drink and talk about stories which I had missed, whenever I visited Korea.

CONTENTS

ONE

INTRODUCTION

1.1 Scattering amplitudes and string theory

Since Rutherford used the scattering of α and β particles to investigate the structure of atoms [1], scattering experiments have been proving grounds for modern theories of elementary particles and their interactions due to their capability on controlling the input data and accessing relatively high energy scale[1]. The most recent and influential example could be the discovery of the spin-zero particle at the Large Hadron Collider (LHC) in CERN, whose existence has been qualitatively predicted by the Standard Model.

The typical input of a scattering experiment is a set of physical observables, such as masses and momenta, of effectively non-interacting incoming particles approaching each other, and the output is again a set of physical observables of non-interacting outgoing particles, generated by local interactions among incoming particles. Those incoming and outgoing particles are often called external particles to distinguish them from internal particles created and annihilated during local interactions.

The theoretical framework for establishing the relation between observables of incoming and outgoing particles relies on computing the quantum probability amplitude between an in- and an out-state [2]. An in-state (or an out-state) is defined by a state decomposable into quantum states representing incoming (or outgoing) particles at the macroscopically far past (or future). The resulting probability amplitude is called the scattering amplitude.

Conventionally scattering amplitudes are computed in the framework of quantum field theories (QFTs), in which particles are realized by zero-dimensional mathematical objects called point particles. The QFTs of point particles have provided a mathematically and physically consistent algorithm for computing scattering amplitudes of elementary particles through the Standard Model if one does not include gravity into the model, which causes uncontrollable ultraviolet (UV) divergences in the quantum corrections to amplitudes.

In an attempt to integrate our understanding of elementary particles in the Standard Model with gravity, several theories have been proposed. Among other candidates, theories labeled by string theory are based on the idea that elementary particles can be realized by one-dimensional objects, shortly strings, instead of points. This idea provides a conjecture that string theory can be free of UV divergences due to the finite size of strings. Moreover, vibration modes of each string can be interpreted as the physical properties of the string, which can be identified with the physical properties of elementary particles.

The idea of string theory has been formulated in various ways. Notably, superstring theory, which emerges from the combination of the idea of string theory and the supersymmetry, has provided a massless sector in its spectrum including gravitons, gauge bosons, and fermions, in which gravitons and gauge bosons are realized by closed strings and open strings respectively. Moreover, it turns out that superstring theory necessarily unifies gravity with gauge interactions since the consistency of interactions among open strings requires the existence of closed string states in its spectrum.

[1] There exist other types of experiments which can access the higher energy scale compared to the biggest collider performing scattering experiments. For example, cosmic rays scattered in the atmosphere of the earth have much higher energy than hadrons in LHC. However, in general, one cannot sufficiently control those rays as an input of a scattering experiment.

Having massless states described above, interactions among those states have been extensively studied while establishing striking interplays with mathematics and other fields of physics. In particular, in addition to the investigation on the UV completion of superstring amplitudes, superstring amplitudes have many implications on the structure of scattering amplitudes of gravity and gauge theories. One of the remarkable outcomes is the discovery of the Kawai-Lewellen-Tye (KLT) relation between closed and open superstring amplitudes at tree level [3], which states that a closed superstring amplitude can be represented by a product of two open superstring amplitudes.

The KLT relation for superstring amplitudes and their low-energy limit furnish the oldest incarnation of the double-copy structure of amplitudes, which has been studied in various contexts not limited to gravity amplitudes. In particular, Bern, Carrasco, and Johansson (BCJ) discovered in [4, 5] that if the kinematic dependence of a gauge theory amplitude mirrors the relation among color factors of the amplitude, named by the BCJ duality, gravity amplitudes can be obtained by replacing those color factors with another copy of the kinematic dependence.

A remarkable feature of the BCJ duality is that one can extend the discussion of the double-copy structure to loop integrands of gauge and gravity amplitudes, although, in string theory, the KLT relation has been found only at tree level. This feature drastically reduces the complexity of the computation of loop level amplitudes of (possibly supersymmetric) gravity compared to the conventional Feynman diagram method, and consequently, UV divergences of gravity theories can be studied in a more systematic and precise manner [6, 7, 8, 9, 10, 11].

At tree level, the double-copy structure of gravity integrands emerged from the BCJ duality of gauge theory integrands has been proved in [12], and a systematic way of obtaining kinematic numerators manifestly satisfying the BCJ duality has been extracted from the n-point open superstring amplitude [13, 14]. These BCJ manifest kinematic numerators have been constructed from multiparticle superfields obtained by employing conformal field theory techniques of the so-called pure spinor (PS) formulation of superstring theory [15].

In this thesis, we simplify the method to build those kinematic numerators of tree level open superstring amplitudes by using a perturbative solution of the ten-dimensional Yang-Mills (10D SYM) equations of motion. More precisely, We use the Lorenz gauge to solve the 10D SYM equations, and show that BCJ satisfying numerators can be constructed by taking a gauge transformation on the solution under the Lorenz gauge. This gauge transformation leads to a perturbative solution under another gauge, which we will call the BCJ gauge.

At loop level, the duality remains conjectural with strong support by examples up to and including five loops [5, 16, 17, 10, 11]. At one loop, an algorithm to obtain BCJ-satisfying kinematic dependences has recently been found in [18, 19], based on the Cachazo-He-Yuan (CHY) representation of amplitudes [20, 21, 22, 23, 24, 25, 26] and the low energy limit of superstring amplitudes. Therefore, as for tree level amplitudes, superstring amplitudes provide an accessible framework for obtaining the BCJ numerators (see [27, 28] for five-point one- and two-loop examples in the PS framework).

However, at one loop, only superstring amplitudes with bosonic insertions have been simplified for any number of external states, and an explicit evaluation of amplitudes with fermions and bosons is not yet accessible both in the PS and the Ramond-Neveu-Schwarz (RNS) formulation of superstring theory beyond seven-point[2]. In computing one-loop amplitudes within the RNS formalism, a significant challenge is to manifest the cancellations between bosons and fermions in the loop due to spacetime supersymmetry. These cancellations arise from different boundary conditions for the world-sheet spinors and are particularly well understood in one-loop amplitudes of massless bosons [32, 33].

[2] See [29, 30, 31] for the most recent development on the computation of one-loop amplitudes in the PS formalism.

In this context, we develop mathematical methods to systematically manifest the supersymmetric cancellations in the RNS superstring theory, which make two-fermion-n-boson amplitudes at one-loop completely accessible and allow for comparison with the respective superspace components of the pure spinor expressions [34, 35, 29, 30, 31]. Similar to the treatment of bosonic one-loop amplitudes in [33], our manipulations of Jacobi-theta functions rely on the Eisenstein-Kronecker series [36, 37, 38] which has a profound relation to number theory.

The structure of this thesis is the following. From the next section to the end of this chapter, we will briefly review the formal structure of scattering amplitudes in string theory. It also includes a short and nontechnical account on formulating the superstring theory. In Chapter 2, we will consider the relation between tree level open superstring amplitudes computed in the PS superstring theory and perturbative solutions of 10D SYM, and present a perturbative solution of 10D SYM generating the BCJ numerators for scattering amplitudes of 10D SYM. Chapter 3 deals with one-loop superstring amplitudes in the RNS formalism. We review the method for computing world-sheet conformal correlators with spin fields in the RNS formulation and develop an algorithm for summing over spin structures of those correlators. The last chapter is devoted to conclusions and outlooks expected from the results of the thesis. Main results of this thesis presented in Chapter 2 and Chapter 3 are based on our previous works [39, 40].

1.2 Formal structure of string scattering amplitudes

String theory has been formulated in various contexts. Before diving into computations of string scattering amplitudes in specific formulations, we will briefly review the formal structure of string scattering amplitudes shared by most of string theory formulations.

1.2.1 String and string action

String theory postulates that quantum states representing elementary particles can be realized by quantum states in the spectrum of a quantized string. Here, a *string* is a physical system whose observables can be continuously parameterized by a real parameter σ in a closed interval $I = [0,l],\ l > 0$[3].

A string and its dynamics are often defined by a *string action*

$$S[A] = \int d\tau \int_0^l d\sigma L(A(\sigma,\tau)) \tag{1.1}$$

where τ is the evolution parameter and L is the Lagrangian depending on the collection $A(\sigma,\tau)$ of string degrees of freedom. The trajectory Σ of the string in the target space, i.e., the space of string degrees of freedom, is called the *world-sheet* of the string.

In this thesis, we only consider the case that $A(\sigma,\tau)$ includes the position $X(\sigma,\tau)$ of the string in a given spacetime $\mathscr{M}$. The spacetime position of the string then defines the spacetime trajectory of the string which is a two-dimensional surface in $\mathscr{M}$. If we further assume that $X(\sigma,\tau)$ is smooth and causal with $\partial_\tau X$ to be timelike or null, and $\partial_\sigma X$ to be spacelike, it is natural to require the action to be invariant under reparametrizations on (σ,τ) since we can relate each parametrization with a local observer in $\mathscr{M}$ which should not have any physical meaning due to the equivalence principle.

One of the simplest formulations of string theory is the bosonic string theory based on the Nambu-Goto action [41, 42]

$$S_{NG}[X] = -\frac{1}{2\pi\alpha'}\int_\Sigma \sqrt{-\det(g)} \tag{1.2}$$

[3]We assume that the space of observables is a topological space.

proportional to the relativistic area of the world-sheet Σ of a causally propagating string in $\mathscr{M}$. In (1.2), g is the induced metric on Σ and α' is a parameter having dimension (length)^2. Also, we take the flat Minkowski metric for a background spacetime $\mathscr{M}$ since the bosonic string theory describes gravity through interactions among gravitons which are hypothetical quantum states relevant to local fluctuations of a fixed spacetime metric. By introducing local coordinates $\sigma^a = (\tau, \sigma)$, $a = 0, 1$ and X^m, $m = 0, 1, \ldots, \dim \mathscr{M} - 1$ for Σ and $\mathscr{M}$ respectively, we have

$$g = g_{ab} d\sigma^a d\sigma^b, \qquad g_{ab} = \partial_a X^m \partial_b X_m, \qquad \text{for } a, b = 0, 1$$

so that the Lagrangian L_{NG} of S_{NG} is locally given by

$$L_{NG} = -\frac{1}{2\pi\alpha'} \sqrt{-\det(\partial_a X^m \partial_b X_m)}. \tag{1.3}$$

1.2.2 Closed and open strings

In addition to usual Euler-Lagrange equations, the string action in (1.2) implies boundary conditions of $A(\sigma, \tau)$ at $\sigma = 0, l$. Especially, if the Lagrangian depends on X and its first derivatives on (σ, τ) as in (1.3), boundary conditions of X at $\sigma = 0, l$ are in the form of

$$0 = \left. \frac{\partial L}{\partial(\partial_\sigma X^m)} \delta X^m \right|_{\sigma = 0, l}.$$

These boundary conditions can be fulfilled by a closed string, i.e., $X^m(\sigma, \tau) = X^m(\sigma + l, \tau)$ for all X^m, or an open string with boundary conditions

$$0 = \frac{\partial L}{\partial(\partial_\sigma X^m)} \quad \text{or} \quad 0 = \delta X^m \text{ at } \sigma = 0, l \tag{1.4}$$

for each X^m. The former in (1.4) is called the Neumann boundary condition and the latter is called the Dirichlet boundary condition.

Imposing the Dirichlet boundary condition to some of the local coordinates of a string end is equivalent to fixing those coordinates, which corresponds to confine the endpoint to a subspace in $\mathscr{M}$. Since the energy and momentum of the string cannot be conserved at the end confined on a subspace, one has to consider the subspace as a dynamical object, called a D-brane, embedded in $\mathscr{M}$ to retain the energy-momentum conservation of the whole system.

Once we adopt that the existence of dynamical D-branes, we can introduce several physically distinguishable D-branes so that the string can have some extra degrees of freedom indicating the D-brane on which its end is confined. In particular, if both ends of the string have some Dirichlet boundaries, we can introduce N D-branes for each end which force local coordinates $X^m(\sigma, \tau)$ to be $N \times N$ matrix-valued. Thus, X^m can be expanded by using the Hermitian basis $\{t^a | a = 1, \ldots, N^2, (t^a)^\dagger = t^a\}$ of the space of $N \times N$ matrices:

$$X^m(\sigma, \tau) = \sum_{a=1}^{N^2} t^a X^m(\sigma, \tau; a). \tag{1.5}$$

The extra degrees $1 \le a \le N^2$ of freedom in (1.5) are called Chan-Paton factors [43] and phenomenologically important since they can be identified with the color degrees of freedom of an $U(N)$-gauge symmetry carried by an open string.

As a technical remark, for an open string, it is often convenient to extend $\sigma \in [0, l]$ to $[0, 2l]$ such that for $\sigma \in [0, l]$ [44]

$$X^m(l + \sigma) = \begin{cases} X^m(l - \sigma), & \text{if } X^m(0) \text{ and } X^m(l) \text{ are Neuman boundaries} \\ -X^m(l - \sigma) + 2X^m(l), & \text{otherwise} \end{cases} \tag{1.6}$$

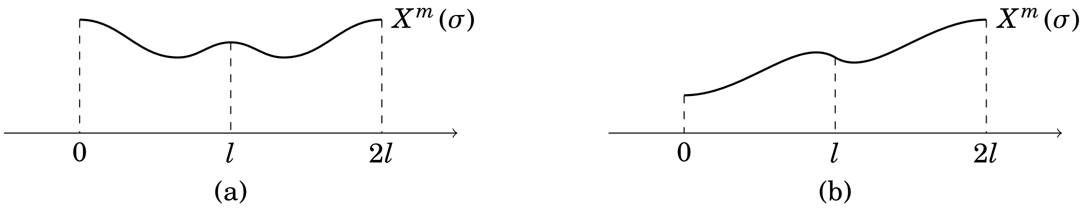


Figure 1.1: The doubling trick on an open string with (a) the Neumann boundary condition for both ends (b) the Dirichlet boundary condition for at least one end.

to avoid complications at string ends. See fig. 1.1. Especially, the doubling trick in (1.6) allows us to handle an open and a closed string in a unified manner since the doubled open string has the periodic boundary condition. In the following discussions, we implicitly rescale $\sigma \to 2\sigma$ whenever we apply the doubling trick to an open string so that the doubled open string is parametrized by $\sigma \in [0,l]$ which maximizes notational simplicity arising from the doubling trick.

1.2.3 Quantization of string action

In order to quantize the string defined by a string action, we have to employ the Hamiltonian analysis on the string action whose detailed exposition depends on the given string action. For instance, the Hamiltonian analysis on S_{NG} begins with the canonical momentum P_m which satisfies two primary first-class constraints

$$0 = \Phi_\pm = \frac{1}{2}(P_m \pm \partial_\sigma X_m)\left(P^m \pm \partial_\sigma X^m\right).$$

Here, we have redefined $X^m \to 2\pi\alpha' X^m$ for notational simplicity. Due to the vanishing canonical Hamiltonian, the complete Hamiltonian is given by

$$H_{NG} = \int_0^l d\sigma \left(u_+(\sigma,\tau)\Phi_+(\sigma,\tau) + u_-(\sigma,\tau)\Phi_-(\sigma,\tau)\right) \tag{1.7}$$

where $u_\pm$ are Lagrange multipliers on Σ and $\Phi_\pm$. For a closed string, $\Phi_\pm$ induces the following constraint algebra:

$$\left[L_m^\pm, L_n^\pm\right] = i(m-n)L_{m+n}^\pm \tag{1.8a}$$
$$\left[L_m^\pm, L_n^\mp\right] = 0. \tag{1.8b}$$

where

$$L_m^\pm = \pm\frac{l}{2}\int_0^l d\sigma e^{-2\pi i m\sigma/l}\Phi_\pm(\sigma,\tau), \quad m \in \mathbb{Z}.$$

This algebra corresponds to the direct sum of two classical Virasoro algebras, so the closed bosonic string theory has the conformal symmetry as its gauge symmetry. For an open string, one can use the doubling trick in (1.6) to combine $\Phi_\pm$ into a single constraint

$$\Phi(\sigma) = \begin{cases} \Phi_+(\sigma) & \text{for } \sigma \in \left[0, \dfrac{l}{2}\right] \\ \Phi_-(\sigma) & \text{for } \sigma \in \left[\dfrac{l}{2}, l\right]. \end{cases}$$

Therefore, the symmetry algebra of an open string corresponds to a single classical Virasoro algebra

$$[L_m, L_n] = i(m-n)L_{m+n} \tag{1.9}$$

A physical spectrum of the string can be then constructed by choosing a suitable Hilbert space. For the bosonic string theory, the most common and practical Hilbert space is the Fock space generated by Fourier modes of $\left(P^M \pm \partial_\sigma X^M\right)(\sigma)$ (or $\left(P^M + \partial_\sigma X^M\right)(\sigma)$ for an open string with the doubling trick). At the quantum level, the constraint (or gauge) algebra in (1.8) or (1.9) is anomalous in the Fock space representation except for $\dim\mathscr{M} = 26$ called the critical dimension. Henceforth we will assume $\dim\mathscr{M} = 26$ for the bosonic string theory to avoid the anomaly.

Having a physical spectrum of the string, one can proceed to compute the quantum evolution operator. In string theory, this can be accomplished by employing the path integral on the Hamiltonian action of the string, and again the details of constructing the consistent path integral are model-dependent. In the bosonic string theory, the Hamiltonian action S_H is given by

$$S_H = \int d\tau \int_0^l d\sigma \left(P_M \dot{X}^M - u_+\Phi_+ - u_-\Phi_-\right).$$

By integrating out the canonical momenta P_M, one obtains the covariant form of S_H as

$$S_P = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma J^{ab}\partial_a X^M \partial_b X_M$$

where

$$(J^{ab}) = \begin{pmatrix} \frac{1}{u_+ + u_-} & \frac{-i(u_- - u_+)}{u_+ + u_-} \\ \frac{-i(u_- - u_+)}{u_+ + u_-} & \frac{4u_+u_-}{u_+ + u_-} \end{pmatrix}$$

and we have retrieved the α' dependence by using the dimensional analysis. Also, we have taken the Wick rotation on τ as $\tau = -i\sigma^2$ to obtain the Euclidean path integral.

The precise geometrical interpretation of J^{ab} can be obtained by considering $J_a{}^b = \epsilon_{ac}J^{cb}$ where ϵ_{ac} is the Levi-Civita symbol with $\epsilon_{12} = 1$. $J_a{}^b$ then satisfies [45]

$$J_a{}^c J_c{}^b = -\delta_a^b.$$

which indicates that $J_a{}^b$ is an almost complex structure. Moreover, the almost complex structure $J = J_a{}^b d\sigma^a \otimes \partial_b$ has the vanishing Nijenhuis tensor defined by [46]

$$N(v,w) = [v,w] + J[Jv,w] + J[v,Jw] - [Jv,Jw]$$

where v and w are vector fields on Σ and [,] is the Lie bracket. Consequently, J induces a complex structure on Σ by the Newlander-Nirenberg theorem [47], and Σ becomes a Riemann surface, i.e., a two-dimensional complex manifold.

Since Σ is a Riemann surface, it is convenient to rewrite S_P based on the complex structure $J_a{}^b$:

$$S_P[X,J,\Sigma] = \frac{1}{8\pi\alpha'}\int_{(\Sigma,J)} \partial X^m \wedge \bar{\partial} X_m \tag{1.10}$$

where ∂ and $\bar{\partial}$ are the holomorphic and anti-holomorphic structure on Σ defined by

$$\partial f = (\partial_a f - iJ_a{}^b\partial_b f)d\sigma^a, \quad \bar{\partial} f = (\partial_a f + iJ_a{}^b\partial_b f)d\sigma^a$$

for any smooth function f on Σ, and $\int_{(\Sigma,J)}$ denotes the integration over the Riemann surface Σ with the complex structure J. The action S_P in (1.10) is often taken as a starting point of constructing the path integral of the bosonic string theory and called the Polyakov action [48, 49, 50].

Since the bosonic string theory has gauge symmetries, we need a gauge-fixed action as well as a proper path integral measure to construct the path integral. A modern gauge-fixing procedure based on the field-antifield formalism [51] is discussed in [52], and we do not repeat here. Also, see [45] for a review on the more traditional approach. The resulting path integral with some local operator insertions denoted by the ellipsis is given by [52, 53]

$$\begin{aligned}&\left\langle\left(\prod_{k=1}^{\dim\mathscr{M}_\Sigma}\Psi_k\right)\dots\right\rangle\\&=\int d^{\dim\mathscr{M}_\Sigma}\mu\int\mathscr{D}X\mathscr{D}b\mathscr{D}c\mathscr{D}\bar{b}\mathscr{D}\bar{c}\left(\prod_{k=1}^{\dim\mathscr{M}_\Sigma}\Psi_k\right)\exp(-S_{GF})\dots\end{aligned}\tag{1.11}$$

where (b,c) and $(\bar{b},\bar{c})$ are the holomorphic and anti-holomorphic ghost-antighost pair, and

$$\begin{aligned}S_{GF}=&\frac{1}{2\pi\alpha'}\int_{(\Sigma,\hat{J})}d^2z\partial_zX^m\partial_{\bar{z}}X_m+\frac{1}{2\pi}\int_{(\Sigma,\hat{J})}d^2z\{b\partial_{\bar{z}}c+\bar{b}\partial_z\bar{c}\}\\\Psi_k=&\frac{-i}{4\pi}\int_{(\Sigma,\hat{J})}d^2z\left(b\frac{\partial\hat{J}^z{}_{\bar{z}}}{\partial\mu^k}-\bar{b}\frac{\partial\hat{J}^{\bar{z}}{}_z}{\partial\mu^k}\right).\end{aligned}\tag{1.12}$$

In (1.12) $\hat{J}$ is a fixed-complex structure resulting from the gauge-fixing procedure and $(z,\bar{z})$ are complex coordinates defined by

$$\bar{\partial}z=0,\quad\partial\bar{z}=0.$$

Finally, $\mathscr{M}_\Sigma$ is the moduli space of the Riemann surface Σ and μ^k are local coordinates of $\mathscr{M}_\Sigma$.

1.2.4 String correspondence and string perturbation theory

From the single quantized string discussed in the above, one may attempt to construct the theory of many-string or shortly the string field theory (SFT) to deal with interactions among string states in a complete manner (see [54] for a recent review on the closed SFT). However, in a conventional string theory having the spacetime position of the string as a dynamical degree of freedom, one can construct a computational framework for the perturbative approach to string interactions in the absence of the SFT. For this, we have to notice that the string length induced by the spacetime position of the string implies a correspondence between the physics of strings and that of point particles obtained by taking the string length to be zero.

In QFTs for point particle interactions, the perturbative expansion of a scattering amplitude can be represented by Feynman diagrams consisting of edges and interaction vertices at which edges can be joined together or split into other edges. Each edge represents the quantum propagation of a single point particle, and an interaction vertex corresponds to a spacetime event dressed with a local insertion which specifies the type of the interaction [55]. The propagation of an external particle of a given scattering process is represented by an infinitely long edge, and each order of the perturbative expansion of a scattering amplitude can be obtained by summing over all possible Feynman diagrams with some infinitely long edges having the same number of closed paths, called loops.

The string correspondence, thus, implies that the perturbative expansion of a string scattering amplitude can also be realized by diagrams, called string diagrams, corresponding to

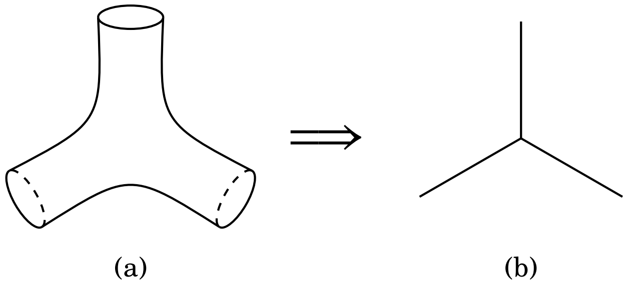


Figure 1.2: (a) A simple string diagram for the interaction of three closed strings and (b) the corresponding Feynman diagram.

Feynman diagrams under the point particle limit. See fig. 1.2 for a simple three-string diagram and the corresponding Feynman diagram. A string diagram can be constructed by joining and splitting of cylinders (for closed strings) and strips (for open strings) representing propagations of string states. Similar to Feynman diagrams with external particles, the propagation of an external string state is represented by an infinitely long cylinder or strip.

In contrast to point particle interactions, however, a joining or splitting process in a string diagram cannot be localized into a spacetime event, since there is no intrinsic notion of the spacetime event at which strings are joined or split. Consequently, the propagation of a string state in the middle of the interaction is locally equivalent to that of a free string, so the string diagram corresponds to the quantum evolution of a free string on the world-sheet whose spacetime projection is homeomorphic to the string diagram.

Having a framework for constructing and computing string diagrams, each order of the perturbative expansion of the scattering amplitude of given external strings can be obtained by summing over all possible string diagrams with some of infinitely long cylinders and strips, and the same number of handles (or holes for a diagram with some boundary), called the genus, corresponding to the number of loops in a Feynman diagram. See fig. 1.3 for an example of the perturbative expansion of an amplitude with three open strings. For an explicit description, let $|\Psi_{+,i}\rangle,\ i=1,\ldots,n_+$ be incoming string states and $|\Psi_{-,i}\rangle,\ i=1,\ldots,n_-$ be outgoing string states of a scattering process. Also, let $(\sigma_{\pm,i},\tau_{\pm,i}),\ i=1,\ldots,n_\pm$ are local coordinates for world-sheets corresponding to the propagation of an incoming or an outgoing string state. Then, a genus g scattering amplitude $\mathscr{A}_g(n_+\to n_-)$ of $n_+ + n_-$ string states can be formally written as

$$\begin{aligned}\mathscr{A}_g(n_-\to n_+) = \sum_k \int \exp(-\frac{i}{\hbar}S_H[A,\Sigma_k])\\ \times \Psi_{+,1}(A;\sigma_{+,1},\tau_{+,1}\to-\infty)\ldots\Psi_{+,n_+}(A;\sigma_{+,n_+},\tau_{+,n_+}\to-\infty)\\ \times \Psi^*_{-,1}(A;\sigma-,1,\tau_{-,1}\to+\infty)\ldots\Psi^*_{-,n_-}(A;\sigma_{-,n_-},\tau_{-,n_-}\to+\infty)\end{aligned}$$

where we have used the following notations:

(1) The world-sheet, denoted by Σ_k, carries an discrete index k which labels inequivalent world-sheets homeomorphic to the given string diagram.

(2) $S_H[A,\Sigma_k]$ is an action for the path integral arising from the string action $S[A,\Sigma_k]$ in (1.1). For the bosonic string theory, S_H corresponds to S_{GF} in (1.12).

(3) $\int$ denotes the path integral over A with a proper measure as in (1.11).

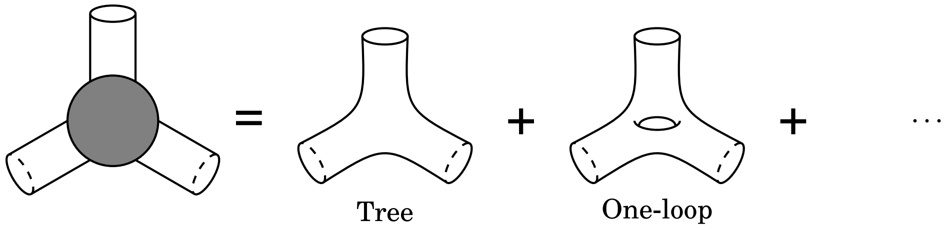


Figure 1.3: The perturbative expansion of a scattering amplitude for three closed strings by using string diagrams.

(4) $\Psi_{\pm,j}(A;\sigma,\tau)$ are wave-functions of external states $\Psi_{\pm,j}$ formally defined by

$$\Psi_{\pm,j}(A;\sigma,\tau) = \langle A(\sigma,\tau)|\Psi_{\pm,j}\rangle.$$

For an open string state, the wave-function is matrix-valued which can be expanded as

$$\Psi_{\pm,j}(A;\sigma,\tau) = \sum_a t^a \Psi_{\pm,j,a}(A;\sigma,\tau)$$

and products among $\Psi_{\pm,j}(A;\sigma,\tau)$ become

$$\mathrm{Tr}(t^{a_1}\dots t^{a_{n_++n_-}})\Psi_{+,1,a_1}(A;\sigma_{+,1},\tau_{+,1})\dots\Psi^*_{-,n_-,a_{n_++n_-}}(A;\sigma_{-,n_-},\tau_{-,n_-})$$

since the joining and splitting process require relevant ends of open strings to be located at the same D-brane. Often the trace part is separated from the amplitude since it is independent of world-sheet degrees of freedom and the remaining part is called the color-ordered amplitude.

1.2.5 String theory as a conformal field theory

As we have seen in the bosonic string theory, a string action may possess the conformal symmetry as a gauge symmetry, so the action defines a two-dimensional conformal field theory (CFT), i.e., a two-dimensional QFT with the conformal symmetry. In the CFT framework, the string spectrum constructed from the Fock space representation can be embedded into a representation consisting of Verma modules, which are highest weight representations of the Virasoro algebra defined by

$$[L_m,L_n] = (m-n)L_{m+n} + \frac{c}{12}m\left(m^2-1\right)\delta_{m+n,0}, \qquad [L_m,c]=0 \qquad \text{for } m,n\in\mathbb{Z}. \tag{1.13}$$

corresponding to the central extension of the classical Virasoro algebra in (1.8) or (1.9) by the central element c. Conventionally, a weight in a Verma module corresponds to an eigenvalue of the representation of L_0, and a highest weight state is called a primary state.

A particularly useful implication of the conformal symmetry is the conformal mapping which maps a world-sheet with infinitely long cylinders or strips to a world-sheet with marked points, called punctures, (see fig. 1.4) so that a string scattering amplitude $\mathscr{A}_g(n_-\to n_+)$ can be rewritten as

$$\mathscr{A}_g(n_-\to n_+) = \sum_k \int \exp(-\frac{i}{\hbar}S_H[A,\Sigma_k])\Psi_{+,1}(A;\sigma_{+,1},\tau_{+,1})\dots\Psi_{+,n_+}(A;\sigma_{+,n_+},\tau_{+,n_+})$$
$$\times\Psi_{-,1}(A;\sigma_{-,1},\tau_{-,1})\dots\Psi_{-,n_-}(A;\sigma_{-,n_-},\tau_{-,n_-})$$

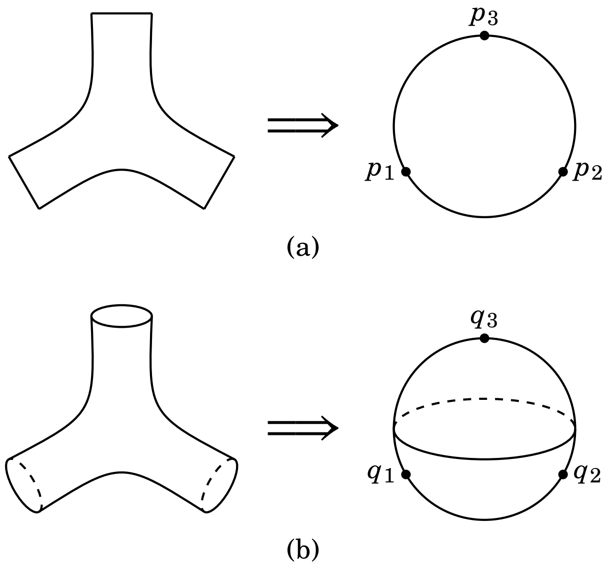


Figure 1.4: (a) The conformal mapping of a string diagram with three open strings into a disk with three punctures (p_1, p_2, p_3) on its boundary, and (b) the conformal mapping of a string diagram with three closed strings into a sphere with three punctures (q_1, q_2, q_3).

where $(\sigma_{\pm,j}, \tau_{\pm,j})$ are local coordinates of punctures. In trun, the computation of the path integral becomes that of a correlator of a 2D CFT on a nontrivial background with some local insertions, i.e.,

$$\begin{aligned}\mathscr{A}_g(n_- \to n_+) \sim \sum_k \big\langle & V_{+,1}(\sigma_{+,1}, \tau_{+,1}) \dots V_{+,n_+}(\sigma_{+,n_+}, \tau_{+,n_+}) \\ & \times V_{-,1}(\sigma_{-,1}, \tau_{-,1}) \dots V_{-,n_-}(\sigma_{-,n_-}, \tau_{-,n_-}) \big\rangle_{\Sigma_k} \end{aligned} \tag{1.14}$$

where $V_{\pm,j}$ are local operators, called vertex operators, representing external states $|\Psi_{\pm,j}\rangle$ and we have omitted the insertion of the measure of the path integral. The relation between a state and a vertex operator emphasized in (1.14) is called the state-operator correspondence. In the bosonic string theory, the vertex operator for an external closed string state can be found in the form of

$$c(z)\bar{c}(\bar{z})V(z,\bar{z})$$

where $V(z,\bar{z})$ is a local operator expressed in matter fields X^m only. The simplest example is the tachyonic vertex operator at the momentum k

$$c(z)\bar{c}(\bar{z})e^{ik\cdot X}(z,\bar{z})$$

which has the negative mass square $-\frac{1}{4\alpha'}$.

The computation of correlators in a 2D CFT is highly constrained again due to the conformal symmetry. In particular, the local behavior of two- and three-point correlators among primary operators, which are vertex operators correspond to primary states, is completely determined by the conformal symmetry up to the normalization. Consequently, the local behavior of two- and three-point correlators of descendant operators corresponding to descendant states constructed from primary states is also governed by the conformal symmetry since correlators of descendant operators are dictated by correlators of primary operators. See [56] for more details. The local information of two- and three-point correlators together with the completeness of the space of local operators guaranteed by the state-operator correspondence then defines the expansion of the product of two operators as a linear combination of local operators, called the operator product expansion (OPE), which establishes the local behavior of n-point correlators.

Finally, it should be emphasized that a CFT can be abstractly defined by a collection of Verma modules and OPEs among primary operators, and does not require any specific Lagrangian formulation [56]. Therefore, a CFT may allow several different realizations which have their benefit for computing correlators. A well-known example is the bosonization of a fermionic system, which is a description of the fermionic system based on bosonic operators. In chapter 3, we will extensively use this bosonization technique to compute correlators among fermionic fields.

1.3 Superstring theory

1.3.1 RNS superstring theory

The string spectrum of the bosonic string theory is insufficient to embed the Hilbert space of elementary particle states since it contains no fermionic state. The *Ramond-Neveu-Schwarz (RNS) superstring theory* is an appealing resolution of this problem proposed by [57, 58, 59], which extends the bosonic string theory with world-sheet fermionic degrees of freedom ψ^m. Here, we provide a short survey on core ideas underpinning the RNS superstring theory, and for more comprehensive discussions we refer to reviews [45, 53].

We begin with an observation that for a point particle, one can obtain a fermionic spectrum by enlarging the phase space consisting of the spacetime position x^m and the corresponding canonical momentum p_m of a point particle with a set of dynamical variables γ^m satisfying the Clifford algebra

$$\{\gamma^m, \gamma^n\}_{PB} = \eta^{mn},$$

through the Poisson bracket $\{\cdot,\cdot\}_{PB}$, and a constraint

$$\gamma^m p_m = 0$$

equivalent to Dirac equations for a massless fermion.

In [57], the world-sheet analogue of the construction of the fermionic spectrum of a point particle has been proposed by introducing a set of world-sheet fields $\psi^m(\sigma)$ which form an affinization of the Clifford algebra

$$\{\psi^m(\sigma), \psi^n(\sigma')\} = \eta^{mn}\delta(\sigma-\sigma'), \tag{1.15}$$

and satisfy a constraint either

$$G_+ = \left(P^m + \partial_\sigma X^m\right)\psi_m(\sigma) = 0 \qquad \text{or} \qquad G_- = \left(P^m - \partial_\sigma X^m\right)\psi_m(\sigma) = 0$$

for a closed string, and

$$G = \left(P^m + \partial_\sigma X^m\right)\psi_m(\sigma) = 0$$

for an open string subject to the doubling trick. For a closed and a doubled open string, the Poisson bracket structure in (1.15) indicates that $\psi^m(\sigma)$ has the periodic boundary condition called the Ramond (R) boundary condition.

Also, it has been noticed in [58, 59] that world-sheet fermions $\psi^m(\sigma)$ with the anti-periodic boundary condition, called the Neveu-Schwarz (NS) boundary condition, $\psi^m(\sigma + l) = -\psi^m(\sigma)$

can generate the spectrum of bosonic states alternative to the spectrum of the bosonic string theory. Accordingly, the world-sheet fermionic extension of the bosonic string theory leads us to a formulation of string theory which naturally incorporates with interactions among spacetime bosons and fermions.

However, it turns out that the naive spectrum of the RNS superstring theory signals an instability of the vacuum through the presence of a tachyonic state. In fact, the mathematical consistency of string interactions in the RNS superstring theory requires truncation of the spectrum through the so-called GSO projection [60, 61], which also eliminates the tachyon. Moreover, the truncated spectrum and interactions among states in the spectrum admit an implicit symmetry called the spacetime supersymmetry relating bosonic and fermionic states.

The dynamics of the RNS superstring can be defined by the Hamiltonian H_{RNS} of the system

$$H_{RNS} = \begin{cases} \int_0^l d\sigma \left(u_+ \Phi_+ + u_- \Phi_- + u_\psi \Phi_\psi + v_+ G_+ \right) & \text{for a closed string,} \\ \frac{1}{2} \int_0^l d\sigma \left(u \Phi + + u_\psi \Phi_\psi + v G \right) & \text{for a doubled open string,} \end{cases}$$

where $u_\pm$, u, u_ψ, v_+ and v are Lagrange multipliers and

$$\Phi_\psi = \psi^m \partial_\sigma \psi_m$$

is a secondary constraint arising from the standard Dirac procedure. Also, it is straightforward to show that $\Phi_\pm$ (or Φ), Φ_ψ and G_+ (or G) are first-class, so they are generators of gauge transformations. More precisely, $\Phi_\pm$ (or Φ) and Φ_ψ are generators of the conformal symmetry, and the symmetry generated by G_+ (or G) is called the world-sheet supersymmetry to distinguish from the spacetime supersymmetry existing in the superstring spectrum. The conformal symmetry generated by above constraints has a quantum anomaly in the Fock space representation, which can be removed by choosing $u_+ = u_\psi$ (or $u = u_\psi$) and $\dim \mathscr{M} = 10$. As in the bosonic string theory, we will assume these conditions in the following discussion.

On the basis of H_{RNS} one can proceed to construct the quantum evolution of a superstring state by employing the path integral in the same spirit of the bosonic string theory. Again the path integral can be systematically defined by employing the covariant action obtained by integrating out the canonical momentum P_m in the Hamiltonian action [62]

$$S_H = \begin{cases} \int_\Sigma d^2\sigma \left\{ P_m \dot{X}^m + \psi^m \dot{\psi}_m - u_+ \left(\Phi_+ + \Phi_\psi \right) - u_- \Phi_- - v_+ G_+ \right\} & \text{for a closed string,} \\ \frac{1}{2} \int_\Sigma d^2\sigma \left\{ P_m \dot{X}^m + \psi^m \dot{\psi}_m - u \left(\Phi_+ + \Phi_\psi \right) - v G \right\} & \text{for a doubled open string.} \end{cases} \tag{1.16}$$

The explicit form of the resulting covariant action can be found in [48, 49, 63], and we do not present here.

The superstring spectrum and the quantum evolution of a string state then enable us to construct perturbative superstring amplitudes whose computation relies on the CFT framework defined by the superstring action. In chapter 3, we will develop some technical tools which can be applied to compute one-loop superstring amplitudes of the RNS superstring theory.

As a side remark, for a closed string, one can extend the bosonic string theory with two sets world-sheet fermions $\psi^m_\pm$ with two constraints

$$G_\pm = (P^m \pm \partial_\sigma X^m)\psi_{\pm m}(\sigma) = 0$$

which leads to a different extension of the bosonic string theory. The extension with a single set of world-sheet fermions is categorized by the heterotic superstring theory and with two sets of world-sheet fermions is categorized by the type II superstring theory.

1.3.2 Pure spinor superstring theory

The *pure spinor (PS) formalism* is an alternative formulation of the superstring theory where spacetime supersymmetry is manifest. It has been firstly proposed by [15] and extensively applied to compute superstring amplitudes due to the explicit spacetime supersymmetric form of resulting amplitudes.

Arguably, the simplest way of constructing the PS superstring action is taking it as a world-sheet extension of the supersymmetric point particle action relevant to the low energy theory of the superstring theory. For this, we begin with equations of motion of the 10D SYM which is the low energy effective theory for the massless states of the open superstring theory. Let m,n again denote spacetime vector indices and $\alpha,\beta = 1,\ldots,16$ denote spacetime spinor indices. We define supercovariant derivatives [64, 65],

$$\nabla_\alpha \equiv D_\alpha - \mathbb{A}_\alpha, \qquad \nabla_m \equiv \partial_m - \mathbb{A}_m \tag{1.17}$$

with a Lie algebra-valued spinor and vector potential $\mathbb{A}_\alpha$ and $\mathbb{A}_m$. The fermionic differential operators

$$D_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\left(\gamma^m\theta\right)_\alpha \partial_m, \qquad \left\{D_\alpha, D_\beta\right\} = \gamma^m_{\alpha\beta}\partial_m \tag{1.18}$$

involve the 16×16 Pauli matrices $\gamma^m_{\alpha\beta} = \gamma^m_{\beta\alpha}$ subject to the Clifford algebra $\gamma^{(m}_{\alpha\beta}\gamma^{n)\beta\gamma} = 2\eta^{mn}\delta^\gamma_\alpha$, and the convention for (anti)symmetrizing indices does not include $\frac{1}{2}$. The constraint equations $\left\{\nabla_\alpha, \nabla_\beta\right\} = \gamma^m_{\alpha\beta}\nabla_m$ together with Bianchi identities then lead to the equations of motion of the 10D SYM [65],

$$\left\{\nabla_\alpha, \nabla_\beta\right\} = \gamma^m_{\alpha\beta}\nabla_m, \tag{1.19a}$$

$$\left[\nabla_\alpha, \nabla_m\right] = -\left(\gamma_m \mathbb{W}\right)_\alpha, \tag{1.19b}$$

$$\left\{\nabla_\alpha, \mathbb{W}^\beta\right\} = \frac{1}{4}\left(\gamma^{mn}\right)_\alpha{}^\beta \mathbb{F}_{mn}, \tag{1.19c}$$

$$\left[\nabla_\alpha, \mathbb{F}^{mn}\right] = \left(\mathbb{W}^{[m}\gamma^{n]}\right)_\alpha, \tag{1.19d}$$

for a spinor field $\mathbb{W}_\alpha$ and

$$\mathbb{F}_{mn} \equiv -\left[\nabla_m, \nabla_n\right], \qquad \mathbb{W}^\alpha_m \equiv \left[\nabla_m, \mathbb{W}^\alpha\right].$$

It is straightforward to check that equations in (1.19) are invariant under the gauge transformations

$$\begin{aligned} \delta_\Omega \mathbb{A}_\alpha &= [\nabla_\alpha, \Omega], & \delta_\Omega \mathbb{A}_m &= [\nabla_m, \Omega], \\ \delta_\Omega \mathbb{W}^\alpha &= \left[\Omega, \mathbb{W}^\alpha\right], & \delta_\Omega \mathbb{F}^{mn} &= \left[\Omega, \mathbb{F}^{mn}\right], \end{aligned} \tag{1.20}$$

with a Lie algebra-valued gauge parameter $\Omega = \Omega(x,\theta)$.

