Automated Cryptocurrency Portfolios: Portfolio Optimization, an Empirical Study

Master's Thesis submitted

to

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Abstract

This thesis revisits the portfolio optimization theory: the mathematical formulation of the problem, its derivations (risk minimization formulation) and assumptions, its limitations, as well as some improvements and extensions of the existing framework. The aim of the thesis is also to simulate and implement in Python: Markowitz (Global Minimum Variance, maximum Sharpe), Hierarchical Risk Parity and three simple portfolios: equally weighted, inverse volatility and inverse variance in the novel asset class of cryptocurrencies. The CRyptocurrency IndeX, CRIX, is used as benchmark. Portfolio optimization is computed using 120 days of daily historical data with portfolio rebalancing taking place every 7 days and 30 days. Portfolios are long-short fully invested with no leverage and improvements in the covariance matrix are applied by means of random matrix theory eigenvalues clipping.

Keywords: portfolio optimization, Markowitz, modern portfolio theory, hierarchical risk parity, random matrix theory, eigenvalue clipping, cryptocurrencies, CRIX

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List of Abbreviations

ADA	Cardano	ATOM	Cosmos
BCH	Bitcoin Cash	BNB	Binance Coin
BSV	Bitcoin SV	BTC	Bitcoin
CRO	Crypto.com Chain	EOS	EOS
ETH	Ethereum	LEO	LEO Token
LINK	Chainlink	LTC	Litecoin
NEO	NEO	TRX	TRON
USDC	USD Coin	USDT	Tether
XLM	Stellar	XMR	Monero
XRP	Ripple	XTZ	Tezos

CC	Cryptocurrency	CRIX	CRyptocurrency IndeX
RMT	Random Matrix Theory	ETF	Exchange Traded Fund
GMV	Global Minimum Variance	HRP	Hierarchical Risk Parity
MVO	Mean-Variance Optimization	CAPM	Capital Asset Pricing Model
CML	Capital Market Line	FOC	First Order Condition
VaR	Value at Risk	CVaR	Conditional Value at Risk

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1 Introduction

Portfolio optimization is an area of interest that has gained attention during the last decades. Financial services and products have grown in number and sophistication. Nowadays most trades are executed by a computer, retail investors have access to robo-advisors or even by themselves with their smartphones or a computer through an online broker like Interactive Brokers worldwide or Scalable Capital in Germany.

Technology catalyzed the emergence of new asset classes like Exchange Traded Funds (ETFs) and cryptocurrencies. According to etfgi.com, the assets of global ETFs grew from merely 203.4 billion U.S. dollars in 2003 to 6.181 trillion U.S. dollars in 2019. Through the innovations in Blockchain,the asset of cryptocurrencies was developed. As of, September 23^{rd} 2020, the total capitalization of the 5884 cryptocurrencies is of 334.8 billion U.S. dollars according to CoinGecko.

Harry Markowitz got powerful insight into the selection of an optimal portfolio for an investor's given risk aversion. In his paper "Portfolio Selection" published in the Journal of Finance in 1952, he considered that investors maximize expected returns and perceive variance as undesirable. He proposed the use of variance as the risk measure of an asset. He describes the feasible efficient surface where a set of efficient portfolios are gotten for different risk profiles along the efficient frontier. Along this efficient frontier we can find portfolios that maximize returns for each additional unit of risk (maximum Sharpe ratio portfolio) and the Global Minimum Variance (GMV) portfolio.

The idea behind the risk-reward trade-off is that of diversification. Exemplified in the classical problem of choosing to invest between two businesses, one which fares well during sunny days while the other during rainy days. Markowitz approached the concept of diversification through the variance of assets and covariances between them in a portfolio.

After the framework was introduced, acceleration in the field took place. Treynor, Sharpe, Lintner, and Tobin arrived to the CAPM after the introduction of a risk-free asset. Other considerations were introduced, like Pogue analyzing the effects on our objective function subject to transaction costs or fees. Cost impact literature flourished. Domowitz and Beardsley (2002) analyzed the liquidity cost and the dynamics between supply and demand comovements. Bikker et al. (2007) focused on market timing and market disruption for a big pension fund. Barber and Odean (2004) found that many individual investors could improve their after-tax performance.

The Markowitz framework misses to capture all the behaviour of data with his model, since only the variance and the mean are considered. Mandelbrot (1963) pointed out the fact that empirical price changes are too 'peaked' to be considered as Gaussian. Fama (1965) argued that empirical evidence strongly supports the random walk model. Davies et al. (2009) take into account the covariance, co-skewness, and co-kurtosis for a polynomial goal programming model; where the aim is as usual to maximize the return and minimize the variance, with the additional objectives to maximize skewness (since we are keen for positive returns), and minimize kurtosis (tail risk is undesired).

It is of great importance to have good data to work with. However, estimates of risks and returns in practice are noisy. Market conditions change through time and expected returns display significant time variation, i.e. not stationary. MVO is sensitive to its inputs, as small changes in expected return can strongly affect the weights. Jobson and Korkie (1981) argues that mean and variance are reliable predictors and that naive portfolios like equally weighted can outperform optimization. Litterman and Winkelmann (2000) proposes to use a weighted data, and the idea that the most recent information should be more important than long past observations. Ledoit and Wolf (2004) proposes a transformation to the covariance matrix call shrinkage. Since the matrix tend to contain a lot of positive error, the aim is to pull the extreme values downwards to more central values.

Until now, only a myopic view of the market is considered. The portfolios are optimized at one point and fail to get the whole picture and challenges affecting the construction of optimal portfolios. Under the premise that markets are non stationary and that they have dynamic and uncertainty (stochastic) components, a new class of models emerge. First, is intuitive that a finite market must implement feedback as it cannot grow forever. As it is a hard task to find the absolute feedback that governs the market, it is easier to analyze the data by the feedbacks that regulate the gains or losses of one asset with respect to another. Another dynamic effect is momentum, which is the permanence of uncertainty in the market, characterized by periods of low and high volatility. To include feedback and momentum, many models have been proposed and the most widely used are dynamic linear models. Autoregressive Moving Average (ARMA) and Vector Autoregressive Moving Average (VARMA) consider that the realizations are linear functions of previous values and a random disturbance. Engle (1982) further proposed a model where volatility is modeled by an autoregressive process and a regressive process where white-noise is scaled by the volatility, giving place to the Autoregressive Conditional Heteroskedasticity (ARCH), where the variance is allowed to be dynamic. It was extended and generalized by Bollerslev (1986)

In the next section, I'll get more into the derivation of the optimal portfolios within the Markowitz efficient frontier, another optimization algorithm will be discussed: the hierarchical risk parity, and also an introduction to random matrix theory and its application to 'clean' the covariance matrix from the noise, finalizing with some comments about the specific methodology followed in the thesis. Afterwards, the data of cryptocurrencies will be presented, as well as the performance of the models and the performance of the different strategies will be compared. Where we find optimization as good at realizing low out-of-sample volatility and better performance than the naive portfolios: equally weighted, inverse volatility, and inverse variance.

