Bull. London Math. Soc. 53 (2021) 906–916 doi:10.1112/blms.12470

On holomorphic matrices on bordered Riemann surfaces

Jürgen Leiterer

Abstract

Let $\mathbb{D}$ be the unit disk. Kutzschebauch and Studer (*Bull. Lond. Math. Soc.* 51 (2019) 995–1004) recently proved that, for each continuous map $A : \overline{\mathbb{D}} \to \mathrm{SL}(2, \mathbb{C})$, which is holomorphic in $\mathbb{D}$, there exist continuous maps $E, F : \overline{\mathbb{D}} \to \mathfrak{sl}(2, \mathbb{C})$, which are holomorphic in $\mathbb{D}$, such that $A = e^E e^F$. Also they asked if this extends to arbitrary compact bordered Riemann surfaces. We prove that this is possible.

1. *Introduction*

Let $\overline{X}$ be a compact bordered Riemann surface[†], and let X be the interior of $\overline{X}$. Denote by $\mathrm{SL}(2, \mathbb{C})$ the group of complex 2×2 matrices with determinant 1, and by $\mathfrak{sl}(2, \mathbb{C})$ its Lie algebra of complex 2×2 matrices with trace zero. We prove the following.

Theorem 1.1. *Let $A : \overline{X} \to \mathrm{SL}(2, \mathbb{C})$ be a continuous map, which is holomorphic in X. Then there exist continuous maps $E, F : \overline{X} \to \mathfrak{sl}(2, \mathbb{C})$, which are holomorphic in X, such that $A = e^E e^F$ on $\overline{X}$.*

Let $\overline{\mathbb{D}}$ be the closed unit disk in $\mathbb{C}$. For $\overline{X} = \overline{\mathbb{D}}$, Theorem 1.1 was recently proved by Kutzschebauch and Studer [**11**, Theorem 2]. In [**11**] also, the question is asked if Theorem 1.1 is true in general, and it is noted that there is some problem to adapt in a straightforward way the proof of [**11**] to the general case. The problem is that $\overline{X}$ need not be simply connected. Our proof of Theorem 1.1 is nevertheless some adaption of the proof given in [**11**] for the case $\overline{X} = \overline{\mathbb{D}}$.

Let $\mathcal{A}(\overline{X})$ be the algebra of complex-valued functions which are continuous on $\overline{X}$ and holomorphic in X. The first step in our proof of Theorem 1.1 is the following.

Lemma 1.2. *Let $a, b \in \mathcal{A}(\overline{X})$ with $\{a = 0\} \cap \{b = 0\} = \emptyset$ and, moreover, $\{a = 0\} \neq \overline{X}$. Then there exist $g, h \in \mathcal{A}(\overline{X})$ such that $b + ga = e^h$.*

Recall that (by definition) the Bass stable rank of a commutative unital ring R is equal to 1, if, for all $a, b \in R$ with $aR + bR = R$, there exists $g \in R$ such that $b + ga$ is invertible. Although not used in the present paper, let us note the following immediate corollary of Lemma 1.2.

Received 4 November 2020; published online 17 February 2021.

2020 *Mathematics Subject Classification* 30F99, 47A56, 15A54, 15A16 (primary).

[Correction added on 17 May 2021, after first online publication: The spelling of the author name in reference 2 was corrected.]

[†]In the sense of [**1**, II.3A], which includes that $\overline{X}$ is connected. For example, $\overline{X}$ can be the closure of a bounded smooth domain X in the complex plane.

COROLLARY 1.3. *The Bass stable rank of* $\mathcal{A}(\overline{X})$ *is equal to* 1.[†]

That the Bass stable rank of $\mathcal{A}(\overline{\mathbb{D}})$ is one is an important ingredient of the proof of Theorem 1.1 given in [**11**] for $\overline{X} = \overline{\mathbb{D}}$. As pointed out there, this makes it possible to limit to matrices of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $\{a = 0\} = \emptyset$. In the same way, Lemma 1.2 makes it possible to limit to matrices of the form $\left(\begin{smallmatrix} e^h & b \\ c & d \end{smallmatrix}\right)$, and, for matrices of this form, it is possible to adapt the proof from [**11**] to the case of non-simply connected $\overline{X}$.

Let $\mathrm{M}(2,\mathbb{C})$ be the algebra of all complex 2×2 matrices, and $\mathrm{GL}(2,\mathbb{C})$ the group of its invertible elements. Then, in the same way as in [**11**, Corollary 1], the following corollary can be deduced from Theorem 1.1.

COROLLARY 1.4. *Let* $A : \overline{X} \to \mathrm{GL}(2,\mathbb{C})$ *be continuous on* $\overline{X}$*, holomorphic in* X*, and null-homotopic. Then there exist continuous maps* $E, F : \overline{X} \to \mathrm{M}(2,\mathbb{C})$*, which are holomorphic in* X*, such that* $A = e^E e^F$ *on* $\overline{X}$*.*

The study of the question 'how many exponentials factors are necessary to represent a given holomorphic matrix' was started by Mortini and Rupp [**14**]. In the case of an invertible 2×2 matrix with entries from $\mathcal{A}(\overline{\mathbb{D}})$, they proved that four exponentials are sufficient [**14**, Theorem 7.1]. Then Doubtsov and Kutzschebauch [**6**, Proposition 3] improved this to three exponentials. Eventually Kutzschebauch and Studer obtained that two exponentials are sufficient, which cannot be further improved, by an example Mortini and Rupp [**14**, Example 6.4]. This example shows that, under the hypotheses of Theorem 1.1 or Corollary 1.4, in general there does not exist a continuous $B : \overline{X} \to \mathrm{M}(2,\mathbb{C})$ with $A = e^B$. As noted in [**6**], to find such B with values in $\mathfrak{sl}(2,\mathbb{C})$ is impossible already by the fact that not every matrix in $\mathrm{SL}(2,\mathbb{C})$ has a logarithm in $\mathfrak{sl}(2,\mathbb{C})$.