In order to find a point particle relevant to the 10D SYM, we note that the constraint equations (1.19a) are equivalent to the integrability condition on $\mathbb{F}_{\alpha\beta} := \left\{\nabla_\alpha, \nabla_\beta\right\} - \gamma^m_{\alpha\beta}\nabla_m$ along a pure spinor line defined by [66]

$$x^m = x^m_0 + t\lambda^\alpha\gamma^m_{\alpha\beta}\bar{\lambda}^\beta, \qquad \theta^\alpha = \theta^\alpha_0 + \bar{\zeta}\lambda^\alpha - \zeta\bar{\lambda}^\alpha$$

where λ^α is a commuting pure spinor subject to the pure spinor condition $\lambda^\alpha\gamma^m_{\alpha\beta}\lambda^\beta = 0$ and $(t,\zeta,\bar{\zeta})$ are parameters for the pure spinor line with respect to the reference point $\left(x^m_0, \theta^\alpha_0\right)$. The integrability condition corresponds the BRST formalism of twistor-like constraints [67, 68]

$$\phi_\alpha = p_m\gamma^m_{\alpha\beta}\lambda^\beta = 0 \tag{1.21}$$

on the extended phase space of a point particle whose BRST charge Q_B is given by

$$Q_B = \lambda^\alpha \left(p_\alpha + p_m \gamma^m_{\alpha\beta}\theta^\beta\right)$$

with the ghosts θ^α and the antighosts p_α related to constraints in (1.21). By choosing an appropriate gauge one can obtain a covariant Hamiltonian action [67]

$$S = \int d\tau \left(\frac{1}{2}\dot{x}_m \dot{x}^m + \pi_\alpha \dot{\lambda}^\alpha + p_\alpha \dot{\theta}^\alpha\right) \tag{1.22}$$

called the pure spinor superparticle action.

The string action of the PS open superstring theory can be then found as the world-sheet extension of (1.22) in the same way as the Nambu-Goto action (1.2) of the bosonic string theory is the world-sheet extension of the world-line action. Explicitly, the PS open superstring action under the doubling trick is given by [15]

$$S_{PS} = \frac{1}{2\pi}\int d^2 z \left(\frac{1}{2}\partial X^m \bar{\partial} X_m + p_\alpha \bar{\partial}\theta^\alpha - \omega_\alpha \bar{\partial}\lambda^\alpha\right)$$

with world-sheet fields $\{X^m(z,\bar{z}), \theta^\alpha(z), p_\alpha(z), \lambda^\alpha(z), \omega_\alpha(z)\}$ which are world-sheet counterparts of dynamical variables in (1.22).

The physical spectrum is defined by the cohomology of the world-sheet extension Q of Q_B

$$Q = \oint \lambda^\alpha(z) d_\alpha(z), \qquad d_\alpha(z) = p_\alpha - \frac{1}{2}(\gamma^m\theta)_\alpha \partial X_m - \frac{1}{8}(\gamma^m\theta)_\alpha(\theta\gamma_m\partial\theta)$$

and contains the massless open superstring state corresponding to the vertex operator

$$V = \lambda^\alpha A_\alpha(X,\theta) = \sum_a \lambda^\alpha t^a A_\alpha(X,\theta;a) \tag{1.23}$$

where t^a are generators of the given Lie algebra as in (1.5). The BRST-closedness enforces $A_\alpha(X,\theta)$ to be the linearized field of the spinor potential $\mathbb{A}_\alpha$ in (1.17) satisfying linearized equations of motion obtained by discarding the quadratic terms in (1.19)

$$\left\{D_{(\alpha}, A_{\beta)}\right\} = \gamma^m_{\alpha\beta} A_m, \tag{1.24a}$$
$$[D_\alpha, A_m] = (\gamma_m W)_\alpha + [\partial_m, A_\alpha], \tag{1.24b}$$
$$\left\{D_\alpha, W^\beta\right\} = \frac{1}{4}\left(\gamma^{mn}\right)_\alpha{}^\beta F_{mn}, \tag{1.24c}$$
$$[D_\alpha, F_{mn}] = \left[\partial_{[m}, (\gamma_{n]} W)_\alpha\right] . \tag{1.24d}$$

where superfields $\{A_m, W^\alpha, F_{mn}\}$ are linearized fields of $\{\mathbb{A}_m, \mathbb{W}^\alpha, \mathbb{F}_{mn}\}$. These equations are invariant under the linearized gauge transformations

$$\delta_\Omega A_\alpha = \left[D_\alpha, \Omega\right], \qquad \delta_\Omega A_m = \left[\partial_m, \Omega\right], \qquad \delta_\Omega W^\alpha = 0, \qquad \delta_\Omega F^{mn} = 0$$

for a Lie algebra-valued gauge parameter Ω, which render $\delta_\Omega V$ BRST exact. Also, linearized superfields $\{A_\alpha, A^m, W^\alpha, F_{mn}\}$ can be expanded in θ^α by using the Harnad-Shnider (HS) gauge

$\theta^\alpha A_\alpha = 0$ [69, 70],

$$A_\alpha(\theta) = \frac{1}{2}(\theta\gamma_m)_\alpha e^m + \frac{1}{3}(\theta\gamma_m)_\alpha(\theta\gamma^m\chi) - \frac{1}{32}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)f_{np} \tag{1.25a}$$
$$+ \frac{1}{60}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)(\chi\gamma_p\theta)k_n + \frac{1}{1152}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)(\theta\gamma^{pqr}\theta)f^{qr}k^n + \mathcal{O}\left(\theta^6\right)$$
$$A^m(\theta) = e^m + (\theta\gamma^m\chi) - \frac{1}{8}(\theta\gamma^{mpq}\theta)f^{pq} + \frac{1}{12}(\theta\gamma^{mnp}\theta)(\chi\gamma^p\theta)k^n \tag{1.25b}$$
$$+ \frac{1}{192}(\theta\gamma^m{}_{nr}\theta)(\theta\gamma^r{}_{pq}\theta)f^{pq}k^n - \frac{1}{480}(\theta\gamma^m{}_{nr}\theta)(\theta\gamma^r{}_{pq}\theta)(\chi\gamma^q\theta)k^n k^p + \mathcal{O}\left(\theta^6\right)$$
$$W^\alpha(\theta) = \chi^\alpha + \frac{1}{4}(\theta\gamma^{mn})^\alpha f_{mn} - \frac{1}{4}(\theta\gamma_{mn})^\alpha(\chi\gamma^n\theta)k^m - \frac{1}{48}(\theta\gamma_m{}^q)^\alpha(\theta\gamma_{qnp}\theta)f^{np}k^m \tag{1.25c}$$
$$+ \frac{1}{96}(\theta\gamma_m{}^q)^\alpha(\theta\gamma_{qnp}\theta)(\chi\gamma^p\theta)k^m k^n - \frac{1}{1920}(\theta\gamma_m{}^r)^\alpha(\theta\gamma_{nr}{}^s\theta)(\theta\gamma_{spq}\theta)f^{pq}k^m k^n + \mathcal{O}\left(\theta^6\right)$$
$$F^{mn}(\theta) = f^{mn} - k^{[m}(\chi\gamma^{n]}\theta) + \frac{1}{8}(\theta\gamma_{pq}{}^{[m}\theta)k^{n]}f^{pq} - \frac{1}{12}(\theta\gamma_{pq}{}^{[m}\theta)k^{n]}k^p(\chi\gamma^q\theta) \tag{1.25d}$$
$$- \frac{1}{192}(\theta\gamma_{ps}{}^{[m}\theta)k^{n]}k^p f^{qr}(\theta\gamma^s{}_{qr}\theta) + \frac{1}{480}(\theta\gamma^{[m}{}_{ps}\theta)k^{n]}(\chi\gamma^r\theta)k^p k^q(\theta\gamma^s{}_{qr}\theta) + \mathcal{O}\left(\theta^6\right)$$

These θ-expansions are understood to be accompanied by plane waves, e.g. $A_\alpha(x,\theta) = A_\alpha(\theta)e^{k\cdot x}$. The bosonic and fermionic polarizations e_m and χ^α correspond to gluons and gluinos, respectively, and we denote the linearized gluon field strength by $f_{mn} = k_m e_n - k_n e_m$.

As in other formulations of string theory, scattering amplitudes in the PS superstring theory are then defined by conformal correlators with vertex operator insertions as well as the proper measure insertion. In the next chapter, we will discuss tree level superstring amplitudes of massless states computed by inserting vertex operators in the form of (1.23).

TWO

TREE LEVEL OPEN SUPERSTRING AMPLITUDES AND 10D SYM

In this chapter, we discuss the computation of tree level scattering amplitudes of massless open superstrings in the PS superstring theory with a particular emphasis on the role of the 10D SYM equations for the computation. In the following discussions, we often take $2\alpha' = 1$ unless otherwise specified.

2.1 Tree level superstring amplitudes in the PS superstring theory

In the PS superstring theory, a world-sheet relevant to the tree level string diagram of N open strings can be conformally mapped to a disk with N punctures at the boundary of the disk. The latter can be further mapped to the upper-half plane with the real axis corresponding to the boundary of the disk as in fig. 2.1. Also, the insertion of the path integral measure can be taken into account by inserting the $N-3$ integrated form of the vertex operator. For the massless open string vertex operator given in (1.23) the integrated vertex operator has the form of

$$U = \int dz \left(\partial\theta^\alpha A_\alpha + \Pi^m A_m + d_\alpha W^\alpha + \frac{1}{2}F_{mn}N^{mn}\right), \qquad \text{for} \begin{cases} N^{mn} = \frac{1}{2}\left(\lambda\gamma^{mn}\omega\right), \\ \Pi^m = \partial X^m + \frac{1}{2}(\theta\gamma^m\partial\theta) \end{cases} \tag{2.1}$$

where the integral is taken over the real axis. Consequently, the N-point tree level superstring amplitude for massless open strings is given by

$$\begin{aligned} \mathscr{A}_N &= \left\langle V(z_1)V(z_{N-1})V(z_N)\prod_{i=2}^{N-2} U \right\rangle \\ &= \mathrm{Tr}\left(t^{a_1}\dots t^{a_N}\right)\left\langle V_{a_1}(z_1)V_{a_{N-1}}(z_{N-1})V_{a_N}(z_N)\prod_{i=2}^{N-2} U_{a_i}\right\rangle \end{aligned} \tag{2.2}$$

where a_i denote color degrees of freedom. It is important to note that the bracket $\langle\dots\rangle$ is normalized as

$$\left\langle \left(\lambda\gamma^m\theta\right)\left(\lambda\gamma^n\theta\right)\left(\lambda\gamma^p\theta\right)\left(\theta\gamma_{mnp}\theta\right)\right\rangle = 2880 \tag{2.3}$$

due to zero modes of λ^α and θ^α.

As we have discussed in section 1.2.5, the computation of $\mathscr{A}_N$ in (2.2) can be implemented by employing the CFT framework defined by the PS superstring action. The relevant CFT can be summarized as the following:

(1) The energy-momentum tensor $T(z)$ given by

$$T(z) = -\frac{1}{2}\Pi^m\Pi_m - d_\alpha\partial\theta^\alpha + \omega_\alpha\partial\lambda^\alpha.$$

(2) Conformal primaries relevant to superstring amplitudes in the PS formulation generated by

$$\left\{e^{ik\cdot X}, \Pi^m, d_\alpha, \theta^\alpha, \omega_\alpha, \lambda^\alpha\right\}$$

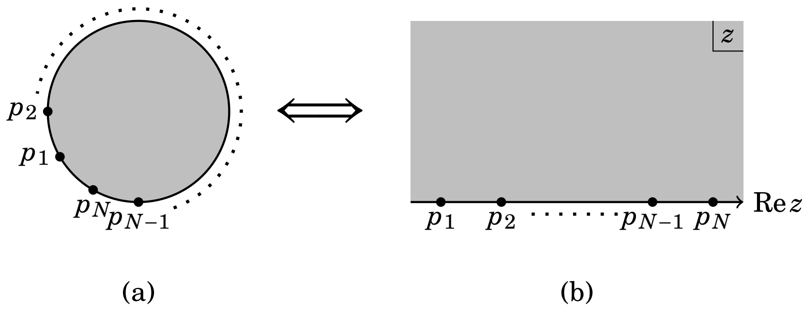


Figure 2.1: (a) A disk with N punctures $(p_1, p_2, \ldots, p_{N-1}, p_N)$ on its boundary and (b) the corresponding upper half-plane with the real axis.

whose OPEs are given by [71, 15]

$$\begin{aligned}
e^{ik_1\cdot X(z,\bar{z})}e^{ik_2\cdot X(w,\bar{w})} &= |z-w|^{2k_1\cdot k_2}e^{i(k_1+k_2)\cdot X(w,\bar{w})}(1+O(z-w,\bar{z}-\bar{w}))\\
\Pi^m(z)e^{ik\cdot X(w,\bar{w})} &= -i(z-w)^{-1}k^m e^{ik\cdot X(w,\bar{w})}+\ldots\\
d_\alpha(z)e^{ik\cdot X}(w) &= \frac{i}{2}(z-w)^{-1}(\gamma^m\theta)k_m e^{ik\cdot X}+\ldots\\
\Pi^m(z)\Pi^n(w) &= -\frac{\eta^{mn}}{(z-w)^2}+\ldots\\
\Pi^m(z)d_\alpha(w) &= -(z-w)^{-1}(\gamma^m\partial\theta)_\alpha+\ldots\\
d_\alpha(z)d_\beta(w) &= -(z-w)^{-1}\gamma^m_{\alpha\beta}\Pi_m+\ldots\\
d_\alpha(z)\theta^\beta(w) &= (z-w)^{-1}\delta^\beta_\alpha+\ldots\\
\omega_\alpha(z)\lambda^\beta(w) &= (z-w)^{-1}\delta^\beta_\alpha+\ldots
\end{aligned}$$

where we have omitted non-singular OPEs.

By using the CFT above, the color-ordered part of (2.2) has been computed in the series of works [72, 73, 74] for $N=4,5,6$ and a closed form of the aribtrary N have been found in [14] as

$$A_N = \prod_{i=2}^{N-2}\int_{z_{i-1}}^{1} dz_i \prod_{\substack{j,k=1\\ j<k}}^{N}\left|z_{jk}\right|^{-s_{jk}}\left(\prod_{l=2}^{N-2}\sum_{m=1}^{l-1}\frac{s_{ml}}{z_{ml}}A_{SYM}(1,2,\ldots,N)+\mathscr{P}(2,3,\ldots,N-2)\right)$$

where (z_1, z_{N-1}, z_N) are taken to be $(0,1,\infty)$ with $z_{ij}=z_i-z_j$ for complex coordinates z_i ($i=2,\ldots,N-2$) of integrated vertex operators, and $s_{12\ldots j}=\alpha'\left(k_1+k_2+\cdots+k_j\right)^2$ for external momenta k_i. Also, $1,2,\ldots,N$ stand for shorthanded notations for momentum-polarization pairs of vertex operator insertions and $\mathscr{P}(2,3,\ldots,N-2)$ denotes the summation over all possible permutations of $\{2,3,\ldots,N-2\}$. $A_{SYM}(1,2,\ldots,N)$ denotes the z_i-independent kinematic factor called the color-ordered N-point SYM subamplitude, since under the limit $\alpha'\to 0$,

$$A_{SYM}(1,\rho(2),\rho(3),\ldots,\rho(N-2),N-1,N), \qquad \rho\in S_{N-3}$$

form a basis of the N-point color-ordered 10D SYM amplitude [4, 75, 76].

2.1.1 SYM amplitudes in pure-spinor superspace

The explicit form of $A_{SYM}(1,2,\ldots,N)$ has been obtained by using successive OPEs between an unintegrated vertex operator [14] and an integrated operator, and further simplified in [77] by using OPEs between integrated vertex operators. Both computations lead to the expression

$$A_{SYM}(1,2,\ldots,N) = \sum_{j=1}^{N-2} \langle M_{12\ldots j} M_{j+1 j+2\ldots N-1} M_N \rangle \tag{2.5}$$

with $M_{12\ldots j} = \lambda^\alpha \mathscr{A}_\alpha^{12\ldots j}(\theta)$ for some non-local multiparticle superfields $\mathscr{A}_\alpha^{12\ldots j}$.

The multiparticle superfields $\mathscr{A}_\alpha^{P=12\ldots p}$ satisfy the following equations [14]

$$D_{(\alpha}\mathscr{A}^P_{\beta)} = \gamma^m_{\alpha\beta}\mathscr{A}^P_m + \sum_{XY=P}\left(\mathscr{A}^X_\alpha \mathscr{A}^Y_\beta - \mathscr{A}^Y_\alpha \mathscr{A}^X_\beta\right) \tag{2.6a}$$

which induce further non-local multiparticle superfields $\{\mathscr{A}^P_m, \mathscr{W}^\alpha_P, \mathscr{F}^P_{mn}\}$ satisfying

$$D_\alpha \mathscr{A}^P_m = k^P_m \mathscr{A}^P_\alpha + \left(\gamma_m \mathscr{W}_P\right)_\alpha + \sum_{XY=P}\left(\mathscr{A}^X_\alpha \mathscr{A}^Y_m - \mathscr{A}^Y_\alpha \mathscr{A}^X_m\right) \tag{2.6b}$$

$$D_\alpha \mathscr{W}^\beta_P = \tfrac{1}{4}(\gamma^{mn})_\alpha{}^\beta \mathscr{F}^P_{mn} + \sum_{XY=P}\left(\mathscr{A}^X_\alpha \mathscr{W}^\beta_Y - \mathscr{A}^Y_\alpha \mathscr{W}^\beta_X\right) \tag{2.6c}$$

$$\begin{aligned} D_\alpha \mathscr{F}^{mn}_P &= k^{[m}_P(\gamma^{n]}\mathscr{W}_P)_\alpha + \sum_{XY=P}\left(\mathscr{A}^X_\alpha \mathscr{F}^{mn}_Y - \mathscr{A}^Y_\alpha \mathscr{F}^{mn}_X\right) \\ &\quad + \sum_{XY=P}\left(\mathscr{A}^{[n}_X(\gamma^{m]}\mathscr{W}_Y)_\alpha - \mathscr{A}^{[n}_Y(\gamma^{m]}\mathscr{W}_X)_\alpha\right). \end{aligned} \tag{2.6d}$$

Here, the multiparticle momentum k^m_P is defined by

$$k^m_P \equiv k^m_1 + \cdots + k^m_p$$

and the summation over multiparticle labels $XY = P$ instructs to deconcatenate $P = 1\ldots p$ into non-empty words $X = 1\ldots j$ and $Y = j+1\ldots p$ with $j = 1,\ldots,p-1$. Equations in (2.6) are invariant under non-linear gauge transformations

$$\delta\mathscr{A}^P_\alpha = D_\alpha\Omega_P + \sum_{XY=P}\Omega_X\mathscr{A}^Y_\alpha, \qquad \delta\mathscr{A}^P_m = \partial_m\Omega_P + \sum_{XY=P}\Omega_X\mathscr{A}^Y_m,$$
$$\delta\mathscr{W}^\alpha_P = \sum_{XY=P}\Omega_X\mathscr{W}^\alpha_Y, \qquad \delta\mathscr{F}^P_{mn} = \sum_{XY=P}\Omega_X\mathscr{F}^Y_{mn}.$$

for a multiparticle gauge parameter Ω_P.

Also, the multiparticle index P of $\mathscr{K}_P \in \{\mathscr{A}^P_\alpha, \mathscr{A}^P_m, \mathscr{W}^\alpha_P, \mathscr{F}^P_{mn}\}$ satisfies the shuffle relation

$$\mathscr{K}_{A\sqcup\!\sqcup B} = 0, \qquad \forall A, B \neq \emptyset \text{ and } P = AB$$

where $\sqcup\!\sqcup$ denotes the shuffle product[1], so $\mathscr{K}_P$ uniquely define the Lie algebra-valued generating series $\mathbb{K} \in \{\mathbb{A}_\alpha, \mathbb{A}^m, \mathbb{W}^\alpha, \mathbb{F}^{mn}\}$

$$\begin{aligned} \mathbb{K} &= \mathscr{K}_{a_1}t^{a_1} + \mathscr{K}_{a_1a_2}t^{a_1}t^{a_2} + \mathscr{K}_{a_1a_2a_3}t^{a_1}t^{a_2}t^{a_3} + \ldots \\ &= \mathscr{K}_{a_1}t^{a_1} + \tfrac{1}{2}\mathscr{K}_{a_1a_2}[t^{a_1}, t^{a_2}] + \tfrac{1}{3}\mathscr{K}_{a_1a_2a_3}[[t^{a_1}, t^{a_2}], t^{a_3}] + \ldots \\ &= \sum_{p=1}^{\infty}\sum_{a_1,a_2,\ldots,a_p}\frac{1}{p}\,\mathscr{K}_{a_1a_2\ldots a_p}\left[t^{a_1},\left[t^{a_2},\ldots,\left[t^{a_{p-1}}, t^{a_p}\right]\right]\ldots\right]. \end{aligned}$$

[1]The shuffle product $\sqcup\!\sqcup$ between the words $A = a_1a_2\ldots a_{|A|}$ and $B = b_1b_2\ldots b_{|B|}$ is defined recursively by

$$\emptyset \sqcup\!\sqcup A = A \sqcup\!\sqcup \emptyset = A, \qquad A \sqcup\!\sqcup B \equiv a_1(a_2\ldots a_{|A|} \sqcup\!\sqcup B) + b_1(b_2\ldots b_{|B|} \sqcup\!\sqcup A),$$

and $\emptyset$ denotes the empty word.

Nonlinear equations of motion in (2.6) can be then cast into the form of [78]

$$\{D_{(\alpha},\mathbb{A}_{\beta)}\} = \frac{1}{4}\gamma^m_{\alpha\beta}\mathbb{A}_m + \{\mathbb{A}_\alpha,\mathbb{A}_\beta\}, \tag{2.7a}$$

$$[D_\alpha,\mathbb{A}_m] = [\partial_m,\mathbb{A}_\alpha] + (\gamma_m\mathbb{W})_\alpha + [\mathbb{A}_\alpha,\mathbb{A}_m], \tag{2.7b}$$

$$\left\{D_\alpha,\mathbb{W}^\beta\right\} = \frac{1}{4}(\gamma^{mn})_\alpha{}^\beta\mathbb{F}_{mn} + \left\{\mathbb{A}_\alpha,\mathbb{W}^\beta\right\}, \tag{2.7c}$$

$$\left[D_\alpha,\mathbb{F}^{mn}\right] = \left(\mathbb{W}^{[m}\gamma^{n]}\right)_\alpha + \left[\mathbb{A}_\alpha,\mathbb{F}^{mn}\right], \tag{2.7d}$$

which are equations of motion of the 10D SYM obtained by inserting definitions in (1.17) into (1.19). The non-local multiparticle superfields $\mathscr{K}_P$ are often called the supersymmetric Berends-Giele currents of the 10D SYM since they are the supersymmetric analogue of the original Berends-Giele currents of the Yang-Mills theory [79].

2.2 Perturbative solutions of the 10D SYM and Berernds-Giele currents in the Lorenz gauge

In section 2.1 we have seen that the generating series of supersymmetric Berends-Giele currents form a perturbative solution of the equations of motion for the 10D SYM. This observation together with the gauge invariance of $A_{SYM}(1,\ldots,N)$ indicates that we can construct those Berends-Giele currents by perturbatively solving the 10D SYM equations instead of relying on the CFT framework. For this, we note that

$$\Box\mathbb{K} = \left[\partial_m,\left[\partial^m,\mathbb{K}\right]\right]$$

for any Lie-algebra valued function $\mathbb{K}$, so by Jacobi identities and repeated use of $\partial^m = \nabla^m + \mathbb{A}^m$, we have the wave equation of the 10D SYM as

$$\begin{aligned}\Box\mathbb{K} &= [\nabla^m + \mathbb{A}^m,[\partial_m,\mathbb{K}]]\\ &= [[\nabla^m,\partial_m],\mathbb{K}] + [\mathbb{A}^m,[\partial_m,\mathbb{K}]] + [\mathbb{A}^m,[\nabla_m,\mathbb{K}]] + [\nabla^m,[\nabla_m,\mathbb{K}]].\end{aligned} \tag{2.8}$$

We then impose the Lorenz gauge $[\partial_m,\mathbb{A}^m]=0$ so that the first term in the second line vanishes, and by inserting $\mathbb{K}\to\{\nabla_\alpha,\nabla_m,\mathbb{W}^\alpha\}$ into (2.8), we find wave equations for $\mathbb{A}_\alpha,\mathbb{A}_m,\mathbb{W}^\alpha$ as

$$\Box\mathbb{A}_\alpha = \left[\mathbb{A}_m,\left[\partial^m,\mathbb{A}_\alpha\right]\right] + \left[\mathbb{A}_m,(\gamma^m\mathbb{W})_\alpha\right] \tag{2.9a}$$

$$\Box\mathbb{A}_m = \left[\mathbb{A}_p,\left[\partial^p,\mathbb{A}^m\right]\right] + \left[\mathbb{F}^{mp},\mathbb{A}_p\right] + \gamma^m_{\alpha\beta}\left\{\mathbb{W}^\alpha,\mathbb{W}^\beta\right\} \tag{2.9b}$$

$$\Box\mathbb{W}^\alpha = \left[\partial_m,\left[\mathbb{A}_n,(\gamma^m\gamma^n\mathbb{W})^\alpha\right]\right]. \tag{2.9c}$$

One can see that the right-hand sides of (2.9) contain only non-linear terms in superfields, thus they can be solved in a perturbative manner.

In order to find a perturbative solution of (2.9), we expand $\mathbb{K}\in\{\mathbb{A}_\alpha,\mathbb{A}_m,\mathbb{W}^\alpha\}$ by $\mathscr{K}_P\in\left\{\mathscr{A}^P_\alpha,\mathscr{A}^P_m,\mathscr{W}^\alpha_P\right\}$ as [78]

$$\mathbb{K} = \sum_{p=1}^{\infty}\sum_{a_1,a_2,\ldots,a_p}\frac{1}{p}\,\mathscr{K}_{a_1a_2\ldots a_p}\left[t^{a_1},\left[t^{a_2},\ldots,\left[t^{a_{p-1}},t^{a_p}\right]\right]\ldots\right]. \tag{2.10}$$

Equations in (2.9) then imply Berends-Giele recursions in the form of

$$\mathscr{K}_P \equiv \frac{1}{s_P}\sum_{XY=P}\mathscr{K}_{[X,Y]}, \qquad s_P = \frac{1}{2}k_P^2 \tag{2.11}$$

where

$$\mathscr{A}_\alpha^{[P,Q]} \equiv -\frac{1}{2}\Big[\mathscr{A}_\alpha^P(k^P\cdot\mathscr{A}^Q)+\mathscr{A}_m^P(\gamma^m\mathscr{W}^Q)_\alpha-(P\leftrightarrow Q)\Big] \tag{2.12a}$$

$$\mathscr{A}_m^{[P,Q]} \equiv -\frac{1}{2}\Big[\mathscr{A}_m^P(k^P\cdot\mathscr{A}^Q)+\mathscr{A}_n^P\mathscr{F}_{mn}^Q-(\mathscr{W}^P\gamma_m\mathscr{W}^Q)-(P\leftrightarrow Q)\Big] \tag{2.12b}$$

$$\mathscr{W}^\alpha_{[P,Q]} \equiv \frac{1}{2}\left(k_P^m+k_Q^m\right)\gamma_m^{\alpha\beta}\left[\mathscr{A}_P^n(\gamma_n\mathscr{W}_Q)_\beta-(P\leftrightarrow Q)\right] \tag{2.12c}$$

as well as

$$\mathscr{F}_P^{mn} \equiv k_P^m\mathscr{A}_P^n-k_P^n\mathscr{A}_P^m-\sum_{XY=P}\left(\mathscr{A}_X^m\mathscr{A}_Y^n-\mathscr{A}_X^n\mathscr{A}_Y^m\right) \tag{2.12d}$$

obtained from $\mathscr{F}_{mn}=-[\nabla_m,\nabla_n]$. It is straightforward to show by induction that these solutions obey the equations of motion (2.7) by assuming $\{\mathscr{A}_\alpha^i,\mathscr{A}_m^i,\mathscr{W}_i^\alpha,\mathscr{F}_{mn}^i\}$ satisfy the linearized equations (1.24).

The perturbative nature of the solution in the above is preserved by gauge transformation generated by Lie algebra-valued gauge parameters in the form of

$$\Omega=\sum_{p=1}^{\infty}\sum_{a_1,a_2,\ldots,a_p}\frac{1}{p}\Omega_{a_1a_2\ldots a_p}\left[t^{a_1},\left[t^{a_2},\ldots,\left[t^{a_{p-1}},t^{a_p}\right]\right]\ldots\right],\qquad \Omega_{A\sqcup\!\sqcup B}=0\ \forall A,B\neq\emptyset, \tag{2.13}$$

which lead to another generating series of Berends-Giele currents. Non-linear gauge transformations of the generating series (2.10) of multiparticle superfields reparametrize the SYM amplitudes by moving terms between different cubic diagrams. They can therefore be viewed as an example of the "generalized gauge freedom" of [4, 12, 5]. In the remainder of this chapter we will exploit the effects of different gauge parameters Ω_P.

2.3 Non-linear superfields and Berends–Giele currents in BCJ gauge

In [77], supersymmetric Berends–Giele currents were constructed in a totally different fashion originated from the CFT defined by the PS superstring action. Starting with a *local* representation of multiparticle superfields

$$K_P\in\left\{A_\alpha^P,A_m^P,W_P^\alpha,F_{mn}^P\right\},$$

redefinitions were employed in order to enforce the symmetries of nested commutators $\left[\left[t^1,t^2\right],t^3\right]$ in a Lie algebra such as $K_{123}+K_{231}+K_{312}=0$. Their Berends-Giele currents $\mathscr{K}_P^{BCJ}$ were constructed by adjoining propagators, i.e., inverse Mandelstam invariants s_P in (2.11), to Lie symmetry-satisfying numerators, following an intuitive mapping to cubic graphs compatible with the ordering of the external legs. Incidentally, the family $K_P^{\rm BCJ}$ of local multiparticle superfields forming $\mathscr{K}_P^{\rm BCJ}$ satisfies the same "generalized Lie symmetries" [80] as a string of structure constants in $[t^a,t^b]=f^{abc}t^c$,

$$\text{“kinematics"}\ K_{12\ldots p}^{\rm BCJ}\ \longleftrightarrow f^{12a_3}f^{a_33a_4}f^{a_44a_5}\ldots f^{a_ppa_{p+1}}\ \text{“color"}. \tag{2.14}$$

The construction of $\mathscr{K}_P^{BCJ}$ in [77] is motivated by the BCJ conjecture [4] on a duality between color and kinematics: The kinematic factors N_i of scattering amplitudes can be arranged to satisfy the same Jacobi identity as their associated color factors C_i, see [12] for the striking impact on gravity amplitudes, [5] for the loop-level formulation of the conjecture and [81] for a review.

Despite their different construction, the Berends-Giele currents $\mathscr{K}_P^{\rm BCJ}$ of [77] or those in the *Lorenz gauge* $\mathscr{K}_P^{\rm L}\equiv\mathscr{K}_P$ constructed in the previous section give rise to identical tree level amplitudes, since these different currents are in fact related by a non-linear gauge transformation as verified below up to multiplicity five. Accordingly, the currents $\mathscr{K}_P^{\rm BCJ}$ are said to be in *BCJ gauge*.

2.3.1 Recursive definition of local superfields in Lorenz gauge

The definition of local superfields $\hat{K}_{[P,Q]}$ in Lorenz gauge[2] is given by

$$\hat{A}_{\alpha}^{[P,Q]} = -\frac{1}{2}\big[\hat{A}_{\alpha}^{P}(k^{P}\cdot\hat{A}^{Q}) + \hat{A}_{m}^{P}(\gamma^{m}\hat{W}^{Q})_{\alpha} - (P\leftrightarrow Q)\big] \tag{2.15a}$$

$$\hat{A}_{m}^{[P,Q]} = -\frac{1}{2}\big[\hat{A}_{m}^{P}(k^{P}\cdot\hat{A}^{Q}) + \hat{A}_{n}^{P}\hat{F}_{mn}^{Q} - (\hat{W}^{P}\gamma_{m}\hat{W}^{Q}) - (P\leftrightarrow Q)\big] \tag{2.15b}$$

$$\hat{W}_{[P,Q]}^{\alpha} = \frac{1}{2}(k_{P}^{m}+k_{Q}^{m})\gamma_{m}^{\alpha\beta}\big[\hat{A}_{P}^{n}(\gamma_{n}\hat{W}_{Q})_{\beta} - (P\leftrightarrow Q)\big], \tag{2.15c}$$

it amounts to picking up the numerator on top of various inverse s_X in the recursions (2.12a) to (2.12c) for Berends–Giele currents. We will often use a simplified notation for brackets $[P,Q]$ when one of P,Q is of single-particle type,

$$\hat{K}_{12\ldots p} \equiv \hat{K}_{[12\ldots p-1,p]}.$$

In this topology, the field-strength appearing above is given by

$$\hat{F}_{mn}^{12\ldots p} \equiv k_{m}^{12\ldots p}\hat{A}_{n}^{12\ldots p} - k_{n}^{12\ldots p}\hat{A}_{m}^{12\ldots p} + \sum_{j=2}^{p}\sum_{\delta\in P(\beta_j)}(k_{12\ldots j-1}\cdot k_j)\hat{A}_{[n}^{12\ldots j-1,\{\delta\}}\hat{A}_{m]}^{j,\{\beta_j\setminus\delta\}}, \tag{2.15d}$$

where $\beta_j \equiv \{j+1, j+2,\ldots,p\}$ and $P(\beta_j)$ denotes its power set. For $p=2,3,4$ we have

$$\begin{aligned}
\hat{F}_{mn}^{12} &= k_{m}^{12}\hat{A}_{n}^{12} - k_{n}^{12}\hat{A}_{m}^{12} + (k_1\cdot k_2)\hat{A}_{[n}^{1}\hat{A}_{m]}^{2}\\
\hat{F}_{mn}^{123} &= k_{m}^{123}\hat{A}_{n}^{123} - k_{n}^{123}\hat{A}_{m}^{123} + (k_1\cdot k_2)\Big(\hat{A}_{[n}^{1}\hat{A}_{m]}^{23} + \hat{A}_{[n}^{13}\hat{A}_{m]}^{2}\Big) + (k_{12}\cdot k_3)\hat{A}_{[n}^{12}\hat{A}_{m]}^{3}\\
\hat{F}_{mn}^{1234} &= k_{m}^{1234}\hat{A}_{n}^{1234} - k_{n}^{1234}\hat{A}_{m}^{1234} + (k_1\cdot k_2)\Big(\hat{A}_{[n}^{1}\hat{A}_{m]}^{234} + \hat{A}_{[n}^{13}\hat{A}_{m]}^{24} + \hat{A}_{[n}^{14}\hat{A}_{m]}^{23} + \hat{A}_{[n}^{134}\hat{A}_{m]}^{2}\Big)\\
&\quad + (k_{12}\cdot k_3)\Big(\hat{A}_{[n}^{12}\hat{A}_{m]}^{34} + \hat{A}_{[n}^{124}\hat{A}_{m]}^{3}\Big) + (k_{123}\cdot k_4)\hat{A}_{[n}^{123}\hat{A}_{m]}^{4}.
\end{aligned}$$

2.3.2 Review of generalized Lie symmetries for multiparticle superfields

The approach of [77] to Berends–Giele currents in BCJ gauge $\mathcal{K}_{P}^{\mathrm{BCJ}}$ is based on local superfields $K_{12\ldots p}$ satisfying all generalized Lie symmetries $\pounds_k$ up to $k=p$,

$$\begin{aligned}
0 &= \pounds_k \circ K_{12\ldots p} \qquad (k=2,\ldots,p)\\
&= \begin{cases} K_{12\ldots n+1[n+2[\ldots[2n-1[2n,2n+1]]\ldots]]} - K_{2n+1\ldots n+2[n+1[\ldots[3[21]]\ldots]]} = 0 & \text{for } k=2n+1,\\ K_{12\ldots n[n+1[\ldots[2n-2[2n-1,2n]]\ldots]]} + K_{2n\ldots n+1[n[\ldots[3[21]]\ldots]]} = 0 & \text{for } k=2n. \end{cases}
\end{aligned} \tag{2.16}$$

For example,

$$\begin{aligned}
&\pounds_2 \circ K_{12} = K_{12} + K_{21} = 0, \qquad \pounds_3 \circ K_{123} = K_{123} + K_{231} + K_{321} = 0,\\
&\pounds_4 \circ K_{1234} = K_{1234} - K_{1243} + K_{3412} - K_{3421} = 0,
\end{aligned}$$

and so forth. These symmetries leave $(p-1)!$ independent permutations of $K_{12\ldots p}$ and are also obeyed by nested commutators $[\ldots[[t^1,t^2],t^3],\ldots,t^p]$ and the color factors in

$$K_{12\ldots p} \longleftrightarrow f^{12a_3}f^{a_3 3 a_4}f^{a_4 4 a_5}\ldots f^{a_p p a_{p+1}}.$$

Therefore the local superfields K_P admit the following diagrammatic interpretation in fig. 2.2.

[2]Starting from rank four, the superfields denoted by $\{\hat{A}_{\alpha}^{P},\hat{A}_{m}^{P},\hat{W}_{P}^{\alpha},\hat{F}_{P}^{mn}\}$ in this work and [77] do not match.

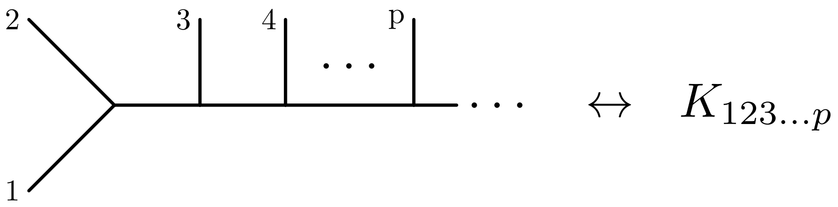


Figure 2.2: The diagrammatic interpretation of local superfields $K_{12\ldots p}$.

2.3.3 Recursive definition of local superfields in BCJ gauge

The recursively defined superfields $\hat{K}_{12\ldots p}$ in (2.15) do not yet satisfy the Lie symmetries (2.16). However, this can be compensated by redefinitions $K_{12\ldots p} = \hat{K}_{12\ldots p} + \ldots$ via superfields $\hat{H}_{12\ldots p} \equiv \hat{H}_{[12\ldots p-1,p]}$ which amount to a non-linear gauge transformation of their corresponding generating series. Starting from $\hat{H}_i = \hat{H}_{ij} = 0$, the superfields $\hat{H}_{12\ldots p}$ at multiplicity p enter through the following recursive system of equations [77]

$$K_{[12\ldots p-1,p]} \equiv \hat{K}_{[12\ldots p-1,p]} - \sum_{j=2}^{p} \sum_{\delta \in P(\beta_j)} (k^{1\ldots j-1} \cdot k^j) \big[\hat{H}_{1\ldots j-1,\{\delta\}} \hat{K}_{j,\{\beta_j \setminus \delta\}} - (1\ldots j-1 \leftrightarrow j)\big] - \begin{cases} D_\alpha \hat{H}_{[12\ldots p-1,p]} & \text{for } K = A_\alpha \\ k^m_{12\ldots p} \hat{H}_{[12\ldots p-1,p]} & \text{for } K = A^m \\ 0 & \text{for } K = W^\alpha \end{cases} \tag{2.17}$$

and will be introduced separately in the next subsection.

The redefinitions in (2.17) have been originally designed in a two-step procedure which yields the expressions for $\hat{H}_{12\ldots p}$ in a constructive manner[3] [77]. As a result, the superfields $K_{12\ldots p}$ defined by (2.17) as well as

$$F^{12\ldots p}_{mn} \equiv k^{12\ldots p}_m A^{12\ldots p}_n - k^{12\ldots p}_n A^{12\ldots p}_m + \sum_{j=2}^{p} \sum_{\delta \in P(\beta_j)} (k_{12\ldots j-1} \cdot k_j) A^{12\ldots j-1,\{\delta\}}_{[n} A^{j,\{\beta_j \setminus \delta\}}_{m]}$$

satisfy all the Lie symmetries $\pounds_2, \pounds_3, \ldots$ in (2.16) up to and including $\pounds_p$. For example, since $\hat{H}_i = \hat{H}_{ij} = 0$, the definitions in (2.17) yield

$$K_1 = \hat{K}_1, \qquad K_{12} = \hat{K}_{12}, \qquad \forall\, K \in \{A_\alpha, A^m, W^\alpha, F^{mn}\}, \tag{2.18}$$

and the first non-trivial redefinition occurs at multiplicity three with

$$A^{123}_\alpha = \hat{A}^{[12,3]}_\alpha - D_\alpha \hat{H}_{[12,3]}, \qquad A^m_{123} = \hat{A}^m_{[12,3]} - k^m_{123} \hat{H}_{[12,3]}, \qquad W^\alpha_{123} = \hat{W}^\alpha_{[12,3]}. \tag{2.19}$$

A rank-four sample of the redefinitions (2.17) is provided by

$$\begin{aligned} A^m_{1234} =& \hat{A}^m_{[123,4]} - (k^{123} \cdot k^4) \hat{H}_{[12,3]} A^m_4 - (k^{12} \cdot k^3) \hat{H}_{[12,4]} A^m_3 \\ & - (k^1 \cdot k^2) \big(\hat{H}_{[13,4]} A^m_2 - \hat{H}_{[23,4]} A^m_1\big) - k^m_{1234} \hat{H}_{[123,4]}. \end{aligned} \tag{2.20}$$

[3] As discussed in [77], an intermediate step of the redefinition procedure gives rise to redefined superfields $A'^m_{12\ldots p}$ which determine the definition of $H_{[12\ldots p-1,p]}$ via $\pounds_p \circ A'^m_{[12\ldots p-1,p]} \equiv p\, k^m_{12\ldots p} H_{[12\ldots p-1,p]}$. For this definition to work, the overall momentum $k^m_{12\ldots p}$ must factorize in the sum dictated by $\pounds_p \circ A'^m_{[12\ldots p-1,p]}$, providing a strong consistency check of the setup. The relation between $H_{12\ldots p}$ and $\hat{H}_{12\ldots p}$ will be given in (2.21).