2 Theory

2.1 Markowitz Framework

Markowitz' classical portfolio optimization has applications in asset allocation, deciding how to split an investment between the asset classes, and in portfolio optimization, splitting an investment between securities. For our empirical study, the only asset class considered is that of cryptocurrencies. However, our specific goal is in portfolio optimization to find our vector w of optimal weights.

First, consider a vector $s = (s_1, ..., s_N)$ consisting of the assets from i to N. Let the expected return and the individual security risk be represented respectively by the mean, μ_i, and the standard deviation, σ_i, for the asset s_1. To be able to find the targeted efficient portfolios along the efficient frontier, it is worthy to formalize the problem and express it in mathematical terms. The inputs μ and Σ and our variable of interest w are expressed as the following in matricial form:

$$\Sigma = \begin{bmatrix} \sigma_{1,1} & \dots & \sigma_{1,N} \\ \vdots & \ddots & \vdots \\ \sigma_{N,1} & \dots & \sigma_{N,N} \end{bmatrix}, \mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_N \end{bmatrix}, w = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$$

where $\sigma_{i,j} = \rho_{i,j} * \sigma_i * \sigma_j$ for $i \neq j$ and $\rho_{i,j}$ is the correlation between securities i and j , such that $\sigma_{i,i} = \sigma_i^2$ and therefore $\rho_{i,i} = 1$.

Thus, the portfolio p expected return and variance are as follows

$$E\left[p\right] = \mu_1 w_1 +, ..., +\mu_N w_N = \mu^\top w$$

$$V\left[p\right] = \sum_i^N \sum_j^N \rho_{i,j} \sigma_i \sigma_j w_i w_j = w^\top \Sigma w$$

The weights vector, w, is constrained to be fully invested. It means that the addition of all the weights for each security will add up to one:

$$\sigma_{i=1}^N w_i = 1 = w^\top \mathbf{1}$$

The calculation of variance is always nonnegative, then the variance of the portfolio $w^\top \Sigma w \geq 0$ for any given w. We assume a positive definite Σ, there's no redundancies between assets s. Given that assumption, the variance is a strictly convex function of the variables in the portfolio, which leads to a unique solution. This problem can be expressed in 3 different ways:

- Risk minimization formulation - It solves for the portfolio with the least exposure to variance that satisfies at least the target return, μ_0, chosen by the investor.

$$\begin{aligned} \underset{w}{\text{minimize}} \quad & w^\top \Sigma w \\ \text{subject to} \quad & w^\top \mu \geq \mu_0, \\ & Ax = b, \\ & Cx \geq d \end{aligned} \tag{1}$$

- Expected return maximization formulation - alternatively, the problem would be to find the portfolio with the highest expected return for a given maximum level of variance, σ_0^2, willing to take.

$$\begin{aligned} \underset{w}{\text{maximize}} \quad & \mu^\top w \\ \text{subject to} \quad & w^\top \Sigma w \leq \sigma_0^2, \\ & Ax = b, \\ & Cx \geq d \end{aligned} \tag{2}$$

- Risk aversion formulation - an explicit model for the trade-off between risk and return. The objective is to maximize the expected return, penalized by the variance of the portfolio for a determined risk-aversion coefficient λ. For a small λ, the objective function is slightly penalized; leading to riskier portfolios. The opposite happens for big values of λ.

$$\begin{aligned} \underset{w}{\text{maximize}} \quad & \mathrm{w}^\top \mu - \lambda w^\top \Sigma w \\ \text{subject to} \quad & w^\top \mathbf{1} = 1, \\ & Ax = b, \\ & Cx \geq d \end{aligned} \tag{3}$$

Since Σ is positive definite, it is possible to solve the problem through the Lagrange multiplier method. we have a $\lambda = (\lambda_1, ..., \lambda_{c-1})$, where c is the number of constraints and can have a maximum value of *degrees of freedom - 1* in order to solve the linear equations from the First Order Conditions (FOCs).

Quadratic programming is a class of nonlinear optimization problems and deals with the problem of minimizing a quadratic function, like the variance of the portfolio, subject to linear equality and inequality constraints. Only the expected return maximization formulation doesn't lead to quadratic programming, since it has a convex quadratic constraint. Since Σ is positive semidefinite , the other two models stated are convex problems, which means that the local solution is, the global solution as well. The objective function is a convex function of w.

We will encounter further what happens to the optimal portfolio once a r_f asset is introduced. Other important consideration to find the optimal portfolio construction are transaction costs. Transactions costs impact our strategy and we should consider it in the utility function since the investor should be sensitive to costs and try to avoid them. Costs can be explicit like: fees, commisions, bid-ask spreads, or taxes; while the implicit costs are delay cost, price movement risk, market impact costs, timing risk, and opportunity cost. Pogue (1970) was aware of these implications and included another λ term accounting for the transaction costs in the risk aversion formulation.

Since the MVO framework for optimization is myopic, in practice we have other considerations like rebalancing. Rebalancing is needed since the real strategy is dynamic and the optimal weights should change given the new realized returns. Rebalancing can normally be done by: calendar (setting the frequency), threshold (setting a deviation threshold from the optimal weight), or range (setting the asset allocation target mix and a tolerance from the desired allocation).

Another important concept to introduce is the tracking error. It is the divergence of a portfolio with some benchmark. It helps to find how the out-of-sample portfolio is doing relative to the benchmark. At any given point in the future, after the optimal portfolio has been calculated, the new information embedded in the prices will cause the weights of the portfolio to deviate from that of the previously computed optimum. This will be taken in consideration for the rebalancing step, since one has to choose the trade-off of incurring into high costs and very accurate trackability of the benchmark, or lax tracking and fewer rebalances, i.e. fewer costs.

Minimum Variance Optimization

According to the risk minimization problem from equation (1) we formulate the Lagrangian in the following way, where 1 is a vector with size N filled with ones.

$$\begin{aligned} \underset{w}{\text{minimize}} \quad & \mathrm{w}^\top \Sigma w \\ \text{subject to} \quad & w^\top \mu = \mu_0, \\ & w^\top \mathbf{1} = 1 \end{aligned} \tag{4}$$

it constraints the maximization with a target expected return μ_0 and that all weights add up to one

$$L(w, \lambda_1, \lambda_2) = w^\top \Sigma w - \lambda_1 (w^\top \mu - \mu_0) - \lambda_2 (w^\top \mathbf{1} - 1) \tag{5}$$

Since $\frac{\partial x^\top B x}{\partial x} = (B + B^\top)x$ and $\frac{\partial x^\top a}{\partial x} = \frac{\partial a^\top x}{\partial x} = a$, we take the partial derivatives of the variables and equal them to zero, to get either the maximum or minimum. Due to the fact that Σ is a symmetric matrix, then $(B + B^\top) = 2B$. We proceed by taking the partial derivatives of the Lagrangian in equation (5) with respect to our variables w, λ_1, and λ_2.