NOTE: After this paper was written and the preprint was posted in the arXiv [**12**], I got to know the preprint [**2**, Theorem 1.3] with a substantial generalization of Theorem 1.1. This generalization, in particular, contains Theorem 1.1 with $\mathrm{SL}(n,\mathbb{C})$ in place of $\mathrm{SL}(2,\mathbb{C})$, for arbitrary $n \geqslant 2$ (see [**2**, Example 1.4 (1)]).

2. *A sufficient criterion for the existence of a logarithm*

A matrix $\Phi \in \mathrm{M}(2,\mathbb{C})$ will be often considered as the linear operator in $\mathbb{C}^2$ defined by multiplication from the left by Φ (considering the vectors in $\mathbb{C}^2$ as column vectors). The kernel and the image of this operator will be denoted by $\operatorname{Ker}\Phi$ and $\operatorname{Im}\Phi$, respectively. For $\Phi \in \mathrm{M}(2,\mathbb{C})$ and $\lambda \in \mathbb{C}$, we often write $\lambda - \Phi$ instead of $\lambda I - \Phi$. A matrix $\Phi \in \mathrm{M}(2,\mathbb{C})$ will be called a *projection*, if it is a linear projection as an operator, that is, if $\Phi^2 = \Phi$.

LEMMA 2.1. *Let* X *be a topological space and let* $B : X \to \mathrm{SL}(2,\mathbb{C})$ *be continuous. Suppose there exists a continuous complex-valued function* λ *on* X *such that, for all* $\zeta \in X$*:*

(a) $e^{\lambda(\zeta)}$ *is an eigenvalue of* $B(\zeta)$*;*
(b) $e^{\lambda(\zeta)} \neq e^{-\lambda(\zeta)}$.

[†] If $\overline{X}$ is the closure of a bounded smooth domain in $\mathbb{C}$, this was proved by Corach and Suaréz [**5**, Theorem 2.3]. Actually they proved the stronger result that, if K is an arbitrary compact subset of $\mathbb{C}$, then the Bass stable rank of the algebra of functions which are continuous on K and holomorphic in the inner points of K is equal to 1. That the Bass stable rank of $\mathcal{A}(\overline{\mathbb{D}})$ is equal to 1 was obtained before in [**4, 10**]. I do not know if there already exists a published proof of Corollary 1.3 for arbitrary compact bordered Riemann surfaces.

Then there exists a uniquely determined map $F : X \to \mathfrak{sl}(2,\mathbb{C})$ such that $B = e^F$ on X and, for all $\zeta \in X$, $\lambda(\zeta)$ is an eigenvalue of $F(\zeta)$. This map is continuous. If X is a complex space[†] *and B, λ are holomorphic, then F is even holomorphic.*

Proof. *Existence:* Since $e^{\lambda(\zeta)}$ is an eigenvalue of $B(\zeta)$ and $\det B(\zeta) = 1$, $e^{-\lambda(\zeta)}$ is the other eigenvalue of $B(\zeta)$, which is distinct from $e^{\lambda(\zeta)}$, by condition (b). Therefore

$$\mathbb{C}^2 = \operatorname{Ker}\left(e^{\lambda(\zeta)} - B(\zeta)\right) \oplus \operatorname{Ker}\left(e^{-\lambda(\zeta)} - B(\zeta)\right) \quad \text{for all} \quad \zeta \in X,$$

where '$\oplus$' means 'direct sum' (not necessarily orthogonal). Let $P : X \to \mathrm{M}(2,\mathbb{C})$ be the map which assigns to each $\zeta \in X$ the linear projection from $\mathbb{C}^2$ onto $\operatorname{Ker}(e^{\lambda(\zeta)} - B(\zeta))$ along $\operatorname{Ker}(e^{-\lambda(\zeta)} - B(\zeta))$. Then

$$B = e^{\lambda}P + e^{-\lambda}(I - P), \tag{2.1}$$

which implies

$$P = \frac{1}{e^{\lambda} - e^{-\lambda}}B - \frac{e^{-\lambda}}{e^{\lambda} - e^{-\lambda}}I. \tag{2.2}$$

This shows that P is continuous on X and, if X is a complex space and B, λ are holomorphic, then P is even holomorphic on X. Now

$$F := \lambda P - \lambda(I - P) \tag{2.3}$$

has the desired properties.

Uniqueness: Let $\zeta \in X$ and $\Theta \in \mathfrak{sl}(2,\mathbb{C})$ such that $e^{\Theta} = B(\zeta)$, and $\lambda(\zeta)$ is an eigenvalue of Θ. Then Θ and $B(\zeta)$ commute. By (2.2), also Θ and $P(\zeta)$ commute. Therefore $\Theta = \alpha P(\zeta) + \beta(I - P(\zeta))$ for some numbers $\alpha, \beta \in \mathbb{C}$, which then are the eigenvalues of Θ, that is, either $\alpha = \lambda(\zeta)$ and $\beta = -\lambda(\zeta)$, or $\alpha = -\lambda(\zeta)$ and $\beta = \lambda(\zeta)$. $\alpha = -\lambda(\zeta)$ and $\beta = \lambda(\zeta)$ is not possible, since otherwise, by condition (b) and by (2.1), we would have

$$e^{\Theta} = e^{-\lambda(\zeta)}P + e^{\lambda(\zeta)}(I - P(\zeta)) \neq e^{\lambda(\zeta)}P(\zeta) + e^{-\lambda(\zeta)}(I - P(\zeta)) = B(\zeta).$$

Therefore $\alpha = \lambda(\zeta)$ and $\beta = -\lambda(\zeta)$. Hence, by (2.3),

$$\Theta = \lambda(\zeta)P(\zeta) - \lambda\zeta(I - P(\zeta)) = F(\zeta).$$ □

3. *Proof of Lemma* 1.2 *and Theorem* 1.1

In this section, $\overline{X}$ is a compact bordered Riemann surface, where we assume (as always possible[‡]) that X is a bounded smooth domain in some larger open Riemann surface $\widetilde{X}$, and $\overline{X}$ is the closure of X in $\widetilde{X}$. The boundary of $\overline{X}$ will be denoted by ∂X. If we speak about an *open subset U of $\overline{X}$*, then we always mean that U is a subset of $\overline{X}$ which is open in the topology of $\overline{X}$ (and in general not open in $\widetilde{X}$). For $K \subseteq \overline{X}$, let $\overline{K}$ be the closure of K (in $\overline{X}$ or in $\widetilde{X}$).