2.3.4 Explicit form of the redefinitions $\hat{H}$

One can show that expressions for $\hat{H}_{[12\dots p-1,p]}$ can be conveniently summarized by

$$\hat{H}_{[A,B]} \equiv H_{[A,B]} - \frac{1}{2}\big[\hat{H}_A(k_A \cdot A_B) - (A \leftrightarrow B)\big] \tag{2.21}$$
$$H'_{A,B,C} \equiv H_{A,B,C} + \frac{1}{2}\big[H_{[A,B]}(k_{AB} \cdot A_C) + \text{cyclic}(A,B,C)\big],$$

with the central building block

$$H_{A,B,C} \equiv -\frac{1}{4}A^m_A A^n_B F^{mn}_C + \frac{1}{2}(W_A \gamma_m W_B)A^m_C + \text{cyclic}(A,B,C). \tag{2.22}$$

In particular, the redefinitions up to multiplicity five are captured by

$$H_{[12,3]} = \frac{1}{3}H_{1,2,3} \tag{2.23a}$$
$$H_{[123,4]} = \frac{1}{4}\big(H'_{12,3,4} + H'_{34,1,2}\big) \tag{2.23b}$$
$$H_{[12,34]} = \frac{1}{4}\big(-2H'_{12,3,4} + 2H'_{34,1,2}\big) \tag{2.23c}$$
$$H_{[1234,5]} = \frac{1}{5}\big(H'_{123,4,5} - H'_{543,2,1} + H'_{12,3,45}\big) \tag{2.23d}$$
$$H_{[123,45]} = \frac{1}{5}\big(-3H'_{123,4,5} - 2H'_{543,2,1} + 2H'_{12,3,45}\big). \tag{2.23e}$$

The treatment and significance of the additional topologies $H_{[12,34]}$ and $H_{[123,45]}$ is explained around (2.31) and in appendix A. Higher-rank versions of H_P are under investigation, and it would be interesting to extend the simple expressions in (2.23) to arbitrary multiplicity[4]. The expressions above are sufficient to identify the redefinitions up to and including multiplicity five as originating from a non-linear gauge transformation.

It is worth mentioning a remarkable feature of $H_{A,B,C}$ in (2.22): Upgrading the polarization vectors and spinors in the color-ordered SYM three-point amplitude at tree level[5],

$$A_{SYM}(1,2,3) = -\frac{1}{2}e^m_1 e^n_2 f^{mn}_3 + (\chi_1 \gamma_m \chi_2)e^m_3 + \text{cyclic}(1,2,3) \tag{2.24}$$

to superfields according to $e^m_i \to A^m_i(\theta)$, $\chi^\alpha_i \to W^\alpha_i(\theta)$ and $f^{mn}_i = k^{[m}_i e^{n]}_i \to F^{mn}_i(\theta)$, the amplitude (2.24) can be rewritten as

$$A_{SYM}(1,2,3) = 2H_{1,2,3}(\theta = 0).$$

2.3.5 Supersymmetric Berends–Giele currents in BCJ gauge

In this subsection, we will justify the terminology of Lorenz and BCJ gauge for the representations $\mathcal{K}^{\text{L}}_P$ and $\mathcal{K}^{\text{BCJ}}_P$ of Berends–Giele currents. It will be verified up to multiplicity five that they are indeed related by a non-linear gauge transformation, i.e.

$$\mathbb{A}^{\text{BCJ}}_m = \mathbb{A}^{\text{L}}_m - [\partial_m, \mathbb{H}] + [\mathbb{A}^{\text{L}}_m, \mathbb{H}], \tag{2.25}$$

[4]Noting that $H_{[12\dots p-1,p]}$ here corresponds to $H_{12\dots p}$ from [77], the expression of $H_{[123,4]}$ presented in (2.23) considerably simplifies the expression of H_{1234} given in the appendix C of [77].

[5]See (1.25) for the appearance of polarization vectors and spinors in θ-expansions of linearized superfields.

translating into

$$\mathscr{A}_P^{m,\mathrm{BCJ}} = \mathscr{A}_P^{m,\mathrm{L}} - k_P^m \mathscr{H}_P + \sum_{XY=P} (\mathscr{A}_X^{m,\mathrm{L}} \mathscr{H}_Y - \mathscr{A}_Y^{m,\mathrm{L}} \mathscr{H}_X). \tag{2.26}$$

Clearly, (2.25) is a special case of a non-linear gauge transformation (1.20) with $\Omega \to -\mathbb{H}$. The generating series of gauge parameters

$$\mathbb{H} \equiv \sum_{a_1,a_2,a_3} \mathscr{H}_{a_1a_2a_3} t^{a_1}t^{a_2}t^{a_3} + \sum_{a_1,a_2,a_3,a_4} \mathscr{H}_{a_1a_2a_3a_4} t^{a_1}t^{a_2}t^{a_3}t^{a_4} + \cdots \tag{2.27}$$

is built from Berends–Giele currents $\mathscr{H}_P$ of the superfields $\hat{H}_{[A,B]}$. As before, the Berends–Giele symmetry $\mathscr{H}_{A\sqcup B} = 0$ implies Lie algebra-valuedness of the series (2.27) [82]. Of course, the same $\mathbb{H}$ describes the transformation of the remaining series $\mathbb{A}_\alpha$, $\mathbb{W}^\alpha$, $\mathbb{F}^{mn}$, see (1.20). We will focus on the transformation between the currents $\mathscr{A}_P^{m,\mathrm{BCJ}}$ and $\mathscr{A}_P^{m,\mathrm{L}}$ of the vector potential since the remaining superfields follow the same or simpler lines.

In the following discussion we will construct Berends–Giele currents up to rank four using the mapping between planar binary trees and nested brackets [77], see appendix A for rank five. By (2.18), the two gauge choices are identical at multiplicities one and two,

$$\mathscr{K}_1^{\mathrm{BCJ}} = \mathscr{K}_1^{\mathrm{L}}, \qquad \mathscr{K}_{12}^{\mathrm{BCJ}} = \mathscr{K}_{12}^{\mathrm{L}},$$

reflecting the vanishing of the simplest redefinitions,

$$\hat{H}_1 = \hat{H}_{12} = 0 \quad \Rightarrow \quad \mathscr{H}_1 = \mathscr{H}_{12} = 0, \tag{2.28}$$

and justifying the absence of single-particle and two-particle contributions in the series (2.27).

Rank three At multiplicity three, the two binary trees displayed in fig. 2.3 lead to

$$\mathscr{K}_{123}^{\mathrm{BCJ}} = \frac{K_{[12,3]}}{s_{12}s_{123}} + \frac{K_{[1,23]}}{s_{23}s_{123}}, \qquad \mathscr{K}_{123}^{\mathrm{L}} = \frac{\hat{K}_{[12,3]}}{s_{12}s_{123}} + \frac{\hat{K}_{[1,23]}}{s_{23}s_{123}},$$

with $\hat{K}_{[P,Q]} = -\hat{K}_{[Q,P]}$ from (2.15a) to (2.15c). Hence, the relation (2.19) between the local superfields in the two gauges is sufficient to determine the corresponding relation between their Berends–Giele currents. For example, $A^m_{[12,3]} = \hat{A}^m_{[12,3]} - k^m_{123}\hat{H}_{[12,3]}$ implies that

$$\mathscr{A}_{123}^{m,\mathrm{BCJ}} = \mathscr{A}_{123}^{m,\mathrm{L}} - k_{123}^m \mathscr{H}_{123}, \qquad \mathscr{H}_{123} = \frac{\hat{H}_{[12,3]}}{s_{12}s_{123}} + \frac{\hat{H}_{[1,23]}}{s_{23}s_{123}},$$

where (2.28) allows to restore a vanishing deconcatenation term $0 = \mathscr{A}_1^{m,\mathrm{L}}\mathscr{H}_{23} + \mathscr{A}_{12}^{m,\mathrm{L}}\mathscr{H}_3 - \mathscr{A}_{23}^{m,\mathrm{L}}\mathscr{H}_1 - \mathscr{A}_3^{m,\mathrm{L}}\mathscr{H}_{12}$ and to verify (2.26) at $P = 123$.

Rank four Similar calculations at multiplicity four lead to the relation

$$\mathscr{A}_{1234}^{m,\mathrm{BCJ}} = \mathscr{A}_{1234}^{m,\mathrm{L}} - k_{1234}^m \mathscr{H}_{1234} + \mathscr{A}_1^m \mathscr{H}_{234} - \mathscr{A}_4^m \mathscr{H}_{123} \tag{2.29}$$

required by (2.26), where (2.28) identifies the last two terms on the right-hand side as a perfect deconcatenation $\sum_{XY=1234}(\mathscr{A}_X^{m,\mathrm{L}}\mathscr{H}_Y - \mathscr{A}_Y^{m,\mathrm{L}}\mathscr{H}_X)$. The Berends–Giele currents comprise the five binary trees depicted in fig. 2.3,

$$\begin{aligned}
\mathscr{A}_{1234}^{m,\mathrm{BCJ}} &= \frac{1}{s_{1234}}\Big(\frac{A^m_{[123,4]}}{s_{12}s_{123}} + \frac{A^m_{[321,4]}}{s_{23}s_{123}} + \frac{A^m_{[12,34]}}{s_{12}s_{34}} + \frac{A^m_{[342,1]}}{s_{34}s_{234}} + \frac{A^m_{[324,1]}}{s_{23}s_{234}}\Big)\\
\mathscr{A}_{1234}^{m,\mathrm{L}} &= \frac{1}{s_{1234}}\Big(\frac{\hat{A}^m_{[123,4]}}{s_{12}s_{123}} + \frac{\hat{A}^m_{[321,4]}}{s_{23}s_{123}} + \frac{\hat{A}^m_{[12,34]}}{s_{12}s_{34}} + \frac{\hat{A}^m_{[342,1]}}{s_{34}s_{234}} + \frac{\hat{A}^m_{[324,1]}}{s_{23}s_{234}}\Big)\\
\mathscr{H}_{1234} &= \frac{1}{s_{1234}}\Big(\frac{\hat{H}_{[123,4]}}{s_{12}s_{123}} + \frac{\hat{H}_{[321,4]}}{s_{23}s_{123}} + \frac{\hat{H}_{[12,34]}}{s_{12}s_{34}} + \frac{\hat{H}_{[342,1]}}{s_{34}s_{234}} + \frac{\hat{H}_{[324,1]}}{s_{23}s_{234}}\Big),
\end{aligned} \tag{2.30}$$

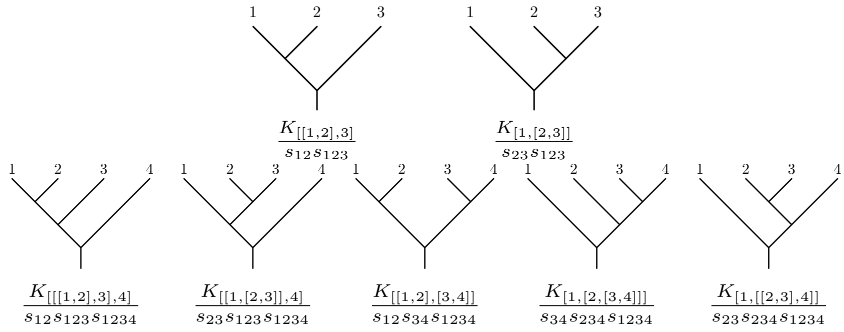


Figure 2.3: The planar binary trees used to define $\mathscr{K}_{123}$ and $\mathscr{K}_{1234}$.

where four of the five numerators in (2.30) belong to the topology of (2.20).

However, the third term representing the middle diagram in fig. 2.3 follows the separate conversion rule

$$\begin{aligned} A^m_{[12,34]} =& \hat{A}^m_{[12,34]} - k^m_{1234}\hat{H}_{[12,34]} \\ &+ (k^1\cdot k^2)\big(\hat{H}_{[13,4]}A^m_2 - \hat{H}_{[23,4]}A^m_1\big) + (k^3\cdot k^4)\big(\hat{H}_{[12,4]}A^m_3 - \hat{H}_{[12,3]}A^m_4\big) \end{aligned} \tag{2.31}$$

between Lorenz gauge and BCJ gauge. As a defining property of BCJ gauge, the left-hand side can be expressed in terms of the basic topology (2.17) via $A^m_{[12,34]} = A^m_{1234} - A^m_{1243}$. The new topology $\hat{H}_{[12,34]}$ of redefining fields (see [83]) is determined by (2.31) whose solution can be found in (2.23).

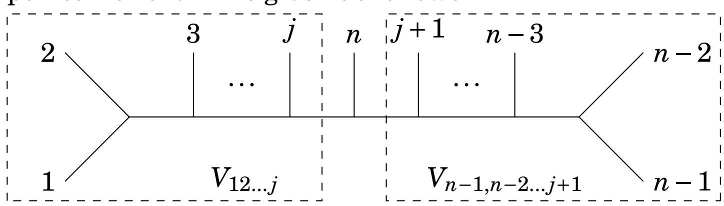
Upon insertion into (2.30), contributions of the form $\hat{H}_{[12,3]}A^m_4$ in (2.20) and (2.31) conspire to the desired deconcatenation term in (2.29), verifying the mediation of a non-linear gauge transformation between $\mathscr{A}^{m,\mathrm{BCJ}}_{1234}$ and $\mathscr{A}^{m,\mathrm{L}}_{1234}$. The analogous analysis of the gauge transformation at multiplicity five is relegated to appendix A.

2.3.6 BCJ master numerators from multiparticle superfields

With the local multiparticle superfields (2.17) corresponding to Berends–Giele currents in BCJ gauge, one can construct kinematic numerators that manifestly satisfy the BCJ duality. Each cubic diagram is associated with a local superspace expression made from trilinears in multiparticle vertex operators

$$V_{12\dots p} = \lambda^\alpha A^{12\dots p}_\alpha\,, \tag{2.32}$$

and it is sufficient to specify the $(n-2)!$ master numerators which are independent under kinematic Jacobi relations: The kinematic numerator of the depicted half-ladder diagrams with fixed endpoints 1 and $n-1$ is given as follows



$\longleftrightarrow \quad \langle V_{12\dots j}V_n V_{n-1,n-2\dots j+1}\rangle$

The kinematic factors for any other graph different from the figure and its $(n-2)!$ permutations in $2,3,\dots,n-2,n$ can be reached by a sequence of Jacobi relations. The last leg n always enters through a single-particle vertex operator V_n. This representation agrees with the field-theory limit of the open superstring amplitude and yields the amplitudes (2.5) of 10D SYM [13, 84]. At the end of the next section 2.4, we will give a compact formula for the bosonic and fermionic components $\langle V_A V_B V_C \rangle$ of the master numerators, based on a combination of BCJ- and Harnad-Shnider gauge.

2.4 Theta-expansions in Harnad–Shnider gauge

In the last section we have identified a particular gauge transformation $\mathbb{H}$ which relates the Berends–Giele currents in the BCJ gauge to their counterparts in the Lorenz gauge. Similarly, we will now construct another gauge transformation

$$\mathbb{L} \equiv \sum_{a_1,a_2} \mathscr{L}_{a_1a_2} t^{a_1} t^{a_2} + \sum_{a_1,a_2,a_3} \mathscr{L}_{a_1a_2a_3} t^{a_1} t^{a_2} t^{a_3} + \cdots \tag{2.33}$$

whose expansion starts at multiplicity two and is designed to simplify the theta-expansions of the multiparticle superfields.

2.4.1 Generating series of Harnad–Shnider gauge variations

A convenient gauge choice to expand the superfields of ten-dimensional SYM in theta is the Harnad–Shnider (HS) gauge [69],

$$\theta^\alpha \mathbb{A}^{\rm HS}_\alpha = 0 \tag{2.34}$$

At the linearized level, the gauge $\theta^\alpha A^i_\alpha = 0$ has been used in [70] to obtain the theta-expansions of the single-particle superfields to arbitrary order. However, the recursive definition (2.12a) of multiparticle Berends–Giele currents $\mathscr{A}^P_\alpha$ in Lorenz gauge does not preserve linearized HS gauge, e.g.

$$\theta^\alpha A^i_\alpha = 0 \ \Rightarrow \ \theta^\alpha \mathscr{A}^{12}_\alpha = \frac{1}{2s_{12}} \big[A^2_m (\theta\gamma^m W_1) - (1 \leftrightarrow 2) \big] \neq 0.$$

Still, there is a non-linear gauge transformation $\mathbb{L}$ which brings the currents from Lorenz gauge into HS gauge via

$$\mathbb{A}^{\rm HS}_\alpha = \mathbb{A}^{\rm L}_\alpha - [D_\alpha, \mathbb{L}] + [\mathbb{A}^{\rm L}_\alpha, \mathbb{L}]. \tag{2.35}$$

It can determined recursively by contracting with θ^α:

$$[\mathscr{D}, \mathbb{L}] = \theta^\alpha \mathbb{A}^{\rm L}_\alpha + [\theta^\alpha \mathbb{A}^{\rm L}_\alpha, \mathbb{L}], \tag{2.36}$$

where the Euler operator

$$\mathscr{D} \equiv \theta^\alpha D_\alpha = \theta^\alpha \frac{\partial}{\partial \theta^\alpha}$$

weights the $k^{\rm th}$ order in θ by a factor of k. At the level of multiparticle components in (2.33), this translates into

$$\mathscr{D}\mathscr{L}_P = \theta^\alpha \mathscr{A}^P_\alpha + \sum_{XY=P} \big(\theta^\alpha \mathscr{A}^X_\alpha \mathscr{L}_Y - \theta^\alpha \mathscr{A}^Y_\alpha \mathscr{L}_X \big), \tag{2.37}$$

where the Berends–Giele currents $\mathscr{L}_X, \mathscr{L}_Y$ on the right hand side have lower multiplicity than $\mathscr{L}_P$ on the left hand side. Hence, (2.37) is a recursion w.r.t. multiplicity in the Lie-series expansion (2.33). The currents $\mathscr{A}^P_\alpha$ are understood to follow the Lorenz-gauge setup in (2.11) to (2.12d). Using $\theta^\alpha A^i_\alpha = \mathscr{L}_i = 0$ at the linearized level, we have for instance

$$\mathscr{D}\mathscr{L}_{12} = \theta^\alpha \mathscr{A}^{12}_\alpha, \quad \mathscr{D}\mathscr{L}_{123} = \theta^\alpha \mathscr{A}^{123}_\alpha, \quad \mathscr{D}\mathscr{L}_{1234} = \theta^\alpha \mathscr{A}^{1234}_\alpha + \theta^\alpha \mathscr{A}^{12}_\alpha \mathscr{L}_{34} - \theta^\alpha \mathscr{A}^{34}_\alpha \mathscr{L}_{12}.$$

By imposing $\mathbb{L}(\theta = 0) = 0$, we arrive at explicit theta-expansions such as

$$\mathscr{L}_{12} = \frac{1}{2s_{12}}\Big((\theta\gamma_m\chi_1)e_2^m + \frac{1}{8}(\theta\gamma_{mnp}\theta)e_1^m f_2^{np} + \frac{1}{12}(\theta\gamma_{mnp}\theta)(\theta\gamma^m\chi_1)k_{12}^n e_2^p - (1\leftrightarrow 2) + \cdots\Big)e^{k_{12}x},$$

with terms of order $\theta^{\geq 4}$ in the ellipsis and analogous expressions for $\mathscr{L}_{12\ldots p}$ at $p \geq 3$.

2.4.2 Multiparticle theta-expansions in Harnad-Shnider gauge

The theta-expansion of non-linear fields in HS gauge (2.34) can be elegantly captured by means of higher mass dimension superfields [78],

$$\mathbb{W}^{m_1\ldots m_k\alpha} \equiv [\nabla^{m_1}, \mathbb{W}^{m_2\ldots m_k\alpha}]\,, \quad \mathbb{F}^{m_1\ldots m_k|pq} \equiv [\nabla^{m_1}, \mathbb{F}^{m_2\ldots m_k|pq}]\,, \tag{2.38}$$

subject to non-linear gauge transformations [78]

$$\delta_\Omega \mathbb{W}^{m_1\ldots m_k\alpha} = \big[\Omega, \mathbb{W}^{m_1\ldots m_k\alpha}\big], \quad \delta_\Omega \mathbb{F}^{m_1\ldots m_k|pq} = \big[\Omega, \mathbb{F}^{m_1\ldots m_k|pq}\big]\,. \tag{2.39}$$

In the subsequent, we assume that the superfields have been brought to HS gauge via (1.20) through the transformation $\Omega \to \mathbb{L}$ constructed from (2.37). For ease of notation, the accompanying $^{\rm HS}$ superscripts as in (2.35) will henceforth be suppressed. Contracting the non-linear equations of motion (2.7) with θ^α yields [69]

$$\big(\mathscr{D}+1\big)\mathbb{A}_\beta = (\theta\gamma^m)_\beta\mathbb{A}_m, \qquad \mathscr{D}\mathbb{A}_m = (\theta\gamma_m\mathbb{W})$$
$$\mathscr{D}\mathbb{W}^\beta = \frac{1}{4}(\theta\gamma^{mn})^\beta\mathbb{F}_{mn}, \qquad \mathscr{D}\mathbb{F}^{mn} = -(\mathbb{W}^{[m}\gamma^{n]}\theta)$$

by virtue of HS gauge. This can be used to reconstruct the entire theta-expansion of any superfield from their zeroth orders $\mathbb{K}(\theta = 0)$ [69],

$$\begin{aligned} [\mathbb{A}_\alpha]_k &= \frac{1}{k+1}(\theta\gamma^m)_\alpha[\mathbb{A}_m]_{k-1}\,, \quad [\mathbb{A}_m]_k = \frac{1}{k}(\theta\gamma_m[\mathbb{W}]_{k-1}) \\ [\mathbb{W}^\alpha]_k &= \frac{1}{4k}(\theta\gamma^{mn})^\alpha[\mathbb{F}_{mn}]_{k-1}\,, \quad [\mathbb{F}^{mn}]_k = -\frac{1}{k}([\mathbb{W}^{[m}]_{k-1}\gamma^{n]}\theta)\,, \end{aligned} \tag{2.40}$$

where the notation $[\ldots]_k$ instructs to only keep terms of order $(\theta)^k$ of the enclosed superfields. The analogous expressions for superfields at higher mass dimensions are

$$\begin{aligned} [\mathbb{W}_m^\alpha]_k &= \frac{1}{k}\Big\{\frac{1}{4}(\theta\gamma^{pq})^\alpha[\mathbb{F}_{m|pq}]_{k-1} - (\theta\gamma_m)_\beta\sum_{l=0}^{k-1}\{[\mathbb{W}^\beta]_l, [\mathbb{W}^\alpha]_{k-l-1}\}\Big\} \\ [\mathbb{F}^{m|pq}]_k &= -\frac{1}{k}\Big\{([\mathbb{W}^{m[p}]_{k-1}\gamma^{q]}\theta) + (\theta\gamma^m)_\alpha\sum_{l=0}^{k-1}[[\mathbb{W}^\alpha]_l, [\mathbb{F}^{pq}]_{k-l-1}]\Big\} \\ [\mathbb{W}_{mn}^\alpha]_k &= \frac{1}{k}\Big\{\frac{1}{4}(\theta\gamma^{pq})^\alpha[\mathbb{F}_{mn|pq}]_{k-1} + (\theta\gamma_m)_\beta\sum_{l=0}^{k-1}\{[\mathbb{W}^\beta]_l, [\mathbb{W}_n^\alpha]_{k-l-1}\} \\ &\quad + (\theta\gamma_n)_\beta\sum_{l=0}^{k-1}\Big(\{[\mathbb{W}_m^\beta]_l, [\mathbb{W}^\alpha]_{k-l-1}\} + \{[\mathbb{W}^\beta]_l, [\mathbb{W}_m^\alpha]_{k-l-1}\}\Big)\Big\}\,, \end{aligned} \tag{2.41}$$

see [78] for the underlying equations of motion and (B.2) for generalizations to higher mass dimension.

The component wavefunctions The theta-independent terms $[\mathbb{K}]_0$ initiate the above recursions in the order of theta, and their multiparticle components $[\mathcal{K}_P]_0$ at lowest mass dimensions

$$[\mathcal{A}_P^m]_0 \equiv \mathfrak{e}_P^m e^{k_P x}, \qquad [\mathcal{W}_P^\alpha]_0 \equiv \mathcal{X}_P^\alpha e^{k_P x} \tag{2.42}$$

are shown in [84] to supersymmetrize the Berends–Giele currents in [79], e.g.

$$s_{12}\mathfrak{e}_{12}^m = e_2^m(k_2\cdot e_1) - e_1^m(k_1\cdot e_2) + \frac{1}{2}(k_1^m - k_2^m)(e_1\cdot e_2) + (\chi_1\gamma^m\chi_2)$$
$$s_{12}\mathcal{X}_{12}^\alpha = \frac{1}{2}k_{12}^m\gamma_m^{\alpha\beta}\left[e_1^n(\gamma_n\chi_2)_\beta - e_2^n(\gamma_n\chi_1)_\beta\right].$$

Note that Lorenz gauge for the superfields $\mathcal{A}_P^m$ propagates to the component currents $\mathfrak{e}_P^m$,

$$(k_P\cdot\mathfrak{e}_P) = (k_P\cdot[\mathcal{A}_P]_0) = 0, \tag{2.43}$$

since the transformation towards HS gauge in (2.36) is chosen with $\mathbb{L}(\theta=0)=0$.

At higher mass dimensions, the wavefunctions in

$$[\mathcal{W}_P^{m_1\ldots m_k\alpha}]_0 \equiv \mathcal{X}_P^{m_1\ldots m_k\alpha}e^{k_P x}, \qquad [\mathcal{F}_P^{m_1\ldots m_k|pq}]_0 \equiv \mathfrak{f}_P^{m_1\ldots m_k|pq}e^{k_P x} \tag{2.44}$$

with $k=0,1,2,\ldots$ inherit the recursive expressions from (2.38) such that

$$\mathfrak{f}_P^{mn} \equiv k_P^m\mathfrak{e}_P^n - k_P^n\mathfrak{e}_P^m - \sum_{XY=P}(\mathfrak{e}_X^m\mathfrak{e}_Y^n - \mathfrak{e}_X^n\mathfrak{e}_Y^m)$$
$$\mathcal{X}_P^{m_1\ldots m_k\alpha} \equiv k_P^{m_1}\mathcal{X}_P^{m_2\ldots m_k|pq} - \sum_{XY=P}(\mathfrak{e}_X^{m_1}\mathcal{X}_Y^{m_2\ldots m_k\alpha} - \mathcal{X}_X^{m_2\ldots m_k\alpha}\mathfrak{e}_Y^{m_1}), \qquad k=1,2,\ldots$$
$$\mathfrak{f}_P^{m_1\ldots m_k|pq} \equiv k_P^{m_1}\mathfrak{f}_P^{m_2\ldots m_k|pq} - \sum_{XY=P}(\mathfrak{e}_X^{m_1}\mathfrak{f}_Y^{m_2\ldots m_k|pq} - \mathfrak{f}_X^{m_2\ldots m_k|pq}\mathfrak{e}_Y^{m_1}), \qquad k=1,2,\ldots\,. \tag{2.45}$$

The theta-expansion Using the notation $\mathcal{K}_P(x,\theta)\equiv\mathcal{K}_P(\theta)e^{k_P\cdot x}$ one can show that the recursions (2.40) and (2.41) lead to the following multiparticle theta-expansions,

$$\begin{aligned}\mathcal{A}_\alpha^P(\theta) =& \frac{1}{2}(\theta\gamma_m)_\alpha\mathfrak{e}_P^m + \frac{1}{3}(\theta\gamma_m)_\alpha(\theta\gamma^m\mathcal{X}_P) - \frac{1}{32}(\theta\gamma_m)^\alpha(\theta\gamma^{mnp}\theta)\mathfrak{f}_{np}^P\\ &+\frac{1}{60}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)(\mathcal{X}_n^P\gamma_p\theta) + \frac{1}{1152}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)(\theta\gamma^{pqr}\theta)\mathfrak{f}_P^{n|qr}\\ &+\sum_{XY=P}[\mathcal{A}_\alpha^{X,Y}]_5 + \ldots\end{aligned} \tag{2.46a}$$

$$\begin{aligned}\mathcal{A}_P^m(\theta) =& \mathfrak{e}_P^m + (\theta\gamma^m\mathcal{X}_P) - \frac{1}{8}(\theta\gamma^{mpq}\theta)\mathfrak{f}_P^{pq} + \frac{1}{12}(\theta\gamma^{mnp}\theta)(\mathcal{X}_P^n\gamma^p\theta)\\ &+\frac{1}{192}(\theta\gamma^m{}_{nr}\theta)(\theta\gamma^r{}_{pq}\theta)\mathfrak{f}_P^{n|pq} - \frac{1}{480}(\theta\gamma^m{}_{nr}\theta)(\theta\gamma^r{}_{pq}\theta)(\mathcal{X}_P^{np}\gamma^q\theta)\\ &+\sum_{XY=P}\Big([\mathcal{A}_{X,Y}^m]_4 + [\mathcal{A}_{X,Y}^m]_5\Big) + \ldots\end{aligned} \tag{2.46b}$$

$$\begin{aligned}\mathcal{W}_P^\alpha(\theta) =& \mathcal{X}_P^\alpha + \frac{1}{4}(\theta\gamma^{mn})^\alpha\mathfrak{f}_{mn}^P - \frac{1}{4}(\theta\gamma_{mn})^\alpha(\mathcal{X}_P^m\gamma^n\theta) - \frac{1}{48}(\theta\gamma_m{}^q)^\alpha(\theta\gamma_{qnp}\theta)\mathfrak{f}_P^{m|np}\\ &+\frac{1}{96}(\theta\gamma_m{}^q)^\alpha(\theta\gamma_{qnp}\theta)(\mathcal{X}_P^{mn}\gamma^p\theta) - \frac{1}{1920}(\theta\gamma_m{}^r)^\alpha(\theta\gamma_{nr}{}^s\theta)(\theta\gamma_{spq}\theta)\mathfrak{f}_P^{mn|pq}\\ &+\sum_{XY=P}\Big([\mathcal{W}_{X,Y}^\alpha]_3 + [\mathcal{W}_{X,Y}^\alpha]_4 + [\mathcal{W}_{X,Y}^\alpha]_5\Big) + \ldots\end{aligned} \tag{2.46c}$$

$$\begin{aligned}\mathcal{F}_P^{mn}(\theta) =& \mathfrak{f}_P^{mn} - (\mathcal{X}_P^{[m}\gamma^{n]}\theta) + \frac{1}{8}(\theta\gamma_{pq}{}^{[m}\theta)\mathfrak{f}_P^{n]|pq} - \frac{1}{12}(\theta\gamma_{pq}{}^{[m}\theta)(\mathcal{X}_P^{n]p}\gamma^q\theta)\\ &-\frac{1}{192}(\theta\gamma_{ps}{}^{[m}\theta)\mathfrak{f}_P^{n]p|qr}(\theta\gamma^s{}_{qr}\theta) + \frac{1}{480}(\theta\gamma^{[m}{}_{ps}\theta)(\mathcal{X}_P^{n]pq}\gamma^r\theta)(\theta\gamma^s{}_{qr}\theta)\\ &+\sum_{XY=P}\Big([\mathcal{F}_{X,Y}^{mn}]_2 + [\mathcal{F}_{X,Y}^{mn}]_3 + [\mathcal{F}_{X,Y}^{mn}]_4 + [\mathcal{F}_{X,Y}^{mn}]_5\Big) + \sum_{XYZ=P}[\mathcal{F}_{X,Y,Z}^{mn}]_5 + \ldots\end{aligned} \tag{2.46d}$$

with terms of order $\theta^{\geq 6}$ in the ellipsis. The non-linearities of the form $\sum_{XY=P}[\mathcal{K}_{X,Y}]_l$ can be traced back to the quadratic expressions in (2.41), e.g.

$$[\mathcal{A}_\alpha^{X,Y}]_5 = \frac{1}{144}(\theta\gamma_m)_\alpha(\theta\gamma^{mnp}\theta)(\mathcal{X}^X\gamma_n\theta)(\mathcal{X}^Y\gamma_p\theta)$$
$$[\mathcal{A}^m_{X,Y}]_4 = \frac{1}{24}(\theta\gamma^m{}_{np}\theta)(\mathcal{X}^X\gamma^n\theta)(\mathcal{X}^Y\gamma^p\theta)$$
$$[\mathcal{W}^\alpha_{X,Y}]_3 = -\frac{1}{6}(\theta\gamma_{mn})^\alpha(\mathcal{X}_X\gamma^m\theta)(\mathcal{X}_Y\gamma^n\theta)$$
$$[\mathcal{F}^{mn}_{X,Y}]_2 = -(\mathcal{X}_X\gamma^{[m}\theta)(\mathcal{X}_Y\gamma^{n]}\theta)\ ,$$

and further instances as to make the complete orders $\theta^{\leq 5}$ available are spelt out in appendix B. It is easy to see that these non-linear terms vanish in the single-particle case, and one recovers the linearized expansions of (1.25).

Analogous theta-expansions for superfields (2.38) of higher mass dimensions start with

$$\mathcal{W}_P^{m\alpha}(x,\theta) = e^{k_P x}\Big(\mathcal{X}_P^{m\alpha} + \frac{1}{4}(\theta\gamma_{np})^\alpha \mathfrak{f}_P^{m|np} + \sum_{XY=P}\big[(\mathcal{X}_X\gamma^m\theta)\mathcal{X}_Y^\alpha - (X\leftrightarrow Y)\big] + \ldots\Big) \tag{2.47a}$$
$$\mathcal{F}_P^{m|pq}(x,\theta) = e^{k_P x}\Big(\mathfrak{f}_P^{m|pq} - (\mathcal{X}_P^{m[p}\gamma^{q]}\theta) + \sum_{XY=P}\big[(\mathcal{X}_X\gamma^m\theta)\mathfrak{f}_Y^{pq} - (X\leftrightarrow Y)\big] + \ldots\Big) \tag{2.47b}$$

where the lowest two orders $\sim \theta^2, \theta^3$ in the ellipsis along with generalizations to higher mass dimensions are spelt out in appendix B.

2.4.3 Combining HS gauge with BCJ gauge

The steps in (2.35) and (2.36) towards HS gauge can be literally repeated when starting with BCJ gauge:

$$\mathbb{A}_\alpha^{\text{BCJ}-HS} = \mathbb{A}_\alpha^{\text{BCJ}} - [D_\alpha, \mathbb{L}'] + [\mathbb{A}_\alpha^{\text{BCJ}}, \mathbb{L}']$$
$$[\mathcal{D}, \mathbb{L}'] = \theta^\alpha \mathbb{A}_\alpha^{\text{BCJ}} + [\theta^\alpha \mathbb{A}_\alpha^{\text{BCJ}}, \mathbb{L}']\ .$$

The multiparticle expansion of the gauge parameter $\mathbb{L}'$ can be constructed along the lines of (2.37) where we again set $\mathbb{L}'(\theta=0)=0$. The resulting gauge combines the benefits of a simplified theta-expansion due to

$$\theta^\alpha \mathbb{A}_\alpha^{\text{BCJ-HS}} = 0$$

with a manifestation of the BCJ duality in cubic-diagram numerators subject to Lie symmetries. The arguments of subsection (2.4.2) give rise to theta-expansions completely analogous to HS gauge, see (2.46) and appendix B. The only difference is a redefinition of the component Berends–Giele currents according to

$$\mathfrak{e}_P^m \to \mathcal{A}_P^{m,\text{BCJ}}(\theta=0) = \mathfrak{e}_P^m + \sum_{XY=P}(\mathfrak{e}_X^m\mathfrak{h}_Y - \mathfrak{e}_Y^m\mathfrak{h}_X) - k_P^m\mathfrak{h}_P \tag{2.48a}$$
$$\mathcal{X}_P^\alpha \to \mathcal{W}_P^{\alpha,\text{BCJ}}(\theta=0) = \mathcal{X}_P^\alpha + \sum_{XY=P}(\mathcal{X}_X^\alpha\mathfrak{h}_Y - \mathcal{X}_Y^\alpha\mathfrak{h}_X), \tag{2.48b}$$

where the multiparticle gauge parameters contribute through their $\theta=0$ order,

$$\mathfrak{h}_P \equiv \mathcal{H}_P(\theta=0).$$

The redefinitions in (2.48) propagate to their counterparts at higher mass dimension via (2.45). Since BCJ gauge already violates the Lorenz-gauge condition at the three-particle level, e.g.

$k^{123}_m \mathscr{A}^{m,\text{BCJ}}_{123} = -2s_{123}\mathscr{H}_{123}$, transversality (2.43) of the modified current $\mathfrak{e}^m_P \to \mathscr{A}^{m,\text{BCJ}}_P(\theta=0)$ will no longer hold.

Similarly, the theta-expansions of higher-mass dimension Berends–Giele currents given in (2.47) and appendix B preserve their structure after the replacements in (2.48). As mentioned earlier, the BCJ gauge appears naturally in the context of string amplitudes due to the redefinitions induced by the double poles in OPE contractions [74, 14]. Hence, BCJ-HS gauge is particularly convenient for an accelerated approach to component amplitudes of the pure spinor superstring.

2.4.4 Application of Berends–Giele currents in Harnad-Shnider gauge

In this section, we sketch applications of multiparticle superfields in HS gauge to tree level open superstring scattering amplitudes in the PS superstring theory discussed in section 2.1. In the PS open superstring theory and the 10D SYM tree level kinematics can be expressed in terms of the building block

$$\langle M_A M_B M_C \rangle, \qquad M_A \equiv \lambda^\alpha \mathscr{A}^A_\alpha \tag{2.49}$$

as in (2.5). Indeed, BRST-invariant combinations of the building block (2.49) descend from a generating series of color-dressed tree level amplitudes $\mathscr{M}^{\text{SYM}}(1,2,\ldots,N)$ [78],

$$\frac{1}{3}\text{Tr}\,\langle \mathbb{V}\mathbb{V}\mathbb{V}\rangle = \sum_{N=3}^{\infty}(N-2)\sum_{i_1<i_2<\ldots<i_N}\mathscr{M}^{\text{SYM}}(i_1,i_2,\ldots,i_N), \qquad \mathbb{V}\equiv\lambda^\alpha \mathbb{A}_\alpha\,. \tag{2.50}$$

Since (2.50) is invariant under non-linear gauge transformations, the components of (2.49) can be equivalently evaluated in HS gauge for arbitrary multiplicity,

$$\left\langle M^{\text{HS}}_A M^{\text{HS}}_B M^{\text{HS}}_C \right\rangle = \frac{1}{2}\mathfrak{e}^m_A \mathfrak{e}^n_B \mathfrak{f}^C_{mn} + (\mathscr{X}_A\gamma_m\mathscr{X}_B)\mathfrak{e}^m_C + \text{cyc}(A,B,C)\,. \tag{2.51}$$

The component currents $\mathfrak{e}^m_A, \mathscr{X}^\alpha_A$ and $\mathfrak{f}^{mn}_A$ defined in (2.42) and (2.44) can be obtained by truncating the superspace recursion (2.11) to (2.12d) to $\theta=0$. By the theta-expansions in (2.46), this component extraction involves no tensor structures $\sim \lambda^3\theta^5$ other than

$$\begin{aligned}
\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma_r\theta)(\theta\gamma^{pqr}\theta)\rangle &= 32(\delta^{mp}\delta^{nq}-\delta^{mq}\delta^{np})\\
\langle(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\gamma_n\theta)_\alpha(\gamma_p\theta)_\beta\rangle &= -18\gamma^m_{\alpha\beta},
\end{aligned}$$

and elegantly settles the building blocks for components of tree level amplitudes. In [84], it has been demonstrated that (2.51) reproduces the Berends-Giele formula for bosonic tree amplitudes [79] along with its supersymmetric completion from the pure spinor superspace formula [85].

As mentioned at the end of section 2.3, the component expression in (2.51) allows us to obtain BCJ numerators by inserting multiparticle Berends–Giele currents $A^{P,\text{BCJ}}_\alpha$ in the BCJ gauge to the building block in (2.51). Explicitly, we have

$$\left\langle M^{\text{BCJ-HS}}_A M^{\text{BCJ-HS}}_B M^{\text{BCJ-HS}}_C \right\rangle = \frac{1}{2}\mathfrak{e}^m_{A,\text{BCJ}}\mathfrak{e}^n_{B,\text{BCJ}}\mathfrak{f}^{C,\text{BCJ}}_{mn} + (\mathscr{X}_{A,\text{BCJ}}\gamma_m\mathscr{X}_{B,\text{BCJ}})\mathfrak{e}^m_{C,\text{BCJ}} + \text{cyc}(A,B,C). \tag{2.52}$$

Moreover, since components currents $\mathfrak{e}^m_{A,\text{BCJ}}, \mathscr{X}^\alpha_{A,\text{BCJ}}$ and $\mathfrak{f}^{mn}_{A,\text{BCJ}}$ are zeroth order components of θ-expansions of corresponding multiparticle Berends-Giele currents $\mathscr{A}^m_{A,\text{BCJ}}, \mathscr{W}^\alpha_{A,\text{BCJ}}$ and $\mathscr{F}^{mn}_{A,\text{BCJ}}$, they can be obtained by following the description in section 2.3 with replacing multiparticle Berends-Giele currents to their zeroth order components. See [28] for more details.

