Aspects of the geometry of Prym varieties and their moduli

Dissertation
zur Erlangung des akademischen Grades

doctor rerum naturalium
(Dr. rer. nat.)

im Fach: Mathematik

eingereicht an der

Mathematisch-Naturwissenschaftlichen Fakultät
der Humboldt-Universität zu Berlin

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Tag der mündlichen Prüfung: 15.07.2021

A mis padres,
por su apoyo incondicional

Abstract

In this thesis, we study several moduli spaces of Prym pairs, Prym varieties, and spin curves. After the appropriate theoretical framework is introduced, we obtain new results concerning two different aspects of their geometry, which we describe across two corresponding chapters.

In Chapter 1, we consider the universal Prym variety over the moduli space $\mathcal{R}_g$ of Prym pairs of genus g, and determine its unirationality for $g = 3$. To do this, we build an explicit rational parametrization of the universal 2-fold Prym curve over $\mathcal{R}_3$, which dominates the universal Prym variety through the global version of the Abel-Prym map. Furthermore, we adapt the proof to the setting of Nikulin surfaces and show that the universal double Nikulin surface over $\mathcal{F}_3^{\mathfrak{N}}$ is also unirational.

In Chapter 2, we explore the interaction between $\overline{\mathcal{R}}_g$ and the moduli space $\overline{\mathcal{S}}_g$ of (stable) spin curves of genus g. When the divisor of curves equipped with a vanishing theta-null is moved from $\overline{\mathcal{S}}_g^+$ to $\overline{\mathcal{R}}_g$, it yields two geometric divisors $\overline{\mathcal{P}}_{\mathrm{null}}^+$ and $\overline{\mathcal{P}}_{\mathrm{null}}^-$ of (stable) Prym curves with a vanishing theta-null. We use test curve techniques to compute the classes of $\overline{\mathcal{P}}_{\mathrm{null}}^+$ and $\overline{\mathcal{P}}_{\mathrm{null}}^-$ in $\mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$ for $g \geq 5$, and evaluate these (Prym-null) classes on some more families of curves in order to analyse their vanishing theta-nulls.

In addition, at the end of Chapter 2 we discuss a potential compactification of the moduli space of curves carrying a double square root. We then examine the boundary of the moduli space $\overline{\mathcal{RS}}_g$ of (stable) Prym-spin curves of genus g and check the Prym-null classes against the diagram $\overline{\mathcal{R}}_g \leftarrow \overline{\mathcal{RS}}_g \rightarrow \overline{\mathcal{S}}_g$. Finally, we propose an extension of the product of roots, defined over smooth curves by the tensor product, to an operation on stable double roots.

Zusammenfassung

In dieser Doktorarbeit untersuchen wir einige Modulräume der Prym-Paaren, Prym-Varietäten und Spin-Kurven. Nachdem der passende theoretische Rahmen eingeführt wird, erhalten wir neue Ergebnisse zu zwei verschiedenen Aspekten ihrer Geometrie, die wir in zwei entsprechenden Kapiteln beschreiben.

In Kapitel 1 betrachten wir die universelle Prym-Varietät über dem Modulraum $\mathcal{R}_g$ der Prym-Paaren vom Geschlecht g und bestimmen ihre Unirationalität für $g = 3$. Dazu bilden wir eine explizite rationale Parametrisierung der universellen 2-fachen Prym-Kurve über $\mathcal{R}_3$, die die universelle Prym-Varietät durch die globale Version der Abel-Prym-Abbildung dominiert. Darüber hinaus passen wir den Beweis an den Rahmen von Nikulin-Flächen an und zeigen, dass die universelle doppelte Nikulin-Fläche über $\mathcal{F}_3^{\mathfrak{N}}$ ebenfalls unirational ist.

In Kapitel 2 untersuchen wir die Wechselwirkung zwischen $\overline{\mathcal{R}}_g$ und dem Modulraum $\overline{\mathcal{S}}_g$ der (stabilen) Spin-Kurven vom Geschlecht g. Wenn man den Divisor der Kurven, die mit einem verschwindenden Thetanull ausgestattet sind, von $\overline{\mathcal{S}}_g^+$ nach $\overline{\mathcal{R}}_g$ versetzt, erhält man zwei geometrische Divisoren $\overline{\mathcal{P}}^+_{\mathrm{null}}$ und $\overline{\mathcal{P}}^-_{\mathrm{null}}$ der (stabilen) Prym-Kurven mit einem verschwindenden Thetanull. Wir verwenden Testkurventechniken, um die Klassen von $\overline{\mathcal{P}}^+_{\mathrm{null}}$ und $\overline{\mathcal{P}}^-_{\mathrm{null}}$ in $\mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$ für $g \geq 5$ zu berechnen, und werten diese (Prymnull-)Klassen auf einigen weiteren Familien von Kurven aus, um ihre verschwindenden Thetanulls zu analysieren.

Darüber hinaus diskutieren wir am Ende von Kapitel 2 eine mögliche Kompaktifizierung des Modulraums der Kurven, die eine doppelte Quadratwurzel tragen. Anschließend untersuchen wir den Rand des Modulraums $\overline{\mathcal{RS}}_g$ der (stabilen) Prym-Spin-Kurven vom Geschlecht g und überprüfen die Prymnull-Klassen anhand des Diagramms $\overline{\mathcal{R}}_g \leftarrow \overline{\mathcal{RS}}_g \to \overline{\mathcal{S}}_g$. Zum Schluss schlagen wir eine Erweiterung des Produkts von Wurzeln, das über glatten Kurven durch das Tensorprodukt definiert ist, zu einer Operation auf stabilen Doppelwurzeln vor.

Acknowledgements

If I had to pinpoint the most important aspect of these last few years as a doctoral student, it would not be one of geometrical nature. It would no doubt be the many people that have made this journey possible. My appreciation for them knows no bounds, and I would like to take this opportunity to express at least a small part of it.

To begin with, I wish to thank my adviser Gavril Farkas for his continuous support and guidance. Since our first interaction in 2017, he has brought many problems to my attention, solved many others, and time and again pointed me in the right direction.

I am also grateful to the two institutions that have kindly hosted me during this period. The Humboldt University of Berlin has given me a chance to work and learn within its walls, and the Berlin Mathematical School has funded my research and welcomed me into the city. Special thanks go to Shirley, from the One-Stop Office, for her patience and invaluable assistance.

Many fellow mathematicians have made my Ph.D. experience better, and I would like to mention some of them. Mara, Jason and Leonardo have helped me survive the working routine. Andrei has been a lovely office mate, and a source of compelling conversation: in particular, I want to acknowledge his instructive remarks on the rational Picard group of the Prym moduli space. Finally, Emre and İrfan have warmly offered me support, advice and inspiration, which I am truly grateful for. Moreover, Emre's work on double spin roots is the backbone of the last section of this thesis.

Outside of academia, I would like to extend my sincere thanks to Mar, my therapist. She is one of the many pieces that have allowed this project to come to fruition, and I really am glad to have contacted her.

"A friend in need is a friend indeed", the saying goes. I am keenly aware of how lucky I am in this regard, and of the meaningful role that many of my old and not-so-old friends play in my life. As a small token of appreciation, I want to address some of them. To Fer, Rober and Miguel, thank you for making me feel that things are just like they were the last time we met. To Miriam, thank you for the bond we have come to build over the years, which fears no borders: I wish nothing else but to see it grow stronger. To Víctor, thank you for being my brother-in-arms, and a wonderful one at that: I cherish dearly the influence our friendship has on me, both at a personal and professional level. To Samual, thank you for being a brilliant (flat)mate, and an overall decent lad: I am very excited for Camilla and you, and wish you guys the best in your new adventure.

To Berta, thank you for your kind heart and contagious energy: my enjoyment of these last four years would have been so much less rich without you in them, and I look forward to the many stories that the future holds. To Bruch, Pablo, Amelia and Vicky, thank you for making this eclectic city all the more fun and welcoming for me, no matter the time or the circumstances. And to Pedro and Álvar, thank you for being not just my friends, but family.

On the topic of family, I cannot even begin to express my deepest gratitude for mine. Their unwavering support throughout the years has allowed me to be who I am, and is a driving force behind any project that I pursue. My parents, Miguel and Pilar, are a constant source of guidance and inspiration: they push me to be better, and my love of mathematics is clearly rooted in them, and the upbringing that they fondly gave me. I would like to dedicate this dissertation to them, and also to my grandmothers, who are a shining light in this world of ours. In addition, I want to thank everyone in my (extended) family for always being there for me, and especially Pedro and Lola for their help in proofreading the German version of the abstract.

There is one last, but certainly not least, person that deserves my warmest acknowledgment. Ελένη, you are the sweetest, kindest, coolest partner in crime that I could ever dream of. It could be argued that this work owes as much to you as it does to me, and I would not be one to disagree. Since we met for the first time, my life is brighter and my days are happier. From the bottom of my heart: thank you.

For the sake of the many Spanish-speaking people mentioned here, I believe it appropriate to provide a (somewhat liberal) translation of the above text:

Agradecimientos

Si tuviera que señalar el aspecto más importante de mis últimos años como estudiante de doctorado, no sería uno de naturaleza geométrica. Sería, sin lugar a dudas, la mucha gente que me ha acompañado en este camino, cuya presencia lo ha hecho transitable. Por ello, quisiera aprovechar estas líneas para expresar al menos una pequeña parte de mi profundo agradecimiento.

En primer lugar, me gustaría dar las gracias a mi supervisor Gavril Farkas por su apoyo y orientación, incluyendo su planteamiento de problemas de gran interés para la tesis y su inestimable ayuda con todo tipo de obstáculos.

También estoy agradecido a mi universidad (HU) y mi escuela de doctorado (BMS) por su acogida y financiación; y a Shirley, por su infinita paciencia.

Gracias a Mara, Jason, Leonardo, Andrei, Emre, İrfan y otros muchos compañeros de profesión, por hacer mi rutina más agradable y llevadera.

Fuera del ámbito académico, quisiera dar mi sincero agradecimiento a Mar, mi psicóloga. Ella es una de las razones por las cuales este proyecto ha podido llegar a buen puerto, y realmente me alegro de haber buscado su ayuda.

"Amigo en la adversidad es amigo de verdad", dice el refrán. Soy muy consciente de la suerte que tengo en este sentido, y del valioso papel que muchas de mis viejas y no tan viejas amistades desempeñan en mi vida. En particular, me gustaría hablar sobre algunas de ellas. A Fer, Rober y Miguel, gracias por hacer que cada vez que nos vemos todo sea como la última vez. A Miriam, gracias por el vínculo que hemos construido a lo largo de los años, que trasciende cualquier frontera: nada me alegra más que verlo seguir creciendo. A Vic, gracias por ser el mejor hermano de armas que pudiera imaginar: sin nuestras conversaciones, no sería quien soy ni como persona ni como matemático. A Sam, ya va siendo hora de aprender español más allá de contar hasta cinco. A Berta, gracias por tu amistad y por la alegría que me trae haberte conocido: si te lees la parte en inglés, me ahorras trabajo. A Bruch, Pablo, Amelia y Vicky, gracias por hacer de esta ciudad un lugar todavía más entretenido y acogedor. Y a Pedro y Álvar, gracias por ser tanto mis amigos como parte de la familia.

Hablando de familias, es complicado resumir mi enorme gratitud por la mía. Su apoyo firme e incondicional me ha permitido convertirme en la persona que soy hoy, y embarcarme en proyectos de cualquier calibre. Mis padres, Miguel y Pilar, son una fuente inagotable de guía e inspiración, y un impulso constante a seguir hacia adelante. Mi amor por las matemáticas brota claramente de ellos, y del cariño y la educación que tan generosamente me han dado. Por ese motivo, esta tesis doctoral está dedicada a ellos, y también a mis queridas abuelas, que son un radiante foco de luz que ilumina mi mundo. Además, quisiera extender mi agradecimiento a todos los demás miembros de mi familia ampliada, por su aliento y ayuda, y especialmente a Pedro y Lola, por revisar el precario alemán de mi Zusammenfassung.

Hay una última, pero para nada menos importante, persona que merece ser mencionada. Eleni, ni en mis más dulces sueños podría esperar contar con una compañera de fatigas tan increíble como tú. Podría decirse que este trabajo te debe tanto a ti como a mí, y no sería yo quien lo rebatiera. Tu presencia en mi vida hace que mi corazón se sienta cálido y que mis ojos sonrían. Por todo ello, y por mucho más: gracias.

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Notation (general)

We work over the field of complex numbers. Although most of the following conventions are clarified when they are first mentioned, we compile them below for the sake of readability.

variety	algebraic, reduced, separated scheme over $\mathbb{C}$
curve	variety of dimension 1
surface	variety of dimension 2
p.p.a.v.	principally polarized abelian variety
$\underline{\mathrm{Spec}}(\mathcal{A})$	relative Spec (spectrum) of a sheaf $\mathcal{A}$ of algebras
$\underline{\mathrm{Proj}}(\mathcal{A})$	relative Proj (homogeneous spectrum) of $\mathcal{A}$
$V^\vee$	dual space $\mathrm{Hom}(V, \mathbb{C})$ of a vector space V
$G(d, V)$	Grassmannian of d-dimensional linear subspaces of V
$\mathbb{P}(V)$	for V vector space: $G(1, V)$, i.e. projective space of lines of V
$\mathbb{P}(\mathcal{E})$	for $\mathcal{E}$ quasi-coherent sheaf: $\underline{\mathrm{Proj}}(\mathrm{Sym}\,\mathcal{E})$, i.e. $\mathbb{P}(\mathcal{E})_x = \mathbb{P}(\mathcal{E}\|_x{}^\vee)$
$\mathrm{Pic}(X)$	Picard group of a space X
$\mathrm{Pic}^d(X)$	component of $\mathrm{Pic}(X)$ of line bundles of degree d
$\mathrm{Pic}(X)_\mathbb{Q}$	rational Picard group of X, i.e. $\mathrm{Pic}(X)_\mathbb{Q} = \mathrm{Pic}(X) \otimes_\mathbb{Z} \mathbb{Q}$
$\mathcal{O}(D)$	line bundle associated to a divisor D
$\|L\|$	linear series $\mathbb{P}H^0(L)$ induced by a line bundle L
L_x	stalk of a bundle L at a point x
$L\|_x$	fiber of a bundle L at a point x, i.e. $L\|_x = L_x \otimes_{\mathcal{O}_x} \kappa(x)$
$h^i(L)$	$\dim H^i(L)$
$g(X)$	arithmetic genus $p_a(X)$ of a variety X
$p_g(X)$	geometric genus of X
$\overline{A}$	closure of a subspace $A \hookrightarrow X$
$\mathcal{I}_Z$	ideal sheaf of a closed subspace $Z \hookrightarrow X$
$\mathrm{Bl}_Z(X)$	blow-up of X at Z
$\mathrm{Sing}(X)$	set of singular points of X

Notation (specific)

$\sqrt{N}$	set of square roots of a line bundle N
$\mathcal{O}_C$	trivial bundle of a curve C
ω_C	canonical bundle of C
$\#U$	order of a set U
η	Prym root of a curve C, with $\eta^{\otimes 2} \cong \mathcal{O}_C$
(C, η)	Prym pair, equivalent to (π, ι) or (C', ι)
$R_g(C)$	set of Prym roots of C
$J(C)$	Jacobian variety of a curve C, with $J(C) = \mathrm{Pic}^0(C)$
$P(C, \eta)$	Prym variety associated to a Prym pair (C, η)
$(-)^\circ$	connected component containing the identity element
θ	theta characteristic of a curve C, with $\theta^{\otimes 2} \cong \omega_C$
(C, θ)	spin curve, of even $(+)$ or odd $(-)$ parity
$S_g(C)$	set of theta characteristics of C
$S_g^{\mathrm{x}}(C)$	parity-based subsets of $S_g(C)$, with $\mathrm{x} \in \{+, -\}$
s_g, s_g^+, s_g^-	orders of $S_g(C)$, $S_g^+(C)$ and $S_g^-(C)$, respectively
$S_\eta^{\mathrm{x},\mathrm{y}}(C)$	set of theta characteristics $\theta \in S_g^{\mathrm{x}}(C)$ with $\theta \otimes \eta \in S_g^{\mathrm{y}}(C)$
N_g^+, N_g^-, $\mathrm{N}_g^\pm$	orders of $S_\eta^{+,+}(C)$, $S_\eta^{-,-}(C)$ and $S_\eta^{+,-}(C)$, respectively
$\mathrm{st}(X)$	stable model of a quasistable curve X
B_X	$\overline{X - E(X)}$ with $E(X) \subset X$ exceptional components
B_{pq}	irreducible 1-nodal curve (obtained by gluing $p, q \in B$)
(C, η, θ)	Prym-spin curve, with $\eta \in R_g(C)$ and $\theta \in S_g(C)$
$(L_1, L_2)_{\mathrm{sync}}$	double limit root of (N_1, N_2)
$\mathcal{D}(N_1, N_2)$	difference (line) bundle $N_1 \otimes N_2^\vee$
V_i, $V(L_i)$	largest open set where $\rho_i \colon X \cong X_i$ is an isomorphism
$\circledast$	"limit product" $(L_1, L_2)_{\mathrm{sync}} \mapsto (L_1, L_2, L_1 \circledast L_2)_{\mathrm{sync}}$

Notation (moduli spaces)

$\mathcal{M}$	moduli stack, usually not proper (e.g. $\mathcal{M} = \mathcal{M}_g, \mathcal{R}_g, \mathcal{S}_g, \ldots$)
M	coarse moduli space for $\mathcal{M}$, with $\mathcal{M} \to M$ initial, M scheme
$\overline{\mathcal{M}}$	compactification of $\mathcal{M}$, with $\mathcal{M} \hookrightarrow \overline{\mathcal{M}}$ open, $\overline{\mathcal{M}}$ proper
$\rotatebox[origin=c]{-90}{$\rightsquigarrow$}$	
$\mathcal{M}_g$	moduli stack of smooth curves of genus g
$\overline{\mathcal{M}}_g$	moduli stack of stable curves of genus g
$\mathcal{A}_g$	moduli stack of p.p.a.v. of dimension g
$\mathcal{R}_g$	Prym moduli space (moduli stack of Prym pairs of genus g)
$\overline{\mathcal{R}}_g$	moduli stack of stable Prym curves of genus g
$\mathcal{S}_g$, $\mathcal{S}_g^+$, $\mathcal{S}_g^-$	moduli stacks of (even, odd) spin curves of genus g
$\overline{\mathcal{S}}_g$, $\overline{\mathcal{S}}_g^+$, $\overline{\mathcal{S}}_g^-$	moduli stacks of stable (even, odd) spin curves of genus g
$\mathcal{Y}_g \to \mathcal{R}_g$	universal Prym variety over $\mathcal{R}_g$, with $(\mathcal{Y}_g)_{(C,\eta)} \cong P(C,\eta)$
$(\mathcal{C}')^2 \to \mathcal{R}_g$	universal 2-fold Prym curve, with $\mathcal{C}' \cong \mathcal{R}_g \times_{\mathcal{M}_{2g-1}} \mathcal{C}_{2g-1}$
$\mathcal{F}_{g,2}^{\mathfrak{N}} \to \mathcal{F}_g^{\mathfrak{N}}$	universal double Nikulin surface (of genus g)
Δ	divisor of $\overline{\mathcal{M}}$ or $\overline{M}$
δ	divisor class $\mathcal{O}_{\overline{\mathcal{M}}}(\Delta) \in \operatorname{Pic}(\overline{\mathcal{M}})$ in the moduli stack
$[\Delta]$	divisor class $\mathcal{O}_{\overline{M}}(\Delta) \in \operatorname{Pic}(\overline{M})$ in the coarse moduli space
$\rotatebox[origin=c]{-90}{$\rightsquigarrow$}$	
Δ_0, Δ_i	boundary divisors of $\overline{\mathcal{M}}_g$, with $i \in \{1, \ldots, \lfloor g/2 \rfloor\}$
Δ_0^{x}, Δ_i^{y}	boundary divisors of $\overline{\mathcal{R}}_g$, with $\mathrm{x} \in \{\mathrm{t}, \mathrm{p}, \mathrm{b}\}$, $\mathrm{y} \in \{\mathrm{n}, \mathrm{t}, \mathrm{p}\}$
Δ_0^{x}, Δ_i^{y}	boundary divisors of $\overline{\mathcal{S}}_g^+$, with $\mathrm{x} \in \{\mathrm{n}, \mathrm{b}\}$, $\mathrm{y} \in \{+, -\}$
λ	Hodge class, corresponding to $\bigwedge^g \phi_*(\omega_\phi)$
Θ_{null}	theta-null divisor (on $\mathcal{S}_g^+$)
$\mathcal{P}_{\mathrm{null}}^+$, $\mathcal{P}_{\mathrm{null}}^-$	even and odd Prym-null divisors (on $\mathcal{R}_g$)
$\varrho_{\mathrm{null}}^+$, $\varrho_{\mathrm{null}}^-$	Prym-null classes (divisor classes of $\overline{\mathcal{P}}_{\mathrm{null}}^+$ and $\overline{\mathcal{P}}_{\mathrm{null}}^-$)
$\mathcal{S}_{\mathrm{lim}}^2(\mathcal{N}_1, \mathcal{N}_2)$	moduli space of double limit roots of $(\mathcal{N}_1, \mathcal{N}_2)$
$\overline{\mathcal{RS}}_g$	moduli space of stable Prym-spin curves, $\hookrightarrow \mathcal{S}_{\mathrm{lim}}^2(\mathcal{O}_{\overline{\mathcal{C}}_g}, \omega_\phi)$

Introduction

There are no two moduli spaces quite as studied in Algebraic Geometry as those of algebraic curves and abelian varieties. Their shared history dates back to the 19th century[1] and has continued to evolve to this very day, giving rise to a sprawling theory and many open questions. It is precisely between these two spaces that we find the moduli space $\mathcal{R}_g$ of *Prym pairs*, or *Prym moduli space*, parametrizing points (C, η) such that C is a smooth curve and η is a nontrivial square root of its trivial bundle $\mathcal{O}_C$, in the sense of:

$$\eta \in \mathrm{Pic}^0(C) - \{\mathcal{O}_C\}, \qquad \eta^{\otimes 2} \cong \mathcal{O}_C$$

From a geometrical point of view, this *Prym root* η is the algebraic version of a nontrivial étale double cover of C, characterised as:

$$\pi \colon C' \to C, \qquad C' = \underline{\mathrm{Spec}}(\mathcal{O}_C \oplus \eta)$$

The involution induced by π on the Jacobian variety $J(C')$ brings to the table a subgroup of (-1)-eigenvectors, whose central connected component

$$P(C, \eta) \hookrightarrow J(C')$$

naturally inherits the structure of a principally polarized abelian variety[2]. In his landmark paper [Mum74], Mumford refers to $P(C, \eta)$ as the *Prym variety* of a double cover, in recognition of the German mathematician Friedrich Prym. An account of Prym's life and contributions, included in [Far12] Section 1, reveals his instrumental role as "Riemann's interpreter", and is a highly recommended read that provides further historical context for Prym curves and varieties.

When the Prym construction is applied to $\mathcal{R}_g$, it results in a diagram

$$\begin{tikzcd} & \mathcal{R}_g \arrow[dl] \arrow[dr] & \\ \mathcal{M}_g & & \mathcal{A}_{g-1} \end{tikzcd} \qquad \begin{tikzcd} & (C,\eta) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ C & & P(C,\eta) \end{tikzcd}$$

connecting the aforementioned moduli spaces. This fleshes out the relevance of Prym pairs within Algebraic Geometry, as they offer an important insight into

[1] When Riemann inadvertently obtained the dimension of $\mathcal{M}_g$ through an elegant *moduli count*, as featured in [Rie57], and Abel and Jacobi laid the foundations for Jacobian varieties to be regarded as abelian varieties that make up a distinguished locus in $\mathcal{A}_g$.

[2] Readers averse to lengthy expressions may find relief in the standard acronym p.p.a.v.

the behaviour of abelian varieties from the perspective of curves (especially for genus $g \leq 6$, where the Prym map $\mathcal{R}_g \to \mathcal{A}_{g-1}$ remains dominant). Additional interest in Prym curves stems from their interaction with *theta characteristics*, that is, square roots θ of the canonical bundle ω_C, given by:

$$\theta \in \mathrm{Pic}^{g-1}(C), \qquad \theta^{\otimes 2} \cong \omega_C$$

The moduli space $\mathcal{S}_g$ of *spin curves*, or pairs (C, θ) such that θ is a theta characteristic, splits into two components $\mathcal{S}_g^+$, $\mathcal{S}_g^-$ according to the *parity* of θ. Then the tensor product yields a map

$$\mathcal{R}_g \times_{\mathcal{M}_g} \mathcal{S}_g \to \mathcal{S}_g, \qquad (C, \eta, \theta) \mapsto (C, \theta \otimes \eta)$$

which can be considered in relation to the components $\mathcal{S}_g^+$ and $\mathcal{S}_g^-$, leading to a decomposition of $\mathcal{R}_g \times_{\mathcal{M}_g} \mathcal{S}_g$ into four spaces

$$\mathcal{RS}_g^{++} \sqcup \mathcal{RS}_g^{+-} \sqcup \mathcal{RS}_g^{-+} \sqcup \mathcal{RS}_g^{--}$$

where $\mathcal{RS}_g^{+-}$ parametrizes triplets (C, η, θ) with $\theta \in \mathcal{S}_g^+(C)$ and $\theta \otimes \eta \in \mathcal{S}_g^-(C)$, and so on. Spin curves and Prym pairs have long been close partners, but it is this very specific interplay that we hope to explore.

In view of the above, the aim of this thesis is twofold. First, we add to the research on the Prym moduli space in low genus by determining the birational geometry of the *universal Prym variety* $\mathcal{Y}_g \to \mathcal{R}_g$ for genus $g = 3$. This moduli stack, parametrizing points (C, η, L) such that

$$(C, \eta) \in \mathcal{R}_g, \qquad L \in P(C, \eta)$$

should be understood as the (universal) family of all of the Prym varieties that arise from Prym pairs of genus g, which are recovered as the fibers of $\mathcal{Y}_g \to \mathcal{R}_g$. In particular, the universal Prym variety $\mathcal{Y}_3$ sits in the middle of a diagram

$$(\mathcal{C}')^2 \xrightarrow{\mathfrak{ap}} \mathcal{Y}_3 \longrightarrow \mathcal{R}_3$$

of dominant maps, where $(\mathcal{C}')^2 = \mathcal{C}' \times_{\mathcal{R}_g} \mathcal{C}'$ is the 2-fold product over $\mathcal{R}_g$ of the universal Prym cover, and $\mathfrak{ap}$ is the universal *Abel-Prym map*. By studying the geometry of general Prym pairs of genus 3, we build a rational parametrization $\mathbb{P}^{12} \approx G(3, 7) \to (\mathcal{C}')^2$, and so obtain the following results:

Theorem A. *In genus 3, it holds that:*

(i) *The universal 2-fold Prym curve $(\mathcal{C}')^2 \to \mathcal{R}_3$ is unirational.*

(ii) *The universal Prym variety $\mathcal{Y}_3 \to \mathcal{R}_3$ is unirational.*

(iii) *The universal double Nikulin surface $\mathcal{F}^{\mathfrak{N}}_{3,2} \to \mathcal{F}^{\mathfrak{N}}_3$ is unirational.*

In the main body of work, this corresponds to theorem 1.4.6, corollary 1.4.7 and theorem 1.5.8. Note that the statement on Nikulin surfaces is an extension of the first result, as the proof can be adapted from curves to surfaces.

Our second objective is related to the link between $\mathcal{R}_g$ and $\mathcal{S}_g$. On the even component $\mathcal{S}_g^+$ of the latter space, we have the well-known geometric divisor of curves with a *vanishing theta-null*, or *theta-null divisor*:

$$\Theta_{\text{null}} \;=\; \{(C,\theta) \in \mathcal{S}_g^+ \;/\; h^0(C,\theta) \geq 2\}$$

The theta-null divisor can be moved to the Prym moduli space by means of the two commutative diagrams

$$\begin{tikzcd} & \mathcal{R}_g \arrow[dr] & \\ \mathcal{RS}_g^{++} \arrow[ur] \arrow[rr] \arrow[dr] & & \mathcal{M}_g \\ & \mathcal{S}_g^+ \arrow[ur] & \end{tikzcd} \quad \Bigg| \quad \begin{tikzcd} & \mathcal{R}_g \arrow[dr] & \\ \mathcal{RS}_g^{+-} \arrow[ur] \arrow[rr] \arrow[dr] & & \mathcal{M}_g \\ & \mathcal{S}_g^+ \arrow[ur] & \end{tikzcd}$$

hence producing a *Prym-null divisor* $\mathcal{P}_{\text{null}} \subset \mathcal{R}_g$ that splits into two irreducible components $\mathcal{P}^+_{\text{null}}$ and $\mathcal{P}^-_{\text{null}}$, namely

$$\begin{aligned} \mathcal{P}^+_{\text{null}} &= \{(C,\eta) \in \mathcal{R}_g \;/\; \exists\, \theta \in \Theta_{\text{null}}(C) \text{ with } \theta \otimes \eta \in S_g^+(C)\} \;\subset\; \mathcal{R}_g \\ \mathcal{P}^-_{\text{null}} &= \{(C,\eta) \in \mathcal{R}_g \;/\; \exists\, \theta \in \Theta_{\text{null}}(C) \text{ with } \theta \otimes \eta \in S_g^-(C)\} \;\subset\; \mathcal{R}_g \end{aligned}$$

which we call *even and odd Prym-null divisors*. After examining how the parity of spin curves changes when tensored by a Prym curve, we start a computation of the classes of $\overline{\mathcal{P}}^+_{\text{null}}$ and $\overline{\mathcal{P}}^-_{\text{null}}$ in the rational Picard group $\text{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$. With the help of the *method of test curves* and the theory of *limit linear series* on curves of compact type, this process culminates in the following result:

Theorem B. *For $g \geq 5$, the rational divisor classes of $\overline{\mathcal{P}}^{+}_{\mathrm{null}}$ and $\overline{\mathcal{P}}^{-}_{\mathrm{null}}$ in*

$$\mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}} \;=\; \left\langle \lambda,\, \delta_0'',\, \delta_0',\, \delta_0^{\mathrm{ram}},\, \{\delta_i,\, \delta_{g-i},\, \delta_{i:g-i}\}_{i=1,\dots,\lfloor g/2\rfloor} \right\rangle_{\mathbb{Q}}$$

are given by:

$$\begin{aligned}
\overline{\mathcal{P}}^{+}_{\mathrm{null}} \;\equiv\; 2^{g-3}\Bigg(& (2^{g-1}+1)\,\lambda \;-\; \frac{1}{4}\left(2^{g-2}\,\delta_0' + (2^{g-1}+1)\,\delta_0^{\mathrm{ram}}\right) \\
& - \;\sum \Big((2^{i-1}-1)(2^{g-i}-1)\,\delta_i + (2^i-1)(2^{g-i-1}-1)\,\delta_{g-i} \,+ \\
& \qquad + (2^{g-1}-2^{i-1}-2^{g-i-1}+1)\,\delta_{i:g-i} \Big)\Bigg)
\end{aligned}$$

$$\begin{aligned}
\overline{\mathcal{P}}^{-}_{\mathrm{null}} \;\equiv\; 2^{g-3}\Bigg(& 2^{g-1}\,\lambda \;-\; \frac{1}{4}\left(2^{g-1}\,\delta_0'' + 2^{g-2}\,\delta_0' + (2^{g-1}-1)\,\delta_0^{\mathrm{ram}}\right) \\
& - \;\sum \Big(2^{i-1}\,(2^{g-i}-1)\,\delta_i + (2^i-1)\,2^{g-i-1}\,\delta_{g-i} \,+ \\
& \qquad + (2^{g-1}-2^{i-1}-2^{g-i-1})\,\delta_{i:g-i} \Big)\Bigg)
\end{aligned}$$

In the main body of work, this corresponds to theorem 2.3.14. Note that the notation for the generating classes of $\mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$ is the traditional one here[3], but that a personal variant is used anywhere else in this thesis. The reason behind this change in notation concerns the study of a compactification of $\mathcal{R}_g \times_{\mathcal{M}_g} \mathcal{S}_g$, which became our focus during the final period of my doctoral research. It is a difficult task to describe the boundary of such a space, and the new notation is intended to make this possible.

While developing theorem B, it became clear to us that it would be efficient (and enlightening) to approach the problem from the perspective of *Prym-spin curves* (C, η, θ). Unfortunately, there seem to be no compactifications of $\mathcal{RS}_g^{++}$ and $\mathcal{RS}_g^{+-}$ in the literature, although the moduli space of multiple spin curves has recently been compactified by Sertöz (in [Ser17] and [Ser19]). If his results could be adapted to spaces of multiple roots of different bundles, then not only would we be able to provide a much more essential proof of theorem B, but the doors would also be wide open for a beautiful new theory to blossom. The time constraints that a Ph.D. project entails, coupled with the fact that this course of action was not in our original plans, prevented us from working towards this goal with the fidelity that it deserves. However, we still arrived at several ideas which appear promising, and are thus included in this thesis. Due to their role

[3] As detailed in [FL10] Section 1, Ex. 1.3 & 1.4, or [Far12] Section 6, Ex. 6.5 & 6.6.

in our understanding of theorem B, they are collected in the final section of the second chapter. We briefly summarize them next.

Following closely the efforts of Sertöz, we propose a candidate space

$$\mathcal{S}^2_{\mathrm{lim}}(\mathcal{N}_1, \mathcal{N}_2) \to \overline{\mathcal{M}}_g$$

for the compactification of the moduli space of *double roots* of $(\mathcal{N}_1, \mathcal{N}_2)$, whose points are referred to as *double limit roots*; this is definition 2.4.7. We have no rigorous proof that the arguments from the multiple spin setting apply to ours, but we fully believe this to be the case; see remark 2.4.9. Under the (generous) assumption that they do, we consider the compactification

$$\overline{\mathcal{RS}}_g \quad = \quad \mathcal{S}^2_{\mathrm{lim}}(\mathcal{O}_{\overline{\mathcal{C}}_g}, \omega_\phi) - \overline{\mathcal{S}}_g$$

of the space $\mathcal{R}_g \times_{\mathcal{M}_g} \mathcal{S}_g$, which splits into four connected components

$$\overline{\mathcal{RS}}^{++}_g \sqcup \overline{\mathcal{RS}}^{+-}_g \sqcup \overline{\mathcal{RS}}^{-+}_g \sqcup \overline{\mathcal{RS}}^{--}_g$$

After describing the boundaries of $\overline{\mathcal{RS}}^{++}_g$ and $\overline{\mathcal{RS}}^{+-}_g$, we derive both Prym-null classes in an alternative fashion, and obtain the same formulas as in theorem B. Finally, we further propose a candidate notion of "product of limit roots" that seeks to extend the tensor product of square roots, in the form of a map

$$\circledast\colon \quad \mathcal{S}^2_{\mathrm{lim}}(N_1, N_2) \quad \to \quad \mathcal{S}^3_{\mathrm{lim}}(N_1, N_2, N_1 \otimes N_2)$$

which we build at the level of points; this is proposition 2.4.16. Observe that if this map could be defined at the level of families, it would have many potential applications, such as the extension of the isomorphisms

$$\mathcal{RS}^{\mathrm{xy}}_g \cong \mathcal{S}^{\mathrm{x}}_g \times_{\mathcal{M}_g} \mathcal{S}^{\mathrm{y}}_g, \qquad (C, \eta, \theta) \leftrightarrow (C, \theta, \theta \otimes \eta)$$

to the corresponding compactifications $\overline{\mathcal{RS}}^{\mathrm{xy}}_g$ and $\overline{\mathcal{S}}^{\mathrm{xy}}_g$, for all $\mathrm{x}, \mathrm{y} \in \{+, -\}$.

Before finishing this introduction, we point out that theorem B has received in recent times an independent proof by Rojas; see [Roj21]. His argument, also developed as part of his ongoing doctoral research, relies on largely similar test curve techniques, although a clever use of pushforwards removes the need for a final family of test curves. In addition, he puts together an explicit adaptation of Teixidor's irreducibility analysis for $\overline{\mathcal{P}}^+_{\mathrm{null}}$ and $\overline{\mathcal{P}}^-_{\mathrm{null}}$, which we previously were addressing with a reference to [TiB88]; this is now remark 2.2.3.

We conclude with a succinct breakdown of the contents of the thesis.

Content breakdown. This dissertation is comprised of two chapters:

(1) Chapter 1: Birational geometry of the universal Prym variety.

Structured around theorem A, this chapter is divided into five sections. Sections 1.1, 1.2 and 1.3 focus on preliminary theory, while sections 1.4 and 1.5 focus on novel contributions.

(1.1) Section 1.1 is the cornerstone of the thesis, introducing square roots of arbitrary line bundles, their interpretation as possibly branched double covers, and the particular cases of Prym pairs and spin curves.

(1.2) Section 1.2 continues to lay the groundwork, with the construction of the Prym variety first as an abelian variety, secondly as a principally polarized abelian variety, and finally as a subgroup of $\mathrm{Pic}^{2g-2}(C')$. Moreover, the Abel-Prym map is introduced.

(1.3) Section 1.3 recounts the theory of fine and coarse moduli spaces and moduli stacks, and then uses it to define the Prym moduli space, the universal Prym variety, and other relative versions of earlier notions. Birational geometry of moduli spaces, including Grassmannians, is discussed at the end.

(1.4) Section 1.4 describes the geometry of Prym pairs of genus 3, and then of 2-pointed pairs. This allows for a parametrization of the universal 2-fold Prym curve to be built explicitly, thus proving the first part of theorem A.

(1.5) Section 1.5 introduces the basics of the theory of (polarized) K3 and Nikulin surfaces, together with the universal double Nikulin surface. Then the proof of the previous section is adapted to the Nikulin case, thus completing theorem A.

(2) Chapter 2: Prym curves with a vanishing theta-null.

Structured around theorem B, this chapter is divided into four sections. Section 2.1 focuses on preliminary theory, section 2.2 plays an intermediate role and sections 2.3 and 2.4 focus on novel contributions.

(2.1) Section 2.1 provides the necessary background on compactifications of moduli spaces of curves, defining stable and quasistable curves, describing the boundary divisor classes and rational Picard groups of $\overline{\mathcal{M}}_g$, $\overline{\mathcal{R}}_g$ and $\overline{\mathcal{S}}_g$, and discussing several families of test curves used in the later computation.

(2.2) Section 2.2 introduces the even and odd Prym-null divisors, as well as the class expansion of the sum $\overline{\mathcal{P}}^+_{\text{null}} + \overline{\mathcal{P}}^-_{\text{null}}$. In addition, the change in parity induced on spin curves by a Prym curve is studied, both in a smooth and a singular setting.

(2.3) Section 2.3 explores the behaviour of the Prym-null divisors in relation to the different collections of test curves that were previously introduced. This requires some standard facts on limit linear series, which are given at the beginning. Once the class computation is completed, theorem B is derived and applied to three more families of curves.

(2.4) Section 2.4 defines double limit roots (of the same bundle) and suggests how to extend this notion to double limit roots of different bundles. A compactification of the Prym-spin moduli space is then considered, offering a different perspective from which to look at theorem B. Finally, a notion of "product of limit roots" is built at the level of points.

Chapter 1

Birational geometry of the universal Prym variety

1.1 Square roots of line bundles

A crucial aspect of the theory of algebraic curves is the fact that there are several particularly relevant bundles naturally associated to any curve, such as the trivial or the canonical bundle. The study of square roots of these bundles (with respect to the tensor product) turns out to be a fruitful pursuit, and some of its many facets can be traced back to the 19th century. A modern algebraic treatment of such (*Prym* and *spin*) structures finally starts in the 1970s, fueled by the influential papers [Mum74] and [Mum71]. In this section, we discuss the notion of square roots of line bundles, its connection to double covers of curves, and the particular cases of Prym pairs and spin curves.

We work over the complex field, with the following conventions: a “variety” is an algebraic, reduced, separated scheme over $\mathbb{C}$, and a “curve” is a variety of dimension 1. Note that we allow varieties to be reducible.

1.1.1 Square roots and double covers

Let C be a smooth, integral curve of genus g, and let N be a line bundle on C of even degree d.

Definition 1.1.1. A *square root* of N is a line bundle L on C equipped with an isomorphism $\alpha\colon L^{\otimes 2} \cong N$. An isomorphism $(L_1, \alpha_1) \cong (L_2, \alpha_2)$ of square roots is an isomorphism $\psi\colon L_1 \cong L_2$ such that $\alpha_1 = \alpha_2 \circ \psi^{\otimes 2}$. The set of isomorphism classes of square roots of N will be denoted by $\sqrt{N}$.

It is easy to see that two square roots are isomorphic if and only if they are isomorphic as line bundles, which translates into an injection

$$\sqrt{N} \hookrightarrow \mathrm{Pic}^{d/2}(C), \quad [(L, \alpha)] \mapsto [L]$$

As is standard practice with line bundles, we will abuse notation and also refer to the isomorphism class $[(L, \alpha)] \equiv [L]$ as the *square root* $L \in \sqrt{N}$.

Remark 1.1.2. $J_2(C) = \sqrt{\mathcal{O}_C}$ is the 2-torsion of the Jacobian, which is a finite group of order 2^{2g} by the standard theory of abelian varieties (see for example [Mum08] Chapter II, Section 6). In addition, the tensor product of a 2-torsion element $T \in J_2(C)$ and a root $L \in \sqrt{N}$ is again a root $L \otimes T \in \sqrt{N}$. This gives rise to a group action of $J_2(C)$ on every space of square roots $\sqrt{N}$, so that $\sqrt{N}$ is in bijection with $J_2(C)$ and is also of order 2^{2g}.

Square roots of effective bundles have a natural interpretation as (possibly branched) double covers. Indeed, fix an effective divisor B on C of degree d and take $N = \mathcal{O}_C(B)$. Let $L \in \sqrt{N}$ and observe that $L^\vee \in \sqrt{N^\vee}$. Then we can use the morphism $\beta\colon (L^\vee)^{\otimes 2} \cong N^\vee \cong \mathcal{O}_C(-B) \hookrightarrow \mathcal{O}_C$ induced by $\alpha\colon L^{\otimes 2} \cong N$ to make $\mathcal{O}_C \oplus L^\vee$ into an $\mathcal{O}_C$-algebra with the product

$$(a, s) \cdot (b, t) = (ab + \beta(s \otimes t), at + bs)$$

which in turn yields a curve $C' = \underline{\mathrm{Spec}}(\mathcal{O}_C \oplus L^\vee)$ over C.