If U is an open subset of $\overline{X}$, then we denote by $\mathcal{A}(U)$ the algebra of continuous complex valued functions on U which are holomorphic in $U \cap X$.

To prove Theorem 1.1, we begin with the observation that

$$\Theta e^{\Phi}\Theta^{-1} = e^{\Theta\Phi\Theta^{-1}} \qquad \text{for all } \Theta, \Phi \in \mathrm{GL}(2,\mathbb{C}). \tag{3.1}$$

[†] By a complex space we mean a *reduced* complex space in the terminology of [**8**], which is the same as an analytic space in the terminology of [**3, 13**]. For example, each Riemann surface is a complex space.

[‡] One can take for $\widetilde{X}$ a non-compact open neighborhood of $\overline{X}$ in the double of $\overline{X}$ (for the definition of the double of $\overline{X}$, see, for example, [**1**, II. 3E]).

This shows that conjugation does not change the number of exponential factors needed to represent a given matrix. As in [11], we will use this observation several times.

Next we recall some known facts (Lemma 3.1, its Corollary 3.2 and Lemma 3.3), for completeness with proofs.

LEMMA 3.1. *Let α be a continuous* (0,1)*-form on $\overline{X}$ (that is, a continuous section over $\overline{X}$ of the holomorphic cotangential bundle of $\widetilde{X}$) which is $\mathcal{C}^\infty$ in X. Then there exists a continuous function $u : \overline{X} \to \mathbb{C}$ which is $\mathcal{C}^\infty$ in X such that $\overline{\partial} u = \alpha$ in X.*

Proof. As observed by Forstneric, Fornæss and Wold in [7, Section 2, formula (8)] (together with corresponding references), to solve the $\overline{\partial}$-equation on Riemann surfaces, one can use the following know fact: There exists a 1-form, ω, defined and holomorphic on $(\widetilde{X} \times \widetilde{X}) \setminus \Delta$, where Δ is the diagonal in $\widetilde{X} \times \widetilde{X}$, such that, if $h : U \to \mathbb{C}$ is a holomorphic coordinate on some open set $U \subseteq \widetilde{X}$, then, on $(U \times U) \setminus \Delta$, ω is of the form

$$\omega(\zeta, \eta) = \left(\frac{1}{h(\zeta) - h(\eta)} + \theta_h(\zeta, \eta) \right) dh(\zeta), \quad (\zeta, \eta) \in (U \times U) \setminus \Delta, \tag{3.2}$$

where θ_h is a holomorphic function on $U \times U$. Since $\overline{X}$ is compact, and α is continuous on $\overline{X}$, then it is clear that the function $u : \widetilde{X} \to \mathbb{C}$ defined by

$$u(\eta) = \frac{1}{2\pi i} \int_{\zeta \in X} \omega(\zeta, \eta) \wedge \alpha(\zeta), \quad \eta \in \widetilde{X},$$

is continuous on $\widetilde{X}$. To prove that, in X, u is $\mathcal{C}^\infty$ and solves the equation $\overline{\partial} u = \alpha$, we consider a point $\xi \in X$ and take an open neighborhoods V and U of ξ such that $\overline{V} \subseteq U$, $U \subseteq X$ and there exists a holomorphic coordinate $h : U \to \mathbb{C}$ of $\widetilde{X}$. Further choose a $\mathcal{C}^\infty$-function $\chi : \widetilde{X} \to [0,1]$ such that $\chi = 1$ in a neighborhood $\overline{V}$. Then $u = u_1 + u_2 + u_3$, where

$$u_1(\eta) = \frac{1}{2\pi i} \int_{\zeta \in V} \omega(\zeta, \eta) \wedge \alpha(\zeta),$$

$$u_2(\eta) = \frac{1}{2\pi i} \int_{\zeta \in X \setminus V} \chi(\zeta) \omega(\zeta, \eta) \wedge \alpha(\zeta),$$

$$u_3(\eta) = \frac{1}{2\pi i} \int_{\zeta \in X \setminus V} (1 - \chi(\zeta)) \omega(\zeta, \eta) \wedge \alpha(\zeta).$$

Then u_2 and u_3 are holomorphic in V. Therefore it remains to prove that u_1 is $\mathcal{C}^\infty$ and $\overline{\partial} u_1 = \alpha$, in V. By (3.2), $u_1 = u_1' + u_1''$, where

$$u_1'(\eta) = \frac{1}{2\pi i} \int_{\zeta \in V} \frac{dh(\zeta) \wedge \alpha(\zeta)}{h(\zeta) - h(\eta)} \quad \text{and} \quad u_1''(\eta) = \int_{\zeta \in V} \theta_h(\zeta, \eta) dh(\zeta) \wedge \alpha(\zeta).$$

Since θ_h is holomorphic, u_1'' is holomorphic. Further

$$(u_1' \circ h^{-1})(w) = \frac{1}{2\pi i} \int_{z \in h(V)} \frac{dz \wedge \big((h^{-1})^* \alpha\big)(z)}{w - z} \quad \text{for } w \in h(V).$$

Therefore, as is well known (see, for example, [9, Theorem 1.2.2]), $u_1' \circ h^{-1}$ is $\mathcal{C}^\infty$ and $\overline{\partial}(u_1' \circ h^{-1}) = (h^{-1})^* \alpha$, in $h(V)$, which implies that u_1' is $\mathcal{C}^\infty$ and $\overline{\partial} u_1' = \alpha$, in V. □

COROLLARY 3.2. *Let U_1, U_2 be non-empty open subsets of $\overline{X}$ with $U_1 \cup U_2 = \overline{X}$, and let $f \in \mathcal{A}(U_1 \cap U_2)$. Then there exist $f_1 \in \mathcal{A}(U_1)$ and $f_2 \in \mathcal{A}(U_2)$ with $f = f_1 - f_2$ on $U_1 \cap U_2$.*