The generating series (2.50) found appearance in [67] as a superspace action for the 10D SYM. The component evaluation in (2.51) is compatible with the component action of SYM in the sense that

$$\frac{1}{3}\text{Tr }\langle \mathbb{V}\mathbb{V}\mathbb{V}\rangle = \text{Tr }\left(\frac{1}{4}\mathbb{F}_{mn}\mathbb{F}^{mn} + (\mathbb{W}\gamma^m\nabla_m\mathbb{W})\right)\Big|_{\theta=0}.$$

The fermionic coupling vanishes on-shell by the Dirac equation

$$\nabla_m\left(\gamma^m\mathbb{W}\right)_\alpha = 0$$

and a total derivative ∂_m has been discarded to relate

$$(\partial_m\mathbb{A}_n)\mathbb{F}^{mn} = \partial_m\left(\mathbb{A}_n\mathbb{F}^{mn}\right) - \mathbb{A}_n\left(\left[\mathbb{A}_m,\mathbb{F}^{mn}\right] + \gamma^n_{\alpha\beta}\left\{\mathbb{W}^\alpha,\mathbb{W}^\beta\right\}\right)$$

through the expression for $\partial_m\mathbb{F}^{mn}$ in

$$\left[\nabla_m,\mathbb{F}^{mp}\right] = \gamma^p_{\alpha\beta}\left\{\mathbb{W}^\alpha,\mathbb{W}^\beta\right\}.$$

2.4.5 Application of Harnad-Shnider gauge to BCJ master numerators

One can also define local multiparticle superfields in Harnad-Shnider gauge such as $A^{12\ldots p}_{\alpha,\text{HS}}$ by adapting the recursions (2.40) and (2.41) to local fields[6]. Their θ-expansion is of the form in (2.46), i.e. the components due to multiparticle vertex operators $V^{\text{HS}}_{12\ldots p} = \lambda^\alpha A^{12\ldots p}_{\alpha,\text{HS}}$ in Harnad-Shnider gauge follow the structure of (2.52),

$$\left\langle V^{\text{HS}}_A V^{\text{HS}}_B V^{\text{HS}}_C\right\rangle = \frac{1}{2}e^m_A e^n_B f^C_{mn} + (\chi_A\gamma_m\chi_B)e^m_C + \text{cyc}(A,B,C). \tag{2.53}$$

The local multiparticle polarizations on the right-hand side are defined as the θ^0 order of the local multiparticle superfields of section 2.3,

$$[A^m_P]_0 = e^m_P e^{k_P\cdot x}\ , \quad [W^\alpha_P]_0 = \chi^\alpha_P e^{k_P\cdot x}\ , \quad [F^{mn}_P]_0 = f^{mn}_P e^{k_P\cdot x}. \tag{2.54}$$

By applying (2.53) to the BCJ master numerators in section 2.3.6, we obtain a compact formula for their bosonic and fermionic components [84]. Note that similar multiparticle polarizations and BCJ numerators have been given [86] for matrix elements of YM with higher-mass-dimension deformations F^3,F^4 that preserve the color-kinematics duality [87].

[6]Alternatively, the map between Berends–Giele currents and multiparticle superfields exemplified in (2.29) can be adapted to Harnad-Shnider-gauge currents.

THREE

ONE-LOOP SUPERSTRING AMPLITUDES AND DOUBLY-PERIODIC FUNCTIONS

The PS superstring theory in the previous chapter can be applied to compute loop amplitudes by taking world-sheets as Riemann surfaces of genus $g \geq 1$. This framework has significantly extended the computational reach for superstring amplitudes, see [88, 89, 90, 91, 92, 93] for multiloop results.

However, in contrast to the bosonic string action in (1.10) the PS superstring action has no explicit dependence on the complex structure of a given world-sheet, and instead the complex structure can be expressed as a composition of dynamical variables of the action. The composite nature of the complex structure is then translated into the complication of the insertion of the path integral measure [94], which poses difficulties in the direct evaluation of loop amplitudes with six and more external legs, although indirect methods have been successfully applied to pinpoint the complete one-loop six- and seven-point results [35, 29, 30, 31].

On the other hand, the covariant action of the RNS superstring theory arising from the Hamiltonian action in (1.16) reveals that world-sheets of the RNS superstring theory are super Riemann surfaces which are supersymmetric extensions of Riemann surfaces. Also, the RNS superstring action depends explicitly on the super complex structure of a super Riemann surface, which leads to the path integral formulation for tree and loop superstring scattering amplitudes from first principles.

However, the computation of superstring amplitudes in the RNS superstring theory has the two-fold difficulty originating from the mathematical complication of the moduli space of super Riemann surfaces and the non-manifestation of the spacetime supersymmetry in the formulation. In particular, the latter often requires extra manipulations to reveal the consequences of spacetime supersymmetry in amplitude computations.

At one-loop, the moduli space of world-sheets relevant to the one-loop string diagram is relatively simple compared to multiloop cases, but one has to sum over different boundary conditions of world-sheet fermions, called *spin structures*, to demonstrate the supersymmetric cancellation in superstring amplitudes. The summation over spin structures is well understood for one-loop RNS amplitudes with any number of external bosons [32, 95, 33], and in this chapter we advance the method to implement spin sums for n-boson-two-fermion (nB-2F) amplitudes at one-loop.

3.1 Massless vertex operators in the RNS formalism

In the RNS superstring theory, massless bosons and fermions enter the amplitude prescription through vertex operators constructed from the set of local operators

$$\left\{c(z), \psi^{\mu}(z), S_{a}(z), e^{q\phi}(z), e^{ipX}(z,\bar{z}) \middle| \mu = 0, \dots, \dim \mathscr{M} - 1,\ a = 1, \dots, 16,\ \text{and } q \in \mathbb{Z}/2\right\} \tag{3.1}$$

together with an anti-holomorphic operator for a closed string. Here, μ denotes a spacetime vector index, and a is a Weyl spinor index. The integer or the half-integer q of the operator $e^{q\phi}$ is called the superghost number due to its close relation with the ghost-antighost pair of the world-sheet supersymmetry. Also, we set $\alpha' = 2$ for notational simplicity.

The vertex operator of a massless open string state with momentum p (under the doubling trick) is then given by

$$V_{\text{open}}^{(-1)\text{ or }(-1/2)}(p;z) = \begin{cases} e^{\mu}(p)\, c(z)\, \psi_{\mu}(z)\, e^{ipX-\phi}(z) & \text{for a NS bosonic state} \\ \chi^{a}(p)\, c(z)\, S_{a}(z)\, e^{ipX-\phi/2}(z) & \text{for a R fermionic state} \end{cases} \tag{3.2a}$$

where e^{μ} is a spacetime vector and χ^{a} is a Majonara-Weyl spinor satisfying on-shell conditions

$$p_{\mu}e^{\mu} = 0 \qquad \text{and} \qquad p_{\mu}\gamma^{\mu}_{ab}\chi^{b} = 0 \tag{3.2b}$$

imposed by the BRST condition. In contrast to chapter 2, the 16×16 Pauli matrices γ^{μ}_{ab} satisfy the Clifford algebra $\gamma^{\mu}_{ab}\gamma^{\nu bc} + \gamma^{\nu}_{ab}\gamma^{\mu bc} = -2\eta^{\mu\nu}\delta^{c}_{b}$ with the minus sign.

For a massless closed string, the vertex operator with momentum p can be written in the form of

$$V_{\text{closed}}^{(1)\text{ or }(-1/2)}(p;z,\bar z) = \begin{cases} e^{\mu}(p)c(z)\psi_{\mu}(z)\bar V(p;\bar z)e^{ipX-\phi}(z,\bar z) \\ \chi^{a}(p)c(z)S_{a}(z)\bar V(p;\bar z)e^{ipX-\phi/2}(z,\bar z) \end{cases} \tag{3.2c}$$

where $\bar V$ is an anti-holomorphic vertex operator, and its specific form is determined by the type of the RNS superstring theory. In the type II superstring theory, it can be constructed from anti-holomorphic copies of operators in (3.1), and in the heterotic superstring theory it can be built from world-sheet currents. A simple example is the graviton vertex operator in the type II superstring theory

$$V_{\text{graviton}}(p;z,\bar z) = h_{\mu\nu}(p)c(z)\psi^{\mu}(z)\bar c(\bar z)\bar\psi^{\nu}(\bar z)e^{ipX-\phi-\bar\phi}(z,\bar z)$$

where $h_{\mu\nu}$ is the symmetric and traceless part of $e_{\mu}\bar e_{\nu}$ corresponding a metric fluctuation satisfying the transverse condition $p^{\mu}h_{\mu\nu} = 0$. Henceforth, we only consider the heterotic case for the closed superstring theory.

Local operators in (3.1) are conformal primaries having conformal weights

$$\begin{gathered} h(c) = (-1,0), \qquad h\left(\psi^{\mu}\right) = \left(\frac{1}{2},0\right), \qquad h\left(S_{a}\right) = \left(\frac{5}{8},0\right), \\ h\left(e^{q\phi}\right) = \left(-\frac{1}{2}q^{2}-q,0\right), \qquad h\left(e^{ipX}\right) = \left(\frac{p^{2}}{2},\frac{p^{2}}{2}\right) \end{gathered} \tag{3.3}$$

and following OPEs [96, 97]

$$\psi^{\mu}(z)\psi^{\nu}(0) = z^{-1}\eta^{\mu\nu} + \psi^{\mu}\psi^{\nu}(0) + \mathcal{O}\left(z^{1}\right) \tag{3.4a}$$

$$\psi^{\mu}(z)S_{a}(0) = \frac{1}{\sqrt{2}}z^{-1/2}\gamma^{\mu}_{ab}S^{b}(0) + \mathcal{O}(z^{1/2}) \tag{3.4b}$$

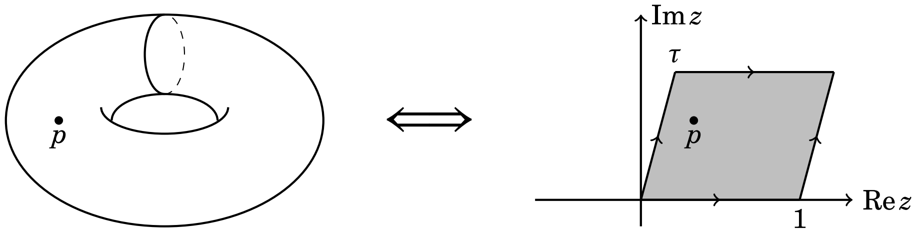
$$S_{a}(z)S^{b}(0) = z^{-5/4}\delta^{b}_{a} + \frac{1}{4}z^{-1/4}\left(\gamma_{\mu\nu}\right)_{a}{}^{b}\psi^{\mu}\psi^{\nu}(0) + \mathcal{O}\left(z^{3/4}\right) \tag{3.4c}$$

$$S_{a}(z)S_{b}(0) = \frac{1}{\sqrt{2}}z^{-3/4}\gamma_{\mu ab}\psi^{\mu}(0) + \mathcal{O}\left(z^{1/4}\right) \tag{3.4d}$$

$$e^{q_1\phi}(z)e^{q_2\phi}(0) = z^{-q_1q_2}e^{(q_1+q_2)\phi}(0) + z^{-q_1q_2+1}q_2\partial\phi e^{(q_1+q_2)\phi}(0) + \mathcal{O}\left(z^{-q_1q_2+2}\right) \tag{3.4e}$$

$$e^{ip_1X}(z,\bar z)e^{ip_2X}(0,0) = |z|^{2p_1\cdot p_2}e^{i(p_1+p_2)X}(0) + \mathcal{O}\left(|z|^{2p_1\cdot p_2+1}\right) \tag{3.4f}$$

where we have omitted OPEs irrelevant to amplitude computations. Conformal weights in (3.3) indicate that holomorphic parts of vertex operators for massless bosons and fermions have the conformal weight 0, which is required by the BRST condition. Also The OPE (3.4b) indicates that the insertion point of a spin field S_a corresponds to the endpoint of the branch cut for each ψ^{μ} insertion, so the interaction between S_a and ψ^{μ} is non-local [98].



(a) (b)

Figure 3.1: (a) A torus with a puncture p and (b) its embedding on the complex plane.

3.2 One-loop amplitudes in RNS superstring theory with two fermions

3.2.1 World-sheets for one-loop RNS superstring amplitudes

As mentioned at the beginning of this chapter, world-sheets of the RNS superstring theory are super Riemann surfaces. Roughly speaking, a super Riemann surface $\Sigma^{1|1}$ is an extension of an ordinary Riemann surface which locally admits supercomplex coordinates (z,θ) in $\mathbb{C}^{1|1}$ where θ is an anti-commuting variable [99]. For two open sets U_α and U_β in $\Sigma^{1|1}$ with the non-empty overlap, local coordinates (z_α,θ_α) of U_α and (z_β,θ_β) of U_β are superholomorphically related to each other [99]

$$\begin{aligned} z_\alpha =& u_{\alpha\beta}(z_\beta)+\theta_\beta\eta_{\alpha\beta}(z_\beta)\sqrt{u'_{\alpha\beta}(z_\beta)},\\ \theta_\alpha =& \eta_{\alpha\beta}(z_\beta)+\theta_\beta\sqrt{u'_{\alpha\beta}(z_\beta)+\eta_{\alpha\beta}(z_\beta)\eta'_{\alpha\beta}(z_\beta)} \end{aligned}$$

where $u_{\alpha\beta}$ is an even holomorphic function on z_β and $\eta_{\alpha\beta}$ is an odd holomorphic function on z_β.

For the case that $\eta_{\alpha\beta}=0$, the super-holomorphic condition is reduced to

$$z_\alpha = u_{\alpha\beta}(z_\beta), \qquad \theta_\alpha = \theta_\beta\sqrt{u'_{\alpha\beta}(z_\beta)}. \tag{3.6}$$

If we project out the odd coordinate, the super-holomorphic condition becomes the holomorphic condition on an ordinary Riemann surface Σ_{red}, so the space obtained by the above reduced super-holomorphic condition becomes a line bundle over Σ_{red} whose fibre is locally trivialized by θ_α. For a genus g Σ_{red}, the square root in (3.6) implies 2^{2g} choices of the sign, which are called spin structures. Those spin structures can be classified by even or odd, depending on whether the space of global holomorphic sections of the line bundle over Σ_{red} has the even or odd number of dimensions [99], and realized by boundary conditions of world-sheet fermions ψ^m in the RNS superstring action.

At one-loop, superstring amplitudes in the RNS superstring theory can be computed from correlators with vertex operator insertions and the measure insertion defined on a super Riemann surface $\Sigma^{1|1}$ whose reduced Riemann surface Σ_{red} is a punctured genus one Riemann surface. For closed strings, a punctured genus one Riemann surface is a torus with marked points[1], which can be embedded into the complex plane z through the identification

$$z \simeq z+m+n\tau, \qquad m,n\in\mathbb{Z},\ \tau\in\mathbb{C} \text{ with } \mathrm{Im}(\tau)>0$$

as depicted in fig. 3.1. A punctured genus one Riemann surface for an open superstring ampli-

[1]Throughout this chapter, we only consider oriented surfaces.

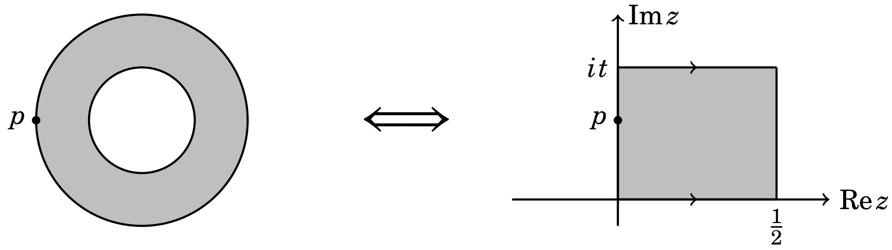


Figure 3.2: (a) An annulus with a puncture p at its boundary and (b) the corresponding embedding on the complex plane.

tude is an annulus with marked points located at its boundary, which can be embedded into the complex plane through [100] (See fig. 3.2.)

$$0 \le \mathrm{Re}\, z \le \frac{1}{2}, \qquad z \simeq z + it, \qquad t \in \mathbb{R}.$$

Upon the doubling trick, an annulus on the complex plane can be interpreted as a torus with $\tau = it$, so in the most of amplitude computations, there is no need to distinguish open string world-sheets from closed string world-sheets.

3.2.2 One-loop RNS superstring amplitudes with two fermions

The contribution of the insertion of the path integral measure together with the integration over odd variables in one-loop superstring amplitudes results in the modification of the form of some vertex operator insertions. Those modified vertex operators have the form of [97, 101]

$$V_{\text{open}}^{(0)\text{ or }(+1/2)}(p;z) = \begin{cases} e^{\mu}(p)c(z)\left(p^{\nu}\psi_{\nu}\psi_{\mu} + i\partial X_{\mu}\right)(z)e^{ipX}(z), \\ \chi^{a}c(z)\left(\frac{1}{\sqrt{2}}\gamma^{\mu}_{ab}i\partial X_{\mu}S^{b} + p_{\mu}S^{\mu}_{a}\right)(z)(z)e^{ipX+\phi/2}(z) \end{cases} \tag{3.7a}$$

for an open string, and

$$V_{\text{closed}}^{(0)\text{ or }(+1/2)}(p;z,\bar{z}) = \begin{cases} e^{\mu}(p)c(z)\left(p^{\nu}\psi_{\nu}\psi_{\mu} + i\partial X_{\mu}\right)(z)\bar{V}(p;\bar{z})e^{ipX}(z,\bar{z}), \\ \chi^{a}(p)c(z)\left(\frac{1}{\sqrt{2}}\gamma^{\mu}_{ab}i\partial X_{\mu}S^{b} + p_{\mu}S^{\mu}_{a}\right)\bar{V}(p;\bar{z})e^{ipX+\phi/2}(z,\bar{z}), \end{cases} \tag{3.7b}$$

for a closed string, where $\bar{V}$ is again an anti-holomorphic operator may or may not be modified by the measure insertion, and

$$S^{\mu}_{a}(z) \propto \oint_{z} dw\,(w-z)^{-1}\psi^{\mu}\psi^{\nu}(w)\gamma_{\nu ab}S^{b}(z). \tag{3.8}$$

are excited spin field operators satisfying the irreducibility constraint

$$\gamma^{ab}_{m}S^{\mu}_{b} = 0 \tag{3.9}$$

due to the on-shell condition on χ^{a}. Also, we need integrated versions of vertex operators in (3.2) and (3.7) obtained by replacing

$$c(z) \quad \leftrightarrow \quad \int$$

where $\int$ is the integral over a torus for a closed string and over the boundary of an annulus for an open string. In the following discussion, the integrand of an integrated vertex operator will be denoted by U instead of V in the above.

One-loop amplitudes of the open RNS superstring are then computed from correlation functions with vanishing overall superghost charge. For n external bosons and two external fermions, we have

$$\mathscr{A}^{\text{1-loop}}_{\text{open}} = \int_{\mathscr{M}_{1,n+2}} \sum_{\nu=1}^{4} (-1)^{n+1} \left\langle \prod_{i=1}^{M} U^{(0)}_{\text{open}}(e_i,p_i;z_i) U^{(-1/2)}_{\text{open}}(\chi,p_A;z_A) U^{(+1/2)}_{\text{open}}\left(\bar{\chi},p_B;z_B\right) \right\rangle_\nu \tag{3.10a}$$

where $\int_{\mathscr{M}_{1,n+2}}$ denotes the integral over the moduli space of genus-one Riemann surfaces with $n+2$ punctures, and $\nu = 1,2,3,4$ are spin structures corresponding to the four boundary conditions of the world-sheet spinors ψ^μ which may be independently chosen as periodic or antiperiodic under translations around the A- and B-cycle of a torus. Similarly, the closed superstring amplitude for n bosons and 2 fermions also can be written as

$$\mathscr{A}^{\text{1-loop}}_{\text{closed}} = \int_{\mathscr{M}_{1,n+2}} \sum_{\nu=1}^{4} (-1)^{n+1} \left\langle \prod_{i=1}^{n} U^{(0)}_{\text{closed}}(p_i;z_i,\bar{z}_i) U^{(-1/2)}_{\text{closed}}(p_A;z_A,\bar{z}_A) U^{(+1/2)}_{\text{closed}}(p_B;z_B,\bar{z}_B) \right\rangle_\nu . \tag{3.10b}$$

The correlation function in (3.10) factorizes into contributions from the decoupled CFT sectors of the superghost fields $e^{q\phi}$, the $\{\psi^\mu, S_a, S^\mu_b\}$ system, world-sheet bosons X^μ as well as the anti-holomorphic sector for closed string amplitudes, and by ignoring the anti-holomorphic sector, only the former two depend on the spin structure ν. Correlation functions of the world-sheet bosonic sector in (3.10) can be expressed in the form of

$$\left\langle \prod_{j=1}^{m} i\partial X^{\mu_j}\left(z_j\right) \prod_{k=1}^{n} e^{ip_k\cdot X}(z_k,\bar{z}_k) \right\rangle \qquad \text{for } m \le n,$$

and the general spin structure dependent correlation function is given by either

$$\left\langle e^{-\phi/2}(z_A) e^{+\phi/2}(z_B)\right\rangle_\nu \left\langle \prod_{i=1}^{n} \psi^{\mu_i}\psi^{\nu_i}(z_i) S_a(z_A) S^b(z_B) \right\rangle_\nu \tag{3.11a}$$

or

$$\left\langle e^{-\phi/2}(z_A) e^{+\phi/2}(z_B)\right\rangle_\nu \left\langle \prod_{i=1}^{n} \psi^{\mu_i}\psi^{\nu_i}(z_i) S_a(z_A) S^\lambda_b(z_B) \right\rangle_\nu . \tag{3.11b}$$

We will often use the notation

$$\langle\langle \psi^\mu(z_1)\ldots S_a(z_2)\rangle\rangle_\nu = \left\langle e^{-\phi/2}(z_A) e^{+\phi/2}(z_B)\right\rangle_\nu \langle \psi^\mu(z_1)\ldots S_a(z_2)\rangle_\nu \tag{3.12}$$

to denote spin structure dependent correlators.

In the RNS formalism, the two-fermion amplitudes (3.10) with $n = 2$ bosons has firstly been computed in [102] (also see [103, 104] for work on $n = 3$). For higher numbers of bosons, however, the challenges from the correlators in (3.10) and the sum over spin structures have never been addressed prior to [40].

As a technical remark, the superghost pictures of the above vertex operators partially depart from the prescription of [53, 105] on the distribution of superghost charges near a world-sheet degeneration. This can be balanced by relocating the superghost pictures which will generically introduce boundary terms in moduli space (see appendix A of [35] for explicit examples in the PS formalism). Such boundary terms are likely to vanish with a large amount of supersymmetry in ten-dimensional flat spacetime, but they might play a role in compactifications with reduced supersymmetry. It would be interesting to pinpoint the onset of such boundary terms.

In the following sections, we will briefly review computations of the genus-one correlators of the world-sheet bosons, the superghosts and the combination of Lorentz currents $\psi^\lambda \psi^\rho$ with unexcited spin fields $S_a(z_A)S^b(z_B)$. New results to be given in section 3.3.5 and section 3.5 include the genus-one correlators involving excited spin fields as well as the spin sum over the ν-dependent correlators of (3.10).

3.3 Computations of correlation functions

3.3.1 Structure of the correlators

By construction, correlation functions in (3.10) are doubly-periodic under

$$z \to z + m + n\tau, \qquad m, n \in \mathbb{Z}$$

for each insertion point. Elementary mathematical objects for representing such doubly-periodic functions are Jacobi theta functions $\theta_\nu(z,\tau)$, $\nu = 1,2,3,4$ defined by

$$\theta_\nu(z,\tau) = \begin{cases} -i \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{2\pi i (n+\frac{1}{2})z} & \text{for } \nu = 1, \\ \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} e^{2\pi i (n+\frac{1}{2})z} & \text{for } \nu = 2, \\ \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} e^{2\pi i n z} & \text{for } \nu = 3, \\ \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} e^{2\pi i n z} & \text{for } \nu = 4, \end{cases}$$

where the dependence on the second argument via $q \equiv e^{2\pi i\tau}$ will often be suppressed in the subsequent. They are quasi-periodic under $z \to z+1$ and $z \to z+\tau$

$$\begin{aligned} \theta_1(z) &= -\theta_1(z+1) = -q^{1/2} e^{2i\pi z} \theta_1(z+\tau) \\ \theta_2(z) &= -\theta_2(z+1) = q^{1/2} e^{2i\pi z} \theta_2(z+\tau) \\ \theta_3(z) &= \theta_3(z+1) = q^{1/2} e^{2i\pi z} \theta_3(z+\tau) \\ \theta_4(z) &= \theta_4(z+1) = -q^{1/2} e^{2i\pi z} \theta_4(z+\tau), \end{aligned}$$

and related to each other by the half-periodicity

$$\theta_1(z) = -\theta_2\left(z+\frac{1}{2}\right) = -iq^{1/8} e^{i\pi z} \theta_4\left(z+\frac{\tau}{2}\right) = -iq^{1/8} e^{i\pi z} \theta_3\left(z+\frac{1}{2}+\frac{\tau}{2}\right). \tag{3.14}$$

Bosonic correlators of the free fields X^μ can be straightforwardly computed from the two-point function on the torus

$$\left\langle iX^\mu(z) iX^\lambda(0) \right\rangle = \eta^{\mu\lambda} \left[\log\left|\frac{\theta_1(z)}{\theta_1'(0)}\right|^2 - \frac{2\pi}{\mathrm{Im}(\tau)} [\mathrm{Im}(z)]^2 \right] \equiv \eta^{\mu\lambda} G(z)$$

via Wick-contractions, e.g.[2]

$$\left\langle i\partial X^\mu(z_1)\prod_{j=1}^N e^{ip_jX(z_j)}\right\rangle = \sum_{l=2}^N p_l^\mu\left(\partial\log\theta_1(z_{1l})+2\pi i\frac{\text{Im}(z_{1l})}{\text{Im}(\tau)}\right)\prod_{i<j}^N e^{p_i\cdot p_jG(z_{ij})} \tag{3.15}$$

with $z_{ij}\equiv z_i-z_j$ and additional Wick contractions $i\partial X^\mu(z)i\partial X^\lambda(0)\sim\eta^{\mu\lambda}\partial^2G(z)$ between multiple insertions of ∂X^μ.

The CFT sectors which are sensitive to the spin structure involve the prime form

$$E(z,w)=\frac{\theta_1(z-w)}{\theta_1'(0)}$$

raised to some fractional powers. By design of the GSO projection, the powers of the prime form always conspire to integers when combining the individual correlators of the superghost system [108]

$$\left\langle e^{-\phi/2}(z_A)e^{+\phi/2}(z_B)\right\rangle_\nu = \frac{\theta_1'(0)E(z_A,z_B)^{1/4}}{\theta_\nu\left(\frac{1}{2}(z_B-z_A)\right)} \tag{3.16}$$

and the $\{\psi^\mu,S_a,S^\mu_b\}$ system, starting with the two-point function of the spin field [108]

$$\left\langle S_a(z_A)S^b(z_B)\right\rangle_\nu = \frac{\delta^b_a\,\theta_\nu\left(\frac{1}{2}(z_B-z_A)\right)^5}{\theta_1'(0)^5E(z_A,z_B)^{5/4}}. \tag{3.17}$$

In slight abuse of notation, the combined partition function $\left(\frac{\theta_\nu(0)}{\theta_1'(0)}\right)^4$ of ten worldsheet bosons and fermions as well as the respective ghosts has been absorbed into the normalization of the ν-dependent correlators in (3.16), (3.17) and (3.12). This is useful for a unified treatment of the odd spin structure $\nu=1$ and the even ones $\nu=2,3,4$ such that their contributions to the amplitude (3.10) can be efficiently combined. In particular, this choice of normalization bypasses spurious indeterminates of the form $\frac{0}{0}$ from the formally vanishing $\theta_\nu(0)$ in the partition function of the odd spin structure.

3.3.2 Bosonization

The interacting nature of spin fields as reflected in their OPE (3.4) with the worldsheet spinors ψ^μ renders $SO(1,9)$-covariant correlation functions inaccessible to free-field methods. In other words, correlators cannot be obtained from a naive sum over Wick contractions as in (3.15), and the computation of higher-point instances becomes a nontrivial problem, see [109, 110]. However, a free-field description in even spacetime dimensions $D=2n$ can be found by representing the $\{\psi^\mu,S_a,S^\mu_b\}$-system via n free bosons. These redefinitions are known as bosonization [111] and break the $SO(1,9)$ symmetry to an $SU(5)$ subgroup.

Let $\mathbf{H}$ denote an $SU(n)$ vector of free chiral bosons $\{H^j,\ j=1,2,\ldots,n\}$ subject to normalization $H^j(z)H^k(0)\sim-\delta^{jk}\ln(z)+\ldots$, then its exponentials $e^{i\mathbf{p}\cdot\mathbf{H}}$ are conformal primaries of weight $\frac{1}{2}\mathbf{p}^2$ with OPEs[3]

$$e^{i\mathbf{p}\cdot\mathbf{H}}(z)e^{i\mathbf{q}\cdot\mathbf{H}}(0)\sim z^{\mathbf{p}\cdot\mathbf{q}}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{H}}(0)\ +\ \ldots\ .$$

[2] In order to obtain a double-copy representation of closed-string correlators, one can follow the prescription of chiral splitting [106, 45, 107] and exclude the contributions from the joint zero modes of ∂X^μ and $\bar\partial X^\mu$ from the Wick contractions. This simplifies the contractions in (3.15) to the meromorphic expression $\partial X^\mu(z_1)e^{ip_jX(z_j)}\to p^\mu_j\partial\log\theta_1(z_{1j})e^{ip_jX(z_j)}$.

[3] To simplify the notation, we neglect Jordan-Wigner cocycle factors [112, 113] in our discussion. These are additional algebraic objects accompanying the exponentials to ensure that $e^{\pm iH^j}$ and $e^{\pm iH^{k\neq j}}$ associated with different bosons anticommute. It suffices to remember that they are implicitly present and that the bosonized representation of ψ^μ still obeys fermi statistics.

The OPE among the worldsheet spinors, $\psi^\mu(z)\psi^\nu(0) \sim \eta^{\mu\nu} z^{-1} + \dots$ can be reproduced from the dictionary

$$\psi^{\pm j}(z) \equiv \frac{1}{\sqrt{2}}\left(\psi^{2j-2}(z) \pm i\psi^{2j-1}(z)\right) \equiv e^{\pm iH^j(z)} \ , \tag{3.18}$$

where $j \in \{1,2,\dots,n\}$. One can notice that $\psi^{\pm j}$ form the Cartan–Weyl basis for the fundamental representation of $SO(1,2n-1)$.

Spinor components of $SO(1,2n-1)$ can be labelled by their eigenvalues $\pm\frac{1}{2}$ under the n simultaneously diagonalized Lorentz generators $\frac{1}{2}\gamma^{\mu\nu}$ which are most conveniently chosen as $\frac{1}{2}\gamma^{2i-2,2i-1}$ with $i = 1,2,\dots,n$ in the $SU(n)$ setting. This suggests to identify spinor indices with n-component lattice vectors $\left(\pm\frac{1}{2},\pm\frac{1}{2},\dots,\pm\frac{1}{2}\right)$ from the (anti-)spinor conjugacy classes of $SO(1,2n-1)$. The chiral irreducibles can be disentangled by counting the number of negative entries:

$$S^{a=(\pm\frac{1}{2},\dots,\pm\frac{1}{2})} \leftrightarrow \text{left-handed spinor} \leftrightarrow a \text{ has an even number of '}-\text{' signs}$$
$$S_{a=(\pm\frac{1}{2},\dots,\pm\frac{1}{2})} \leftrightarrow \text{right-handed spinor} \leftrightarrow a \text{ has an odd number of '}-\text{' signs} \ .$$

Given this dictionary between spinor indices and lattice vectors, we can make the bosonization of spin fields more precise: The S_a, S^a are represented as an exponential of bosons $\mathbf{H}$ contracted into a vector a in the weight lattice of (the Lie algebra of) $SO(1,2n-1)$:

$$S_a(z), S^a(z) \ \equiv \ e^{ia\cdot\mathbf{H}(z)}, \ \ a \ \in \ \left\{(a^1,a^2,\dots,a^n) \mid a^j = \pm\frac{1}{2}, \ j = 1,\dots,n\right\} \ . \tag{3.19}$$

Accordingly, vector indices μ are identified with lattice vectors $(0,\dots,0,\pm1,0,\dots,0)$ of $SO(1,2n-1)$ from the vector conjugacy class with one nonzero entry ±1 such that (3.18) can be written as $\psi^\mu = e^{i\mu\cdot\mathbf{H}}$.

Bosonization of ψ^μ and S_a, S^a allows us to relate other conformal primaries to their bosonized expressions, in particular the excited spin fields S_a^μ in (3.8)[4]:

$$S^\mu_a(z)\big|_{\mu_j+a_j=\pm\frac{3}{2}} = e^{i(a+\mu)\cdot\mathbf{H}(z)} = e^{\pm\frac{i}{2}H^1\pm\dots\pm\frac{i}{2}H^{j-1}\pm\frac{3}{2}iH^j\pm\frac{i}{2}H^{j+1}\pm\dots\pm\frac{i}{2}H^n}(z). \tag{3.20}$$

Therefore, in the bosonization scheme, S_a^μ can be taken as independent primaries involving a factor of $e^{\pm\frac{3}{2}iH^j}$ which capture the gamma-traceless components of the composite operators $\sim \psi^\mu\psi_\nu\gamma^\nu_{ab}S^b$.

The Cartan–Weyl basis has the remarkable advantage that entries of gamma- and charge conjugation matrices can be written as delta functions for the lattice vectors of $SO(1,2n-1)$ associated with the vector- and the spinor indices. Up to a complex phase (which can in principle be determined by keeping track of all the cocycles [112, 113]), one has

$$\delta_a^b \sim \delta(a+b) \ , \qquad \eta^{\mu\nu} \sim \delta(\mu+\nu) \ , \qquad \gamma^\mu_{ab} \sim \sqrt{2}\delta(\mu+a+b) \ . \tag{3.21}$$

The relations in (3.21) admit a derivation of covariant OPEs (3.4) in bosonized language, see appendix C for details.

[4]Obviously, the $n2^n$ bosonized fields of the form (3.20) do no exhaust the $(2n-1)2^n$ independent components of the excited spin field S_a^μ in a spin-3/2 representation of $SO(1,9)$. Still, the $n2^n$ components in (3.20) are sufficient to infer the Lorentz-covariant correlators in the next subsections.

3.3.3 Loop level correlators from bosonization

Correlation functions involving free bosons are well known on surfaces of arbitrary genus [114], and their genus-one instances are given by [108]

$$\left\langle \prod_{j=1}^{N} e^{iq_j H(z_j)} \right\rangle_\nu = \frac{1}{\theta_1'(0)} \delta(\sum_{j=1}^{N} q_j) \theta_\nu(\sum_{k=1}^{N} q_k z_k) \prod_{l<m}^{N} E(z_l, z_m)^{q_l q_m}. \tag{3.22}$$

where we again normalize the correlator such as to absorb the partition function of two worldsheet supermultiplets X^μ, ψ^μ. A general account on bosonization at nonzero genus including the role of spin structures can be found in [115, 116, 117], also see [118] for bosonization of odd-spin structure amplitudes.

For a given choice of the weight vectors μ, a, b, one-loop correlators of the fields $\{\psi^\mu, S_a, S_b^\mu\}$ can be straightforwardly reduced to products of the free-field correlator (3.22) by virtue of their bosonization (3.18), (3.19) and (3.20). Once a sufficient number of such "component" results is available, they can be combined into Lorentz covariant expressions. The idea is to make an ansatz for the correlator with all admissible Lorentz tensors involving products of $\eta^{\mu\lambda}, \delta_a^b, \gamma_{ab}^\mu$ whose index structure is compatible with the $\{\psi^\mu, S_a, S_b^\mu\}$ insertions. Each linearly independent Lorentz tensor in the ansatz is accompanied by a spin structure dependent function of the insertion points z_i and the modular parameter τ, which remains to be determined.

Then, for each component result computed via (3.22), one can use the delta-function representations (3.21) of $\eta^{\mu\nu}, \delta_a^b$ and γ_{ab}^μ to identify the tensor structures compatible with the given choice of lattice vectors. Each choice yields an equation among the unknown functions of z_j and τ along with the Lorentz tensors in the ansatz. In [102, 119, 110], this procedure is applied to construct higher-point correlation functions involving ψ^μ, S_a, some of which are reviewed in section 3.3.4.

Given that the delta-function representation (3.21) of $\eta^{\mu\nu}, \delta_a^b$ and γ_{ab}^μ is only fixed up to complex phases, covariant OPEs such as (3.4) and

$$S_a(z) S_b^\mu(0) = \frac{\gamma_{ab}^\nu \psi_\nu \psi^\mu(0)}{\sqrt{2} z^{5/4}} + \mathcal{O}\left(z^{1/4}\right) \tag{3.23a}$$

$$\psi^\mu \psi^\nu(z) \psi^\lambda \psi^\rho(0) = \frac{\eta^{\lambda[\nu} \eta^{\mu]\rho}}{z^2} + \frac{\eta^{\lambda[\nu} \psi^{\mu]} \psi^\rho(0) - \eta^{\rho[\nu} \psi^{\mu]} \psi^\lambda(0)}{z} + \mathcal{O}\left(z^0\right) \tag{3.23b}$$

are required to determine the signs in correlators, where we remind of the antisymmetrization conventions $\eta^{\lambda[\nu} \eta^{\mu]\rho} = \eta^{\lambda\nu} \eta^{\mu\rho} - \eta^{\lambda\mu} \eta^{\nu\rho}$.

3.3.4 Correlators involving spin fields

The spin-field correlators in the last line of (3.11) computed by the bosonization in the previous section can be assembled from the results of [110] for any number of $\psi^\lambda \psi^\rho$ insertions.

Lower-point example In the notation of (3.12), the simplest generalization of the two-point function

$$\left\langle\left\langle S_a(z_A) S^b(z_B) \right\rangle\right\rangle_\nu = \frac{\delta_a^b \theta_\nu\left(\frac{1}{2}(z_B - z_A)\right)^4}{\theta_1'(0)^4 E(z_A, z_B)} \tag{3.24}$$

is given by

$$\left\langle\left\langle \psi^\lambda \psi^\rho(z_1) S_a(z_A) S^b(z_B) \right\rangle\right\rangle_\nu = \frac{\gamma^{\lambda\rho}{}_a{}^b \theta_\nu\left(\frac{1}{2} z_{AB}\right)^2 \theta_\nu\left(\frac{1}{2}(z_{A1} + z_{B1})\right)^2}{2\theta_1'(0)^4 E_{1A} E_{1B}}, \tag{3.25}$$

where we used the shorthands

$$z_{ij} \equiv z_i - z_j, \qquad E_{ij} \equiv E(z_i, z_j).$$

In the subsequent cases with multiple insertions of $\psi^\lambda \psi^\rho$, it is convenient to introduce the notation

$$T_{ij}^{\nu} \equiv \frac{E_{iA}E_{jB}\theta_\nu\left(z_{ij}+\frac{1}{2}z_{AB}\right)+E_{jA}E_{iB}\theta_\nu(z_{ji}+\frac{1}{2}z_{AB})}{E_{ij}E_{AB}\theta_\nu\left(\frac{1}{2}z_{AB}\right)}, \qquad t_i^\nu \equiv \frac{\theta_\nu\left(\frac{1}{2}(z_{Ai}+z_{Bi})\right)}{\theta_\nu\left(\frac{1}{2}z_{AB}\right)} \tag{3.26}$$

for the coefficients of the tensor structures:

$$\begin{aligned}\left\langle\!\left\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)S_a(z_A)S^b(z_B)\right\rangle\!\right\rangle_\nu &= \frac{E_{AB}\theta_\nu\left(\frac{1}{2}z_{AB}\right)^4}{4\theta_1'(0)^4E_{1A}E_{1B}E_{2A}E_{2B}} \\ \times\Big[\gamma^{\mu_1\nu_1\mu_2\nu_2}{}_a{}^b\left(t_1^\nu t_2^\nu\right)^2 &+ \eta^{\nu_1[\mu_2}\eta^{\nu_2]\mu_1}\delta_a^b\left(T_{12}^\nu\right)^2+\left(\eta^{\mu_2[\nu_1}\gamma^{\mu_1]\nu_2}{}_a{}^b-\eta^{\nu_2[\nu_1}\gamma^{\mu_1]\mu_2}{}_a{}^b\right)T_{12}^\nu t_1^\nu t_2^\nu\Big]\end{aligned} \tag{3.27}$$

The relative signs in the second line depend on the conventions for the Clifford algebra, and we follow [109, 119, 110] with a minus sign on the right-hand side of the anticommutator $\{\gamma^\mu,\gamma^\nu\} = -2\eta^{\mu\nu}$.

Given that vector indices are antisymmetrized with the normalization convention $\eta^{\nu_1[\mu_2}\eta^{\nu_2]\mu_1} = \eta^{\nu_1\mu_2}\eta^{\nu_2\mu_1}-\eta^{\nu_1\nu_2}\eta^{\mu_2\mu_1}$, each tensor in the $[\ldots]$ bracket of (3.27) and the subsequent equation appears with a prefactor of ± 1. We note that the building blocks in (3.26) are related via $t_i^\nu = T_{Bi}^\nu$.