Proposition 1.1.3. *The projection $\pi\colon C' \to C$ is a flat double cover branched over B fitting into the following diagram:*

$$\begin{tikzcd} C' = \underline{\mathrm{Spec}}(\mathcal{O}_C \oplus L^\vee) \arrow[rr, hook] \arrow[dr, "\pi", "2:1"'] & \circlearrowleft & \mathbb{P}(\mathcal{O}_C \oplus L^\vee) \arrow[dl, "\mathbb{P}^1"] \\ & C & \end{tikzcd}$$

Furthermore, π restricts to a bijection $\mathrm{Sing}(C') \to \mathrm{Sing}(B)$*; in particular, C' is smooth if and only if B has no multiple points.*

Proof. Consider the global section $1_B \in H^0(C, \mathcal{O}_C(B))$ whose zero locus is the divisor B, given by the natural inclusion

$$H^0(C, \mathcal{O}_C) \hookrightarrow H^0(C, \mathcal{O}_C(B)), \quad 1 \mapsto 1_B$$

Take an affine local trivialization $\{U_i, v_i\}$ of L such that $\alpha(v_i^{\otimes 2}) = 1_B|_{U_i}$, which in turn trivializes $\mathcal{O}_C(B)|_{U_i} \cong (L|_{U_i})^{\otimes 2} \cong \mathcal{O}_{U_i}$ via α^{-1}. On the one hand, if we define $f_i \in H^0(U_i, \mathcal{O}_C(-B)) \hookrightarrow H^0(U_i, \mathcal{O}_C)$ to be the section corresponding to the morphism

$$f_i\colon \mathcal{O}_C(B)|_{U_i} = 1_B|_{U_i}\, \mathcal{O}_{U_i} \longrightarrow \mathcal{O}_{U_i}, \quad \lambda \cdot 1_B|_{U_i} \longmapsto \lambda$$

then B is, by construction, the Cartier divisor $\{(U_i, f_i)\}$. On the other hand, if we let $s_i \in H^0(U_i, L^\vee)$ be the section

$$s_i\colon L|_{U_i} = v_i\, \mathcal{O}_{U_i} \longrightarrow \mathcal{O}_{U_i}, \quad \lambda\, v_i \longmapsto \lambda$$

induced by $v_i \in H^0(U_i, L)$, i.e. with $s_i(v_i) = 1$, then it holds that

$$\alpha(v_i^{\otimes 2}) = 1_B|_{U_i} \Leftrightarrow \beta(s_i^{\otimes 2}) = f_i$$

In particular $L^\vee|_{U_i} \cong s_i\, \mathcal{O}_{U_i}$ and $(a, \lambda\, s_i) \cdot (b, \mu\, s_i) = (ab + \lambda\mu\, f_i, (a\mu + b\lambda)\, s_i)$, so we have an isomorphism of $\mathcal{O}_{U_i}$-algebras

$$\begin{array}{ccccc} (\mathcal{O}_C \oplus L^\vee)|_{U_i} & \cong & \mathcal{O}_{U_i} \oplus L^\vee|_{U_i} & \cong & \mathcal{O}_{U_i}[\mathsf{t}]/(\mathsf{t}^2 - f_i) \\ & & (a, \lambda\, s_i) & \mapsto & a + \lambda\, \mathsf{t} \end{array}$$

showing that $\pi \colon C' \to C$ is a flat double cover branched over the zeroes of $\{f_i\}$, that is to say, branched over B. Moreover, from the local description it follows that a point $y \in \pi^{-1}(U_i) \subset C'$ is singular if and only if both f_i and its Jacobian matrix vanish at $\pi(y)$, which amounts to $\pi(y)$ being a zero of f_i of order ≥ 2. Therefore C' is smooth if and only if B is also smooth.

Finally, the diagram above stems from the closed embedding

$$\underline{\mathrm{Spec}}(\mathcal{O}_C \oplus L^\vee) \cong \underline{\mathrm{Proj}}\,(\mathcal{O}_C \oplus L^\vee)[\mathsf{x}] \hookrightarrow \mathbb{P}(\mathcal{O}_C \oplus L^\vee) \to C$$

induced by $\mathrm{Sym}_{\mathcal{O}_C}(\mathcal{O}_C \oplus L^\vee) \twoheadrightarrow (\mathcal{O}_C \oplus L^\vee)[\mathsf{x}]$. $\square$

With the notation we have introduced to prove the proposition, it is easy to characterise the image $C' \hookrightarrow \mathbb{P}(\mathcal{O}_C \oplus L^\vee)$. For $x \in C$, take $f_x = f_i(x) \in \kappa(x)$ and $h_x \in \kappa(x)$ such that $(h_x)^2 = f_x$. Then the fibers of π are:

$$\pi^{-1}(x) = \mathrm{Spec}\,\kappa(x)[\mathsf{t}]/(\mathsf{t}^2 - f_x) = \begin{cases} \{(\mathsf{t} - h_x), (\mathsf{t} + h_x)\} & = \{x_1, x_2\} \quad \text{if } x \notin B \\ \{(\mathsf{t})\} & = \{q\} \quad\quad\;\; \text{if } x \in B \end{cases}$$

Choosing the basis $\{(1,0), (0, v_x)\} \subset \kappa(x) \oplus L|_x$ with $v_x = v_i(x) \in L|_x$, we can compute the image of $\pi^{-1}(x)$ and $\pi^{-1}(p)$ for $x \notin B$ and $p \in B$:

$$\begin{array}{rclcccc} x_1 & = & (\mathsf{t} - h_x) & \mapsto & (1 : h_x) & \in & \mathbb{P}(\mathcal{O}_C \oplus L^\vee)_x \\ x_2 & = & (\mathsf{t} + h_x) & \mapsto & (1 : -h_x) & \in & \mathbb{P}(\mathcal{O}_C \oplus L^\vee)_x \\ q & = & (\mathsf{t}) & \mapsto & (1 : 0) & \in & \mathbb{P}(\mathcal{O}_C \oplus L^\vee)_p \end{array}$$

To conclude, if we set $\omega_x = h_x\, v_x \in L|_x$ and consider the restriction

$$\begin{array}{cccccc} \alpha \colon & (L|_x)^{\otimes 2} & \cong & \mathcal{O}_C(B)|_x & \cong & \kappa(x) \\ & v_x^{\otimes 2} & \mapsto & 1_x & \mapsto & 1 \\ & \omega_x^{\otimes 2} & \mapsto & f_x \cdot 1_x & \mapsto & f_x \end{array}$$

then it is clear that the rational points of $C' \hookrightarrow \mathbb{P}(\mathcal{O}_C \oplus L^\vee)$ are of the form

$$C' \;\cong\; \{(x; 1, \omega_x) \;/\; \omega_x \in L|_x,\ \alpha(\omega_x^{\otimes 2}) = f_x \in \kappa(x)\} \;\subset\; \mathbb{P}(\mathcal{O}_C \oplus L^\vee)$$

where $f_x = f_i(x) = 0$ if and only if $x \in B$.

Furthermore, for every affine open U in C it holds that, by definition of the global Spec, $\pi^{-1}(U) \cong \operatorname{Spec} H^0(U, \mathcal{O}_C \oplus L^\vee)$. In other words:

$$H^0(U, \pi_*\mathcal{O}_{C'}) = H^0(\pi^{-1}(U), \mathcal{O}_{C'}) \cong H^0(U, \mathcal{O}_C \oplus L^\vee)$$

so we have a decomposition $\pi_*\mathcal{O}_{C'} \cong \mathcal{O}_C \oplus L^\vee$. This is in fact the case for any double cover of C.

Proposition 1.1.4. *Let $\pi\colon C' \to C$ be a flat double cover branched over B and ramified over $R = \pi^{-1}(B)$. Then the exact sequence of $\mathcal{O}_C$-modules*

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{\pi^\#} \pi_*\mathcal{O}_{C'} \longrightarrow \operatorname{coker}(\pi^\#) \cong L^\vee \longrightarrow 0 \qquad (\star)$$

splits, and $L = \operatorname{coker}(\pi^\#)^\vee$ is a square root of $\mathcal{O}_C(B)$ such that $\pi^*L \cong \mathcal{O}_{C'}(R)$.

Proof. Consider the trace map $\operatorname{tr}\colon \pi_*\mathcal{O}_{C'} \to \mathcal{O}_C$ associated to π, so that $\operatorname{tr} \circ \pi^\#$ corresponds to multiplication by 2. It follows that the map $\operatorname{tr}/2\colon \pi_*\mathcal{O}_{C'} \to \mathcal{O}_C$ is a retraction of $\pi^\#\colon \mathcal{O}_C \hookrightarrow \pi_*\mathcal{O}_{C'}$, which causes $(\star)$ to split.

Let us focus next on $\pi^*L \cong \mathcal{O}_{C'}(R)$. If ω_π is the relative dualizing sheaf for π, then there is an isomorphism $\mathcal{O}_{C'}(R) \cong \omega_\pi \cong \underline{\operatorname{Hom}}_{\mathcal{O}_{C'}}(\mathcal{O}_{C'}, \omega_\pi)$, and thus

$$\pi_*(\mathcal{O}_{C'}(R)) \cong \pi_*(\omega_\pi) \cong \pi_*\underline{\operatorname{Hom}}_{\mathcal{O}_{C'}}(\mathcal{O}_{C'}, \omega_\pi) \cong \underline{\operatorname{Hom}}_{\mathcal{O}_C}(\pi_*\mathcal{O}_{C'}, \mathcal{O}_C) = (\pi_*\mathcal{O}_{C'})^\vee$$

by definition of ω_π. Moreover, observe that the projection formula yields

$$\pi_*\pi^*M \cong \pi_*(\mathcal{O}_{C'} \otimes \pi^*M) \cong \pi_*\mathcal{O}_{C'} \otimes M$$

for any line bundle M on C. Keeping these results in mind, we can modify the split exact sequence $(\star)$ through either dualizing or tensoring by L in order to obtain the following commutative diagram:

$$\begin{tikzcd}
(\star)^\vee \ : & 0 \arrow[r] & L \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & (\pi_*\mathcal{O}_{C'})^\vee \cong \pi_*(\mathcal{O}_{C'}(R)) \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}_C \arrow[r] \arrow[d, equal] & 0 \\
(\star) \otimes L \ : & 0 \arrow[r] & L \arrow[r] & \pi_*\mathcal{O}_{C'} \otimes L \cong \pi_*\pi^*L \arrow[r] & \mathcal{O}_C \arrow[r] & 0
\end{tikzcd}$$

As both exact sequences remain split, they induce an isomorphism

$$\pi_*(\mathcal{O}_{C'}(R)) \cong \pi_*\pi^*L \quad \Rightarrow \quad H^0(C', \mathcal{O}_{C'}(R)) \cong H^0(C', \pi^*L), \quad 1_R \mapsto v$$

where 1_R is the global section of $\mathcal{O}_{C'}(R)$ whose zero locus is R, in keeping with the notation of proposition 1.1.3. We have then found a global section v of π^*L defining the ramification divisor R, and thus $\pi^*L \cong \mathcal{O}_{C'}(R)$ as expected.

To show that $L \in \sqrt{\mathcal{O}_C(B)}$, we use a similar trick. Note that

$$\pi_*\mathcal{O}_{C'}(B) \cong \pi_*\pi^*(\mathcal{O}_C(B)) \cong \pi_*(\mathcal{O}_{C'}(2R)) \cong \pi_*(\mathcal{O}_{C'}(R)^{\otimes 2}) \cong \pi_*\pi^*L^{\otimes 2}$$

and tensor $(\star)$ by $L^{\otimes 2}$ and $\mathcal{O}_C(B)$ to produce two split exact sequences:

$$\begin{array}{rl} (\star)\otimes L^{\otimes 2} \ \ : & 0 \longrightarrow L^{\otimes 2} \longrightarrow \pi_*\mathcal{O}_{C'}\otimes L^{\otimes 2} \cong \pi_*\pi^* L^{\otimes 2} \longrightarrow L \longrightarrow 0 \\ (\star)\otimes \mathcal{O}_C(B) : & 0 \longrightarrow \mathcal{O}_C(B) \longrightarrow \pi_*\mathcal{O}_{C'}(B) \cong \pi_*\pi^* L^{\otimes 2} \longrightarrow L^\vee(B) \longrightarrow 0 \end{array}$$

By the Krull-Schmidt theorem (as in [Ati56] Th. 2), the decompositions

$$\pi_*\pi^* L^{\otimes 2} \cong L^{\otimes 2} \oplus L \cong \mathcal{O}_C(B) \oplus L^\vee(B)$$

have to agree, which is only possible if $L^{\otimes 2} \cong \mathcal{O}_C(B)$ and $L \cong L^\vee(B)$. □

If we now consider the isomorphism $\alpha\colon L^{\otimes 2} \cong \mathcal{O}_C(B)$ built in the proof, or equivalently its counterpart $\beta\colon (L^\vee)^{\otimes 2} \cong \mathcal{O}_C(-B)$, we see that they arise from compositions

$$\begin{array}{ccccccccc} L^{\otimes 2} & \longrightarrow & \pi_*\pi^* L^{\otimes 2} & \cong & \pi_*(\mathcal{O}_{C'}(2R)) & \cong & \pi_*\mathcal{O}_{C'}(B) & \longrightarrow & \mathcal{O}_C(B) \\ (L^\vee)^{\otimes 2} & \longrightarrow & \pi_*\pi^*(L^\vee)^{\otimes 2} & \cong & \pi_*(\mathcal{O}_{C'}(-2R)) & \cong & \pi_*\mathcal{O}_{C'}(-B) & \xrightarrow{\mathrm{tr}/2} & \mathcal{O}_C(-B) \end{array}$$

The latter isomorphism has an alternate description which is useful to mention. Let $\psi\colon L^\vee \cong \mathrm{coker}(\pi^\#) \cong \ker(\mathrm{tr}/2) \hookrightarrow \pi_*\mathcal{O}_{C'}$ be the section splitting

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{\pi^\#} \pi_*\mathcal{O}_{C'} \longrightarrow L^\vee \longrightarrow 0 \qquad (\star)$$

and use the product of $\pi_*\mathcal{O}_{C'}$ to define a map

$$\psi_2\colon (L^\vee)^{\otimes 2} \to \pi_*\mathcal{O}_{C'}, \quad s\otimes t \mapsto \psi(s)\cdot\psi(t)$$

A local computation shows that $\beta = (\mathrm{tr}/2)\circ\psi_2$, that is to say, we have:

$$\begin{array}{ccccccc} \psi(s)\cdot\psi(t) & \in & \pi^\#(\mathcal{O}_C(-B)) & \subset & \pi_*\mathcal{O}_{C'}(-B) & \hookrightarrow & \pi_*\mathcal{O}_{C'} \\ \big\downarrow{\scriptstyle \mathrm{tr}/2} & & \wr\| & \swarrow{\scriptstyle \mathrm{tr}/2} & & & \big\downarrow{\scriptstyle \mathrm{tr}/2} \\ \beta(s\otimes t) & \in & \mathcal{O}_C(-B) & & & \hookrightarrow & \mathcal{O}_C \end{array}$$

for all $s, t \in L^\vee$. Hence the isomorphism of $\mathcal{O}_C$-modules induced by $(\star)$, namely

$$\pi_*\mathcal{O}_{C'} \cong \mathcal{O}_C \oplus L^\vee, \quad \psi(s) \mapsto (0, s), \quad \psi(s)\cdot\psi(t) \mapsto (\beta(s\otimes t), 0)$$

becomes an isomorphism of $\mathcal{O}_C$-algebras with respect to the previously defined $\mathcal{O}_C$-algebra structure induced on $\mathcal{O}_C \oplus L^\vee$ by the root $L \in \sqrt{\mathcal{O}_C(B)}$.

Additionally, by construction of $v \in H^0(C', \pi^*L)$ we have

$$\begin{array}{ccc} \mathcal{O}_C & \hookrightarrow & \pi_*\pi^*L\,, \\ 1 & \mapsto & v \end{array} \qquad \begin{array}{ccccc} \pi^*L^{\otimes 2} & \cong & \mathcal{O}_{C'}(2R) & \cong & \pi^*\mathcal{O}_C(B) \\ v^{\otimes 2} & \mapsto & 1_R^{\otimes 2} & \mapsto & \pi^\#(1_B) \end{array}$$

That is, even if the section $1_B \in H^0(C, \mathcal{O}_C(B)) \cong H^0(C, L^{\otimes 2})$ does not admit a global square root on C, when we pull back to C' this global root always exists and defines the ramification divisor of the double cover.

Remark 1.1.5. In the setting of proposition 1.1.3, it is easy to describe locally the section $v \in H^0(C', \pi^*L)$ and check that $v|_{\pi^{-1}(U_i)}{}^{\otimes 2} = \pi^\#(v_i)^{\otimes 2} \mapsto \pi^\#(1_B|_{U_i})$. The family of roots $\{v_i\}$, or $\{\pi^\#(v_i)\}$, may not glue, but $\{v|_{\pi^{-1}(U_i)}\}$ does.

Remark 1.1.6. In both of the preceding propositions, π comes equipped with an involution $\iota\colon C' \to C'$ with fixed locus $R = \pi^{-1}(B)$, which is induced by the automorphism $(a, s) \mapsto (a, -s)$ of $\mathcal{O}_C \oplus L^\vee$ and interchanges the points in each fiber of π.

As might be expected by this point, the previous structures are in fact equivalent, which we compile in the following statement.

Theorem 1.1.7. *Let C be a smooth, integral curve of genus g together with an effective divisor B on C of even degree d. Then there are natural bijective maps between the sets of:*

(i) *Square roots $L \in \sqrt{\mathcal{O}_C(B)}$ as in definition 1.1.1.*

(ii) *Pairs (π, ι) such that $\pi\colon C' \to C$ is a flat double cover branched over B and $\iota\colon C' \to C'$ is the involution interchanging the points in each fiber of π, with fixed locus $R = \pi^{-1}(B)$.*

(iii) *Pairs (C', ι) where C' is a curve equipped with an involution $\iota\colon C' \to C'$ with fixed locus R such that there is an isomorphism $C'/\langle\iota\rangle \cong C$ which restricts to $R \cong B$ and $\operatorname{Sing}(C') \cong \operatorname{Sing}(B)$.*

In both (ii) *and* (iii)*, there is a bijection $\operatorname{Sing}(C') \to \operatorname{Sing}(B)$; in particular, the curve C' is smooth if and only if the divisor B is smooth. When this is the case and furthermore $L \not\cong \mathcal{O}_C$, then C' is integral and has genus $2g - 1 + d/2$.*

Proof. We have already described the map (i) $\to$ (ii), namely:

$$L \mapsto (\pi\colon \underline{\operatorname{Spec}}(\mathcal{O}_C \oplus L^\vee) \to C, \iota)$$

which is well-defined by proposition 1.1.3 and remark 1.1.6, and whose inverse

$$(\pi, \iota) \mapsto \operatorname{coker}(\pi^\#\colon \mathcal{O}_C \hookrightarrow \pi_*\mathcal{O}_{C'})^\vee$$

is given by proposition 1.1.4 and the subsequent observations.

As for (ii) $\to$ (iii), the obvious map $(\pi, \iota) \mapsto (C', \iota)$ is also well-defined and bijective, since a pair (C', ι) as in (iii) yields a flat double cover

$$\pi\colon C' \to C'/\langle\iota\rangle \cong C$$

branched over B, and so a point (π, ι) as in (ii).

Finally, note that the curve C' can only disconnect if the double cover π, or equivalently the square root L, is trivial. Since smooth curves are irreducible if and only if they are connected, integrality of C' follows from $L \not\cong \mathcal{O}_C$, while its genus is determined by the Riemann-Hurwitz formula for π:

$$2g(C') - 2 = 2\,(2g-2) + \deg(R) \;\Rightarrow\; g(C') = 2g - 1 + d/2$$

as we have $R \cong B$ and thus $\deg(R) = \deg(B) = d$. □

Corollary 1.1.8. *Let L be a square root of $N = \mathcal{O}_C(B)$, and let $\pi\colon C' \to C$ be the flat double cover associated to it by theorem 1.1.7. Then it holds that*

$$\pi_*\pi^*M \cong M \oplus (M \otimes L^\vee)$$

for any $M \in \operatorname{Pic}(C)$, and the kernel of the pullback $\pi^*\colon \operatorname{Pic}(C) \to \operatorname{Pic}(C')$ is:

$$\ker(\pi^*) = \begin{cases} \{\mathcal{O}_C\} & \text{if } B \neq 0 \text{ or } L \cong \mathcal{O}_C \\ \{\mathcal{O}_C, L\} & \text{if } B = 0 \text{ and } L \not\cong \mathcal{O}_C \end{cases}$$

Proof. Combining different parts of proposition 1.1.4, we get

$$\pi_*\pi^*M \cong \pi_*\mathcal{O}_{C'} \otimes M \cong (\mathcal{O}_C \oplus L^\vee) \otimes M \cong M \oplus (M \otimes L^\vee)$$

Assume L to be nontrivial. Then $\pi^*M \cong \mathcal{O}_{C'}$ if and only if $\deg(\pi^*M) = 0$ and $h^0(C', \pi^*M) = 1$, and furthermore

$$\begin{aligned} \deg(\pi^*M) = 0 \quad &\Leftrightarrow \quad \deg(M) = 0 \\ h^0(\pi^*M) = h^0(M) + h^0(M \otimes L^\vee) = 1 \quad &\Leftrightarrow \quad h^0(M) = 1 \text{ or } h^0(M \otimes L^\vee) = 1 \end{aligned}$$

Since $\deg(M \otimes L^\vee) = \deg(M) - d/2$, we can see that $\pi^*M \cong \mathcal{O}_{C'}$ if and only if either $M \cong \mathcal{O}_C$ or $M \cong L$, the latter case requiring $d = 0$, $B = 0$. □

1.1.2 Prym pairs and spin curves

Throughout the thesis, we will be particularly interested in square roots of either the trivial or the canonical bundle. Let C be a smooth, integral curve of genus g, and let us start with the choice $N = \mathcal{O}_C$, which leads to the definition of Prym roots and Prym pairs:

Definition 1.1.9. A *Prym root* of C is a nontrivial square root of $\mathcal{O}_C$, that is, a line bundle $\eta \ncong \mathcal{O}_C$ of degree zero equipped with an isomorphism $\eta^{\otimes 2} \cong \mathcal{O}_C$. A *Prym pair* is a pair (C, η) such that η is a Prym root of C. The set of Prym roots of C is denoted by $R_g(C) \hookrightarrow \mathrm{Pic}^0(C) - \{\mathcal{O}_C\}$.

Since Prym roots correspond to nontrivial 2-torsion points of the Jacobian, we have $R_g(C) = J_2(C) - \{\mathcal{O}_C\}$, and thus $\# R_g(C) = 2^{2g} - 1$. By remark 1.1.2, the tensor product of a square root $L \in \sqrt{N}$ and a Prym root η is again a root $L \otimes \eta$ of N, different from L.

By virtue of theorem 1.1.7, Prym pairs can be naturally understood as nontrivial étale double covers. The isomorphism $\beta \colon \eta^{\otimes 2} \cong \mathcal{O}_C$ identifies η with its dual $\eta^\vee$, simplifying the overall picture, which we briefly summarize next.

Lemma 1.1.10. *The projection $\pi \colon C' = \underline{\mathrm{Spec}}(\mathcal{O}_C \oplus \eta) \to C$ is a nontrivial étale double cover fitting into the following diagram:*

$$\begin{tikzcd} \begin{array}{rcl} C' & = & \underline{\mathrm{Spec}}(\mathcal{O}_C \oplus \eta) \\ & \cong & \{(x; 1, v_x) \;/\; v_x \in \eta|_x,\ \beta(v_x^{\otimes 2}) = 1\} \end{array} \arrow[rr, hook] \arrow[dr, "\pi", "2:1"'] & \arrow[d, phantom, "\circlearrowleft"] & \mathbb{P}(\mathcal{O}_C \oplus \eta) \arrow[dl, "\mathbb{P}^1"] \\ & C & \end{tikzcd}$$

Proof. This is proposition 1.1.3, as flat and unramified implies étale. Moreover, the construction only works locally (as globally we have $\eta \ncong \mathcal{O}_C$), so it cannot produce the trivial cover. $\square$

Recall that $\mathcal{O}_C \oplus \eta$ is an $\mathcal{O}_C$-algebra with the product

$$(a, s) \cdot (b, t) = (ab + \beta(s \otimes t), at + bs)$$

Specializing proposition 1.1.3, we get $1_B = 1 \in H^0(C, \mathcal{O}_C)$, $f_i = 1$, and

$$\begin{array}{ccccc} (\mathcal{O}_C \oplus \eta)|_{U_i} & \cong & \mathcal{O}_{U_i} \oplus \eta|_{U_i} & \cong & \mathcal{O}_{U_i}[\mathsf{t}]/(\mathsf{t}^2 - 1) \\ & & (a, \lambda\, v_i) & \mapsto & a + \lambda\, \mathsf{t} \end{array}$$

for an affine local trivialization $\{U_i, v_i\}$ of η such that $\beta(v_i^{\otimes 2}) = 1$. In turn, the fibers of π correspond to:

$$\pi^{-1}(x) = \operatorname{Spec} \kappa(x)[\mathsf{t}]/(\mathsf{t}^2 - 1) = \{(\mathsf{t}-1), (\mathsf{t}+1)\} = \{x_1, x_2\}$$

Conversely, η can be recovered as a cokernel:

Lemma 1.1.11. *Let $\pi\colon C' \to C$ be a nontrivial étale double cover. Then there is a Prym root η fitting into the following split exact sequence:*

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{\pi^\#} \pi_*\mathcal{O}_{C'} \longrightarrow \eta \longrightarrow 0$$

which results in an isomorphism $\pi_*\mathcal{O}_{C'} \cong \mathcal{O}_C \oplus \eta$ of $\mathcal{O}_C$-algebras.

Proof. This is just proposition 1.1.4 and the ensuing discussion. □

In this setting, the trace map can be locally described as

$$\operatorname{tr}\colon H^0(U, \pi_*\mathcal{O}_{C'}) \cong H^0(U, \mathcal{O}_C)^{\oplus 2} \to H^0(U, \mathcal{O}_C), \quad (f, g) \mapsto f + g$$

for $U \subsetneq C$, whereas $\pi^\#\colon \mathcal{O}_C \hookrightarrow \pi_*\mathcal{O}_{C'}$ is given by $f \mapsto (f, f)$. Therefore a local section $s \in H^0(U, \eta)$ is of the form $(f, -f)$ for some $f \in H^0(U, \mathcal{O}_C)$, i.e.

$$\psi\colon \eta \cong \operatorname{coker}(\pi^\#) \cong \ker(\operatorname{tr}/2) \hookrightarrow \pi_*\mathcal{O}_{C'}, \quad s \mapsto \psi(s) = (f, -f)$$

Consequently we have $\psi(s) \cdot \psi(t) = (f, -f) \cdot (g, -g) = (fg, fg) \in \pi^\#(\mathcal{O}_C)$ for $s, t \in H^0(U, \eta)$ such that $\psi(s) = (f, -f)$ and $\psi(t) = (g, -g)$, yielding

$$\beta\colon \eta^{\otimes 2} \cong \pi^\#(\mathcal{O}_C) \cong \mathcal{O}_C, \quad s \otimes t \longmapsto \psi(s) \cdot \psi(t) \xmapsto{\operatorname{tr}/2} fg$$

as the isomorphism making η into a Prym root.

Remark 1.1.12. Following remark 1.1.6, π is equipped with a fixed-point-free involution $\iota\colon C' \to C'$, which is induced by the automorphism $(a, s) \mapsto (a, -s)$ of $\mathcal{O}_C \oplus \eta$ and interchanges the points in each fiber of π.

Proposition 1.1.13. *Fix a smooth, integral curve C of genus g. Then there are natural bijective maps between the sets of:*

(i) *Prym pairs (C, η) as in definition 1.1.9, that is, with $\eta \in R_g(C)$.*

(ii) *Pairs (π, ι) such that $\pi\colon C' \to C$ is a nontrivial étale double cover and $\iota\colon C' \to C'$ is the involution interchanging the points in each fiber of π.*

(iii) *Pairs (C', ι) where C' is a smooth, integral curve of genus $2g - 1$ with a fixed-point-free involution $\iota\colon C' \to C'$ such that $C'/\langle\iota\rangle \cong C$.*

In particular, these sets have order $2^{2g} - 1$.

Proof. The statement is a direct consequence of theorem 1.1.7, when interpreted through the lemmas above. □

In summary, the identification (i) ⇄ (ii) of Prym pairs with nontrivial étale double covers is given by

$$\begin{array}{rcl} (C,\eta) & \mapsto & (\pi\colon \underline{\mathrm{Spec}}(\mathcal{O}_C\oplus\eta)\to C,\iota) \\ (C,\mathrm{coker}(\pi^{\#}\colon \mathcal{O}_C\hookrightarrow \pi_*\mathcal{O}_{C'})) & \leftarrow\!\shortmid & (\pi\colon C'\to C,\iota) \end{array}$$

where $\pi_*\mathcal{O}_{C'}$ and $\mathcal{O}_C\oplus\eta$ are isomorphic as $\mathcal{O}_C$-algebras. Furthermore, observe that corollary 1.1.8 provides a decomposition $\pi_*\pi^*L\cong L\oplus(L\otimes\eta)$ for any line bundle L on C, and as a result an isomorphism

$$H^0(C',\pi^*L)\cong H^0(C,L)\oplus H^0(C,L\otimes\eta)$$

Accordingly, the pullback $\pi^*\colon\ \mathrm{Pic}(C)\to\mathrm{Pic}(C')$ has kernel $\{\mathcal{O}_C,\eta\}$ and factors through a group monomorphism

$$\pi^*\colon\ \mathrm{Pic}(C)/\{\mathcal{O}_C,\eta\}\hookrightarrow\mathrm{Pic}(C')$$

where $\{\mathcal{O}_C,\eta\}\cong\mathbb{Z}/2\mathbb{Z}$.

As a brief interlude, we shall now discuss the choice $N=\omega_C$, which brings us to the notion of theta characteristics and spin curves:

Definition 1.1.14. A *theta characteristic* of C is a square root of ω_C, that is, a line bundle θ of degree $g-1$ equipped with an isomorphism $\theta^{\otimes 2}\cong\omega_C$. A *spin curve* is a pair (C,θ) such that θ is a theta characteristic of C. The set of theta characteristics of C is denoted by $S_g(C)=\sqrt{\omega_C}\hookrightarrow\mathrm{Pic}^{g-1}(C)$.

Remark 1.1.2 implies that $\#S_g(C)=2^{2g}$, while theorem 1.1.7 turns into:

Corollary 1.1.15. *Let C be a smooth, integral curve of genus g together with a smooth, effective canonical divisor $K_C=p_1+\ldots+p_{2g-2}\in|\omega_C|\cong\mathbb{P}^{g-1}$. Then there are natural bijective maps between the sets of:*

(i) *Theta characteristics $\theta\in S_g(C)$ as in definition 1.1.14.*

(ii) *Pairs (π,ι) where $\pi\colon C'\to C$ is a flat double cover branched over K_C and $\iota\colon C'\to C'$ is the associated involution fixing $\pi^{-1}(K_C)$.*

(iii) *Pairs (C',ι) where C' is a smooth, integral curve of genus $3g-2$ and $\iota\colon C'\to C'$ is an involution equipped with an isomorphism $C'/\langle\iota\rangle\cong C$ which maps the fixed points of ι to K_C.*

An essential property of spin curves is their *parity*, whose importance stems from its invariance in families. We have thus two types of spin curves:

Definition 1.1.16. A theta characteristic θ of C is said to be *even* (resp. *odd*) if the dimension $h^0(C,\theta)$ of the space of global sections $H^0(C,\theta)$ is even (resp. odd). The parity-based decomposition of $S_g(C)$ is written as

$$S_g(C) = S_g^+(C) \sqcup S_g^-(C)$$

with $+$ (resp. $-$) denoting even parity (resp. odd).

As shown in [Mum71], the parity of $h^0(C,\theta)$ is stable under deformations of C and θ, which in turn facilitates the computation of

$$\#S_g^+(C) = 2^{g-1}(2^g+1), \qquad \#S_g^-(C) = 2^{g-1}(2^g-1)$$

by reducing it to the hyperelliptic case; see *ibid.* Section 4.

Remark 1.1.17. A question that might arise is how tensoring by a fixed Prym root modifies the parity of theta characteristics. This issue will be discussed in the following chapter, as it proves to be very relevant to the problem addressed there. Specifically, the answer is provided by proposition 2.2.12.

1.2 The Prym construction

Given a smooth, integral curve C, the *Jacobian construction* is a canonical way of building a principally polarized abelian variety out of C, namely:

$$J(C) \quad = \quad (\mathrm{Pic}^0(C), \Theta_C)$$

whose dimension is the genus of the curve:

$$\dim J(C) \quad = \quad \dim \mathrm{Pic}^0(C) \quad = \quad h^1(C,\mathcal{O}_C) \quad = \quad h^0(C,\omega_C) \quad = \quad g(C) \quad = \quad g$$

After translating by a certain degree $g-1$ line bundle, the theta divisor Θ_C of $J(C)$ can be set-theoretically characterised as:

$$\begin{array}{rcccl} J(C) \;=\; \mathrm{Pic}^0(C) & \cong & \mathrm{Pic}^{g-1}(C) & & \\ \Theta_C & \cong & W_{g-1}(C) & = & \{L \in \mathrm{Pic}^{g-1}(C) \;/\; h^0(L) \geq 1\} \\ \mathrm{Sing}(\Theta_C) & \cong & W^1_{g-1}(C) & = & \{L \in \mathrm{Pic}^{g-1}(C) \;/\; h^0(L) \geq 2\} \end{array}$$

If the curve C is further endowed with a nontrivial étale double cover, then the covering curve C' is also smooth and integral, and one might wonder how its

Jacobian variety $J(C')$ differs from that of C. Morally, this difference takes the shape of a new principally polarized abelian variety, known as the *Prym variety* associated to the double cover, which we describe formally in this section.

1.2.1 The Prym variety as an abelian variety

Let (C, η) be a Prym pair, with corresponding nontrivial étale double cover $\pi\colon C' \to C$ and involution ι, as in proposition 1.1.13. The pullback map

$$\pi^*\colon \operatorname{Pic}(C) \to \operatorname{Pic}(C'), \quad L = \mathcal{O}_C(D) \mapsto \pi^* L = \mathcal{O}_{C'}(\pi^* D)$$

restricts over each connected component of $\operatorname{Pic}(C)$ to maps

$$\pi^*\colon \operatorname{Pic}^d(C) \to \operatorname{Pic}^{2d}(C'), \qquad \pi^*\colon J(C) \to J(C')$$

for $d \in \mathbb{Z}$. Since $H_0 = \ker(\pi^*) = \{\mathcal{O}_C, \eta\} \cong \mathbb{Z}/2\mathbb{Z}$ is a finite subgroup of $J(C)$, the pullback map $J(C) \to J(C')$ factors through an isogeny $J(C) \twoheadrightarrow J(C)/H_0$ into an embedding of abelian varieties

$$\pi^*\colon J(C)/H_0 \hookrightarrow J(C'), \qquad H_0 = \{\mathcal{O}_C, \eta\} \subset J_2(C) \cong (\mathbb{Z}/2\mathbb{Z})^{2g}$$

of dimensions g and $2g-1$, respectively. As abelian subvarieties admit complements, there exists some abelian subvariety P of $J(C')$ such that

$$\pi^*(J(C)/H_0) + P = J(C'), \qquad \dim(\pi^*(J(C)/H_0) \cap P) = 0$$

so in particular P is of dimension $g-1$. Let us construct this variety.

Definition 1.2.1. The *norm map* associated to π is the group epimorphism

$$\operatorname{Nm}_\pi\colon \operatorname{Pic}(C') \to \operatorname{Pic}(C), \quad L' = \mathcal{O}_{C'}(D') \mapsto \operatorname{Nm}_\pi L' = \mathcal{O}_C(\pi_* D')$$

where $\pi_* D'$ is the standard pushforward of divisors.

Since $\deg(\pi_* D') = \deg(D')$, we have restrictions

$$\operatorname{Nm}_\pi\colon \operatorname{Pic}^d(C') \to \operatorname{Pic}^d(C), \qquad \operatorname{Nm}_\pi\colon J(C') \to J(C)$$

for $d \in \mathbb{Z}$. Moreover $\pi_*\pi^* D = 2D$ for any divisor D on C, so the composition

$$\begin{array}{rrclrcl} \operatorname{Nm}_\pi \circ \pi^*\colon & \operatorname{Pic}(C) & \to & \operatorname{Pic}(C), & \mathcal{O}_C(D) & \mapsto & \mathcal{O}_C(2D) \\ & J(C) & \to & J(C), & L & \mapsto & L^{\otimes 2} \end{array}$$

not just restricts to degree zero, but actually corresponds to the square map of line bundles. This is far from the only connection between pullback and norm, however: for instance, if we fix rational points $x_0 \in C$, $x_0' \in C'$ with $\pi(x_0') = x_0$ and consider the Abel-Jacobi embeddings

$$\begin{array}{rcccccc} \mathsf{aj}\colon & C & \hookrightarrow & J(C), & x & \mapsto & \mathcal{O}_C(x - x_0) \\ \mathsf{aj}'\colon & C' & \hookrightarrow & J(C'), & x' & \mapsto & \mathcal{O}_C(x' - x_0') \end{array}$$

then we get commutative diagrams

$$\begin{tikzcd} C' \arrow[r, hook, "\mathsf{aj}'"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & J(C') \arrow[d, "\mathrm{Nm}_\pi"] \\ C \arrow[r, hook, "\mathsf{aj}"'] & J(C) \end{tikzcd} \quad \overset{\mathrm{Pic}^0}{\dashrightarrow} \quad \begin{tikzcd} J(C') \arrow[r, phantom, "\cong"] & J(C')^\vee \arrow[r, phantom, "\overset{i'}{\cong}"] & J(C')^\vee \\ J(C) \arrow[u, "\pi^*"] \arrow[r, phantom, "\cong"] \arrow[ur, phantom, "\circlearrowright"] & J(C)^\vee \arrow[u, "\mathrm{Nm}_\pi^*"'] \arrow[r, phantom, "\underset{i}{\cong}"] \arrow[ur, phantom, "\circlearrowright"] & J(C)^\vee \arrow[u, "\mathrm{Nm}_\pi^*"'] \end{tikzcd}$$

where i is the inverse operation of the dual abelian variety $J(C)^\vee = \mathrm{Pic}^0(J(C))$ and the composite isomorphism $\lambda_\Theta = i \circ (\mathsf{aj}^*)^{-1}\colon J(C) \cong J(C)^\vee$ is precisely the principal polarization defined by the theta divisor Θ_C (resp. i', C', $\lambda_{\Theta'}$, aj'). In other words, the maps π^* and Nm_π are dual to one another:

$$\begin{array}{ccccc} \pi^* & : & J(C) & \to & J(C') \\ \| & & \lambda_\Theta \wr\| & \circlearrowright & \|\wr \lambda_{\Theta'} \\ \mathrm{Nm}_\pi^* & : & J(C)^\vee & \to & J(C')^\vee \end{array} \qquad \begin{array}{ccccc} \mathrm{Nm}_\pi & : & J(C') & \to & J(C) \\ \| & & \lambda_{\Theta'} \wr\| & \circlearrowright & \|\wr \lambda_{\Theta} \\ (\pi^*)^* & : & J(C')^\vee & \to & J(C)^\vee \end{array}$$

In turn, the involution $\iota\colon C' \to C'$ induces an involution on $\mathrm{Pic}(C')$:

$$\iota^*\colon \mathrm{Pic}(C') \to \mathrm{Pic}(C'), \quad \mathcal{O}_{C'}(x') \mapsto \mathcal{O}_{C'}(\iota(x'))$$

which preserves degree. We denote by $1+\iota$, $1-\iota$ the following maps:

$$\begin{array}{rcccccc} 1+\iota = \mathrm{Id} + \iota^*\colon & \mathrm{Pic}(C') & \to & \mathrm{Pic}(C'), & L & \mapsto & L \otimes \iota^* L \\ 1-\iota = \mathrm{Id} - \iota^*\colon & \mathrm{Pic}(C') & \to & \mathrm{Pic}(C'), & L & \mapsto & L \otimes \iota^* L^\vee \end{array}$$

Since $\deg(L \otimes \iota^* L) = 2\deg(L)$ and $\deg(L \otimes \iota^* L^\vee) = 0$, in particular we get:

$$1+\iota\colon J(C') \to J(C'), \qquad 1-\iota\colon \mathrm{Pic}(C') \to J(C')$$

Take $x_1 \in C'$ with $\pi(x_1) = x \in C$, $\pi^{-1}(x) = \{x_1, x_2\} \subset C'$, and observe that

$$\pi^*\pi_*(x_1) = \pi^*(x) = x_1 + x_2 = x_1 + \iota(x_1) = (\mathrm{Id} + \iota^*)(x_1)$$

which allows us to describe the composition $\pi^* \circ \mathrm{Nm}_\pi$ as

$$\pi^* \circ \mathrm{Nm}_\pi = 1 + \iota \colon \quad \begin{array}{rcl} \mathrm{Pic}(C') & \to & \mathrm{Pic}(C'), \\ J(C') & \to & J(C'), \end{array} \qquad \begin{array}{rcl} \mathcal{O}_{C'}(D') & \mapsto & \mathcal{O}_{C'}(D' + \iota^* D') \\ L & \mapsto & L \otimes \iota^* L \end{array}$$

In summary, we have obtained:

Lemma 1.2.2. *The pullback map π^* and the norm map Nm_π are related by the following properties:*

(i) $\mathrm{Nm}_\pi \circ \pi^* = (-)^{\otimes 2} \colon J(C) \to J(C)$, $L \mapsto L^{\otimes 2}$

(ii) $\pi^* \circ \mathrm{Nm}_\pi = 1 + \iota \colon J(C') \to J(C')$, $L \mapsto L \otimes \iota^* L$

(iii) $\mathrm{Nm}_\pi^* = \lambda_{\Theta'} \circ \pi^* \circ \lambda_\Theta^{-1}$ *and* $(\pi^*)^* = \lambda_\Theta \circ \mathrm{Nm}_\pi \circ \lambda_{\Theta'}^{-1}$

Proposition 1.2.3. *Consider the maps π^*, Nm_π, $1+\iota$ and $1-\iota$ defined over either $\mathrm{Pic}(C)$ or $\mathrm{Pic}(C')$, as appropriate. Then it holds that:*

$$\begin{array}{lrcccccc} (\mathrm{a}_1) & \ker(\mathrm{Nm}_\pi) & = & \mathrm{im}(1-\iota) & \subset & \ker(1+\iota) & \subset & J(C') \\ (\mathrm{a}_2) & \mathrm{im}(\pi^*) & = & \mathrm{im}(1+\iota) & \subset & \ker(1-\iota) & \subset & \mathrm{Pic}(C') \\ (\mathrm{b}_1) & \ker(\mathrm{Nm}_\pi) & \cap & \mathrm{im}(\pi^*) & = & \pi^*(J_2(C)) & \subset & J_2(C') \\ (\mathrm{b}_2) & \ker(\mathrm{Nm}_\pi) & + & \pi^*(J(C)) & = & J(C') & & \end{array}$$

Proof. Let us start with (a_1). Lemma 1.2.2(ii) and corollary 1.1.8 ensure that

$$\ker(1+\iota) = \mathrm{Nm}_\pi^{-1}(\ker \pi^*) = \mathrm{Nm}_\pi^{-1}(\{\mathcal{O}_C, \eta\}) = \ker(\mathrm{Nm}_\pi) \cup \mathrm{Nm}_\pi^{-1}(\eta)$$

Since the involution ι^* is compatible with pullback and norm (in the sense that $\iota^* \circ \pi^* = \pi^*$ and $\mathrm{Nm}_\pi \circ \iota^* = \mathrm{Nm}_\pi$), we also get

$$\begin{array}{rcccccccc} (1+\iota)\circ(1-\iota) & = & \mathrm{Id} - (\iota^* \circ \iota^*) & = & 0 & \Rightarrow & \mathrm{im}(1-\iota) & \subset & \ker(1+\iota) \\ \mathrm{Nm}_\pi \circ (1-\iota) & = & \mathrm{Nm}_\pi - \mathrm{Nm}_\pi & = & 0 & \Rightarrow & \mathrm{im}(1-\iota) & \subset & \ker(\mathrm{Nm}_\pi) \end{array}$$

It remains to check the inclusion $\ker(\mathrm{Nm}_\pi) \subset \mathrm{im}(1-\iota)$. Let D' be a divisor on C' such that $\mathrm{Nm}_\pi D' \sim 0$. Then the surjectivity of the norm map over principal divisors ensures the existence of another divisor D on C' such that $D \sim D'$ and $\mathrm{Nm}_\pi D = 0$. Consequently, when restricted to any given fiber $\pi^{-1}(x) = \{x_1, x_2\}$, the divisor D must be of the form

$$n_x\, x_1 - n_x\, x_2 = (\mathrm{Id} - \iota^*)(n_x\, x_1) = (\mathrm{Id} - \iota^*)(-n_x\, x_2)$$

for some $n_x \in \mathbb{Z}$. In particular, if we write $D = D^+ - D^-$ with D^+, D^- effective, it follows that $\iota^* D^+ = D^-$ and $\iota^* D^- = D^+$, yielding

$$(\mathrm{Id} - \iota^*) D^+ = (\mathrm{Id} - \iota^*)(-D^-) = D \sim D' \quad \Rightarrow \quad \ker(\mathrm{Nm}_\pi) \subset \mathrm{im}(1-\iota)$$

Similarly, regarding (a_2) we have:

$$\begin{array}{rcccccc} (1-\iota)\circ(1+\iota) & = & \mathrm{Id}-(\iota^*\circ\iota^*) & = 0 & \Rightarrow & \mathrm{im}(1+\iota) & \subset & \ker(1-\iota) \\ (1-\iota)\circ\pi^* & = & \pi^*-\pi^* & = 0 & \Rightarrow & \mathrm{im}(\pi^*) & \subset & \ker(1-\iota) \end{array}$$

As a result of the property $1+\iota = \pi^*\circ \mathrm{Nm}_\pi$ and the fact that the norm map is surjective and preserves degree, we obtain

$$\mathrm{im}(\pi^*) = \mathrm{im}(1+\iota), \qquad \pi^*(J(C)) = (1+\iota)(J(C')) = \mathrm{im}\,(1+\iota)|_{J(C')}$$

which concludes (a_2) and paves the way for (b_2). Indeed, any $L\in J(C')$ can be written as the tensor product

$$L = M^{\otimes 2} = (1-\iota)M\otimes(1+\iota)M$$

for some $M\in J(C')$, as abelian varieties are divisible, and thus

$$\ker(\mathrm{Nm}_\pi)+\pi^*(J(C)) = \mathrm{im}(1-\iota)+\mathrm{im}\,(1+\iota)|_{J(C')} = J(C')$$

Finally, if $L=\pi^*M$ for some $M\in\mathrm{Pic}(C)$, then lemma 1.2.2(i) shows that

$$L\in\ker(\mathrm{Nm}_\pi) \quad\Leftrightarrow\quad \mathcal{O}_C = \mathrm{Nm}_\pi L = \mathrm{Nm}_\pi \pi^*M = M^{\otimes 2} \quad\Leftrightarrow\quad M\in J_2(C)$$

hence $\ker(\mathrm{Nm}_\pi)\cap\mathrm{im}(\pi^*) = \pi^*(J_2(C))$ as stated in (b_1). □

A priori, proposition 1.2.3 makes the subvariety

$$\ker(\mathrm{Nm}_\pi) = \mathrm{im}(1-\iota)\subset J(C')$$

seem like a good candidate for the abelian complement $P\subset J(C')$ of

$$\pi^*(J(C)/H_0) = \pi^*(J(C))$$

that we are trying to build. However, $\ker(\mathrm{Nm}_\pi)$ is not an abelian variety due to lack of connection, and so we have to shift our attention towards its connected components.