Proof. For $K \subseteq \overline{X}$, we denote by $\partial_{\overline{X}} K$ the boundary of K with respect to the topology of $\overline{X}$ (which is, in general, smaller than the boundary in $\widetilde{X}$). Since U_1 and U_2 are open subsets of $\overline{X}$ and $U_1 \cup U_2 = \overline{X}$, we have

$$\overline{U_1 \setminus U_2} \cap \overline{U_2 \setminus U_1} = \emptyset.$$

Therefore we can find a $\mathcal{C}^\infty$ function $\chi : \widetilde{X} \to [0,1]$ with $\chi = 1$ in an $\widetilde{X}$-neighborhood of $\overline{U_1 \setminus U_2}$, and $\chi = 0$ in an $\widetilde{X}$-neighborhood of $\overline{U_2 \setminus U_1}$. Then we have well-defined continuous functions $c_1 : U_1 \to \mathbb{C}$ and $c_2 : U_2 \to \mathbb{C}$ which are $\mathcal{C}^\infty$ in $X \cap U_1$ and $X \cap U_2$, respectively, such that

$$c_1 = \begin{cases} (1-\chi)f & \text{on } U_1 \cap U_2, \\ 0 & \text{on } U_1 \setminus U_2, \end{cases} \quad \text{and} \quad c_2 = \begin{cases} -\chi f & \text{on } U_1 \cap U_2, \\ 0 & \text{on } U_2 \setminus U_1. \end{cases}$$

Then

$$f = c_1 - c_2 \quad \text{on} \quad U_1 \cap U_2, \tag{3.3}$$

$$\overline{\partial} c_1 = -\overline{\partial}\chi f = \overline{\partial} c_2 \quad \text{on} \quad X \cap U_1 \cap U_2. \tag{3.4}$$

Relation (3.4) shows that there is a well-defined continuous (0,1)-form on $\overline{X}$, α, which is $\mathcal{C}^\infty$ in X, such that

$$\alpha = \overline{\partial} c_j \text{ on } X \cap U_j, \quad \text{for} \quad j = 1, 2. \tag{3.5}$$

By the preceding lemma, we can find a continuous function $u : \overline{X} \to \mathbb{C}$ which is $\mathcal{C}^\infty$ in X such that $\overline{\partial} u = \alpha$ in X. Set $f_j = c_j - u$, $j = 1, 2$. Then, by (3.5), $f_j \in \mathcal{A}(U_j)$ and, by (3.3), $f = f_1 - f_2$ on $U_1 \cap U_2$. □

LEMMA 3.3. *For each $a \in \mathcal{A}(\overline{X})$, either $\{a = 0\} = \overline{X}$ or $\partial X \cap \{a = 0\}$ is nowhere dense in ∂X.*

Proof. Assume $\partial X \cap \{a = 0\}$ is not nowhere dense in ∂X. Then there exist $\xi \in \partial X$ and an open subset U of $\overline{X}$ with $\xi \in U$ and $a \equiv 0$ on $U \cap \partial X$. Then (by definition of a bordered Riemann surface), we have an open subset V of $\overline{X}$ with $\xi \in V$, and a homeomorphism $\varphi : V \to \{z \in \mathbb{C} \,|\, |z| < 1, \operatorname{Im} z \geqslant 0\}$, which is biholomorphic from $V \setminus \partial X$ onto $\{z \in \mathbb{C} \,|\, |z| < 1, \operatorname{Im} z > 0\}$ and such that $\varphi(V \cap \partial X) =]-1, 1[$. Then the continuous function $a \circ \varphi^{-1}$ is holomorphic in $\{z \in \mathbb{C} \,|\, |z| < 1, \operatorname{Im} z > 0\}$ and has the real value 0 on $]-1, 1[$. Therefore, by the Schwarz reflection principle, there is a holomorhic function $\widetilde{a}$ on $\{z \in \mathbb{C} \,|\, |z| < 1\}$ with

$$\widetilde{a} = a \circ \varphi^{-1} \quad \text{on} \quad \bigl\{z \in \mathbb{C} \bigm| |z| < 1, \operatorname{Im} z \geqslant 0\bigr\}. \tag{3.6}$$

Since $a = 0$ on $\varphi^{-1}(]-1, 1[) = V \cap \partial X$, from (3.6) we get $\widetilde{a} = 0$ on $]-1, 1[$. Therefore $\widetilde{a} = 0$ on $\{z \in \mathbb{C} \,|\, |z| < 1\}$. Again by (3.6) this implies that $a = 0$ on $V \setminus \partial X$. Hence ($\overline{X}$ is connected) $\{a = 0\} = \overline{X}$. □

The first step in the proof of Lemma 1.2 is the following lemma.

LEMMA 3.4. *Let $a, b \in \mathcal{A}(\overline{X})$ such that $\{a = 0\} \cap \{b = 0\} = \emptyset$. Then there exist finitely many closed subsets $K_1, \ldots, K_\ell$ of $\overline{X}$ such that*

$$K_j \cap K_k = \emptyset \quad \textit{for all} \quad 1 \leqslant j, k \leqslant \ell \textit{ with } j \neq k, \tag{3.7}$$

$$\{a = 0\} \subseteq K_1 \cup \ldots \cup K_\ell, \tag{3.8}$$

and, for some open disks $\mathbb{D}_1, \ldots, \mathbb{D}_\ell$ contained in $\mathbb{C} \setminus \{0\}$,

$$b(K_j) \subseteq \mathbb{D}_j \quad \textit{for } j = 1, \ldots, \ell. \tag{3.9}$$

Proof. If $\{a=0\}=\emptyset$, the claim of the lemma is trivial. Therefore we may assume that $\{a=0\}\neq\emptyset$.

First let $\partial X\cap\{a=0\}=\emptyset$. Since $\overline{X}$ is compact and $\{a=0\}$ has no accumulation points in X, and since $\{a=0\}\neq\emptyset$, then $\{a=0\}$ consists of a finite number of points $\xi_1,\dots,\xi_\ell\in X$. Then $b(\xi_1)\neq 0$, …, $b(\xi_\ell)\neq 0$, and $K_1:=\{\xi_1\}$, …, $K_\ell:=\{\xi_\ell\}$ have the desired properties.