The correlator with one more pair of ψ^μ is given by

$$\begin{aligned}\left\langle\!\left\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)\psi^{\mu_3}\psi^{\nu_3}(z_3)S_a(z_A)S^b(z_B)\right\rangle\!\right\rangle_\nu &= \frac{E_{AB}^2\theta_\nu\left(\frac{1}{2}z_{AB}\right)^4}{8\theta_1'(0)^4\prod_{i=1}^3E_{iA}E_{iB}} \\ \times\Big[\gamma^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}{}_a{}^b\left(t_1^\nu t_2^\nu t_3^\nu\right)^2 &+ \left(\eta^{\nu_1[\mu_2}\eta^{\nu_2][\mu_3}\eta^{\nu_3]\mu_1}-\eta^{\mu_1[\mu_2}\eta^{\nu_2][\mu_3}\eta^{\nu_3]\nu_1}\right)\delta_a^bT_{12}^\nu T_{23}^\nu T_{13}^\nu \\ +\Big\{\left(\eta^{\mu_2[\nu_1}\gamma^{\mu_1]\nu_2\mu_3\nu_3}{}_a{}^b-\eta^{\nu_2[\nu_1}\gamma^{\mu_1]\mu_2\mu_3\nu_3}{}_a{}^b\right)&T_{12}^\nu t_1^\nu t_2^\nu\left(t_3^\nu\right)^2+\eta^{\nu_1[\mu_2}\eta^{\nu_2]\mu_1}\gamma^{\mu_3\nu_3}{}_a{}^b\left(T_{12}^\nu t_3^\nu\right)^2 \\ +\left(\eta^{\mu_3[\nu_2}\eta^{\mu_2][\nu_1}\gamma^{\mu_1]\nu_3}{}_a{}^b-\eta^{\nu_3[\nu_2}\eta^{\mu_2][\nu_1}\gamma^{\mu_1]\mu_3}{}_a{}^b\right)&T_{12}^\nu T_{23}^\nu t_1^\nu t_3^\nu+\text{cyclic}\,(1,2,3)\Big\}\Big]\,,\end{aligned} \tag{3.28}$$

where the cyclic sum in the curly bracket does not extend to the totally symmetric terms in the second line.

The n-point function The above examples of spin-field correlators involving $S_a(z_A)$, $S^b(z_B)$ and $n{\le}3$ currents $\psi^{\mu_j}\psi^{\nu_j}$ point to the generalization to n insertions of $\psi^{\mu_j}\psi^{\nu_j}(z_j)$ (which can be derived from the results of [110]). The structure of the results is captured by

$$\left\langle\!\!\left\langle\prod_{j=1}^n\psi^{\mu_j}\psi^{\nu_j}(z_j)S_a(z_A)S^b(z_B)\right\rangle\!\!\right\rangle_\nu = \frac{E_{AB}^{n-1}\theta_\nu(\frac{1}{2}z_{AB})^4}{2^n\theta_1'(0)^4\prod_{j=1}^nE_{jA}E_{jB}}\sum_i(\ell_{(i)})^{[\mu_i\nu_i]}{}_a{}^b\varphi_\nu^{(i)}(z)\,, \tag{3.29}$$

where the sum over i gathers Lorentz tensors $\ell_{(i)}$ with the index structure of the left-hand side along with spin structure dependent functions of the $n{+}2$ punctures $\varphi_\nu^{(i)}(z) = \varphi_\nu^{(i)}(z_1,z_2,\ldots,z_n,z_A,z_B)$ and τ. The prefactors are ± 1 once the $\ell_{(i)}$ in (3.29) are organized in terms of a single form $(\gamma^{\rho_1\rho_2\ldots\rho_{2k}})_a{}^b$ and products of $\eta^{\cdot\cdot}$ with antisymmetrizations in $\mu_i\leftrightarrow\nu_i$, cf. (3.28).

Most importantly, each Lorentz tensor $\ell_{(i)}$ in (3.29) can be translated into its accompanying function $\varphi_\nu^{(i)}(z)$ through the following dictionary (where $\supset$ is understood as "contains a factor of")

$$\ell_{(i)}\supset(\gamma^{\cdots\mu_j\cdots})_a{}^b \text{ or } (\gamma^{\cdots\nu_j\cdots})_a{}^b \quad\Rightarrow\quad \varphi_\nu^{(i)}(z)\supset t_j^\nu \tag{3.30a}$$

$$\ell_{(i)}\supset\eta^{\mu_j\mu_k} \text{ or } \eta^{\nu_j\nu_k} \text{ or } \eta^{\mu_j\nu_k} \quad\Rightarrow\quad \varphi_\nu^{(i)}(z)\supset T_{jk}^\nu\,, \tag{3.30b}$$

see (3.26) for the definitions of t_j^ν and T_{jk}^ν. The summation range $\sum_i$ in (3.29) involves all Lorentz tensors that can be obtained from partitions of the antisymmetrized pairs of indices $[\mu_1\nu_1],[\mu_2\nu_2],\ldots,[\mu_n\nu_n]$ into a form $(\gamma^{\cdots})_a{}^b$ and products of $\eta^{\cdot\cdot}$.

For each Lorentz tensor $\ell_{(i)}$, the relative prefactor ±1 can be read off by starting with the $2n$-form $\gamma^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\ldots\mu_n\nu_n}$ and moving the pairs of indices entering the given $\eta^{\cdot\cdot}$ to neighboring position. The rule is that the indices in $\eta^{\mu_i\mu_j}$, $\eta^{\nu_i\nu_j}$, $\eta^{\mu_i\nu_j}$ with $i<j$ must be moved into the order $\gamma^{\cdots\mu_i\mu_j\cdots}$, $\gamma^{\cdots\nu_i\nu_j\cdots}$, $\gamma^{\cdots\mu_i\nu_j\cdots}$ and *not* the converse one (such as $\gamma^{\cdots\mu_j\mu_i\cdots}$). Then, the number of transpositions among the μ_i and ν_i required to attain the pairs of neighbors determines the sign of the Lorentz tensor $\ell_{(i)}$ according to the total antisymmetry of the $\gamma^{\cdots}$. The leftover indices of the form must be left in their order after transferring the neighboring pairs to the $\eta^{\cdot\cdot}$.

For instance, the negative sign of $\eta^{\nu_1\mu_2}\eta^{\nu_2\nu_3}\gamma^{\mu_1\mu_3}{}_a{}^b$ in (3.28) can be seen by rearranging $\gamma^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}=-\gamma^{\mu_1\nu_1\mu_2\nu_2\nu_3\mu_3}$ and then removing the neighboring pairs $\nu_1\mu_2\to\eta^{\nu_1\mu_2}$ and $\nu_2\nu_3\to\eta^{\nu_2\nu_3}$, leaving $-\gamma^{\mu_1\nu_1\mu_2\nu_2\nu_3\mu_3}\to-\gamma^{\mu_1\mu_3}$.

3.3.5 Correlators involving excited spin fields

On top of the spin-field correlators in section 3.3.4, the integrand (3.10) for two-fermion amplitudes requires correlators of the form $\langle\langle\prod_j\psi^{\lambda_j}\psi^{\rho_j}(z_j)S_a(z_A)S_b^\mu(z_B)\rangle\rangle_\nu$ with an excited spin field S_b^μ. Again, following the techniques of [108, 111, 114, 110], we will determine the structure of these genus-one correlators using bosonization techniques for any number of $\psi^\lambda\psi^\rho$ insertions.

Three-point example Since the two-point correlator $\langle S_a(z_A)S_b^\mu(z_B)\rangle_\nu$ of primary fields with different conformal weights vanishes, the simplest example involving an excited spin field reads

$$\left\langle\psi^\lambda\psi^\rho(z_1)S_a(z_A)S_b^\mu(z_B)\right\rangle_\nu=(\eta^{\mu\rho}\gamma^\lambda_{ab}-\eta^{\mu\lambda}\gamma^\rho_{ab})r_\nu(z_1,z_A,z_B)\,,\qquad(3.31)$$

with some function r_ν of z_1,z_A,z_B and τ. The tensor on the right-hand side is uniquely determined by the antisymmetry of $\psi^\lambda\psi^\rho=-\psi^\rho\psi^\lambda$ and the irreducibility condition (3.9) of the excited spin field S_b^μ which forbids a "gamma-trace" $\sim\gamma^\mu_{ab}$ as well as the corresponding three-form $\gamma^{\lambda\rho\mu}_{ab}$. By choosing the weight vectors to be

$$\begin{aligned}&\lambda\to(-1,0,0,0,0),\qquad \rho\to(0,+1,0,0,0),\qquad \mu\to(+1,0,0,0,0)\\&a\to\frac{1}{2}(-,-,-,-,-),\qquad b\to\frac{1}{2}(+,-,+,+,+),\end{aligned}$$

one can assemble the function in r_ν in (3.31) via five copies of (3.22):

$$\begin{aligned}\pm\sqrt{2}r_\nu(z_1,z_A,z_B)&=\left\langle e^{-iH^1(z_1)}e^{-\frac{i}{2}H^1(z_A)}e^{\frac{3i}{2}H^1(z_B)}\right\rangle_\nu\left\langle e^{iH^2(z_1)}e^{-\frac{i}{2}H^2(z_A)}e^{-\frac{i}{2}H^2(z_B)}\right\rangle_\nu\\&\quad\times\prod_{j=3}^5\left\langle e^{-\frac{i}{2}H^j(z_A)}e^{\frac{i}{2}H^j(z_B)}\right\rangle_\nu\\&=\frac{\theta_\nu\left(\frac{3}{2}z_B-z_1-\frac{1}{2}z_A\right)\theta_\nu\left(z_1-\frac{1}{2}(z_A+z_B)\right)\theta_\nu^3\left(\frac{1}{2}(z_B-z_A)\right)}{\theta_1'(0)^5E(z_1,z_B)^2E(z_A,z_B)^{5/4}}\end{aligned}$$

The factor of $\sqrt{2}$ on the left-hand side stems from the normalization (3.21) of the gamma-matrices in the Cartan–Weyl basis. By adjoining the correlator (3.16) of the superghosts, the above results combine to

$$\begin{aligned}\left\langle\left\langle\psi^\lambda\psi^\rho(z_1)S_a(z_A)S_b^\mu(z_B)\right\rangle\right\rangle_\nu&=\eta^{\mu[\rho}\gamma^{\lambda]}_{ab}\frac{\theta_\nu\left(\frac{1}{2}(z_{A1}+z_{B1})\right)\theta_\nu\left(z_{1B}+\frac{1}{2}z_{AB}\right)\theta_\nu^2\left(\frac{1}{2}z_{AB}\right)}{\sqrt{2}\theta_1'(0)^4E_{1B}^2E_{AB}}\\&\equiv\frac{\theta_\nu^4\left(\frac{1}{2}z_{AB}\right)}{\sqrt{2}\theta_1'(0)^4E_{1A}E_{1B}E_{AB}}\times\eta^{\mu[\rho}\gamma^{\lambda]}_{ab}t_1^\nu\mathbb{T}_1^\nu\,,\end{aligned}\qquad(3.32)$$

with shorthands $z_{ij} = z_i - z_j$ and $E_{ij} = E(z_i, z_j)$, where the sign can be fixed via Jordan–Wigner cocycles or the covariant OPE (3.23). In passing to the second line of (3.32), we have introduced an additional building block

$$\mathbb{T}_j^\nu \equiv \frac{E_{jA}\theta_\nu\left(z_{jB} + \frac{1}{2}z_{AB}\right)}{E_{jB}\theta_\nu\left(\frac{1}{2}z_{AB}\right)} \tag{3.33}$$

which extends the definitions of T_{ij}^ν and t_i^ν in (3.26) to account for the z-dependence from an excited spin field.

Four-point example As an example with several viable tensor structures, we consider

$$\begin{aligned}\left\langle\left\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)S_a(z_A)S_b^\lambda(z_B)\right\rangle\right\rangle_\nu &= \gamma_{ab}^{[\mu_1}\eta^{\nu_1][\mu_2}\eta^{\nu_2]\lambda}R_\nu^1(z)\\ + \gamma_{ab}^{[\mu_2}\eta^{\nu_2][\mu_1}\eta^{\nu_1]\lambda}R_\nu^2(z) + \gamma_{ab}^{\mu_1\nu_1[\mu_2}\eta^{\nu_2]\lambda}R_\nu^3(z) &+ \gamma_{ab}^{\mu_2\nu_2[\mu_1}\eta^{\nu_1]\lambda}R_\nu^4(z),\end{aligned} \tag{3.34}$$

with $R_\nu^j(z) \equiv R_\nu^j(z_1, z_2, z_A, z_B)$, where $R_\nu^1 \leftrightarrow R_\nu^2$ and $R_\nu^3 \leftrightarrow R_\nu^4$ are related to each other by exchange of z_1 and z_2. In order to see that four tensor structures are sufficient to express the correlator in question, one can verify that the tensor product of the Lorentz representations of $\psi^{\mu_1}\psi^{\nu_1}$, $\psi^{\mu_2}\psi^{\nu_2}$, S_a, S_b^λ contains precisely four scalars.

Starting from $\lambda \to (1,0,0,0,0)$, one can isolate $R_\nu^3(z)$ through the choice

$$\begin{aligned}&\mu_1 \to (0,0,-1,0,0), &&\nu_1 \to (0,0,0,-1,0), &&\mu_2 \to (0,1,0,0,0)\\ &\nu_2 \to (-1,0,0,0,0), &&a \to \frac{1}{2}(-,-,+,+,-), &&b \to \frac{1}{2}(+,-,+,+,+)\end{aligned}$$

of lattice vectors, which specializes (3.34) to

$$\begin{aligned}\pm 2\sqrt{2}R_\nu^3(z) &= \left\langle e^{-\phi/2}(z_A)e^{\phi/2}(z_B)\right\rangle_\nu \prod_{j=3}^4 \left\langle e^{-iH^j(z_1)}e^{\frac{i}{2}H^j(z_A)}e^{\frac{i}{2}H^j(z_B)}\right\rangle_\nu\\ &\times \left\langle e^{-iH^1(z_2)}e^{-\frac{i}{2}H^1(z_A)}e^{\frac{3i}{2}H^1(z_B)}\right\rangle_\nu \left\langle e^{iH^2(z_2)}e^{-\frac{i}{2}H^2(z_A)}e^{-\frac{i}{2}H^2(z_B)}\right\rangle_\nu \left\langle e^{-\frac{i}{2}H^5(z_A)}e^{\frac{i}{2}H^5(z_B)}\right\rangle_\nu\\ &= \frac{\theta_\nu\left(\frac{1}{2}(z_{A1}+z_{B1})\right)^2\theta_\nu\left(\frac{1}{2}(z_{A2}+z_{B2})\right)\theta_\nu\left(z_{2B}+\frac{1}{2}z_{AB}\right)}{\theta_1'(0)^4E_{1A}E_{1B}E_{2B}^2}.\end{aligned} \tag{3.35}$$

The three powers of $\sqrt{2}$ stem from the product of three gamma-matrices in (3.34) along with R_ν^3.

Likewise, combinations of R_ν^1 and R_ν^3 can be addressed via $\lambda \to (1,0,0,0,0)$ and

$$\begin{aligned}&\mu_1 \to (0,0,1,0,0), &&\nu_1 \to (0,\mp 1,0,0,0), &&\mu_2 \to (0,\pm 1,0,0,0)\\ &\nu_2 \to (-1,0,0,0,0), &&a \to \frac{1}{2}(-,-,-,-,-), &&b \to \frac{1}{2}(+,+,-,+,+),\end{aligned} \tag{3.36}$$

which specializes (3.34) to

$$\begin{aligned}&\left\langle e^{-\phi/2}(z_A)e^{\phi/2}(z_B)\right\rangle_\nu \left\langle e^{-iH^1(z_2)}e^{-\frac{i}{2}H^1(z_A)}e^{\frac{3i}{2}H^1(z_B)}\right\rangle_\nu \left\langle e^{\mp iH^2(z_1)}e^{\pm iH^2(z_2)}e^{-\frac{i}{2}H^2(z_A)}e^{\frac{i}{2}H^2(z_B)}\right\rangle_\nu\\ &\times \left\langle e^{iH^3(z_1)}e^{-\frac{i}{2}H^3(z_A)}e^{-\frac{i}{2}H^3(z_B)}\right\rangle_\nu \prod_{j=4}^5 \left\langle e^{-\frac{i}{2}H^j(z_A)}e^{\frac{i}{2}H^j(z_B)}\right\rangle_\nu\\ &= \frac{\theta_\nu\left(\frac{1}{2}z_{AB}\right)\theta_\nu\left(z_{2B}+\frac{1}{2}z_{AB}\right)\theta_\nu\left(\frac{1}{2}(z_{A1}+z_{B1})\right)}{\theta_1'(0)^4E_{12}E_{AB}E_{1A}E_{1B}E_{2B}^2} \times \begin{cases} E_{1A}E_{2B}\theta_\nu(z_{12}+\frac{1}{2}z_{AB}) : \nu_1=(0,-1,0,0,0)\\ E_{1B}E_{2A}\theta_\nu(z_{21}+\frac{1}{2}z_{AB}) : \nu_1=(0,+1,0,0,0)\end{cases}.\end{aligned} \tag{3.37}$$

Both $\gamma_{ab}^{[\mu_1}\eta^{\nu_1][\mu_2}\eta^{\nu_2]\lambda}$ and $\gamma_{ab}^{\mu_1\nu_1[\mu_2}\eta^{\nu_2]\lambda}$ are non-zero for the lattice vectors in (3.36), but they exhibit different symmetry properties under exchange of μ_2 and ν_1: Since $\gamma_{ab}^{[\mu_1}\eta^{\nu_1][\mu_2}\eta^{\nu_2]\lambda}$ is symmetric under $\mu_2 \leftrightarrow \nu_1$, its coefficient must be the sum of the two expressions in (3.37) related by exchange of μ_2 and ν_1. The difference of the two expressions in (3.37) in turn reproduces the coefficient (3.35) of the tensor $\gamma_{ab}^{\mu_1\nu_1[\mu_2}\eta^{\nu_2]\lambda}$ with manifest antisymmetry in $\mu_2 \leftrightarrow \nu_1$, as can be verified through the Fay trisecant identity [120]

$$\begin{aligned}
&E_{12}E_{AB}\theta_\nu\left(\frac{1}{2}(z_1+z_2-z_A-z_B)+z_0\right)\theta_\nu\left(\frac{1}{2}(z_1+z_2-z_A-z_B)-z_0\right)\\
&=E_{1A}E_{2B}\theta_\nu\left(\frac{1}{2}z_{12}+\frac{1}{2}z_{AB}+z_0\right)\theta_\nu\left(\frac{1}{2}z_{12}+\frac{1}{2}z_{AB}-z_0\right)\\
&\quad -E_{1B}E_{2A}\theta_\nu\left(\frac{1}{2}z_{12}-\frac{1}{2}z_{AB}+z_0\right)\theta_\nu\left(\frac{1}{2}z_{12}-\frac{1}{2}z_{AB}-z_0\right)
\end{aligned}$$

at $z_0 \to \frac{1}{2}z_{12}$. After assembling the above results and fixing the signs through covariant OPEs, the correlator of interest is given by

$$\begin{aligned}
\left\langle\!\left\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)S_a(z_A)S_b^\lambda(z_B)\right\rangle\!\right\rangle_\nu &= \frac{\theta_\nu\left(\frac{1}{2}z_{AB}\right)^4}{2\sqrt{2}\theta_1'(0)^4E_{1A}E_{1B}E_{2A}E_{2B}}\\
&\times\left[\gamma_{ab}^{\mu_1\nu_1[\mu_2}\eta^{\nu_2]\lambda}\mathbb{T}_2^\nu(t_1^\nu)^2t_2^\nu+\gamma_{ab}^{[\mu_1}\eta^{\nu_1][\mu_2}\eta^{\nu_2]\lambda}\mathbb{T}_2^\nu T_{12}^\nu t_1^\nu+(1\leftrightarrow 2)\right].
\end{aligned}\tag{3.38}$$

The functions t_i^ν, T_{jk}^ν and $\mathbb{T}_l^\nu$ are defined in (3.26) and (3.33), respectively, and the notation $+(1\leftrightarrow 2)$ instructs to add the image of the previous two terms under $(z_1,\mu_1,\nu_1)\leftrightarrow(z_2,\mu_2,\nu_2)$.

Five-point example The same strategy gives rise to five permutation-inequivalent functions of the punctures in the correlator with three currents $\psi^{\mu_i}\psi^{\nu_i}$,

$$\begin{aligned}
&\left\langle\!\left\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)\psi^{\mu_3}\psi^{\nu_3}(z_3)S_a(z_A)S_b^\lambda(z_B)\right\rangle\!\right\rangle_\nu = \frac{E_{AB}\theta_\nu\left(\frac{1}{2}z_{AB}\right)^4}{4\sqrt{2}\theta_1'(0)^4\prod_{j=1}^3E_{jA}E_{jB}}\\
&\times\Big[\Big\{\gamma_{ab}^{\mu_1\nu_1\mu_2\nu_2[\mu_3}\eta^{\nu_3]\lambda}\mathbb{T}_3^\nu\left(t_1^\nu t_2^\nu\right)^2t_3^\nu+\left(\eta^{\nu_1[\mu_2}\eta^{\nu_2]\mu_1}-\eta^{\mu_1[\mu_2}\eta^{\nu_2]\nu_1}\right)\gamma_{ab}^{[\mu_3}\eta^{\nu_3]\lambda}\mathbb{T}_3^\nu\left(T_{12}^\nu\right)^2t_3^\nu\\
&\quad+\left(\eta^{\mu_2[\nu_1}\gamma_{ab}^{\mu_1]\nu_2[\mu_3}\eta^{\nu_3]\lambda}-\eta^{\nu_2[\nu_1}\gamma_{ab}^{\mu_1]\mu_2[\mu_3}\eta^{\nu_3]\lambda}\right)\mathbb{T}_3^\nu T_{12}^\nu\,t_1^\nu t_2^\nu t_3^\nu+\text{cyc}(1,2,3)\Big\}\\
&\quad+\Big\{\left(\gamma_{ab}^{\mu_1\mu_2\nu_2}\eta^{\nu_1[\mu_3}\eta^{\nu_3]\lambda}-\gamma_{ab}^{\nu_1\mu_2\nu_2}\eta^{\mu_1[\mu_3}\eta^{\nu_3]\lambda}\right)\mathbb{T}_3^\nu T_{13}^\nu\,t_1^\nu\left(t_2^\nu\right)^2\\
&\qquad+\gamma_{ab}^{[\nu_2}\eta^{\mu_2][\mu_1}\eta^{\nu_1][\mu_3}\eta^{\nu_3]\lambda}\mathbb{T}_3^\nu T_{12}^\nu T_{13}^\nu\,t_2^\nu+\text{perm}(1,2,3)\Big\}\Big]\,,
\end{aligned}\tag{3.39}$$

which can be determined by suitable choices of the lattice vectors λ,μ_i,ν_i,a,b along the lines of the previous examples. Note that the explicit correlators in (3.32), (3.38) and (3.39) are sufficient to capture the contributions of the excited spin field to open-string integrands (3.10) with two fermions and $n \le 3$ bosons.

The n-point function Similar to the discussion in section 3.3.4, the above examples of correlators involving $S_a(z_A)S_b^\lambda(z_B)$ strongly suggest their generalization to n insertions of $\psi^{\mu_j}\psi^{\nu_j}$: After stripping off the overall prefactor of

$$\left\langle\!\!\left\langle \prod_{j=1}^n\psi^{\mu_j}\psi^{\nu_j}(z_j)S_a(z_A)S_b^\lambda(z_B)\right\rangle\!\!\right\rangle_\nu = \frac{\sqrt{2}E_{AB}^{n-2}\theta_\nu(\frac{1}{2}z_{AB})^4}{2^n\theta_1'(0)^4\prod_{j=1}^nE_{jA}E_{jB}}\sum_i(\mathcal{L}_{(i)})_{ab}^{[\mu_i\nu_i],\lambda}\Phi_\nu^{(i)}(z)\,,\tag{3.40}$$

the remaining contributions are Lorentz tensors $\mathscr{L}_{(i)}$ with the index structure of the left-hand side and spin structure dependent functions $\Phi^{(i)}_\nu(z)$ of $z_1,z_2,\ldots,z_n,z_A,z_B$ and τ that obey the following rules: First, the tensors $\mathscr{L}_{(i)}$ are antisymmetric in all pairs $\mu_i \leftrightarrow \nu_i$ and cannot involve the vector index of the excited spin field S^λ_b on a gamma-matrix to account for its irreducibility constraint. Second, $\mathscr{L}_{(i)}$ is a sum of products of a single odd-rank form $\gamma^{\rho_1\rho_2\ldots\rho_{2k+1}}_{ab}$ accompanied by $n-k$ factors of $\eta^{\cdot\cdot}$, and each summand has a prefactor ±1 given the choice of normalization in (3.40).

Most importantly, each Lorentz tensor $\mathscr{L}_{(i)}$ in (3.40) can be translated into its accompanying function $\Phi^{(i)}_\nu(z)$ through the following dictionary (where $\supset$ is understood as "contains a factor of"),

$$\mathscr{L}_{(i)} \supset \gamma^{\ldots\mu_j\ldots}_{ab} \text{ or } \gamma^{\ldots\nu_j\ldots}_{ab} \quad\Rightarrow\quad \Phi^{(i)}_\nu(z) \supset t^\nu_j \tag{3.41a}$$

$$\mathscr{L}_{(i)} \supset \eta^{\mu_j\mu_k} \text{ or } \eta^{\nu_j\nu_k} \text{ or } \eta^{\mu_j\nu_k} \quad\Rightarrow\quad \Phi^{(i)}_\nu(z) \supset T^\nu_{jk} \tag{3.41b}$$

$$\mathscr{L}_{(i)} \supset \eta^{\mu_j\lambda} \text{ or } \eta^{\nu_j\lambda} \quad\Rightarrow\quad \Phi^{(i)}_\nu(z) \supset \mathbb{T}^\nu_j, \tag{3.41c}$$

see (3.26) and (3.33) for the building blocks $t^\nu_j, \mathbb{T}^\nu_j$ and T^ν_{jk}. While the first two rules (3.41a) and (3.41b) tie in with those for two unexcited spin fields, see (3.30a) and (3.30b), the additional vector index of the excited spin field is addressed by (3.41c).

The summation range $\sum_i$ in (3.40) involves all Lorentz tensors $\mathscr{L}_{(i)}$ that can be obtained from partitions of the antisymmetrized pairs of indices $[\mu_1\nu_1],[\mu_2\nu_2],\ldots,[\mu_n\nu_n]$ into a form $(\gamma^{\cdots})_{ab}$, products of $\eta^{\cdot\cdot}$ and an additional $\eta^{\cdot\lambda}$ associated with the excited spin field.

Similar to the rules of section 3.3.4 to determine the signs in the correlator with unexcited spin fields, the idea is to start with a reference $(2n+1)$-form $\gamma^{\mu_1\nu_1\mu_2\nu_2\ldots\mu_n\nu_n\lambda}$. Pairs of indices which enter the given product of $\eta^{\cdot\cdot}$ must be moved to neighboring positions $\gamma^{\cdots\mu_i\mu_j\cdots}$, $\gamma^{\cdots\nu_i\nu_j\cdots}$, $\gamma^{\cdots\mu_i\nu_j\cdots}$ with $i<j$ (and not $i>j$) or $\gamma^{\cdots\mu_i\lambda\cdots}$, $\gamma^{\cdots\nu_i\lambda\cdots}$ (with λ on the right of μ_i,ν_i) before transferring them to the metric tensors. For instance, the positive sign of $\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2}\gamma^{\nu_3}\eta^{\mu_3\lambda}$ in (3.39) can be seen by rearranging $\gamma^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\lambda} = (-1)^2\gamma^{\mu_1\mu_2\nu_1\nu_2\nu_3\mu_3\lambda}$ before transferring the pairs $\mu_1\mu_2$, $\nu_1\nu_2$, $\mu_3\lambda$ to the $\eta^{\cdot\cdot}$ and converting $\gamma^{\mu_1\mu_2\nu_1\nu_2\nu_3\mu_3\lambda} \to \gamma^{\nu_3}$.

3.4 Doubly-periodic functions and spin sums

In addition to the doubly-periodicity, spin structure dependent parts of correlation functions in (3.11) are meromorphic functions for each insertion point, so by summing over spin structures they become elliptic functions for those insertion points [108]. Traditionally, an elliptic function has been represented by the Weierstrass $\wp$-function and a linear combination of Weierstrass ζ-functions. However, for superstring amplitudes, it is somewhat useful to introduce a set of doubly-periodic functions $f^{(n)}(z)$, $n \in \mathbb{N}_0$ generated from a non-holomorphic extension of the Kronecker-Eisenstein series [36, 38]

$$F(z,\alpha) = \frac{\theta_1'(0)\theta_1(z+\alpha)}{\theta_1(z)\theta_1(\alpha)}, \qquad \Omega(z,\alpha) = e^{2\pi i\alpha\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}}F(z,\alpha) = \sum_{n=0}^{\infty}\alpha^{n-1}f^{(n)}(z), \tag{3.42}$$

which can be used to generate homotopy invariant iterated integrals over an elliptic curve [38] and therefore enter the definition of elliptic multiple zeta values [121]. The latter have been identified as a convenient language for the α'-expansion of one-loop open-string amplitudes [33], including double-trace contributions [122]. The simplest expansion coefficients read

$$f^{(0)}(z) = 1, \qquad f^{(1)}(z) \equiv \partial\log\theta_1(z) + 2\pi i\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}$$

$$f^{(2)}(z) = \frac{1}{2}\left\{\left(\partial\log\theta_1(z) + 2\pi i\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\right)^2 + \partial^2\log\theta_1(z) - \frac{\theta_1'''(0)}{3\theta_1'(0)}\right\}.$$

3.4.1 Spin sums on bosonic one-loop amplitudes

In order to exemplify the relevance of the doubly-periodic $f^{(n)}$ in (3.42) for spin sums, let us review their instances in the N-gluon amplitudes. From the N vertex operators (3.7), we are led to products of the Szegö kernels

$$P_\nu(z) \equiv \frac{\theta_1'(0)\theta_\nu(z)}{\theta_1(z)\theta_\nu(0)}$$

which ultimately appear in the combinations

$$\mathcal{G}(\vec{z}_n) = \mathcal{G}(z_1, z_2, \ldots, z_n) \equiv \sum_{\nu=2}^{4} (-1)^{\nu+1} \left(\frac{\theta_\nu(0)}{\theta_1'(0)}\right)^4 P_\nu(z_1)P_\nu(z_2)\ldots P_\nu(z_n), \qquad \sum_{j=1}^{n} z_j = 0 \tag{3.43}$$

with $n \leq N$. It is not difficult to show that $\mathcal{G}(\vec{z}_n)$ is elliptic for each z_i, $i = 1, 2, \ldots, n$, therefore it can be represented by a suitable set of elliptic functions on z_i.

All-multiplicity techniques for the simplification of (3.43) by using elliptic functions have been given in [32], also see [95] for an alternative method. In order to represent (3.43) in elliptic functions, we notice that Szegö kernels in (3.43) can be replaced by

$$\prod_{i=1}^{n} P_\nu(z_i) = \Omega(\vec{z}_n, \omega_{\nu-1}) = \prod_{i=1}^{n} \Omega(z_i, \omega_{\nu-1}), \qquad \omega_\nu = \left(\frac{1}{2}, \frac{-1-\tau}{2}, \frac{\tau}{2}\right)$$

due to the relations in (3.14) and $\sum_{i=1}^{n} z_i = 0$. From the definition of Ω in (3.42) one can easily show that $\Omega(\vec{z}_n, y)$ is an elliptic function on y having a pole structure at $y = 0$ as

$$\sum_{k=0}^{n} y^{-n+k} V_k(\vec{z}) = \frac{1}{y^n} + \frac{V_1(\vec{z})}{y^{n-1}} + \frac{V_2(\vec{z})}{y^{n-2}} + \ldots + \frac{V_{n-2}(\vec{z})}{y^2} + V_n(\vec{z}) \tag{3.44}$$

with $V_k(\vec{z}) = \frac{1}{k!}\frac{\partial^k}{\partial y^k}\left\{\prod_{i=1}^{n} y\Omega(z_i, y)\right\}_{y=0}$, which are polynomials of $f^{(n)}(z_i)$, so $\Omega(\vec{z}_n, y)$ can be represented by using Weierstrass $\wp$-functions [123, 124]:

$$\Omega(\vec{z}_n, y) = \sum_{k=0}^{n-2} \frac{(-1)^{n-k}}{(n-k-1)!} \left(\wp^{(n-k-2)}(y) - \hat{G}_{n-k-2}\right) V_k(\vec{z}_n) + V_n(\vec{z}_n).$$

for

$$\hat{G}_{k\geq 2} = \begin{cases} 0, & \text{for } k = 2 \\ G_k(\tau), & \text{otherwise} \end{cases}, \qquad G_k = \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k}, \qquad k \geq 3,$$

which allows us to the representation of $\Omega(\vec{z}_n, \omega_{\nu-1})$ in terms of V_k, Eisenstein series G_k as well as Weierstass invariants $e_i = \wp(\omega_i), i = 1, 2, 3$. By combining this result with (see for instance [125])

$$(-1)^{\nu+1}\left(\frac{\theta_\nu(0)}{\theta_1'(0)}\right)^4 = \begin{cases} \dfrac{1}{(e_1-e_2)(e_1-e_3)} & \text{for } \nu = 2 \\ \dfrac{1}{(e_2-e_1)(e_2-e_3)} & \text{for } \nu = 3 \\ \dfrac{1}{(e_3-e_1)(e_3-e_2)} & \text{for } \nu = 4 \end{cases}$$

the correlation function in (3.43) can be expressed in terms of V_k and symmetric polynomials of (e_1, e_2, e_3) related to Eisenstein series G_k by

$$e_1 + e_2 + e_3 = 0, \qquad e_1e_2 + e_2e_3 + e_1e_3 = -15G_4, \qquad e_1e_2e_3 = 35G_6.$$

For instance the $n \le 9$-point results of [32] translate into [33]

$$\mathcal{G}(\vec{z}_n) = 0, \qquad 1 \le n \le 3, \qquad \mathcal{G}(\vec{z}_n) = V_{n-4}(\vec{z}_n), \qquad 4 \le n \le 7$$
$$\mathcal{G}(\vec{z}_8) = V_4(\vec{z}_8) + 3\mathrm{G}_4, \qquad \mathcal{G}(\vec{z}_9) = V_5(\vec{z}_9) + 3\mathrm{G}_4 V_1(\vec{z}_9),$$

where further simplifications arise in the degeneration limit $\tau \to i\infty$ [19].

Hence, the worldsheet integrand for the N-gluon amplitude comprising spin sums (3.43) and correlators of X^μ can be entirely expressed in terms of $f^{(n)}$ functions in (3.42). This motivates to express the two-fermion amplitudes in (3.10) which are related to external bosons by supersymmetry in the same language, also see [35] for the six-point one-loop amplitude in pure spinor superspace involving $f^{(2)}_{ij}$ & $f^{(1)}_{ij} f^{(1)}_{pq}$.

Note that the same techniques can be used for spin sums in bosonic one-loop N-point amplitudes in orbifold compactifications with reduced supersymmetry [126] (see [127, 128] for earlier work on the four-point function).

3.4.2 A standard form for spin sums

In view of the ultimate goal of this work to sum the above correlators (3.29) and (3.40) over the spin structures $\nu = 1,2,3,4$, we identify a prototype spin sum from the dictionaries (3.30) and (3.41). First, the prefactors of (3.29) and (3.40) along with the ν-dependent minus sign in the amplitude prescription (3.10) suggest to introduce the shorthands

$$Z_\nu\left(\frac{1}{2}y\right) \equiv \frac{(-1)^{\nu+1}\theta_\nu\left(\frac{1}{2}y\right)^4}{\theta_1'(0)^4}, \qquad y \equiv z_{AB} \tag{3.45}$$

where $Z_\nu\left(\frac{1}{2}y\right)$ may be interpreted as a partition function of X^μ and ψ^μ with twisted boundary conditions. All the ν-dependence in the building blocks $t^\nu_j, \mathbb{T}^\nu_j$ and T^ν_{jk} for $\varphi^{(i)}_\nu(z)$ and $\Phi^{(i)}_\nu(z)$ in (3.30) and (3.41) occurs via products of ratios $\frac{\theta_\nu(x\pm\frac{1}{2}y)}{\theta_\nu(\frac{1}{2}y)}$, with x representing some z_{ij} with $i,j \in \{1,2,\ldots,n,A,B\}$. It is particularly convenient to gather such ratios of θ_ν functions via

$$F_\nu(z,w) \equiv \frac{\theta_1'(0)\theta_\nu(z+w)}{\theta_1(z)\theta_\nu(w)} = \frac{\theta_\nu(z+w)}{E(z)\theta_\nu(w)}\,, \tag{3.46}$$

which generalizes the Kronecker–Eisenstein series in (3.42) to even spin structures with $F_{\nu=1}(z,w) = F(z,w)$ and exhibits the following symmetry property,

$$F_\nu(-z,-w) = -F_\nu(z,w)\,.$$

More precisely, the building blocks of the above spin-field correlators can be expressed in terms of the function $F_\nu(x,y)$ by means of

$$T^\nu_{jk} = \frac{E_{jA}E_{kB}F_\nu\left(z_{jk},\frac{1}{2}y\right) - E_{jB}E_{kA}F_\nu\left(z_{kj},\frac{1}{2}y\right)}{E_{AB}}$$
$$t^\nu_j = E_{Bi}F_\nu\left(z_{Bi},\frac{1}{2}y\right)\,, \qquad \mathbb{T}^\nu_j = E_{jA}F_\nu\left(z_{jB},\frac{1}{2}y\right)\,.$$

Then, the most general spin sum we will be concerned with in the next section can be brought into the standard form

$$\mathcal{W}\left(x_1,x_2,\ldots,x_N\,\middle|\,\frac{1}{2}y\right) \equiv \sum_{\nu=1}^{4} Z_\nu(y)F_\nu\left(x_1,x_2,\ldots,x_N,\frac{1}{2}y\right) \tag{3.47}$$

where $F_\nu(x_1,x_2\ldots,x_N)$ denotes $\prod_{i=1}^N F_\nu\left(x_i,\frac{1}{2}y\right)$ and $x_i, i = 1,2,\ldots,N$ are linearly dependent over $(-1,+1)$ i.e., $x_1 \pm x_2 \pm \ldots \pm x_N = 0$ for some choices of signs. The spin sum in (3.47) generalizes the prototype spin sum (3.43) for bosonic one-loop amplitudes.

Examples with unexcited spin fields Let us give the simplest examples of spin-field correlators rewritten in terms of the standard spin sum (3.47) with building blocks (3.45) and (3.46): In presence of unexcited spin fields, the correlators (3.24), (3.25) and (3.27) translate into

$$\sum_{\nu=1}^{4}(-1)^{\nu+1}\left\langle\!\left\langle S_a(z_A)S^b(z_B)\right\rangle\!\right\rangle_\nu = \sum_{\nu=1}^{4}\frac{\delta_a^b Z_\nu\left(\frac{1}{2}y\right)}{E_{AB}} = \frac{\delta_a^b}{E_{AB}}\mathcal{W}\left(-\left|\frac{1}{2}y\right.\right) \tag{3.48}$$

$$\begin{aligned}\sum_{\nu=1}^{4}(-1)^{\nu+1}\left\langle\!\left\langle \psi^\lambda\psi^\rho(z_1)S_a(z_A)S^b(z_B)\right\rangle\!\right\rangle_\nu &= \sum_{\nu=1}^{4}\frac{\gamma^{\lambda\rho}{}_a{}^b E_{1B}}{2E_{1A}} Z_\nu\left(\frac{1}{2}y\right)F_\nu\left(z_{B1},z_{B1},\frac{1}{2}y\right)\\ &= \frac{\gamma^{\lambda\rho}{}_a{}^b E_{1B}}{2E_{1A}}\mathcal{W}\left(z_{B1},z_{B1}\left|\frac{1}{2}y\right.\right)\end{aligned} \tag{3.49}$$

as well as

$$\begin{aligned}&\sum_{\nu=1}^{4}(-1)^{\nu+1}\left\langle\!\left\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi_{\mu_2}\psi_{\nu_2}(z_2)S_a(z_A)S^b(z_B)\right\rangle\!\right\rangle_\nu\\ &= (\gamma^{\mu_1\nu_1}{}_{\mu_2\nu_2})_a{}^b\frac{E_{AB}E_{1B}E_{2B}}{4E_{1A}E_{2A}}\mathcal{W}\left(z_{B1},z_{B1},z_{B2},z_{B2}\left|\frac{1}{2}y\right.\right)\\ &\quad+\delta^{[\nu_1}_{[\mu_2}\gamma^{\mu_1]}{}_{\nu_2]a}{}^b\left\{\frac{E_{2B}}{4E_{2A}}\mathcal{W}\left(z_{12},z_{B1},z_{B2}\left|\frac{1}{2}y\right.\right)-\frac{E_{1B}}{4E_{1A}}\mathcal{W}\left(z_{21},z_{B1},z_{B2}\left|\frac{1}{2}y\right.\right)\right\}\\ &\quad-\delta^{\nu_1}_{[\mu_2}\delta^{\mu_1}_{\nu_2]}\delta^b_a\left\{\frac{1}{2E_{AB}}\mathcal{W}\left(z_{12},z_{21}\left|\frac{1}{2}y\right.\right)\right.\\ &\qquad\left.-\frac{E_{1A}E_{2B}}{4E_{AB}E_{1B}E_{2A}}\mathcal{W}\left(z_{12},z_{12}\left|\frac{1}{2}y\right.\right)-\frac{E_{1B}E_{2A}}{4E_{AB}E_{1A}E_{2B}}\mathcal{W}\left(z_{21},z_{21}\left|\frac{1}{2}y\right.\right)\right\}\end{aligned} \tag{3.50}$$

The generalization to three insertions of $\psi_{\mu_j}\psi_{\nu_j}(z_j)$ can be found in appendix D.1.

Examples with an excited spin field In presence of excited spin fields, the expressions (3.32) and (3.38) for the simplest correlators give rise to

$$\begin{aligned}\sum_{\nu=1}^{4}(-1)^{\nu+1}\left\langle\!\left\langle \psi^\lambda\psi^\rho(z_1)S_a(z_A)S^\mu_b(z_B)\right\rangle\!\right\rangle_\nu &= -\frac{\eta^{\mu[\rho}\gamma^{\lambda]}_{ab}}{\sqrt{2}E_{AB}}\sum_{\nu=1}^{4}Z_\nu\left(\frac{1}{2}y\right)F_\nu\left(z_{1B},z_{B1},\frac{1}{2}y\right)\\ &= -\frac{\eta^{\mu[\rho}\gamma^{\lambda]}_{ab}}{\sqrt{2}E_{AB}}\mathcal{W}\left(z_{1B},z_{B1}\left|\frac{1}{2}y\right.\right)\end{aligned} \tag{3.51}$$

as well as

$$\begin{aligned}&\sum_{\nu=1}^{4}(-1)^{\nu+1}\left\langle\!\left\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)S_a(z_A)S^\lambda_b(z_B)\right\rangle\!\right\rangle_\nu\\ &= -\gamma^{[\mu_1}_{ab}\eta^{\nu_1][\mu_2}\eta^{\nu_2]\lambda}\left\{\frac{1}{2\sqrt{2}E_{AB}}\mathcal{W}\left(z_{12},z_{2B},z_{B1}\left|\frac{1}{2}y\right.\right)-\frac{E_{1B}E_{2A}}{2\sqrt{2}E_{AB}E_{1A}E_{2B}}\mathcal{W}\left(z_{21},z_{2B},z_{B1}\left|\frac{1}{2}y\right.\right)\right\}\\ &\quad-\gamma^{\mu_1\nu_1[\mu_2}_{ab}\eta^{\nu_2]\lambda}\frac{E_{1B}}{2\sqrt{2}E_{1A}}\mathcal{W}\left(z_{2B},z_{B1},z_{B1},z_{B2},\left|\frac{1}{2}y\right.\right)+(z_1,\mu_1,\nu_1)\leftrightarrow(z_2,\mu_2,\nu_2)\,.\end{aligned} \tag{3.52}$$

The generalization to three insertions of $\psi^{\mu_j}\psi^{\nu_j}(z_j)$ can be found in appendix D.2.