Corollary 1.2.4. *The subvariety* $\ker(\mathrm{Nm}_\pi)\subset J(C')$ *disconnects into two components* P^+ *and* P^-*, which correspond to*

$$\begin{array}{ccccccc} \mathcal{O}_{C'} & \in & P^+ & = & \mathrm{im}\,(1-\iota)|_{J(C')} & = & \mathrm{im}\,(1-\iota)|_{\mathrm{Pic}^{2k}(C')} \\ \mathcal{O}_{C'} & \notin & P^- & = & \mathrm{im}\,(1-\iota)|_{\mathrm{Pic}^1(C')} & = & \mathrm{im}\,(1-\iota)|_{\mathrm{Pic}^{2k+1}(C')} \end{array}$$

with $1-\iota\colon\ \mathrm{Pic}(C')\to J(C')$*, for all* $k\in\mathbb{Z}$*.*

Proof. The connected components of $\mathrm{Pic}(C')$ are $\{\mathrm{Pic}^d(C')\}_{d\in\mathbb{Z}}$, meaning that

$$\mathrm{im}\,(1-\iota)|_{\mathrm{Pic}^d(C')} = (1-\iota)(\mathrm{Pic}^d(C')) \subset \mathrm{im}(1-\iota) = \ker(\mathrm{Nm}_\pi)$$

is connected for all $d \in \mathbb{Z}$. Furthermore, we have

$$\mathrm{im}\,(1-\iota)|_{\mathrm{Pic}^d(C')} \cap \mathrm{im}\,(1-\iota)|_{\mathrm{Pic}^e(C')} \neq \varnothing$$

if and only if $d \equiv e \mod 2$. Indeed, take any $k \in \mathbb{Z}$ and a line bundle L on C of degree k. Recalling that $\mathrm{im}(\pi^*) \subset \ker(1-\iota)$ by proposition 1.2.3, we get

$$\begin{array}{lclcl} \mathcal{O}_{C'} & = & (1-\iota)(\pi^* L) & \in & \mathrm{im}\,(1-\iota)|_{\mathrm{Pic}^{2k}(C')} \\ \mathcal{O}_{C'}(x_1 - x_2) & = & (1-\iota)(\pi^* L \otimes \mathcal{O}_{C'}(x_1)) & \in & \mathrm{im}\,(1-\iota)|_{\mathrm{Pic}^{2k+1}(C')} \end{array}$$

for any $x \in C$ with $\pi^{-1}(x) = \{x_1, x_2\}$. Additionally, if D is a divisor on C' such that $D' = (\mathrm{Id} - \iota^*)D \sim 0$, then applying the Weil reciprocity law to D' and the principal divisor E' on C' given by $\pi^*\eta = \mathcal{O}_{C'}(-E') \cong \mathcal{O}_{C'}$ we see that

$$\deg(D^+) + \deg(D^-) \equiv 0 \mod 2 \;\; \Rightarrow \;\; \deg(D) \equiv 0 \mod 2$$

which implies $\mathcal{O}_{C'} \notin \mathrm{im}\,(1-\iota)|_{\mathrm{Pic}^1(C')}$ and finishes the proof. □

As we saw in proposition 1.2.3, any $L \in J(C')$ can be written as

$$L \;=\; M^{\otimes 2} \;=\; (1-\iota)M \otimes (1+\iota)M$$

for $M \in \sqrt{L} \subset J(C')$. Consequently, the connected component P^+ of $\ker(\mathrm{Nm}_\pi)$ is not only an algebraic subgroup of $J(C')$, but also satisfies

$$\begin{cases} P^+ + \pi^*(J(C)) \;=\; \mathrm{im}\,(1-\iota)|_{J(C')} + \mathrm{im}\,(1+\iota)|_{J(C')} \;=\; J(C') \\ P^+ \cap \pi^*(J(C)) \;\subset\; \ker(\mathrm{Nm}_\pi) \cap \mathrm{im}(\pi^*) \;=\; \pi^*(J_2(C)) \end{cases}$$

that is to say, P^+ is an abelian subvariety of $J(C')$ complementary to $\pi^*(J(C))$, of dimension:

$$\dim P^+ \;=\; \dim J(C') - \dim \pi^*(J(C)) \;=\; (2g-1) - g \;=\; g-1$$

In particular, its 2-torsion subgroup P_2^+ has order $2^{2(g-1)} = 2^{2g-2}$. We are still missing some information regarding the intersection of P^+ and $\pi^*(J(C))$, so let us quickly compute it before using P^+ to define our coveted Prym variety. Note that we can rewrite the intersection as:

$$P^+ \cap \pi^*(J(C)) \;=\; P_2^+ \cap \pi^*(J_2(C)) \;\subset\; \pi^*(J_2(C)) \;\subset\; J_2(C')$$

and express it as the pullback of the following subgroup of $J_2(C)$:

$$H_1 \quad = \quad (\pi^*)^{-1}(P_2^+) \cap J_2(C) \quad = \quad \{\eta' \in J_2(C) \;/\; \pi^*\eta' \in P^+\} \quad \subset \quad J_2(C)$$

which maps to P_2^+ via the homomorphism $\pi^* \colon H_1 \to P_2^+$.

Lemma 1.2.5. *The subgroup $H_1 \subset J_2(C)$ has index 2 and fits in a chain*

$$\{\mathcal{O}_C\} \quad \subset \quad H_0 \quad \subset \quad H_1 \quad \subset \quad J_2(C)$$

where $H_0 = \ker(\pi^*) = \{\mathcal{O}_C, \eta\}$. Therefore $\# H_1 = 2^{2g-1}$ and $\pi^* \colon H_1 \to P_2^+$ is an epimorphism yielding $H_1/H_0 \cong P_2^+ = \pi^*(H_1) \subset \pi^*(J_2(C))$.

Proof. Using corollary 1.2.4, it is trivial to check that the map

$$\lambda_\eta \colon \quad J_2(C) \quad \to \quad \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}, \qquad \lambda_\eta(\eta') \quad = \quad \begin{cases} 1 & \text{if } \pi^*\eta' \in P^+ \\ -1 & \text{if } \pi^*\eta' \notin P^+ \end{cases}$$

is a group homomorphism of kernel $H_1 \subset J_2(C)$. Since the pullback gives rise to an embedding $H_1/H_0 \hookrightarrow P_2^+$, we have

$$\#(H_1/H_0) \leq \# P_2^+ = 2^{2g-2} \quad \Rightarrow \quad \# H_1 \leq 2^{2g-1} < 2^{2g} = \# J_2(C)$$

so H_1 is a proper subgroup of $J_2(C)$. Then λ_η is surjective and we get

$$\#(J_2(C)/H_1) = \#(\mathbb{Z}/2\mathbb{Z}) = 2 \quad \Rightarrow \quad \# H_1 = 2^{2g-1}$$

which shows that $H_1/H_0 \hookrightarrow P_2^+$ is in fact an isomorphism. $\square$

As a result of the lemma, $\pi^*(J(C))$ intersects its complement P^+ in all 2^{2g-2} of the 2-torsion points of the latter, as opposed to a strictly smaller subset:

$$P^+ \cap \pi^*(J(C)) \quad = \quad P_2^+ \cap \pi^*(J_2(C)) \quad = \quad \pi^*(H_1) \quad = \quad P_2^+ \quad \cong \quad H_1/H_0$$

If we denote by $H_1' \subset J(C) \oplus P^+$ the amalgamated product of Id_{H_1} and π^*, i.e.

$$\begin{aligned} H_1' \quad = \quad H_1 *_{H_1} P_2^+ \quad &= \quad \{(\eta', \pi^*\eta') \;/\; \eta' \in H_1\} \\ &= \quad J_2(C) \times_{J_2(C')} P_2^+ \qquad \subset \quad J_2(C) \oplus P_2^+ \end{aligned}$$

with $\# H_1' = \# H_1 = 2^{2g-1}$, then we have a decomposition of $J(C')$ as:

$$\begin{array}{ccccc} (J(C) \oplus P^+)/H_1' & \cong & (\pi^*(J(C)) \oplus P^+)/P_2^+ & \cong & J(C') \\ (N, L') & \mapsto & (\pi^* N, L') & \mapsto & \pi^* N \otimes L' \\ (\mathrm{Nm}_\pi M, (1-\iota)M) & \leftarrow\!\shortmid & ((1+\iota)M, (1-\iota)M) & \leftarrow\!\shortmid & L = M^{\otimes 2} \end{array}$$

In conclusion, up to a finite subgroup of 2-torsion points (namely, 2^{2g-1} points), the Jacobian of C' splits into the Jacobian of C and the abelian variety P^+.

Remark 1.2.6. We can now build on proposition 1.2.3(a_2) to obtain

$$\pi^*(J(C)) \quad = \quad \mathrm{im}\,(1+\iota)|_{J(C')} \quad = \quad \ker\,(1-\iota)|_{J(C')}$$

Indeed, if we take $M, N \in J(C')$ such that $(1-\iota)M = \mathcal{O}_{C'}$ and $M = N^{\otimes 2}$, then $(1-\iota)N \in P_2^+ \subset \pi^*(J_2(C))$ by lemma 1.2.5, that is, $(1-\iota)N = \pi^*\eta'$ for some $\eta' \in H_1 \subset J_2(C)$. Hence we can write

$$M = N^{\otimes 2} = (1+\iota)N \otimes (1-\iota)N = \pi^*(\mathrm{Nm}_\pi N \otimes \eta') \in \pi^*(J(C))$$

so that $\ker\,(1-\iota)|_{J(C')} = \pi^*(J(C))$. In particular, this kernel is connected.

Let us use the notation $(-)^\circ$ to refer to the connected component containing the identity element $\mathcal{O}_{C'} \in J(C')$. Then we define:

Definition 1.2.7. The *Prym variety* associated to the Prym pair (C, η), with corresponding double cover $\pi\colon C' \to C$ and involution ι, is the abelian variety

$$P(C,\eta) \quad = \quad P^+ \quad = \quad \ker^\circ(\mathrm{Nm}_\pi) \quad = \quad \mathrm{im}(1-\iota) \quad = \quad \ker^\circ(1+\iota)$$

where $\mathrm{Nm}_\pi\colon J(C') \to J(C)$, $1-\iota\colon J(C') \to J(C')$ and $1+\iota\colon J(C') \to J(C')$.

By the discussion above, the Prym variety $P(C,\eta)$ can also be written as

$$P(C,\eta) \quad = \quad \mathrm{im}\,(1-\iota)|_{\mathrm{Pic}^{2k}(C')}$$

with $1-\iota\colon \mathrm{Pic}(C') \to J(C')$ and $k \in \mathbb{Z}$, and it is an abelian subvariety of $J(C')$ of dimension $g-1$, complementary to

$$J(C)/H_0 \quad \cong \quad \pi^*(J(C)) \quad = \quad \mathrm{im}\,(1+\iota)|_{J(C')} \quad = \quad \ker\,(1-\iota)|_{J(C')}$$

If we denote by $P_2(C,\eta)$ the 2-torsion of $P(C,\eta)$, of order 2^{2g-2}, then we have

$$H_1' \quad = \quad J_2(C) \times_{J_2(C')} P_2(C,\eta) \quad = \quad \{(\eta', \pi^*\eta')\} \quad \subset \quad J_2(C) \oplus P_2(C,\eta)$$

as above, of order 2^{2g-1}, and the decomposition

$$J(C') \quad \cong \quad (J(C) \oplus P(C,\eta))/H_1'$$

which characterises any line bundle $L = M^{\otimes 2} \in J(C')$, up to some 2-torsion, as the product $\pi^*N \otimes L'$ given by $N = \mathrm{Nm}_\pi M \in J(C)$, $L' = (1-\iota)M \in P(C,\eta)$.

1.2.2 The Prym variety as a p.p.a.v.

We have built $P(C,\eta)$ as an abelian variety, but we still need to (principally) polarize it. In order to do this, we can use the natural maps

$$\begin{array}{rcccccc} \pi^*\colon & J(C) & \to & \pi^*(J(C)) & \subset & J(C') \\ j\colon & P(C,\eta) & = & \ker^\circ(\mathrm{Nm}_\pi) & \hookrightarrow & J(C') \end{array}$$

to restrict the theta divisor $\Theta_{C'}$ of $J(C')$ to both $J(C)$ and $P(C,\eta)$, respectively inducing polarizations φ and ρ on these varieties:

$$\begin{array}{rcccccccc} \varphi = \lambda_{(\pi^*)^*\Theta'}\colon & J(C) & \xrightarrow{\pi^*} & J(C') & \overset{\lambda_{\Theta'}}{\cong} & J(C')^\vee & \xrightarrow{(\pi^*)^*} & J(C)^\vee \\ \rho = \lambda_{j^*\Theta'}\colon & P(C,\eta) & \overset{j}{\hookrightarrow} & J(C') & \overset{\lambda_{\Theta'}}{\cong} & J(C')^\vee & \xrightarrow{j^*} & P(C,\eta)^\vee \end{array}$$

By construction, φ and ρ make up the polarization

$$\begin{array}{ccccccc} J(C)\oplus P(C,\eta) & \to & J(C') & \overset{\lambda_{\Theta'}}{\cong} & J(C')^\vee & \to & J(C)^\vee \oplus P(C,\eta)^\vee \\ (N,L') & \mapsto & \pi^*N\otimes L' & & & & \end{array}$$

of $J(C)\oplus P(C,\eta)$, of kernel $H_1' = H_1 *_{H_1} \pi^*(H_1)$. Moreover, the maps

$$\begin{array}{rcccccc} j\colon & P(C,\eta) & \hookrightarrow & J(C') & & \\ 1-\iota\colon & J(C') & \twoheadrightarrow & P(C,\eta) & = & \mathrm{im}\,(1-\iota)|_{J(C')} \end{array}$$

are related by the compositions

$$\begin{array}{rclcccc} (1-\iota)\circ j & = & (-)^{\otimes 2} & : & P(C,\eta) & \to & P(C,\eta) \\ j\circ(1-\iota) & = & 1-\iota & : & J(C') & \to & J(C') \end{array}$$

and so play for $P(C,\eta)$ the same role that π^* and Nm_π did for $J(C)$ in lemma 1.2.2. As a result, φ and ρ are isogenies of exponent 2 such that

$$H_1 \subset \ker(\varphi) = J_2(C), \qquad \pi^*(H_1) = \ker(\rho) = P_2(C,\eta)$$

and, in consonance with [Bea77] Th. 3.7, we obtain factorizations

$$\begin{array}{rcccccccc} \varphi = \lambda_{(\pi^*)^*\Theta'} = \lambda_\Theta \circ (-)^{\otimes 2}\colon & J(C) & \xrightarrow{\pi^*} & J(C') & \xrightarrow{\mathrm{Nm}_\pi} & J(C) & \overset{\lambda_\Theta}{\cong} & J(C)^\vee \\ \rho = \lambda_{j^*\Theta'} = \lambda_\Xi \circ (-)^{\otimes 2}\colon & P(C,\eta) & \overset{j}{\hookrightarrow} & J(C') & \xrightarrow{1-\iota} & P(C,\eta) & \overset{\lambda_\Xi}{\cong} & P(C,\eta)^\vee \end{array}$$

where λ_Θ is the principal polarization of $J(C)$ defined by Θ_C. In particular, we have found a principal polarization of $P(C,\eta)$, namely

$$\lambda_\Xi : \quad P(C,\eta) \;\cong\; P(C,\eta)^\vee, \qquad \lambda_{j^*\Theta'} \;=\; \lambda_\Xi \circ (-)^{\otimes 2}$$

through which the maps j and $1-\iota$ are dual to one another.

Remark 1.2.8. The principal polarization λ_Ξ corresponds to a certain divisor $\Xi_{(C,\eta)}$ on the Prym variety such that $j^*\Theta_{C'}$ is algebraically equivalent to $2\,\Xi_{(C,\eta)}$, in the same way that $(\pi^*)^*\Theta_{C'}$ is algebraically equivalent to $2\,\Theta_C$. Therefore

$$\deg(\Theta_{C'} \cdot P(C,\eta)) \;=\; \deg(j^*\Theta_{C'}) \;=\; 2\,\deg(\Xi_{(C,\eta)})$$

meaning that the intersection of $P(C,\eta)$ and $\Theta_{C'}$ is of even degree.

We can now complete definition 1.2.7:

Definition 1.2.9. The *Prym variety* associated to the Prym pair (C,η) is the principally polarized abelian variety

$$(P(C,\eta),\, \Xi_{(C,\eta)})$$

whose theta divisor $\Xi_{(C,\eta)}$ induces a principal polarization λ_Ξ squaring to

$$\lambda_\Xi \circ (-)^{\otimes 2} \;=\; \rho : \quad P(C,\eta) \overset{j}{\hookrightarrow} J(C') \overset{\lambda_{\Theta'}}{\cong} J(C')^\vee \overset{j^*}{\to} P(C,\eta)^\vee$$

that is, to the polarization that arises when we restrict $\Theta_{C'}$ to $P(C,\eta)$.

From here on, we will abuse the notation $P(C,\eta)$ to refer to both the nonpolarized and principally polarized versions of the Prym variety, depending on the context, as well as denote

$$P(C,\eta) \;=\; P(\pi,\iota) \;=\; P(C',\iota)$$

on account of proposition 1.1.13.

1.2.3 The Prym variety in degree 2g − 2

Similarly to how the Jacobian variety had a useful interpretation after translation to a different degree, i.e.

$$\begin{array}{rcccccl} T_K\colon & \mathrm{Pic}^0(C) & \cong & \mathrm{Pic}^{g-1}(C) & \ni & K, & T_K(L) \;=\; L\otimes K \\ & \Theta_C & \cong & W_{g-1}(C) & = & \{M\in \mathrm{Pic}^{g-1}(C)\;/\;h^0(C,M)\geq 1\} & \\ & L & \mapsto & L\otimes K & / & \mathrm{mult}_L(\Theta_C)=h^0(C,L\otimes K) & \end{array}$$

for a certain $K\in \mathrm{Pic}^{g-1}(C)$, so will the Prym variety when translated to degree $2g-2$. If we consider the norm map

$$\mathrm{Nm}_\pi\colon\quad \mathrm{Pic}^{2g-2}(C') \;\to\; \mathrm{Pic}^{2g-2}(C) \;\ni\; \omega_C$$

then it is shown in [Mum71] that $\mathrm{Nm}_\pi^{-1}(\omega_C)$ disconnects into two components

$$\begin{aligned} \mathrm{Nm}_\pi^{-1}(\omega_C)^+ &= \{M\in \mathrm{Nm}_\pi^{-1}(\omega_C)\;/\;h^0(C',M)\equiv 0 \mod 2\}\\ \mathrm{Nm}_\pi^{-1}(\omega_C)^- &= \{M\in \mathrm{Nm}_\pi^{-1}(\omega_C)\;/\;h^0(C',M)\equiv 1 \mod 2\}\end{aligned}$$

which can be identified with the components P^+, P^- of $\ker(\mathrm{Nm}_\pi)$ described in corollary 1.2.4. Indeed, translating by any element $K_1\in \mathrm{Nm}_\pi^{-1}(\omega_C)^+$ we get:

$$\begin{array}{rcccccc} T_{K_1}\colon & P^+ & \cong & \mathrm{Nm}_\pi^{-1}(\omega_C)^+, & \mathcal{O}_{C'} & \mapsto & K_1\\ T_{K_1}\colon & P^- & \cong & \mathrm{Nm}_\pi^{-1}(\omega_C)^-, & L & \mapsto & L\otimes K_1 \end{array}$$

Conveniently, we have $\Theta_{C'}\cong W_{2g-2}(C')\subset \mathrm{Pic}^{2g-2}(C')$, and furthermore:

Lemma 1.2.10. *There is some $K_1\in \mathrm{Nm}_\pi^{-1}(\omega_C)^+$ yielding isomorphisms*

$$\begin{array}{rccc} T_{K_1}\colon & \mathrm{Pic}^0(C') & \cong & \mathrm{Pic}^{2g-2}(C')\\ & P(C,\eta) & \cong & \mathrm{Nm}_\pi^{-1}(\omega_C)^+\\ & \Xi_{(C,\eta)} & \cong & \Xi \end{array}$$

such that $(\mathrm{Nm}_\pi^{-1}(\omega_C)^+,\,\Xi)$ is a principally polarized abelian variety with

$$W_{2g-2}(C')\cdot \mathrm{Nm}_\pi^{-1}(\omega_C)^+ \;=\; 2\,\Xi$$

Moreover, it holds that $\mathrm{Nm}_\pi^{-1}(\omega_C)^-\subset W_{2g-2}(C')\subset \mathrm{Pic}^{2g-2}(C')$.

Proof. Most of the statement follows directly from the above discussion; for the remainder, see [Mum74] Section III. Regarding $W_{2g-2}(C')\cdot \mathrm{Nm}_\pi^{-1}(\omega_C)^+ = 2\,\Xi$, observe that $\mathrm{mult}_L(\Theta_{C'})=h^0(C',L\otimes K_1)$ is even for all $L\in P(C,\eta)$, and that $\Theta_{C'}\cdot P(C,\eta)$ is algebraically equivalent to $2\,\Xi_{(C,\eta)}$ by remark 1.2.8. □

The equation $W_{2g-2}(C') \cdot \mathrm{Nm}_\pi^{-1}(\omega_C)^+ = 2\,\Xi$ provided by the lemma renders a nice set-theoretical description of the theta divisor, namely:

$$\Xi_{(C,\eta)} \;\cong\; \Xi \;=\; \{M \in \mathrm{Nm}_\pi^{-1}(\omega_C)^+ \;/\; h^0(C', M) \geq 2\}$$

and we can summarize the degree $2g-2$ picture in the following manner:

$$\begin{array}{rcllcl}
\mathrm{Pic}^0(C') & \cong & \mathrm{Pic}^{2g-2}(C'), & & L & \mapsto \; L \otimes K_1 \\
P(C,\eta) & \cong & \mathrm{Nm}_\pi^{-1}(\omega_C)^+ & = & \{M \in \mathrm{Nm}_\pi^{-1}(\omega_C) \;/\; h^0(M) \equiv 0 \mod 2\} \\
\Xi_{(C,\eta)} & \cong & \Xi & = & \{M \in \mathrm{Nm}_\pi^{-1}(\omega_C)^+ \;/\; h^0(M) \geq 2\} \\
& & \text{with} & & W_{2g-2}(C') \cdot \mathrm{Nm}_\pi^{-1}(\omega_C)^+ \;=\; 2\,\Xi
\end{array}$$

This characterization of the Prym variety is helpful in many ways; for instance, it can be used to describe the singularities of Ξ, as in [Mum74].

1.2.4 The Abel-Prym map

Given a fixed, rational point $x_0 \in C$, the family of Abel-Jacobi maps associated to x_0 plays an important role in the theory of Jacobian varieties:

$$\mathsf{aj}\colon C^{(d)} \to J(C), \quad D \mapsto \mathcal{O}_C(D - d\,x_0)$$

with $d \in \mathbb{Z}^+$. In particular, aj is injective for $d = 1$, and generically injective for $d \leq g$. If we now consider these maps on the cover C' and compose them with $1 - \iota\colon J(C') \to P(C,\eta)$, we obtain a new family of maps, relevant to the theory of Prym varieties. The study of these *Abel-Prym* maps is generally well known, but did not seem to properly find its way into the literature until very recently, when it was compiled by Casalaina-Martin in [LZ21] Appendix A.

Definition 1.2.11. Given a fixed, rational point $x_0' \in C'$, the morphism

$$\mathsf{ap}\colon (C')^{(d)} \to P(C,\eta), \quad D \mapsto (1-\iota)(\mathcal{O}_{C'}(D - d\,x_0'))$$

is known as the *Abel-Prym map* associated to x_0' in degree $d \in \mathbb{Z}^+$.

If we denote $D_0 = (\mathrm{Id} - \iota^*)(d\,x_0') = d\,(x_0' - \iota(x_0'))$, we can write

$$\mathsf{ap}(D) \;=\; (1-\iota)(\mathcal{O}_{C'}(D - d\,x_0')) \;=\; \mathcal{O}_{C'}(D - \iota^* D - D_0)$$

and see that the Abel-Prym map fits into the commutative diagram

$$
\begin{tikzcd}
(C')^{(d)} \arrow[rr, "\mathsf{aj}'"] \arrow[d, hook, "\mathrm{Id}\times\iota^*"'] \arrow[drr, "\mathsf{ap}"] & & J(C') \arrow[d, "1-\iota"] \\
(C')^{(d)} \times (C')^{(d)} \arrow[d, "\phi_d"'] & & P(C,\eta) \arrow[d, hook, "j"] \\
J(C') \arrow[rr, "\simeq", "T_{L_0}"'] & & J(C')
\end{tikzcd}
\qquad
\begin{tikzcd}
D \arrow[r, mapsto] \arrow[d, mapsto] & \mathcal{O}_{C'}(D - d\,x'_0) \arrow[dd, mapsto] \\
(D,\, \iota^* D) \arrow[d, mapsto] \arrow[r, phantom, "\circlearrowleft"] & {} \\
\mathcal{O}_{C'}(D - \iota^* D) \arrow[r, mapsto] & \mathsf{ap}(D)
\end{tikzcd}
$$

where the isomorphism T_{L_0} is translation by $L_0 = \mathcal{O}_{C'}(-D_0) \in \operatorname{im}(1-\iota)|_{\operatorname{Pic}^d(C')}$ and the map ϕ_d is the Abel-Jacobi difference map

$$\phi_d\colon (C')^{(d)} \times (C')^{(d)} \to J(C'), \quad (D,\, E) \mapsto \mathcal{O}_{C'}(D-E)$$

as defined in [ACGH85] Chapter V, Ex. D. Moreover, observe that two effective, degree d divisors D, E belong to the same fiber of ap if and only if the effective, degree $2d$ divisors $D + \iota^* E$ and $E + \iota^* D$ are linearly equivalent:

$$
\begin{aligned}
\mathsf{ap}(D) = \mathsf{ap}(E) \quad &\Leftrightarrow \quad D - \iota^* D - D_0 \sim E - \iota^* E - D_0 \\
&\Leftrightarrow \quad D - \iota^* D \sim E - \iota^* E \\
&\Leftrightarrow \quad D + \iota^* E \sim E + \iota^* D
\end{aligned}
$$

If the divisors $D + \iota^* E$ and $E + \iota^* D$ were to be distinct, they would give rise to a complete linear series of type g^r_{2d} with $r \geq 1$, that is, to an element of $W^1_{2d}(C')$. Therefore we will be able to describe better the fibers of ap in instances where we have additional information regarding the existence of such linear series.

Lemma 1.2.12. *If X is a hyperelliptic curve, then its hyperelliptic involution τ commutes with any automorphism of X.*

Proof. The hyperelliptic involution is uniquely determined by the canonical map of X, which in this case is a double cover of the rational normal curve. Let φ be an automorphism of X. Then the conjugate $\varphi^{-1} \circ \tau \circ \varphi$ is an involution whose fixed locus is identified with the fixed locus of τ via φ. Thus $\varphi^{-1} \circ \tau \circ \varphi = \tau$ by uniqueness and τ commutes with φ. $\square$

Lemma 1.2.13. *The Abel-Prym map in degree 1 is injective unless C' is hyperelliptic.*

Proof. Assume C' is not hyperelliptic, i.e. $W^1_2(C') = \emptyset$, and take $D = x$, $E = y$.

Note that $\iota(x) \neq x$, as ι is fixed-point-free (proposition 1.1.13). Then

$$\begin{aligned}\mathsf{ap}(x) = \mathsf{ap}(y) \;&\Leftrightarrow\; x + \iota(y) \sim y + \iota(x) \;\in\; G^0_2(C') \\ &\Leftrightarrow\; x + \iota(y) = y + \iota(x) \\ &\Leftrightarrow\; x = y\end{aligned}$$

so $\mathsf{ap}\colon C' \hookrightarrow P(C,\eta)$ is injective. Conversely, if $\tau\colon C' \to C'$ is the hyperelliptic involution, then ι commutes with τ by lemma 1.2.12 and as a result restricts to a fixed-point-free involution on its fixed locus R_τ. In particular we get

$$x \in R_\tau \;\Rightarrow\; 2\,x \sim 2\,\iota(x) \;\Rightarrow\; x - \iota(x) \sim \iota(x) - x \;\Rightarrow\; \mathsf{ap}(x) = \mathsf{ap}(\iota(x))$$
$$\textit{(actually,}\quad x \in C' \;\Rightarrow\; \iota(\tau(x)) \in \mathsf{ap}^{-1}(\mathsf{ap}(x)) \quad \textit{in general)}$$

but $\iota(x) \neq x$, so injectivity fails. □

Remark 1.2.14. As a side effect of lemma 1.2.12, if C' is hyperelliptic then τ induces an involution on C that causes it to also be hyperelliptic, which can be derived from a fixed-point count (see lemma 1.4.1). In contrast, the converse is not true in general, as shown in [Far76]: although it holds in genus 2, where all 15 possible nontrivial étale double covers of a curve are hyperelliptic, it already breaks down in genus 3, where only 28 of the 63 covers preserve hyperellipticity.

Remark 1.2.15. In the next section we introduce the Prym moduli space, which allows us to talk about a general Prym pair. In genus $g \geq 3$, the difference map for such a pair, namely

$$\delta = \phi_d \circ (\mathrm{Id} \times \iota^*)\colon (C')^{(d)} \to J(C'), \quad D \mapsto \mathcal{O}_{C'}(D - \iota^* D),$$

is generically finite whenever $1 \leq d \leq g-1$; see [FV16]. This is not surprising, as generality comes hand in hand with the dimension of $W^1_{2d}(C')$ remaining low, by virtue of Brill-Noether theory. Since the Abel-Prym map is the composition of a translation and the difference map δ, as mentioned above, it follows that

$$\mathsf{ap}\colon (C')^{(d)} \to P(C,\eta), \quad 1 \leq d \leq g-1, \quad g \geq 3, \quad (C,\eta) \text{ general},$$

is also generically finite, hence $\dim \mathsf{ap}((C')^{(d)}) = \dim (C')^{(d)} = d \leq g-1$. More details and an improved statement can be found in [LZ21] Appendix A.

Lemma 1.2.16. *Assume* $\mathsf{ap}\colon (C')^{(g-1)} \to P(C,\eta)$ *to be generically finite. Then it is dominant.*

Proof. Immediate from $\dim (C')^{(g-1)} = g-1 = \dim P(C,\eta)$. □

Remark 1.2.15 and lemma 1.2.16 ensure that, for a general Prym pair (C,η)

of genus $g \geq 3$, the Abel-Prym map

$$\mathsf{ap}\colon (C')^{(g-1)} \to P(C,\eta), \quad D \mapsto \mathcal{O}_{C'}(D - \iota^* D - D_0)$$

is dominant. Alternatively, note that the difference map

$$\delta = \phi_{g-1} \circ (\mathrm{Id} \times \iota^*)\colon (C')^{(g-1)} \to J(C'), \qquad \mathsf{ap} = T_{L_0} \circ \delta$$

maps directly to $P(C,\eta)$ when g is odd, and thus in this case

$$\delta\colon (C')^{(g-1)} \to P(C,\eta), \quad D \mapsto \mathcal{O}_{C'}(D - \iota^* D)$$

is dominant as well, and equivalent to ap via $T_{L_0}\colon P(C,\eta) \cong P(C,\eta)$.

Later in the chapter we will work with Prym pairs of genus $g = 3$, hence we will be specifically interested in $\mathsf{ap} = T_{L_0} \circ \delta\colon (C')^{(2)} \to P(C,\eta)$, or rather in

$$\delta\colon (C')^{(2)} \to P(C,\eta)$$

In the setting of moduli spaces and universal families, this Abel-Prym map will prove useful to study the birational geometry of the universal Prym variety.

1.3 Prym varieties in families

In the previous sections we looked at Prym pairs and varieties as fixed, individual objects. The next natural step is to arrange them into *moduli spaces*, that is to say, into spaces which not only parametrize them, but which are further endowed with some additional algebro-geometric structure that allows the parametrized objects to vary by moving along the space. With this goal in mind, let us first recall the different notions related to *moduli problems*.

1.3.1 Moduli spaces and stacks

Let A be the family of objects that we want to parametrize, equipped with an equivalence relation $\sim$ (e.g. the family of smooth curves of genus g, equivalent via isomorphisms). Moreover, let Sch be the category of schemes over $\mathbb{C}$ and let $\mathcal{F}$ be a presheaf on Sch (i.e. a contravariant functor from Sch to the category of sets):

$$\mathcal{F}\colon \mathsf{Sch}^{\mathrm{opp}} \rightsquigarrow \mathsf{Sets}$$

together with an equivalence relation $R \subset \mathcal{F} \times \mathcal{F}$, such that

$$\mathcal{F}(\operatorname{Spec}\mathbb{C}) = A, \qquad R(\operatorname{Spec}\mathbb{C}) = \sim$$

The pair $(\mathcal{F}, R)$ induces a new presheaf of schemes $\mathcal{F}/R$, namely

$$\mathcal{F}/R\colon \mathsf{Sch}^{\mathrm{opp}} \rightsquigarrow \mathsf{Sets}, \qquad S \mapsto \mathcal{F}(S)/R(S), \qquad \operatorname{Spec}(\mathbb{C}) \mapsto A/{\sim}$$

which we call the *moduli functor* of the moduli problem $(A, \sim, \mathcal{F}, R)$.

Definition 1.3.1. Let $\mathcal{C}$ be a category together with a presheaf $\mathcal{G}\colon \mathcal{C}^{\mathrm{opp}} \rightsquigarrow \mathsf{Sets}$ on $\mathcal{C}$. We say that $\mathcal{G}$ is *representable* in $\mathcal{C}$ if there exist an object M of $\mathcal{C}$ and a natural isomorphism

$$\Phi\colon \mathcal{G} \cong h_M$$

where $h_M = \operatorname{Hom}(-, M)$ is standard notation for the (contravariant) functor of points of M. In particular, there is a set $U \in \mathcal{G}(M)$ such that $\Phi(U) = \mathrm{Id}_M$ and

$$\Phi^{-1}(f\colon S \to M) \quad = \quad G(f)(U) \quad \in \quad \mathcal{G}(S)$$

for all $f \in h_M(S)$. The set U is usually called the *universal family* over M.

Definition 1.3.2. A pair (M, Φ) is a *fine moduli space* for the moduli problem $(A, \sim, \mathcal{F}, R)$ if the moduli functor $\mathcal{F}/R$ is representable in Sch by (M, Φ), that is, if M is a scheme and Φ is a natural isomorphism

$$\Phi\colon \mathcal{F}/R \cong h_M, \quad U \mapsto \mathrm{Id}_M$$

where $U \in \mathcal{F}(M)/R(M)$ is the universal family over M.

We will regularly drop the natural isomorphism Φ from the notation for fine moduli spaces, especially when working with the universal family over M, whose existence is equivalent to that of $\Phi\colon \mathcal{F}/R \cong h_M$.

Remark 1.3.3. If (M, Φ) is a fine moduli space for $(A, \sim, \mathcal{F}, R)$, then there is an identification between objects $Y \in A/{\sim}$ and rational points of M, and more generally between objects $Y \in \mathcal{F}(S)/R(S)$ over S and S-valued points of M:

$$A/{\sim} \cong \operatorname{Hom}(\operatorname{Spec}\mathbb{C}, M), \qquad \mathcal{F}(S)/R(S) \cong \operatorname{Hom}(S, M)$$

which captures the idea of “parametrizing objects as points of a space”.

Example 1.3.4. Take $\mathbb{C}[x_0, \ldots, x_n]$ and consider the following functor:

$$\mathcal{F}\colon \mathsf{Sch}^{\mathrm{opp}} \rightsquigarrow \mathsf{Sets}, \qquad S \mapsto \{(L, \varphi) \;/\; L \in \operatorname{Pic}(S),\, \varphi\colon \mathcal{O}_S^{n+1} \twoheadrightarrow L\}$$

where $\mathcal{F}(f\colon T \to S)$ is given by the pullback map, and φ produces a map

$$\mathbb{C}^{n+1} \cong \langle x_0, \ldots, x_n \rangle \twoheadrightarrow V_L \subset H^0(S, L), \qquad x_i \mapsto \sigma_i$$

such that L is generated by the global sections $\sigma_0, \ldots, \sigma_n \in H^0(S, L)$. Let R be the equivalence relation

$$(L, \varphi) \sim_R (L', \varphi') \;\Leftrightarrow\; \exists\, \chi\colon L \cong L' \;/\; \chi \circ \varphi = \varphi'$$

in which case $\chi(\sigma_i) = \sigma_i'$ for all $i \in \{0, \ldots, n\}$. Then the moduli functor

$$\mathcal{F}/R\colon \mathsf{Sch}^{\mathrm{opp}} \rightsquigarrow \mathsf{Sets}, \qquad S \mapsto \{(L, \varphi) \;/\; L \in \operatorname{Pic}(S),\, \varphi\colon \mathcal{O}_S^{n+1} \twoheadrightarrow L\}/\!\sim_R$$

aims to parametrize the set of line quotients W of $\mathcal{O}^{n+1}_{\operatorname{Spec}\mathbb{C}} = \mathbb{C}^{n+1}$, or equivalently, the set $\mathbb{P}(\mathbb{C}^{n+1})$ of 1-dimensional vector subspaces $E = \varphi^{\mathrm{t}}(W^\vee) \subset \mathbb{C}^{n+1}$. As such, a natural moduli candidate would be the projective space

$$\mathbb{P}^n_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_n]$$

which turns out to be fit for the role, as it is a scheme equipped with a natural isomorphism

$$(\mathcal{F}/R)(S) \cong \operatorname{Hom}(S, \mathbb{P}^n_{\mathbb{C}}), \quad [(L, \varphi)] \mapsto (\varphi_L\colon S \to \mathbb{P}^n_{\mathbb{C}})$$

where $\varphi_L(s) = \varphi^{\mathrm{t}}(L|_s{}^\vee) = (\sigma_0(s) : \ldots : \sigma_n(s))$. Hence $\mathbb{P}^n_{\mathbb{C}}$ is indeed a fine moduli space for the moduli problem $(\mathbb{P}(\mathbb{C}^{n+1}), =, \mathcal{F}, R)$, with universal family

$$U = \mathcal{O}(1) \in (\mathcal{F}/R)(\mathbb{P}^n_{\mathbb{C}}), \quad \mathcal{O}(1) \in \operatorname{Pic}(\mathbb{P}^n_{\mathbb{C}}), \quad \mathcal{O}^{n+1}_{\mathbb{P}^n_{\mathbb{C}}} \twoheadrightarrow \mathcal{O}(1)$$

since by definition of the tautological bundle $\mathcal{O}(-1) \in \operatorname{Pic}(\mathbb{P}^n_{\mathbb{C}})$ it holds that

$$U|_s{}^\vee = \operatorname{Id}(s) = \mathcal{O}(-1)|_s$$

and thus U is the dual $\mathcal{O}(-1)^\vee = \mathcal{O}(1)$, i.e. the Serre twisting sheaf. Then

$$\operatorname{Hom}(S, \mathbb{P}^n_{\mathbb{C}}) \cong (\mathcal{F}/R)(S), \quad (f\colon S \to \mathbb{P}^n_{\mathbb{C}}) \mapsto [(f^*\mathcal{O}(1), \varphi)]$$

where $\varphi\colon \mathcal{O}_S^{n+1} \twoheadrightarrow f^*\mathcal{O}(1)$ is the pullback of $\mathcal{O}^{n+1}_{\mathbb{P}^n_{\mathbb{C}}} \twoheadrightarrow \mathcal{O}(1)$. In other words, this provides an identification between parametrized objects (line quotients of $\mathcal{O}_S^{n+1}$) and S-valued points of $\mathbb{P}^n_{\mathbb{C}}$, as pointed out in remark 1.3.3.

Example 1.3.4 showcases the interest of moduli spaces, as $\mathbb{P}^n_{\mathbb{C}}$ carries a rich (topological, algebraic, geometric) structure which permeates into the line quotients parametrized by it, yielding for them notions such as pathwise connection, continuous and algebraic deformations, and so on. Unfortunately, many natural moduli problems do not admit a fine moduli space.