Now let $\partial X\cap\{a=0\}\neq\emptyset$. Fix a metric $\rho(\cdot,\cdot)$ on $\widetilde{X}$. For a subset K of $\widetilde{X}$ we denote by $\operatorname{diam} K$ the diameter of K with respect to this metric. Since $\overline{X}$ is compact, a,b are continuous and $\{a=0\}\cap\{b=0\}=\emptyset$, we have

$$\theta:=\min_{\zeta\in\overline{X}}\left(|a(\zeta)|+|b(\zeta)|\right)>0,$$

and we can find $\varepsilon>0$ such that

$$\bigl|b(\zeta)-b(\eta)\bigr|<\theta\quad\text{for all }\zeta,\eta\in\overline{X}\text{ with }\rho(\zeta,\eta)<\varepsilon. \tag{3.10}$$

We call a set $\Lambda\subseteq\partial X$ a *closed Interval* in ∂X if there is a homeomorphic map ψ from [0,1] onto Λ.

Since $\overline{X}$ is compact, ∂X is the union of a finite number of pairwise disjoint Jordan curves.

STATEMENT 1. Let Γ be one of these Jordan curves. Then there exists a finite number of closed intervals $\Lambda_1,\dots,\Lambda_q$ in Γ such that

$$\Lambda_j\cap\Lambda_k=\emptyset\quad\text{for}\quad 1\leqslant j,k\leqslant q\text{ with }j\neq k, \tag{3.11}$$

$$\Gamma\cap\{a=0\}\subseteq\Lambda_1\cup\ldots\cup\Lambda_q, \tag{3.12}$$

$$\Lambda_j\cap\{a=0\}\neq\emptyset\quad\text{for }j=1,\dots,q, \tag{3.13}$$

$$\operatorname{diam}(\Lambda_j)<\varepsilon\quad\text{for}\quad 1\leqslant j\leqslant q, \tag{3.14}$$

Proof of Statement 1. If $\Gamma\cap\{a=0\}=\emptyset$, the claim of the statement is trivial. Therefore we may assume that $\Gamma\cap\{a=0\}\neq\emptyset$.

Since Γ is a Jordan curve, we have a homeomorphism ϕ from $\mathbb{T}:=\{z\in\mathbb{C}\,|\,|z|=1\}$ onto Γ. Since $\{a=0\}\neq\overline{X}$, $\{a=0\}\cap\Gamma$ is nowhere dense in Γ (Lemma 3.3). Therefore we can find $0<t_1<t_2<\ldots<t_p<2\pi$ such that

$$a\bigl(\phi(e^{it_\kappa})\bigr)\neq 0\quad\text{for}\quad\kappa=1,\dots,p, \tag{3.15}$$

and

$$\operatorname{diam}\phi\Bigl(e^{i[t_\kappa,t_{\kappa+1}]}\Bigr)<\varepsilon\text{ for }\kappa=1,\dots,p-1,\text{ and}$$
$$\operatorname{diam}\Bigl(\phi\Bigl(e^{i[t_p,2\pi]}\Bigr)\cup\phi\Bigl(e^{i[0,t_1]}\Bigr)\Bigr)<\varepsilon. \tag{3.16}$$

By (3.15), we can find $\sigma>0$ such that $t_\kappa+\sigma<t_{\kappa+1}$ for $\kappa=1,\dots,p-1$, $t_p+\sigma<2\pi$, and

$$a\bigl(\phi(e^{it})\bigr)\neq 0\text{ for }t_j\leqslant t\leqslant t_j+\sigma\text{ and }\kappa=1,\dots,p. \tag{3.17}$$

Define closed intervals in Γ, $\Delta_1,\dots,\Delta_p$, by

$$\Delta_\kappa=\phi\Bigl(e^{i[t_\kappa+\sigma,t_{\kappa+1}]}\Bigr)\text{ for }\kappa=1,\dots,p-1,\text{ and }\Delta_p=\phi\Bigl(e^{i[t_p+\sigma,2\pi]}\Bigr)\cup\phi\Bigl(e^{i[0,t_1]}\Bigr).$$

Then it is clear that

$$\Delta_\kappa\cap\Delta_\lambda=\emptyset\quad\text{for all }\kappa,\lambda\in\{1,\dots,p\}\text{ with }\kappa\neq\lambda, \tag{3.18}$$

from (3.16) it follows that

$$\operatorname{diam}\Delta_\kappa<\varepsilon\quad\text{for}\quad\kappa=1,\dots,p, \tag{3.19}$$

and from (3.17) it follows that

$$\Gamma \cap \{a=0\} \subseteq \Delta_1 \cup \ldots \cup \Delta_p. \tag{3.20}$$

Let $\{\kappa_1, \ldots, \kappa_q\}$ be the set of all $\kappa \in \{1, \ldots, p\}$ with $\Delta_\kappa \cap \{a=0\} \neq \emptyset$ (such κ exist, as $\Gamma \cap \{a=0\} \neq \emptyset\}$), and define $\Lambda_j = \Delta_{\kappa_j}$ for $j = 1, \ldots, q$. Then (3.11) is clear by (3.18). (3.12) and (3.13) hold by (3.20) and the definition of the set $\{\kappa_1, \ldots, \kappa_q\}$. (3.14) is clear by (3.19). Statement 1 is proved.