$$\frac{\partial L}{\partial w} = 2\Sigma w - \lambda_1 \mu - \lambda_2 \mathbf{1} = 0 \tag{6a}$$

$$\frac{\partial L}{\partial \lambda_1} = -w^\top \mu + \mu_0 = 0 \tag{6b}$$

$$\frac{\partial L}{\partial \lambda_2} = -w^\top \mathbf{1} + 1 = 0 \tag{6c}$$

Now we have a system of three equations with three incognitos. We solve for w in equation (6a)

$$w = \frac{1}{2}\lambda_1 \Sigma^{-1} \mu + \frac{1}{2}\lambda_2 \Sigma^{-1} \mathbf{1} \tag{7}$$

and plug in (6b) and (6c)

$$\begin{aligned} \frac{1}{2}\lambda_1 \mu^\top \Sigma^{-1} \mu + \frac{1}{2}\lambda_2 \mu^\top \Sigma^{-1} \mathbf{1} = \mu_0 \\ \frac{1}{2}\lambda_1 \mathbf{1}^\top \Sigma^{-1} \mu + \frac{1}{2}\lambda_2 \mathbf{1}^\top \Sigma^{-1} \mathbf{1} = 1 \end{aligned}$$

we represent some constants with a, b, and c in a more convenient way to factorize the solution

$$\frac{1}{2}c\lambda_1 + \frac{1}{2}b\lambda_2 = \mu_0$$
$$\frac{1}{2}b\lambda_1 + \frac{1}{2}a\lambda_2 = 1$$

$$a = \mathbf{1}^\top \Sigma^{-1} \mathbf{1} \tag{8a}$$

$$b = \mathbf{1}^\top \Sigma^{-1} \mu = \mu^\top \Sigma^{-1} \mathbf{1} \tag{8b}$$

$$c = \mu^\top \Sigma^{-1} \mu \tag{8c}$$

and solve the system of equation to find λ_1 and λ_2:

$$\lambda_1 = \frac{2(a\mu_0 - b)}{(ac - b^2)} \qquad \lambda_2 = \frac{2(c - b\mu_0)}{(ac - b^2)} \tag{9}$$

then substitute λ in (7) and factorize as much as possible to express the optimal weights such that a target return is met:

$$w = \frac{[\Sigma^{-1}(a\mu - b\mathbf{1})]\mu_0 + [\Sigma^{-1}(c\mathbf{1} - b\mu)]}{ac - b^2} \tag{10}$$

Global Minimum Variance Optimal Weights

To determine the weights of the portfolio with the minimum variance similarly through the risk minimization formulation expressed in equation (1) and with Lagrange multipliers. The constraint sum of weights equal to one holds as well and we build the following Lagrangian.

$$L(w, \lambda_1) = w^\top \Sigma w - \lambda_1(w^\top \mathbf{1} - 1) \tag{11}$$

partial derivatives need to be taken to get the FOCs = 0

$$\frac{\partial L}{\partial w} = 2\Sigma w - \lambda \mathbf{1} = 0 \tag{12a}$$

$$\frac{\partial L}{\partial \lambda} = -(w^\top \mathbf{1} - 1) = 0 \tag{12b}$$

solving for w in equation (12a)

$$w = \frac{1}{2}\lambda_1 \Sigma^{-1}\mathbf{1} \tag{13}$$

plug in equation (12b) and compute the Lagrange multiplier

$$\lambda = 2(\mathbf{1}^\top \Sigma^{-1}\mathbf{1}) \tag{14}$$

when substituted in (13), we finally arrive to the optimal weights for the Global Minimum Variance portfolio:

$$w^{GMV} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^\top \Sigma^{-1}\mathbf{1}} \tag{15}$$

Maximum Sharpe Optimal Weights

Tobin (1958) wondered why would an investor hold cash instead of an interest bearing government debt. Other scientists like Jack Treynor, William Sharpe, and John Lintner followed and formulated the capital asset pricing model. Sharpe (1964) argued that a rational investor could achieve any wanted point along the capital market line (CML). In a plot between risk and return, represented in x and y axis respectively. This line was characterized with its y-intercept in the risk-free rate and a slope of additional expected return per unit of risk.

The CML improved the efficient frontier, after a risk-free asset is considered, CML finds a portfolio more optimal than the minimum variance from the efficient frontier at such level of risk σ_p. Such portfolio, like all portfolios along the CML, are the combination of the risk-free asset and a risky portfolio, called tangent portfolio owing to the fact that it is the line which starts from the r_f rate and is tangent to the efficient frontier. Fama (1970) followed the 'market model' of Markowitz and showed that the market portfolio was equivalent to the tangent portfolio Sharpe-Lintner expected return model, under certain assumptions.

Motivated by the Treynor Index (by Jack Treynor), Sharpe (1966) introduced a measure for mutual funds' performance where it tries to find how much excess return is generated for a given measure of risk. Sharpe used the standard deviation σ instead of β from the CAPM. w denotes the weights for the risky securities.

Sharpe Ratio expressed in matrices and vectors when considering a risk-free asset:

$$SharpeRatio = \frac{w^\top \mu - r_f}{\sqrt{w^\top \Sigma w}}$$

The problem to find the optimal portfolio that maximizes this ratio is the following

$$\begin{aligned} \underset{w}{\text{maximize}} \quad & \frac{w^\top \mu - r_f}{\sqrt{w^\top \Sigma w}} \\ \text{subject to} \quad & w^\top \mathbf{1} = 1 \end{aligned} \tag{16}$$

Although it has a polyhedral feasible surface, the objective function is complicated and possibly not concave leading to a non-convex optimization problem. Under the assumptions that $w^\top \mu - r_f > 0$, we can reduce the problem into a convex quadratic setting and reformulate it into the risk minimization form. Where we want to minimize the variance of the portfolio, but with the condition that the expected portfolio excess return μ_p is equal to the target portfolio excess return $\mu_{0p} = \mu_0 - r_f$. With the consideration of the risk-free asset

$$\mu_p = w^\top (\mu - r_f \mathbf{1}) = w^\top \mu - r_f$$

$$\begin{aligned} \underset{w}{\text{minimize}} \quad & w^\top \Sigma w \\ \text{subject to} \quad & \mu_p = \mu_{0p} \end{aligned} \tag{17}$$

Let $\mu_p = \mu_{0p} \therefore w^\top \mu - r_f = \mu_0 - r_f \therefore w^\top \mu = \mu_0$ and we construct the Lagrangian and proceed similarly with the partial derivatives equal to zero. Also assuming that the risk-free has zero variance and is uncorrelated with the assets.

$$L(w, \lambda) = w^\top \Sigma w - \lambda (w^\top \mu - \mu_0) \tag{18}$$

$$\frac{\partial L}{\partial w} = 2\Sigma w - \lambda \mu = 0 \tag{19a}$$

$$\frac{\partial L}{\partial \lambda} = -(w^\top \mathbf{1} - \mu_0) = 0 \tag{19b}$$