Example 1.3.5. Consider the functor

$$\mathcal{F}\colon \mathsf{Sch}^{\mathrm{opp}} \rightsquigarrow \mathsf{Sets}, \qquad S \mapsto \{\text{smooth, genus } g \text{ curves over } S\}$$

where $\mathcal{F}(f\colon T \to S)(\mathcal{X} \to S) = (\mathcal{X} \times_S T \to T)$, with the equivalence relation

$$(\mathcal{X} \to S) \sim_R (\mathcal{Y} \to S) \;\Leftrightarrow\; \mathcal{X} \cong \mathcal{Y} \text{ as } S\text{-schemes}$$

Then the moduli functor

$$\mathcal{F}/R\colon \mathsf{Sch}^{\mathrm{opp}} \rightsquigarrow \mathsf{Sets}, \qquad S \mapsto \{\text{smooth, genus } g \text{ curves over } S\}/{\sim_R}$$

intends to parametrize the set of isomorphism classes of smooth, genus g curves. Nevertheless, unlike example 1.3.4, this functor fails to be representable due to the existence of curves with nontrivial automorphisms. Let us briefly discuss this. First, take a hyperelliptic, smooth, genus g curve X and let $\tau\colon X \cong X$ be the hyperelliptic involution. If a fine moduli space M_g were to exist, the natural isomorphism $\mathcal{F}/R \cong h_{M_g}$ would be given by

$$(\mathcal{F}/R)(S) \to \mathrm{Hom}(S, M_g), \quad [\mathcal{X} \to S] \mapsto (S \to M_g,\ s \mapsto [\mathcal{X}_s])$$

However, the pair (X, τ) allows us to build curves over, for instance, the multiplicative group scheme $S = \mathfrak{G}_{\mathsf{m}} = \operatorname{Spec} \mathbb{C}[t]_t = (\mathbb{A}^1_{\mathbb{C}} - \{0\}, \cdot)$, namely

$$\begin{array}{rcl} \mathcal{X} = \mathfrak{G}_{\mathsf{m}} \times X & \to & \mathfrak{G}_{\mathsf{m}}, \\ (\lambda,\, x) & \mapsto & \lambda \end{array} \qquad \begin{array}{rcl} \mathcal{Y} = (\mathfrak{G}_{\mathsf{m}} \times X)/{\sim} & \to & \mathfrak{G}_{\mathsf{m}}, \\ [(\lambda,\, x)] & \mapsto & \lambda^2 \end{array}$$

with $(\lambda,\, x) \sim (-\lambda,\, \tau(x))$, which are not isomorphic as S-schemes (as they have different monodromy), but whose fibers are always isomorphic to X:

$$\mathcal{X}_\lambda = \{\lambda\} \times X \cong X, \qquad \mathcal{Y}_\lambda = (X \sqcup X)/{\sim} \cong X$$

with $i_1(x) \sim i_2(\tau(x))$ for $i_1, i_2\colon X \to X \sqcup X$ canonical injections. Hence

$$[\mathcal{X} \to \mathfrak{G}_{\mathsf{m}}] \neq [\mathcal{Y} \to \mathfrak{G}_{\mathsf{m}}] \mapsto (\mathfrak{G}_{\mathsf{m}} \to M_g,\ \lambda \mapsto [X])$$

both correspond to the same constant map $\lambda \mapsto [X]$ and the natural transformation $\mathcal{F}/R \to h_{M_g}$ cannot be an isomorphism.

Intuitively, what stops the moduli problem in example 1.3.5 from admitting a fine moduli space is the fact that nontrivial automorphisms enable the process of gluing fibers nontrivially, which twists the family and alters its global structure (without changing it fiberwise). One possible way of working around this takes advantage of the fact that the natural transformation $\mathcal{F}/R \to h_M$ often exists, even when it fails to be an isomorphism, suggesting it may be useful to slightly relax the requirements for a moduli space.

Definition 1.3.6. A pair (M, Φ) is a *coarse moduli space* for $(A, \sim, \mathcal{F}, R)$ if M is a scheme and $\Phi\colon \mathcal{F}/R \to h_M$ is a natural transformation such that

(i) The map $\Phi(\operatorname{Spec}\mathbb{C})\colon A/\!\sim\, \cong M(\mathbb{C})$ is a bijection.

(ii) The natural transformation Φ is universal among all natural transformations of the form $\mathcal{F}/R \to h_T$ for T scheme, that is, any such transformation factors through Φ via $g_*\colon h_M \to h_T$ for a unique $g\colon M \to T$.

Condition (i) ensures that the rational points of a coarse moduli space still parametrize the desired family of objects $A/\!\sim$, although now the S-valued points may not represent objects over S anymore. However, by virtue of condition (ii), the coarse moduli space is still initial among all schemes which parametrize (in any way) objects of $\mathcal{F}/R$.

Remark 1.3.7. The moduli problem depicted in example 1.3.5 admits a coarse moduli space M_g, which is equipped with the natural transformation

$$\begin{aligned} \{\text{smooth, genus } g \text{ curves over } S\}/\!\sim_R \;&\to\; \operatorname{Hom}(S, M_g) \\ [\mathcal{X} \to S] \;&\mapsto\; (S \to M_g,\ s \mapsto [\mathcal{X}_s]) \end{aligned}$$

and whose rational points parametrize the set of isomorphism classes of smooth, genus g curves. Even if it lacks a universal family, M_g is nevertheless a scheme, a fact that still has significant consequences for the curves it parametrizes.

A notable aspect of the theory of coarse moduli spaces is that they are not in possession of a universal family. While such a family is easy to understand at the level of fibers (e.g. for the moduli space of genus g curves it would look like

$$[U \to M_g] \;\leftrightsquigarrow\; (\mathrm{Id}\colon M_g \to M_g,\ s \mapsto [U_s] = s,\ U_s \cong C_s \in s \text{ repr.})$$

$$\begin{tikzcd} U \arrow[r] & M_g \\ s \ni C_s \cong U_s \arrow[r] \arrow[u] \arrow[ur, phantom, "\square"] & \{s\} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r] & M_g \\ \coprod_{s\in S} C_s \approx U \times_{M_g} S \arrow[r] \arrow[u] \arrow[ur, phantom, "\square"] & S \arrow[u] \end{tikzcd}$$

if it existed), the obstructions to representability of the moduli functor prevent this collection of fibers from acquiring the necessary structure of object of $\mathcal{F}/R$ over M (e.g. of curve over M_g). In many situations, coarse moduli spaces are powerful enough on their own to study the way algebro-geometric objects vary in families – but universal families are convenient tools regardless, and so it makes sense to wonder if they can be recovered in those situations where fine moduli spaces do not exist. This leads us to the notion of *stacks*, which circumvent the automorphism issue by taking a different approach, where we opt to remember the automorphisms instead of modding out by them. In particular, if we recall that a groupoid is a category where every morphism is an isomorphism, the idea is to replace the set $A/\sim$ with the groupoid $(A, \sim)$ such that any two objects X, $Y \in A$ have an arrow $X \to Y$ between them if and only if $X \sim Y$.

Through the Yoneda embedding, schemes can be understood as "sheaves of sets", that is, presheaves

$$h_X \colon \mathsf{Sch}^{\mathrm{opp}} \rightsquigarrow \mathsf{Sets}, \quad S \mapsto \mathrm{Hom}(S, X)$$

fulfilling a certain gluing condition. Morally, stacks are meant to be understood as "sheaves of groupoids (up to 2-isomorphism)":

$$\mathcal{X} \colon \mathsf{Sch}^{\mathrm{opp}} \dashrightarrow \mathsf{Grpd}, \quad S \mapsto \text{“}(\mathrm{Hom}(S, \mathcal{X}), \mathrm{Isom}(\mathrm{Hom}(S, \mathcal{X})))\text{”}$$

where Grpd is the 2-category of groupoids. As evidenced by the clunky notation, the machinery of sheaf theory is not well suited for this construction, and so a new language is needed, that of *fibered categories* and *descent theory*. It would require a whole book to properly introduce these notions (see e.g. [Ols16] for an in-depth dive on the topic), hence we will restrict ourselves to stating no more than the formal definition of (algebraic) stack, and then adhere to the intuitive interpretation.

Definition 1.3.8. A fibered category $\mathcal{X}$ over Sch is a *stack* if:

(i) $\mathcal{X} \rightsquigarrow \mathsf{Sch}$ is fibered in groupoids.

(ii) Every covering in Sch is of effective descent.

Condition (i) provides the "presheaf of groupoids" behaviour,

$$\mathcal{X} \colon \mathsf{Sch}^{\mathrm{opp}} \dashrightarrow \mathsf{Grpd}, \quad S \mapsto \mathcal{X}_S$$

where $\mathcal{X}_S$ denotes the fiber of the scheme S under $\mathcal{X} \rightsquigarrow \mathsf{Sch}$. It is worth noting the existence of a 2-categorical version of the Yoneda lemma, which renders an equivalence of categories further supporting the intuition described above:

$$\mathsf{HOM}_{\mathsf{Sch}}(S, \mathcal{X}) \sim \mathcal{X}_S$$

with $\mathsf{HOM}_{\mathsf{Sch}}$ denoting the category of morphisms of fibered categories over Sch (as objects) and base preserving natural isomorphisms between them (as arrows). At the same time, condition (ii) makes it possible to glue compatible local data, thus providing the sheaf-like behaviour. As a result, definition 1.3.8 succeeds in generalising the notion of a scheme in a way that accounts for isomorphisms of its (S-valued) points.

Schemes, however, are not truly the main objects of study in Algebraic Geometry; algebraic varieties are. Accordingly, stacks are often enriched with additional layers of algebraic structure in order to obtain spaces whose geometry is better behaved than that of a plain stack.

Definition 1.3.9. A stack $\mathcal{X} \rightsquigarrow \mathsf{Sch}$ is *algebraic* (resp. *Deligne-Mumford*) if:

(i) The diagonal $\mathcal{X} \times_{\mathsf{Sch}} \mathcal{X}$ is representable.

(ii) There exists a smooth (resp. étale) surjective morphism $\pi \colon X \twoheadrightarrow \mathcal{X}$ for some scheme X.

The surjection $\pi \colon X \twoheadrightarrow \mathcal{X}$ is usually called an *atlas* of $\mathcal{X}$.

By virtue of their atlas, algebraic stacks are locally modelled after schemes, mirroring how schemes are locally modelled after affine schemes. This feature brings forth important implications, out of which perhaps the most interesting one for our purposes is the fact that many moduli spaces turn out to naturally inherit an algebraic (or even Deligne-Mumford) stack structure.

Remark 1.3.10. Most scheme-theoretical notions can be translated into the language of (algebraic) stacks, such as properties of schemes and morphisms of schemes, quasi-coherent sheaves, the relative $\underline{\mathrm{Spec}}$ and $\underline{\mathrm{Proj}}$ constructions, and so on. From a practical point of view, this will allow us to work with algebraic stacks while still largely using the more familiar language of scheme theory.

If we now go back to the moduli problem $(A, \sim, \mathcal{F}, R)$ from the beginning, we can reinterpret the moduli functor to be a "presheaf of groupoids"

$$\mathcal{F}_R \colon \mathsf{Sch}^{\mathrm{opp}} \dashrightarrow \mathsf{Grpd}, \qquad S \mapsto (\mathcal{F}_R)_S, \qquad \mathrm{Spec}(\mathbb{C}) \mapsto (A, \sim)$$

where $(\mathcal{F}_R)_S$ is the groupoid with objects $\mathcal{F}(S)$ and arrows $X \to Y$ if and only if $X \sim_R Y$. As the previous discussion suggests, $\mathcal{F}_R$ is not really a functor, but rather a fibered category over Sch, with fibers $(\mathcal{F}_R)_S$.

Definition 1.3.11. If the fibered category $\mathcal{F}_R = \mathcal{M} \rightsquigarrow \mathsf{Sch}$ is a stack, we say that it is the *moduli stack* for the moduli problem $(A, \sim, \mathcal{F}, R)$.

Remark 1.3.12. Recall the identification between sections of the moduli functor and points of a fine moduli space given in remark 1.3.3:

$$(\mathcal{F}/R)(S) \cong \mathrm{Hom}(S, M)$$

If $\mathcal{M}$ is a moduli stack for $(A, \sim, \mathcal{F}, R)$, this is simply the 2-Yoneda lemma:

$$(\mathcal{F}_R)_S \sim \mathsf{HOM}_{\mathsf{Sch}}(S, \mathcal{M})$$

so, in a way, moduli stacks are always "fine" by construction, in the sense that morphisms $S \to \mathcal{M}$ with S scheme correspond to objects of $\mathcal{F}_R$ over S. Notice that this occurs without the need for a universal family, as the left-hand side of the equivalence does not make sense for $S = \mathcal{M}$. Nonetheless, if the set $\mathcal{F}(S)$ is given by "schemes of a certain type over S" (e.g. smooth, genus g curves over S, as in example 1.3.5), it is often possible to build a stack $\mathcal{U}$ over $\mathcal{M}$ such that

$$\begin{tikzcd} \mathcal{U} \arrow[r] & \mathcal{M} \\ X \cong \mathcal{U}_X \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \mathrm{Spec}(\mathbb{C}) \arrow[u, "\phi_X"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{U} \arrow[r] & \mathcal{M} \\ \mathcal{X} \cong \mathcal{U} \times_{\mathcal{M}} S \arrow[u] \arrow[r, "\pi"'] \arrow[ur, phantom, "\square"] & S \arrow[u, "\phi_{(\mathcal{X}\to S)}"'] \end{tikzcd}$$

for every morphism $S \to \mathcal{M}$ with S scheme, where we use the notation:

$$\begin{aligned} \mathsf{HOM}_{\mathsf{Sch}}(-, \mathcal{M}) \quad &\sim \quad \mathcal{F}_R \\ \phi_X \quad &\mapsto \quad X \in (\mathcal{F}_R)_{\mathbb{C}} = (A, \sim) \\ \phi_{(\mathcal{X}\to S)} \quad &\mapsto \quad (\pi\colon \mathcal{X} \to S) \in (\mathcal{F}_R)_S \end{aligned}$$

That is to say, $\mathcal{U} \to \mathcal{M}$ is universal in the sense that every object $\mathcal{X} \to S$ of $\mathcal{F}_R$ over S can be obtained as the pullback of $\mathcal{U} \to \mathcal{M}$ by the corresponding morphism $S \to \mathcal{M}$:

$$\phi_{(\mathcal{X}\to S)}{}^* (\mathcal{U} \to \mathcal{M}) \quad = \quad (\mathcal{X} \to S)$$

meaning that we can write the 2-Yoneda equivalence as:

$$\begin{aligned} \mathsf{HOM}_{\mathsf{Sch}}(S, \mathcal{M}) \quad &\sim \quad (\mathcal{F}_R)_S \\ (\phi\colon S \to \mathcal{M}) \quad &\mapsto \quad \phi^*(\mathcal{U} \to \mathcal{M}) \end{aligned}$$

In light of the similarities to definition 1.3.1, we usually refer to $\mathcal{U} \to \mathcal{M}$ as the *universal family* over $\mathcal{M}$. Typically, $\mathcal{U}$ can be realised as a moduli stack which parametrizes pairs (X, x) such that $X \in A$ and $x \in X$, with

$$(X, x) \sim (Y, y) \;\Leftrightarrow\; \exists\, X \to Y \;/\; x \mapsto y$$

so that $\mathcal{U} \to \mathcal{M}$ is given by the forgetful functor; see example 1.3.15 for a more detailed description of this in the case of smooth, genus g curves.

The next proposition shows that some of the algebraic structure of the base $\mathcal{M}$ carries over to the universal family described in remark 1.3.12, as well as to other universal stacks that we will introduce later on.

Proposition 1.3.13. *Let $\mathcal{Y} \to \mathcal{Z}$ be a morphism of stacks over* Sch *such that, for every scheme S and morphism $S \to \mathcal{Z}$, the fiber product $\mathcal{Y} \times_{\mathcal{Z}} S$ is a scheme. If $\mathcal{Z}$ is an algebraic stack, then so is $\mathcal{Y}$.*

Proof. This is a less general version of [Ols16] Prop. 10.2.2. □

Example 1.3.14. The moduli problem from examples 1.3.5 and 1.3.7 admits a moduli stack $\mathcal{M}_g$ over Sch, such that:

(i) An object of $\mathcal{M}_g$ is a smooth, genus g curve $X \to S$.

(ii) An arrow $(X \to S) \to (X' \to S')$ in $\mathcal{M}_g$ is a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[d] \\ S \arrow[r] & S' \end{tikzcd}$$

(iii) $\mathcal{M}_g$ is fibered in groupoids over Sch via the forgetful functor

$$\mathcal{M}_g \rightsquigarrow \mathsf{Sch}, \quad (X \to S) \mapsto S$$

(iv) $\mathcal{M}_g$ is in fact a Deligne-Mumford stack, equipped with a morphism

$$\begin{array}{ccccc} \mathcal{M}_g = \mathcal{F}_R & \to & \mathcal{F}/R & \to & M_g \\ (X \to S) & \mapsto & [X \to S] & \mapsto & (S \to M_g,\ s \mapsto [\mathcal{X}_s]) \end{array}$$

to the coarse moduli space M_g of smooth, genus g curves (as in 1.3.7).

For a reference to the last statement, see [Ols16] Th. 8.4.5 & 11.1.2. Note that, for S scheme, the fiber $(\mathcal{M}_g)_S$ is the groupoid whose objects are smooth, genus g curves over S and whose arrows are isomorphisms of schemes over S. Indeed, if we recall the equivalence relation from 1.3.5, as well as the notion of "moduli functor" for stacks, it is clear that $\mathcal{F}_R = \mathcal{M}_g \rightsquigarrow \mathsf{Sch}$. Hence curves $X \to S$ are naturally identified with morphisms $S \to \mathcal{M}_g$, as in remark 1.3.12:

$$(\mathcal{F}_R)_S = (\mathcal{M}_g)_S \sim \mathsf{HOM}_{\mathsf{Sch}}(S, \mathcal{M}_g)$$

The essential distinction between this picture and the previous one is that now the nonisomorphic curves $\mathcal{X} \to \mathfrak{G}_{\mathsf{m}}$ and $\mathcal{Y} \to \mathfrak{G}_{\mathsf{m}}$ no longer correspond to the same morphism $\mathfrak{G}_{\mathsf{m}} \to \mathcal{M}_g$, and instead sit in the ramification of $\mathcal{M}_g \to M_g$.

With the nontrivial automorphism issues of example 1.3.5 addressed by this new moduli stack construction, universality is swiftly recovered.

Example 1.3.15. The moduli stack $\mathcal{M}_g$ described in example 1.3.14 admits a universal family $\mathcal{C}_g \to \mathcal{M}_g$ (in the sense of remark 1.3.12), such that:

(i) An object of $\mathcal{C}_g$ is a pair $(X \to S,\, \sigma)$ such that $(X \to S) \in (\mathcal{M}_g)_S$ and $\sigma\colon S \to X$ is a section of $X \to S$.

(ii) An arrow $(X \to S,\, \sigma) \to (X' \to S',\, \sigma')$ in $\mathcal{C}_g$ is a pair of diagrams

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[d] \\ S \arrow[r, "g"'] & S' \end{tikzcd} \qquad\qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[dr, phantom, "\circlearrowleft"] & X' \\ S \arrow[r, "g"'] \arrow[u, "\sigma"] & S' \arrow[u, "\sigma'"'] \end{tikzcd}$$

where the left one is cartesian and the right one is commutative.

(iii) $\mathcal{C}_g$ is a stack via the forgetful functor

$$\mathcal{C}_g \rightsquigarrow \mathsf{Sch}, \quad (X \to S,\, \sigma) \mapsto S$$

hence it is a moduli stack that parametrizes smooth, 1-pointed, genus g curves. As such, it is also denoted $\mathcal{M}_{g,1}$.

(iv) $\mathcal{C}_g$ is universal over $\mathcal{M}_g$ by means of the forgetful morphism

$$\mathcal{C}_g = \mathcal{M}_{g,1} \to \mathcal{M}_g, \quad (X \to S,\, \sigma) \mapsto (X \to S)$$

that is, for every scheme S and curve $(X \to S) \in (\mathcal{M}_g)_S$ it holds that

$$\phi_{(X\to S)}{}^{*}\,(\mathcal{C}_g \to \mathcal{M}) \;=\; (X \to S)$$

where $\phi_{(X\to S)}\colon S \to \mathcal{M}_g$ is the morphism corresponding to $X \to S$.

(v) $\mathcal{C}_g$ is in fact an algebraic stack, equipped with a morphism

$$\mathcal{C}_g = \mathcal{M}_{g,1} \to M_{g,1}$$

to the coarse moduli space $M_{g,1}$ of smooth, 1-pointed, genus g curves.

The universality of $\mathcal{C}_g$ follows from the definition of fiber product, which yields, for every curve $\pi\colon X \to S$ and scheme T, the expression

$$(\mathcal{C}_g \times_{\mathcal{M}_g} S)_T \;=\; \{(\alpha\colon T \to X,\, \beta\colon T \to S) \;/\; \beta = \pi \circ \alpha\} \;\cong\; \mathrm{Hom}(T, X)$$

so that $\mathcal{C}_g \times_{\mathcal{M}_g} S$ is represented by the scheme X. Applying proposition 1.3.13 and the fact that $\mathcal{M}_g$ is a Deligne-Mumford stack, we deduce that the stack $\mathcal{C}_g$ is algebraic. $\mathcal{C}_g$ is usually referred to as the *universal curve* over $\mathcal{M}_g$.

In conclusion, the introduction of stacks has enabled us to keep track of all automorphisms and thus recapture the usefulness of universal families, but this has come at the significant cost of a black box of technicality. Nevertheless, we are finally ready to look at previous sections from a modular perspective.

1.3.2 The Prym moduli space

Recalling definition 1.1.9, we now want to set up a moduli problem for the family of Prym pairs (C, η) of genus g, with the standard equivalence relation of isomorphisms of roots:

$$(C, \eta) \cong (C', \eta') \;\Leftrightarrow\; \exists\, \varphi\colon C \cong C' \;/\; \varphi^*(\eta') \cong \eta$$

In order to build an appropriate moduli functor, we first need a relative notion of Prym pair.

Definition 1.3.16. Let S be a scheme. A *family of Prym curves* over the base S, or simply a *Prym curve* over S, is a triplet $(f\colon C \to S,\, \eta,\, \beta)$ such that:

(i) $f\colon C \to S$ is a (smooth, genus g) curve over S.

(ii) $\eta \in \mathrm{Pic}^0(C) - \{\mathcal{O}_C\}$ is a nontrivial line bundle on C of degree zero.

(iii) $\beta\colon \eta^{\otimes 2} \to \mathcal{O}_C$ is a sheaf homomorphism.

(iv) The restriction of $(f\colon C \to S,\, \eta,\, \beta)$ to any fiber $f^{-1}(s) = C_s$ gives rise to a Prym pair $(C_s, \eta_s) \in R_g(C_s)$ with isomorphism $\beta_s\colon \eta_s^{\otimes 2} \cong \mathcal{O}_{C_s}$.

An isomorphism $(C \to S, \eta, \beta) \cong (C' \to S, \eta', \beta')$ is a pair (φ, ψ) where:

(i) $\varphi\colon C \cong C'$ is an isomorphism over S.

(ii) $\psi\colon \varphi^*(\eta') \cong \eta$ is a sheaf isomorphism such that $\varphi^*(\beta') = \beta \circ \psi^{\otimes 2}$.

With minimal changes, we could likewise define *families of spin curves*, or more generally *families of curves carrying a square root of a fixed line bundle*.

Consider the functor

$$\mathcal{F}^{\mathrm{Pr}}\colon \mathsf{Sch}^{\mathrm{opp}} \rightsquigarrow \mathsf{Sch}, \qquad S \mapsto \{\text{smooth, genus } g \text{ Prym curves over } S\}$$

together with the equivalence relation given by isomorphisms of roots. Then

$$(\mathcal{F}^{\mathrm{Pr}}_{\cong})_{\mathbb{C}} = (\{\text{Prym pairs of genus } g\}, \cong)$$

is indeed the family we aim to parametrize.

Definition 1.3.17. The moduli problem $(\mathcal{F}^{\mathrm{Pr}}, \cong)$ admits both a coarse moduli space R_g and a Deligne-Mumford moduli stack $\mathcal{R}_g$, which fit into a diagram

$$\begin{tikzcd} \mathcal{R}_g \arrow[r, "\pi_{\mathcal{R}}"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{M}_g \arrow[d] \\ R_g \arrow[r, "\pi_R"'] & M_g \end{tikzcd}$$

Depending on the context, we refer to R_g or $\mathcal{R}_g$ as the *Prym moduli space*.

Remark 1.3.18. Note that the Prym moduli space can be expressed as a finite quotient of the moduli space of smooth, genus g curves with a level 2 structure. For further details on either of these moduli spaces, see e.g. [DM69], [MFK94], [Bea77], [BCF04] or [FL10].

Remark 1.3.19. We can describe the stack $\mathcal{R}_g$ more explicitly as follows:

(i) An object of $\mathcal{R}_g$ is a smooth, genus g Prym curve $(f\colon C \to S,\, \eta,\, \beta)$.

(ii) An arrow $(f, \eta, \beta) \to (f', \eta', \beta')$ in $\mathcal{R}_g$ is a pair of a cartesian diagram

$$\begin{tikzcd} C \arrow[r, "\varphi"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & C' \arrow[d, "f'"] \\ S \arrow[r] & S' \end{tikzcd}$$

and a sheaf isomorphism $\psi\colon \varphi^*(\eta') \cong \eta$ such that $\varphi^*(\beta') = \beta \circ \psi^{\otimes 2}$.

Since this generalises definition 1.3.16, we get $\mathcal{F}^{\mathrm{Pr}}_{\cong} = \mathcal{R}_g$ as fibered categories.

As an application of the theory introduced earlier in the section, we have an equivalence between Prym curves over S and morphisms $S \to \mathcal{R}_g$, that is:

$$\begin{array}{rcl} (\mathcal{R}_g)_S & \sim & \mathsf{HOM}_{\mathsf{Sch}}(S, \mathcal{R}_g) \\ (f\colon C \to S,\, \eta,\, \beta) & \mapsto & (\phi_{(f,\eta,\beta)}\colon S \to \mathcal{R}_g) \end{array}$$

Similarly, the projection onto the coarse moduli space is given by

$$\begin{array}{rcl} \mathcal{R}_g & \to & R_g \\ (f\colon C \to S,\, \eta,\, \beta) & \mapsto & (S \to R_g,\ s \mapsto [(C_s, \eta_s)]) \end{array}$$

hence it is branched over Prym pairs with nontrivial automorphisms. On the other hand, the forgetful maps

$$\pi_{\mathcal{R}}\colon \mathcal{R}_g \to \mathcal{M}_g, \qquad \pi_R\colon R_g \to M_g$$

are unramified, finite covers of degree $2^{2g} - 1$, through which the Prym moduli space is a multisection of the universal Picard variety

$$\mathrm{Pic}_{0,g} \to \mathcal{M}_g$$

and in particular irreducible. In this way, it can be understood as the *universal set of Prym roots* over $\mathcal{M}_g$, since $\pi_R^{-1}(C) = R_g(C)$ as in definition 1.1.9.

Remark 1.3.20. For the remainder of the chapter, we are always assumed to be working on the moduli stack unless otherwise specified, as it is the natural setting to think of the universal Prym variety. We will accordingly use the stack notation ($\mathcal{R}_g$) from now on, and for the sake of simplicity drop the distinction between coarse moduli spaces and moduli stacks to gather both of them under the umbrella term "moduli space". After all, as far as their birational geometry is concerned, such a distinction is not relevant.

Let us next translate the results of section 1.2 into the language of moduli spaces. We saw that the Prym construction associates to a Prym pair (C, η) of genus g a principally polarized abelian variety of dimension $g - 1$.

Definition 1.3.21. Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g. The Prym construction yields a morphism

$$\mathrm{Pr}_g \colon \mathcal{R}_g \to \mathcal{A}_{g-1}, \quad (C, \eta) \mapsto (P(C, \eta), \Xi_{(C,\eta)})$$

which is called the *Prym map*, or sometimes the *Prym-Torelli map*.

Since $\mathcal{R}_g$ is a finite cover of $\mathcal{M}_g$ and moreover $A_g \cong \mathbb{H}_g/\mathrm{Sp}(2g, \mathbb{Z})$, we have

$$\begin{cases} \dim \mathcal{R}_g & = & \dim \mathcal{M}_g & = & 3g - 3 \\ \dim \mathcal{A}_g & = & \dim \mathrm{Sym}_g(\mathbb{C}) & = & g(g+1)/2 \end{cases}$$

and thus $\dim \mathcal{R}_g \geq \dim \mathcal{A}_{g-1}$ if and only if $g \geq 6$. Specifically, we get a table:

g	2	3	4	5	6	7	8	...
$\dim \mathcal{R}_g$	3	6	9	12	15	18	21	...
$\dim \mathcal{A}_{g-1}$	1	3	6	10	15	21	28	...

suggesting that the Prym map might be dominant for low genera. This is indeed the case.

Proposition 1.3.22. *The Prym map* $\mathrm{Pr}_g \colon \mathcal{R}_g \to \mathcal{A}_{g-1}$ *is:*

(i) *If* $g \leq 6$*, dominant.*

(ii) *If* $g = 6$*, generically finite of degree* 27*.*

(iii) *If* $g \geq 7$*, generically injective, i.e. birational to its image.*

It is, however, never injective.

Proof. See [Bea77] for $g \leq 6$, [DS81] for $g = 6$, [FS82] for $g \geq 7$, and [Don92] for the lack of injectivity. □

In other words, (principally polarized) abelian varieties of dimension ≤ 5 can be expressed as generalized Prym varieties of some double cover, in the sense of [Bea77]. This furthers our understanding of $\mathcal{A}_g$ for low g beyond the Jacobian construction and its corresponding Torelli map

$$t_g \colon \mathcal{M}_g \to \mathcal{A}_g, \quad C \mapsto (J(C), \Theta_C)$$

which is dominant only for $g \leq 3$, even if always injective. Furthermore, general Prym varieties of dimension ≥ 6 uniquely determine the double cover that produces them, in a result akin to Torelli's theorem for Jacobian varieties.

A major aspect of the study of the Prym moduli space is its privileged role as an intermediary between curves and principally polarized abelian varieties:

$$\begin{tikzcd} & \mathcal{R}_g \arrow[dl, "\pi_{\mathcal{R}}"'] \arrow[dr, "\mathrm{Pr}_g"] & \\ \mathcal{M}_g & & \mathcal{A}_{g-1} \end{tikzcd}$$

which is of special importance in low genera, due to the dominance of the Prym map. This leads to a shared effort by many mathematicians to understand better the geometry of $\mathcal{R}_g$ for low g; see [Far12] for an overview of the advances on this and other related topics.

We aim to contribute to this effort by looking at the universal Prym variety in genus $g = 3$.

Definition 1.3.23. The Prym moduli space admits a *universal Prym variety*

$$\mathcal{Y}_g \to \mathcal{R}_g$$

that is, an algebraic stack $\mathcal{Y}_g$ over $\mathcal{R}_g$ such that $(\mathcal{Y}_g)_{(C,\eta)} \cong P(C, \eta)$, i.e.

$$\begin{tikzcd} \mathcal{Y}_g \arrow[r] & \mathcal{R}_g \\ P(C,\eta) \cong (\mathcal{Y}_g)_{(C,\eta)} \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \mathrm{Spec}(\mathbb{C}) \arrow[u, "\phi_{(C,\eta)}"'] \end{tikzcd}$$

for every Prym pair $(C, \eta) \in (\mathcal{R}_g)_{\mathbb{C}}$. The stack $\mathcal{Y}_g$ is obtained by means of the cartesian diagram

$$\begin{tikzcd} \mathcal{X}_{g-1} \arrow[r] & \mathcal{A}_{g-1} \\ \mathcal{Y}_g \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \mathcal{R}_g \arrow[u, "\mathrm{Pr}_g"'] \end{tikzcd}$$

as the pullback of the universal family $\mathcal{X}_{g-1} \to \mathcal{A}_{g-1}$ by the Prym map.

Noting that $\mathcal{X}_{g-1}$ is an algebraic stack by proposition 1.3.13, it follows that the universal Prym variety is indeed an algebraic stack. As a moduli space, $\mathcal{Y}_g$

parametrizes points of the form

$$(\mathcal{Y}_g)_{\mathbb{C}} \;=\; \{(C,\eta,L)\;/\;(C,\eta)\in(\mathcal{R}_g)_{\mathbb{C}},\; L\in P(C,\eta)\}$$

maps to a scheme $Y_g \to M_g$, and is of dimension

$$\dim \mathcal{Y}_g \;=\; \dim \mathcal{R}_g + (g-1) \;=\; 4g-4$$

since the fibers of $\mathcal{Y}_g \to \mathcal{R}_g$ are $(g-1)$-dimensional.

The universal Prym variety is not the only universal object over $\mathcal{R}_g$ that we are interested in.

Definition 1.3.24. The Prym moduli space admits a *universal Prym curve*

$$\mathcal{C}' \to \mathcal{R}_g$$

that is, an algebraic stack $\mathcal{C}'$ over $\mathcal{R}_g$ such that $(\mathcal{C}')_{(C',\iota)} \cong C'$, i.e.

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[dr, phantom, "\square"] & \mathcal{R}_g \\
C' \cong (\mathcal{C}')_{(C',\iota)} \arrow[u] \arrow[r] & \operatorname{Spec}(\mathbb{C}) \arrow[u, "\phi_{(C',\iota)}"']
\end{tikzcd}$$

for every pair $(C',\iota) \in (\mathcal{R}_g)_{\mathbb{C}}$, with the notation of proposition 1.1.13. The stack $\mathcal{C}'$ is obtained as the pullback

$$\begin{tikzcd}
\mathcal{C}_{2g-1} \arrow[r] \arrow[dr, phantom, "\square"] & \mathcal{M}_{2g-1} \\
\mathcal{C}' \arrow[u] \arrow[r] & \mathcal{R}_g \arrow[u]
\end{tikzcd}$$

of the universal curve $\mathcal{C}_{2g-1} \to \mathcal{M}_{2g-1}$ by the map $\mathcal{R}_g \to \mathcal{M}_{2g-1}$, $(C',\iota) \mapsto C'$.

We can consider the d-fold product over $\mathcal{R}_g$ of the universal Prym curve

$$(\mathcal{C}')^d = \mathcal{C}' \times_{\mathcal{R}_g} \overset{(d)}{\cdots} \times_{\mathcal{R}_g} \mathcal{C}' \to \mathcal{R}_g$$

of fibers $(C')^d$, so that $\dim(\mathcal{C}')^d = \dim \mathcal{R}_g + d = 3g-3+d$. This is a moduli space with rational points

$$((\mathcal{C}')^d)_{\mathbb{C}} \;=\; \{(C',\iota,x_1,\dots,x_d)\;/\;(C',\iota)\in(\mathcal{R}_g)_{\mathbb{C}},\; x_1,\dots,x_d\in C'\}$$

and, more importantly, with a natural map to the universal Prym variety.

Definition 1.3.25. For $d \in \mathbb{Z}^+$ even, the *universal Abel-Prym map* in degree d

is the rational map

$$\mathfrak{ap}\colon (\mathcal{C}')^d \to \mathcal{Y}_g, \qquad (C', \iota, x_1, \ldots, x_d) \mapsto \left(C', \iota, \delta\left(\sum_{i=1}^d x_i\right)\right)$$

where $\delta \equiv \mathsf{ap}\colon (C')^{(d)} \to P(C', \iota)$ is the difference map; see remark 1.2.15.

Proposition 1.3.26. *The universal Abel-Prym map* $\mathfrak{ap}\colon (\mathcal{C}')^d \to \mathcal{Y}_g$ *is generically finite for* $d \leq g-1$*, and moreover dominant if* $d = g-1$*.*

Proof. This is the global version of remark 1.2.15 and lemma 1.2.16, or alternatively of the more general corollary A.9 found in [LZ21] Appendix A. □

As a result of the proposition, the dominant map

$$\mathfrak{ap}\colon (\mathcal{C}')^{g-1} \to \mathcal{Y}_g$$

controls some of the birational geometry of $\mathcal{Y}_g$. In particular, if we managed to give a rational parametrization of $(\mathcal{C}')^{(g-1)}$, then the unirationality of $\mathcal{Y}_g$ would follow, as we will discuss below.

Remark 1.3.27. The inspiration for this technique stems from [FV16], where the authors show that $(\mathcal{C}')^5$ is unirational in genus 6. Using a slightly modified version of the universal Abel-Prym map, this implies that $\mathcal{Y}_6$ is unirational. To our knowledge, genus 6 is the only instance of the unirationality of the universal Prym variety that is currently present in the literature.

Ultimately, we will show that $(\mathcal{C}')^2$ and $\mathcal{Y}_3$ are unirational in genus 3.

1.3.3 Birational geometry of moduli spaces

Here we introduce some aspects of the theory of birational invariants while focusing on the study of the "simplest" of varieties: parametrizable ones. First we discuss parametrizations from the point of view of Algebraic Geometry, and then examine them in the setting of moduli spaces of curves. We mostly follow the excellent survey by Verra [Ver13] on the topic.

Let X be an algebraic variety. We write $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}} = \operatorname{Proj} \mathbb{C}[t_0, \ldots, t_n]$.

Definition 1.3.28. A *rational parametrization* of X is a dominant rational map from some projective space onto X, that is:

$$\mathbb{P}^n \to X$$

If such a map exists, X is said to be *unirational*; if furthermore it can be chosen to be birational, X is said to be *rational*. Similarly, a *uniruled parametrization* of X is a dominant rational map

$$Y \times \mathbb{P}^1 \to X$$

for some variety Y of dimension $\dim(X) - 1$. If such a map exists, X is said to be *uniruled*.

Note that X is uniruled if and only if there is a rational curve which passes through a general point of X. Moreover, if $\mathbb{P}^n \to X$ is a parametrization of X, then necessarily $n \geq \dim X$.

Remark 1.3.29. When X is unirational, we can find a rational parametrization $\mathbb{P}^n \to X$ such that $n = \dim X$. Such a map is generically finite, but it may not be generically injective. In fact, even though unirationality and rationality agree on curves and surfaces over $\mathbb{C}$, some threefolds are unirational but not rational.

The geometry of unirational varieties is remarkably easier to navigate than that of other, more general varieties; in the sense that, through the dominant map $\mathbb{P}^n \to X$, almost all of X can be covered with a rational grid and explored with the help of n complex coordinates. This is a very special feature which, for instance, causes the Kodaira dimension of these varieties to drop. Let us briefly elaborate on this; see [Laz04] for further details.

Take ω_X to be the canonical bundle of X. Consider the rational map

$$\phi_m \colon X \to \mathbb{P}H^0(X, \omega_X^{\otimes m})^\vee$$

induced by $\omega_X^{\otimes m}$, and set $N = \{m \geq 0 \ / \ h^0(X, \omega_X^{\otimes m}) \neq 0\} \subset \mathbb{Z}$.

Definition 1.3.30. The *Kodaira dimension* of X, denoted $\kappa(X)$, is defined as:

$$\kappa(X) \;=\; \max_{m \in N} \{\dim \phi_m(X)\} \;\leq\; \dim(X)$$

unless $h^0(X, \omega_X^{\otimes m}) = 0$ for all $m > 0$, in which case $\kappa(X) = -\infty$.

In a way, Kodaira dimension classifies varieties by both complexity and frequency, the "simplest" and most special being those of minimal Kodaira dimension. Some weaker conditions than unirationality also fall into this group.

Definition 1.3.31. We say that X is *rationally connected* if there is a rational curve through any two points of X.

Remark 1.3.32. The previous notions are related by the chain:

$$\text{rational} \;\Rightarrow\; \text{unirational} \;\Rightarrow\; \text{rationally connected} \;\Rightarrow\; \text{uniruled}$$

and they all have Kodaira dimension $-\infty$. Most notably, the converse implications on both extremes are known not to hold, whereas the central one remains an open problem.

On the other side, we have varieties of maximal Kodaira dimension:

Definition 1.3.33. We say that X is *of general type* if $\kappa(X) = \dim(X)$.

Remark 1.3.34. As the name suggests, "most" varieties are of general type (for example, smooth degree d hypersurfaces in $\mathbb{P}^n$ are of general type if and only if $d > n+1$), and their geometry tends to be much harder to work with.

If we shift our attention from abstract varieties to moduli spaces, it is clear that the unirationality of a moduli space is an ideal scenario, in which most of the objects of a certain type can be determined by a fixed amount of complex parameters. This is far from the norm, but for some time it was thought to be: when Severi showed in 1915 that $\mathcal{M}_g$ (the first moduli space to be regarded as such) was unirational for genus $g \leq 10$, he conjectured that this would happen for all genera. It would actually take over 60 years for such a conjecture to be proven false (by Harris, Mumford and Eisenbud; see [HM82] and [EH87]), in an event that would go on to radically change the way we understand the geometry of moduli spaces.

The main reason that Severi's conjecture stood its ground for so long is the fact that looking for rational parametrizations, and even more so for birational ones, is often a really difficult task. Existing techniques are very specific to the moduli space under review, as they mostly rely on the geometry of the objects parametrized by the space, which can vary wildly. Nevertheless, there are ideas that seem to appear with a certain degree of regularity across different moduli spaces, such as the use of curves moving along a surface (e.g. in a linear series) to build a parametrization of the space. Because of this, it is at times possible, even if unlikely, to draw inspiration from one case and adapt its unirationality statements to another. We are particularly interested in $\mathcal{M}_g$ and $\mathcal{R}_g$.

The birational geometry of $\mathcal{M}_g$ has been thoroughly studied, and yet some of the intermediate genera remain inaccessible. We compile the current state of knowledge into a single proposition.

Proposition 1.3.35. *The moduli space $\mathcal{M}_g$ is:*

(i) *For $g \leq 15$, of Kodaira dimension $-\infty$.*

(a) *If $g \leq 6$, rational.*

(b) *If $g \leq 14$, unirational.*

(c) *If $g \leq 15$, rationally connected.*

(ii) *For $g \geq 22$, of maximal Kodaira dimension (i.e. of general type).*

For $16 \leq g \leq 21$, the Kodaira dimension of $\mathcal{M}_g$ is unknown.

Proof. See [Ver13] Chapter 2 and references therein. □

A similar picture occurs for the Prym moduli space, although the increased complexity brings about maximal Kodaira dimension at a lower genus.

Proposition 1.3.36. *The moduli space $\mathcal{R}_g$ is:*

(i) *For $g \leq 8$, of Kodaira dimension $-\infty$.*

(a) *If $g \leq 4$, rational.*

(b) *If $g \leq 7$, unirational.*

(c) *If $g \leq 8$, uniruled.*

(ii) *For $g \geq 12$ and $g \neq 13$, of Kodaira dimension ≥ 0.*

(a) *If $g \geq 14$, of general type.*

For $9 \leq g \leq 11$ or $g = 13$, the Kodaira dimension of $\mathcal{R}_g$ is unknown.

Proof. See [Ver13] Chapter 4, [Far12] Section 6, and references therein. □

One final, rational tool that we need to introduce is the *Grassmannian* of a vector space V, that is, the set

$$G(d, V) \quad = \quad \{U \subset V \;/\; U \text{ is a } d\text{-dimensional subspace of } V\}$$

with $d \leq n = \dim(V)$. The Grassmannian $G(d, V)$ is in fact a fine example of a moduli space, represented by a scheme of dimension $d\,(n-d)$, and has already been discussed for $d = 1$ in example 1.3.4, since $G(1, V) = \mathbb{P}(V)$.

Remark 1.3.37. The rationality of $G(d, V)$ follows from the embedding

$$\psi \colon G(d, V) \hookrightarrow \mathbb{P}(\textstyle\bigwedge^d V)$$

known as the *Plücker embedding*, and the local description of $G(d, V)$ in terms of affine spaces $\mathbb{A}^{d(n-d)}$ that is obtained through ψ. For a good reference on the topic, we direct the reader to [Har92] Part I, Lecture 6.

As a rare instance of a rational moduli space, the Grassmannian will be key to building a successful parametrization of $(\mathcal{C}')^2$, and thus of $\mathcal{Y}_3 \to \mathcal{R}_3$.