From Statement 1, we obtain a finite number of closed intervals $\Lambda_1, \ldots, \Lambda_r$ in ∂X such that

$$\Lambda_j \cap \Lambda_k = \emptyset \quad \text{for} \quad 1 \leqslant j,k \leqslant r \text{ with } j \neq k, \tag{3.21}$$

$$\partial X \cap \{a=0\} \subseteq \Lambda_1 \cup \ldots \cup \Lambda_r, \tag{3.22}$$

$$\Lambda_j \cap \{a=0\} \neq \emptyset \quad \text{for } j = 1, \ldots, r, \tag{3.23}$$

$$\operatorname{diam}(\Lambda_j) < \varepsilon \quad \text{for} \quad j = 1, \ldots, r. \tag{3.24}$$

By (3.21) and (3.24), we can find open subsets U_j of $\overline{X}$, $j = 1, \ldots, r$, with

$$\Lambda_j \subseteq U_j \quad \text{for} \quad 1 \leqslant j \leqslant r, \tag{3.25}$$

$$\overline{U}_j \cap \overline{U}_k = \emptyset \quad \text{for all } 1 \leqslant j,k \leqslant r \text{ with } j \neq k, \tag{3.26}$$

$$\operatorname{diam}\left(\overline{U}_j\right) < \varepsilon \quad \text{for } j = 1, \ldots, r. \tag{3.27}$$

Note that then, by (3.23),

$$U_j \cap \{a=0\} \neq \emptyset \quad \text{for } j = 1, \ldots, r. \tag{3.28}$$

Set $K_j = \overline{U}_j$ for $j = 1, \ldots, r$. Then, by (3.22) and (3.25),

$$\{a=0\} \cap (\partial X \cup K_1 \cup \ldots \cup K_r) = \{a=0\} \cap (K_1 \cup \ldots \cup K_r). \tag{3.29}$$

Statement 2. $N := \{a=0\} \cap (\overline{X} \setminus (\partial X \cup K_1 \cup \ldots \cup K_r))$ is finite.

Proof of Statement 2. Assume N is infinite. Since $\overline{X}$ is compact, then N has an accumulation point $\xi \in \overline{X}$. Since $\{a=0\}$ is closed, $\xi \in \{a=0\}$. As $\{a=0\} \cap X$ is discrete in X, this implies that $\xi \in \partial X \cap \{a=0\}$ and further, by (3.22) and (3.25), that $\xi \in U_1 \cup \ldots \cup U_r$. In particular, with respect to the topology of $\overline{X}$, ξ is an inner point of $\partial X \cup K_1 \cup \ldots \cup K_r$, which is not possible, for ξ is an accumulation point of N and therefore, in particular, an accumulation point of $\overline{X} \setminus (\partial X \cup K_1 \cup \ldots \cup K_r)$. Statement 2 is proved.

Let $\xi_{r+1}, \ldots, \xi_\ell$ the distinct points of N, and define $K_j = \{\xi_j\}$ for $j = r+1, \ldots, \ell$. We claim that $K_1, \ldots, K_\ell$ have the desired properties (3.7)–(3.9).

Indeed, (3.7) follows from (3.26) and the fact that $\xi_{r+1}, \ldots, \xi_\ell$ are pairwise distinct and lie in N and, hence, outside $K_1 \cup \ldots \cup K_r$. By (3.29),

$$\{a=0\} \cap (\partial X \cup K_1 \cup \ldots \cup K_r) \subseteq K_1 \cup \ldots \cup K_r,$$

and, by definition of $K_{r+1}, \ldots, K_\ell$,

$$\{a=0\} \cap \left(\overline{X} \setminus (\partial X \cup K_1 \cup \ldots \cup K_r)\right) = N = K_{r+1} \cup \ldots \cup K_\ell.$$

Together implies (3.8). To prove (3.9), we first note that by (3.28) and the definition of $K_{r+1}, \ldots, K_\ell$, for each $j \in \{1, \ldots, \ell\}$, we have a point $\xi_j \in K_j$ with $a(\xi_j) = 0$. Since, by definition of θ, $|b(\xi_j)| \geqslant \theta$, setting $\mathbb{D}_j = \{z \in \mathbb{C} \mid |z - b(\xi_j)| < \theta\}$, we obtain open disks $\mathbb{D}_1, \ldots, \mathbb{D}_\ell \subseteq \mathbb{C} \setminus \{0\}$. Since $\operatorname{diam} K_j < \varepsilon$ for $j = 1 \ldots, \ell$ (for $1 \leqslant j \leqslant r$ this holds by (3.27), and for $r+1 \leqslant j \leqslant \ell$, we have $\operatorname{diam} K_j = 0$), now (3.9) follows from (3.10). □

Proof of Lemma 1.2. If $\{a=0\}=\emptyset$, we set $g=(1-b)/a$. Then $b+ga=1=e^0$ on X, and the claim of the lemma is proved.

Now let $\{a=0\}\neq\emptyset$. By Lemma 3.4, we can find finitely many closed subsets $K_1,\dots,K_\ell$ of $\overline{X}$ and open disks $\mathbb{D}_1,\dots,\mathbb{D}_\ell$ in $\mathbb{C}\setminus\{0\}$ satisfying (3.7)–(3.9). Choose open subsets $W_1,\dots,W_\ell$ of $\overline{X}$ such that

$$K_j\subseteq W_j\quad\text{for}\quad 1\leqslant j\leqslant\ell, \tag{3.30}$$

$$W_j\cap W_k=\emptyset\quad\text{for all}\quad 1\leqslant j,k\leqslant\ell\text{ with }j\neq k, \tag{3.31}$$

$$b(W_j)\subseteq\mathbb{D}_j\quad\text{for }j=1,\dots,\ell. \tag{3.32}$$

Since $D_j\subseteq\mathbb{C}\setminus\{0\}$, we can find holomorphic functions $\log_j:\mathbb{D}_j\to\mathbb{C}$ with $e^{\log_j z}=z$ for $z\in\mathbb{D}_j$. Set $W=W_1\cup\ldots\cup W_\ell$ and $V=\overline{X}\setminus\{a=0\}$. Then, by (3.30) and (3.8), $V\cup W=\overline{X}$, and, by (3.31) and (3.32), we can define $f\in\mathcal{A}(W)$ setting $f=\log_j\circ b$ on W_j. Then

$$b=e^f\quad\text{on }W. \tag{3.33}$$

Since $a\neq 0$ on V and $f\in\mathcal{A}(W)$, we have $f/a\in\mathcal{A}(V\cap W)$. Therefore, by Corollary 3.2, we can find $v\in\mathcal{A}(V)$ and $w\in\mathcal{A}(W)$ with $f/a=v-w$, that is,