Solving for w leads to

$$w = \frac{\lambda \Sigma^{-1} \mu}{2} \tag{20}$$

substituting in eq. (19b), gives us

$$\lambda = \frac{2\mu_0}{\mu^\top \Sigma^{-1} \mu} \tag{21}$$

then the *lambda* is plugged in equation (20) and we find

$$w = \frac{\mu_0 \Sigma^{-1} \mu}{\mu^\top \Sigma^{-1} \mu} \tag{22}$$

we further assume that $w^\top \mathbf{1} = 1$ and solve for μ_0

$$\mathbf{1}^\top w = \mu_0 \frac{\mathbf{1} \Sigma^{-1} \mu}{\mu^\top \Sigma^{-1} \mu} = 1 \tag{23}$$

$$\mu_0 = \frac{\mu^\top \Sigma^{-1} \mu}{\mathbf{1} \Sigma^{-1} \mu} \tag{24}$$

we substitute again in eq. (22) and finally arrive to the optimal weight

$$w^{sharpe} = \frac{\Sigma^{-1} \mu}{\mathbf{1}^\top \Sigma^{-1} \mu} \tag{25}$$

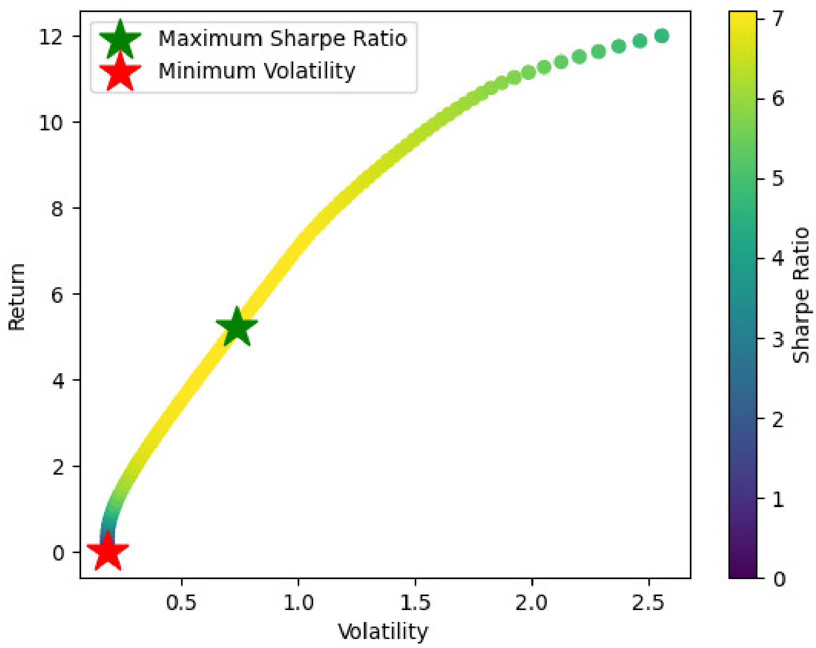

Figure 1: Efficient Frontier of the first time window analyzed, from September 14, 2017 until January 11, 2018. The GMV and maximum Sharpe can be observed

As mentioned in the introduction, MVO faces some challenges that makes us question its effectiveness, like any other method, and Best and Grauer (1992) showed that portfolio composition is extremely sensitive to the changes in expected asset returns. On the other hand, there is uncertainty in the calculations of the expected returns and in practice it's not prude to rely on the estimations and treat them as error-free. Given that, the investor could use some shrinkage and Bayesian estimators. Ben-Tal and Nemirovski (1998) introduced the robust estimation framework. This is applied to convex optimization problems where data is not specified exactly and it is known that it belongs to a given uncertainty set, constraints must hold for all possible values in the uncertainty set. They focused on the uncertainty in the constraints, as opposed to some of the literature dealing with uncertainty in the objective function.

2.2 Simple Approach Portfolios

To challenge the classical Markowitz Framework, we ponder of whether portfolio optimization is needed and if other more simplistic approaches should be used. I also want to consider three simple portfolios, being: the equally weighted ($1/N$) portfolio, the inverse volatility, and the inverse variance portfolios.

We define the following portfolio weight calculations:

- Equally Weighted Portfolio

$$w_i^{EW} = \frac{1}{N}$$

Then the matricial form is as follows:

$$w^{EW} = \frac{\mathbf{1}}{N} \tag{26}$$

with a vector 1 size N

- Inverse Volatility σ_i

$$w_i^{IV} = \frac{\frac{1}{\sigma_i}}{\sum_1^N \frac{1}{\sigma_i}}$$

Let diagonal $\text{diag}[\Sigma] = [\sigma_{i,i}, ..., \sigma_{N,N}]$ and trace $\text{tr}[\Sigma] = \sum_{i,i}^N \Sigma$, hence: $\text{tr}[\Sigma] = \sum_i^N \text{diag}[\Sigma]$. That allows us to formulate in matrices and vectors as

$$w_{IV=\frac{\text{diag}[\Sigma^{1/2}]}{\text{tr}[\text{diag}[\Sigma^{1/2}]]}} \tag{27}$$

- Inverse Variance

$$w_i^{IVar} = \frac{\frac{1}{\sigma_i^2}}{\sum_1^N \frac{1}{\sigma_i^2}}$$

Similar to the inverse volatilty, with the exception that our σ is squared, we obtain

$$w_{IVar=\frac{\text{diag}[\Sigma]}{\text{tr}[\text{diag}[\Sigma]]}} \tag{28}$$

2.3 Hierarchical Risk Parity

López de Prado (2016) proposes another way to calculate the weights with the appliance of graph theory and machine learning.

STAGE 1: Tree Clustering

From the previous empirical variance matrix Σ, we can compute the correlation matrix ρ

$$\rho_{i,j} = \begin{bmatrix} \rho_{1,1} & \cdots & \rho_{1,N} \\ \vdots & \ddots & \vdots \\ \rho_{N,1} & \cdots & \rho_{N,N} \end{bmatrix}$$

such that all elements in the matrix are correlations between asset i and j and the diagonal of matrix ρ is filled with ones, meaning the $\rho_{i,i} = 1$. We further take ρ and calculate the distance matrix d with diagonal elements $d_{i,i} = 0$).

$$d = \sqrt{\frac{1}{2}(1 - \rho_{i,j})} \tag{29}$$

$$\text{i,j} = \begin{bmatrix} 1,1 & ... & 1,N \\ \vdots & \ddots & \vdots \\ N,1 & ... & N,N \end{bmatrix}$$

After the distances between the columns of the ρ are computed, we compute another distance, defined by the Euclidean distances between the columns in the matrix d. Other distances could be included like the Manhattan, maximum, or Mahalanobis. For the purpose of the thesis, I use the Euclidean distance

$$D[d_i, d_j] = \sqrt{\sum_{1}^{N}(d_{n,i} - d_{n,j})^2)} \tag{30}$$

denoted by d_i and d_j to the distance column of asset i and j from matrix d respectively.