1.4 Parametrizing the universal Prym variety

In section 1.3 we have set up both the universal Prym variety $\mathcal{Y}_3 \to \mathcal{R}_3$ and the basic language that is required to study its birational geometry. In order to look for rational parametrizations of $\mathcal{Y}_3$, however, we are still missing one more step, which is to acquire a better understanding of the geometry of the objects parametrized by the universal 2-fold Prym curve $(\mathcal{C}')^2 \to \mathcal{Y}_3$ in genus 3. We fill in this gap by first looking at general Prym pairs of genus 3, and afterwards at the slightly harder case of general 2-pointed pairs. The description offered here is initially motivated by the brief comments on Nikulin surfaces of genus 3 that are included in [FV12] Section 2.

1.4.1 General Prym pairs of genus 3

We start with a lemma extracted from [Mas76] Prop. 2.7, which shows that, in the context of proposition 1.1.13, nonhyperelliptic Prym pairs (C, η) give rise to nonhyperelliptic curves C'.

Lemma 1.4.1. *Let (C, η) be a Prym pair of genus g, and let $\pi\colon C' \to C$ be the associated double cover, with involution ι. If C' is hyperelliptic, then so is C.*

Proof. Let $\tau'\colon C' \to C'$ be the hyperelliptic involution, which commutes with ι by lemma 1.2.12. Then τ' restricts to an involution τ on C, and ι restricts to a fixed-point-free involution on the fixed locus R of τ', of order $2g(C') + 2 = 4g$. As a result, τ fixes at least the $2g$ points of $\pi(R)$, and Riemann-Hurwitz shows that any such involution has to be the hyperelliptic one. □

We now fix $g = 3$, so that C' is a smooth, integral curve of genus $2g - 1 = 5$ which sits outside the trigonal locus, as ensured by the following lemma. Note that the nonhyperelliptic case can be found in [dGP05] Lemme 0.3.

Lemma 1.4.2. *Let (C, η) be a Prym pair of genus 3, and let $\pi\colon C' \to C$ be the associated double cover, with involution ι. Then C' is not trigonal.*

Proof. Assume that C' is trigonal, and let $L \in W^1_3(C')$. On the one hand, if C' is hyperelliptic, with $M \in W^1_2(C')$, then the base-point-free pencil trick implies

$$h^0(C', L \otimes M) \geq 2\, h^0(C', M) = 4$$

and thus $L \otimes M \in W^3_5(C')$, which contradicts Clifford's theorem. On the other hand, if C' is not hyperelliptic, we consider two possibilities:

(a) If $i^*L = L$, then ι is compatible with the triple cover $\phi_L \colon C' \to \mathbb{P}^1$ and induces an involution on $\mathbb{P}^1$, which fixes 2 points by Riemann-Hurwitz. Then the fiber of ϕ_L over any one of the fixed points is stable by ι, yet of order 3, which is not possible since ι is fixed-point-free.

(b) If $L' = i^*L \neq L$, then the image of C' under the map

$$C \quad \to \quad \mathbb{P}H^0(L) \times \mathbb{P}H^0(L') = \mathbb{P}^1 \times \mathbb{P}^1 \quad \subset \quad \mathbb{P}(H^0(L) \otimes H^0(L')) = \mathbb{P}^3$$

is a curve of bidegree $(3,3)$ lying on the quadric $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, hence of arithmetic genus $(3-1)(3-1) = 4$. This contradicts $g(C') = 5$.

Observe that the hyperelliptic argument works for any curve of genus $g(C') \geq 3$, whereas the nonhyperelliptic one relies on the involution and $g(C') = 5$. $\square$

Proposition 1.4.3. *For a general point $(C', \iota) \in \mathcal{R}_3$, it holds that:*

(i) *The canonical map embeds C' in $\mathbb{P}^4$ as the complete intersection of three linearly independent quadrics.*

(ii) *$H^0(\mathbb{P}^4, I_{C'}(2))$ is a 3-dimensional subspace of $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.*

(iii) *There is an involution $\iota \colon \mathbb{P}^4 \to \mathbb{P}^4$ restricting to $\iota \colon C' \to C'$ such that*

$$H^0(\mathbb{P}^4, I_{C'}(2)) \quad \subset \quad \ker(\mathrm{Id} - \iota^*) \quad \subset \quad H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$$

where ι^* is the involution induced on $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$ by $\iota \colon \mathbb{P}^4 \to \mathbb{P}^4$.

Proof. By generality and lemmas 1.4.1 and 1.4.2, the smooth, genus 5 curve C' is neither hyperelliptic nor trigonal, so (i) and (ii) follow from the sequence

$$0 \longrightarrow H^0(\mathbb{P}^4, I_{C'}(2)) \longrightarrow H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \longrightarrow H^0(C', \omega_{C'}^{\otimes 2}) \longrightarrow 0$$

see e.g. [Har77] Chapter IV, Ex. 5.5. Moreover, $\iota \colon C' \to C'$ induces involutions

$$H^0(C', \omega_{C'})^\vee \to H^0(C', \omega_{C'})^\vee, \qquad \mathbb{P}H^0(C', \omega_{C'})^\vee \to \mathbb{P}H^0(C', \omega_{C'})^\vee$$

which after fixing coordinates turn into involutions $\iota \colon \mathbb{C}^5 \to \mathbb{C}^5$ and $\iota \colon \mathbb{P}^4 \to \mathbb{P}^4$. The latter restricts to $\iota \colon C' \to C'$ via the canonical embedding, and the former decomposes $\mathbb{C}^5$ as the direct sum of the eigenspaces

$$E_\Lambda = \ker(\mathrm{Id} - \iota), \quad E_L = \ker(\mathrm{Id} + \iota) \quad \subset \quad \mathbb{C}^5 = E_\Lambda \oplus E_L$$

with $\dim E_\Lambda + \dim E_L = 5$, so that the fixed subspaces of $\iota \colon \mathbb{P}^4 \to \mathbb{P}^4$ are

$$\Lambda = \mathbb{P}(E_\Lambda), \quad L = \mathbb{P}(E_L) \quad \subset \quad \mathbb{P}^4$$

with $\dim \Lambda + \dim L = 3$. Since ι is fixed-point-free over C', we have $C' \cap \Lambda = \emptyset$

and $C' \cap L = \emptyset$, hence neither of them can be a hyperplane; we may assume

$$\dim \Lambda = 2, \quad \dim L = 1$$

Finally, the involution ι^* decomposes $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$ as the direct sum

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = \ker(\mathrm{Id} - \iota^*) \oplus \ker(\mathrm{Id} + \iota^*)$$

and restricts to an involution ι' on $H^0(\mathbb{P}^4, I_{C'}(2))$, accordingly yielding

$$H^0(\mathbb{P}^4, I_{C'}(2)) = \ker(\mathrm{Id} - \iota') \oplus \ker(\mathrm{Id} + \iota')$$

Suppose there exists some $0 \neq q \in \ker(\mathrm{Id} + \iota') = H^0(\mathbb{P}^4, I_{C'}(2)) \cap \ker(\mathrm{Id} + \iota^*)$, and let $Q \subset \mathbb{P}^4$ be the quadric hypersurface defined by the zeroes of q. Then

$$v \in E_\Lambda \quad \Rightarrow \quad q(v) = q(i(v)) = (i^*q)(v) = -q(v) \quad \Rightarrow \quad q(v) = 0$$

i.e. Λ is contained in Q, and so is C' by definition of $I_{C'}(2)$. Therefore

$$\dim(C' \cap \Lambda) \quad \geq \quad \dim C' + \dim \Lambda - \dim Q \quad = \quad 1 + 2 - 3 \quad = \quad 0$$

which contradicts $C' \cap \Lambda = \emptyset$. This concludes the proof, as we get

$$H^0(\mathbb{P}^4, I_{C'}(2)) \quad = \quad \ker(\mathrm{Id} - \iota') \quad = \quad H^0(\mathbb{P}^4, I_{C'}(2)) \cap \ker(\mathrm{Id} - \iota^*)$$

from plugging $\ker(\mathrm{Id} + \iota') = 0$ into the decomposition of $H^0(\mathbb{P}^4, I_{C'}(2))$. $\square$

Given a general Prym pair (C, η) of genus 3, proposition 1.4.3 shows that its associated involution ι may be understood as a projective involution restricted to a complete intersection of three quadrics. Additionally, this involution fixes a plane and a line, while remaining fixed-point-free over the complete intersection. Let us look at an explicit example, which we shall use as a model.

Definition 1.4.4. Consider the involution

$$\begin{array}{rccclcl} \tau\colon & \mathbb{C}^5 & \to & \mathbb{C}^5, & (x_0,\, x_1,\, x_2,\, x_3,\, x_4) & \mapsto & (-x_0,\, -x_1,\, x_2,\, x_3,\, x_4) \\ \tau = \mathbb{P}(\tau)\colon & \mathbb{P}^4 & \to & \mathbb{P}^4, & (x_0 : x_1 : x_2 : x_3 : x_4) & \mapsto & (-x_0 : -x_1 : x_2 : x_3 : x_4) \end{array}$$

which we shall refer to as the *model involution* (on $\mathbb{C}^5$ or $\mathbb{P}^4$, respectively).

If we write the standard basis of $\mathbb{C}^5$ as $\{e_0, \ldots, e_4\}$, that is,

$$e_0 = (1,\, 0,\, 0,\, 0,\, 0), \quad \ldots, \quad e_4 = (0,\, 0,\, 0,\, 0,\, 1) \quad \in \quad \mathbb{C}^5$$

and follow the proof of proposition 1.4.3, the eigenspaces of $\tau\colon \mathbb{C}^5 \to \mathbb{C}^5$ are

$$\ker(\mathrm{Id} - \tau) = \langle e_2, e_3, e_4\rangle, \qquad \ker(\mathrm{Id} + \tau) = \langle e_0, e_1\rangle$$

meaning that the model involution on $\mathbb{P}^4$ fixes a plane and a line, namely

$$\Lambda_\tau = \{x_0 = x_1 = 0\}, \qquad L_\tau = \{x_2 = x_3 = x_4 = 0\}$$

Furthermore, τ induces an involution τ^* on $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$ with eigenspaces

$$\begin{aligned} \ker(\mathrm{Id} - \tau^*) &= \mathbb{C}[x_0, x_1]_2 \oplus \mathbb{C}[x_2, x_3, x_4]_2 && \text{(of dim 9)} \\ \ker(\mathrm{Id} + \tau^*) &= x_0\, \mathbb{C}[x_2, x_3, x_4]_1 \oplus x_1\, \mathbb{C}[x_2, x_3, x_4]_1 && \text{(of dim 6)} \end{aligned}$$

Observe that choosing a complete intersection of three quadrics over which the model involution is stable amounts to choosing a 3-dimensional subspace U of $\ker(\mathrm{Id} - \tau^*)$. Moreover, for a general such U, the curve

$$C' = \mathrm{bs}(\mathbb{P}U) = \bigcap\nolimits_{Q \in \mathbb{P}U} Q \quad \hookrightarrow \quad \mathbb{P}^4$$

does not intersect the plane Λ_τ or the line L_τ.

As will become clear in theorem 1.4.6, the previous discussion is enough to show unirationality for $\mathcal{R}_3$. Nonetheless, there are better ways of reaching this conclusion, since $\mathcal{R}_3$ is in fact rational (recall proposition 1.3.36, or instead see [Dol08] Sections 3 & 4 for a proof of rationality).

1.4.2 The universal Prym variety in genus 3

Up to now, we have focused on understanding how a pair $(C', \iota) \in \mathcal{R}_3$ looks like, but we are interested in $(\mathcal{C}')^2$, so we need to add two points to the picture. Denote $E_i = \langle e_i \rangle \in \mathbb{P}^4$ and $E_{ij} = \langle e_i + e_j \rangle \in \mathbb{P}^4$, and in particular consider

$$E_{02} = (1:0:1:0:0), \quad E_{13} = (0:1:0:1:0) \quad \in \quad \mathbb{P}^4$$

which will play the role of *model points*. Notice that E_{02} and E_{13} are not fixed by the model involution $\tau\colon \mathbb{P}^4 \to \mathbb{P}^4$ described in definition 1.4.4.

Lemma 1.4.5. *After an adequate coordinate change, a general point (C', ι, x_1, x_2) of $(\mathcal{C}')^2$ can be written as $(C', \tau, E_{02}, E_{13})$.*

Proof. Adhering to the notation of proposition 1.4.3, the points $x_1, x_2 \in C' \subset \mathbb{P}^4$

correspond to line subspaces of $\mathbb{C}^5 = E_\Lambda \oplus E_L$, and thus are of the form

$$x_1 = \langle v_1^\Lambda + v_1^L \rangle, \quad x_2 = \langle v_2^\Lambda + v_2^L \rangle$$

where $v_i^\Lambda \in E_\Lambda$, $v_i^L \in E_L$ for $i \in \{1,2\}$. Then the sets $\{v_1^\Lambda, v_2^\Lambda\}$ and $\{v_1^L, v_2^L\}$ are linearly independent by generality, and there is some $u \in E_\Lambda$ such that

$$E_\Lambda = \langle v_1^\Lambda, v_2^\Lambda, u \rangle, \quad E_L = \langle v_1^L, v_2^L \rangle$$

Finally, the automorphism $\varphi\colon \mathbb{P}^4 \to \mathbb{P}^4$ induced by

$$\begin{array}{llll} \varphi\colon \quad \mathbb{C}^5 \to \mathbb{C}^5, & v_1^L \mapsto e_0, & v_2^L \mapsto e_1, & \\ & v_1^\Lambda \mapsto e_2, & v_2^\Lambda \mapsto e_3, & u \mapsto e_4 \end{array}$$

justifies the claim, as $\varphi \circ \iota \circ \varphi^{-1} = \tau$ and $\varphi(x_1) = E_{02}$, $\varphi(x_2) = E_{13}$. $\square$

By virtue of lemma 1.4.5, we are able to characterise the points of $(\mathcal{C}')^2$ in a very tangible manner, so much so that it becomes possible to interpret them as points in a certain Grassmannian. This leads to the desired set of results:

Theorem 1.4.6. *The universal* 2*-fold Prym curve* $(\mathcal{C}')^2 \to \mathcal{R}_3$ *is unirational.*

Proof. Let $(C', \tau, E_{02}, E_{13})$ be a general point of $(\mathcal{C}')^2$, as per lemma 1.4.5, with

$$H^0(\mathbb{P}^4, I_{C'}(2)) \;\subset\; \ker(\mathrm{Id} - \tau^*) \;=\; \mathbb{C}[x_0, x_1]_2 \,\oplus\, \mathbb{C}[x_2, x_3, x_4]_2$$

by proposition 1.4.3. We now have two additional conditions, since

$$E_{02}, E_{13} \in C' \subset Q \qquad \text{for all } Q \in \mathbb{P}H^0(\mathbb{P}^4, I_{C'}(2))$$

If we let $q = \sum \lambda_{ij}\, x_i x_j \in H^0(\mathbb{P}^4, I_{C'}(2))$ be the quadric polynomial defining Q, then $E_{02}, E_{13} \in Q$ translates into

$$\begin{cases} 0 = q(E_{02}) = \lambda_{00} + \lambda_{22} & \Leftrightarrow \quad q = \lambda_{00}\,(x_0^2 - x_2^2) + \dots \\ 0 = q(E_{13}) = \lambda_{11} + \lambda_{33} & \Leftrightarrow \quad q = \lambda_{11}\,(x_1^2 - x_3^2) + \dots \end{cases}$$

so that $H^0(\mathbb{P}^4, I_{C'}(2))$ is a 3-dimensional subspace of the 7-dimensional space

$$\begin{aligned} V_7 \;&=\; \{q \in \ker(\mathrm{Id} - \tau^*) \;/\; q(E_{02}) = q(E_{13}) = 0\} \\ &=\; (x_0^2 - x_2^2)\,\mathbb{C} \,\oplus\, (x_1^2 - x_3^2)\,\mathbb{C} \,\oplus\, x_0 x_1\,\mathbb{C} \,\oplus\, x_2 x_3\,\mathbb{C} \,\oplus\, x_4\,\mathbb{C}[x_2, x_3, x_4]_1 \end{aligned}$$

Under these circumstances, choosing a complete intersection of three quadrics passing through E_{02}, E_{13} over which the model involution is stable amounts to choosing a 3-dimensional subspace U of V_7. Furthermore, for a general such U, the model involution is fixed-point-free over the curve

$$C' = \mathrm{bs}(\mathbb{P}U) = \bigcap\nolimits_{Q\in\mathbb{P}U} Q \quad \hookrightarrow \quad \mathbb{P}^4$$

In other words, there is a dominant rational map

$$\mathbb{P}^{12} \approx G(3,V_7) \to (\mathcal{C}')^2, \qquad U \mapsto (\mathrm{bs}(\mathbb{P}U), \tau, E_{02}, E_{13})$$

as a general (C', ι, x_1, x_2) can be recovered from $H^0(\mathbb{P}^4, I_{C'}(2)) \subset V_7$. Since the Grassmannian is rational, this provides a rational parametrization of $(\mathcal{C}')^2$. □

Corollary 1.4.7. *The universal Prym variety $\mathcal{Y}_3 \to \mathcal{R}_3$ is unirational.*

Proof. Immediate from theorem 1.4.6 and proposition 1.3.26. □

If we look at the parametrization of $(\mathcal{C}')^2$ given in theorem 1.4.6, we can see that its fibers are expected to be 4-dimensional:

$$\begin{cases} \dim G(3,V_7) & = & 3\,(7-3) & = & 12 \\ \dim\,(\mathcal{C}')^2 & = & 3g-3+2 & = & 8 \end{cases}$$

In fact, the general fiber is not only of dimension 4, but also isomorphic to the subgroup $G_4 \subset \mathrm{PGL}(5)$ of projective linear transformations that fix the model involution and both of the points E_{02}, E_{13}. This subgroup can be written as:

$$G_4 = \left\{ \left(\begin{array}{cc|ccc} a_{00} & 0 & & & \\ 0 & a_{11} & & 0 & \\ \hline & & a_{00} & 0 & a_{24} \\ & 0 & 0 & a_{11} & a_{34} \\ & & 0 & 0 & a_{44} \end{array} \right) \in \mathrm{GL}(2)\times\mathrm{GL}(3) \subset \mathrm{GL}(5) \right\} \Big/ \{\mathbb{C}^*\,\mathrm{Id}_5\}$$

Then the quotient $G(3,V_7)/G_4$ is of dimension 8, and we obtain the following:

Corollary 1.4.8. *The rational parametrization $G(3,V_7) \to (\mathcal{C}')^2$ (resp. $\to \mathcal{Y}_3$) factors through $G(3,V_7) \twoheadrightarrow G(3,V_7)/G_4$ into the dominant rational map*

$$G(3,V_7)/G_4 \to (\mathcal{C}')^2 \qquad \textit{(resp. } G(3,V_7)/G_4 \to \mathcal{Y}_3\textit{)}$$

which is generically injective, and thus birational (resp. generically finite).

Proof. Note that $\mathbb{P}U \subseteq |I_{\mathrm{bs}(\mathbb{P}U)}(2)|$, with equality holding for a general U. □

1.5 Extension to Nikulin surfaces

The idea of drawing from surface theory to study the birational geometry of $\mathcal{M}_g$, by way of taking linear series of curves that move along a certain surface, has been present in Algebraic Geometry for quite some time. Families of sextic, octic and nonic curves lying in the projective plane were used by Severi to show that $\mathcal{M}_g$ is unirational when $g \leq 10$, and collections of linear series lying in K3 surfaces are the basis of an unusually uniform, uniruled parametrization of $\mathcal{M}_g$ that works across several genera (namely for $10 \neq g \leq 11$).

As it turns out, the connection between K3 surfaces and algebraic curves is carried over to the realm of Prym pairs, with *Nikulin surfaces* playing a similar role in the birational understanding of the Prym moduli space. This analogous interaction, explored by [FV12], also yields a uniform parametrization of $\mathcal{R}_g$ in low genera (namely for $6 \neq g \leq 7$), among many other results.

In this section, we introduce the basics of K3 and Nikulin theory, and adapt the proof of theorem 1.4.6 to show that the universal double Nikulin surface is unirational in genus 3. We therefore take advantage of the similarities between Nikulin surfaces and Prym pairs in the opposite direction, building on the case of curves to bring section 1.4 into the less hospitable surface world.

1.5.1 K3 and Nikulin surfaces

As usual, we take "surface" to mean algebraic variety of dimension 2. Then we may consider the following types of (polarized) surfaces:

Definition 1.5.1. A *K3 surface* is a complete nonsingular surface S such that $\omega_S \cong \mathcal{O}_S$ and $h^1(S, \mathcal{O}_S) = 0$. Moreover, a *polarized K3 surface* of genus g is a pair $(S, \mathcal{O}_S(H))$ such that S is a $K3$ surface, H is a curve in S with $p_a(H) = g$, and $\mathcal{O}_S(H)$ is very ample and primitive in $\mathrm{Pic}(S)$.

For each $g \geq 2$, the moduli space $\mathcal{F}_g$ of polarized K3 surfaces of genus g is a Deligne-Mumford stack of dimension 19; see e.g. [Huy16] Chapter 5.

Definition 1.5.2. A *Nikulin surface* is a pair (S, e) such that S is a K3 surface and $e \in \mathrm{Pic}(S)$ is a square root of

$$e^{\otimes 2} \cong \mathcal{O}_S(E_1 + \ldots + E_8)$$

where $\{E_i\}_{i=1}^8$ is a set of disjoint, smooth, rational (-2)-curves on S. Moreover, a *polarized Nikulin surface* of genus g is a triplet $(S, e, \mathcal{O}_S(H))$ where (S, e) is a

Nikulin surface, $(S, \mathcal{O}_S(H))$ is a polarized K3 surface of genus g, and $H \cdot E_i = 0$ for all $i \in \{1, \ldots, 8\}$.

Through a generalised version of theorem 1.1.7, the root e corresponds to a double cover branched over the (-2)-curves $E_1, \ldots, E_8$, namely

$$\pi_S \colon S' \to S$$

where S' is a K3 surface and $F_i = \pi_e^{-1}(E_i)$ are (-1)-curves on S'. These can be blown-down to produce another K3 surface Y', which fits into a diagram

$$\begin{tikzcd} S' \arrow[r, "\pi_S"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & S \arrow[d] \\ Y' \arrow[r, "\pi_Y"'] & Y \end{tikzcd} \qquad \begin{tikzcd} F_i \arrow[r] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowleft"] & E_i \arrow[d, mapsto] \\ q_i \arrow[r, mapsto] & p_i \end{tikzcd}$$

such that $\pi_Y \colon Y' \to Y$ is a double cover ramified over the points $q_1, \ldots, q_8$. The involution $\iota \colon Y' \to Y'$ associated to π_Y, with 8 fixed points, is called a *Nikulin involution*. Keeping this diagram in mind, we shall also refer to the pair (Y', ι) as a *Nikulin surface*; see [vGS07] or [FV12] for more details.

The additional conditions imposed by the root e cause the dimension of the moduli space $\mathcal{F}_g^{\mathfrak{N}}$ of polarized Nikulin surfaces of genus g to drop by 8, making it 11-dimensional for each $g \geq 2$. Our interest in this space stems from the next lemma, proven in [Ver13] Lemma 3.6, which reveals that the linear series given by the polarization is in fact a collection of Prym pairs.

Lemma 1.5.3. *Let $(S, e, \mathcal{O}_S(H))$ be a polarized Nikulin surface of genus $g \geq 2$. Then $\eta_C = e \otimes \mathcal{O}_C$ is a Prym root of C for any $C \in |H|$.*

The condition $H \cdot E_i = 0$ for $i \in \{1, \ldots, 8\}$ ensures that the diagram above, when restricted to the curve $C \in |H|$, yields an étale double cover $\pi \colon C' \to C$ and a fixed-point-free involution $\iota \colon C' \to C'$, induced by the Nikulin involution on Y'. The pair (π, ι) corresponds to the Prym pair (C, η_C).

Definition 1.5.4. We say that the *universal double Nikulin surface* $\mathcal{F}_{g,2}^{\mathfrak{N}}$ is the moduli space parametrizing points of the form

$$(\mathcal{F}_{g,2}^{\mathfrak{N}})_{\mathbb{C}} \quad = \quad \{(Y', \iota, \mathcal{O}_S(H), y_1, y_2) \;/\; (Y', \iota, \mathcal{O}_S(H)) \in (\mathcal{F}_g^{\mathfrak{N}})_{\mathbb{C}},\ y_1, y_2 \in Y'\}$$

Since the forgetful projection $\mathcal{F}_{g,2}^{\mathfrak{N}} \to \mathcal{F}_g^{\mathfrak{N}}$ has 4-dimensional fibers, this space is always of dimension 15.

Observe that the universal double Nikulin surface is related to the universal 2-fold Prym curve $(\mathcal{C}')^2 \to \mathcal{R}_3$. Indeed, over $\mathcal{F}_{3,2}^{\mathfrak{N}}$ we can find a useful projective bundle $\mathcal{P}^{\mathfrak{N}}_{3,2} \to \mathcal{F}^{\mathfrak{N}}_{3,2}$, with rational points

$$(Y', \iota, C, y_1, y_2)$$

such that $C \subset S$ is a smooth curve, its induced double cover $C' = \pi_Y^{-1}(C) \subset Y'$ passes through $y_1, y_2 \in Y'$, and $(Y', \iota, \mathcal{O}_S(C), y_1, y_2) \in \mathcal{F}^{\mathfrak{N}}_{3,2}$. Then the maps

$$\begin{tikzcd}[column sep=small] & \mathcal{P}^{\mathfrak{N}}_{3,2} \arrow[dl] \arrow[dr] & \\ \mathcal{F}^{\mathfrak{N}}_{3,2} & & (\mathcal{C}')^2 \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & (Y', \iota, C, y_1, y_2) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ (Y', \iota, \mathcal{O}_S(C), y_1, y_2) & & (C', \iota, y_1, y_2) \end{tikzcd}$$

link the birational geometry of $\mathcal{F}^{\mathfrak{N}}_{3,2}$ and $(\mathcal{C}')^2$, in the sense that it is possible to extend parametrizations of the former to the latter (note that the right map is dominant due to [FV12] Th. 0.2). Ultimately, the existence of this diagram lies at the heart of the striking similarities between sections 1.4 and 1.5.

1.5.2 The universal double Nikulin surface in genus 3

As discussed in the previous section (proposition 1.4.3), general Prym pairs of genus 3 can be expressed as complete intersections of quadric hypersurfaces together with a projective involution. There is a parallel description for general polarized Nikulin surfaces of genus 3, in the following sense:

Proposition 1.5.5. *For a general point $(Y', \iota, \mathcal{O}_S(H)) \in \mathcal{F}^{\mathfrak{N}}_3$, it holds that:*

(i) *The surface Y' is embedded in $\mathbb{P}^5$ as the complete intersection of three linearly independent quadrics, and the collection of Prym pairs induced by $|H|$ is realised as a series of hyperplane sections of $Y' \hookrightarrow \mathbb{P}^5$.*

(ii) *$H^0(\mathbb{P}^5, I_{Y'}(2))$ is a 3-dimensional subspace of $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$.*

(iii) *The involution $\iota\colon Y' \to Y'$ is induced by an involution $\iota\colon \mathbb{C}^6 \to \mathbb{C}^6$, and thus $\iota\colon \mathbb{P}^5 \to \mathbb{P}^5$, with eigenspaces (resp. fixed subspaces):*

$$\begin{aligned} E_\Lambda &= \ker(\mathrm{Id} - \iota), & E_L &= \ker(\mathrm{Id} + \iota) & \subset \quad & \mathbb{C}^6 = E_\Lambda \oplus E_L \\ \Lambda &= \mathbb{P}(E_\Lambda), & L &= \mathbb{P}(E_L) & \subset \quad & \mathbb{P}^5 \end{aligned}$$

such that $\dim \Lambda = 3$, $\dim L = 1$. *Furthermore, there is an inclusion*

$$H^0(\mathbb{P}^5, I_{Y'}(2)) \quad \subset \quad \ker(\mathrm{Id} - \iota^*) \quad \subset \quad H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$$

where ι^* is the involution induced on $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$ by $\iota\colon \mathbb{P}^5 \to \mathbb{P}^5$.

Proof. This characterization can be found in [vGS07] Section 2.6 (for the map to $\mathbb{P}^{2g-1}$ in arbitrary g) and Example 3.7 (for the case $g = 3$). $\square$

We may now define a new model involution for Nikulin surfaces:

Definition 1.5.6. Consider the involution

$$\begin{array}{llll} \tau\colon \mathbb{C}^6 \to \mathbb{C}^6, & (x_0,\, x_1,\, x_2,\, x_3,\, x_4,\, x_5) & \mapsto & (-x_0,\, -x_1,\, x_2,\, x_3,\, x_4,\, x_5) \\ \tau\colon \mathbb{P}^5 \to \mathbb{P}^5, & (x_0 : x_1 : x_2 : x_3 : x_4 : x_5) & \mapsto & (-x_0 : -x_1 : x_2 : x_3 : x_4 : x_5) \end{array}$$

which we shall refer to as the *model involution* (on $\mathbb{C}^6$ or $\mathbb{P}^5$, respectively).

With the notation of section 1.4, the eigenspaces of $\tau\colon \mathbb{C}^5 \to \mathbb{C}^5$ are

$$\ker(\mathrm{Id} - \tau) = \langle e_2, e_3, e_4, e_5\rangle, \qquad \ker(\mathrm{Id} + \tau) = \langle e_0, e_1\rangle$$

the fixed subspaces of $\tau\colon \mathbb{P}^5 \to \mathbb{P}^5$ are

$$\Lambda_\tau = \{x_0 = x_1 = 0\}, \qquad L_\tau = \{x_2 = x_3 = x_4 = x_5 = 0\}$$

and the 21-dimensional vector space $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(2))$ decomposes as the sum of

$$\begin{array}{lcll} \ker(\mathrm{Id} - \tau^*) & = & \mathbb{C}[x_0, x_1]_2 \,\oplus\, \mathbb{C}[x_2, x_3, x_4, x_5]_2 & \text{(of dim 13)} \\ \ker(\mathrm{Id} + \tau^*) & = & x_0\,\mathbb{C}[x_2, x_3, x_4, x_5]_1 \,\oplus\, x_1\,\mathbb{C}[x_2, x_3, x_4, x_5]_1 & \text{(of dim 8)} \end{array}$$

Again, choosing a complete intersection of three quadrics over which the model involution is stable amounts to choosing a point U in the Grassmannian

$$G(3, \ker(\mathrm{Id} - \tau^*)) = G(3, 13) \approx \mathbb{P}^{30}$$

and a general such U yields a surface

$$Y' = \mathrm{bs}(\mathbb{P}U) = \textstyle\bigcap_{Q\in\mathbb{P}U} Q \quad\hookrightarrow\quad \mathbb{P}^5$$

over which the model involution has 8 fixed points. In this setting, we take

$$E_{02} = (1:0:1:0:0:0), \quad E_{13} = (0:1:0:1:0:0) \;\in\; \mathbb{P}^5 - (\Lambda_\tau \cup L_\tau)$$

to be our *model points*, and proceed with an analogue to lemma 1.4.5.

Lemma 1.5.7. *Consider the line bundle $\mathcal{O}_{Y'}(1) \in \mathrm{Pic}(Y')$ induced by $Y' \hookrightarrow \mathbb{P}^5$. After an adequate coordinate change, a general point $(Y', \iota, \mathcal{O}_S(H), y_1, y_2) \in \mathcal{F}^{\mathfrak{N}}_{3,2}$ can be written as $(Y', \tau, \mathcal{O}_{Y'}(1), E_{02}, E_{13})$.*

Proof. As line subspaces of $\mathbb{C}^6$, the points $y_1, y_2 \in Y' \subset \mathbb{P}^5$ are of the form

$$y_1 = \langle v_1^\Lambda + v_1^L \rangle, \quad y_2 = \langle v_2^\Lambda + v_2^L \rangle$$

where $v_i^\Lambda \in E_\Lambda$, $v_i^L \in E_L$ for $i \in \{1, 2\}$. Then $\{v_1^\Lambda, v_2^\Lambda\}$ and $\{v_1^L, v_2^L\}$ are linearly independent by generality and we can complete them to get bases

$$E_\Lambda = \langle v_1^\Lambda, v_2^\Lambda, u, w \rangle, \quad E_L = \langle v_1^L, v_2^L \rangle$$

so that the map

$$\begin{aligned} \varphi\colon \quad \mathbb{C}^6 \to \mathbb{C}^6, \qquad & v_1^L \mapsto e_0, \qquad v_2^L \mapsto e_1, \\ & v_1^\Lambda \mapsto e_2, \qquad v_2^\Lambda \mapsto e_3, \qquad u \mapsto e_4, \qquad w \mapsto e_5 \end{aligned}$$

provides the desired coordinate change. □

As in the Prym framework, lemma 1.5.7 is the final piece that is required to parametrize $\mathcal{F}_{3,2}^{\mathfrak{N}}$ by a suitable Grassmannian.

Theorem 1.5.8. *The universal double Nikulin surface $\mathcal{F}_{3,2}^{\mathfrak{N}}$ is unirational.*

Proof. Let $(Y', \tau, \mathcal{O}_{Y'}(1), E_{02}, E_{13})$ be a general point of $\mathcal{F}_{3,2}^{\mathfrak{N}}$, with

$$H^0(\mathbb{P}^5, I_{Y'}(2)) \quad \subset \quad \ker(\mathrm{Id} - \tau^*) \quad = \quad \mathbb{C}[x_0, x_1]_2 \ \oplus\ \mathbb{C}[x_2, x_3, x_4, x_5]_2$$

As every $Q \in |I_{Y'}(2)|$ passes through E_{02} and E_{13}, we see that $H^0(\mathbb{P}^5, I_{Y'}(2))$ is in fact a 3-dimensional subspace of the 11-dimensional space

$$\begin{aligned} V_{11} \quad &= \quad \{q \in \ker(\mathrm{Id} - \tau^*) \ / \ q(E_{02}) = q(E_{13}) = 0\} \\ &= \quad (x_0^2 - x_2^2)\,\mathbb{C} \ \oplus\ (x_1^2 - x_3^2)\,\mathbb{C} \ \oplus\ x_0x_1\,\mathbb{C} \ \oplus\ x_2x_3\,\mathbb{C} \ \oplus \\ &\qquad \oplus\ x_4\,\mathbb{C}[x_2, x_3, x_4]_1 \ \oplus\ x_5\,\mathbb{C}[x_2, x_3, x_4, x_5]_1 \end{aligned}$$

Once more, choosing a complete intersection of three quadrics passing through E_{02}, E_{13} over which the model involution is stable amounts to choosing a point $U \in G(3, V_{11})$. In addition, for a general such U, the model involution restricts to a Nikulin involution over the surface

$$Y' = \mathrm{bs}(\mathbb{P}U) = \textstyle\bigcap_{Q \in \mathbb{P}U} Q \quad \hookrightarrow \quad \mathbb{P}^5$$

Finally, the above discussion gives rise to a dominant rational map

$$\mathbb{P}^{24} \approx G(3, V_{11}) \to \mathcal{F}_{3,2}^{\mathfrak{N}}, \qquad U \mapsto (\mathrm{bs}(\mathbb{P}U), \tau, \mathcal{O}(1), E_{02}, E_{13})$$

since a general $(Y', \iota, \mathcal{O}_S(H), y_1, y_2)$ can be recovered from $H^0(\mathbb{P}^5, I_{Y'}(2)) \subset V_{11}$, implying that $\mathcal{F}^{\mathfrak{N}}_{3,2}$ is indeed unirational. □

Remark 1.5.9. This result offers an alternative, even if more convoluted, proof that $(\mathcal{C}')^2$ and $\mathcal{Y}_3$ are unirational, on account of the diagram

$$\begin{tikzcd}
 & \mathcal{P}^{\mathfrak{N}}_{3,2} \arrow[dl] \arrow[dr] & & \\
\mathcal{F}^{\mathfrak{N}}_{3,2} & & (\mathcal{C}')^2 \arrow[r, "\mathfrak{ap}"] & \mathcal{Y}_3
\end{tikzcd}$$

Observe that restricting V_{11} to the hyperplane $\{x_5 = 0\} \subset \mathbb{P}^5$ brings us back to the space V_7 from theorem 1.4.6.

If we look at the parametrization $G(3, V_{11}) \to \mathcal{F}^{\mathfrak{N}}_{3,2}$, we now have:

$$\begin{cases} \dim G(3, V_{11}) & = & 3\,(11-3) & = & 24 \\ \dim \mathcal{F}^{\mathfrak{N}}_{3,2} & = & 11 + 2\cdot 2 & = & 15 \end{cases}$$

The general fiber is 9-dimensional and isomorphic to the subgroup $G_9 \subset \mathrm{PGL}(6)$ of projective linear transformations fixing the model involution and both of the points E_{02}, E_{13}, given by

$$\left\{ \left(\begin{array}{cc|cccc} a_{00} & 0 & & & & \\ 0 & a_{11} & & \multicolumn{2}{c}{0} & \\ \hline & & a_{00} & 0 & a_{24} & a_{25} \\ & & 0 & a_{11} & a_{34} & a_{35} \\ \multicolumn{2}{c|}{0} & 0 & 0 & a_{44} & a_{45} \\ & & 0 & 0 & a_{54} & a_{55} \end{array} \right) \in \mathrm{GL}(2) \times \mathrm{GL}(4) \subset \mathrm{GL}(6) \right\} \Big/ \{\mathbb{C}^* \, \mathrm{Id}_6\}$$

Then the quotient $G(3, V_{11})/G_9$ is of dimension 15, and birational to $\mathcal{F}^{\mathfrak{N}}_{3,2}$.

Corollary 1.5.10. *There is a birational equivalence*

$$G(3, V_{11})/G_9 \xrightarrow{\approx} \mathcal{F}^{\mathfrak{N}}_{3,2}$$

induced by the rational parametrization $G(3, V_{11}) \to \mathcal{F}^{\mathfrak{N}}_{3,2}$.

Chapter 2

Prym curves with a vanishing theta-null

2.1 Compactifications of moduli spaces

If we move away from birational geometry, it is just a matter of time before we set foot on the boundary of our moduli space. Indeed, looking at families of curves reveals that objects in the boundary appear very naturally (as limits of such families), so it is desirable to work in a proper setting that supports them. This is provided by a suitable *compactification* of the moduli space: namely, a larger, proper space which includes these additional deformations or limits, but whose points still have a modular meaning.

In this section, we describe some of these compactifications, as well as their boundaries. We conclude by introducing several specific families of curves that are frequently tested against interesting divisors in order to study their divisor class. As a basic source text on this general topic, we refer to [HM98].

2.1.1 Stable, semistable and quasistable curves

Given a connected, δ-nodal curve X with irreducible components $X_1, \ldots, X_n$, the arithmetic genus of X can be computed by the formula:

$$g(X) \quad = \quad p_a(X) \quad = \quad \textstyle\sum_{i=1}^n (g_i - 1) + \delta + 1 \quad = \quad \textstyle\sum_{i=1}^n g_i + (\delta - n + 1)$$

where $g_i = p_g(X_i) = p_a(\tilde{X}_i)$ is the geometric genus of X_i, that is, the arithmetic genus of its normalization $\tilde{X}_i$. Furthermore, we can associate a graph Γ_X to X, called the *dual graph of the curve*, by means of the following dictionary:

X	Γ_X
irreducible component X_i	vertex X_i with weight g_i
node $z_k \in X_i \cap X_j$	edge z_k connecting X_i and X_j
node z_k in self-intersection of X_i	loop z_k on vertex X_i

Then the formula above can be rewritten as:

$$g(X) \quad = \quad p_a(X) \quad = \quad \textstyle\sum_{i=1}^n g_i + (\delta - n + 1) \quad = \quad w(\Gamma_X) + b_1(\Gamma_X)$$

where $w(\Gamma_X) = \sum_{i=1}^n g_i$ is the total weight of the vertex-weighted dual graph, while $b_1(\Gamma_X) = \delta - n + 1$ is its first Betti number.

Definition 2.1.1. Let X be a complete, connected, nodal curve. We say that X is *stable* (resp. *semistable*) if every smooth rational component of X meets the other components of X in at least 3 points (resp. at least 2 points).

In terms of their dual graph, stable curves (resp. semistable) can be defined by the property that all 0-weighted vertices are endpoints of at least 3 (resp. 2) possibly repeated edges. Therefore chains of smooth rational components that connect two non-rational components are allowed in semistable curves, but not in stable ones.

As shown by Deligne and Mumford in [DM69], stable curves are the natural objects that arise when we aim to compactify the moduli space of curves while still preserving a modular interpretation.

Definition 2.1.2. Let S be a scheme. A *family of stable curves* of genus g over the base S, or simply a *stable curve* of genus g over S, is a curve $f\colon X \to S$ of genus g such that every geometric fiber $f^{-1}(s) = X_s$ is a stable curve.

Remark 2.1.3. More generally, if we have a property P of curves, we shall say that a *family of P-curves* over S, or simply a *P-curve* over S, is a curve over S such that every geometric fiber has property P (e.g. semistability).

With definition 2.1.2 and the theory of section 1.3, we can set up a moduli problem for stable curves of genus g, which admits both a coarse moduli space $\overline{M}_g$ and a Deligne-Mumford moduli stack $\overline{\mathcal{M}}_g$. This moduli space is the central object of study in Deligne and Mumford's seminal paper, leading up to:

Theorem 2.1.4. *The moduli space $\overline{\mathcal{M}}_g$ of stable, genus g curves is proper, and contains the moduli space $\mathcal{M}_g$ of smooth, genus g curves as an open subset.*

Proof. See [DM69] Th. 5.2. □

Remark 2.1.5. Since $\overline{\mathcal{M}}_g$ both compactifies $\mathcal{M}_g$ and parametrizes meaningful algebro-geometric objects at the same time, it is often referred to as a *modular compactification* of $\mathcal{M}_g$. There are many examples of interesting moduli spaces which are not closed, but admit this type of compactification.

In light of remark 2.1.5, we want to introduce modular compactifications of several spaces relevant to our work, such as the Prym moduli space $\mathcal{R}_g$ (recall definition 1.3.17), or the moduli space of even spin curves $\mathcal{S}_g^+$ (resp. odd, $\mathcal{S}_g^-$). The addition of square roots to the compactification turns out to require a new class of curves, slighty broader than that of stable ones.

Definition 2.1.6. Let E be an irreducible component of a semistable curve X. Then E is said to be *exceptional* if it is smooth, rational, and meets the other components in exactly 2 points.

Note that stable curves can then be characterised as semistable curves with no exceptional components. Let us relax this condition.

Definition 2.1.7. Let X be a semistable curve. We say that X is *quasistable* if any two distinct exceptional components are disjoint. In turn, the *stable model* of a quasistable curve X is the stable curve $\mathrm{st}(X)$ obtained by contracting each exceptional component to a point.

In other words, quasistable curves are semistable curves which do not allow chains of smooth rational components, but still allow non-rational components to be connected by a single exceptional one.

Definition 2.1.8. A *stable Prym curve* is a triplet (X, η, β) where:

(i) X is a quasistable curve (of genus g).

(ii) $\eta \in \mathrm{Pic}^0(X)$ is a nontrivial line bundle of total degree 0 on X such that $\eta|_E = \mathcal{O}_E(1)$ for every exceptional component E of X.

(iii) $\beta\colon \eta^{\otimes 2} \to \mathcal{O}_X$ is a sheaf homomorphism such that the restriction $\beta|_A$ is generically non-zero for every non-exceptional component A of X.

Similarly, a *stable even spin curve* is a triplet (X, θ, α) where:

(i) X is a quasistable curve (of genus g).

(ii) $\theta \in \mathrm{Pic}^{g-1}(X)$ is a line bundle of total degree $g-1$ on X with $h^0(X, \theta)$ even, and $\theta|_E = \mathcal{O}_E(1)$ for every exceptional component E of X.