$$f+aw=av\quad\text{on}\quad V\cap W.$$

Therefore, we have a function $h\in\mathcal{A}(\overline{X})$ with

$$h=f+aw\quad\text{on}\quad W. \tag{3.34}$$

The series $\sum_{\mu=0}^{\infty}\frac{a^\mu w^\mu}{\mu!}\frac{bw}{\mu+1}$ converges uniformly on the compact subsets of W to some $s\in\mathcal{A}(W)$, and, by (3.34) and (3.33), we have

$$e^h-b=e^{f+aw}-b=be^{aw}-b=b(e^{aw}-1)\quad\text{on}\quad W.$$

Together this implies that, on $V\cap W=W\setminus\{a=0\}$,

$$\frac{e^h-b}{a}=\frac{b}{a}\sum_{\mu=1}^{\infty}\frac{a^\mu w^\mu}{\mu!}=\frac{b}{a}\sum_{\mu=0}^{\infty}\frac{a^{\mu+1}w^{\mu+1}}{(\mu+1)!}=\sum_{\mu=0}^{\infty}\frac{a^\mu w^\mu}{\mu!}\frac{bw}{\mu+1}=s.$$

Therefore, we have a function $g\in\mathcal{A}(\overline{X})$ with $g=\frac{e^h-b}{a}$ on V and $g=s$ on W. Then, on $V=\overline{X}\setminus\{a=0\}$, it is clear that

$$b+ga=b+\frac{e^h-b}{a}a=e^h.$$

Since $\{a=0\}$ is nowhere dense in $\overline{X}$, it follows by continuity that $b+ga=e^h$ on all of $\overline{X}$. □

Proof of Theorem 1.1. For $f\in\mathcal{A}(\overline{X})$, we denote by $\operatorname{Re}f$ and $|f|$ the functions $\overline{X}\ni\zeta\to\operatorname{Re}f(\zeta)$, and $\overline{X}\ni\zeta\to|f(\zeta)|$, respectively. By $\mathcal{A}^{\mathrm{SL}(2,\mathbb{C})}(\overline{X})$ and $\mathcal{A}^{\mathfrak{sl}(2,\mathbb{C})}(\overline{X})$, we denote the sets of continuous maps from $\overline{X}$ to $\mathrm{SL}(2,\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{C})$, respectively, which are holomorphic in X.

Now let $A\in\mathcal{A}^{\mathrm{SL}(2,\mathbb{C})}(\overline{X})$ be given.

If $A\equiv I$ or $A\equiv -I$, the claim of Theorem 1.1 is trivial. Therefore it is sufficient to consider the following three cases.

(I) A is of the form $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ with $\{c=0\}\neq\overline{X}$.

(II) A is of the form $\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)$ with $\{b=0\}\neq\overline{X}$.

(III) A is of the form $\left(\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right)$ where neither $\{a=1\}=\{d=1\}=\overline{X}$ nor $\{a=-1\}=\{d=-1\}=\overline{X}$.

By observation (3.1), Case (II) can be reduced to Case (I), since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} d & 0 \\ b & a \end{pmatrix}.$$

Consider Case (III). Since $\det A \equiv 1$, then $a \neq 0$ and $d = a^{-1}$ on $\overline{X}$. Moreover, then $\{a - a^{-1} = 0\} \neq \overline{X}$, for otherwise we would have $\{a^2 = 1\} = \overline{X}$, that is, either $\{a = 1\} = \{d = 1\} = \overline{X}$ or $\{a = -1\} = \{d = -1\} = \overline{X}$. As

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & 0 \\ a - a^{-1} & a^{-1} \end{pmatrix},$$

this shows, again by (3.1), that also Case (III) can be reduced to Case (I).

So, we may assume that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ where $\{c = 0\} \neq \overline{X}$. Since also $\{c = 0\} \cap \{a = 0\} = \emptyset$ (the values of A are invertible), then we can apply Lemma 1.2, which gives $g, h \in \mathcal{A}(\overline{X})$ with $a + gc = e^h$ on $\overline{X}$. Then

$$\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} A \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} e^h & * \\ * & * \end{pmatrix}.$$

Therefore, again by observation (3.1), finally we see that $A = \left(\begin{smallmatrix} e^h & b \\ c & d \end{smallmatrix}\right)$ with $h, b, c, d \in \mathcal{A}(\overline{X})$ can be assumed.

The remaining part of the proof is an adaption of the proof given in [**11**] for $\overline{X} = \overline{\mathbb{D}}$. Chose $\delta > 0$ so large that, on $\overline{X}$,

$$\operatorname{Re}\left(e^{\delta} + e^{h-\delta} d\right) > 0, \tag{3.35}$$

$$\left|(1 + e^{h-2\delta} d)^2 - 4e^{-2\delta} - 1\right| < \frac{1}{2}, \tag{3.36}$$

and define

$$E = \begin{pmatrix} h - \delta & 0 \\ 0 & \delta - h \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} e^{\delta} & e^{\delta - h} b \\ e^{h-\delta} c & e^{h-\delta} d \end{pmatrix}.$$

Then

$$E \in \mathcal{A}^{\mathfrak{sl}(2,\mathbb{C})}(\overline{X}), \quad B \in \mathcal{A}^{\mathrm{SL}(2,\mathbb{C})}(\overline{X}), \quad \text{and} \quad A = e^E B \text{ on } \overline{X}. \tag{3.37}$$

It follows from (3.36) that $\log((1 + e^{h-2\delta} d)^2 - 4e^{-2\delta})$ is well defined, where, since $|\log z| < 1$ if $|z - 1| < 1/2$,

$$\left|\log\left((1 + e^{h-2\delta} d)^2 - 4e^{-2\delta}\right)\right| < 1 \quad \text{on} \quad \overline{X}. \tag{3.38}$$

Since

$$\frac{(\operatorname{tr} B)^2}{4} - 1 = \frac{e^{2\delta}}{4}\left(\left(1 + e^{h-2\delta} d\right)^2 - 4e^{-2\delta}\right),$$

this implies that also $\log(\frac{(\operatorname{tr} B)^2}{4} - 1)$ is well defined, where

$$\log\left(\frac{(\operatorname{tr} B)^2}{4} - 1\right) = 2\delta - \log 4 + \log\left(\left(1 + e^{h-2\delta} d\right)^2 - 4e^{-2\delta}\right) \quad \text{on} \quad \overline{X}. \tag{3.39}$$