In order to cluster the columns of assets in similar risk profiles, we need to choose a linkage criterion, which is the distance between a newly formed cluster and the other elements. There exists complete-linkage, unweighted average linkage, weighted average linkage, but here I consider only to implement the single-linkage clustering: $D_{i,u} = \arg\min_{i,j}(d_{i,j})$, where $D_{i,u}$ is the distance between a column in matrix d and the cluster u. The cluster appends the nearest point and dropped the columns and rows for the appended asset,in the set $i, j, ..., N$. It is done recursively, until all assets are clustered: until the $N - 1^{th}$ iteration.

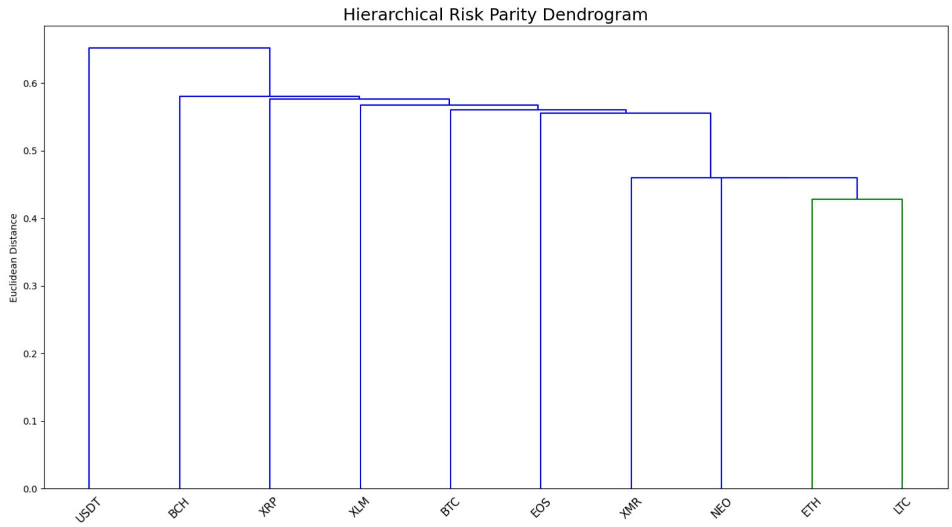

Figure 2: Dendogram of the first time window analyzed, from September 14, 2017 until January 11, 2018. It denotes the hierarchical structure of the assets and shows different colors for the two clusters found for the CCs

STAGE 2: Quasi-Diagonalization Matrix seriation is the procedure, which helps re-arrange data and shows clusters where similar investments are placed together. The large variances are along the diagonal, surrounded by the other smaller variances. Since the smaller variances off the diagonal are not completely zero, hence the name quasi-diagonalization.

STAGE 3: Recursive Bisection Finally, bisection is performed top-down between the clusters identified through the clustering tree and each cluster is getting a weight with respect to its inverse variance, as shown in eq. (28). López de Prado (2016) proves that it is an optimal solution for the variance minimization when the covariance matrix is diagonal, i.e. all off-diagonal elements are equivalent to zero.

2.4 Random Matrix Theory and Eigenvalue Clipping

The empirical determination of a correlation matrix C results in a complicated task. For a set of N different assets, each representing a time series of length N, it is expected that C is noisy and somehow reminiscent of a random variable. Using this correlation matrix in practice, one should wonder if it dominated by measurement risk. Small eigenvalues in the matrix are the most responsive to noise and happen to be the ones that determine the least risky portfolios. Hence, the notion that the correlation matrix carries real information that we need to take into account.

To find a way to reduce the noise in the correlation matrix, one should be able to discern between random noise and information. Laloux et al. (2000) formulated a method to do this. They based on the premise that returns are independent, identically distributed random variables.

The density function $\rho_C(\lambda) = \frac{\partial n(\lambda)}{N \partial \lambda}$ of the eigenvalues of a random matrix was already studied by Marchenko-Pastur. They found the theoretical asymptotic of the eigenvalue distribution of such matrices. Given $N \to \infty$, $T \to \infty$, and $Q = T/NT \geq 1$:

$$\rho_C(\lambda) = \frac{Q}{2\pi\sigma^2} \frac{\sqrt{(\lambda_{max} - \lambda)(\lambda - \lambda_{min})}}{\lambda} \tag{31a}$$

$$\lambda_{min}^{max} = \sigma^2(1 + 1/Q \pm 2\sqrt{1/Q}) \tag{31b}$$

Laloux et al. (2000) proposed the method of eigenvalue clipping, where the eigenvalues that have a higher value than that of the theoretical distribution in eq. (31a and 31b) are deemed as carrying valuable information and the eigenvalues below the Marchenko-Pastur edge are discarded.

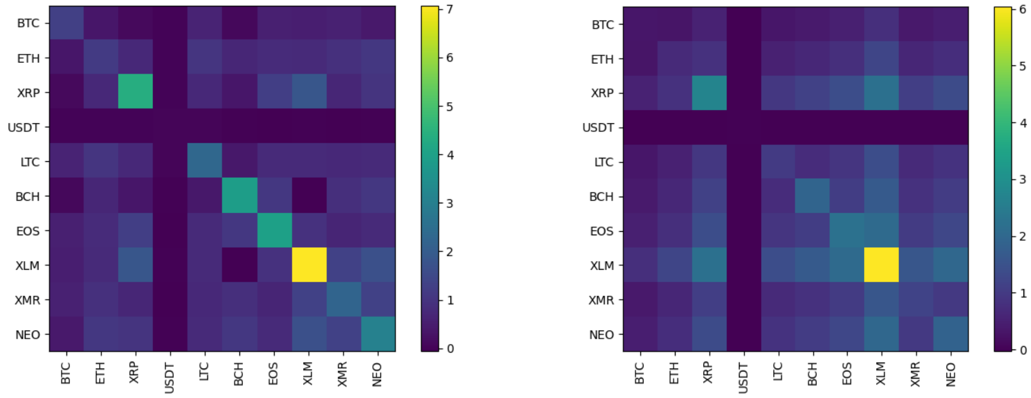

Figure 3: Heatmap of the Σ before (left) and after (right) the eigenvalue clipping RMT method from the first time window analyzed, from September 14, 2017 until January 11, 2018.

2.5 Method

For the purpose of constructing and comparing the portfolios, I used the unconstrained minimum variance and unconstrained maximum Sharpe in order to find the optimal values. Imposing constraints would lead to sub-optimal portfolios and although short selling might be complicated in practice, I did not want to restrict the optimization. Since the maximum Sharpe ratio portfolio depends on the forecasts of the expected returns and some of the analyzed 120-days time windows had negative expected return, the allocation couldn't be done through this optimization method. I used the r_f of zero, since it simplifies the problem and we are in all-time low levels of interest rates in the developed economies. However, the codes in the model have consideration for the inclusion of a risk-free asset

When setting the quadratic programming problem, given the assumption that the portfolio weights add up to one, the unconstrained portfolios could assign weights that would give place to leverage, i.e. more than 100% exposure. To avoid comparing portfolios with no leverage limit, I scaled back the weights of the portfolios by dividing all the weights by the sum of the weights' absolute values in the following way

$$w_i^{UL} = \frac{w_i}{\sum_1^N |w_i|} = \frac{w}{|w|^\top \mathbf{1}} \tag{32}$$

The Markowitz models used were computed by the solutions derived through the Lagrange multipliers, although the quadratic programming versions of the GMV and maximum

Sharpe are also provided. Besides, another optimization algorithm was applied: the HRP. To this three aforementioned portfolios, I also computed them with the variant of having 'cleaned' covariance by eigenvalue clipping.