(iii) $\alpha\colon \eta^{\otimes 2} \to \omega_X$ is a sheaf homomorphism such that the restriction $\alpha|_A$ is generically non-zero for every non-exceptional component A of X.

For a *stable odd spin curve*, simply take $h^0(X, \theta)$ odd.

If we denote by $E_1, \ldots, E_r$ the exceptional components of X, then

$$B_X \quad = \quad \overline{X - (E_1 \cup \ldots \cup E_r)} \quad \subset \quad X$$

is a closed subcurve of X, over which sheaf isomorphisms are recovered:

$$\begin{aligned} \beta|_{B_X}\colon \quad \eta^{\otimes 2}|_{B_X} \quad &\cong \quad \mathcal{O}_{B_X}(-p_1 - q_1 - \ldots - p_r - q_r) \\ \alpha|_{B_X}\colon \quad \theta^{\otimes 2}|_{B_X} \quad &\cong \quad \omega_X|_{B_X}(-p_1 - q_1 - \ldots - p_r - q_r) \end{aligned}$$

where $B_X \cap E_i = \{p_i, q_i\}$ for all i. When X is stable, we get $B_X = X$.

Compare these notions with definitions 1.1.9 and 1.1.14, their smooth counterparts. In the same way, we may extend definition 1.3.16.

Definition 2.1.9. Let S be a scheme. A *family of stable Prym curves* over the base S, or a *stable Prym curve* over S, is a triplet $(f\colon X \to S,\ \eta,\ \beta)$ such that:

(i) $f\colon X \to S$ is a quasistable (genus g) curve over S, as in remark 2.1.3.

(ii) $\eta \in \mathrm{Pic}^0(X)$ is a line bundle on X.

(iii) $\beta\colon \eta^{\otimes 2} \to \mathcal{O}_X$ is a sheaf homomorphism.

(iv) The restriction of $(f\colon X \to S,\ \eta,\ \beta)$ to any fiber $f^{-1}(s) = X_s$ gives rise to a stable Prym curve (X_s, η_s, β_s).

An isomorphism $(X \to S, \eta, \beta) \cong (X' \to S, \eta', \beta')$ is a pair (φ, ψ) where:

(i) $\varphi\colon X \cong X'$ is an isomorphism over S.

(ii) $\psi\colon \varphi^*(\eta') \cong \eta$ is a sheaf isomorphism such that $\varphi^*(\beta') = \beta \circ \psi^{\otimes 2}$.

With minimal changes, we could likewise define *families of stable spin curves*.

The resulting moduli problems all admit proper moduli spaces, namely $\overline{\mathcal{R}}_g$, $\overline{\mathcal{S}}_g^+$ and $\overline{\mathcal{S}}_g^-$, which respectively compactify $\mathcal{R}_g$, $\mathcal{S}_g^+$ and $\mathcal{S}_g^-$. Further details on these can be found in [Cor89], for the compactification of $\mathcal{S}_g$, and [BCF04], for the Cornalba-inspired compactification of $\mathcal{R}_g$. It is also important to highlight [Bea77], where a different but earlier compactification of $\mathcal{R}_g$ was built through the use of admissible double covers of stable curves.

Observe that stabilising a quasistable curve preserves its genus, as the same number of 0-weighted vertices and edges are removed from its dual graph, and consider the natural maps

$$\begin{array}{rccclcl} \pi_{\mathcal{R}}\colon & \overline{\mathcal{R}}_g & \to & \overline{\mathcal{M}}_g, & (X,\eta,\beta) & \mapsto & \mathrm{st}(X) \\ \pi_{\mathcal{S}}\colon & \overline{\mathcal{S}}_g & \to & \overline{\mathcal{M}}_g, & (X,\theta,\alpha) & \mapsto & \mathrm{st}(X) \end{array}$$

where $\mathrm{st}(X)$ is the stable model of X. These maps are finite and ramified over the boundary, and extend the finite, unramified covers $\mathcal{R}_g \to \mathcal{M}_g$, $\mathcal{S}_g \to \mathcal{M}_g$.

2.1.2 Boundaries and Picard groups

Let us now look at the boundary that has been added to our moduli spaces in order to compactify them. First we study the boundary divisors of $\overline{\mathcal{M}}_g$, and then use them to describe the boundary divisors of the covers $\overline{\mathcal{R}}_g$, $\overline{\mathcal{S}}_g$.

Since the curves of $\partial\overline{\mathcal{M}}_g = \overline{\mathcal{M}}_g - \mathcal{M}_g$ are nodal, it stands to reason that we could classify them in terms of the type of nodes they carry.

Definition 2.1.10. Let X be a semistable curve of genus g. We say that a node $z \in \mathrm{Sing}(X)$ is *of type* $i \in \{1, \dots, \lfloor g/2 \rfloor\}$ if it is in the intersection of a genus i component of X with a genus $g-i$ component of X, and *of type* 0 if it is in the self-intersection of an irreducible component of X.

With respect to the dual graph, nodes of type $i \geq 1$ correspond to edges that connect subgraphs of weights i and $g-i$, whereas nodes of type 0 correspond to loops. Furthermore, we can consider the closure of the loci

$$\Delta_i^{\text{set}} \quad = \quad \{\text{curves with a node of type } i\} \quad \subset \quad \partial\overline{\mathcal{M}}_g \quad \subset \quad \overline{\mathcal{M}}_g$$

to obtain divisors Δ_i of $\overline{\mathcal{M}}_g$ such that $\Delta_i \subset \partial\overline{\mathcal{M}}_g$ and

$$\partial\overline{\mathcal{M}}_g \quad = \quad \Delta_0 \,\cup\, \Delta_1 \,\cup\, \ldots \,\cup\, \Delta_{\lfloor g/2 \rfloor}$$

Here it is essential to make a distinction between the moduli stack $\overline{\mathcal{M}}_g$ and the coarse moduli space $\overline{M}_g$. If we choose to work in the latter space, we can define divisors Δ_i of $\overline{M}_g$ in the same way, and obtain a diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_g \arrow[r, "\rho"] & \overline{M}_g \\ \Delta_i \arrow[u, phantom, "\cup"] \arrow[r, "\rho_i"'] \arrow[ur, phantom, "\circlearrowleft"] & \Delta_i \arrow[u, phantom, "\cup"] \end{tikzcd}$$

with $\deg(\rho_i) = \#\,\mathrm{Aut}(X)$ for $X \in \Delta_i$ general; see example 1.3.14. In particular

$$\deg(\rho_i) \quad = \quad \begin{cases} 1 & \text{if } i \neq 1 \\ 2 & \text{if } i = 1 \end{cases}$$

due to the existence of nontrivial involutions on elliptic tails. This difference is easy to miss, but necessary to track, especially when divisor classes and Picard groups are brought into the picture.

Proposition 2.1.11. *The pullback map ρ^* sets up an isomorphism between the rational Picard groups of $\overline{M}_g$ and $\overline{\mathcal{M}}_g$, namely*

$$\rho^* \colon \quad \mathrm{Pic}(\overline{M}_g)_{\mathbb{Q}} \quad \cong \quad \mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$$

such that

$$\rho^*[\Delta_i] \quad = \quad \deg(\rho_i) \cdot \delta_i \quad = \quad \begin{cases} \delta_i & \text{if } i \neq 1 \\ 2\,\delta_i & \text{if } i = 1 \end{cases}$$

where we use the standard notation

$$\begin{array}{rcccl} [\Delta_i] & = & \mathcal{O}_{\overline{M}_g}(\Delta_i) & \in & \mathrm{Pic}(\overline{M}_g)_{\mathbb{Q}} \\ \delta_i & = & \mathcal{O}_{\overline{\mathcal{M}}_g}(\Delta_i) & \in & \mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}} \end{array}$$

for the classes of Δ_i in $\overline{M}_g$ and $\overline{\mathcal{M}}_g$, respectively.

Proof. The statement is a compilation of results from [HM98] Section 3.D, with a slight change of notation fostered by the application of stack theory. $\square$

Remark 2.1.12. In general, if $\mathcal{M} \to M$ is the projection from a moduli stack to its coarse moduli space, then the pullback $\operatorname{Pic}(M)_{\mathbb{Q}} \to \operatorname{Pic}(\mathcal{M})_{\mathbb{Q}}$ is injective, and further becomes an isomorphism if M has only *finite quotient singularities*. This is the case not just for $\overline{M}_g$, but also for $\overline{R}_g$, $\overline{S}_g^+$ or $\overline{S}_g^-$.

As suggested in proposition 2.1.11, the locus of stable curves with a node of type i naturally gives rise to two different rational divisor classes

$$\rho^*[\Delta_i], \quad \delta_i \quad \in \quad \operatorname{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$$

We shall mostly work with the ones defined directly on the moduli stack, i.e.

$$\delta_0,\, \delta_1,\, \ldots,\, \delta_{\lfloor g/2 \rfloor}$$

since their degree over 1-parameter families of curves is easier to calculate.

Remark 2.1.13. If $X \to B$ is a stable curve of genus g over a smooth base B of dimension 1, with fibers $X_b \in \overline{\mathcal{M}}_g$ for each $b \in B$, then the degree

$$\deg \delta_i(X \to B) \quad = \quad X \cdot \Delta_i \quad \in \quad \mathbb{Z}$$

of δ_i over $X \to B$ morally reflects the "number of fibers X_b lying in Δ_i", in the sense of [HM98] Section 3.D, Prop. 3.91. Somewhat abusively, we often denote this degree by $X \cdot \Delta_i = X \cdot \delta_i$ in order to highlight the stacky framework.

Remark 2.1.14. Any rational divisor class ξ in $\overline{\mathcal{M}}_g$ can be written as

$$\xi \quad = \quad j_* j^* \xi + b_0\, \delta_0 + \ldots + b_{\lfloor g/2 \rfloor}\, \delta_{\lfloor g/2 \rfloor} \quad \in \quad \operatorname{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$$

for some $b_0, \ldots, b_{\lfloor g/2 \rfloor} \in \mathbb{Q}$ and the embedding $j\colon \mathcal{M}_g \hookrightarrow \overline{\mathcal{M}}_g$. Indeed, observe that the class $\xi - j_* j^* \xi$ is trivial over $\mathcal{M}_g$, and that the divisors $\Delta_0, \ldots, \Delta_{\lfloor g/2 \rfloor}$ are the irreducible components of the boundary $\partial \overline{\mathcal{M}}_g = \overline{\mathcal{M}}_g - \mathcal{M}_g$.

According to remark 2.1.14, rational divisor classes in $\overline{\mathcal{M}}_g$ are generated by rational divisor classes in $\mathcal{M}_g$ (or rather, their pushforward) and the boundary classes δ_i. Fortunately, the group $\operatorname{Pic}(\mathcal{M}_g)_{\mathbb{Q}}$ has a simple description, which we can obtain readily thanks to the work of Harer [Har83].

The universal curve $\phi\colon \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ carries a relative dualizing sheaf

$$\omega_\phi \quad \in \quad \operatorname{Pic}(\overline{\mathcal{C}}_g)$$

whose direct image $\phi_*(\omega_\phi)$, the *Hodge bundle*, can be used to define the classes

$$\begin{array}{rcccl} & & \lambda & = & \bigwedge^g \phi_*(\omega_\phi) & \in & \mathrm{Pic}(\overline{\mathcal{M}}_g) \\ \lambda & = & j^*\lambda & = & \bigwedge^g \phi_*(\omega_{\mathcal{C}_g|\mathcal{M}_g}) & \in & \mathrm{Pic}(\mathcal{M}_g) \end{array}$$

For $g \geq 3$, Harer's theorem implies that $\mathrm{Pic}(\mathcal{M}_g)_{\mathbb{Q}} = \lambda\,\mathbb{Q}$, and therefore

$$\mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}} \;=\; \lambda\,\mathbb{Q} \,\oplus\, \delta_0\,\mathbb{Q} \,\oplus\, \ldots \,\oplus\, \delta_{\lfloor g/2 \rfloor}\,\mathbb{Q}$$

as the classes λ, δ_0, ..., $\delta_{\lfloor g/2 \rfloor}$ are linearly independent. This discussion may be found in [AC87], together with a stronger result on the structure of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$.

Theorem 2.1.15. *For any $g \geq 3$, it holds that*

$$\begin{array}{rcl} \mathrm{Pic}(\mathcal{M}_g)_{\mathbb{Z}} & = & \lambda\,\mathbb{Z} \\ \mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Z}} & = & \lambda\,\mathbb{Z} \,\oplus\, \delta_0\,\mathbb{Z} \,\oplus\, \ldots \,\oplus\, \delta_{\lfloor g/2 \rfloor}\,\mathbb{Z} \end{array}$$

With our current understanding of the rational Picard group of $\overline{\mathcal{M}}_g$, we are ready to shift our attention towards $\overline{\mathcal{R}}_g$ and $\overline{\mathcal{S}}_g$.

In order to study the boundary of $\overline{\mathcal{R}}_g$, we can take advantage of the map

$$\pi_{\mathcal{R}}\colon \quad \overline{\mathcal{R}}_g \;\to\; \overline{\mathcal{M}}_g, \qquad (X,\eta,\beta) \;\mapsto\; \mathrm{st}(X)$$

which turns the decomposition in irreducible components

$$\partial\overline{\mathcal{M}}_g \;=\; \Delta_0 \,\cup\, \Delta_1 \,\cup\, \ldots \,\cup\, \Delta_{\lfloor g/2 \rfloor}$$

into a building block for the corresponding decomposition of $\partial\overline{\mathcal{R}}_g$.

First, note that a general point $Y \in \Delta_i$ is of the form:

$$\begin{array}{lll} (i > 0) \quad & Y = C \cup_{p\sim q} D \quad & \text{with} \;\; (C,p) \in \mathcal{M}_{i,1},\; (D,q) \in \mathcal{M}_{g-i,1} \\ (i = 0) \quad & Y = B_{pq} \quad & \text{with} \;\; (B,p,q) \in \mathcal{M}_{g-1,2} \end{array}$$

where B_{pq} denotes the irreducible 1-nodal curve obtained from B by gluing the points p and q. Let us describe the fibers $\pi_{\mathcal{R}}^{-1}(Y)$ for each i.

Example 2.1.16 $(i > 0)$**.** Let $(X,\eta,\beta) \in \overline{\mathcal{R}}_g$ with $\mathrm{st}(X) = Y = C \cup_{p\sim q} D$. The existence of β prevents X from having exceptional components, i.e.

$$X = \mathrm{st}(X) = Y = C \cup_{p\sim q} D, \qquad \beta\colon \eta^{\otimes 2} \cong \mathcal{O}_Y = (\mathcal{O}_C, \mathcal{O}_D)$$

Then η is a nontrivial element of $J_2(C) \oplus J_2(D)$, and we have three irreducible components over Δ_i, characterised by their general point (X,η,β):

(Δ_i^{n})	Condition:	$\eta = (\eta_C, \mathcal{O}_D)$ with $\eta_C \in R_i(C)$.
	Notation:	$\Delta_i^{\mathrm{n}} \subset \overline{\mathcal{R}}_g$ (for *nontrivial on* i), or traditionally Δ_i.
	Degree:	$\deg(\Delta_i^{\mathrm{n}} \vert \Delta_i) = 2^{2i} - 1$.
(Δ_i^{t})	Condition:	$\eta = (\mathcal{O}_C, \eta_D)$ with $\eta_D \in R_{g-i}(D)$.
	Notation:	$\Delta_i^{\mathrm{t}} \subset \overline{\mathcal{R}}_g$ (for *trivial on* i), or traditionally Δ_{g-i}.
	Degree:	$\deg(\Delta_i^{\mathrm{t}} \vert \Delta_i) = 2^{2(g-i)} - 1$.
(Δ_i^{p})	Condition:	$\eta = (\eta_C, \eta_D)$ with $\eta_C \in R_i(C)$, $\eta_D \in R_{g-i}(D)$.
	Notation:	$\Delta_i^{\mathrm{p}} \subset \overline{\mathcal{R}}_g$ (for *Prym*), or traditionally $\Delta_{i:g-i}$.
	Degree:	$\deg(\Delta_i^{\mathrm{p}} \vert \Delta_i) = (2^{2i} - 1)(2^{2(g-i)} - 1)$.

The pullback of $\Delta_i \subset \overline{\mathcal{M}}_g$ can be written as

$$\pi_{\mathcal{R}}^*(\Delta_i) \quad = \quad \Delta_i^{\mathrm{n}} + \Delta_i^{\mathrm{t}} + \Delta_i^{\mathrm{p}}$$

and, with the notation of proposition 2.1.11, we have relations

$$\pi_{\mathcal{R}}^*(\delta_i) \quad = \quad \delta_i^{\mathrm{n}} + \delta_i^{\mathrm{t}} + \delta_i^{\mathrm{p}}$$

for $1 \leq i \leq \lfloor g/2 \rfloor$ and $\delta_i^{\mathrm{x}} = \mathcal{O}_{\overline{\mathcal{R}}_g}(\Delta_i^{\mathrm{x}}) \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$, $\mathrm{x} \in \{\mathrm{n}, \mathrm{t}, \mathrm{p}\}$. Observe that

$$\deg(\Delta_i^{\mathrm{n}}|\Delta_i) + \deg(\Delta_i^{\mathrm{t}}|\Delta_i) + \deg(\Delta_i^{\mathrm{p}}|\Delta_i) \quad = \quad 2^{2g} - 1 \quad = \quad \deg(\pi_{\mathcal{R}})$$

as expected. A small exception to the above is the special case of $g = 2i$, where the components $\Delta_i^{\mathrm{n}} = \Delta_i^{\mathrm{t}}$ coincide and have degree $2(2^{2i} - 1)$ over Δ_i.

Example 2.1.17 ($i = 0$)**.** Let $(X, \eta, \beta) \in \overline{\mathcal{R}}_g$ with $\mathrm{st}(X) = Y = B_{pq}$. There are two possibilities for X, depending on whether it contains or not an exceptional component. If it does not, i.e.

$$X = \mathrm{st}(X) = Y = B_{pq}, \qquad \beta \colon \eta^{\otimes 2} \cong \mathcal{O}_Y$$

then the normalization $\nu \colon B \to B_{pq}$ induces an exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow J_2(B_{pq}) \xrightarrow{\nu^*} J_2(B) \longrightarrow 0, \qquad \eta_B = \nu^*\eta \in J_2(B)$$

and the potential triviality of $\eta_B = \nu^*\eta$ determines two irreducible components:

(Δ_0^{t})	Condition:	$\eta_B = \mathcal{O}_B$, hence $\eta \in (\nu^*)^{-1}(\eta_B) - \{\mathcal{O}_Y\}$ unique.
	Notation:	$\Delta_0^{\mathrm{t}} \subset \overline{\mathcal{R}}_g$ (for *trivial*), or traditionally Δ_0''.
	Degree:	$\deg(\Delta_0^{\mathrm{t}} \vert \Delta_0) = 1$.

(Δ_0^{p}) | Condition: $\eta_B \in R_{g-1}(B)$, hence $\eta \in (\nu^*)^{-1}(\eta_B) \cong \mathbb{Z}_2$.
Notation: $\Delta_0^{\mathrm{p}} \subset \overline{\mathcal{R}}_g$ (for *Prym*), or traditionally Δ_0'.
Degree: $\deg(\Delta_0^{\mathrm{p}}|\Delta_0) = 2\,(2^{2(g-1)} - 1)$.

On the other hand, if X has an exceptional component E, then we can project it onto Y as a sort of "exceptional blow-up", i.e. there is a map

$$X = B \cup_{p\sim 0,\, q\sim\infty} E \longrightarrow \mathrm{st}(X) = Y = B_{pq}$$

induced by $\nu\colon B \to B_{pq}$, $E \mapsto z = \nu(p) = \nu(q)$. Then we have

$$B_X = \overline{X - E} \cong B, \qquad \beta\colon \eta_B^{\otimes 2} \cong \mathcal{O}_B(-p-q)$$

for $\eta_B = \eta|_B \in \mathrm{Pic}(B)$, and Mayer-Vietoris yields an exact sequence

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{Pic}(X) \xrightarrow{\xi} \mathrm{Pic}(B) \oplus \mathrm{Pic}(E) \longrightarrow 0, \qquad \xi(\eta) = (\eta_B, \mathcal{O}_E(1))$$

This way, we obtain one last irreducible component:

(Δ_0^{b}) | Condition: $\eta_B \in \sqrt{\mathcal{O}_B(-p-q)}$.
Notation: $\Delta_0^{\mathrm{b}} \subset \overline{\mathcal{R}}_g$ (for *blown-up*), or traditionally Δ_0^{ram}.
Degree: $\deg(\Delta_0^{\mathrm{b}}|\Delta_0) = 2^{2(g-1)}$.

Due to the appearance of an exceptional component over the node $z \in B_{pq}$, the divisor Δ_0^{b} is in fact the ramification divisor of $\pi_{\mathcal{R}}\colon \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$. The pullback of $\Delta_0 \subset \overline{\mathcal{M}}_g$ can accordingly be written as

$$\pi_{\mathcal{R}}^*(\Delta_0) \;=\; \Delta_0^{\mathrm{t}} + \Delta_0^{\mathrm{p}} + 2\,\Delta_0^{\mathrm{b}}$$

and, with the notation of proposition 2.1.11, we have the relation

$$\pi_{\mathcal{R}}^*(\delta_0) \;=\; \delta_0^{\mathrm{t}} + \delta_0^{\mathrm{p}} + 2\,\delta_0^{\mathrm{b}}$$

for $\delta_0^{\mathrm{x}} = \mathcal{O}_{\overline{\mathcal{R}}_g}(\Delta_0^{\mathrm{x}}) \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$, $\mathrm{x} \in \{\mathrm{t}, \mathrm{p}, \mathrm{b}\}$. Observe that

$$\deg(\Delta_0^{\mathrm{t}}|\Delta_0) + \deg(\Delta_0^{\mathrm{p}}|\Delta_0) + 2\,\deg(\Delta_0^{\mathrm{b}}|\Delta_0) \;=\; 2^{2g} - 1 \;=\; \deg(\pi_{\mathcal{R}})$$

as expected.

Remark 2.1.18. In example 2.1.17, note that $\deg(\Delta_0^{\mathrm{b}}|\Delta_0)$ is finite because, for $\eta_B \in \sqrt{\mathcal{O}_B(-p-q)}$ fixed, any two line bundles

$$\lambda,\, \mu \;\in\; \xi^{-1}(\eta_B, \mathcal{O}_E(1)) \;\cong\; \mathbb{C}^*$$

even if nonisomorphic as bundles, induce triplets

$$(X, \lambda, \beta_\lambda) \;\cong\; (X, \mu, \beta_\mu) \;\in\; \overline{\mathcal{R}}_g$$

that are always isomorphic as stable Prym curves; see [BCF04] Lemma 2.

We can now repeat the process for $\overline{\mathcal{S}}_g$, or rather its irreducible components $\overline{\mathcal{S}}_g^+$, $\overline{\mathcal{S}}_g^-$. Recall the projection

$$\pi_{\mathcal{S}}\colon \quad \overline{\mathcal{S}}_g \;\to\; \overline{\mathcal{M}}_g, \qquad (X, \theta, \alpha) \;\mapsto\; \mathrm{st}(X) = Y$$

whose fibers $\pi_{\mathcal{S}}^{-1}(Y)$ we describe for $Y \in \Delta_i$ general, $0 \leq i \leq \lfloor g/2 \rfloor$.

Example 2.1.19 ($i > 0$)**.** Let $(X, \theta, \alpha) \in \overline{\mathcal{S}}_g$ with $\mathrm{st}(X) = Y = C \cup_{p\sim q} D$. The existence of α forces X to have an exceptional component, i.e. there is a map

$$X = C \cup_{p\sim 0} E \cup_{q\sim\infty} D \longrightarrow \mathrm{st}(X) = Y = C \cup_{p\sim q} D$$

induced by $E \mapsto z = [p] = [q]$, and we get

$$B_X = \overline{X - E} \cong C \sqcup D, \qquad \alpha\colon (\theta_C, \theta_D)^{\otimes 2} \cong \omega_X|_{B_X}(-p-q) = (\omega_C, \omega_D)$$

for $(\theta_C, \theta_D) = \theta|_{B_X} \in \mathrm{Pic}(C) \oplus \mathrm{Pic}(D)$. Therefore, θ is determined by a pair

$$(\theta_C, \theta_D) \in S_i(C) \oplus S_{g-i}(D), \qquad \theta = (\theta_C, \mathcal{O}_E(1), \theta_D) \in \mathrm{Pic}(X)$$

In particular, notice that the (even, odd) parity of θ is subject to the (identical, alternating) character of the parities of θ_C and θ_D, since we have the relation

$$h^0(X, \theta) \;=\; h^0(C, \theta_C) + h^0(D, \theta_D)$$

by Mayer-Vietoris. As a result, out of the four irreducible components that are obtained over each Δ_i, two lie in $\overline{\mathcal{S}}_g^+$ and two lie in $\overline{\mathcal{S}}_g^-$. The even ones are:

(Δ_i^+)	Condition:	$\theta_C \in S_i^+(C)$, $\theta_D \in S_{g-i}^+(D)$.
	Notation:	$\Delta_i^+ = \Delta_{g-i}^+ \subset \overline{\mathcal{S}}_g^+$ (for *even on* i), or traditionally A_i^+.
	Degree:	$\deg(\Delta_i^+ \vert \Delta_i) = 2^{g-1}(2^i+1)(2^{g-i}+1)$.
(Δ_i^-)	Condition:	$\theta_C \in S_i^-(C)$, $\theta_D \in S_{g-i}^-(D)$.
	Notation:	$\Delta_i^- = \Delta_{g-i}^- \subset \overline{\mathcal{S}}_g^+$ (for *odd on* i), or traditionally B_i^+.
	Degree:	$\deg(\Delta_i^- \vert \Delta_i) = 2^{g-1}(2^i-1)(2^{g-i}-1)$.

Similarly, the odd ones are (abusing notation):

(Δ_i^+) | Condition: $\theta_C \in S_i^+(C)$, $\theta_D \in S_{g-i}^-(D)$.
Notation: $\Delta_i^+ = \Delta_{g-i}^- \subset \overline{\mathcal{S}}_g^-$ (for *even on* i), or traditionally A_i^-.
Degree: $\deg(\Delta_i^+|\Delta_i) = 2^{g-1}(2^i+1)(2^{g-i}-1)$.

(Δ_i^-) | Condition: $\theta_C \in S_i^-(C)$, $\theta_D \in S_{g-i}^+(D)$.
Notation: $\Delta_i^- = \Delta_{g-i}^+ \subset \overline{\mathcal{S}}_g^-$ (for *odd on* i), or traditionally B_i^-.
Degree: $\deg(\Delta_i^-|\Delta_i) = 2^{g-1}(2^i-1)(2^{g-i}+1)$.

Observe that a factor of 2 has to be considered in the computation

$$\begin{aligned}\Delta_i^+ \subset \overline{\mathcal{S}}_g^+, \qquad \deg(\Delta_i^+|\Delta_i) &= 2\cdot \#S_i^+(C)\cdot \#S_{g-i}^+(D)\\ &= 2\cdot 2^{i-1}(2^i+1)\cdot 2^{g-i-1}(2^{g-i}+1)\\ &= 2\cdot 2^{g-2}(2^i+1)(2^{g-i}+1)\end{aligned}$$

to account for the nontrivial automorphism of (X,θ,α) that arises from scaling by -1 on the exceptional component. Over the coarse moduli space $\overline{S}_g^+ \to \overline{M}_g$, this factor is not present. Consequently, the pullback of $\Delta_i \subset \overline{\mathcal{M}}_g$ (resp. $\subset \overline{M}_g$) by $\pi_+ \colon \overline{\mathcal{S}}_g^+ \to \overline{\mathcal{M}}_g$ (resp. $\pi_+ \colon \overline{S}_g^+ \to \overline{M}_g$) can be written as

$$\pi_+^*(\Delta_i) \;=\; 2\,\Delta_i^+ + 2\,\Delta_i^- \qquad (\text{resp.} \quad \pi_+^*(\Delta_i) \;=\; \Delta_i^+ + \Delta_i^-\ )$$

and, with the notation of proposition 2.1.11, we have relations

$$\pi_+^*(\delta_i) \;=\; 2\,\delta_i^+ + 2\,\delta_i^- \qquad (\text{resp.} \quad \pi_+^*[\Delta_i] \;=\; [\Delta_i^+] + [\Delta_i^-]\ )$$

for $1 \le i \le \lfloor g/2 \rfloor$ and $\delta_i^{\mathrm{x}} = \mathcal{O}_{\overline{\mathcal{S}}_g^+}(\Delta_i^{\mathrm{x}}) \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+)$, $\mathrm{x} \in \{+,-\}$. The same analysis works for $\overline{\mathcal{S}}_g^-$ and $\overline{S}_g^-$, although we get $\Delta_i^+ = \Delta_i^- \subset \overline{\mathcal{S}}_g^-$ when $g = 2i$.

Example 2.1.20 $(i=0)$**.** Let $(X,\theta,\alpha) \in \overline{\mathcal{S}}_g$ with $\mathrm{st}(X) = Y = B_{pq}$. There are again two possibilities for X. If it has no exceptional components, i.e.

$$X = \mathrm{st}(X) = Y = B_{pq}, \qquad \alpha\colon \theta^{\otimes 2} \cong \omega_Y$$

then the normalization $\nu\colon B \to B_{pq}$ induces a double cover

$$\nu^*\colon \sqrt{\omega_Y} \longrightarrow \sqrt{\omega_B(p+q)}, \qquad (\nu^*\theta)^{\otimes 2} \cong \nu^*\omega_Y \cong \omega_B(p+q)$$

so that θ is determined by a square root $\theta_B \in \sqrt{\omega_B(p+q)}$ and a choice on how to glue its fibers $\theta_B|_p$ and $\theta_B|_q$. Only two such gluings are possible, one making $h^0(X,\theta)$ even and the other one making it odd. We describe the component Δ_0^{n} obtained in this way only for $\overline{\mathcal{S}}_g^+$, as its $\overline{\mathcal{S}}_g^-$ counterpart is very similar.

(Δ_0^{n}) | Condition: $\theta_B \in \sqrt{\omega_B(p+q)}$ with even gluing.
Notation: $\Delta_0^{\mathrm{n}} \subset \overline{\mathcal{S}}_g^+$ (for <u>n</u>*ot blown-up*), or traditionally A_0^+.
Degree: $\deg(\Delta_0^{\mathrm{n}}|\Delta_0) = 2^{2(g-1)}$.

On the other hand, if X has an exceptional component E, then

$$X = B \cup_{p\sim 0,\, q\sim\infty} E \longrightarrow \mathrm{st}(X) = Y = B_{pq}, \qquad B_X = \overline{X - E} \cong B$$

and we have $\theta_B = \theta|_B \in S_{g-1}(B)$, since α gives rise to an isomorphism

$$\alpha\colon \theta_B^{\otimes 2} \cong \omega_X|_B(-p-q) \cong \nu^*\omega_Y(-p-q) \cong \omega_B$$

Moreover, recall the exact sequence

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{Pic}(X) \xrightarrow{\xi} \mathrm{Pic}(B) \oplus \mathrm{Pic}(E) \longrightarrow 0, \qquad \xi(\theta) = (\theta_B, \mathcal{O}_E(1))$$

and note that $h^0(X,\theta) = h^0(B,\theta_B)$, again by Mayer-Vietoris. In conclusion, we get the remaining irreducible component of $\partial\overline{\mathcal{S}}_g^+$, and similarly for $\partial\overline{\mathcal{S}}_g^-$, as:

(Δ_0^{b}) | Condition: $\theta_B \in S_{g-1}^+(B)$.
Notation: $\Delta_0^{\mathrm{b}} \subset \overline{\mathcal{S}}_g^+$ (for <u>b</u>*lown-up*), or traditionally B_0^+.
Degree: $\deg(\Delta_0^{\mathrm{b}}|\Delta_0) = 2^{g-2}(2^{g-1}+1)$.

The pullback of $\Delta_0 \subset \overline{\mathcal{M}}_g$ by $\pi_+\colon \overline{\mathcal{S}}_g^+ \to \overline{\mathcal{M}}_g$ can be written as

$$\pi_+^*(\Delta_0) \quad = \quad \Delta_0^{\mathrm{n}} + 2\,\Delta_0^{\mathrm{b}}$$

and, with the notation of proposition 2.1.11, we have the relation

$$\pi_+^*(\delta_0) \quad = \quad \delta_0^{\mathrm{n}} + 2\,\delta_0^{\mathrm{b}}$$

for $\delta_0^{\mathrm{x}} = \mathcal{O}_{\overline{\mathcal{S}}_g^+}(\Delta_0^{\mathrm{x}}) \in \mathrm{Pic}(\overline{\mathcal{S}}_g^+)$, $\mathrm{x} \in \{\mathrm{n}, \mathrm{b}\}$. Finally, the divisor

$$\Delta_0^{\mathrm{b}} + \textstyle\sum(\Delta_i^+ + \Delta_i^-) \qquad (\text{resp. } \Delta_0^{\mathrm{b}})$$

is the ramification divisor of $\pi_+\colon \overline{\mathcal{S}}_g^+ \to \overline{\mathcal{M}}_g$ (resp. $\pi_+\colon \overline{S}_g^+ \to \overline{M}_g$).

Remark 2.1.21. In the previous examples, it makes sense to denote

$$\begin{array}{lllll} (\text{over } \overline{\mathcal{R}}_g) & \Delta_i^{\mathrm{n}} = \Delta_{g-i}^{\mathrm{t}} & \text{and} & \Delta_i^{\mathrm{p}} = \Delta_{g-i}^{\mathrm{p}} \\ (\text{over } \overline{\mathcal{S}}_g^+) & \Delta_i^+ = \Delta_{g-i}^+ & \text{and} & \Delta_i^- = \Delta_{g-i}^- \\ (\text{over } \overline{\mathcal{S}}_g^-) & \Delta_i^+ = \Delta_{g-i}^- & & \end{array}$$

for all $1 \le i \le g-1$. These conventions nicely simplify future calculations.

Examples 2.1.16 and 2.1.17 provide us with a collection of boundary classes of $\overline{\mathcal{R}}_g$, while examples 2.1.19 and 2.1.20 follow suit with $\overline{\mathcal{S}}_g^+$ (and $\overline{\mathcal{S}}_g^-$):

$$\begin{array}{rcll} \delta_0^{\mathrm{t}},\, \delta_0^{\mathrm{p}},\, \delta_0^{\mathrm{b}}, \quad \delta_i^{\mathrm{t}},\, \delta_i^{\mathrm{n}},\, \delta_i^{\mathrm{p}} & \in & \mathrm{Pic}(\overline{\mathcal{R}}_g), & \qquad 1 \leq i \leq \lfloor g/2 \rfloor \\ \delta_0^{\mathrm{n}},\, \delta_0^{\mathrm{b}}, \qquad \delta_i^{+},\, \delta_i^{-} & \in & \mathrm{Pic}(\overline{\mathcal{S}}_g^+), & \qquad 1 \leq i \leq \lfloor g/2 \rfloor \end{array}$$

For $g \geq 5$, we then get

$$\mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}} \;=\; \lambda\,\mathbb{Q} \,\oplus\, \delta_0^{\mathrm{t}}\,\mathbb{Q} \,\oplus\, \delta_0^{\mathrm{p}}\,\mathbb{Q} \,\oplus\, \delta_0^{\mathrm{b}}\,\mathbb{Q} \,\oplus\, \bigoplus_{i=1}^{\lfloor g/2 \rfloor} (\delta_i^{\mathrm{t}}\,\mathbb{Q} \,\oplus\, \delta_i^{\mathrm{n}}\,\mathbb{Q} \,\oplus\, \delta_i^{\mathrm{p}}\,\mathbb{Q})$$

and similarly

$$\mathrm{Pic}(\overline{\mathcal{S}}_g^+)_{\mathbb{Q}} \;=\; \lambda\,\mathbb{Q} \,\oplus\, \delta_0^{\mathrm{n}}\,\mathbb{Q} \,\oplus\, \delta_0^{\mathrm{b}}\,\mathbb{Q} \,\oplus\, \bigoplus_{i=1}^{\lfloor g/2 \rfloor} (\delta_i^{+}\,\mathbb{Q} \,\oplus\, \delta_i^{-}\,\mathbb{Q})$$

where λ denotes the pullback of $\lambda \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ to $\overline{\mathcal{R}}_g$ and $\overline{\mathcal{S}}_g^+$, respectively.

Remark 2.1.22. As the Hodge bundle construction used to build $\lambda \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ commutes with base change, the class λ in $\overline{\mathcal{R}}_g$ or $\overline{\mathcal{S}}_g^+$ can likewise be defined by means of the Hodge bundle associated to each of these spaces.

Remark 2.1.23. For the decompositions of $\mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$ and $\mathrm{Pic}(\overline{\mathcal{S}}_g^+)_{\mathbb{Q}}$ to hold, it is enough to see that $\mathrm{Pic}(\mathcal{R}_g)_{\mathbb{Q}}$ and $\mathrm{Pic}(\mathcal{S}_g^+)_{\mathbb{Q}}$ are infinite cyclic, as discussed in remark 2.1.14 for $\overline{\mathcal{M}}_g$. In the case of $\mathcal{R}_g$, we have finite maps

$$\mathcal{M}_g(2) \to \mathcal{R}_g \to \mathcal{M}_g, \qquad (C, \eta_1, \dots, \eta_g) \mapsto (C, \eta_1) \mapsto C$$

where $\mathcal{M}_g(2)$ is the moduli of curves with a level 2 structure, that is, a basis of the 2-torsion of their Jacobian. As a result, we get injective pullback maps

$$\mathrm{Pic}(\mathcal{M}_g)_{\mathbb{Q}} \hookrightarrow \mathrm{Pic}(\mathcal{R}_g)_{\mathbb{Q}} \hookrightarrow \mathrm{Pic}(\mathcal{M}_g(2))_{\mathbb{Q}}$$

Since Putman's work [Put12a, Put12b] shows that $\mathrm{Pic}(\mathcal{M}_g(2))_{\mathbb{Q}} \cong \mathbb{Q}$ for $g \geq 5$, it follows that $\mathrm{Pic}(\mathcal{R}_g)_{\mathbb{Q}} \cong \mathbb{Q}$ in this range. The corresponding result for $\mathcal{S}_g^+$ is due to Harer [Har93] for $g \geq 9$, and Putman [Put12a] for $g \geq 5$.

Knowing the generating classes of a rational Picard group opens the door to a simple, yet challenging question: how do we express important divisor classes in terms of these generators? One possible answer is offered by the technique of intersecting each divisor with specific families of *test curves*.

2.1.3 Test curves on the Prym moduli space

If we want to compute the class expansion of some divisor of $\overline{\mathcal{R}}_g$ in terms of the generating classes of $\mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$, we can take an empirical approach. Clearly, the intersection of any family of Prym curves with either the divisor or its class expansion must produce the same result. Consequently, choosing well-behaved families for which both of these intersections may be independently determined will turn out a series of linear relations between the class expansion coefficients. Once enough linearly independent linear relations are obtained, the coefficients (and thus the whole class expansion) can be extracted from them. Since we are, in a sense, testing the divisor against different families of curves, we often refer to these tools as *test curves*.

Let us take the most basic families of test curves on $\overline{\mathcal{M}}_g$ and examine ways of lifting them to $\overline{\mathcal{R}}_g$. In the following examples, we denote

$$\pi\colon \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g$$

instead of $\pi_{\mathcal{R}}$, as we will not work with spin curves here. However, descriptions of common test curves on $\overline{\mathcal{S}}_g$ can be found in [Far10] or [FV14].

Example 2.1.24 (reducible nodal curves)**.** For each integer $2 \le i \le g-1$, we fix general curves $C \in \mathcal{M}_i$ and $(D,q) \in \mathcal{M}_{g-i,1}$ and consider the test curve

$$\mathcal{C}^i = (C \times C) \cup_{\Delta_C \sim C\times\{q\}} (C \times D) \longrightarrow C$$

corresponding to the family of reducible nodal curves

$$\mathcal{C}^i \;\; \equiv \;\; \{C \cup_{y\sim q} D\}_{y\in C} \;\; \subset \;\; \Delta_i \;\; \subset \;\; \overline{\mathcal{M}}_g$$

Using the standard test curve techniques of [HM98] Chapter 3, we can see that the intersection numbers of $\mathcal{C}^i$ with the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$ given earlier in the section are described by the table:

	λ	δ_i	$\delta_{(j\neq i)}$
$\mathcal{C}^i$	0	$2-2i$	0

We now fix two Prym roots $\eta_C \in R_i(C)$, $\eta_D \in R_{g-i}(D)$ and lift $\mathcal{C}^i$ to test curves

F_i, G_i, $H_i \to C$, as follows[1]:

$$\begin{array}{rclcccc} F_i & \equiv & \{(C \cup_{y\sim q} D,\, (\eta_C, \mathcal{O}_D))\}_{y\in C} & \subset & \Delta_i^{\mathrm{n}} & \subset & \overline{\mathcal{R}}_g \\ G_i & \equiv & \{(C \cup_{y\sim q} D,\, (\mathcal{O}_C, \eta_D))\}_{y\in C} & \subset & \Delta_i^{\mathrm{t}} & \subset & \overline{\mathcal{R}}_g \\ H_i & \equiv & \{(C \cup_{y\sim q} D,\, (\eta_C, \eta_D))\}_{y\in C} & \subset & \Delta_i^{\mathrm{p}} & \subset & \overline{\mathcal{R}}_g \end{array}$$

Observe that $\pi_*(F_i) = \pi_*(G_i) = \pi_*(H_i) = \mathcal{C}^i$. Then

$$\begin{array}{rcccccc} F_i \cdot \delta_i^{\mathrm{n}} & = & F_i \cdot \pi^*\delta_i & = & \mathcal{C}^i \cdot \delta_i & = & 2-2i \\ G_i \cdot \delta_i^{\mathrm{t}} & = & G_i \cdot \pi^*\delta_i & = & \mathcal{C}^i \cdot \delta_i & = & 2-2i \\ H_i \cdot \delta_i^{\mathrm{p}} & = & H_i \cdot \pi^*\delta_i & = & \mathcal{C}^i \cdot \delta_i & = & 2-2i \end{array}$$

and all other intersection numbers are 0, which is collected in the table:

	λ	δ_0^{t}	δ_0^{p}	δ_0^{b}	δ_i^{n}	δ_i^{t}	δ_i^{p}	$\delta_{(j\neq i)}$
F_i	0	0	0	0	$2-2i$	0	0	0
G_i	0	0	0	0	0	$2-2i$	0	0
H_i	0	0	0	0	0	0	$2-2i$	0

Note the exception of $g = 2i$, where we have $F_i \cdot \delta_i^{\mathrm{n}} = G_i \cdot \delta_i^{\mathrm{n}} = 2-2i$.