From the discussion in the next section, one can find that most of the spin sums in (3.48) to (3.52) vanish, except for the case with $\mathcal{W}\left(z_{12},z_{2B},z_{B1}\left|\frac{1}{2}y\right.\right)$. The latter leads to the non-vanishing four-point amplitude among two bosons and two fermions which has been first computed in [102].

3.5 Evaluating spin sums in two-fermion amplitudes

In this section, we present a method to evaluate the prototype spin sum (3.47) for two-fermion amplitudes in terms of the doubly-periodic functions $f^{(n)}$ in (3.42).

3.5.1 Quasi-periodicity of $\mathscr{W}$

As we have seen in (3.47), the arguments $x_i, i = 1,2,\ldots,N$ of the prototype spin sum

$$\mathscr{W}\left(x_1,x_2,\ldots,x_N\left|\frac{y}{2}\right.\right)$$

are linearly dependent over $\{\pm 1\}$. Thus, one can always rearrange x_i such that

$$\sum_{i=1}^{n} x_i = \sum_{i=n+1}^{N} x_i \ . \tag{3.53}$$

for some $n \le N$. Then, by denoting $x_{n+i} = x'_i$, the quasi-periodicity of $\mathscr{W}$ can be easily obtained from the definition of $\mathscr{W}$ as

$$\mathscr{W}\left(\vec{x}_n,\vec{x}'_{n'}\left|\frac{1}{2}(y+1)\right.\right) = -\mathscr{W}\left(\vec{x}_n,\vec{x}'_{n'}\left|\frac{y}{2}\right.\right) \tag{3.54a}$$

$$\mathscr{W}\left(\vec{x}_n,\vec{x}'_{n'}\left|\frac{1}{2}(y+\tau)\right.\right) = -q^{-1/2}e^{-2\pi i(y+\sum_{i=1}^{n'}x'_i)}\mathscr{W}\left(\vec{x}_n,\vec{x}'_{n'}\left|\frac{y}{2}\right.\right) \tag{3.54b}$$

$$\mathscr{W}\left(x_1,x_2,\ldots,x_i+1,\ldots,x_n,\vec{x}'_n\left|\frac{y}{2}\right.\right) = \mathscr{W}\left(\vec{x}_n,\vec{x}'_{n'}\left|\frac{y}{2}\right.\right) \tag{3.54c}$$

$$\mathscr{W}\left(x_1,x_2,\ldots,x_i+\tau,\ldots,x_n,\vec{x}'_n\left|\frac{y}{2}\right.\right) = \mathscr{W}\left(\vec{x}_n,\vec{x}'_{n'}\left|\frac{y}{2}\right.\right) \tag{3.54d}$$

$$\mathscr{W}\left(\vec{x}_n,x'_1,x'_2,\ldots,x'_i+1,\ldots,x'_{n'}\left|\frac{y}{2}\right.\right) = \mathscr{W}\left(\vec{x}_n,\vec{x}'_{n'}\left|\frac{y}{2}\right.\right) \tag{3.54e}$$

$$\mathscr{W}\left(\vec{x}_n,x'_1,x'_2,\ldots,x'_i+\tau,\ldots,x'_{n'}\left|\frac{y}{2}\right.\right) = e^{-2\pi i y}\mathscr{W}\left(\vec{x}_n,\vec{x}'_{n'}\left|\frac{y}{2}\right.\right) \tag{3.54f}$$

where we introduce $\vec{x}_n = (x_1,x_2,\ldots,x_n)$ and $\vec{x}'_{n'} = (x'_1,x'_2,\ldots,x'_{n'})$, $n' = N-n$ for notational simplicity.

3.5.2 Elliptic representation of $\mathscr{W}$

$\mathbf{n}' = \mathbf{0}$ We begin with the simplest case of spin sums having $n' = 0$:

$$\frac{\mathscr{W}\left(\vec{x}_n\left|\frac{y}{2}\right.\right)}{E(y)} = \sum_{\nu=1}^{4}\frac{Z_\nu\left(\frac{y}{2}\right)}{E(y)}F_\nu\left(\vec{x}_n,\frac{y}{2}\right). \tag{3.55}$$

In order to represent (3.55) in elliptic functions, we consider

$$\frac{Z_1(\alpha)}{E(2\alpha)}F_1(\vec{x}_n,\alpha)$$

which is related to each summand in (3.55) by $\alpha = \frac{y}{2}+s_\nu,\ s_\nu = (0,\omega_{\nu-1})$ due to (3.14). One can easily see that $\frac{Z_1(\alpha)}{E(2\alpha)}$ is an elliptic function on α having simple poles at $\alpha = \omega_{\nu-1},\ \nu = 2,3,4$, so it can be expressed as

$$\frac{Z_1(\alpha)}{E(2\alpha)} = \frac{1}{2}\sum_{\nu=2}^{4}Z_\nu(0)\left(\zeta(\omega_{\nu-1}-\alpha)-\eta_{\nu-1}\right)$$

where ζ is the Weierstrass ζ-function and $\eta_\nu = \zeta(\omega_\nu)$ for $\nu = 1,2,3$. Also, we note that $F_1(\vec{x}_n,\alpha)$ can be replaced by $\Omega(\vec{x}_n,\alpha)$ which can be expanded as in (3.44). Therefore, by examining the pole structure we have

$$\frac{Z_1(\alpha)}{E(2\alpha)}F_1(\vec{x}_n,\alpha) = \frac{1}{2}\sum_{\nu=2}^{4} Z_\nu(0)\big(\zeta(\omega_{\nu-1}-\alpha)-\eta_{\nu-1}\big)F_\nu(\vec{x}_n,0)+\zeta(\alpha)\sum_{l=0}^{n-1} g_{n-l-1}V_l(\vec{x}_n)$$
$$+\sum_{l=0}^{n} g_{n-l}V_l(\vec{x}_n)+\sum_{k=0}^{n-2}\sum_{l=0}^{n-k-2}\left(\frac{(-1)^k}{(k+1)!}\wp^{(k)}(\alpha)-\hat{G}_{k+2}\right)g_{n-k-l-2}V_l(\vec{x}_n) \tag{3.56}$$

where

$$g_0 = 0, \qquad g_{k\geq 1} = \sum_{\nu=2}^{4}\frac{1}{2k!}Z_\nu(0)\wp^{(k-1)}(\omega_{\nu-1}), \qquad \hat{G}_{k\geq 2} = \begin{cases} 0, & \text{for } k=2 \\ G_k & \text{otherwise,}\end{cases}$$

and $g_{k\geq 1}$ can be computed as in section 3.4.1. For some lower k we have

$$g_{k\leq 2} = 0, \qquad g_3 = \frac{1}{2}, \qquad g_{4\leq k\leq 6} = 0, \qquad g_7 = \frac{3G_4}{2}, \qquad g_8 = 0, \qquad g_9 = 5G_6.$$

Now, by inserting $\alpha = \frac{y}{2}+s_\nu$ to (3.56) and using (see appendix E for a proof)

$$2\zeta(2z) = \sum_{\nu=1}^{4}\zeta(z+s_\nu), \qquad 2^{k+2}\wp^{(k)}(2z) = \sum_{\nu=1}^{4}\wp^{(k)}(z+s_\nu)$$

we obtain an elliptic representation of (3.55)

$$\sum_{\nu=1}^{4}\frac{Z_\nu\left(\frac{1}{2}y\right)}{E(y)}F_\nu(\vec{x}_n,\alpha)$$
$$= 4\sum_{l=0}^{n} g_{n-l}V_l(\vec{x}_n)+\sum_{k=0}^{n-2}\sum_{l=0}^{n-k-2}\left(\frac{(-1)^k 2^{k+2}}{(k+1)!}\wp^{(k)}(y)-4G_{k+2}\right)g_{n-k-l-2}V_l(\vec{x}_n). \tag{3.57}$$

$\mathbf{n'\neq 0}$ For $n'\neq 0$, the quasi-periodicity of $\mathcal{W}$ implies that the following expression

$$\frac{\mathcal{W}\left(\vec{x}_n,\vec{x}'_n\,\middle|\,\frac{1}{2}y\right)}{E(y)F(\vec{x}'_n,y)} \tag{3.58}$$

is an elliptic function on x_i, x'_j and y. Moreover, each x_i (resp. x'_i) has a simple pole at $x_i = 0$ (resp. $x'_i + y = 0$) whose residue is in the form of

$$\frac{\mathcal{W}\left(\vec{x}_{n-1},\vec{x}'_n\,\middle|\,\frac{1}{2}y\right)}{E(y)F(\vec{x}'_n,y)}$$

which is recursively related to the target spin sum in (3.58). Thus, one can expect that V_k again provide a convenient basis for representing (3.58) since V_k are recursively related to V_{k-1} by simple poles of V_k.

We can deduce an ansatz for (3.58) by considering the case $n=0$:

$$\frac{\mathcal{W}\left(\vec{x}'_{n'}\,\middle|\,\frac{1}{2}y\right)}{E(y)F(\vec{x}'_{n'},y)} = \frac{\sum_{k=1}^{n'-3} w_{n'-k}(y)V_k(\vec{x}'_{n'})+w_{n'}(y)}{F(\vec{x}'_{n'},y)} \tag{3.59}$$

where we have rewritten $\mathcal{W}\left(\vec{x}_{n'}\,\middle|\,\frac{1}{2}y\right)$ in (3.57) as

$$\mathcal{W}\left(\vec{x}_{n'}\,\middle|\,\frac{1}{2}y\right) = \sum_{k=1}^{n'-3} w_{n'-k}(y)V_k(\vec{x}_{n'})+w_{n'}(y),$$

by collecting elliptic functions $w_k(y)$ on y represented by $\wp^{(k-2)}$ and G_k for V_l. (3.59) indicates that $F(x_i, y)$ and $F(x'_i, y)$ can appear in addition to V_k, so we claim that for $n > 0$,

$$\frac{\mathcal{W}\left(\vec{x}_n, \vec{x}'_{n'} \,\middle|\, \frac{1}{2}y\right)}{E(y)F(\vec{x}'_{n'}, y)} = \frac{F(\vec{x}_n, y)\sum_{p=1}^{n'-3} c_{n,n'-p}(y)V_p(\vec{x}'_{n'}, -\vec{x}_n - \vec{y}_n, \vec{y}_n) + c_{n,n'}(y)}{F(\vec{x}'_{n'}, y)}$$
$$+ \sum_{p=1}^{n-3} d_{n-p,n'}(y)V_p(\vec{x}_n, -\vec{x}'_{n'} - \vec{y}_n, \vec{y}_n) + d_{n,n'}(y) \tag{3.60}$$

where $\vec{y}_n = (\overbrace{y, y, \ldots, y}^{n})$. $c_{n,n'}(y)$ and $d_{n,n'}(y)$ are elliptic functions on y satisfying

$$c_{n,n'}(y) = c_{n-1,n'}(y) - \sum_{p=1}^{n'-3} c_{n,n'-p}(y)V_p(y, -y) \tag{3.61a}$$
$$d_{n,n'}(y) = d_{n,n'-1}(y) - \sum_{p=1}^{n-3} c_{n-p,n'}(y)V_p(y, -y)\ . \tag{3.61b}$$

where

$$V_1(y, -y) = 0, \qquad V_2(y, -y) = -\wp(y), \qquad V_{n>3}(y, -y) = (-1)^n(n-1)G_n$$

by Proposition E.3.

The claim can be proven by induction. It is obvious that at either $n = 0$ or $n' = 0$, the claim is true by imposing

$$c_{n,0}(y) = 0,\ d_{n,0} = w_n(y),\ c_{0,n'}(y) = w_{n'}(y),\ d_{0,n'}(y) = 0\ .$$

At $n = n' = 1$, by using the Riemann identity

$$\sum_{\nu=1}^{4}(-1)^{\nu+1}\theta_\nu(z_1)\theta_\nu(z_2)\theta_\nu(z_3)\theta_\nu(z_4) = \theta_1(z'_1)\theta_1(z'_2)\theta_1(z'_3)\theta_1(z'_4)$$
$$z'_1 \equiv \frac{1}{2}(z_1 + z_2 + z_3 + z_4), \quad z'_2 \equiv \frac{1}{2}(z_1 + z_2 - z_3 - z_4)\ ,$$
$$z'_3 \equiv \frac{1}{2}(z_1 - z_2 - z_3 + z_4), \quad z'_4 \equiv \frac{1}{2}(z_1 - z_2 + z_3 - z_4)\ ,$$

one can easily show that

$$\frac{\mathcal{W}\left(x, x \,\middle|\, \frac{1}{2}y\right)}{F(x, y)E(y)} = 0$$

which provides the starting point for induction.

Now, suppose that for $n < m$ and $n' < m'$ the claim is true. For $n = m$, the left hand side of (3.60) has the residue at $x_m = 0$ as

$$\frac{\mathcal{W}\left(\vec{x}_{m-1}, \vec{x}'_{m'} \,\middle|\, \frac{1}{2}y\right)}{E(y)F(\vec{x}'_{m'}, y)} = \frac{F(\vec{x}_{m-1}, y)\sum_{p=1}^{m'-3} c_{m-1,m'-p}(y)V_p(\vec{x}'_{m'}, -\vec{x}_{m-1} - \vec{y}_{m-1}, \vec{y}_{m-1}) + c_{m-1,m'}(y)}{F(\vec{x}'_{m'}, y)}$$
$$+ \sum_{p=1}^{m-4} d_{m-1,m'-p}(y)V_p(\vec{x}_{m-1}, -\vec{x}'_{m'} - \vec{y}_{m'}, \vec{y}_{m'}) + d_{m-1,m'}(y)\ . \tag{3.62}$$

The right hand side of (3.60) has the residue at $x_m = 0$ as

$$\frac{F(\vec{x}_{m-1},y)\sum_{p=1}^{m'-3}\sum_{q=0}^{p} c_{m,m'-p}(y)V_q(y,-y)V_{p-q}(\vec{x}'_{m'},-\vec{x}_{m-1}-\vec{y}_{m-1},\vec{y}_{m-1})+c_{m,m'}(y)}{F(\vec{x}'_{m'},y)}$$
$$+\sum_{p=1}^{m-3} d_{m,m'-p}(y)V_{p-1}(\vec{x}_{m-1},-\vec{x}'_{m'}-\vec{y}_{m'},\vec{y}_{m'})$$
$$=\frac{F(\vec{x}_{m-1},y)\sum_{r=0}^{m'-3}\sum_{p=0}^{m'-r-3} c_{m,m'-r-p}(y)V_p(y,-y)V_r(\vec{x}'_{m'},-\vec{x}_{m-1}-\vec{y}_{m-1},\vec{y}_{m-1})}{F(\vec{x}'_{m'},y)}$$
$$+\sum_{p=0}^{m-4} d_{m,m'-1-p}(y)V_p(\vec{x}_{m-1},-\vec{x}'_{m'}-\vec{y}_{m'},\vec{y}_{m'})\ . \tag{3.63}$$

Then, by (3.61)

$$\sum_{p=0}^{m'-3-r} c_{m,m',p+r}(y)V_p(y,-y) = c_{m-1,m',r}\ ,$$

so (3.62) and (3.63) are same.

In a similar manner, one can show that both sides of (3.60) have a common residue at $x'_{m'}+y = 0$. Consequently, the difference between both sides is an elliptic function on y. The difference can be determined by inserting $x'_{n'} = 0$ into both sides and one can easily show that it vanishes again due to (3.61).

By multiplying $F(x'_{n'},y)$ to both sides of (3.60), the elliptic representation in (3.60) can be expressed more symmetrically in $\vec{x}_n$ and $\vec{x}'_{n'}$:

$$\begin{aligned}\frac{\mathcal{W}\left(\vec{x}_n,\vec{x}'_{n'}\,\middle|\,\tfrac{1}{2}y\right)}{E(y)} =& F(\vec{x}_n,y)\sum_{p=1}^{n'-3} c_{n,n'-p}(y)V_p(\vec{x}'_{n'},-\vec{x}_n-\vec{y}_n,\vec{y}_n)+c_{n,n'}(y)\\ &+F(\vec{x}'_{n'},y)\sum_{p=1}^{n-3} d_{n-p,n'}(y)V_p(\vec{x}_n,-\vec{x}'_{n'}-\vec{y}_n,\vec{y}_n)+d_{n,n'}(y).\end{aligned}$$

3.5.3 Cleaning up the prime forms

In the elliptic representation of a spin sum in (3.60) V_k carry arguments $-x_i-y$ and $-x'_j-y$. As we have seen in section 3.4.2, these arguments are expressed as $-z_{kl}-z_{AB}$ and it is often desired to split those arguments into z_{kl}, z_{AB}, z_{lA} and z_{kB} which are more suitable to integrate. For this, consider

$$F(z_{ij},z_{AB})V_n(-z_{ij}-z_{AB},x_1,\ldots,x_n) = \frac{1}{n!}\frac{\partial^n}{\partial\alpha^n}\left(F(z_{ij},z_{AB})\alpha F(-z_{ij}-z_{AB},\alpha)\prod_{i=1}^{n}F(x_i,\alpha)\right).$$

where $\sum_{i=1}^n x_i = z_{ij}+z_{AB}$. By combining

$$F(z_{ij},z_{AB})F(-z_{ij}-z_{AB},\alpha) = -\frac{E(-z_{ij}-z_{AB}+\alpha)}{E(z_{ij})E(z_{AB})E(\alpha)}.$$

with the Fay's identity in (E.4)

$$E(-z_{ij}-z_{AB}+\alpha)E(\alpha)\frac{E(z_{jA})E(z_{Bi})}{E(z_{ji})E(z_{jA})E(z_{Bi})E(z_{BA})} = \det\begin{pmatrix}\frac{E(z_{ji}+\alpha)}{E(z_{ji})} & \frac{E(z_{jA}+\alpha)}{E(z_{jA})}\\ \frac{E(z_{Bi}+\alpha)}{E(z_{Bi})} & \frac{E(z_{BA}+\alpha)}{E(z_{BA})}\end{pmatrix}$$

we have

$$\frac{E(-z_{ij}-z_{AB}+\alpha)}{E(\alpha)}\frac{E(z_{jA})E(z_{Bi})}{E(z_{ji})E(z_{jA})E(z_{Bi})E(z_{BA})}=\det\begin{pmatrix} F(z_{ji},\alpha) & F(z_{jA},\alpha) \\ F(z_{Bi},\alpha) & F(z_{BA},\alpha)\end{pmatrix}$$

which implies

$$F(z_{ij},z_{AB})V_n(-z_{ij}-z_{AB},\dots)=\frac{E(z_{jA})E(z_{Bi})}{E(z_{jA})E(z_{Bi})}\frac{1}{n!}\frac{\partial^n}{\partial\alpha^n}\left(\alpha\det\begin{pmatrix} F(z_{ji},\alpha) & F(z_{jA},\alpha) \\ F(z_{Bi},\alpha) & F(z_{BA},\alpha)\end{pmatrix}\dots\right).$$

3.5.4 Worked out examples

While the procedure of the previous section can be applied to evaluate spin sums of arbitrary multiplicity, we shall now present its simplest applications covering spin sums in three- to five-point amplitudes. For notational simplicity, we use the notation $V_k(i_1,\dots,i_n)=V_k\left(z_{i_1i_2},\dots,z_{i_ni_1}\right)$ in the following.

$\mathbf{n'=0}$

$$\begin{aligned}
\frac{\mathcal{W}\left(z_{12},z_{21}\,\middle|\,\tfrac{1}{2}y\right)}{E(y)}&=0,\\
\frac{\mathcal{W}\left(z_{12},z_{23},z_{31}\,\middle|\,\tfrac{1}{2}y\right)}{E(y)}&=0,\\
\frac{\mathcal{W}\left(z_{12},z_{23},z_{34},z_{41}\,\middle|\,\tfrac{1}{2}y\right)}{E(y)}&=2V_1(1,2,3,4),\\
\frac{\mathcal{W}\left(z_{12},z_{23},z_{34},z_{45},z_{51}\,\middle|\,\tfrac{1}{2}y\right)}{E(y)}&=2V_2(1,2,3,4,5)-2V_2(A,B).
\end{aligned}$$

$\mathbf{n'=1}$

$$\begin{aligned}
\frac{E_{2B}E_{1A}}{E_{2A}E_{1B}}\frac{\mathcal{W}\left(z_{12},z_{12}\,\middle|\,\tfrac{1}{2}y\right)}{E(y)}&=0,\\
\frac{E_{3B}E_{1A}}{E_{3A}E_{1B}}\frac{\mathcal{W}\left(z_{12},z_{23},z_{13}\,\middle|\,\tfrac{1}{2}y\right)}{E(y)}&=0,\\
\frac{E_{4B}E_{1A}}{E_{4A}E_{1B}}\frac{\mathcal{W}\left(z_{12},z_{23},z_{34},z_{14}\,\middle|\,\tfrac{1}{2}y\right)}{E(y)}&=2V_1(1,B,A,4),\\
\frac{E_{5B}E_{1A}}{E_{5A}E_{1B}}\frac{\mathcal{W}\left(z_{12},z_{23},z_{34},z_{45},z_{15}\,\middle|\,\tfrac{1}{2}y\right)}{E(y)}&=2V_2(1,2,3,4,5)+2V_2(A,B)-2V_2(1,2,3,4,5,A,B).
\end{aligned}$$

$\mathbf{n'=2}$

$$\frac{E_{2B}E_{1A}E_{4B}E_{3A}}{E_{2A}E_{1B}E_{4A}E_{3B}}\frac{\mathscr{W}\left(z_{12},z_{34},z_{12},z_{34}\,\big|\,\frac{1}{2}y\right)}{E(y)}=0,$$

$$\frac{E_{3B}E_{1A}E_{5B}E_{4A}}{E_{3A}E_{1B}E_{5A}E_{4B}}\frac{\mathscr{W}\left(z_{12},z_{23},z_{45},z_{13},z_{45}\,\big|\,\frac{1}{2}y\right)}{E(y)}=2V_1(1,B,A,3)V_1(4,B,A,5),$$

$$\begin{aligned}&\frac{E_{4B}E_{1A}E_{6B}E_{5A}}{E_{4A}E_{1B}E_{6A}E_{5B}}\frac{\mathscr{W}\left(z_{12},z_{23},z_{34},z_{56},z_{14},z_{56}\,\big|\,\frac{1}{2}y\right)}{E(y)}\\&=2V_1(1,2,3,4)(V_2(5,6)+2V_2(A,B)-V_2(5,6,A,B)-V_2(A,B))\\&\quad-2V_2(1,2,3,4)V_1(5,6,A,B)\\&\quad-2V_1(1,2,3,4,A,B)(V_2(5,6)+V_2(A,B)-V_2(5,6,A,B))\\&\quad+2V_2(1,2,3,4,A,B)V_1(5,6,A,B)\end{aligned}$$

3.5.5 Examples of spin-summed correlators

In this section, we assemble the expressions for various spin-summed correlators in fermionic one-loop amplitudes.

Two unexcited spin fields The vanishing of the spin sums in (3.48) to (3.50) immediately propagates to

$$\sum_{\nu=1}^{4}(-1)^{\nu+1}\left\langle\left\langle S_a(z_A)S^b(z_B)\right\rangle\right\rangle_\nu=0$$

$$\sum_{\nu=1}^{4}(-1)^{\nu+1}\left\langle\left\langle \psi^{\mu_1\nu_1}(z_1)S_a(z_A)S^b(z_B)\right\rangle\right\rangle_\nu=0$$

$$\sum_{\nu=1}^{4}(-1)^{\nu+1}\left\langle\left\langle \psi^{\mu_1\nu_1}(z_1)\psi^{\mu_2\nu_2}(z_2)S_a(z_A)S^b(z_B)\right\rangle\right\rangle_\nu=0.$$

The first non-vanishing spin sums occur in the five-point correlator (D.1) such that

$$\begin{aligned}&\sum_{\nu=1}^{4}(-1)^{\nu+1}\left\langle\left\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)\psi^{\mu_3}\psi^{\nu_3}(z_3)S_a(z_A)S^b(z_B)\right\rangle\right\rangle_\nu\\&=(\gamma^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3})_a{}^b\,h^{(0)}_{\varnothing}\;+\;\eta^{\nu_1\mu_2}(\gamma^{\mu_1\nu_2\mu_3\nu_3})_a{}^b\,h^{(0)}_{[12]}\\&\quad+\eta^{\mu_1\nu_2}\eta^{\nu_1\mu_2}(\gamma^{\mu_3\nu_3})_a{}^b\,h^{(0)}_{(12)}+\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}(\gamma^{\mu_1\nu_3})_a{}^b\,h^{(0)}_{12,23}\\&\quad+\;\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\mu_1\nu_3}\delta^b_a\,h^{(0)}_{[123]}+\text{permutations}\end{aligned}$$

with

$$h^{(0)}_{\varnothing}=h^{(0)}_{12,23}=-h^{(0)}_{(12)}=\frac{1}{2}\,,\qquad h^{(0)}_{[12]}=h^{(0)}_{[123]}=0\,,$$

or equivalently

$$\begin{aligned}&\sum_{\nu=1}^{4}(-1)^{\nu+1}\left\langle\left\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)\psi^{\mu_3}\psi^{\nu_3}(z_3)S_a(z_A)S^b(z_B)\right\rangle\right\rangle_\nu\\&=\frac{1}{2}\Big\{(\gamma^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3})_a{}^b-\left[(\eta^{\mu_1\nu_2}\eta^{\nu_1\mu_2}-\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2})(\gamma^{\mu_3\nu_3})_a{}^b+\text{cyc}(1,2,3)\right]\\&\quad+\left[\eta^{\mu_2[\nu_1}(\gamma^{\mu_1][\nu_3})_a{}^b\eta^{\mu_3]\nu_2}-\eta^{\nu_2[\nu_1}(\gamma^{\mu_1][\nu_3})_a{}^b\eta^{\mu_3]\mu_2}+\text{cyc}(1,2,3)\right]\Big\}\,.\end{aligned}\tag{3.67}$$

The spin sums in the corresponding six-point correlator evaluate to

$$
\begin{aligned}
&4\sum_{\nu=1}^{4}(-1)^{\nu+1}\Big\langle\Big\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)\psi^{\mu_3}\psi^{\nu_3}(z_3)\psi^{\mu_4}\psi^{\nu_4}(z_4)S_a(z_A)S^b(z_B)\Big\rangle\Big\rangle_\nu \\
&=(\gamma^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4\nu_4})_a{}^b\, h^{(1)}_{\emptyset}+\eta^{\nu_1\mu_2}(\gamma^{\mu_1\nu_2\mu_3\nu_3\mu_4\nu_4})_a{}^b\, h^{(1)}_{[12]} \\
&\quad+\eta^{\mu_1\nu_2}\eta^{\nu_1\mu_2}(\gamma^{\mu_3\nu_3\mu_4\nu_4})_a{}^b\, h^{(1)}_{(12)}+\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}(\gamma^{\mu_1\nu_3\mu_4\nu_4})_a{}^b\, h^{(1)}_{12,23} \\
&\quad+\eta^{\nu_1\mu_2}\eta^{\nu_3\mu_4}(\gamma^{\mu_1\nu_2\mu_3\nu_4})_a{}^b\, h^{(1)}_{[12],[34]}+\eta^{\mu_1\nu_2}\eta^{\nu_1\mu_2}\eta^{\nu_3\mu_4}(\gamma^{\mu_3\nu_4})_a{}^b\, h^{(1)}_{(12),[34]} \\
&\quad+\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\mu_1\nu_3}(\gamma^{\mu_4\nu_4})_a{}^b\, h^{(1)}_{[123]}+\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\nu_3\mu_4}(\gamma^{\mu_1\nu_4})_a{}^b\, h^{(1)}_{12,23,34} \\
&\quad+\eta^{\mu_1\nu_2}\eta^{\nu_1\mu_2}\eta^{\nu_3\mu_4}\eta^{\mu_3\nu_4}\delta_a^b\, h^{(1)}_{(12),(34)}+\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\nu_3\mu_4}\eta^{\mu_1\nu_4}\delta_a^b\, h^{(1)}_{(1234)} \\
&\quad+\text{permutations}
\end{aligned}
\tag{3.68}
$$

with doubly-periodic functions $h^{(1)}_{\dots}\equiv h^{(1)}_{\dots}(z_j,z_A,z_B)$ given by

$$
\begin{aligned}
h^{(1)}_{\emptyset}(z_j,z_A,z_B) &= \sum_{i=1}^{4}V_1(i,A,B)\\
h^{(1)}_{[12]}(z_j,z_A,z_B) &= V_1(1,2,A,B)-V_1(2,1,A,B)\\
h^{(1)}_{(12)}(z_j,z_A,z_B) &= \sum_{i=1}^{2}V_1(i,A,B)-\sum_{i=3}^{4}V_1(i,A,B)\\
h^{(1)}_{12,23}(z_j,z_A,z_B) &= -V_1(2,A,4,B)\\
h^{(1)}_{[12],[34]}(z_j,z_A,z_B) &= 0\\
h^{(1)}_{(12),[34]}(z_j,z_A,z_B) &= -V_1(3,4,A,B)-V_1(3,4,B,A)\\
h^{(1)}_{[123]}(z_j,z_A,z_B) &= -2V_1(1,2,3)\\
h^{(1)}_{12,23,34}(z_j,z_A,z_B) &= \sum_{i=1}^{3}\Big[V_1(i,i{+}1,A,B)+V_1(i,i{+}1,B,A)\Big]\\
h^{(1)}_{(12),(34)}(z_j,z_A,z_B) &= -\sum_{i=1}^{4}V_1(i,A,B)\\
h^{(1)}_{(1234)}(z_j,z_A,z_B) &= \frac{1}{2}\big[V_1(1,2,A,B)-V_1(1,2,B,A)+\text{cyc}(1,2,3,4)\big]\,.
\end{aligned}
$$

The analogous seven-point correlator can be found in appendix F.1, and the sum over permutations in (3.68) can be reconstructed from section 3.3.4.

One excited spin field Again, the vanishing of the relevant spin sums leads to

$$\sum_{\nu=1}^{4}(-1)^{\nu+1}\Big\langle\Big\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)S_a(z_A)S^\lambda_b(z_B)\Big\rangle\Big\rangle_\nu=0\,,$$

resulting in a vanishing three-point amplitude. The first non-vanishing spin-summed correlator with an excited spin field requires two insertions of $\psi^{\mu_i}\psi^{\nu_i}(z_i)$,

$$
\begin{aligned}
&\frac{1}{\sqrt{2}}\sum_{\nu=1}^{4}(-1)^{\nu+1}\Big\langle\Big\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)S_a(z_A)S^\lambda_b(z_B)\Big\rangle\Big\rangle_\nu\\
&=(\gamma^{\mu_1\nu_1\mu_2})_{ab}\eta^{\nu_2\lambda}H^{(0)}_{\emptyset}+\eta^{\nu_1\mu_2}(\gamma^{\mu_1})_{ab}\eta^{\nu_2\lambda}H^{(0)}_{12}+\text{permutations,}
\end{aligned}
\tag{3.70}
$$

with

$$H^{(0)}_{\emptyset} = H^{(0)}_{12} = -\frac{1}{2} ,$$

or equivalently

$$\begin{aligned}
&\sum_{\nu=1}^{4} (-1)^{\nu+1} \Big\langle\Big\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)S_a(z_A)S_b^{\lambda}(z_B)\Big\rangle\Big\rangle_\nu \\
&= \frac{1}{\sqrt{2}}\big[(\gamma^{\mu_1\nu_1[\mu_2})_{ab}\eta^{\nu_2]\lambda} + (\gamma^{[\mu_1})_{ab}\eta^{\nu_1][\mu_2}\eta^{\nu_2]\lambda} + (1\leftrightarrow 2)\big] \\
&= \frac{1}{\sqrt{2}}\big[\eta^{\lambda[\nu_2}(\gamma^{\mu_2]}\gamma^{\mu_1\nu_1})_{ab} + (1\leftrightarrow 2)\big] .
\end{aligned}$$

The corresponding five-point correlator

$$\begin{aligned}
&2\sqrt{2}\sum_{\nu=1}^{4} (-1)^{\nu+1} \Big\langle\Big\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)\psi^{\mu_3}\psi^{\nu_3}(z_3)S_a(z_A)S_b^{\lambda}(z_B)\Big\rangle\Big\rangle_\nu \\
&= (\gamma^{\mu_1\nu_1\mu_2\nu_2\mu_3})_{ab}\eta^{\lambda\nu_3}\ H^{(1)}_{\emptyset} + \eta^{\nu_1\mu_2}(\gamma^{\mu_1\nu_2\mu_3})_{ab}\eta^{\lambda\nu_3}\ H^{(1)}_{[12]} \\
&\quad + \eta^{\nu_1\mu_2}\eta^{\mu_1\nu_2}(\gamma^{\mu_3})_{ab}\eta^{\lambda\nu_3}\ H^{(1)}_{(12)} + (\gamma^{\mu_1\mu_2\nu_2})_{ab}\eta^{\nu_1\mu_3}\eta^{\lambda\nu_3}\ H^{(1)}_{13} \\
&\quad + \eta^{\mu_1\mu_2}(\gamma^{\nu_2})_{ab}\eta^{\nu_1\mu_3}\eta^{\lambda\nu_3}\ H^{(1)}_{12,13} + \text{permutations}
\end{aligned} \tag{3.71}$$

involves the following doubly-periodic functions $H^{(1)}_{\dots} \equiv H^{(1)}_{\dots}(z_j, z_A, z_B)$:

$$\begin{aligned}
H^{(1)}_{\emptyset}(z_j,z_A,z_B) &= -V_1(1,A,B) - V_1(2,A,B) \\
H^{(1)}_{[12]}(z_j,z_A,z_B) &= -V_1(1,2,A,B) - V_1(1,2,B,A) \\
H^{(1)}_{(12)}(z_j,z_A,z_B) &= -V_1(1,2,A,B) + V_1(1,2,B,A) \\
H^{(1)}_{13}(z_j,z_A,z_B) &= -V_1(1,3,A,2,B) - V_1(1,3,B,2,A) - V_1(2,A,3,B) \\
H^{(1)}_{12,13}(z_j,z_A,z_B) &= -2V_1(1,2,A,B,3) - V_1(1,A,2,B) .
\end{aligned}$$

The analogous six-point correlator is presented in appendix F.2, and the sum over permutations in (3.70) and (3.71) can be reconstructed from (3.38) and (3.39).

Note that (3.67) and (3.71) yield an expression for the worldsheet integrand of the five-point amplitude (3.10) in terms of the $f^{(1)}_{ij}$ functions with $i,j \in \{1,2,3,A,B\}$. It would be interesting to relate its factorization properties to the general considerations of [53, 105] on the distributions of superghost picture numbers at the boundary of (super-)moduli space.

FOUR

CONCLUSION AND OUTLOOK

In this thesis, we have presented various tools for computing scattering amplitudes of superstring theory at tree level and one-loop. Here, we summarize the main results of this work and anticipate potential outlooks.

4.1 Tree level superstring amplitudes

In addition to the review on the formal structure of string amplitudes in chapter 1, in chapter 2 we have provided a method to construct supersymmetric Berends–Giele currents of 10D SYM by perturbatively solving equations of motion of 10D SYM. This method allows us to find kinematic factors of tree level open superstring amplitudes without relying on the CFT techniques of the PS superstring theory.

Also, we have considered two different gauge choices on Berends–Giele currents, called the BCJ gauge and the HS gauge. We have shown that through the BCJ gauge, one can arrive at a set of Berends–Giele currents whose kinematic numerators manifestly satisfying the BCJ color-kinematics duality. Consequently, we have established a systematic method to generate BCJ satisfying kinematic numerators, which are essential to study the double-copy structure of tree level scattering amplitudes. In section 2.4, we have introduced the HS gauge which simplifies the θ-expansion of a given superfield.

One of obvious future research directions relevant to chapter 2 is the application of supersymmetric Berends–Giele currents to loop amplitudes. In the same way as the building block (2.49) is specific to tree amplitudes, any loop order singles out specific scalar combinations of multiparticle superfields which are BRST invariant at the linearized level, e.g.

$$\begin{aligned} M_A(\lambda\gamma_m \mathscr{W}_B)(\lambda\gamma_n \mathscr{W}_C)\mathscr{F}_D^{mn} &\leftrightarrow \text{1-loop [88, 34, 77]} \\ (\lambda\gamma_{mnpqr}\lambda)(\lambda\gamma_s \mathscr{W}_A)\mathscr{F}_B^{mn}\mathscr{F}_C^{pq}\mathscr{F}_D^{rs} &\leftrightarrow \text{2-loop [89, 28]} \\ (\lambda\gamma_m \mathscr{W}_A^n)(\lambda\gamma_n \mathscr{W}_B^p)(\lambda\gamma_p \mathscr{W}_C^m) &\leftrightarrow \text{3-loop [78].} \end{aligned} \tag{4.1}$$

They describe the low-energy limit in string theory and are motivated by the zero-mode saturation rules of the pure spinor formalism [15, 88]. Moreover, they are believed to represent box, double-box and Mercedes-star diagrams in SYM amplitudes to arbitrary multiplicity, see [27, 28]. Again, HS gauge as well as the theta-expansions in (2.46), (2.47) and Appendix B greatly simplify their component evaluation via (2.3).

In contrast to tree level, loop amplitudes in SYM and superstring theory additionally involve tensorial building blocks contracting the loop momenta where HS gauge yields comparable benefits in the component evaluation. One-loop kinematic factors generalizing (4.1) to arbitrary tensor rank have been constructed in [83], and some of them have been defined in terms of the superfields $H_{12\ldots p}$ from the transformation to BCJ gauge. As will be described elsewhere, kinematic factors with explicit reference to gauge parameters will require extra care when adapted to different non-linear gauges. At any rate, HS gauge for Berends-Giele currents sets new scales for the computational effort in component evaluations.

4.2 One-loop superstring amplitudes

In chapter 3, we have studied the correlation functions of two fermionic and any number of bosonic vertex operators on the torus, with particular emphasis on the cancellations between different spin structures reflecting spacetime supersymmetry. These correlators form the worldsheet integrands for the respective massless one-loop amplitudes of the open RNS superstring, and their double copy yields closed-string amplitudes involving up to two Ramond–Ramond forms, gravitinos or dilatinos.

Among other things, the resulting fermionic RNS amplitudes are useful to test the equivalence with the pure spinor formalism in more advanced situations. For example, the explicit correlators in section 3.5.5 and appendix F.2 are suitable for comparison with the five- [34] and six-point [35] results in pure spinor superspace.

Moreover, the $\tau \to i\infty$ limit of the present results extends the RNS ambitwistor-string setup [23, 24] to CHY formulae for one-loop SYM amplitudes with external fermions and the corresponding supergravity amplitudes. In particular, the tensor structure of our correlators at $\tau \to i\infty$ can be converted to explicit and local BCJ numerators using the techniques of [19]. Finally, we hope that our results are useful to study the forward-limit relations between ambitwistor-string correlators at different loop orders and the application of the gluing operators in [129].

While a detailed investigation of the resulting string and field-theory amplitudes is relegated to the future, the major novelties of Chapter 3 are

(i) the one-loop correlation functions involving one excited spin field from the fermion vertex in the $+\frac{1}{2}$ picture and any number of Lorentz currents

(ii) an algorithmic method to systematically perform and simplify the sum over spin structures for the one-loop integrand of two-fermion amplitudes

The n-point correlator (i) can be found in section 3.3.5, and the mathematical techniques for the spin sums (ii) are presented in section 3.5, see in particular section 3.5.5 and appendix F for explicit $n \leq 6$-point expressions.

A mild generalization of the techniques which led to the main results (i) and (ii) can be applied to one-loop correlators involving any number of fermion pairs. And we expect that several of the mathematical tools developed in this work are helpful for higher-genus amplitudes, for instance to extend the two-loop spin sums of [130, 124] for bosonic external states to fermionic amplitudes.

On the one hand, the pure spinor formalism bypasses the spin sums, gathers all component amplitudes into supersymmetric expressions and held the key to the first explicit three-loop calculation [92]. On the other hand, the form of the RNS spin sums at genus one given in [32, 33] pinpointed the ubiquity of doubly-periodic functions $f^{(n)}(z,\tau)$ (see section 3.4) in multiparticle correlators which is crucial to construct the latter from an ansatz in both RNS- and pure spinor variables. Hence, we expect that explicit control over RNS spin sums provides valuable inspiration for the design of multiparticle correlators at genus one [27, 28, 29] and higher genus and appropriate generalizations of the $f^{(n)}(z,\tau)$ functions.

Another kind of follow-up question concerns the extension of the present results to string compactifications with reduced supersymmetry, see e.g. [131] for a review. Higher-genus correlators involving two spin fields and an arbitrary number of NS fermions were found to be robust under dimensional reduction [110], and the same is expected for excited spin fields, see [132] for tree level evidence. It remains to incorporate the fingerprints of the compactification geometry on the fermionic vertex operators where universal statements for a given number of supersymmetries can be made from [96, 133, 134, 135].

For bosonic one-loop amplitudes, the spin sums in half-maximally and quarter-maximally supersymmetric setups could be identified as specializations of maximally supersymmetric spin sums with two additional legs [126]. Upon extrapolation to external fermions, the spin-summed five- and six-point correlators in the maximally supersymmetric setup of this work should admit a similar map to spin summed three- and four-point correlators with reduced supersymmetry.