All the models consider perfect market assumptions, where the laws of one price hold. I just consider one price per day and there's no bid-ask spread. Furthermore, for matter of simplification, I didn't use any of the mentioned costs. I inclined for a calendar rebalancing of 7 and 30 days, to find out which is more suitable. Although in practice could be costly to rebalance the positions of the portfolio with weekly frequency. Transaction costs should be accounted for in real applications.

The codes are available in: https://github.com/morishig/Automated-Cryptocurrency-Portfolios.git

3 Data

The data consists of the daily prices of the 20 cryptocurrencies with the largest market capitalization in CoinGecko as of August 28th 2020. The daily prices corresponds to the period from September 14th 2017 until June 2nd 2020. The data is sub-sampled every 7 days and considers a rolling window of 120 days historic past data.

Our asset universe consists of: ADA (Cardano), ATOM (Cosmos), BCH (Bitcoin Cash), BNB (Binance Coin), BSV (Bitcoin SV), BTC (Bitcoin), CRO (Crypto.com Chain), EOS (EOS), ETH (Ethereum), LEO (LEO Token), LINK (Chainlink), LTC (Litecoin), NEO (NEO), TRX (TRON), USDC (USD Coin), USDT (Tether), XLM (Stellar), XMR (Monero), XRP (Ripple), and XTZ (Tezos). Additionally to the 20 time series, the portfolios created from the cryptocurrencies price data are compared to the CRyptocurrency IndeX (CRIX)[1], developed by Härdle and Trimborn (2015).

We can see more information of the data itself and its distribution in the following Table (1). What we can first notice from the data provided is that the N of the data are different. This is because the method of choosing the CCs with the highest capitalization *ex-post* the portfolio starting date. This will bring in bias. It is called survivorship bias, which in essence means, that some of the cryptocurrencies in the top 20 by market capitalization when the portfolio started (January 11, 2018. 120 days after the first data point of the time series) are not in the top 20 today. In this case, the sample is upwardly biased. In the case of CCs, it is complicated in practice to find historic data and select the constituents in this completely new asset class.

It is striking that there are some extremely high levels of excess kurtosis, this is characterized by the fact that the period analyzed has been a boom and bust cycle of the CCs. The data begins as Bitcoin was building up its way to its highest all-time value around December 2017 and then crashing. The median an mean are close to zero as expected, but the minimum and maximum values can tell us about the very irrational movement and volatility of this risky asset class.

[1] see the methodology in appendix A

	N	Min	$Q_{0.1}$	$Q_{0.25}$	Median	Mean	$Q_{0.75}$	$Q_{0.9}$	Max	Volatility	Variance	Skewness	Kurtosis
BNB	990	-100,244	-6,2193	-2,6872	0,0303	0,5133	3,2227	7,4865	326,994	14,5261	211,007	11,0629	263,817
LINK	936	-66,083	-7,5092	-3,8149	-0,0750	0,3174	3,7373	8,6966	47,607	7,8163	61,095	0,0824	9,459
CRO	517	-51,028	-5,3233	-2,4172	-0,0848	0,2886	2,1521	5,9626	80,128	7,3809	54,478	2,7973	36,593
TRX	936	-55,440	-7,2896	-2,9588	-0,0399	0,2052	2,9510	7,5922	79,908	8,6118	74,163	2,3092	23,721
XLM	992	-43,936	-6,7194	-2,8535	-0,1321	0,1986	2,7518	7,1728	67,142	7,1867	51,649	1,3221	13,041
BSV	570	-64,311	-6,1510	-2,8462	-0,0687	0,1791	2,1824	6,8924	88,660	9,6760	93,626	1,3273	23,850
EOS	992	-48,871	-7,0152	-2,6791	-0,0087	0,1510	2,6050	7,4761	36,889	7,1740	51,466	0,3106	6,153
ADA	958	-52,440	-7,0973	-2,9881	0,0447	0,1150	2,7182	6,9227	87,216	7,5207	56,562	2,2871	28,314
BTC	992	-43,371	-4,4506	-1,6253	0,1264	0,1133	1,9074	4,6836	28,710	4,4306	19,630	-0,7735	12,520
CRIX	992	-44,664	-4,6592	-1,6463	0,1443	0,0970	2,1040	5,0030	19,854	4,4595	19,887	-1,3149	12,746
LEO	378	-7,410	-2,3393	-1,1264	0,0921	0,0296	0,9416	2,2805	12,159	2,4636	6,069	0,6336	4,361
XRP	992	-42,040	-5,7424	-2,2673	-0,0775	0,0215	1,9720	5,1758	59,307	6,2576	39,158	1,4665	18,396
ETH	992	-56,308	-5,6116	-2,1253	-0,0521	0,0104	2,4046	5,7062	26,258	5,4009	29,170	-1,1864	13,555
LTC	992	-47,138	-6,0255	-2,7659	-0,1572	0,0091	2,6902	5,7959	38,430	5,8292	33,979	0,2702	9,496
ATOM	464	-62,069	-7,3278	-3,2237	-0,1731	0,0080	3,5246	7,3961	50,921	8,7846	77,169	-0,2154	12,003
USDT	992	-28,334	-0,4447	-0,1682	0,0045	0,0000	0,1545	0,3905	12,654	1,4485	2,098	-6,3268	168,609
USDC	606	-2,096	-0,4466	-0,1777	0,0162	-0,0011	0,1898	0,3630	2,537	0,4349	0,189	0,1355	6,864
XTZ	700	-62,542	-6,7529	-2,8299	-0,0174	-0,0020	2,9055	7,2431	27,488	6,5824	43,328	-1,1501	13,913
NEO	992	-50,455	-7,1292	-3,2593	0,0060	-0,0190	2,9749	7,2554	35,077	6,5936	43,475	-0,1110	6,454
XMR	992	-51,200	-6,3281	-2,5084	-0,0067	-0,0221	2,8214	6,0689	27,987	5,7912	33,539	-0,7418	8,452
BCH	992	-57,987	-6,4885	-3,1077	-0,2987	-0,0381	2,7417	6,8945	42,188	7,1750	51,481	0,0339	9,209

Table 1: Descriptive statistics of logarithmic returns for the period from September 14, 2017 to June 2, 2020. See the formulas in the appendix B for more details.

Moreover, we can see in the following plots, how each cryptocurrency performed during the studied period. I consider the case of having N portfolios, fully invested in each.