Example 2.1.25 (elliptic tails)**.** We fix a general curve $(C,p) \in \mathcal{M}_{g-1,1}$ and a general pencil $f\colon \mathrm{Bl}_9(\mathbb{P}^2) \to \mathbb{P}^1$ of plane cubics, with fibers

$$\{E_\lambda = f^{-1}(\lambda)\}_{\lambda\in\mathbb{P}^1} \quad \subset \quad \overline{\mathcal{M}}_1$$

together with a section $\sigma\colon \mathbb{P}^1 \to \mathrm{Bl}_9(\mathbb{P}^2)$ induced by one of the basepoints. We may then glue the curve (C,p) to the pencil f along σ, thus producing a pencil of stable curves

$$\mathcal{C}^0 = (C \times \mathbb{P}_1) \cup_{\{p\}\times\mathbb{P}^1 \sim \sigma(\mathbb{P}^1)} \mathrm{Bl}_9(\mathbb{P}^2) \longrightarrow \mathbb{P}^1$$

which corresponds to

$$\mathcal{C}^0 \quad \equiv \quad \{C \cup_{p\sim\sigma(\lambda)} E_\lambda\}_{\lambda\in\mathbb{P}^1} \quad \subset \quad \Delta_1 \quad \subset \quad \overline{\mathcal{M}}_g$$

As in the previous example, [HM98] shows that the intersection numbers of the

[1] Recall the conventions $\Delta_i^{\mathrm{n}} = \Delta_{g-i}^{\mathrm{t}}$ and $\Delta_i^{\mathrm{p}} = \Delta_{g-i}^{\mathrm{p}}$ set in remark 2.1.21.

pencil $\mathcal{C}^0$ with the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$ are given by the table:

	λ	δ_0	δ_1	$\delta_{(j\geq 2)}$
$\mathcal{C}^0$	1	12	-1	0

If we now fix a Prym root $\eta_C \in R_{g-1}(C)$, then the degree 3 branched covering

$$\gamma_1 \colon \overline{\mathcal{R}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$$

allows us to lift $\mathcal{C}^0$ to test curves F_0, G_0, H_0, as follows:

$$\begin{array}{lcllll}
F_0 & \equiv & \{(C \cup_{p\sim\sigma(\lambda)} E_\lambda,\, (\eta_C, \mathcal{O}_{E_\lambda}))\}_{\lambda\in\mathbb{P}^1} & \subset & \Delta_1^{\mathrm{t}} & \subset \overline{\mathcal{R}}_g \\
G_0 & \equiv & \{(C \cup_{p\sim\sigma(\lambda)} E_\lambda,\, (\mathcal{O}_C, \eta_{E_\lambda}))\ /\ \eta_{E_\lambda} \in \gamma_1^{-1}(E_\lambda)\}_{\lambda\in\mathbb{P}^1} & \subset & \Delta_1^{\mathrm{n}} & \subset \overline{\mathcal{R}}_g \\
H_0 & \equiv & \{(C \cup_{p\sim\sigma(\lambda)} E_\lambda,\, (\eta_C, \eta_{E_\lambda}))\ /\ \eta_{E_\lambda} \in \gamma_1^{-1}(E_\lambda)\}_{\lambda\in\mathbb{P}^1} & \subset & \Delta_1^{\mathrm{p}} & \subset \overline{\mathcal{R}}_g
\end{array}$$

Observe that $\pi_*(F_0) = \mathcal{C}^0$ and $\pi_*(G_0) = \pi_*(H_0) = 3\,\mathcal{C}^0$, so in particular

$$\begin{array}{lclclcl}
F_0 \cdot \delta_1^{\mathrm{t}} & = & F_0 \cdot \pi^*\delta_1 & = & \mathcal{C}^0 \cdot \delta_1 & = & -1 \\
G_0 \cdot \delta_1^{\mathrm{n}} & = & G_0 \cdot \pi^*\delta_1 & = & 3\,\mathcal{C}^0 \cdot \delta_1 & = & -3 \\
H_0 \cdot \delta_1^{\mathrm{p}} & = & H_0 \cdot \pi^*\delta_1 & = & 3\,\mathcal{C}^0 \cdot \delta_1 & = & -3
\end{array}$$

Looking at the 12 points $\lambda_\infty \in \mathbb{P}^1$ that correspond to singular fibers of $\mathcal{C}^0$ and blowing up the node of the rational component $E_{\lambda_\infty} \in \Delta_0$, we see that, for F_0, the pullback of $\eta_{\lambda_\infty} = (\eta_C, \mathcal{O}_{E_{\lambda_\infty}})$ is $(\eta_C, \mathcal{O}_{\mathbb{P}^1})$, which is nontrivial. As discussed in example 2.1.17, this implies that $F_{0,\lambda_\infty} \in \Delta_0^{\mathrm{p}}$, hence

$$F_0 \cdot \delta_0^{\mathrm{p}} \;=\; F_0 \cdot \pi^*\delta_0 \;=\; \mathcal{C}^0 \cdot \delta_0 \;=\; 12$$

Furthermore, the covering $\gamma_1 \colon \overline{\mathcal{R}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$ is branched over E_{λ_∞}, and thus the fiber $\gamma_1^{-1}(E_{\lambda_\infty})$ consists of two elements: one lying in the ramification divisor of γ_1, which we denote by $\eta^{\mathrm{b}}_{E_{\lambda_\infty}}$, and one outside, which we denote by $\eta^{\mathrm{t}}_{E_{\lambda_\infty}}$. Then the pullback of $(\mathcal{O}_C, \eta^{\mathrm{t}}_{E_{\lambda_\infty}})$ is $(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}^1})$, that is, $(\mathcal{O}_C, \eta^{\mathrm{t}}_{E_{\lambda_\infty}}) \in \Delta_0^{\mathrm{t}}$, and we get

$$\begin{array}{lclcl}
G_0 \cdot \delta_0^{\mathrm{t}} & = & \mathcal{C}^0 \cdot \delta_0 & = & 12 \\
G_0 \cdot \delta_0^{\mathrm{b}} & = & \mathcal{C}^0 \cdot \delta_0 & = & 12
\end{array}$$

Finally, the pair $(\eta_C, \eta^{\mathrm{t}}_{E_{\lambda_\infty}})$ pulls back to the nontrivial pair $(\eta_C, \mathcal{O}_{\mathbb{P}^1})$, and so it belongs to Δ_0^{p}, yielding

$$\begin{array}{lclcl}
H_0 \cdot \delta_0^{\mathrm{p}} & = & \mathcal{C}^0 \cdot \delta_0 & = & 12 \\
H_0 \cdot \delta_0^{\mathrm{b}} & = & \mathcal{C}^0 \cdot \delta_0 & = & 12
\end{array}$$

All other intersection numbers are 0, except for $F_0 \cdot \lambda = 1$, $G_0 \cdot \lambda = H_0 \cdot \lambda = 3$. In summary, we obtain a table:

	λ	δ_0^{t}	δ_0^{p}	δ_0^{b}	δ_1^{n}	δ_1^{t}	δ_1^{p}	$\delta_{(j\geq 2)}$
F_0	1	0	12	0	0	-1	0	0
G_0	3	12	0	12	-3	0	0	0
H_0	3	0	12	12	0	0	-3	0

Note that the formulas

$$\begin{aligned} \pi_*(G_0)\cdot\delta_0 &= G_0\cdot\pi^*\delta_0 &= G_0\cdot(\delta_0^{\mathrm{t}} + 2\,\delta_0^{\mathrm{b}}) \\ \pi_*(H_0)\cdot\delta_0 &= H_0\cdot\pi^*\delta_0 &= H_0\cdot(\delta_0^{\mathrm{p}} + 2\,\delta_0^{\mathrm{b}}) \end{aligned}$$

both hold.

Example 2.1.26 (irreducible nodal curves)**.** In keeping with the notation used in example 2.1.17, we fix a general curve $(B,p) \in \mathcal{M}_{g-1,1}$ and consider the test curve obtained by gluing p to a varying point $y \in B$, namely

$$\mathcal{Y} = \mathrm{Bl}_{(p,p)}(B\times B)/(\Delta_B \sim B\times\{p\}) \longrightarrow B$$

This corresponds to a family

$$\mathcal{Y} \;\equiv\; \{B_{py}\}_{y\in B} \;\subset\; \Delta_0 \;\subset\; \overline{\mathcal{M}}_g$$

where B_{py} is an irreducible nodal curve for $y \neq p$ and B_{pp} is a copy of B with a *pigtail* attached to p, in the sense of [HM98] Section 3.C. Again, we can readily see that the intersection table of $\mathcal{Y}$ with the generators of $\mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$ is:

	λ	δ_0	δ_1	$\delta_{(j\geq 2)}$
$\mathcal{Y}$	0	$2-2g$	1	0

Pulling back $\mathcal{Y}$ by the map $\Delta_0^{\mathrm{t}} \to \Delta_0$, we lift it to a test curve Y_0 such that:

$$Y_0 \;\equiv\; \{(B_{py},\,\eta_y^{\mathrm{t}}) \,/\, \eta_y^{\mathrm{t}} \in \Delta_0^{\mathrm{t}}(B_{py})\}_{y\in B} \;\subset\; \Delta_0^{\mathrm{t}} \;\subset\; \overline{\mathcal{R}}_g$$

Since $\deg(\Delta_0^{\mathrm{t}}|\Delta_0) = 1$, we have $\pi_*(Y_0) = \mathcal{Y}$, hence

$$Y_0\cdot\delta_0^{\mathrm{t}} \;=\; Y_0\cdot\pi^*\delta_0 \;=\; \mathcal{Y}\cdot\delta_0 \;=\; 2-2g$$

In addition, the special fiber η_p^{t} lies in Δ_1^{n}, as it pulls back to the trivial bundle $(\mathcal{O}_B, \mathcal{O}_{\mathbb{P}^1})$ on the normalization $B\times\mathbb{P}^1$ of B_{pp}, and thus is trivial over B. Then the last non-zero intersection number standing is

$$Y_0 \cdot \delta_1^{\mathrm{n}} \quad = \quad Y_0 \cdot \pi^* \delta_1 \quad = \quad \mathcal{Y} \cdot \delta_1 \quad = \quad 1$$

and we get a table:

	λ	δ_0^{t}	δ_0^{p}	δ_0^{b}	δ_1^{n}	δ_1^{t}	δ_1^{p}	$\delta_{(j\geq 2)}$
Y_0	0	$2-2g$	0	0	1	0	0	0

Note that we could have also pulled back by $\Delta_0^{\mathrm{p}} \to \Delta_0$ or $\Delta_0^{\mathrm{b}} \to \Delta_0$.

2.2 Prym curves and vanishing theta-nulls

The interaction between Prym curves and theta characteristics leads to the definition of a divisor $\mathcal{P}_{\mathrm{null}}$ on $\mathcal{R}_g$ closely related to an important divisor of $\mathcal{S}_g^+$, the *theta-null divisor*. In this section, we introduce the divisor $\mathcal{P}_{\mathrm{null}}$ and set out on a journey to study its closure in $\overline{\mathcal{R}}_g$ by way of computing the divisor classes of its two irreducible components. Our first detour involves delving deeper into the aforementioned interaction; more specifically, into how the parity of a theta characteristic changes when it is tensored by a Prym root.

2.2.1 The divisor $\mathcal{P}_{\mathrm{null}}$ and its irreducible components

Recall definition 1.1.14, and let C be a smooth, integral curve of genus g.

Definition 2.2.1. An even theta characteristic θ on C with $h^0(C,\theta) \neq 0$ (that is, with $h^0(C,\theta) \geq 2$ and $h^0(C,\theta) \equiv 0 \mod 2$) is called a *vanishing theta-null*.

The terminology here may seem confusing, as vanishing theta-nulls are even theta characteristics with *non-vanishing* global sections. This is justified by the classical theory of theta functions, whose *Thetanullwert* vanishes only when the associated even theta characteristic is a vanishing theta-null; see [Bea13].

The locus of curves with a vanishing theta-null, namely

$$\Theta_{\mathrm{null}} \quad = \quad \{(C,\theta) \in \mathcal{S}_g^+ \ / \ h^0(C,\theta) \geq 2\} \quad = \quad \mathcal{S}_g^+ \cap \mathcal{W}^2_{g-1,\,g}$$

gives rise to the *theta-null divisor* Θ_{null} on $\mathcal{S}_g^+$, as well as its closure $\overline{\Theta}_{\mathrm{null}}$ in $\overline{\mathcal{S}}_g^+$. This divisor plays an important role in the study of the geometry of $\overline{\mathcal{S}}_g^+$, due to its intrinsic nature and geometric characterization: for example, a computation of the class of $\overline{\Theta}_{\mathrm{null}}$ allows [Far10] to prove that $\overline{\mathcal{S}}_g^+$ is of general type for $g \geq 9$, and of non-negative Kodaira dimension if $g = 8$.

The theta-null divisor can be pushed forward by $\pi_+ \colon \mathcal{S}_g^+ \to \mathcal{M}_g$ to obtain

$$\mathcal{M}_g^{\mathrm{null}} \quad = \quad \{C \in \mathcal{M}_g \ / \ \exists\, \theta \in S_g^+(C) \text{ with } h^0(C,\theta) \geq 2\} \quad \subset \quad \mathcal{M}_g$$

whose closure $\overline{\mathcal{M}}_g^{\mathrm{null}}$ in $\overline{\mathcal{M}}_g$ is described by [TiB88]. In turn, pulling back $\mathcal{M}_g^{\mathrm{null}}$ by $\pi_{\mathcal{R}} \colon \mathcal{R}_g \to \mathcal{M}_g$ results in a divisor

$$\mathcal{P}_{\mathrm{null}} \quad = \quad \{(C,\eta) \in \mathcal{R}_g \ / \ \exists\, \theta \in S_g^+(C) \text{ with } h^0(C,\theta) \geq 2\} \quad \subset \quad \mathcal{R}_g$$

As discussed in section 1.1, the line bundle $\theta \otimes \eta$ is again a theta characteristic, different from θ, which may therefore be even or odd. Moreover, [TiB87] shows that the projection $\Theta_{\mathrm{null}} \to \mathcal{M}_g^{\mathrm{null}}$ is generically finite of degree 1, hence we can construct a rational map

$$\mathcal{P}_{\mathrm{null}} \to \mathcal{S}_g = \mathcal{S}_g^+ \sqcup \mathcal{S}_g^-, \qquad (C,\eta) \mapsto (C, \theta \otimes \eta)$$

where $\theta \in \Theta_{\mathrm{null}}(C)$. Then, with the temporary notation

$$\bar{\theta} = \theta \otimes \eta \in S_g(C), \qquad \theta = \bar{\theta} \otimes \eta \in \Theta_{\mathrm{null}}(C)$$

we may rewrite the defining condition of $\mathcal{P}_{\mathrm{null}}$ as

$$\mathcal{P}_{\mathrm{null}} \quad = \quad \{(C,\eta) \in \mathcal{R}_g \ / \ \exists\, \bar{\theta} \in S_g(C) \text{ with } \bar{\theta} \otimes \eta \in \Theta_{\mathrm{null}}(C)\} \quad \subset \quad \mathcal{R}_g$$

and deduce that the parity of $\bar{\theta} = \theta \otimes \eta$ yields a decomposition

$$\mathcal{P}_{\mathrm{null}} \quad = \quad \mathcal{P}_{\mathrm{null}}^+ \sqcup \mathcal{P}_{\mathrm{null}}^-$$

Dropping the bar for the sake of simplicity, we get the following:

Definition 2.2.2. We refer to the divisor $\mathcal{P}_{\mathrm{null}}$ on $\mathcal{R}_g$ as the *Prym-null divisor*. Accordingly, its irreducible components $\mathcal{P}_{\mathrm{null}}^+$ and $\mathcal{P}_{\mathrm{null}}^-$, namely

$$\begin{aligned} \mathcal{P}_{\mathrm{null}}^+ \quad &= \quad \{(C,\eta) \in \mathcal{R}_g \ / \ \exists\, \theta \in S_g^+(C) \text{ with } \theta \otimes \eta \in \Theta_{\mathrm{null}}(C)\} \quad \subset \quad \mathcal{R}_g \\ \mathcal{P}_{\mathrm{null}}^- \quad &= \quad \{(C,\eta) \in \mathcal{R}_g \ / \ \exists\, \theta \in S_g^-(C) \text{ with } \theta \otimes \eta \in \Theta_{\mathrm{null}}(C)\} \quad \subset \quad \mathcal{R}_g \end{aligned}$$

with $\mathcal{P}_{\mathrm{null}} = \mathcal{P}_{\mathrm{null}}^+ + \mathcal{P}_{\mathrm{null}}^-$, are called the *even* and *odd Prym-null divisors*.

Remark 2.2.3. The irreducibility of $\mathcal{P}_{\mathrm{null}}^+$ and $\mathcal{P}_{\mathrm{null}}^-$ is derived from the analysis included in [TiB88] Section 2, when appropriately adapted to the even and odd Prym-null divisors. This argument has been explicitly realised by Rojas as part of his ongoing doctoral research, which contains an independent study of these divisors that effectively complements our work; see [Roj21] Section 4.

Since the Prym-null divisors are natural, geometric divisors on $\mathcal{R}_g$, our goal is to compute the class of their closures $\overline{\mathcal{P}}^+_{\mathrm{null}}$, $\overline{\mathcal{P}}^-_{\mathrm{null}}$ in $\overline{\mathcal{R}}_g$. Such a computation would build upon the findings of [TiB88] and [Far10], where the classes of both $\overline{\mathcal{M}}_g^{\mathrm{null}}$ and $\overline{\Theta}_{\mathrm{null}}$ are respectively expressed in terms of the generating classes of $\mathrm{Pic}(\overline{\mathcal{M}}_g)_{\mathbb{Q}}$ and $\mathrm{Pic}(\overline{\mathcal{S}}^+_g)_{\mathbb{Q}}$. In particular, write

$$\overline{\mu}_g^{\mathrm{null}} = \mathcal{O}_{\overline{\mathcal{M}}_g}(\overline{\mathcal{M}}_g^{\mathrm{null}}) \in \mathrm{Pic}(\overline{\mathcal{M}}_g), \qquad \overline{\vartheta}_{\mathrm{null}} = \mathcal{O}_{\overline{\mathcal{S}}^+_g}(\overline{\Theta}_{\mathrm{null}}) \in \mathrm{Pic}(\overline{\mathcal{S}}^+_g)$$

for the aforementioned classes, and consider the notation $\lambda, \delta_i, \delta_i^{\mathrm{x}}$ introduced in the previous section[2]. Then [TiB88] and [Far10] provide formulas

$$\begin{aligned} \overline{\mu}_g^{\mathrm{null}} &= 2^{g-3}\Big((2^g+1)\,\lambda - 2^{g-3}\,\delta_0 - \sum_{i=1}^{\lfloor g/2\rfloor}(2^i-1)(2^{g-i}-1)\,\delta_i\Big) \\ \overline{\vartheta}_{\mathrm{null}} &= \frac{1}{4}\lambda - \frac{1}{16}\delta_0^{\mathrm{n}} - \sum_{i=1}^{\lfloor g/2\rfloor}\delta_i^- \end{aligned}$$

the latter of which implies the former, as the class $[\overline{\mathcal{M}}_g^{\mathrm{null}}]$ can also be obtained by pushing forward the class $[\overline{\Theta}_{\mathrm{null}}]$ by the coarse moduli map $\overline{S}^+_g \to \overline{M}_g$.

Remark 2.2.4. The expression for $\overline{\vartheta}_{\mathrm{null}}$ appears different than the original one from [Far10] Th. 0.2, since it is stated in the language of moduli stacks. When we translate it back into the language of coarse moduli spaces, it becomes clear that both formulas agree:

$$[\overline{\Theta}_{\mathrm{null}}] = \frac{1}{4}\lambda - \frac{1}{16}[\Delta_0^{\mathrm{n}}] - \frac{1}{4}[\Delta_1^-] - \frac{1}{2}\sum_{i=2}^{\lfloor g/2\rfloor}[\Delta_i^-] \in \mathrm{Pic}(\overline{S}^+_g)_{\mathbb{Q}}$$

Observe that the class $[\Delta_1^-]$ in *loc. cit.* is implicitly divided by 2 to account for the elliptic tail automorphisms, which we have made explicit here.

Let us write the classes of $\overline{\mathcal{P}}_{\mathrm{null}}$, $\overline{\mathcal{P}}^+_{\mathrm{null}}$ and $\overline{\mathcal{P}}^-_{\mathrm{null}}$ as

$$\begin{aligned} &\varrho_{\mathrm{null}} = \mathcal{O}_{\overline{\mathcal{R}}_g}(\overline{\mathcal{P}}_{\mathrm{null}}) \in \mathrm{Pic}(\overline{\mathcal{R}}_g), \qquad \varrho_{\mathrm{null}} = \varrho^+_{\mathrm{null}} + \varrho^-_{\mathrm{null}} \\ &\varrho^+_{\mathrm{null}} = \mathcal{O}_{\overline{\mathcal{R}}_g}(\overline{\mathcal{P}}^+_{\mathrm{null}}) \in \mathrm{Pic}(\overline{\mathcal{R}}_g) \\ &\varrho^-_{\mathrm{null}} = \mathcal{O}_{\overline{\mathcal{R}}_g}(\overline{\mathcal{P}}^-_{\mathrm{null}}) \in \mathrm{Pic}(\overline{\mathcal{R}}_g) \end{aligned}$$

and recall the notation $\delta_0^{\mathrm{t}}, \delta_0^{\mathrm{p}}, \delta_0^{\mathrm{b}}, \delta_i^{\mathrm{n}}, \delta_i^{\mathrm{t}}, \delta_i^{\mathrm{p}}$ from examples 2.1.16 and 2.1.17. The

[2] See proposition 2.1.11, theorem 2.1.15, and examples 2.1.19 and 2.1.20.

sum $\varrho_{\rm null}$ can be directly computed as the pullback of $\overline{\mu}_g^{\rm null}$ by the map

$$\pi_{\mathcal{R}}\colon \overline{\mathcal{R}}_g \to \overline{\mathcal{M}}_g, \qquad \begin{array}{rcl} \pi_{\mathcal{R}}^*(\lambda) & = & \lambda \\ \pi_{\mathcal{R}}^*(\delta_0) & = & \delta_0^{\rm t} + \delta_0^{\rm p} + 2\,\delta_0^{\rm b} \\ \pi_{\mathcal{R}}^*(\delta_i) & = & \delta_i^{\rm n} + \delta_i^{\rm t} + \delta_i^{\rm p} \end{array}$$

with $1 \leq i < g/2$, and moreover $\pi_{\mathcal{R}}^*(\delta_{g/2}) = \delta_{g/2}^{\rm n} + \delta_{g/2}^{\rm p}$ for even g.

Proposition 2.2.5. *The class of $\overline{\mathcal{P}}_{\rm null}$ in $\mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$ is given by*

$$\begin{array}{rcl} \varrho_{\rm null} & = & 2^{g-3}\Big((2^g+1)\,\lambda \;-\; 2^{g-3}\,(\delta_0^{\rm t} + \delta_0^{\rm p} + 2\,\delta_0^{\rm b}) \\ & & \qquad -\; \displaystyle\sum_{i=1}^{k}(2^i-1)(2^{g-i}-1)(\delta_i^{\rm n} + \delta_i^{\rm t} + \delta_i^{\rm p}) \\ & & \qquad -\; \psi(g)\cdot(2^{g/2}-1)^2(\delta_{g/2}^{\rm n} + \delta_{g/2}^{\rm p})\Big) \end{array}$$

where the upper bound k and the parity-checking function $\psi(g)$, defined as

$$\begin{array}{rcccl} k & = & \lceil g/2 \rceil - 1 & = & \begin{cases} \lfloor g/2 \rfloor & \text{if } g \text{ odd} \\ \lfloor g/2 \rfloor - 1 & \text{if } g \text{ even} \end{cases} \\ \psi(g) & = & \dfrac{1+(-1)^g}{2} & = & \begin{cases} 0 & \text{if } g \text{ odd} \\ 1 & \text{if } g \text{ even} \end{cases} \end{array}$$

account for the slight variation in pullback that occurs when $g = 2i$.

Proof. Follows from $\pi_{\mathcal{R}}^*(\overline{\mu}_g^{\rm null}) = \varrho_{\rm null}$ and the formulas above. □

Remark 2.2.6. Once the classes $\varrho_{\rm null}^+$ and $\varrho_{\rm null}^-$ are computed, proposition 2.2.5 offers a quick double-check of their accuracy, by virtue of $\varrho_{\rm null}^+ + \varrho_{\rm null}^- = \varrho_{\rm null}$.

Remark 2.2.7. With the notation of proposition 2.2.5, we may write

$$\begin{aligned}
\varrho^{+}_{\text{null}} &= \lambda^{+}\cdot\lambda - \left(\delta_0^{\text{t},+}\cdot\delta_0^{\text{t}} + \delta_0^{\text{p},+}\cdot\delta_0^{\text{p}} + \delta_0^{\text{b},+}\cdot\delta_0^{\text{b}}\right)\\
&\quad - \sum_{i=1}^{k}(\delta_i^{\text{n},+}\cdot\delta_i^{\text{n}} + \delta_i^{\text{t},+}\cdot\delta_i^{\text{t}} + \delta_i^{\text{p},+}\cdot\delta_i^{\text{p}})\\
&\quad - \psi(g)\cdot(\delta_{g/2}^{\text{n},+}\cdot\delta_{g/2}^{\text{n}} + \delta_{g/2}^{\text{p},+}\cdot\delta_{g/2}^{\text{p}})\\
\varrho^{-}_{\text{null}} &= \lambda^{-}\cdot\lambda - \left(\delta_0^{\text{t},-}\cdot\delta_0^{\text{t}} + \delta_0^{\text{p},-}\cdot\delta_0^{\text{p}} + \delta_0^{\text{b},-}\cdot\delta_0^{\text{b}}\right)\\
&\quad - \sum_{i=1}^{k}(\delta_i^{\text{n},-}\cdot\delta_i^{\text{n}} + \delta_i^{\text{t},-}\cdot\delta_i^{\text{t}} + \delta_i^{\text{p},-}\cdot\delta_i^{\text{p}})\\
&\quad - \psi(g)\cdot(\delta_{g/2}^{\text{n},-}\cdot\delta_{g/2}^{\text{n}} + \delta_{g/2}^{\text{p},-}\cdot\delta_{g/2}^{\text{p}})
\end{aligned}$$

and subsequently aim our efforts at determining the rational coefficients

$$\lambda^{+},\ \delta_0^{\text{t},+},\ \delta_0^{\text{p},+},\ \delta_0^{\text{b},+},\ \delta_i^{\text{n},+},\ \delta_i^{\text{t},+},\ \delta_i^{\text{p},+}\ \in\ \mathbb{Q}\qquad (\text{resp. }-)$$

for $1\leq i\leq \lfloor g/2\rfloor$. To that end, the assortment of test curves introduced earlier in the chapter will prove to be most useful.

Since the dividing line between $\mathcal{P}^{+}_{\text{null}}$ and $\mathcal{P}^{-}_{\text{null}}$ hinges upon how the parity of a vanishing theta-null is affected by its interaction with a fixed Prym root, this phenomenon must be explored before any further analysis is pursued.

2.2.2 Parity change under tensoring by a Prym root

Our objective now is simple: given a Prym root η on a curve C, we wish to understand how the parity of a theta characteristic θ of C relates to the parity of the tensor product $\theta\otimes\eta$. As it turns out, this amounts to a standard count, since the behaviour is homogeneous across all curves of the same type.

Definition 2.2.8. Let (C,η) be a Prym pair of genus g. Consider the subsets

$$\begin{aligned}
S_\eta^{+,+}(C) &= \{\theta\in S_g^{+}(C)\ /\ \theta\otimes\eta\in S_g^{+}(C)\} \subset S_g^{+}(C)\\
S_\eta^{+,-}(C) &= \{\theta\in S_g^{+}(C)\ /\ \theta\otimes\eta\in S_g^{-}(C)\} \subset S_g^{+}(C)\\
S_\eta^{-,+}(C) &= \{\theta\in S_g^{-}(C)\ /\ \theta\otimes\eta\in S_g^{+}(C)\} \subset S_g^{-}(C)\\
S_\eta^{-,-}(C) &= \{\theta\in S_g^{-}(C)\ /\ \theta\otimes\eta\in S_g^{-}(C)\} \subset S_g^{-}(C)
\end{aligned}$$

into which $S_g(C)$ decomposes as a disjoint union.

Remark 2.2.9. Note that $S_\eta^{+,-}(C) \cong S_\eta^{-,+}(C)$, $\theta \mapsto \theta \otimes \eta$. This leaves us with three distinct sets that we want to study.

As discussed in remark 1.1.2, for any smooth, integral, genus g curve C, the group $J_2(C)$ acts on $S_g(C)$ by means of the map

$$J_2(C) \times S_g(C) \longrightarrow S_g(C), \qquad (\eta, \theta) \longmapsto \theta \otimes \eta$$

whose associated difference map can be written as

$$\mathrm{diff}\colon\ S_g(C) \times S_g(C) \longrightarrow J_2(C), \qquad (\theta_1, \theta_2) \longmapsto \theta_1 \otimes \theta_2^{-1}$$

If we remove the diagonal $\Delta = \mathrm{diff}^{-1}(\mathcal{O}_X)$, we get a map

$$\mathrm{diff}_{\neq}\colon\ S_g(C) \times S_g(C) - \Delta \longrightarrow R_g(C)$$

whose fibers, of order 2^{2g}, reflect how many ways there are of writing a Prym root η as a difference of theta characteristics $\theta_1 \otimes \theta_2^{-1}$, that is, with $\theta_1 = \theta_2 \otimes \eta$. Since we aim to keep track of the parity of $\theta = \theta_2$ and $\theta \otimes \eta = \theta_1$, we just need to consider the restrictions

$$\begin{array}{llcl}
\mathrm{diff}_+\colon & S_g^+(C) \times S_g^+(C) - \Delta & \longrightarrow & R_g(C) \\
\mathrm{diff}_-\colon & S_g^-(C) \times S_g^-(C) - \Delta & \longrightarrow & R_g(C) \\
\mathrm{diff}_\pm\colon & S_g^+(C) \times S_g^-(C) & \longrightarrow & R_g(C)
\end{array}$$

Finally, we may recall from definitions 1.1.9 and 1.1.16 that

$$\#R_g(C) = 2^{2g} - 1, \qquad \begin{array}{l} \#S_g^+(C) = 2^{g-1}(2^g + 1) \\ \#S_g^-(C) = 2^{g-1}(2^g - 1) \end{array}$$

which enables us to count the fibers of these difference maps.

Lemma 2.2.10. *With the previous notation, it holds that*

$$\begin{array}{rcl}
\#\mathrm{diff}_+^{-1}(\eta) & = & 2^{g-1}(2^{g-1} + 1) \\
\#\mathrm{diff}_-^{-1}(\eta) & = & 2^{g-1}(2^{g-1} - 1) \\
\#\mathrm{diff}_\pm^{-1}(\eta) & = & 2^{2g-2}
\end{array}$$

for any Prym pair (C, η) of genus g.

Proof. These numbers follow from the computation:

$$\begin{aligned}
\#\mathrm{diff}_{+}^{-1}(\eta) &= \frac{\#S_g^{+}(C)\cdot(\#S_g^{+}(C)-1)}{\#R_g(C)} = 2^{g-1}(2^{g-1}+1)\\
\#\mathrm{diff}_{-}^{-1}(\eta) &= \frac{\#S_g^{-}(C)\cdot(\#S_g^{-}(C)-1)}{\#R_g(C)} = 2^{g-1}(2^{g-1}-1)\\
\#\mathrm{diff}_{\pm}^{-1}(\eta) &= \frac{\#S_g^{+}(C)\cdot\#S_g^{-}(C)}{\#R_g(C)} = 2^{2g-2}
\end{aligned}$$

Notice that this process depends only on the genus g of the curve. □

Definition 2.2.11. We denote

$$\begin{aligned}
\mathrm{N}_g^{+} &= \#\mathrm{diff}_{+}^{-1}(\eta) = 2^{g-1}(2^{g-1}+1)\\
\mathrm{N}_g^{-} &= \#\mathrm{diff}_{-}^{-1}(\eta) = 2^{g-1}(2^{g-1}-1)\\
\mathrm{N}_g^{\pm} &= \#\mathrm{diff}_{\pm}^{-1}(\eta) = 2^{2g-2}
\end{aligned}$$

for any positive integer $g \in \mathbb{Z}^{+}$.

The numbers N_g^{+}, N_g^{-} and $\mathrm{N}_g^{\pm}$ clearly capture the parity-changing behaviour that we are interested in, revealing the order of each set in definition 2.2.8.

Proposition 2.2.12. *Let (C, η) be a Prym pair of genus g. Under the map*

$$S_g(C) \to S_g(C), \qquad \theta \mapsto \theta \otimes \eta$$

that is, when tensoring by η, there are:

(i) $\mathrm{N}_g^{+} = 2^{g-1}(2^{g-1}+1)$ *even theta characteristics on C that remain even.*

(ii) $\mathrm{N}_g^{-} = 2^{g-1}(2^{g-1}-1)$ *odd theta characteristics on C that remain odd.*

(iii) $\mathrm{N}_g^{\pm} = 2^{2g-2}$ *even theta characteristics on C that become odd.*

(iv) $\mathrm{N}_g^{\pm} = 2^{2g-2}$ *odd theta characteristics on C that become even.*

In particular, $\#S_g(C) = \mathrm{N}_g^{+} + \mathrm{N}_g^{-} + 2\,\mathrm{N}_g^{\pm} = 2^{2g}$.

Proof. As suggested above, we have

$$\begin{aligned}
\#S_\eta^{+,+}(C) &= \#\{\theta \in S_g^{+}(C) \;/\; \theta\otimes\eta \in S_g^{+}(C)\}\\
&= \#\{(\theta_1,\theta_2) \in S_g^{+}(C)\times S_g^{+}(C) \;/\; \theta_1 = \theta_2\otimes\eta \in S_g^{+}(C)\}\\
&= \#\{(\theta_1,\theta_2) \in S_g^{+}(C)\times S_g^{+}(C) - \Delta \;/\; \theta_1\otimes\theta_2^{-1} = \eta\}\\
&= \#\mathrm{diff}_{+}^{-1}(\eta) = \mathrm{N}_g^{+}
\end{aligned}$$

and similarly $\#S_\eta^{-,-}(C) = \mathrm{N}_g^{-}$ and $\#S_\eta^{+,-}(C) = \#S_\eta^{-,+}(C) = \mathrm{N}_g^{\pm}$. □

Even though the smooth case has been addressed, we are not quite done. It will also be helpful for our future endeavours to examine the first singular cases from examples 2.1.17 and 2.1.20, namely $\Delta_0^{\mathrm{t}} \subset \overline{\mathcal{R}}_g$ and $\Delta_0^{\mathrm{n}} \subset \overline{\mathcal{S}}_g$, and determine how Prym and spin curves in these divisors interact with one another.

Let $(B, p, q) \in \mathcal{M}_{g-1,2}$ and take the irreducible nodal curve $X = B_{pq} \in \overline{\mathcal{M}}_g$ obtained from B by gluing the points p and q, with normalization $\nu\colon B \to B_{pq}$. The dualizing bundle ω_X is the subbundle of $\nu_*(\omega_B(p+q))$ fulfilling the residue condition, that is, such that the following diagram commutes:

$$\begin{tikzcd}
 & \omega_X \arrow[dl, "\mathrm{res}_p"'] \arrow[dr, "\mathrm{res}_q"] \arrow[d, phantom, "\circlearrowleft"] & \\
\kappa(p) \arrow[rr, "\cong", "-1"'] & {} & \kappa(q)
\end{tikzcd}$$

In this particular case, we actually have $H^0(X, \omega_X) = H^0(B, \omega_B(p+q))$, since

$$h^0(\omega_B(p+q)) \;=\; 2g-2-(g-1)+1 \;=\; g \;=\; h^0(\omega_X)$$

by Riemann-Roch and duality. As mentioned in example 2.1.20, a spin curve

$$(X, \theta, \alpha) \in \overline{\mathcal{S}}_g, \qquad \alpha\colon \theta^{\otimes 2} \cong \omega_X, \qquad (\nu^*\theta)^{\otimes 2} \cong \omega_B(p+q)$$

is given by a root $\theta_B \in \sqrt{\omega_B(p+q)}$ and a suitable gluing $\varphi\colon \theta_B|_p \cong \theta_B|_q$, which by the above discussion is bound to a condition $\varphi^{\otimes 2} \equiv -1$ corresponding to the commutativity of the following diagram:

$$\begin{tikzcd}
\omega_B(p+q)|_p \arrow[r, "\varphi^{\otimes 2}", "\cong"'] \arrow[d, "\mathrm{res}_p"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \omega_B(p+q)|_q \arrow[d, "\cong"', "\mathrm{res}_q"] \\
\kappa(p) \arrow[r, "\cong", "-1"'] & \kappa(q)
\end{tikzcd}$$

Specifically, consider the canonical isomorphism ψ induced by the diagram

$$\begin{tikzcd}
 & & & \theta_B|_p \arrow[dr] \arrow[dd, "\cong"', "\psi"] & \\
0 \arrow[r] & \theta_B(-p-q) \arrow[r] & \theta_B \arrow[ur] \arrow[dr] \arrow[r, phantom, "\circlearrowleft"] & {} & 0 \\
 & & & \theta_B|_q \arrow[ur] &
\end{tikzcd}$$

where $\theta_B|_p$ and $\theta_B|_q$ are expressed as cokernels of the same map. Let us give an explicit description of ψ. On the one hand, Riemann-Roch and duality yield

$$h^0(\theta_B) - h^0(\theta_B(-p-q)) \;=\; g-1-(g-1)+1 \;=\; 1$$

so we can write $\theta_B|_p = \langle \sigma(p) \rangle$ and $\theta_B|_q = \langle \sigma(q) \rangle$ for any section

$$\sigma \ \in \ H^0(B, \theta_B) - H^0(B, \theta_B(-p-q))$$

and see that ψ is, by definition, the isomorphism

$$\psi \colon \theta_B|_p \cong \theta_B|_q \,, \qquad \sigma(p) \mapsto \sigma(q)$$

On the other hand, we have $\sigma^{\otimes 2} \in H^0(B, \omega_B(p+q)) = H^0(X, \omega_X)$, hence

$$\mathrm{res}_p(\sigma^{\otimes 2}) + \mathrm{res}_q(\sigma^{\otimes 2}) = 0$$

and it is clear that $\psi^{\otimes 2} \equiv -1$, in the sense of:

$$\begin{tikzcd}
\omega_B(p+q)|_p \arrow[r, "\psi^{\otimes 2}", "\cong"'] \arrow[d, "\mathrm{res}_p"', "\cong"] \arrow[dr, phantom, "\circlearrowleft"] & \omega_B(p+q)|_q \arrow[d, "\cong"', "\mathrm{res}_q"] \\
\kappa(p) \arrow[r, "\cong", "-1"'] & \kappa(q)
\end{tikzcd}
\qquad
\begin{tikzcd}
\sigma^{\otimes 2}(p) \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, phantom, "\circlearrowleft"] & \sigma^{\otimes 2}(q) \arrow[d, mapsto] \\
\mathrm{res}_p(\sigma^{\otimes 2}) \arrow[r, mapsto] & \mathrm{res}_q(\sigma^{\otimes 2})
\end{tikzcd}$$

If we also consider the opposite isomorphism

$$-\psi \colon \theta_B|_p \cong \theta_B|_q \,, \qquad \sigma(p) \mapsto -\sigma(q) \,, \qquad (-\psi)^{\otimes 2} \equiv -1$$

then ψ and $-\psi$ are the only possible ways of gluing p and q to make θ_B into a square root of ω_X. The resulting bundles on X, which we denote by

$$(\theta_B, \psi), \ (\theta_B, -\psi) \ \in \ \sqrt{\omega_X}$$

are thus the two elements in the fiber $(\nu^*)^{-1}(\theta_B)$ of the double cover

$$\nu^* \colon \sqrt{\omega_X} \longrightarrow \sqrt{\omega_B(p+q)}$$

Furthermore, observe that the 1-dimensional space of sections $\langle \sigma \rangle \subset H^0(B, \theta_B)$ is preserved under the gluing ψ, but lost under the gluing $-\psi$. As a result, the dimension of the glued global sections is given by

$$\begin{aligned} h^0(X, (\theta_B, \psi)) &= h^0(B, \theta_B) \\ h^0(X, (\theta_B, -\psi)) &= h^0(B, \theta_B) - 1 \end{aligned}$$

so that (θ_B, ψ) and $(\theta_B, -\psi)$ always have different parity.

Finally, let η^{t} be the single Prym root of X lying in the divisor $\Delta_0^{\mathrm{t}} \subset \overline{\mathcal{R}}_g$, as defined in example 2.1.17. In other words, we have

$$\eta^{\mathrm{t}} \neq \mathcal{O}_X, \qquad \nu^* \eta^{\mathrm{t}} = \mathcal{O}_B$$

It follows that tensoring by η^{t} permutes the elements of $(\nu^*)^{-1}(\theta_B)$, since

$$(\theta_B, \psi) \otimes \eta^{\mathrm{t}} \neq (\theta_B, \psi), \qquad \nu^*((\theta_B, \psi) \otimes \eta^{\mathrm{t}}) = \theta_B$$

and similarly for $(\theta_B, -\psi)$. This corresponds to a change in parity:

Proposition 2.2.13. *With the notation above, let $(X, \theta, \alpha) \in \overline{\mathcal{S}}_g^+$ be a general point of Δ_0^{n}. Then tensoring by $(X, \eta^{\mathrm{t}}, \beta) \in \Delta_0^{\mathrm{t}} \subset \overline{\mathcal{R}}_g$ induces a new spin curve $(X, \theta \otimes \eta^{\mathrm{t}}, \alpha \otimes \beta)$ in $\Delta_0^{\mathrm{n}} \subset \overline{\mathcal{S}}_g^-$, of opposite parity (resp. $\overline{\mathcal{S}}_g^-$, $\overline{\mathcal{S}}_g^+$).*

Propositions 2.2.12 and 2.2.13 effectively supply the tools required to study the intersection of the Prym-null divisors with different test curves, paving the way for a better insight into their geometry.

2.3 The geometry of the Prym-null divisors

Our set-up is essentially ready: if we intersect the test curves from section 2.1 with the Prym-null divisors $\overline{\mathcal{P}}_{\mathrm{null}}^+$ and $\overline{\mathcal{P}}_{\mathrm{null}}^-$ from section 2.2, then we can derive the class expansions of $\varrho_{\mathrm{null}}^+$ and $\varrho_{\mathrm{null}}^-$ from the resulting linear system. In order to describe these intersections, we mainly take advantage of the theory of limit linear series on curves of compact type, as developed by [EH86].

2.3.1 Over reducible nodal curves

Recall the test curves from example 2.1.24, that is

$$\begin{array}{rclcccc} F_i & \equiv & \{(C \cup_{y\sim q} D,\ (\eta_C, \mathcal{O}_D))\}_{y\in C} & \subset & \Delta_i^{\mathrm{n}} & \subset & \overline{\mathcal{R}}_g \\ G_i & \equiv & \{(C \cup_{y\sim q} D,\ (\mathcal{O}_C, \eta_D))\}_{y\in C} & \subset & \Delta_i^{\mathrm{t}} & \subset & \overline{\mathcal{R}}_g \\ H_i & \equiv & \{(C \cup_{y\sim q} D,\ (\eta_C, \eta_D))\}_{y\in C} & \subset & \Delta_i^{\mathrm{p}} & \subset & \overline{\mathcal{R}}_g \end{array}$$

$$\begin{array}{rl} \text{with} & C \in \mathcal{M}_i,\ (D, q) \in \mathcal{M}_{g-i,1}\ \text{ general,} \\ \text{and} & \eta_C \in R_i(C),\ \eta_D \in R_{g-i}(D)\ \text{ arbitrary.} \end{array}$$

If $g \neq 2i$, their intersection table is:

	λ	δ_0^{t}	δ_0^{p}	δ_0^{b}	δ_i^{n}	δ_i^{t}	δ_i^{p}	$\delta_{(j\neq i)}$
F_i	0	0	0	0	$2-2i$	0	0	0
G_i	0	0	0	0	0	$2-2i$	0	0
H_i	0	0	0	0	0	0	$2-2i$	0

If $g = 2i$, we have $F_i \cdot \delta_i^{\mathrm{n}} = G_i \cdot \delta_i^{\mathrm{n}} = 2 - 2i$ instead.