The resulting expressions for fermionic one-loop RNS amplitudes with reduced supersymmetry will provide helpful cross-checks and guidance to supersymmetrize their bosonic counterparts [127, 128, 126]: They are important in comparing RNS results with one-loop amplitudes in the hybrid formalism with four or eight supercharges manifest [136, 137, 138, 139]. While one-loop hybrid amplitudes with maximally supersymmetric multiplets in the loop have been computed in [140], it remains to derive their generalizations to spectra with reduced supersymmetry.

Appendix A

BCJ GAUGE VERSUS LORENZ GAUGE AT RANK FIVE

In this appendix, we verify that the supersymmetric Berends–Giele currents at rank five in BCJ gauge and Lorenz gauge are related by a non-linear gauge transformation as in (2.26). Straightforward but tedious calculations lead to the following translation between local superfields in BCJ and Lorenz gauge,

$$\begin{aligned}
A^m_{[1234,5]} =& \hat{A}^m_{[1234,5]} - k^m_{12345}\hat{H}_{[1234,5]} \\
&- (k^1\cdot k^2)(\hat{H}_{[134,5]}A^m_2 + \hat{H}_{[14,5]}A^m_{23} + \hat{H}_{[13,5]}A^m_{24} + \hat{H}_{[13,4]}A^m_{25} - (1\leftrightarrow 2)) \\
&- (k^{12}\cdot k^3)(\hat{H}_{[124,5]}A^m_3 + \hat{H}_{[12,5]}A^m_{34} + \hat{H}_{[12,4]}A^m_{35} - \hat{H}_{[34,5]}A^m_{12}) \\
&- (k^{123}\cdot k^4)(\hat{H}_{[123,5]}A^m_4 + \hat{H}_{[12,3]}A^m_{45}) - (k^{1234}\cdot k^5)(\hat{H}_{[123,4]}A^m_5) \\
A^m_{[123,45]} =& \hat{A}^m_{[123,45]} - k^m_{12345}\hat{H}_{[123,45]} - (k^1\cdot k^2)(\hat{H}_{[13,45]}A^m_2 + \hat{H}_{[45,2]}A^m_{13} - (1\leftrightarrow 2)) \\
&- (k^{12}\cdot k^3)(\hat{H}_{[12,45]}A^m_3 + \hat{H}_{[45,3]}A^m_{12}) - (k^{123}\cdot k^{45})(\hat{H}_{[12,3]}A^m_{45}) \\
&- (k^4\cdot k^5)(\hat{H}_{[123,4]}A^m_5 - \hat{H}_{[123,5]}A^m_4) \\
A^m_{[[12,34],5]} =& \hat{A}^m_{[[12,34],5]} - k^m_{12345}\hat{H}_{[[12,34],5]} \\
&- (k^1\cdot k^2)(\hat{H}_{[34,2]}A^m_{15} - \hat{H}_{[34,1]}A^m_{25} + \hat{H}_{[342,5]}A^m_1 - \hat{H}_{[341,5]}A^m_2) \\
&- (k^3\cdot k^4)(\hat{H}_{[12,3]}A^m_{45} - \hat{H}_{[12,4]}A^m_{35} + \hat{H}_{[123,5]}A^m_4 - \hat{H}_{[124,5]}A^m_3) \\
&- (k^{12}\cdot k^{34})(\hat{H}_{[12,5]}A^m_{34} - \hat{H}_{[34,5]}A^m_{12}) - (k^{1234}\cdot k^5)(\hat{H}_{[12,34]}A^m_5),
\end{aligned}$$

where the second and third equations can be regarded as the definitions of $\hat{H}_{[123,45]}$ and $\hat{H}_{[[12,34],5]}$. The solution of the former is given in (2.23) and (2.21) while the latter is

$$\hat{H}_{[[12,34],5]} = H_{[1234,5]} - H_{[1243,5]} - \frac{1}{2}H_{[12,34]}(k_{1234}\cdot A_5)\,.$$

Plugging the above equations into the generic definition of the rank-five Berends–Giele current as displayed in fig. A.1, namely,

$$\begin{aligned}
s_{12345}\mathscr{K}_{12345} =& \frac{K_{[1,4532]}}{s_{2345}s_{345}s_{45}} - \frac{K_{[1,3452]}}{s_{2345}s_{345}s_{34}} - \frac{K_{[1,3425]}}{s_{2345}s_{234}s_{34}} + \frac{K_{[1,2345]}}{s_{2345}s_{234}s_{23}} - \frac{K_{[12,453]}}{s_{345}s_{12}s_{45}} \\
&+ \frac{K_{[12,345]}}{s_{345}s_{12}s_{34}} + \frac{K_{[45,231]}}{s_{123}s_{23}s_{45}} - \frac{K_{[45,123]}}{s_{123}s_{12}s_{45}} + \frac{K_{[3421,5]}}{s_{1234}s_{234}s_{34}} - \frac{K_{[2341,5]}}{s_{1234}s_{234}s_{23}} \\
&- \frac{K_{[2314,5]}}{s_{1234}s_{123}s_{23}} + \frac{K_{[1234,5]}}{s_{1234}s_{123}s_{12}} + \frac{K_{[1,[23,45]]}}{s_{2345}s_{23}s_{45}} - \frac{K_{[5,[12,34]]}}{s_{1234}s_{12}s_{34}},
\end{aligned}$$

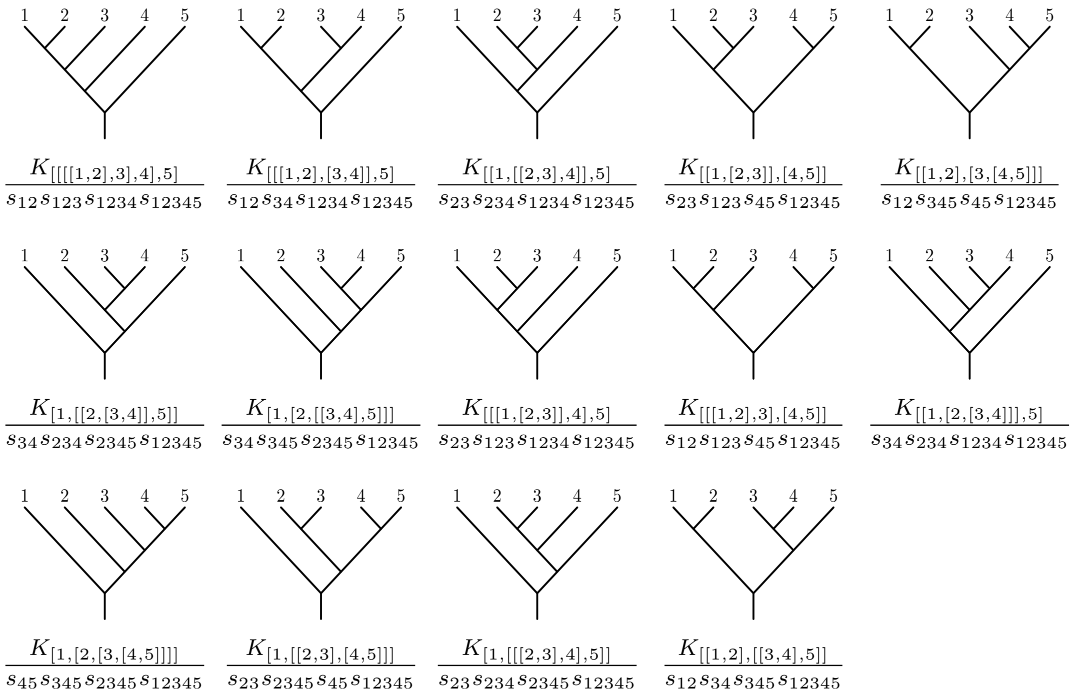


Figure A.1: The fourteen binary trees used in the definition of $\mathscr{K}_{12345}$.

leads to

$$\mathscr{A}^{m,\mathrm{BCJ}}_{12345} = \mathscr{A}^{m,\mathrm{L}}_{12345} - k^m_{12345}\mathscr{H}_{12345} + \mathscr{A}^m_1\mathscr{H}_{2345} + \mathscr{A}^m_{12}\mathscr{H}_{345} - \mathscr{A}^m_5\mathscr{H}_{1234} - \mathscr{A}^m_{45}\mathscr{H}_{123}.$$

By the vanishing of $\mathscr{H}_i$ and $\mathscr{H}_{ij}$, this reproduces the non-linear gauge transformation (2.26) at multiplicity five.

Appendix B

THETA-EXPANSIONS IN HARNAD–SHNIDER GAUGE

B.1 Theta-expansions of $\mathcal{A}^P_\alpha, \mathcal{A}^m_P, \mathcal{W}^\alpha_P, \mathcal{F}^{mn}_P$

The component prescription (2.3) in pure spinor superspace requires the theta-expansion of the enclosed superfields up to the order θ^5. The expansions up to θ^5 of the Berends–Giele currents $\mathcal{A}^P_\alpha, \mathcal{A}^m_P, \mathcal{W}^\alpha_P, \mathcal{F}^{mn}_P$ in HS gauge can be found in (2.46) up to deconcatenation terms. These are now spelt out:

$$
\begin{aligned}
[\mathcal{A}^m_{X,Y}]_5 &= \frac{1}{320}(\theta\gamma^{mnr}\theta)(\theta\gamma_{rpq}\theta)(\mathcal{X}_X\gamma_n\theta)\mathfrak{f}^{pq}_Y - (X \leftrightarrow Y)\\
[\mathcal{W}^\alpha_{X,Y}]_4 &= -\frac{1}{64}(\theta\gamma_m{}^q)^\alpha(\theta\gamma_{qnp}\theta)(\mathcal{X}_X\gamma^m\theta)\mathfrak{f}^{np}_Y - (X \leftrightarrow Y)\\
[\mathcal{W}^\alpha_{X,Y}]_5 &= \frac{1}{120}(\theta\gamma_m{}^q)^\alpha(\theta\gamma_{npq}\theta)(\mathcal{X}_X\gamma^m\theta)(\mathcal{X}^n_Y\gamma^p\theta) + \frac{1}{240}(\theta\gamma_n{}^q)^\alpha(\theta\gamma_{mpq}\theta)(\mathcal{X}_X\gamma^m\theta)(\mathcal{X}^n_Y\gamma^p\theta)\\
&\quad - \frac{1}{1280}(\theta\gamma^{rs})^\alpha(\theta\gamma_{mnr}\theta)(\theta\gamma_{pqs}\theta)\mathfrak{f}^{mn}_X\mathfrak{f}^{pq}_Y - (X \leftrightarrow Y)\\
[\mathcal{F}^{mn}_{X,Y}]_3 &= \frac{1}{8}(\theta\gamma_{pq}{}^{[m}\theta)(\mathcal{X}_X\gamma^{n]}\theta)\mathfrak{f}^{pq}_Y - (X \leftrightarrow Y)\\
[\mathcal{F}^{mn}_{X,Y}]_4 &= -\frac{1}{12}(\theta\gamma_{pq}{}^{[m}\theta)(\mathcal{X}_X\gamma^{n]}\theta)(\mathcal{X}^p_Y\gamma^q\theta) - \frac{1}{24}(\theta\gamma^{pq[m}\theta)(\mathcal{X}_X\gamma_p\theta)(\mathcal{X}^{n]}_Y\gamma_q\theta)\\
&\quad - \frac{1}{128}(\theta\gamma^{[m}{}_{pq}\theta)(\theta\gamma^{n]}{}_{rs}\theta)\mathfrak{f}^{pq}_X\mathfrak{f}^{rs}_Y - (X \leftrightarrow Y)\\
[\mathcal{F}^{mn}_{X,Y}]_5 &= -\frac{1}{192}(\theta\gamma^{[m}{}_{ps}\theta)(\mathcal{X}_X\gamma^{n]}\theta)\mathfrak{f}^{p|qr}_Y(\theta\gamma^s{}_{qr}\theta) - \frac{1}{320}(\mathcal{X}_X\gamma^p\theta)(\theta\gamma_{ps}{}^{[m}\theta)\mathfrak{f}^{n]|qr}_Y(\theta\gamma^s{}_{qr}\theta)\\
&\quad - \frac{1}{320}(\theta\gamma_{ps}{}^{[m}\theta)(\mathcal{X}^{n]}_X\gamma^p\theta)\mathfrak{f}^{qr}_Y(\theta\gamma^s{}_{qr}\theta) + \frac{1}{96}(\theta\gamma^{[m}{}_{pq}\theta)(\theta\gamma^{n]}{}_{rs}\theta)(\mathcal{X}^p_X\gamma^q\theta)\mathfrak{f}^{rs}_Y - (X \leftrightarrow Y)\\
[\mathcal{F}^{mn}_{X,Y,Z}]_5 &= -\frac{1}{24}(\theta\gamma_{pq}{}^{[m}\theta)(\mathcal{X}_X\gamma^{n]}\theta)(\mathcal{X}_Y\gamma^p\theta)(\mathcal{X}_Z\gamma^q\theta) + (X \leftrightarrow Z).
\end{aligned}
$$

B.2 Theta-expansions of the simplest higher-mass dimension superfields

For the simplest superfields of higher mass dimension, the theta-expansion in HS gauge that starts as in (2.47) and has the following second and third order:

$$
\begin{aligned}
[\mathcal{W}^{m\alpha}_P]_2 &= -\frac{1}{4}(\theta\gamma_{np})^\alpha(\mathcal{X}^{mn}_P\gamma^p\theta) + \sum_{XY=P}\Big[\frac{1}{4}(\theta\gamma_{np})^\alpha(\mathcal{X}_X\gamma^m\theta)\mathfrak{f}^{np}_Y\\
&\quad - \frac{1}{8}(\theta\gamma^m{}_{np}\theta)\mathcal{X}^\alpha_X\mathfrak{f}^{np}_Y - (X \leftrightarrow Y)\Big]\\
[\mathcal{W}^{m\alpha}_P]_3 &= -\frac{1}{48}(\theta\gamma_n{}^r)^\alpha(\theta\gamma_{rpq}\theta)\mathfrak{f}^{mn|pq}_P + \sum_{XY=P}\Big[-\frac{1}{4}(\theta\gamma_{np})^\alpha(\mathcal{X}_X\gamma^m\theta)(\mathcal{X}^n_Y\gamma^p\theta)\\
&\quad - \frac{1}{6}(\theta\gamma_{np})^\alpha(\mathcal{X}_X\gamma^n\theta)(\mathcal{X}^m_Y\gamma^p\theta) - \frac{1}{12}(\theta\gamma^m{}_{np}\theta)(\mathcal{X}^n_X\gamma^p\theta)\mathcal{X}^\alpha_Y\\
&\quad - \frac{1}{32}(\theta\gamma_{np})^\alpha(\theta\gamma^m{}_{qr}\theta)\mathfrak{f}^{np}_X\mathfrak{f}^{qr}_Y - (X \leftrightarrow Y)\Big]
\end{aligned}
$$

$$
\begin{aligned}
[\mathscr{F}_P^{m|pq}]_2 = &-\frac{1}{8}\mathfrak{f}^{m[p}{}_{|nr}(\theta\gamma^{q]nr}\theta) - \sum_{XY=P}\big[(\mathscr{X}_X\gamma^m\theta)(\mathscr{X}_Y^{[p}\gamma^{q]}\theta) \\
&+(\mathscr{X}_X^m\gamma^{[p}\theta)(\mathscr{X}_Y\gamma^{q]}\theta) + \frac{1}{8}(\theta\gamma^m{}_{nr}\theta)\mathfrak{f}_X^{pq}\mathfrak{f}_Y^{nr} - (X\leftrightarrow Y)\big] \\
[\mathscr{F}_P^{m|pq}]_3 = &\frac{1}{12}(\mathscr{X}_B^{m[p}{}_n\gamma_r\theta)(\theta\gamma^{q]nr}\theta) + \sum_{XY=P}\big[\frac{1}{8}(\mathscr{X}_X\gamma^m\theta)(\theta\gamma^{[p}{}_{nr}\theta)\mathfrak{f}^{q]|nr} \\
&+\frac{1}{8}(\theta\gamma_{nr}{}^{[p}\theta)(\mathscr{X}_X\gamma^{q]}\theta)\mathfrak{f}_Y^{m|nr} - \frac{1}{8}(\mathscr{X}_X^m\gamma^{[p}\theta)(\theta\gamma^{q]}{}_{nr}\theta)\mathfrak{f}_Y^{nr} \\
&+\frac{1}{8}(\theta\gamma^m{}_{nr}\theta)(\mathscr{X}_X^{[p}\gamma^{q]}\theta)\mathfrak{f}_Y^{nr} - \frac{1}{12}(\theta\gamma^m{}_{nr}\theta)(\mathscr{X}_X^n\gamma^r\theta)\mathfrak{f}_Y^{pq} - (X\leftrightarrow Y)\big] \\
&+\sum_{XYZ=P}\big[(\mathscr{X}_X\gamma^{[p}\theta)(\mathscr{X}_Y\gamma^{q]}\theta)(\mathscr{X}_Z\gamma^m\theta) + (X\leftrightarrow Z)\big]\ .
\end{aligned}
$$

B.3 Theta-expansions of generic higher-mass dimension superfields

For superfields of higher mass dimension as defined in (2.38), the theta-expansion in HS gauge is governed by the recursion

$$
[\mathbb{W}^{N\alpha}]_k = \frac{1}{k}\Big\{\frac{1}{4}(\theta\gamma_{pq})^\alpha[\mathbb{F}^{N|pq}]_{k-1} + \sum_{\substack{M\in P(N)\\ M\neq 0}}\sum_{l=0}^{k-1}\big\{([\mathbb{W}]_l\gamma\theta)^M\,,[\mathbb{W}^{(N\setminus M)\alpha}]_{k-l-1}\big\}\Big\} \tag{B.2a}
$$

$$
[\mathbb{F}^{N|pq}]_k = -\frac{1}{k}\Big\{([\mathbb{W}^{N[p}]_{k-1}\gamma^{q]}\theta) - \sum_{\substack{M\in P(N)\\ M\neq 0}}\sum_{l=0}^{k-1}\big[([\mathbb{W}]_l\gamma\theta)^M\,,[\mathbb{F}^{(N\setminus M)|pq}]_{k-l-1}\big]\Big\}. \tag{B.2b}
$$

We are using multi-index notation $N \equiv n_1 n_2 \dots n_k$, where the power set $P(N)$ consists of the 2^k ordered subsets of N, and $(\mathbb{W}\gamma)^N \equiv (\mathbb{W}^{n_1\dots n_{k-1}}\gamma^{n_k})$. Their resulting theta-expansion to subleading order is given by

$$
\begin{aligned}
\mathscr{W}_P^{N\alpha}(\theta) = &\mathscr{X}_P^{N\alpha} + \frac{1}{4}(\theta\gamma_{pq})^\alpha\mathfrak{f}_P^{N|pq} \\
&+\sum_{XY=P}\sum_{\substack{M\in P(N)\\ M\neq 0}}\big[(\mathscr{X}_X\gamma\theta)^M\mathscr{X}_Y^{(N\setminus M)\alpha} - (\mathscr{X}_Y\gamma\theta)^M\mathscr{X}_X^{(N\setminus M)\alpha}\big] + \dots \\
\mathscr{F}_P^{N|pq}(\theta) = &\mathfrak{f}_P^{N|pq} - (\mathscr{X}^{N[p}\gamma^{q]}\theta) \\
&+\sum_{XY=P}\sum_{\substack{M\in P(N)\\ M\neq 0}}\big[(\mathscr{X}_X\gamma\theta)^M\mathfrak{f}_Y^{(N\setminus M)|pq} - (\mathscr{X}_Y\gamma\theta)^M\mathfrak{f}_X^{(N\setminus M)|pq}\big] + \dots.
\end{aligned}
$$

Appendix C

OPES AND BOSONIZATION

The bosonization technique discussed in section 3.3.2 renders the OPEs among ψ^μ and spin fields S_a, S^b of $SO(D = 2n)$ accessible to free-field methods. For example, (3.18) and (3.19) give rise to

$$\psi^\mu(z) S_a(0) = e^{i\mu\cdot\mathbf{H}(z)}\, e^{ia\cdot\mathbf{H}(0)} \sim z^{\mu\cdot a} e^{i(\mu+a)\cdot\mathbf{H}(0)} \Big(1 + z i\mu\cdot\partial\mathbf{H}(0) + \ldots\Big) . \tag{C.1}$$

Since $\mu = (0,\ldots,0,\pm1,0,\ldots,0)$ and $a = (\pm\frac{1}{2},\pm\frac{1}{2},\ldots,\pm\frac{1}{2})$, the exponent $\mu\cdot a$ of z is either $-\frac{1}{2}$ or $+\frac{1}{2}$. Therefore, one can split (C.1) into (up to the subleading order)

$$\psi^\mu(z) S_a(0) \sim \begin{cases} \dfrac{1}{z^{1/2}} e^{i(\mu+a)\cdot\mathbf{H}(0)} + z^{1/2} i\mu\cdot\partial\mathbf{H} e^{i(\mu+a)\cdot\mathbf{H}(0)} & \text{if } \mu\cdot a = -\dfrac{1}{2} \\ z^{1/2} e^{i(\mu+a)\cdot\mathbf{H}(0)} & \text{if } \mu\cdot a = +\dfrac{1}{2} \, . \end{cases}$$

The subleading term $i\mu\cdot\partial\mathbf{H}e^{i(\mu+a)\cdot\mathbf{H}(0)}$ can be further decomposed into a primary and a descendant part with respect to the energy-momentum tensor $T(z) = -\frac{1}{2}\partial\mathbf{H}\cdot\partial\mathbf{H}$ of the bosonized system,

$$i\mu\cdot\partial\mathbf{H}e^{i(\mu+a)\cdot\mathbf{H}(0)} = \frac{4}{D} i(\mu+a)\cdot\partial\mathbf{H}e^{i(\mu+a)\cdot\mathbf{H}(0)} + i\left(\frac{D-4}{D}\mu - \frac{4}{D}a\right)\cdot\partial\mathbf{H}e^{i(\mu+a)\cdot\mathbf{H}(0)} \, .$$

Thus, we have primary fields $S^\mu_a(z)$ defined by

$$S^\mu_a(z) = \delta\left(\mu\cdot a + \frac{1}{2}\right)\left(\frac{D-4}{D}\mu - \frac{4}{D}a\right)\cdot i\partial\mathbf{H}e^{i(\mu+a)\cdot\mathbf{H}(z)} + \delta\left(\mu\cdot a - \frac{1}{2}\right) e^{i(\mu+a)\cdot\mathbf{H}(z)} \tag{C.2}$$

at the subleading order in the OPE (C.1). Although the first term of (C.2) could in principle be used in section 3.3.3 to evaluate components of the correlators involving S^μ_a, we found the second term $e^{i(\mu+a)\cdot\mathbf{H}(z)}$ more convenient to extract the small number of required examples.

Moreover, if $\mu\cdot a = -\frac{1}{2}$, the resulting lattice vector $\mu + a = b$ refers to a spin field S^b of opposite chirality. Therefore, the OPE (C.1) can be written as

$$\begin{aligned} \psi^\mu(z) S_a(0) &\sim \sum_{b\in(\pm\frac{1}{2},\ldots,\pm\frac{1}{2})} \frac{\delta(\mu+a-b)}{z^{1/2}} \left\{ e^{ib\cdot\mathbf{H}(0)} + z\frac{4}{D}\partial e^{ib\cdot\mathbf{H}(0)} \right\} + z^{1/2} S^\mu_a(0) \\ &\equiv \frac{\gamma^\mu_{ab}}{\sqrt{2} z^{1/2}} \left\{ S^b(0) + z\frac{4}{D}\partial S^b(0) \right\} + z^{1/2} S^\mu_a(0) \, . \end{aligned}$$

In passing to the last line, we have used the definition (3.21) of gamma-matrices in the Cartan–Weyl basis, where the sign of b is flipped by the contraction through the charge-conjugation matrix in $\gamma^\mu_{ab} S^b$. The computation above exemplifies how Lorentz covariance can be a posteriori restored in results obtained from bosonization. In [102, 119, 110], this procedure is applied to construct higher-point correlation functions involving ψ^μ and S_a.

EXAMPLES FOR THE STANDARD FORM OF SPIN SUMS

This appendix complements the discussion in section 3.4.2 by identifying the standard form (3.47) of spin sums in correlators with three insertions of $\psi^{\mu_j}\psi^{\nu_j}(z_j)$. The evaluation of the spin sums is addressed in section 3.5.

D.1 Unexcited spin fields

With two unexcited spin fields, the five-point correlator

$$\begin{aligned}
&8\sum_{\nu=1}^{4}(-1)^{\nu+1}\Big\langle\Big\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)\psi^{\mu_3}\psi^{\nu_3}(z_3)S_a(z_A)S^b(z_B)\Big\rangle\Big\rangle_\nu \\
&= (\eta^{\nu_1[\mu_2}\eta^{\nu_2][\mu_3}\eta^{\nu_3]\mu_1} - \eta^{\mu_1[\mu_2}\eta^{\nu_2][\mu_3}\eta^{\nu_3]\nu_1})\delta_a{}^b\ \xi^{(1)}(z_1,z_2,z_3,z_A,z_B) \\
&\quad + \Big[(\eta^{\mu_2[\nu_1}(\gamma^{\mu_1]\nu_2\mu_3\nu_3})_a{}^b - \eta^{\nu_2[\nu_1}(\gamma^{\mu_1]\mu_2\mu_3\nu_3})_a{}^b)\ \xi^{(2)}(z_1,z_2,z_3,z_A,z_B) \\
&\qquad + (\eta^{\mu_3[\nu_2}\eta^{\mu_2][\nu_1}(\gamma^{\mu_1]\nu_3})_a{}^b - \eta^{\nu_3[\nu_2}\eta^{\mu_2][\nu_1}(\gamma^{\mu_1]\mu_3})_a{}^b)\ \xi^{(3)}(z_1,z_2,z_3,z_A,z_B) \\
&\qquad + (\eta^{\nu_1[\mu_2}\eta^{\nu_2]\mu_1} - \eta^{\mu_1[\mu_2}\eta^{\nu_2]\nu_1})(\gamma^{\mu_3\nu_3})_a{}^b\ \xi^{(4)}(z_1,z_2,z_3,z_A,z_B) + \mathrm{cyc}(1,2,3)\Big] \\
&\quad + (\gamma^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3})_a{}^b\xi^{(5)}(z_1,z_2,z_3,z_A,z_B)
\end{aligned} \tag{D.1}$$

involves spin sums

$$\begin{aligned}
\xi^{(1)}(z_i,z_j,z_k,z_A,z_B) =& -\frac{1}{E_{AB}}\mathcal{W}\left(z_{ij},z_{jk},z_{ki}\,\middle|\,\frac{1}{2}y\right) + \frac{E_{iA}E_{kB}}{E_{AB}E_{iB}E_{kA}}\mathcal{W}\left(z_{ij},z_{jk},z_{ik}\,\middle|\,\frac{1}{2}y\right) \\
&+ \frac{E_{jB}E_{kA}}{E_{AB}E_{jA}E_{kB}}\mathcal{W}\left(z_{ij},z_{ki},z_{kj}\,\middle|\,\frac{1}{2}y\right) - \frac{E_{iA}E_{jB}}{E_{AB}E_{iB}E_{jA}}\mathcal{W}\left(z_{ij},z_{kj},z_{ik}\,\middle|\,\frac{1}{2}y\right) \\
&+ \frac{E_{iB}E_{jA}}{E_{AB}E_{iA}E_{jB}}\mathcal{W}\left(z_{jk},z_{ki},z_{ji}\,\middle|\,\frac{1}{2}y\right) - \frac{E_{jA}E_{kB}}{E_{AB}E_{jB}E_{kA}}\mathcal{W}\left(z_{jk},z_{ji},z_{ik}\,\middle|\,\frac{1}{2}y\right) \\
&- \frac{E_{iB}E_{kA}}{E_{AB}E_{iA}E_{kB}}\mathcal{W}\left(z_{ki},z_{ji},z_{kj}\,\middle|\,\frac{1}{2}y\right) + \frac{1}{E_{AB}}\mathcal{W}\left(z_{ji},z_{kj},z_{ik}\,\middle|\,\frac{1}{2}y\right) \\
\xi^{(2)}(z_i,z_j,z_k,z_A,z_B) =& \frac{1}{E_{AB}}\mathcal{W}\left(z_{ij},z_{jk},z_{ki}\,\middle|\,\frac{1}{2}y\right) + \frac{E_{iA}E_{kB}}{E_{AB}E_{iB}E_{kA}}\mathcal{W}\left(z_{ij},z_{jk},z_{ik}\,\middle|\,\frac{1}{2}y\right) \\
&+ \frac{E_{jB}E_{kA}}{E_{AB}E_{jA}E_{kB}}\mathcal{W}\left(z_{ij},z_{ki},z_{kj}\,\middle|\,\frac{1}{2}y\right) + \frac{E_{iA}E_{jB}}{E_{AB}E_{iB}E_{jA}}\mathcal{W}\left(z_{ij},z_{kj},z_{ik}\,\middle|\,\frac{1}{2}y\right) \\
&- \frac{E_{iB}E_{jA}}{E_{AB}E_{iA}E_{jB}}\mathcal{W}\left(z_{jk},z_{ki},z_{ji}\,\middle|\,\frac{1}{2}y\right) - \frac{E_{jA}E_{kB}}{E_{AB}E_{jB}E_{kA}}\mathcal{W}\left(z_{jk},z_{ji},z_{ik}\,\middle|\,\frac{1}{2}y\right) \\
&- \frac{E_{iB}E_{kA}}{E_{AB}E_{iA}E_{kB}}\mathcal{W}\left(z_{ki},z_{ji},z_{kj}\,\middle|\,\frac{1}{2}y\right) - \frac{1}{E_{AB}}\mathcal{W}\left(z_{ji},z_{kj},z_{ik}\,\middle|\,\frac{1}{2}y\right)
\end{aligned}$$

$$\begin{aligned}\xi^{(3)}(z_i,z_j,z_k,z_A,z_B) = &-\frac{1}{E_{AB}}\mathcal{W}\left(z_{ij},z_{jk},z_{ki}\middle|\frac{1}{2}y\right) - \frac{E_{iA}E_{kB}}{E_{AB}E_{iB}E_{kA}}\mathcal{W}\left(z_{ij},z_{jk},z_{ik}\middle|\frac{1}{2}y\right)\\ &+\frac{E_{jB}E_{kA}}{E_{AB}E_{jA}E_{kB}}\mathcal{W}\left(z_{ij},z_{ki},z_{kj}\middle|\frac{1}{2}y\right) + \frac{E_{iA}E_{jB}}{E_{AB}E_{iB}E_{jA}}\mathcal{W}\left(z_{ij},z_{kj},z_{ik}\middle|\frac{1}{2}y\right)\\ &+\frac{E_{iB}E_{jA}}{E_{AB}E_{iA}E_{jB}}\mathcal{W}\left(z_{jk},z_{ki},z_{ji}\middle|\frac{1}{2}y\right) + \frac{E_{jA}E_{kB}}{E_{AB}E_{jB}E_{kA}}\mathcal{W}\left(z_{jk},z_{ji},z_{ik}\middle|\frac{1}{2}y\right)\\ &-\frac{E_{iB}E_{kA}}{E_{AB}E_{iA}E_{kB}}\mathcal{W}\left(z_{ki},z_{ji},z_{kj}\middle|\frac{1}{2}y\right) - \frac{1}{E_{AB}}\mathcal{W}\left(z_{ji},z_{kj},z_{ik}\middle|\frac{1}{2}y\right)\end{aligned}$$

$$\begin{aligned}\xi^{(4)}(z_i,z_j,z_k,z_A,z_B) = &-\frac{E_{kB}}{E_{kA}}\mathcal{W}\left(z_{Bk},z_{ij},z_{ji},z_{Bk}\middle|\frac{1}{2}y\right) + \frac{E_{kB}E_{iA}E_{jB}}{E_{iB}E_{jA}E_{kA}}\mathcal{W}\left(z_{Bk},z_{ij},z_{Bk},z_{ij}\middle|\frac{1}{2}y\right)\\ &+\frac{E_{kB}E_{iB}E_{jA}}{E_{iA}E_{jB}E_{kA}}\mathcal{W}\left(z_{Bk},z_{ji},z_{Bk},z_{ji}\middle|\frac{1}{2}y\right) - \frac{E_{kB}}{E_{kA}}\mathcal{W}\left(z_{Bk},z_{Bk},z_{ji},z_{ij}\middle|\frac{1}{2}y\right)\end{aligned}$$

$$\begin{aligned}\xi^{(5)}(z_i,z_j,z_k,z_A,z_B) = &\frac{1}{E_{AB}}\mathcal{W}\left(z_{ij},z_{jk},z_{ki}\middle|\frac{1}{2}y\right) + \frac{E_{iA}E_{kB}}{E_{AB}E_{iB}E_{kA}}\mathcal{W}\left(z_{ij},z_{jk},z_{ik}\middle|\frac{1}{2}y\right)\\ &+\frac{E_{jB}E_{kA}}{E_{AB}E_{jA}E_{kB}}\mathcal{W}\left(z_{ij},z_{ki},z_{kj}\middle|\frac{1}{2}y\right) + \frac{E_{iA}E_{jB}}{E_{AB}E_{iB}E_{jA}}\mathcal{W}\left(z_{ij},z_{kj},z_{ik}\middle|\frac{1}{2}y\right)\\ &+\frac{E_{iB}E_{jA}}{E_{AB}E_{iA}E_{jB}}\mathcal{W}\left(z_{jk},z_{ki},z_{ji}\middle|\frac{1}{2}y\right) + \frac{E_{jA}E_{kB}}{E_{AB}E_{jB}E_{kA}}\mathcal{W}\left(z_{jk},z_{ji},z_{ik}\middle|\frac{1}{2}y\right)\\ &+\frac{E_{iB}E_{kA}}{E_{AB}E_{iA}E_{kB}}\mathcal{W}\left(z_{ki},z_{ji},z_{kj}\middle|\frac{1}{2}y\right) + \frac{1}{E_{AB}}\mathcal{W}\left(z_{ji},z_{kj},z_{ik}\middle|\frac{1}{2}y\right) .\end{aligned}$$

The notation +cyc(1,2,3) in (D.1) refers to cyclic permutations of both the Lorentz indices and the punctures including for instance $\{(z_1,\mu_1,\nu_1),(z_2,\mu_2,\nu_2),(z_3,\mu_3,\nu_3)\} \to \{(z_2,\mu_2,\nu_2),(z_3,\mu_3,\nu_3),(z_1,\mu_1,\nu_1)\}$.

D.2 Excited spin field

In case of an excited spin field, the five-point correlator

$$\begin{aligned}&4\sqrt{2}\sum_{\nu=1}^{4}(-1)^{\nu+1}\left\langle\left\langle\psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)\psi^{\mu_3}\psi^{\nu_3}(z_3)S_a(z_A)S_b^{\lambda}(z_B)\right\rangle\right\rangle_\nu\\ &=\Big[(\gamma^{\mu_1\nu_1\mu_2\nu_2[\mu_3})_{ab}\eta^{\nu_3]\lambda}\ \Xi^{(1)}(z_1,z_2,z_3,z_A,z_B)\\ &\quad+(\eta^{\mu_2[\nu_1}(\gamma^{\mu_1]\nu_2[\mu_3})_{ab}-\eta^{\nu_2[\nu_1}(\gamma^{\mu_1]\mu_2[\mu_3})_{ab})\eta^{\nu_3]\lambda}\ \Xi^{(2)}(z_1,z_2,z_3,z_A,z_B)\\ &\quad+(\eta^{\mu_2[\nu_1}\eta^{\mu_1]\nu_2}-\eta^{\nu_2|[\nu_1}\eta^{\mu_1]\mu_2})(\gamma^{[\mu_3})_{ab}\eta^{\nu_3]\lambda}\ \Xi^{(3)}(z_1,z_2,z_3,z_A,z_B)+\mathrm{cyc}(1,2,3)\Big]\\ &\ +\Big[(\gamma^{\mu_1\nu_1[\mu_2})_{ab}\eta^{\nu_2][\mu_3}\eta^{\nu_3]\lambda}\ \Xi^{(4)}(z_1,z_2,z_3,z_A,z_B)\\ &\quad+(\gamma^{[\mu_1})_{ab}\eta^{\nu_1][\mu_2}\eta^{\nu_2][\mu_3}\eta^{\nu_3]\lambda}\ \Xi^{(5)}(z_1,z_2,z_3,z_A,z_B)+\mathrm{perm}(1,2,3)\Big]\end{aligned}$$

involves spin sums

$$\begin{aligned}\Xi^{(1)}(z_i,z_j,z_k,z_A,z_B) = &-\tfrac{1}{E_{AB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{ij},z_{jk}\middle|\frac{1}{2}y\right) - \tfrac{E_{jB}E_{kA}}{E_{AB}E_{jA}E_{kB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{ij},z_{kj}\middle|\frac{1}{2}y\right)\\ &-\tfrac{E_{iB}E_{jA}}{E_{AB}E_{iA}E_{jB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{jk},z_{ji}\middle|\frac{1}{2}y\right) - \tfrac{E_{iB}E_{kA}}{E_{AB}E_{iA}E_{kB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{ji},z_{kj}\middle|\frac{1}{2}y\right)\\ \Xi^{(2)}(z_i,z_j,z_k,z_A,z_B) = &-\tfrac{1}{E_{AB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{ij},z_{jk}\middle|\frac{1}{2}y\right) - \tfrac{E_{jB}E_{kA}}{E_{AB}E_{jA}E_{kB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{ij},z_{kj}\middle|\frac{1}{2}y\right)\\ &+\tfrac{E_{iB}E_{jA}}{E_{AB}E_{iA}E_{jB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{jk},z_{ji}\middle|\frac{1}{2}y\right) + \tfrac{E_{iB}E_{kA}}{E_{AB}E_{iA}E_{kB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{ji},z_{kj}\middle|\frac{1}{2}y\right)\end{aligned}$$

$$
\begin{aligned}
\Xi^{(3)}(z_i,z_j,z_k,z_A,z_B) = {} & \tfrac{1}{E_{AB}}\mathcal{W}\left(z_{ij},z_{ji},z_{Bk},z_{kB}\,\middle|\,\frac{1}{2}y\right) - \tfrac{E_{iA}E_{jB}}{E_{iB}E_{jA}E_{AB}}\mathcal{W}\left(z_{ij},z_{Bk},z_{kB},z_{ij}\,\middle|\,\frac{1}{2}y\right) \\
& - \tfrac{E_{iB}E_{jA}}{E_{iA}E_{jB}E_{AB}}\mathcal{W}\left(z_{ji},z_{Bk},z_{kB},z_{ji}\,\middle|\,\frac{1}{2}y\right) + \tfrac{1}{E_{AB}}\mathcal{W}\left(z_{Bk},z_{kB},z_{ji},z_{ij}\,\middle|\,\frac{1}{2}y\right) \\
\Xi^{(4)}(z_i,z_j,z_k,z_A,z_B) = {} & -\tfrac{1}{E_{AB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{ij},z_{jk}\,\middle|\,\frac{1}{2}y\right) + \tfrac{E_{jB}E_{kA}}{E_{AB}E_{jA}E_{kB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{ij},z_{kj}\,\middle|\,\frac{1}{2}y\right) \\
& - \tfrac{E_{iB}E_{jA}}{E_{AB}E_{iA}E_{jB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{jk},z_{ji}\,\middle|\,\frac{1}{2}y\right) + \tfrac{E_{iB}E_{kA}}{E_{AB}E_{iA}E_{kB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{ji},z_{kj}\,\middle|\,\frac{1}{2}y\right) \\
\Xi^{(5)}(z_i,z_j,z_k,z_A,z_B) = {} & -\tfrac{1}{E_{AB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{ij},z_{jk}\,\middle|\,\frac{1}{2}y\right) + \tfrac{E_{jB}E_{kA}}{E_{AB}E_{jA}E_{kB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{ij},z_{kj}\,\middle|\,\frac{1}{2}y\right) \\
& + \tfrac{E_{iB}E_{jA}}{E_{AB}E_{iA}E_{jB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{jk},z_{ji}\,\middle|\,\frac{1}{2}y\right) - \tfrac{E_{iB}E_{kA}}{E_{AB}E_{iA}E_{kB}}\mathcal{W}\left(z_{Bi},z_{kB},z_{ji},z_{kj}\,\middle|\;\right).
\end{aligned}
$$

Appendix E

SOME IDENTITIES OF ELLIPTIC FUNCTIONS

In this section, we present some identities which are useful for implementing spin sums in chapter 2. We begin with recalling two elementary theorems of elliptic functions whose proofs can be found in the standard literature on elliptic functions such as [141].

THEOREM E.1. *The sum of residues over an irreducible set of poles of an elliptic function vanishes.*

THEOREM E.2. *An elliptic function with an empty irreducible set of poles is a constant function.*

On the basis of these two theorems we state the following identities.

PROPOSITION E.3. *Let $F(x,y)$ be a Eisenstein-Kronecker series defined by*

$$F(x,y) = \frac{\theta_1'(0)\theta_1(x+y)}{\theta_1(x)\theta_1(y)}$$

which has the quasi-periodicity as

$$F(x+1,y) = F(x,y), \quad F(x+\tau,y) = e^{-2\pi i y}F(x,y). \tag{E.1}$$

Then,

$$F(x,y)F(-x,y) = \wp(y) - \wp(x)$$

where $\wp(z)$ is the Weierstrass $\wp$-function defined by

$$\wp(z) = -\partial_z^2 \log\theta_1(z) + \frac{\partial_z^3\theta_1(0)}{3\partial_z\theta(0)}.$$

Proof. It is obvious from the quasi-periodicity of F in (E.1) that $F(x,y)F(x,-y)$ is an elliptic function on both x and y with a pole with multiplicity two at $x=0$ and $y=0$. Therefore, one can deduce that

$$F(x,y)F(-x,y) - \wp(y) + \wp(x) \tag{E.2}$$

has no irreducible pole, so it is a constant by the theorem E.2. By inserting $x=y$ one can also show that the constant is zero, so the statement is true. □

PROPOSITION E.4. *Let $\zeta(x)$ and $\wp(x)$ be the Weierstrass ζ- and $\wp$-function and $s_\nu = \left(0, \frac{1}{2}, \frac{-1-\tau}{2}, \frac{\tau}{2}\right)$ where 1 and τ are elliptic periods of ζ and $\wp$. Then*

$$2\zeta(2x) = \sum_{\nu=1}^{4}\zeta(x+s_\nu), \qquad 2^{k+2}\wp(2x) = \sum_{\nu=1}^{4}\wp^{(k)}(x+s_\nu).$$

Proof. One can easily see that the summation $\sum_{\nu=1}^{4}\zeta(x+s_\nu)$ has simple poles at $x=\pm s_\nu$, $\nu = 1,2,3,4$ and under $x \to x+s_\nu, \nu = 2,3,4$ it transforms as

$$\sum_{\nu=1}^{4}\zeta(x+s_\nu) \to \sum_{\nu=1}^{4}\zeta(x+s_\nu) + 8\eta_\nu$$

where $\eta_\nu = \zeta(s_\nu)$ for $\nu = 2,3,4$. Therefore,

$$2\zeta(2x) - \sum_{\nu=1}^{4} \zeta(x+s_\nu) \tag{E.3}$$

is a constant due to the Liouville theorem. Then by inserting $x=0$ to (E.3), we find that

$$2\zeta(2x) - \sum_{\nu=1}^{4} \zeta(x+s_\nu) = 0$$

since $\eta_2+\eta_3+\eta_4 = 0$. The second identity for $\wp$ can be then proven by noticing

$$\wp(x) = -\zeta'(z).$$

□

We also present the Fay trisecant identities whose proof can be found in [120].