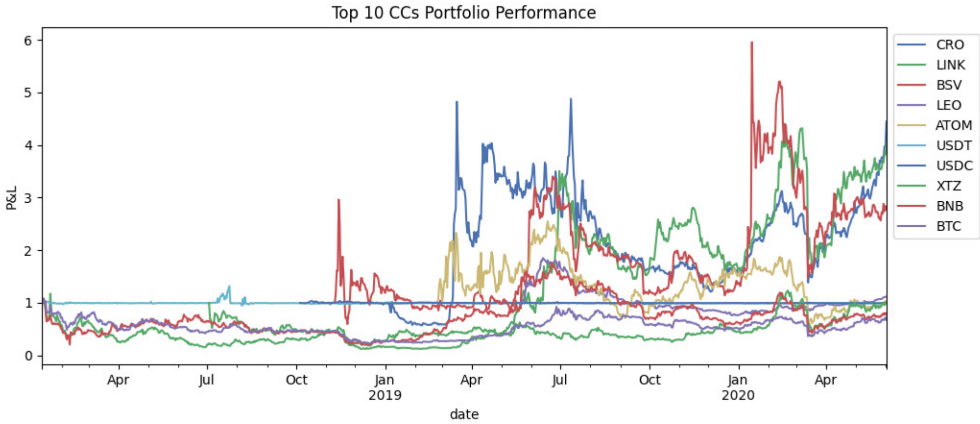

Figure 4: Profit & Loss of the top 10 performing CCs from an initial investment of 1€ for the period from January 11, 2018 until June 02, 2020

Confirming the fact of the bias induced by the selection process, most of the cryptocurrencies which didn't have prices on the day that the portfolio started made it into the top 10 CCs. In fact, 7 of the 20 CCs that didn't have any price information on the portfolio start date made it to the 'top 10' performing CCs, as seen in Fig. (4).

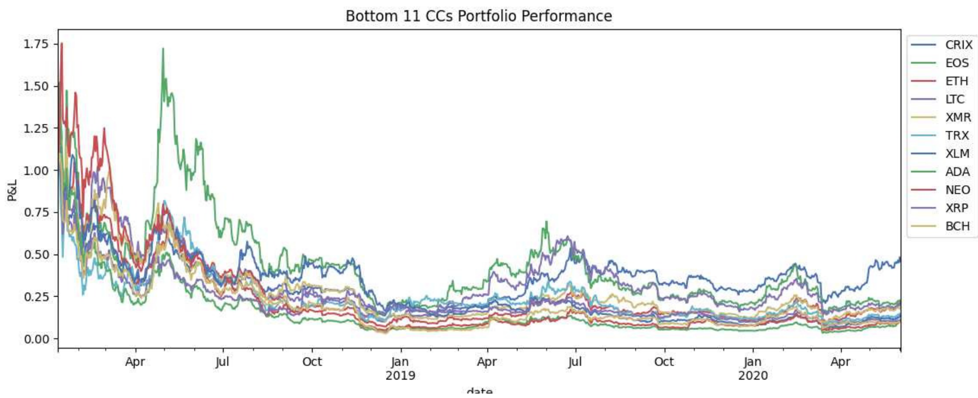

Figure 5: Profit & Loss of the bottom 11 performing CCs from an initial investment of 1€ for the period from January 11, 2018 until June 02, 2020

We could use CRIX as a proxy of the market and we can appreciate from Fig (5) that a calmer period reigned the year 2019, after prices crashed violently during Q1 2018. The moment when the portfolios begin was characterized by bear signals and relatively still high price levels.

4 Results

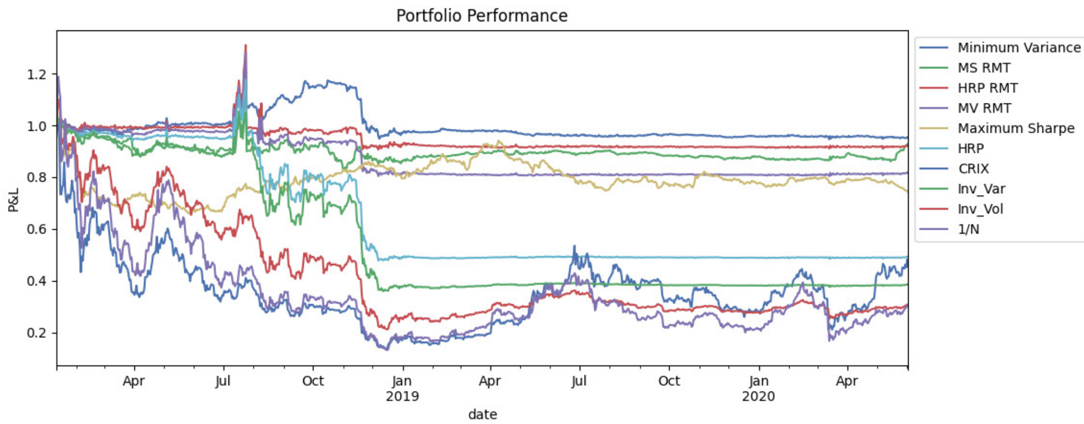

Figure 6: Profit & Loss of the optimization strategies, the naive strategies and the benchmark from an initial investment of 1€ for the period from January 11, 2018 until June 02, 2020. RMT stands for Random Matrix Theory and implies the eigenvalue clipping of the covariance matrix. MV denotes minimum variance and MS maximum Sharpe respectively.

The first thing to notice is that none of the portfolios realized a positive return. Second, all the optimized portfolios performed better than CRIX and the HRP did just slightly better than the CRIX. On the other hand, CRIX outperformed the simple naive portfolios. All the naive portfolios performed quite poorly, i.e. none of he naive could beat the benchmark, opposite to the optimized. It can be appreciated that the inverse variance portfolio and the HRP strategy almost go in tandem from the beginning of year 2019, this is due to that the HRP portfolio relies on inverse variance portfolios within the different hierarchies identified in the cluster. The GMV portfolio had an anomaly good Q4 in the year 2018, where it capitalized some gains before the market stabilized and most strategies plateaued.

I don't believe that the risk measures VaR and CVaR are appropriate, because their computations assume normality of the data, and as we can see through the higher moments of skewness and excess kurtosis they are not normal. However, they still gives us a good idea of the distribution of the returns for the given portfolio.