Set

$$\varphi = \exp\left(\frac{1}{2}\log\left(\frac{(\operatorname{tr} B)^2}{4} - 1\right)\right) \quad \text{on} \quad \overline{X}.$$

Then, by (3.39),

$$\varphi = \exp\left(\delta - \frac{\log 4}{2}\right) \exp\left(\frac{1}{2}\log\left(\left(1+e^{h-2\delta}d\right)^2 - 4e^{-2\delta}\right)\right).$$

Since $|e^z - 1| < 1$ if $|z| < 1/2$ and therefore, by (3.38),

$$\left|\exp\left(\frac{1}{2}\log\left(\left(1+e^{h-2\delta}d\right)^2 - 4e^{-2\delta}\right)\right) - 1\right| < 1,$$

this shows that

$$\operatorname{Re}\varphi > 0 \quad \text{on} \quad \overline{X}. \tag{3.40}$$

Since $\varphi^2 = \frac{(\operatorname{tr} B)^2}{4} - 1$, we see that, for each $\zeta \in \overline{X}$,

$$\theta_+(\zeta) := \frac{\operatorname{tr} B(\zeta)}{2} + \varphi(\zeta) \quad \text{and} \quad \theta_-(\zeta) := \frac{\operatorname{tr} B(\zeta)}{2} - \varphi(\zeta)$$

are the eigenvalues of $B(\zeta)$, where $\theta_+(\zeta) \neq \theta_-(\zeta)$ (as $\varphi(\zeta) \neq 0$). Since $\det B(\zeta) = 1$ and therefore $\theta_-(\zeta) = \theta_+(\zeta)^{-1}$, it follows that $\theta_+(\zeta) \neq \theta_+(\zeta)^{-1}$ for all $\zeta \in \overline{X}$. Since, by (3.35), also $\operatorname{Re}(\operatorname{tr} B) > 0$, it follows from (3.40) that $\operatorname{Re}\theta_+ > 0$ on $\overline{X}$. Therefore $\lambda = \log\theta_+$ is well defined. So, we have found a function $\lambda \in \mathcal{A}(\overline{X})$ with the property that, for all $\zeta \in \overline{X}$, $e^{\lambda(\zeta)}$ $(= \theta_+(\zeta))$ is an eigenvalue of $B(\zeta)$ and $\lambda(\zeta) \neq -\lambda(\zeta)$ (as $\theta_+(\zeta) \neq \theta_+(\zeta)^{-1}$). This implies by Lemma 2.1 that there exists $F \in \mathcal{A}^{\mathfrak{sl}(2,\mathbb{C})}(\overline{X})$ with $B = e^F$. By (3.37) this completes the proof of Theorem 1.1.

Acknowledgements. In the version of the paper first sent to the publisher, Corollary 1.3 was mentioned only in passing, because it is not used in the present paper and, except for $\overline{X} = \overline{\mathbb{D}}$, I did not know if it is already known (in the sense that there exists a published proof). I want to thank a referee who encouraged me to state Corollary 1.3 explicitly, for, to his 'best knowledge [...] this result is previously not published'. This referee also turned my attention to the paper [**5**] of Corach and Suárez, where, in particular, a proof of Corollary 1.3 can be found if $\overline{X}$ is the closure of a bounded smooth domain in $\mathbb{C}$ (see footnote 2).

Open access funding enabled and organized by Projekt DEAL.

References

1. L. Ahlfors and L. Sario, *Riemann surfaces* (Princeton University Press, Princeton, NJ, 1960).

2. A. Brudnyi, 'On exponential factorizations of matrices over Banach algebras', Preprint, 2010, arXiv:2010.12903.

3. H. Cartan, 'Espaces fibrés analytiques', *Symposium International de Topologia Algebraica* (Universidad National Autónomica de Mexico and UNESCO, 1958) 97–121.

4. G. Corach and F. D. Suárez, 'Stable range in holomoprhic functions algebras', *Illinois J. Math.* 29 (1985) 627–639.

5. G. Corach and F. D. Suárez, 'Extension problems and stable rank in commutative Banach algebras', *Topology Appl.* 21 (1985) 1–8.

6. E. Doubtsov and F. Kutzschebauch, 'Factorization by elementary matrices, null-homotopy and products of exponentials for invertible matrices over rings', *Anal. Math. Phys.* 9 (2019) 1005–1018.

7. J. E. Fornæss, F. Forstnerič and E. F. Wold, 'Holomorphic approximation: the legacy of Weierstrass, Runge, Oka-Weil, and Mergelyan', *Advancements in complex analysis. from theory to practice* (eds D. Breaz and M. Th. Rassias; Springer, Cham, 2020) 133–192.

8. H. Grauert and R. Remmert, *Coherent analytic sheaves* (Springer, Berlin 1984).

9. L. Hörmander, *An introduction to complex analysis in several variables*, 3rd edn (North-Holland, Amsterdam, 1990).

10. P. W. Jones, D. Marshall and T. H. Wolff, 'Stable rank of the disc algebra', *Proc. Amer. Math. Soc.* 96 (1986) 603–604.

11. F. Kutzschebauch and L. Studer, 'Exponential factorizations of holomorphic maps', *Bull. Lond. Math. Soc.* 51 (2019) 995–1004.

12. J. Leiterer, 'On holomorphic matrices on bordered Riemann surfaces', Preprint, 2010, arXiv:2010.02581.

13. S. Łojasiewicz, *Introduction to complex analytic geometry* (Birkhäuser, Basel, 1991).
14. R. Mortini and R. Rupp, 'Logarithms and exponentials in the matrix algebra $\mathcal{M}_2(A)$', *Comput. Methods Funct. Theory* 18 (2018) 53–87.

Jürgen Leiterer
Institut für Mathematik
Humboldt-Universität zu Berlin
Rudower Chaussee 25
Berlin D-12489
Germany

leiterer@mathematik.hu-berlin.de