THEOREM E.5. *Let $\theta_\nu(z) \neq 0$ and consider complex variables $\{x_1,x_2,\dots,x_n\}$ and $\{y_1,y_2,\dots,y_n\}$, where x_j and y_k are pairwise different for $j,k = 1,2,\dots,n$. Then, the following Fay trisecant identities hold for $\nu = 1,2,3,4$ [120],*

$$\theta_\nu\left(\sum_{j=1}^{n}(x_j-y_j)+z\right)\theta_\nu(z)^{n-1}\frac{\prod_{j<k}^{n} E(x_j,x_k)E(y_k,y_j)}{\prod_{j,k=1}^{n} E(x_j,y_k)} = \det_{j,k}\left[\frac{\theta_\nu(x_j-y_k+z)}{E(x_j,y_k)}\right], \tag{E.4}$$

where the determinant refers to the $n\times n$ matrix with entries $\frac{\theta_\nu(x_j-y_k+z)}{E(x_j,y_k)}$.

Appendix F

EXAMPLES FOR SPIN-SUMMED CORRELATORS

This appendix adds further examples of spin-summed correlators to section 3.5.5.

F.1 Unexcited spin-fields

The seven-point generalization of the correlators in section 3.5.5 is given by

$$\begin{aligned}
&8\sum_{\nu=1}^{4}(-1)^{\nu+1}\left\langle\left\langle\psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)\psi^{\mu_3}\psi^{\nu_3}(z_3)\psi^{\mu_4}\psi^{\nu_4}(z_4)\psi^{\mu_5}\psi^{\nu_5}(z_5)S_a(z_A)S^b(z_B)\right\rangle\right\rangle_\nu \\
&=(\gamma^{\mu_1\nu_1\mu_2\nu_2\ldots\mu_5\nu_5})_a{}^b h^{(2)}_{\emptyset}+\eta^{\nu_1\mu_2}(\gamma^{\mu_1\nu_2\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5})_a{}^b h^{(2)}_{[12]} \\
&+\eta^{\nu_1\mu_2}\eta^{\mu_1\nu_2}(\gamma^{\mu_3\nu_3\mu_4\nu_4\mu_5\nu_5})_a{}^b h^{(2)}_{(12)}+\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}(\gamma^{\mu_1\nu_3\mu_4\nu_4\mu_5\nu_5})_a{}^b h^{(2)}_{12,23} \\
&+\eta^{\nu_1\mu_2}\eta^{\nu_3\mu_4}(\gamma^{\mu_1\nu_2\mu_3 nu_4\mu_5\nu_5})_a{}^b h^{(2)}_{[12],[34]}+\eta^{\nu_1\mu_2}\eta^{\mu_1\nu_2}\eta^{\nu_3\mu_4}(\gamma^{\mu_3\nu_4\mu_5\nu_5})_a{}^b h^{(2)}_{(12),[34]} \\
&+\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\mu_1\nu_3}(\gamma^{\mu_4\nu_4\mu_5\nu_5})_a{}^b h^{(2)}_{[123]}+\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\nu_3\mu_4}(\gamma^{\mu_1\nu_4\mu_5\nu_5})_a{}^b h^{(2)}_{12,23,34} \\
&+\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\nu_4\mu_5}(\gamma^{\mu_1\nu_3\mu_4\nu_5})_a{}^b h^{(2)}_{12,23,45}+\eta^{\nu_1\mu_2}\eta^{\mu_1\nu_2}\eta^{\nu_3\mu_4}\eta^{\mu_3\nu_4}(\gamma^{\mu_5\nu_5})_a{}^b h^{(2)}_{(12),(34)} \\
&+\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\mu_1\nu_3}\eta^{\nu_4\mu_5}(\gamma^{\mu_4\nu_5})_a{}^b h^{(2)}_{[123],[45]}+\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\nu_3\mu_4}\eta^{\mu_1\nu_4}(\gamma^{\mu_5\nu_5})_a{}^b h^{(2)}_{(1234)} \\
&+\eta^{\nu_1\mu_2}\eta^{\mu_1\nu_2}\eta^{\nu_3\mu_4}\eta^{\nu_4\mu_5}(\gamma^{\mu_3\nu_5})_a{}^b h^{(2)}_{(12),34,45}+\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\nu_3\mu_4}\eta^{\nu_4\mu_5}(\gamma^{\mu_1\nu_5})_a{}^b h^{(2)}_{12,23,34,45} \\
&+\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\nu_3\mu_1}\eta^{\nu_4\mu_5}\eta^{\nu_5\mu_4}\delta_a^b h^{(2)}_{[123],(45)}+\eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\nu_3\mu_4}\eta^{\nu_4\mu_5}\eta^{\mu_1\nu_5}\delta_a^b h^{(2)}_{(12345)} \\
&+\text{permutations}
\end{aligned} \tag{F.1}$$

with the following doubly-periodic functions $h^{(2)}_{\ldots}\equiv h^{(2)}_{\ldots}(z_j,z_A,z_B)$:

$$h^{(2)}_{\emptyset}(z_j,z_A,z_B)=-\sum_{i=1}^{5}\sum_{j=i+1}^{5}V_2(i,A,j,B)-16V_2(A,B) \tag{F.2a}$$

$$h^{(2)}_{[12]}(z_j,z_A,z_B)=\sum_{i=3}^{5}V_2(1,2,B,i,A)+4V_2(1,2,A,B)-(A\leftrightarrow B) \tag{F.2b}$$

$$h^{(2)}_{(12)}(z_j,z_A,z_B)=\sum_{i=3}^{5}\sum_{j=i+1}^{5}V_2(i,B,j,A)-\sum_{i=1}^{2}\sum_{j=i+1}^{5}V_2(i,B,j,A)-4V_2(1,2) \tag{F.2c}$$

$$\begin{aligned}
h^{(2)}_{12,23}(z_j,z_A,z_B)=&\sum_{i=1,i\neq 2}^{5}V_2(2,A,i,B)-V_2(1,A,3,B)-V_2(4,A,5,B)\\
&-2\{V_2(2,A,1,3)+V_2(2,A,3,1)+(A\leftrightarrow B)\}
\end{aligned} \tag{F.2d}$$

$$h^{(2)}_{[12],[34]}(z_j,z_A,z_B)=V_2(1,2,A,3,4,B)-(1\leftrightarrow 2)-(3\leftrightarrow 4)+(1\leftrightarrow 2,3\leftrightarrow 4) \tag{F.2e}$$

$$\begin{aligned}
h^{(2)}_{(12),[34]}(z_j,z_A,z_B)=&(V_1(1,A,B)+V_1(2,A,B)-V_1(5,A,B))\\
&\times(V_1(3,4,A,B)+V_1(3,4,B,A))\\
&-V_2(3,4,A,B)+V_2(3,4,B,A)
\end{aligned} \tag{F.2f}$$

$$h^{(2)}_{[123]}(z_j,z_A,z_B) = V_2(1,A,2,3,B) + \sum_{i=4}^{5} V_2(1,2,A,i,B) + \text{cyc}(1,2,3) - (A \leftrightarrow B) \tag{F.2g}$$

$$h^{(2)}_{12,23,34}(z_j,z_A,z_B) = -V_2(2,A,3,4,B) - V_2(3,A,1,2,B) - \sum_{i=1}^{3} V_2(i,i+1,A,5,B) - (A \leftrightarrow B) \tag{F.2h}$$

$$h^{(2)}_{12,23,45}(z_j,z_A,z_B) = -V_2(2,A,4,5,B) + V_2(2,B,4,5,A) \tag{F.2i}$$

$$h^{(2)}_{(12),(34)}(z_j,z_A,z_B) = V_2(1,A,2,B) + V_2(3,A,4,B) + 4V_2(1,2) + 4V_2(3,4) + \sum_{i=1}^{4} V_2(i,A,5,B) - \sum_{i=1}^{2}\sum_{j=3}^{4} V_2(i,A,j,B) \tag{F.2j}$$

$$h^{(2)}_{[123],[45]}(z_j,z_A,z_B) = -2V_1(1,2,3)(V_1(4,5,A,B) + V_1(4,5,B,A)) \tag{F.2k}$$

$$h^{(2)}_{(1234)}(z_j,z_A,z_B) = V_2(1,A,3,B) + V_2(2,A,4,B) - \sum_{i=1}^{4} V_2(i,A,5,B) - 4V_2(1,2,3,4) \tag{F.2l}$$

$$\begin{aligned} h^{(2)}_{(12),34,45}(z_j,z_A,z_B) = {} & V_2(1,B,4,A) + V_2(2,B,4,A) - V_2(1,B,2,A) - 4V_2(1,2) \\ & - V_2(3,B,4,A) - V_2(4,B,5,A) + V_2(3,B,5,A) \\ & - 2V_2(3,A,5,4) - 2V_2(3,B,5,4) \end{aligned} \tag{F.2m}$$

$$\begin{aligned} h^{(2)}_{12,23,34,45}(z_j,z_A,z_B) = {} & V_2(1,A,2,B) - V_2(2,A,4,B) + V_2(4,A,5,B) \\ & - V_2(1,A,5,B) + 2V_2(1,2,3,4,5,A) + 2V_2(1,2,3,4,5,B) \end{aligned} \tag{F.2n}$$

$$h^{(2)}_{[123],(45)}(z_j,z_A,z_B) = (V_2(1,2,A,B) + \text{cyc}(1,2,3)) - V_1(1,2,3)\sum_{i=1}^{5} V_1(i,A,B) - (A \leftrightarrow B) \tag{F.2o}$$

$$\begin{aligned} h^{(2)}_{(12345)}(z_j,z_A,z_B) = {} & \big(V_2(1,A,2,3,B) + V_2(1,A,3,4,B) + V_2(1,A,4,5,B) \\ & + 4V_2(1,2,A,B) + \text{cyc}(1,2,3,4,5)\big) - (A \leftrightarrow B)\,. \end{aligned} \tag{F.2p}$$

F.2 One excited spin-fields

The six-point generalization of the correlators in section 3.5.5 reads

$$\begin{aligned} & 4\sqrt{2}\sum_{\nu=1}^{4}(-1)^{\nu+1}\Big\langle\Big\langle \psi^{\mu_1}\psi^{\nu_1}(z_1)\psi^{\mu_2}\psi^{\nu_2}(z_2)\psi^{\mu_3}\psi^{\nu_3}(z_3)\psi^{\mu_4}\psi^{\nu_4}(z_4)S_a(z_A)S_b^{\lambda}(z_B)\Big\rangle\Big\rangle_\nu \\ & = (\gamma^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3\mu_4})_{ab}\eta^{\nu_4\lambda}H^{(2)}_{\varnothing} + \eta^{\nu_1\mu_2}(\gamma^{\mu_1\nu_2\mu_3\nu_3\mu_4})_{ab}\eta^{\nu_4\lambda}H^{(2)}_{[12]} \\ & + \eta^{\nu_1\mu_2}\eta^{\mu_1\nu_2}(\gamma^{\mu_3\nu_3\mu_4})_{ab}\eta^{\nu_4\lambda}H^{(2)}_{(12)} + \eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}(\gamma^{\mu_1\nu_3\mu_4})_{ab}\eta^{\nu_4\lambda}H^{(2)}_{12,23} \\ & + \eta^{\nu_1\mu_2}\eta^{\nu_2\mu_3}\eta^{\mu_1\nu_3}(\gamma^{\mu_4})_{ab}\eta^{\nu_4\lambda}H^{(2)}_{[123]} + (\gamma^{\mu_1\mu_2\nu_2\mu_3\nu_3})_{ab}\eta^{\nu_1\mu_4}\eta^{\nu_4\lambda}H^{(2)}_{14} \\ & + \eta^{\nu_2\mu_3}(\gamma^{\mu_1\mu_2\nu_3})_{ab}\eta^{\nu_1\mu_4}\eta^{\nu_4\lambda}H^{(2)}_{[23],14} + \eta^{\nu_2\mu_3}\eta^{\mu_2\nu_3}(\gamma^{\mu_1})_{ab}\eta^{\nu_1\mu_4}\eta^{\nu_4\lambda}H^{(2)}_{(23),14} \\ & + (\gamma^{\nu_2\mu_3\nu_3})_{ab}\eta^{\mu_1\mu_2}\eta^{\nu_1\mu_4}\eta^{\nu_4\lambda}H^{(2)}_{12,14} + \eta^{\nu_2\mu_3}(\gamma^{\nu_3})_{ab}\eta^{\mu_1\mu_2}\eta^{\nu_1\mu_4}\eta^{\nu_4\lambda}H^{(2)}_{12,23,14} \\ & + \text{permutations} \end{aligned} \tag{F.3}$$

with doubly-periodic functions $H^{(2)}_{\dots} \equiv H^{(2)}_{\dots}(z_j, z_A, z_B)$ given by

$$H^{(2)}_{\emptyset}(z_j,z_A,z_B) = \big[V_2(1,A,2,B) + \text{cyc}(1,2,3)\big] - 2V_2(4,B) + 8V_2(A,B) \tag{F.4a}$$

$$H^{(2)}_{[12]}(z_j,z_A,z_B) = V_2(1,2,A,3,B) - 2V_2(1,2,A,B) - (A \leftrightarrow B) \tag{F.4b}$$

$$H^{(2)}_{(12)}(z_j,z_A,z_B) = \big[V_2(1,B,2,A) + \text{cyc}(1,2,3)\big] + 4V_2(1,2) + 2V_2(4,B) \tag{F.4c}$$

$$\begin{aligned} H^{(2)}_{12,23}(z_j,z_A,z_B) = &-V_2(1,B,2,A) - V_2(2,B,3,A) + V_2(1,B,3,A) \\ &- 2V_2(1,A,3,2) - 2V_2(1,B,3,2) - 2V_2(4,B) \end{aligned} \tag{F.4d}$$

$$\begin{aligned} H^{(2)}_{[123]}(z_j,z_A,z_B) = &-2V_2(2,3,A,B) - 2V_2(1,2,A,B,3) - V_2(1,A,2,3,B) \\ &- (A \leftrightarrow B) \end{aligned} \tag{F.4e}$$

$$\begin{aligned} H^{(2)}_{14}(z_j,z_A,z_B) = &\Big[-2V_2(1,4,A,B) - \sum_{i=2}^{3} V_2(i,A,1,4,B) - (A \leftrightarrow B)\Big] \\ &- \sum_{i=2}^{3} V_2(i,B,4,A) + V_2(2,B,3,A) + 2V_2(4,B) \end{aligned} \tag{F.4f}$$

$$\begin{aligned} H^{(2)}_{[23],14}(z_j,z_A,z_B) = &\sum_{i=2}^{3} (V_2(1,A,i,B) + V_2(4,A,i,B)) - 2V_2(1,A,4,B) \\ &- 2V_2(2,A,3,B) - 2V_2(1,4,A,2,3,B) - 2V_2(1,4,B,2,3,B) \\ &+ V_2(4,A,2,3,B) - V_2(4,B,2,3,A) \end{aligned} \tag{F.4g}$$

$$\begin{aligned} H^{(2)}_{(23),14}(z_j,z_A,z_B) = &\Big[-\sum_{i=2}^{3} (V_2(i,A,1,4,B)) - 2V_2(1,4,A,B) - (A \leftrightarrow B)\Big] \\ &- \sum_{i=2}^{3} V_2(4,A,i,B) + 4V_2(2,3) + V_2(2,B,3,A) - 2V_2(4,B) \end{aligned} \tag{F.4h}$$

$$\begin{aligned} H^{(2)}_{12,14}(z_j,z_A,z_B) = &-2V_2(2,A,1,4,B) + 2V_2(3,A,1,4,B) + 2V_2(1,B,2,A) \\ &- V_2(3,A,1,2,B) + V_2(3,B,1,2,A) + 4V_2(4,B,2,1) \end{aligned} \tag{F.4i}$$

$$\begin{aligned} H^{(2)}_{12,23,14}(z_j,z_A,z_B) = &-\big(2V_2(1,2,A,B) + 2V_2(1,4,B,A) + 2V_2(2,3,A,B) \\ &\quad + V_2(1,A,2,3,B) - (A \leftrightarrow B)\big) - 2V_2(2,B,4,A) \\ &- 2V_2(1,4,A,3,2) - 2V_2(1,4,B,3,2) + 2V_2(2,A,1,4,B) \\ &+ 2V_2(4,A,2,1,B) + 2V_2(4,A,3,2,B) + 2V_2(4,B)\,. \end{aligned} \tag{F.4j}$$

Bibliography

[1] E. Rutherford, "*The scattering of α and β particles by matter and the structure of the atom*," London, Edinburgh, Dublin Philos. Mag. J. Sci. **21** (1911) 669–688.

[2] S. Weinberg, *The Quantum Theory of Fields I: Foundations*. Cambridge University Press, 1995.

[3] H. Kawai, D. Lewellen and S.-H. Tye, "*A relation between tree amplitudes of closed and open strings*," Nucl. Phys. B **269** (1986) 1–23.

[4] Z. Bern, J. J. M. Carrasco and H. Johansson, "*New relations for gauge-theory amplitudes*," Phys. Rev. D **78** (2008) 085011, [arXiv:0805.3993].

[5] Z. Bern, J. J. M. Carrasco and H. Johansson, "*Perturbative Quantum Gravity as a Double Copy of Gauge Theory*," Phys. Rev. Lett. **105** (2010) 061602, [arXiv:1004.0476].

[6] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, "*Simplifying Multiloop Integrands and Ultraviolet Divergences of Gauge Theory and Gravity Amplitudes*," Phys. Rev. D **85** (2012) 105014, [arXiv:1201.5366].

[7] Z. Bern, S. Davies, T. Dennen and Y.-t. Huang, "*Absence of Three-Loop Four-Point Divergences in N=4 Supergravity*," Phys. Rev. Lett. **108** (2012) 201301, [arXiv:1202.3423].

[8] Z. Bern, S. Davies, T. Dennen, A. V. Smirnov and V. A. Smirnov, "*The Ultraviolet Properties of N=4 Supergravity at Four Loops*," Phys. Rev. Lett. **111** (2013) 231302, [arXiv:1309.2498].

[9] Z. Bern, S. Davies and T. Dennen, "*Enhanced Ultraviolet Cancellations in N = 5 Supergravity at Four Loop*," Phys. Rev. D **90** (2014) 105011, [arXiv:1409.3089].

[10] Z. Bern, J. J. M. Carrasco, W.-M. Chen, H. Johansson, R. Roiban and M. Zeng, "*The Five-Loop Four-Point Integrand of N=8 Supergravity as a Generalized Double Copy*," Phys. Rev. D **96** (2017) 126012, [arXiv:1708.06807].

[11] Z. Bern, J. J. M. Carrasco, W.-M. Chen, A. Edison, H. Johansson, J. Parra-Martinez et al., "*Ultraviolet Properties of N = 8 Supergravity at Five Loops*," Phys. Rev. D **98** (2018) 086021, [arXiv:1804.09311].

[12] Z. Bern, T. Dennen, Y.-t. Huang and M. Kiermaier, "*Gravity as the square of gauge theory*," Phys. Rev. D **82** (2010) 065003, [arXiv:1004.0693].

[13] C. R. Mafra, O. Schlotterer and S. Stieberger, "*Explicit BCJ numerators from pure spinors*," JHEP **1107** (2011) 092, [arXiv:1104.5224].

[14] C. R. Mafra, O. Schlotterer and S. Stieberger, "*Complete N-Point Superstring Disk Amplitude I. Pure Spinor Computation*," Nucl. Phys. B **873** (2011) 419–460, [arXiv:1106.2645].

[15] N. Berkovits, "*Super-Poincare covariant quantization of the superstring*," JHEP **0004** (2000) 018, [hep-th/0001035].

[16] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, "*Simplifying multiloop integrands and ultraviolet divergences of gauge theory and gravity amplitudes*," Phys. Rev. D **85** (2012) 105014, [arXiv:1201.5366].

[17] Z. Bern, J. J. M. Carrasco, W.-M. Chen, H. Johansson and R. Roiban, "*Gravity Amplitudes as Generalized Double Copies of Gauge-Theory Amplitudes*," Phys. Rev. Lett. **118** (2017) 181602, [arXiv:1701.02519].

[18] Y. Geyer and R. Monteiro, "*Gluons and gravitons at one loop from ambitwistor strings*," JHEP **1803** (2018) 068, [arXiv:1711.09923].

[19] S. He, O. Schlotterer and Y. Zhang, "*New BCJ representations for one-loop amplitudes in gauge theories and gravity*," Nucl. Phys. B **930** (2018) 328–383, [arXiv:1706.00640].

[20] F. Cachazo, S. He and E. Y. Yuan, "*Scattering equations and Kawai-Lewellen-Tye orthogonality*," Phys. Rev. D **90** (2014) 065001, [arXiv:1306.6575].

[21] F. Cachazo, S. He and E. Y. Yuan, "*Scattering of Massless Particles in Arbitrary Dimensions*," Phys. Rev. Lett. **113** (2014) 171601, [arXiv:1307.2199].

[22] T. Adamo, E. Casali and D. Skinner, "*Ambitwistor strings and the scattering equations at one loop*," JHEP **1404** (2014) 104, [arXiv:1312.3828].

[23] Y. Geyer, L. Mason, R. Monteiro and P. Tourkine, "*Loop Integrands for Scattering Amplitudes from the Riemann Sphere*," Phys. Rev. Lett. **115** (2015) 121603, [arXiv:1507.00321].

[24] Y. Geyer, L. Mason, R. Monteiro and P. Tourkine, "*One-loop amplitudes on the Riemann sphere*," JHEP **1603** (2016) 114, [arXiv:1511.06315].

[25] Y. Geyer, L. Mason, R. Monteiro and P. Tourkine, "*Two-loop scattering amplitudes from the Riemann sphere*," Phys. Rev. D **94** (2016) 125029, [arXiv:1607.08887].

[26] Y. Geyer and R. Monteiro, "*Two-loop scattering amplitudes from ambitwistor strings: from genus two to the nodal Riemann sphere*," JHEP **1811** (2018) 008, [arXiv:1805.05344].

[27] C. R. Mafra and O. Schlotterer, "*Towards one-loop SYM amplitudes from the pure spinor BRST cohomology*," Fortschritte der Phys. **63** (2015) 105–131, [arXiv:1410.0668].

[28] C. R. Mafra and O. Schlotterer, "*Two-loop five-point amplitudes of super Yang-Mills and supergravity in pure spinor superspace*," arXiv:1505.02746.

[29] C. R. Mafra and O. Schlotterer, "*Towards the n-point one-loop superstring amplitude I: Pure spinors and superfield kinematics*," arXiv:1812.10969.

[30] C. R. Mafra and O. Schlotterer, "*Towards the n-point one-loop superstring amplitude II: Worldsheet functions and their duality to kinematics*," arXiv:1812.10970.

[31] C. R. Mafra and O. Schlotterer, "*Towards the n-point one-loop superstring amplitude III: One-loop correlators and their double-copy structure*," arXiv:1812.10971.

[32] A. Tsuchiya, "*More on one-loop massless amplitudes of superstring theories*," Phys. Rev. D **39** (1989) 1626–1631.

[33] J. Broedel, C. R. Mafra, N. Matthes and O. Schlotterer, "*Elliptic multiple zeta values and one-loop superstring amplitudes*," JHEP **1507** (2015) 112, [arXiv:1412.5535].

[34] C. R. Mafra and O. Schlotterer, "*The Structure of n-Point One-Loop Open Superstring Amplitudes*," JHEP **1408** (2014) 099, [arXiv:1203.6215].

[35] C. R. Mafra and O. Schlotterer, "*One-loop superstring six-point amplitudes and anomalies in pure spinor superspace*," JHEP **1604** (2016) 148, [arXiv:1603.04790].

[36] L. Kronecker, "*Zur Theorie der elliptischen Funktionen*," Math. Werke **IV** (1881) 313.

[37] D. Zagier, "*Periods of modular forms and Jacobi theta functions*," Invent. Math. **104** (1991) 449–465.

[38] F. Brown and A. Levin, "*Multiple Elliptic Polylogarithms*," arXiv:1110.6917.

[39] S. Lee, C. R. Mafra and O. Schlotterer, "*Non-linear gauge transformations in D = 10 SYM theory and the BCJ duality*," JHEP **1603** (2016) 090, [arXiv:arXiv:1510.08843v1].

[40] S. Lee and O. Schlotterer, "*Fermionic one-loop amplitudes of the RNS superstring*," JHEP **1803** (2018) 190, [arXiv:1710.07353].

[41] Y. Nambu, "*Quark model and the factorization of the Veneziano amplitude*," in *Symmetries Quark Model.* (R. Chand, ed.), (New York, NY), p. 269, Gordon and Breach Science Publishers, 1970.

[42] T. Goto, "*Relativistic Quantum Mechanics of One-Dimensional Mechanical Continuum and Subsidiary Condition of Dual Resonance Model*," Prog. Theor. Phys. **46** (1971) 1560–1569.

[43] J. E. Paton and H. M. Chan, "*Generalized Veneziano model with isospin*," Nucl. Phys. B **10** (1969) 516–520.

[44] L. Brink and M. Henneaux, *Principles of String Theory*. Springer US, 1988.

[45] E. D'Hoker and D. H. Phong, "*The geometry of string perturbation theory*," Rev. Mod. Phys. **60** (1988) 917–1065.

[46] A. Nijenhuis and W. B. Woolf, "*Some Integration Problems in Almost-Complex and Complex Manifolds*," Ann. Math. **77** (1963) 424.

[47] A. Newlander and L. Nirenberg, "*Complex Analytic Coordinates in Almost Complex Manifolds*," Ann. Math. **65** (1957) 391.

[48] S. Deser and B. Zumino, "*A complete action for the spinning string*," Phys. Lett. B **65** (1976) 369–373.

[49] L. Brink, P. Di Vecchia and P. S. Howe, "*A locally supersymmetric and reparametrization invariant action for the spinning string*," Phys. Lett. B **65** (1976) 471–474.

[50] A. Polyakov, "*Quantum geometry of bosonic strings*," Phys. Lett. B **103** (1981) 207–210.

[51] I. A. Batalin and G. Vilkovisky, "*Gauge algebra and quantization*," Phys. Lett. B **102** (1981) 27–31.

[52] B. Craps and K. Skenderis, "*Comments on BRST quantization of strings*," JHEP **0505** (2005) 001, [hep-th/0503038].

[53] E. Witten, "*Superstring Perturbation Theory Revisited*," arXiv:1209.5461.

[54] T. Erler, "*Four Lectures on Closed String Field Theory*," arXiv:1905.06785.

[55] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory, Volume 1: Introduction*. Cambridge University Press, 2012.

[56] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*. Springer New York, 1997.

[57] P. Ramond, "*Dual Theory for Free Fermions*," Phys. Rev. D **3** (1971) 2415–2418.

[58] A. Neveu and J. H. Schwarz, "*Factorizable dual model of pions*," Nucl. Phys. B **31** (1971) 86–112.

[59] A. Neveu and J. H. Schwarz, "*Quark Model of Dual Pions*," Phys. Rev. D **4** (1971) 1109–1111.

[60] F. Gliozzi, J. Scherk and D. Olive, "*Supergravity and the spinor dual model*," Phys. Lett. B **65** (1976) 282–286.

[61] F. Gliozzi, J. Scherk and D. Olive, "*Supersymmetry, supergravity theories and the dual spinor model*," Nucl. Phys. B **122** (1977) 253–290.

[62] H. Aratyn, R. Ingermanson and A. J. Niemi, "*Covariant Canonical Superstrings*," Mod. Phys. Lett. A **02** (1987) 957–965.

[63] P. S. Howe, "*Super Weyl transformations in two dimensions*," J. Phys. A. Math. Gen. **12** (1979) 393–402.

[64] W. Siegel, "*Superfields in higher-dimensional spacetime*," Phys. Lett. B **80** (1979) 220–223.

[65] E. Witten, "*Twistor-like transform in ten dimensions*," Nucl. Phys. B **266** (1986) 245–264.

[66] P. S. Howe, "*Pure spinor lines in superspace and ten-dimensional supersymmetric theories*," Phys. Lett. B **258** (1991) 141–144.

[67] N. Berkovits, "*Covariant quantization of the superparticle using pure spinors*," JHEP **0109** (2001) 016, [hep-th/0105050].

[68] N. Berkovits, "*Origin of the pure spinor and Green-Schwarz formalisms*," JHEP **1507** (2015) 091, [arXiv:1503.03080].

[69] J. Harnad and S. Shnider, "*Constraints and field equations for ten dimensional super Yang-Mills theory*," Commun. Math. Phys. **106** (1986) 183–199.

[70] G. Policastro and D. Tsimpis, "*R^4, purified*," Class. Quantum Gravity **23** (2006) 4753–4779, [hep-th/0603165].

[71] W. Siegel, "*Classical superstring mechanics*," Nucl. Phys. B **263** (1986) 93–104.

[72] C. R. Mafra, "*Pure spinor superspace identities for massless four-point kinematic factors*," JHEP **0804** (2008) 093, [arXiv:0801.0580].

[73] C. R. Mafra, "*Simplifying the tree-level superstring massless five-point amplitude*," JHEP **1001** (2010) 007, [arXiv:0909.5206].

[74] C. R. Mafra, O. Schlotterer, S. Stieberger and D. Tsimpis, "*Six open string disk amplitude in pure spinor superspace*," Nucl. Phys. B **846** (2011) 359–393, [arXiv:1011.0994].

[75] N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, "*Minimal Basis for Gauge Theory Amplitudes*," Phys. Rev. Lett. **103** (2009) 161602, [arXiv:0907.1425].

[76] S. Stieberger, "*Open & Closed vs. Pure Open String Disk Amplitudes*," arXiv:0907.2211.

[77] C. R. Mafra and O. Schlotterer, "*Multiparticle SYM equations of motion and pure spinor BRST blocks*," JHEP **1407** (2014) 153, [arXiv:1404.4986].

[78] C. R. Mafra and O. Schlotterer, "*Solution to the nonlinear field equations of ten dimensional supersymmetric Yang-Mills theory*," Phys. Rev. D **92** (2015) 066001, [arXiv:1501.05562].

[79] F. Berends and W. Giele, "*Recursive calculations for processes with n gluons*," Nucl. Phys. B **306** (1988) 759–808.

[80] C. Reutenauer, *Free Lie Algebras*. London Mathematical Society Monographs, 1993.

[81] J. J. M. Carrasco, "*Gauge and Gravity Amplitude Relations*," Journeys Through Precis. Front. Amplitudes Colliders (2015) 477–557, [arXiv:1506.00974].

[82] R. Ree, "*Lie Elements and an Algebra Associated With Shuffles*," Ann. Math. **68** (1958) 210.

[83] C. R. Mafra and O. Schlotterer, "*Cohomology foundations of one-loop amplitudes in pure spinor superspace*," arXiv:1408.3605.

[84] C. R. Mafra and O. Schlotterer, "*Berends-Giele recursions and the BCJ duality in superspace and components*," JHEP **1603** (2016) 097, [arXiv:1510.08846].

[85] C. R. Mafra, O. Schlotterer, S. Stieberger and D. Tsimpis, "*A recursive method for SYM n-point tree amplitudes*," Phys. Rev. D **83** (2010) 126012, [arXiv:1012.3981].

[86] L. M. Garozzo, L. Queimada and O. Schlotterer, "*Berends-Giele currents in Bern-Carrasco-Johansson gauge for F3- and F4-deformed Yang-Mills amplitudes*," JHEP **1902** (2019) 078, [arXiv:1809.08103].

[87] J. Broedel and L. J. Dixon, "*Color-kinematics duality and double-copy construction for amplitudes from higher-dimension operators*," JHEP **1210** (2012) 091, [arXiv:1208.0876].

[88] N. Berkovits, "*Multiloop Amplitudes and Vanishing Theorems using the Pure Spinor Formalism for the Superstring*," JHEP **0409** (2004) 047, [hep-th/0406055].

[89] N. Berkovits, "*Super-Poincaré covariant two-loop superstring amplitudes*," JHEP **0601** (2006) 005, [hep-th/0503197].

[90] N. Berkovits and C. R. Mafra, "*Equivalence of Two-Loop Superstring Amplitudes in the Pure Spinor and Ramond-Neveu-Schwarz Formalisms*," Phys. Rev. Lett. **96** (2006) 011602, [hep-th/0509234].

[91] H. Gomez and C. R. Mafra, "*The overall coefficient of the two-loop superstring amplitude using pure spinors*," JHEP **1005** (2010) 017, [arXiv:1003.0678].

[92] H. Gomez and C. R. Mafra, "*The closed-string 3-loop amplitude and S-duality*," JHEP **1310** (2013) 217, [arXiv:1308.6567].

[93] H. Gomez, C. R. Mafra and O. Schlotterer, "*The two-loop superstring five-point amplitude and S-duality*," Phys. Rev. D **93** (2015) 045030, [arXiv:1504.02759].

[94] N. Berkovits, "*Pure spinor formalism as an N = 2 topological string*," JHEP **0510** (2005) 089, [hep-th/0509120].

[95] S. Stieberger and T. Taylor, "*Non-Abelian Born–Infeld action and type I–heterotic duality (II): Nonrenormalization theorems*," Nucl. Phys. B **648** (2003) 3–34, [hep-th/0209064].

[96] D. Friedan, E. Martinec and S. Shenker, "*Conformal invariance, supersymmetry and string theory*," Nucl. Phys. B **271** (1986) 93–165.

[97] V. A. Kostelecký, O. Lechtenfeld, W. Lerche, S. Samuel and S. Watamura, "*Conformal techniques, bosonization and tree-level string amplitudes*," Nucl. Phys. B **288** (1987) 173–232.

[98] J. Cohn, D. Friedan, Z. Qiu and S. Shenker, "*Covariant quantization of supersymmetric string theories: The spinor field of the Ramond-Neveu-Schwarz model*," Nucl. Phys. B **278** (1986) 577–600.

[99] E. Witten, "*Notes On Super Riemann Surfaces And Their Moduli*," arXiv:1209.2459.

[100] J. Polchinski, *String Theory, volume 2*. Cambridge University Press, 1998.

[101] I. Koh, W. Troost and A. Van Proeyen, "*Covariant higher spin vertex operators in the Ramond sector*," Nucl. Phys. B **292** (1987) 201–221.

[102] J. J. Atick and A. Sen, "*Covariant one-loop fermion emission amplitudes in closed string theories*," Nucl. Phys. B **293** (1987) 317–347.

[103] Z.-H. Lin, L. Clavelli and S. T. Jones, "*Five-point function in the covariant formulation of the type-I superstring theory*," Nucl. Phys. B **294** (1987) 83–92.

[104] Z.-H. Lin, "*One-loop closed string 5-particle fermion amplitudes in the covariant formulation*," Int. J. Mod. Phys. A **05** (1990) 299–327.

[105] A. Sen, "*Off-shell Amplitudes in Superstring Theory*," Fortschritte der Phys. **63** (2015) 149–188, [arXiv:1408.0571].

[106] E. Verlinde and H. Verlinde, "*Chiral bosonization, determinants and the string partition function*," Nucl. Phys. B **288** (1987) 357–396.

[107] E. D'Hoker and D. H. Phong, "*Conformal scalar fields and chiral splitting on super Riemann surfaces*," Commun. Math. Phys. **125** (1989) 469–513.

[108] J. J. Atick and A. Sen, "*Correlation functions of spin operators on a torus*," Nucl. Phys. B **286** (1987) 189–210.

[109] D. Härtl, O. Schlotterer and S. Stieberger, "*Higher point spin field correlators in superstring theory*," Nucl. Phys. B **834** (2010) 163–221, [arXiv:0911.5168].

[110] D. Härtl and O. Schlotterer, "*Higher loop spin field correlators in various dimensions*," Nucl. Phys. B **849** (2011) 364–409, [arXiv:1011.1249].

[111] V. Alan Kostelecký, O. Lechtenfeld, W. Lerche, S. Samuel and S. Watamura, "*Conformal techniques, bosonization and tree-level string amplitudes*," Nucl. Phys. B **288** (1987) 173–232.

[112] P. Jordan and E. Wigner, "*Über das Paulische Äquivalenzverbot*," Zeitschrift fr Phys. Phys. **47** (1928) 631–651.

[113] I. B. Frenkel and V. G. Kac, "*Basic representations of affine Lie algebras and dual resonance models*," Invent. Math. **62** (1980) 23–66.

[114] J. J. Atick and A. Sen, "*Spin field correlators on an arbitrary genus Riemann surface and non-renormalization theorems in string theories*," Phys. Lett. B **186** (1987) 339–346.

[115] L. Alvarez-Gaumé, J.-B. Bost, G. Moore, P. Nelson and C. Vafa, "*Bosonization on higher genus Riemann surfaces*," Commun. Math. Phys. **112** (1987) 503–552.

[116] L. Alvarez-Gaumé, G. Moore, P. Nelson, C. Vafa and J. Bost, "*Bosonization in arbitrary genus*," Phys. Lett. B **178** (1986) 41–47.

[117] L. Alvarez-Gaumé, G. Moore and C. Vafa, "*Theta functions, modular invariance, and strings*," Commun. Math. Phys. **106** (1986) 1–40.

[118] L. Clavelli, P. H. Cox, B. Harms and H. Konno, "*Bosonization of odd-spin-structure amplitudes*," Phys. Rev. D **43** (1991) 3998–4005.

[119] O. Schlotterer, "*Higher loop spin field correlators in D = 4 superstring theory*," JHEP **1009** (2010) 050, [arXiv:1001.3158].

[120] J. D. Fay, *Theta Functions on Riemann Surfaces*. Springer Berlin Heidelberg, 1973.

[121] B. Enriquez, "*Analogues elliptiques des nombres multizétas*," Bull. Soc. Math. Fr. **144** (2016) 395–427, [arXiv:1301.3042].

[122] J. Broedel, N. Matthes, G. Richter and O. Schlotterer, "*Twisted elliptic multiple zeta values and non-planar one-loop open-string amplitudes*," J. Phys. A Math. Theor. **51** (2018) 285401, [arXiv:1704.03449].

[123] A. G. Tsuchiya, "*On the pole structures of the disconnected part of hyper elliptic g loop M point super string amplitudes*," arXiv:1209.6117.

[124] A. G. Tsuchiya, "*On new theta identities of fermion correlation functions on genus g Riemann surfaces*," arXiv:1710.00206.

[125] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*. Cambridge University Press, 1996.

[126] M. Berg, I. Buchberger and O. Schlotterer, "*From maximal to minimal supersymmetry in string loop amplitudes*," JHEP **1704** (2017) 163, [arXiv:1603.05262].

[127] M. Bianchi and A. V. Santini, "*String predictions for near future colliders from one-loop scattering amplitudes around D-brane worlds*," JHEP **0612** (2006) 010, [hep-th/0607224].

[128] M. Bianchi and D. Consoli, "*Simplifying one-loop amplitudes in superstring theory*," JHEP **1601** (2016) 043, [arXiv:1508.00421].

[129] K. A. Roehrig and D. Skinner, "*A gluing operator for the ambitwistor string*," JHEP **1801** (2018) 069, [arXiv:1709.03262].

[130] E. D'Hoker and D. Phong, "*Two-loop superstrings VI Nonrenormalization theorems and the 4-point function*," Nucl. Phys. B **715** (2005) 3–90, [hep-th/0501197].

[131] R. Blumenhagen, B. Körs, D. Lüst and S. Stieberger, "*Four-dimensional string compactifications with D-branes, orientifolds and fluxes*," Phys. Rep. **445** (2007) 1–193, [hep-th/0610327].

[132] O. Schlotterer, "*Higher spin scattering in superstring theory*," Nucl. Phys. B **849** (2011) 433–460, [arXiv:1011.1235].

[133] T. Banks, L. J. Dixon, D. Friedan and E. Martinec, "*Phenomenology and conformal field theory or can string theory predict the weak mixing angle?*," Nucl. Phys. B **299** (1988) 613–626.

[134] T. Banks and L. J. Dixon, "*Constraints on string vacua with spacetime supersymmetry*," Nucl. Phys. B **307** (1988) 93–108.

[135] S. Ferrara, D. Lüst and S. Theisen, "*World-sheet versus spectrum symmetries in heterotic and type II superstrings*," Nucl. Phys. B **325** (1989) 501–525.

[136] N. Berkovits, "*Covariant quantization of the Green-Schwarz superstring in a Calabi-Yau background*," Nucl. Phys. B **431** (1994) 258–272, [hep-th/9404162].

[137] N. Berkovits, "*A New Description of the Superstring*," hep-th/9604123.

[138] N. Berkovits, C. Vafa and E. Witten, "*Conformal field theory of AdS background with Ramond-Ramond flux*," JHEP **9903** (1999) 018, [hep-th/9902098].

[139] N. Berkovits, "*Quantization of the superstring with manifest U(5) super-Poincaré invariance*," Phys. Lett. B **457** (1999) 94–100, [hep-th/9902099].

[140] N. Berkovits and B. Carlini Vallilo, "*One-loop N-point superstring amplitudes with manifest d=4 supersymmetry*," Nucl. Phys. B **624** (2002) 45–62, [hep-th/0110168].

[141] D. F. Lawden, *Elliptic Functions and Applications*. Springer New York, 1989.