Incorporating the monthly portfolio, was interesting since some of the portfolios had

	N	V_F	CVaR	VaR	$Q_{0.05}$	Median	Mean	Vola	Skw.	Kur.	Sh.R.
MV 30	873	1,0565	-3,4165	-2,9829	-0,6795	0,0045	0,0063	1,2795	-5,9239	151,02	0,0940
MV	873	0,9511	-2,1041	-1,8358	-0,7555	-0,0018	-0,0057	0,7916	-3,1665	63,71	-0,1387
MS RMT	873	0,9293	-3,2012	-2,7931	-1,0935	-0,0087	-0,0084	1,2042	-8,2082	183,18	-0,1333
HRP RMT	873	0,9186	-3,9107	-3,4122	-0,6745	0,0044	-0,0097	1,4709	-6,8555	170,35	-0,1263
HRP RMT 30	873	0,8693	-3,9894	-3,4801	-0,7479	-0,0015	-0,0160	1,5029	-6,3435	163,21	-0,2039
MV RMT 30	873	0,8241	-4,0058	-3,4937	-0,8652	-0,0024	-0,0222	1,5113	-6,0338	154,83	-0,2801
MV RMT	873	0,8148	-3,7249	-3,2483	-0,7942	-0,0032	-0,0235	1,4064	-7,8036	180,81	-0,3187
MS RMT 30	873	0,8064	-1,6228	-1,4133	-1,0030	-0,0027	-0,0246	0,6181	-0,8529	13,83	-0,7619
MS	873	0,7400	-2,9315	-2,5544	-1,2238	-0,0027	-0,0345	1,1129	-3,5042	53,25	-0,5921
MS 30	873	0,7026	-3,3939	-2,9573	-1,3412	-0,0093	-0,0404	1,2886	-3,0767	45,20	-0,5994
HRP 30	873	0,5497	-5,0279	-4,3799	-1,3898	0,0023	-0,0685	1,9122	-3,7889	61,95	-0,6848
HRP	873	0,4906	-4,0781	-3,5492	-1,4186	0,0041	-0,0816	1,5607	-3,5839	44,34	-0,9986
CRIX	873	0,4567	-11,6236	-10,1343	-6,5601	0,0602	-0,0898	4,3949	-1,5817	14,56	-0,3902
Inv Var	873	0,3842	-4,4674	-3,8855	-1,8895	0,0048	-0,1096	1,7173	-2,1366	22,03	-1,2191
Inv Vol	873	0,3076	-6,6836	-5,8166	-4,6070	0,0501	-0,1351	2,5584	-1,1864	6,89	-1,0085
1/N	873	0,3067	-11,8365	-10,3143	-7,4248	0,1240	-0,1354	4,4919	-1,6376	14,10	-0,5759

Table 2: Some descriptive statistics and risk measures of the logarithmic returns for the portfolios. The number 30 indicates that the portfolio was considering monthly rebalancing. Normal distribution and a confidence interval of 99% are assumed. See the formulas in the appendix C

positive effect while the other half had a negative one. Most portfolios were robust and didn't deviate much in matters of returns from their weekly counterparts. However, the maximum Sharpe's final portfolio value V_F deteriorated from the 30 days rebalancing. However, we need to take into account that the weekly-rebalanced portfolio incurs higher transaction costs in practice.

5 Conclusions

It seems like the best option for the period being would have been to store the investment under the mattress. It is hard for me to believe that only forecasting expected returns and variances will allow us to arrive to an optimal portfolio. My analysis parted from only data prices, maybe incorporating other variables would give more insight into the information transmission mechanism of the CC market. Other variables needed to be considered in order to extract the information embedded in the prices. For future work, I would consider other models like the factors model that is widely used in practice. The need of incorporating other information to find the drivers of prices is imperative since there could be other correlated variables, the problem lies in finding trust-worthy data and its relationship with risk-reward.

Although portfolio optimization seems to be a good alternative, since most of the optimized portfolios realized lower volatility than the benchmarks; however, in these optimized portfolios the higher moments are extreme. MVO seems to fail into capturing the market psychology and detect the market cycles, which are dynamic.

Extending the models to incorporate transaction costs would be a sensitive thing to do since there are liquidity constraints in this asset class. In such a way, it can be shown or rejected, the superiority in choice of a monthly rebalancing period and find consistency with the literature Trimborn et al. (2017).

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A CRIX Methodology

According to the data disclosed on the *thecrix.de* and Trimborn and Härdle (2015), the index is built as follows:

It is motivated by the index of Lapeyres. Defined by:

$$INDEX(t)_{Laspeyres} = \frac{\sum_i^k P_{i,t} Q_{i,0}}{\sum_i^k P_{i,0} Q_{i,0}}$$

where $P_{i,t}$ is the price of crypto i at time t; therefore, at point 0 $P_{i,0}$ is the price with an amount $Q_{i,0}$.

Trimborn and Härdle (2015) denote the CRIX as:

$$CRIX(k)_t = \frac{\sum_i^k MV_{i,t} AW_{i,t}}{Divisor}$$

where k is the number of constituents and $MV_{i,t}$ is the market capitalization of the crypto i at time t.

$$AW_{i,t} = \frac{CW_{i,t}}{W_{i,t}}$$

$CW_{i,t}$ is the capped weight, whenever a crypto i has a weight $W_{i,t} = \frac{MV_{i,t}}{\sum_i^k MV_{i,t}}$ of 50% or more in CRIX. $Divisor = \frac{\sum_i^k MV_i}{1000}$ is its starting value so the constituents are not affected by changes in prices.

It is adjusted when necessary:

$$\frac{\sum_i^k MV_{i,t-1}}{Divisor_{t-1}} = CRIX_{t-1} = CRIX_t = \frac{\sum_i^k MV_{i,t}}{Divisor_t}$$

B Descriptive Statistics

The first four moments of the distribution of logarithmic returns vectors x_i for $i...N$ assets, were computed as follows:

Mean: An arithmetic mean was performed

$$\mu = \frac{\sum_1^N x_i}{N}$$

Variance: unbiased empirical variance

$$\Sigma = \frac{1}{N-1}\sum_1^N (x_i - \mu)^2$$

Skewness: Fisher-Pearson coefficient of skewness

$$Skewness = \frac{\frac{1}{N}\sum_1^N (x_i - \mu)^3}{[\frac{1}{N}\sum_1^N (x_i - \mu)^2]^{3/2}}$$

Kurtosis: standardized sample excess kurtosis

$$Kurtosis = \frac{\frac{1}{N}\sum_1^N (x_i - \mu)^4}{[\frac{1}{N}\sum_1^N (x_i - \mu)^2]^2} - 3$$

C Portfolio Risk Measures

The formulas of the value at risk VaR and the conditional value at risk (CVaR, also known as expected shortfall ES) assume a normal distribution, which could not be as good measures given the empirical realized skewness and kurtosis. For the purpose of the thesis the alpha is 0.01, such that the confidence interval is that of 99%.

Value at Risk (VaR):

$$VaR_\alpha(X) = -F_X^{-1}(\alpha)$$

where X is our logarithmic returns distribution, F_X is the cumulative distribution function (cdf), and $(1 - \alpha)$ is equivalent to the confidence interval. VaR computes the quantile when the cdf $= \alpha$.

Conditional Value at Risk (CVaR, also known as expected shortfall):

$$CVaR_\alpha(X) = -\frac{1}{\alpha}\int_0^\alpha VaR_\alpha(X)d\alpha$$

Denoted when both the VaR and the CVaR have the same confidence interval. Intuitively CVaR computes the average of the tail values contained in the α part of the cdf.

Declaration of Authorship

I hereby confirm that I have authored this Master's thesis independently and without use of others than the indicated sources. All passages which are literally or in general matter taken out of publications or other sources are marked as such.

Berlin, September 30, 2020

Guillermo Masayuki Morishige Takane