Out of all the possible intersections, we introduce $F_i \cap \overline{\mathcal{P}}^+_{\mathrm{null}}$ as a model case, indicating in detail the techniques and guidelines that will apply to most of the other test curves in this section.

Remark 2.3.1. If a stable Prym curve

$$F_{i,\,y} \quad = \quad (C \cup_{y\sim q} D,\, (\eta_C, \mathcal{O}_D)) \quad \in \quad F_i$$

lies in $\overline{\mathcal{P}}^+_{\mathrm{null}}$, then it can be expressed as the limit of a smooth family in $\mathcal{P}^+_{\mathrm{null}}$, in the following sense. First, let us write

$$\begin{aligned} (X_y, \eta_y) \quad &= \quad (C \cup_{y\sim 0} E \cup_{q\sim\infty} D,\, (\eta_C, \mathcal{O}_E, \mathcal{O}_D)) \\ \mathrm{st}(X_y, \eta_y) \quad &= \quad (C \cup_{y\sim q} D,\, (\eta_C, \mathcal{O}_D)) \quad = \quad F_{i,\,y} \quad \in \quad \overline{\mathcal{P}}^+_{\mathrm{null}} \end{aligned}$$

to account for the exceptional component $E \cong \mathbb{P}^1$ which appears when working with stable spin structures on $C \cup_{y\sim q} D$. Then there exist families

$$\begin{aligned} f\colon \mathcal{X} \to \mathrm{Spec}(R) = \{\xi,\, p_0\} &\qquad \text{of quasistable curves,} \\ (\mathrm{st}(f),\, \eta,\, \beta) \in \overline{\mathcal{R}}_g &\qquad \text{of stable Prym curves, and} \\ (f,\, \theta,\, \alpha) \in \overline{\mathcal{S}}^+_g &\qquad \text{of stable (even) spin curves,} \end{aligned}$$

such that:

(i) $\mathcal{X}$ is a smooth surface.

(ii) R is a discrete valuation ring with maximal ideal $\mathfrak{m}$, whose spectrum is composed of a special point $p_0 \equiv \mathfrak{m}$ and a generic point $\xi \equiv (0)$.

(iii) On the special fiber $\mathcal{X}_0 = f^{-1}(p_0)$, it holds that

$$(\mathcal{X}_0, \eta|_{\mathcal{X}_0}) = (X_y, \eta_y), \qquad\qquad \mathrm{st}(\mathcal{X}_0, \eta|_{\mathcal{X}_0}) = F_{i,\,y} \in \overline{\mathcal{P}}^+_{\mathrm{null}}$$

(iv) On the generic fiber $\mathcal{X}_\xi = f^{-1}(\xi) = \mathrm{st}(f)^{-1}(\xi)$, it holds that

$$(\mathcal{X}_\xi, \eta_\xi) \in \mathcal{P}^+_{\mathrm{null}}, \qquad\qquad (\mathcal{X}_\xi, \theta_\xi \otimes \eta_\xi) \in \Theta_{\mathrm{null}}$$

or equivalently $(\theta_\xi \otimes \eta_\xi)^{\otimes 2} \simeq \omega_{\mathcal{X}_\xi}$ and $h^0(\mathcal{X}_\xi, \theta_\xi \otimes \eta_\xi) \geq 2,\ \equiv 0 \mod 2$.

If we recall example 2.1.19 and use the notation

$$\theta|_{\mathcal{X}_0} \quad = \quad \theta^+_y \quad = \quad (\theta_C, \mathcal{O}_E(1), \theta_D) \quad \in \quad \overline{\mathcal{S}}^+_g(X_y)$$

then it is clear that θ_C and θ_D must have the same parity. In addition, since C and D are general, the dimension of the global sections of θ_C and θ_D is at most one, and thus we get

$$h^0(C,\theta_C) \quad = \quad h^0(D,\theta_D) \quad \in \quad \{0,1\}$$

Observe that if $F_{i,y}$ were to lie in $\overline{\mathcal{P}}^-_{\mathrm{null}}$, then $\theta|_{\mathcal{X}_0} = \theta^-_y$ would be odd instead of even and these theta characteristics would have opposite parity.

As described in [EH86] Section 2, the data given in remark 2.3.1 produces a limit linear series of type $\mathfrak{g}^1_{g-1}$ on $C \cup_{y\sim q} D$, namely

$$\ell \quad = \quad \Big(\ell_C = (L_C, V_C),\ \ell_D = (L_D, V_D)\Big) \quad \in \quad G^1_{g-1}(C) \times G^1_{g-1}(D)$$

where the line bundles L_C and L_D are obtained by looking at the equality

$$\theta|_{\mathcal{X}_0} \otimes \eta|_{\mathcal{X}_0} \quad = \quad \theta^+_y \otimes \eta_y \quad = \quad (\theta_C \otimes \eta_C, \mathcal{O}_E(1), \theta_D)$$

and twisting with y and q to adjust the degrees to $g-1$, so that

$$\begin{cases} L_C & = & \theta_C \otimes \eta_C\left((g-i)y\right) \\ L_D & = & \theta_D\left(iq\right) \end{cases}$$

Since $\theta_\xi \otimes \eta_\xi \in \Theta_{\mathrm{null}}(\mathcal{X}_\xi)$ is even and parity is constant in families, we get

$$h^0(\theta_C \otimes \eta_C) + h^0(\theta_D) \quad = \quad h^0(\theta^+_y \otimes \eta_y) \quad \equiv \quad h^0(\theta_\xi \otimes \eta_\xi) \quad \equiv \quad 0 \quad \mod 2$$

In particular, $\theta_C \otimes \eta_C$ and θ_D must have the same parity, and the dimension of their global sections is again either 0 or 1 due to generality. This results in two distinct possibilities for the $\overline{\mathcal{P}}^+_{\mathrm{null}}$ setting, and two more for the $\overline{\mathcal{P}}^-_{\mathrm{null}}$ one:

	$h^0(\theta_C)$	$h^0(\theta_D)$	$h^0(\theta_C \otimes \eta_C)$		
$\overline{\mathcal{P}}^+_{\mathrm{null}}$	0	0	0	$\rightsquigarrow$	$(\mathrm{F}, +, 0)$
	1	1	1	$\rightsquigarrow$	$(\mathrm{F}, +, 1)$
$\overline{\mathcal{P}}^-_{\mathrm{null}}$	1	0	0	$\rightsquigarrow$	$(\mathrm{F}, -, 0)$
	0	1	1	$\rightsquigarrow$	$(\mathrm{F}, -, 1)$

In order to study each of these cases, we first need to recall a basic property of the vanishing sequence of a linear series.

Remark 2.3.2. Given a linear series $(L,V) \in G^r_d(C)$ on a smooth curve C and a point $p \in C$, we can find an ordered basis

$$V = \langle s_0, \dots, s_r \rangle \subset H^0(C, L)$$

such that, if we write $a_i(p) = \mathrm{ord}_p(s_i)$ for all $i \in \{0, \dots, r\}$, then

$$a_0(p) < \ldots < a_r(p)$$

is the vanishing sequence of (L, V) at p. Taking any $b \in \mathbb{Z}^+$ and observing that $V(-bp)$ is the subspace of sections $s \in V$ such that $\mathrm{ord}_p(s) \geq b$, we can extract a basis of $V(-bp)$ out of $\langle s_0, \dots, s_r \rangle$, namely

$$V(-bp) = \langle s_j, \dots, s_r \rangle \subset H^0(C, L(-bp))$$

where the index $j \in \{0, \dots, r+1\}$ is determined by the inequalities

$$a_j(p) = \mathrm{ord}_p(s_j) \geq b, \qquad a_{j-1}(p) = \mathrm{ord}_p(s_{j-1}) < b$$

whenever they make sense. Finally, the fact that there are $(r+1) - j$ elements in such a basis leads to the useful relation

$$\dim V(-bp) = r + 1 - j \quad \Leftrightarrow \quad a_{j-1}(p) < b \leq a_j(p)$$

which we will systematically use in the subsequent discussion. For example, we can apply it to $L_C = \theta_C \otimes \eta_C\left((g-i)y\right)$ and deduce that

$$h^0(\theta_C \otimes \eta_C) = h^0(L_C(-(g-i)y) = 2 - j \quad \Leftrightarrow \quad a_{j-1}^{\ell_C}(y) < g - i \leq a_j^{\ell_C}(y)$$

with $j \in \{1, 2\}$ depending on the parity of $\theta_C \otimes \eta_C$.

Let us start by analysing the two possibilities related to the even Prym-null divisor, labelled as in the table above.

Possibility $(\mathrm{F}, +, 0)$**.** In this case, we have

$$\begin{aligned} h^0(\theta_C \otimes \eta_C) = 0 \quad &\Rightarrow \quad a_0^{\ell_C}(y) < a_1^{\ell_C}(y) < g - i \quad \Rightarrow \quad a_0^{\ell_C}(y) \leq g - i - 2 \\ h^0(\theta_D) = 0 \quad &\Rightarrow \quad \qquad a_1^{\ell_D}(q) < i \qquad\qquad \Rightarrow \quad a_1^{\ell_D}(q) \leq i - 1 \end{aligned}$$

Combining these upper bounds, we immediately reach a contradiction with one of the limit $\mathfrak{g}^1_{g-1}$ compatibility conditions

$$g - 1 \leq a_0^{\ell_C}(y) + a_1^{\ell_D}(q) \leq g - 3 \qquad (!!)$$

which therefore prevents this type of intersection from taking place.

Possibility $(\mathrm{F}, +, 1)$**.** In this case, we have

$$\begin{aligned} h^0(\theta_C \otimes \eta_C) = 1 \quad &\Rightarrow \quad a_0^{\ell_C}(y) < g - i \leq a_1^{\ell_C}(y) \\ h^0(\theta_D) = 1 \quad &\Rightarrow \quad a_0^{\ell_D}(q) < i \leq a_1^{\ell_D}(q) \end{aligned}$$

On the one hand, (D, q) is general, so we may assume that $q \notin \mathrm{supp}(\theta_D)$. Then the vanishing sequence of ℓ_D at q can be bounded further:

$$\begin{aligned} h^0(\theta_D\,(-q)) = h^0(\theta_D) - 1 = 0 \;\;&\Rightarrow\;\; a_1^{\ell_D}(q) < i+1 \\ h^0(\theta_D\,(q)) = h^0(\theta_D) = 1 \;\;&\Rightarrow\;\; a_0^{\ell_D}(q) < i-1 \leq a_1^{\ell_D}(q) \end{aligned}$$

We thus get $a_1^{\ell_D}(q) = i$ and, by the limit $\mathfrak{g}^1_{g-1}$ condition, $a_0^{\ell_C}(y) = g-i-1$. On the other hand, C is general, so $\mathrm{supp}(\theta_C \otimes \eta_C)$ consists of $i-1$ distinct points. As a result, we obtain a tight upper bound for $a_1^{\ell_C}(y)$, namely

$$\begin{aligned} \mathrm{div}(\theta_C \otimes \eta_C) \not\geq 2y \;\;&\Rightarrow\;\; \mathrm{div}(s) \not\geq (g-i+2)y \quad \forall\, s \in H^0(L_C) \\ &\Rightarrow\;\; a_1^{\ell_C}(y) \leq g-i+1 \end{aligned}$$

which together with the condition $a_0^{\ell_D}(q) + a_1^{\ell_C}(y) \geq g-1$ yields $a_0^{\ell_D}(q) = i-2$ and $a_1^{\ell_C}(y) = g-i+1$. In turn, this means that $y \in \mathrm{supp}(\theta_C \otimes \eta_C)$, and that ℓ is a refined limit $\mathfrak{g}^1_{g-1}$ of the form

$$\begin{cases} \ell_C \;=\; |\theta_C \otimes \eta_C\,(y)| + (g-i-1)y \;\in\; G^1_{g-1}(C) \\ \ell_D \;=\; |\theta_D\,(2q)| + (i-2)q \;\in\; G^1_{g-1}(D) \end{cases}$$

with vanishing sequences $(g-i-1,\, g-i+1)$ and $(i-2,\, i)$.

Remark 2.3.3. The transversality of the intersection $F_i \cap \overline{\mathcal{P}}^+_{\mathrm{null}}$ is a by-product of the fact that the restriction $\overline{\mathcal{P}}^+_{\mathrm{null}}|_{F_i}$ is isomorphic to a scheme

$$\mathfrak{I}^1_{g-1}(F_i) \to F_i$$

parametrizing those limit linear series of type $\mathfrak{g}^1_{g-1}$ on stable Prym curves of F_i that adhere to the previous characterization. In the vein of [EH86] Th. 3.3, we then have a decomposition

$$\mathfrak{I}^1_{g-1}(F_i) \;\;\cong\;\; \{\ell_C\} \times \{\ell_D\}$$

where the right-hand side is everywhere reduced, since it corresponds to

$$\{(\theta_C, y) \;/\; \theta_C \in S^{-,-}_{\eta_C}(C),\, y \in \mathrm{supp}(\theta_C \otimes \eta_C)\} \times \{\theta_D \;/\; \theta_D \in S^-_{g-i}(D)\}$$

Therefore $\overline{\mathcal{P}}^+_{\mathrm{null}}|_{F_i} \cong \mathfrak{I}^1_{g-1}(F_i)$ is everywhere reduced and the intersection of $\overline{\mathcal{P}}^+_{\mathrm{null}}$ and F_i is transverse, as expected. A breakdown of this argument may be found in the proof of [Far10] Th. 2.2.

In conclusion, for each pair of theta characteristics θ_C, θ_D fulfilling $(\mathrm{F}, +, 1)$, that is, such that $\theta_C \in S^-_i(C)$, $\theta_D \in S^-_{g-i}(D)$ and $\theta_C \otimes \eta_C \in S^-_i(C)$, then every

$y \in \operatorname{supp}(\theta_C \otimes \eta_C)$ yields a limit $\mathfrak{g}^1_{g-1}$ as above, and these limit linear series are the only ones contributing to the intersection $F_i \cap \overline{\mathcal{P}}^+_{\text{null}}$. Consequently, we need to count such pairs of theta characteristics.

Fortunately, we already have all the necessary tools to do this.

Lemma 2.3.4. *For all $i \in \{2, \dots, g-1\}$, it holds that*

$$F_i \cdot \overline{\mathcal{P}}^+_{\text{null}} \quad = \quad 2^{g-2}(2^{i-1}-1)(2^{g-i}-1)(i-1)$$

Proof. In light of the previous considerations, we may split the count into three parts. Specifically, we want to compute the number of:

(i) *Theta characteristics $\theta_C \in S^-_i(C)$ such that $\theta_C \otimes \eta_C \in S^-_i(C)$.*
According to proposition 2.2.12, this is $\mathrm{N}^-_i = 2^{i-1}(2^{i-1}-1)$.

(ii) *Theta characteristics $\theta_D \in S^-_{g-i}(D)$.*
According to definition 1.1.16, this is $\#S^-_{g-i}(D) = 2^{g-i-1}(2^{g-i}-1)$.

(iii) *Once θ_C is fixed, points y in the support of $\theta_C \otimes \eta_C$.*
Since $\theta_C \otimes \eta_C \in S_i(C)$, there are $\deg(\theta_C \otimes \eta_C) = i-1$ such points.

Altogether, we obtain

$$\begin{aligned} F_i \cdot \overline{\mathcal{P}}^+_{\text{null}} &= \# \left\{ \begin{array}{c} (\theta_C, \theta_D, y) \in S^-_i(C) \times S^-_{g-i}(D) \times C \ / \\ \theta_C \otimes \eta_C \in S^-_i(C),\ y \in \operatorname{supp}(\theta_C \otimes \eta_C) \end{array} \right\} \\ &= \mathrm{N}^-_i \cdot \#S^-_{g-i}(D) \cdot (i-1) \\ &= 2^{i-1}(2^{i-1}-1) \cdot 2^{g-i-1}(2^{g-i}-1) \cdot (i-1) \\ &= 2^{g-2}(2^{i-1}-1)(2^{g-i}-1)(i-1) \end{aligned}$$

as stated above. □

In order to determine $F_i \cap \overline{\mathcal{P}}^-_{\text{null}}$ we follow the same argument, with the only difference being that θ_C and θ_D now have opposite parity (remark 2.3.1). Since this brings about minimal variations, we merely outline the situation and carry out the corresponding count. There are again two possibilities to tackle.

Possibility $(\mathrm{F}, -, 0)$**.** Similar contradiction to that of $(\mathrm{F}, +, 0)$.

Possibility $(\mathrm{F}, -, 1)$**.** As with its even counterpart, we are able to build a limit linear series contributing to $F_i \cdot \overline{\mathcal{P}}^-_{\text{null}}$ whenever $y \in \operatorname{supp}(\theta_C \otimes \eta_C)$. In this case, however, we have $\theta_C \in S^+_i(C)$.

We thus get

$$\begin{aligned} F_i \cdot \overline{\mathcal{P}}^-_{\mathrm{null}} &= \# \left\{ \begin{array}{c} (\theta_C,\, \theta_D,\, y) \in S^+_i(C) \times S^-_{g-i}(D) \times C \ / \\ \theta_C \otimes \eta_C \in S^-_i(C),\, y \in \mathrm{supp}(\theta_C \otimes \eta_C) \end{array} \right\} \\ &= \mathrm{N}^{\pm}_i \cdot \# S^-_{g-i}(D) \cdot (i-1) \\ &= 2^{2i-2} \cdot 2^{g-i-1}(2^{g-i}-1) \cdot (i-1) \\ &= 2^{g+i-3}(2^{g-i}-1)(i-1) \end{aligned}$$

which completes the review of the first family of test curves.

The procedure we have employed to study the intersection of F_i and each of the Prym-null divisors works with G_i and H_i as well. Nonetheless, we still need to carefully track the small changes that happen along the way.

Let us briefly do this. If a stable Prym curve

$$G_{i,y} \quad = \quad (C \cup_{y \sim q} D,\, (\mathcal{O}_C, \eta_D)) \quad \in \quad G_i$$

lies in $\overline{\mathcal{P}}^+_{\mathrm{null}}$ (resp. $\overline{\mathcal{P}}^-_{\mathrm{null}}$), we can produce a limit $\mathfrak{g}^1_{g-1}$ on $C \cup_{y\sim q} D$ such that

$$\left\{ \begin{array}{rcl} L_C &=& \theta_C\left((g-i)y\right) \\ L_D &=& \theta_D \otimes \eta_D\,(iq) \end{array} \right.$$

with $h^0(\theta_C) + h^0(\theta_D \otimes \eta_D) \equiv 0 \mod 2$, where θ_C and θ_D have the same parity (resp. opposite parity). Then θ_C and $\theta_D \otimes \eta_D$ have the same parity and we get the following possibilities:

	$h^0(\theta_C)$	$h^0(\theta_D)$	$h^0(\theta_D \otimes \eta_D)$	
$\overline{\mathcal{P}}^+_{\mathrm{null}}$	0	0	0	$\rightsquigarrow$ $(\mathrm{G},+,0)$: contradiction
	1	1	1	$\rightsquigarrow$ $(\mathrm{G},+,1)$: $y \in \mathrm{supp}(\theta_C)$
$\overline{\mathcal{P}}^-_{\mathrm{null}}$	0	1	0	$\rightsquigarrow$ $(\mathrm{G},-,0)$: contradiction
	1	0	1	$\rightsquigarrow$ $(\mathrm{G},-,1)$: $y \in \mathrm{supp}(\theta_C)$

With the only contribution of $(\mathrm{G},+,1)$ and $(\mathrm{G},-,1)$ to their respective intersections, we obtain

$$\begin{aligned} G_i \cdot \overline{\mathcal{P}}^+_{\text{null}} &= \# \left\{ \begin{array}{c} (\theta_C,\, \theta_D,\, y) \in S^-_i(C) \times S^-_{g-i}(D) \times C \,/ \\ \theta_D \otimes \eta_D \in S^-_{g-i}(D),\; y \in \operatorname{supp}(\theta_C) \end{array} \right\} \\ &= \# S^-_i(C) \cdot \mathrm{N}^-_{g-i} \cdot (i-1) \\ &= 2^{i-1}(2^i - 1) \cdot 2^{g-i-1}(2^{g-i-1} - 1) \cdot (i-1) \\ &= 2^{g-2}(2^i - 1)(2^{g-i-1} - 1)(i-1) \end{aligned}$$

and similarly

$$\begin{aligned} G_i \cdot \overline{\mathcal{P}}^-_{\text{null}} &= \# \left\{ \begin{array}{c} (\theta_C,\, \theta_D,\, y) \in S^-_i(C) \times S^+_{g-i}(D) \times C \,/ \\ \theta_D \otimes \eta_D \in S^-_{g-i}(D),\; y \in \operatorname{supp}(\theta_C) \end{array} \right\} \\ &= \# S^-_i(C) \cdot \mathrm{N}^\pm_{g-i} \cdot (i-1) \\ &= 2^{i-1}(2^i - 1) \cdot 2^{2g-2i-2} \cdot (i-1) \\ &= 2^{2g-i-3}(2^i - 1)(i-1) \end{aligned}$$

Note that there are equalities

$$F_i \cdot \overline{\mathcal{P}}^+_{\text{null}} \,/\, (i-1) \;=\; G_{g-i} \cdot \overline{\mathcal{P}}^+_{\text{null}} \,/\, (g-i-1) \qquad\qquad (\text{resp. } \overline{\mathcal{P}}^-_{\text{null}})$$

for all $i \in \{2, \ldots, g-2\}$, which is not surprising, given the construction of the families F_i and G_{g-i}.

Remark 2.3.5. The contradiction in $(\mathrm{G}, +, 0)$ and $(\mathrm{G}, -, 0)$ is again

$$g - 1 \leq a_0^{\ell_C}(y) + a_1^{\ell_D}(q) \leq g - 3 \qquad (!!)$$

In general, this condition will fail every time we try to use theta characteristics without global sections to build a limit $\mathfrak{g}^1_{g-1}$ on our reducible nodal curve, so in the future we will refrain from detailing it any further.

Finally, if a stable Prym curve

$$H_{i,\,y} \;=\; (C \cup_{y \sim q} D,\, (\eta_C, \eta_D)) \;\in\; H_i$$

lies in $\overline{\mathcal{P}}^+_{\text{null}}$ (resp. $\overline{\mathcal{P}}^-_{\text{null}}$), we can produce a limit $\mathfrak{g}^1_{g-1}$ on $C \cup_{y \sim q} D$ such that

$$\left\{ \begin{array}{lcl} L_C &=& \theta_C \otimes \eta_C\,((g-i)y) \\ L_D &=& \theta_D \otimes \eta_D\,(iq) \end{array} \right.$$

with $h^0(\theta_C \otimes \eta_C) + h^0(\theta_D \otimes \eta_D) \equiv 0 \mod 2$, where θ_C and θ_D have the same parity (resp. opposite parity). Then $\theta_C \otimes \eta_C$ and $\theta_D \otimes \eta_D$ have the same parity and we get the following possibilities:

	$h^0(\theta_C)$	$h^0(\theta_D)$	$h^0(\theta_C \otimes \eta_C)$	$h^0(\theta_D \otimes \eta_D)$		
	0	0	0	0	$\rightsquigarrow$	contradiction
$\overline{\mathcal{P}}^+_{\text{null}}$	1	1	0	0	$\rightsquigarrow$	contradiction
	0	0	1	1	$\rightsquigarrow$	$y \in \operatorname{supp}(\theta_C \otimes \eta_C)$
	1	1	1	1	$\rightsquigarrow$	$y \in \operatorname{supp}(\theta_C \otimes \eta_C)$
	0	1	0	0	$\rightsquigarrow$	contradiction
$\overline{\mathcal{P}}^-_{\text{null}}$	1	0	0	0	$\rightsquigarrow$	contradiction
	0	1	1	1	$\rightsquigarrow$	$y \in \operatorname{supp}(\theta_C \otimes \eta_C)$
	1	0	1	1	$\rightsquigarrow$	$y \in \operatorname{supp}(\theta_C \otimes \eta_C)$

The count has now grown in complexity, but not by much. We have

$$\begin{aligned}
H_i \cdot \overline{\mathcal{P}}^+_{\text{null}} &= \# \left\{ \begin{array}{c} (\theta_C,\, \theta_D,\, y) \in (S_i^+(C) \times S_{g-i}^+(D) \times C) \cup \\ \qquad\qquad \cup\, (S_i^-(C) \times S_{g-i}^-(D) \times C) \;/ \\ \theta_C \otimes \eta_C \in S_i^-(C),\, \theta_D \otimes \eta_D \in S_{g-i}^-(D), \\ y \in \operatorname{supp}(\theta_C \otimes \eta_C) \end{array} \right\} \\
&= (\mathrm{N}_i^{\pm}\mathrm{N}_{g-i}^{\pm} + \mathrm{N}_i^-\mathrm{N}_{g-i}^-) \cdot (i-1) \\
&= (2^{2i-2} \cdot 2^{2g-2i-2} + 2^{i-1}(2^{i-1}-1) \cdot 2^{g-i-1}(2^{g-i-1}-1)) \cdot (i-1) \\
&= 2^{g-2}(2^{g-1} - 2^{i-1} - 2^{g-i-1} + 1)(i-1)
\end{aligned}$$

and similarly

$$\begin{aligned}
H_i \cdot \overline{\mathcal{P}}^-_{\text{null}} &= \# \left\{ \begin{array}{c} (\theta_C,\, \theta_D,\, y) \in (S_i^+(C) \times S_{g-i}^-(D) \times C) \cup \\ \qquad\qquad \cup\, (S_i^-(C) \times S_{g-i}^+(D) \times C) \;/ \\ \theta_C \otimes \eta_C \in S_i^-(C),\, \theta_D \otimes \eta_D \in S_{g-i}^-(D), \\ y \in \operatorname{supp}(\theta_C \otimes \eta_C) \end{array} \right\} \\
&= (\mathrm{N}_i^{\pm}\mathrm{N}_{g-i}^{-} + \mathrm{N}_i^-\mathrm{N}_{g-i}^{\pm}) \cdot (i-1) \\
&= (2^{2i-2} \cdot 2^{g-i-1}(2^{g-i-1}-1) + 2^{i-1}(2^{i-1}-1) \cdot 2^{2g-2i-2}) \cdot (i-1) \\
&= 2^{g-2}(2^{g-1} - 2^{i-1} - 2^{g-i-1})(i-1)
\end{aligned}$$

which brings our analysis of standard reducible nodal test curves to a close.

To summarize, we compile the simplified expressions for all three collections of intersections into the following extension of lemma 2.3.4.

Lemma 2.3.6. *For all $i \in \{2, \dots, g-1\}$, we have intersection numbers*

	$\overline{\mathcal{P}}^{+}_{\mathrm{null}}$	$\overline{\mathcal{P}}^{-}_{\mathrm{null}}$
F_i	$(2^{i-1}-1)(2^{g-i}-1)\, r$	$2^{i-1}(2^{g-i}-1)\, r$
G_i	$(2^{i}-1)(2^{g-i-1}-1)\, r$	$(2^{i}-1)\, 2^{g-i-1}\, r$
H_i	$(2^{g-1}-2^{i-1}-2^{g-i-1}+1)\, r$	$(2^{g-1}-2^{i-1}-2^{g-i-1})\, r$

where $r = 2^{g-2}(i-1) = -2^{g-3}(2-2i)$.

Remark 2.3.7. A quick computation shows that these numbers pass the check suggested by remark 2.2.6. Indeed, we can easily see that

$$\begin{aligned}
F_i \cdot \overline{\mathcal{P}}^{+}_{\mathrm{null}} + F_i \cdot \overline{\mathcal{P}}^{-}_{\mathrm{null}} &= 2^{g-2}(2^{i}-1)(2^{g-i}-1)(i-1) = F_i \cdot \overline{\mathcal{P}}_{\mathrm{null}} \\
G_i \cdot \overline{\mathcal{P}}^{+}_{\mathrm{null}} + G_i \cdot \overline{\mathcal{P}}^{-}_{\mathrm{null}} &= 2^{g-2}(2^{i}-1)(2^{g-i}-1)(i-1) = G_i \cdot \overline{\mathcal{P}}_{\mathrm{null}} \\
H_i \cdot \overline{\mathcal{P}}^{+}_{\mathrm{null}} + H_i \cdot \overline{\mathcal{P}}^{-}_{\mathrm{null}} &= 2^{g-2}(2^{i}-1)(2^{g-i}-1)(i-1) = H_i \cdot \overline{\mathcal{P}}_{\mathrm{null}}
\end{aligned}$$

where we have used example 2.1.24 and proposition 2.2.5.

If we combine lemma 2.3.6 with the intersection table of example 2.1.24, we can derive the first batch of coefficients from the formulas in remark 2.2.7.

Proposition 2.3.8. *Fix integers $g \geq 5$ and $i \in \{1, \dots, \lfloor g/2 \rfloor\}$. The generating classes δ_i^{n}, δ_i^{t}, $\delta_i^{\mathrm{p}} \in \mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$ have coefficients*

$$\varrho^{+}_{\mathrm{null}} \left| \begin{aligned}
\delta_i^{\mathrm{n},+} &= 2^{g-3}(2^{i-1}-1)(2^{g-i}-1) \\
\delta_i^{\mathrm{t},+} &= 2^{g-3}(2^{i}-1)(2^{g-i-1}-1) \\
\delta_i^{\mathrm{p},+} &= 2^{g-3}(2^{g-1}-2^{i-1}-2^{g-i-1}+1)
\end{aligned} \right.$$

$$\varrho^{-}_{\mathrm{null}} \left| \begin{aligned}
\delta_i^{\mathrm{n},-} &= 2^{g-3}\, 2^{i-1}(2^{g-i}-1) \\
\delta_i^{\mathrm{t},-} &= 2^{g-3}(2^{i}-1)\, 2^{g-i-1} \\
\delta_i^{\mathrm{p},-} &= 2^{g-3}(2^{g-1}-2^{i-1}-2^{g-i-1})
\end{aligned} \right.$$

in the rational expansions of the Prym-null classes in genus g.

Proof. Every family of test curves generates a linear relation between the coefficients of each expansion. Due to the simplicity of the intersection tables of F_i, G_i and H_i with the generators of $\mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$, their corresponding linear relations directly determine one coefficient (sometimes the same one, if two relations are linearly dependent). For the sake of brevity, we shall describe this computation simply in the F_i case, as all others are analogous. To begin with, we have

$$\deg \varrho^{+}_{\mathrm{null}}(F_i) = F_i \cdot \overline{\mathcal{P}}^{+}_{\mathrm{null}} = -2^{g-3}(2^{i-1}-1)(2^{g-i}-1)(2-2i)$$

by lemma 2.3.6. Furthermore, remark 2.2.7 and example 2.1.24 show that

$$\deg \varrho^{+}_{\mathrm{null}}(F_i) \quad = \quad -\delta_i^{\mathrm{n},+} \deg \delta_i^{\mathrm{n}}(F_i) \quad = \quad -\delta_i^{\mathrm{n},+}(2-2i)$$

with the convention $\delta_i^{\mathrm{n},+} = \delta_{g-i}^{\mathrm{t},+}$ for all i. Since only one coefficient survives, we can immediately extract it from the resulting equation:

$$F_i \quad \rightsquigarrow \quad \delta_i^{\mathrm{n},+} \quad = \quad 2^{g-3}(2^{i-1}-1)(2^{g-i}-1)$$

For $2 \leq i \leq \lfloor g/2 \rfloor$, each coefficient can be similarly computed by means of:

$$F_i \quad \text{or} \quad G_{g-i} \quad \rightsquigarrow \quad \begin{cases} \delta_i^{\mathrm{n},+} & = \quad 2^{g-3}(2^{i-1}-1)(2^{g-i}-1) \\ \delta_i^{\mathrm{n},-} & = \quad 2^{g-3}\, 2^{i-1}(2^{g-i}-1) \end{cases}$$

$$G_i \quad \text{or} \quad F_{g-i} \quad \rightsquigarrow \quad \begin{cases} \delta_i^{\mathrm{t},+} & = \quad 2^{g-3}(2^{i}-1)(2^{g-i-1}-1) \\ \delta_i^{\mathrm{t},-} & = \quad 2^{g-3}(2^{i}-1)\, 2^{g-i-1} \end{cases}$$

$$H_i \quad \text{or} \quad H_{g-i} \quad \rightsquigarrow \quad \begin{cases} \delta_i^{\mathrm{p},+} & = \quad 2^{g-3}(2^{g-1}-2^{i-1}-2^{g-i-1}+1) \\ \delta_i^{\mathrm{p},-} & = \quad 2^{g-3}(2^{g-1}-2^{i-1}-2^{g-i-1}) \end{cases}$$

For $i = 1$, we do not have families F_1, G_1 and H_1. Nevertheless, we can use:

$$G_{g-1} \quad \rightsquigarrow \quad \begin{cases} \delta_1^{\mathrm{n},+} & = \quad 0 \\ \delta_1^{\mathrm{n},-} & = \quad 2^{g-3}(2^{g-1}-1) \end{cases}$$

$$F_{g-1} \quad \rightsquigarrow \quad \begin{cases} \delta_1^{\mathrm{t},+} & = \quad 2^{g-3}(2^{g-2}-1) \\ \delta_1^{\mathrm{t},-} & = \quad 2^{g-3}\, 2^{g-2} \end{cases}$$

$$H_{g-1} \quad \rightsquigarrow \quad \begin{cases} \delta_1^{\mathrm{p},+} & = \quad 2^{g-3}\, 2^{g-2} \\ \delta_1^{\mathrm{p},-} & = \quad 2^{g-3}(2^{g-2}-1) \end{cases}$$

that is to say, the above formulas hold for $i = 1$ as well. □

Remark 2.3.9. Observe that $\delta_1^{\mathrm{n},+} = 0$ is a consequence of the fact that genus 1 curves do not have nontrivial odd theta characteristics, hence $\mathrm{N}_1^- = 0$.

We have obtained all coefficients except for λ^+, $\delta_0^{\mathrm{t},+}$, $\delta_0^{\mathrm{p},+}$ and $\delta_0^{\mathrm{b},+}$ (resp. $-$), so four more linear relations are needed for each divisor. Coincidentally, this is also the number of test curves that we have yet to study.

2.3.2 Over curves with elliptic tails

Recall the test curves from example 2.1.25, that is

$$\begin{array}{rcllll} F_0 & \equiv & \{(C\cup_{p\sim\sigma(\lambda)} E_\lambda,\, (\eta_C, \mathcal{O}_{E_\lambda}))\}_{\lambda\in\mathbb{P}^1} & \subset & \Delta_1^{\mathrm{t}} & \subset \overline{\mathcal{R}}_g \\ G_0 & \equiv & \{(C\cup_{p\sim\sigma(\lambda)} E_\lambda,\, (\mathcal{O}_C, \eta_{E_\lambda}))\ /\ \eta_{E_\lambda}\in\gamma_1^{-1}(E_\lambda)\}_{\lambda\in\mathbb{P}^1} & \subset & \Delta_1^{\mathrm{n}} & \subset \overline{\mathcal{R}}_g \\ H_0 & \equiv & \{(C\cup_{p\sim\sigma(\lambda)} E_\lambda,\, (\eta_C, \eta_{E_\lambda}))\ /\ \eta_{E_\lambda}\in\gamma_1^{-1}(E_\lambda)\}_{\lambda\in\mathbb{P}^1} & \subset & \Delta_1^{\mathrm{p}} & \subset \overline{\mathcal{R}}_g \end{array}$$

$$\begin{array}{rl} \text{with} & (C,p)\in\mathcal{M}_{g-1,1}\ \text{ general,} \\ & \{E_\lambda\}_{\lambda\in\mathbb{P}^1}\subset\overline{\mathcal{M}}_1\ \text{ general pencil of plane cubics with basepoint } \sigma, \\ & \eta_C\in R_{g-1}(C)\ \text{ arbitrary,} \\ \text{and} & \gamma_1: \overline{\mathcal{R}}_{1,1}\to\overline{\mathcal{M}}_{1,1}\ \text{ forgetful degree 3 branched covering.} \end{array}$$

Our goal is to expand their intersection table to include the Prym-null divisors, as we did in lemma 2.3.6 for the previous families of test curves. In this case, it turns out that the new intersection numbers are much less imposing:

	λ	δ_0^{t}	δ_0^{p}	δ_0^{b}	δ_1^{n}	δ_1^{t}	δ_1^{p}	$\delta_{(j\geq 2)}$	$\overline{\mathcal{P}}^+_{\mathrm{null}}$	$\overline{\mathcal{P}}^-_{\mathrm{null}}$
F_0	1	0	12	0	0	-1	0	0	0	0
G_0	3	12	0	12	-3	0	0	0	0	0
H_0	3	0	12	12	0	0	-3	0	0	0

Let us check the veracity of this claim.

First, we determine $F_0\cap\overline{\mathcal{P}}^+_{\mathrm{null}}$ (resp. $\overline{\mathcal{P}}^-_{\mathrm{null}}$). If a stable Prym curve

$$F_{0,\lambda} \;=\; (C\cup_{p\sim e} E_\lambda,\, (\eta_C, \mathcal{O}_{E_\lambda})) \;\in\; F_0$$

lies in $\overline{\mathcal{P}}^+_{\mathrm{null}}$ (resp. $\overline{\mathcal{P}}^-_{\mathrm{null}}$), we can produce a limit $\mathfrak{g}^1_{g-1}$ on $C\cup_{p\sim e} E_\lambda$ such that

$$\begin{cases} L_C &= \theta_C\otimes\eta_C\,(p) \\ L_{E_\lambda} &= \theta_{E_\lambda}\,((g-1)e) \end{cases}$$

with $h^0(\theta_C\otimes\eta_C)+h^0(\theta_{E_\lambda})\equiv 0 \mod 2$, where θ_C and θ_{E_λ} have the same parity (resp. opposite parity). Then $\theta_C\otimes\eta_C$ and θ_{E_λ} have the same parity and we get the following possibilities:

	$h^0(\theta_C)$	$h^0(\theta_{E_\lambda})$	$h^0(\theta_C\otimes\eta_C)$		
$\overline{\mathcal{P}}^+_{\mathrm{null}}$	0	0	0	$\rightsquigarrow$	contradiction
	1	1	1	$\rightsquigarrow$	$(\mathrm{F}_0, +, 1)$
$\overline{\mathcal{P}}^-_{\mathrm{null}}$	1	0	0	$\rightsquigarrow$	contradiction
	0	1	1	$\rightsquigarrow$	$(\mathrm{F}_0, -, 1)$

Note that all four theta characteristics of a genus 1 curve are of degree zero, hence the nontrivial ones have no global sections other than zero. In particular, the dimension of the global sections of θ_{E_λ} is given by

$$h^0(\theta_{E_\lambda}) \quad = \quad \begin{cases} 1 & \text{if } \theta_{E_\lambda} = \mathcal{O}_{E_\lambda} \\ 0 & \text{otherwise} \end{cases}$$

for all $\lambda \in \mathbb{P}^1$, meaning that the table above is comprehensive. Since half of the scenarios are already covered by remark 2.3.5, we just need to look at $(\mathrm{F}_0, +, 1)$ and $(\mathrm{F}_0, -, 1)$ to conclude. This can be done in one fell swoop.

Possibilities $(\mathrm{F}_0, +, 1)$, $(\mathrm{F}_0, -, 1)$**.** In both of these cases, we have

$$h^0(\theta_C \otimes \eta_C) = 1 \;\; \Rightarrow \;\; a_0^{\ell_C}(p) < 1 \leq a_1^{\ell_C}(p)$$
$$h^0(\theta_{E_\lambda}) = 1 \;\; \Rightarrow \;\; a_0^{\ell_{E_\lambda}}(e) < g-1 \leq a_1^{\ell_{E_\lambda}}(e)$$

Now (C, p) is general, so we may assume that $p \notin \operatorname{supp}(\theta_C \otimes \eta_C)$. Therefore

$$h^0(\theta_C \otimes \eta_C\,(p)) = h^0(\theta_C \otimes \eta_C) = 1 \;\; \Rightarrow \;\; a_0^{\ell_C}(p) < 0 \leq a_1^{\ell_C}(p)$$

Furthermore, E_λ is a genus 1 curve, so $\deg(\theta_{E_\lambda}) = 0$. As a result, we get

$$h^0(\theta_{E_\lambda}\,(-e)) = 0 \;\; \Rightarrow \;\; a_1^{\ell_{E_\lambda}}(e) < g$$

that is, $a_1^{\ell_{E_\lambda}}(e) = g-1$, which contradicts the limit $\mathfrak{g}^1_{g-1}$ condition

$$g-1 \leq a_0^{\ell_C}(p) + a_1^{\ell_{E_\lambda}}(e) < g-1 \qquad (!!)$$

(Alternatively, from $p \notin \operatorname{supp}(\theta_C \otimes \eta_C)$ we may also deduce

$$h^0(\theta_C \otimes \eta_C\,(-p)) = h^0(\theta_C \otimes \eta_C) - 1 = 0 \;\; \Rightarrow \;\; a_1^{\ell_C}(p) < 2$$

which yields $a_1^{\ell_C}(p) = 1$ and, via the limit $\mathfrak{g}^1_{g-1}$ condition, $a_0^{\ell_{E_\lambda}}(e) = g-2$. This should prevent θ_{E_λ} from being trivial, due to the implication

$$h^0(\theta_{E_\lambda}\,(e)) = h^0(\mathcal{O}_{E_\lambda}\,(e)) = 1 \;\; \Rightarrow \;\; a_0^{\ell_{E_\lambda}}(e) < g-2 \leq a_1^{\ell_{E_\lambda}}(e)$$

However, the triviality of θ_{E_λ} is ensured by $h^0(\theta_{E_\lambda}) = 1$ and $\deg(\theta_{E_\lambda}) = 0$.)

As every possibility leads to a contradiction, the intersections $F_0 \cap \overline{\mathcal{P}}^+_{\text{null}}$ and $F_0 \cap \overline{\mathcal{P}}^-_{\text{null}}$ are both empty, and thus $F_0 \cdot \overline{\mathcal{P}}^+_{\text{null}} = F_0 \cdot \overline{\mathcal{P}}^-_{\text{null}} = 0$.

Next we study $G_0 \cap \overline{\mathcal{P}}^+_{\text{null}}$ (resp. $\overline{\mathcal{P}}^-_{\text{null}}$). If a stable Prym curve

$$G_{0,\lambda} \quad = \quad (C \cup_{p \sim e} E_\lambda,\, (\mathcal{O}_C, \eta_{E_\lambda})) \quad \in \quad G_0$$

lies in $\overline{\mathcal{P}}^+_{\text{null}}$ (resp. $\overline{\mathcal{P}}^-_{\text{null}}$), we can produce a limit $\mathfrak{g}^1_{g-1}$ on $C \cup_{p\sim e} E_\lambda$ such that

$$\begin{cases} L_C &= \theta_C\,(p) \\ L_{E_\lambda} &= \theta_{E_\lambda} \otimes \eta_{E_\lambda}\,((g-1)e) \end{cases}$$

with $h^0(\theta_C) + h^0(\theta_{E_\lambda} \otimes \eta_{E_\lambda}) \equiv 0 \mod 2$, where θ_C and θ_{E_λ} have the same parity (resp. opposite parity). Then θ_C and $\theta_{E_\lambda} \otimes \eta_{E_\lambda}$ have the same parity and we get the following possibilities:

	$h^0(\theta_C)$	$h^0(\theta_{E_\lambda})$	$h^0(\theta_{E_\lambda} \otimes \eta_{E_\lambda})$		
$\overline{\mathcal{P}}^+_{\text{null}}$	0	0	0	$\rightsquigarrow$	contradiction
	1	1	1	$\rightsquigarrow$	$(\mathrm{G}_0, +, 1)$
$\overline{\mathcal{P}}^-_{\text{null}}$	0	1	0	$\rightsquigarrow$	contradiction
	1	0	1	$\rightsquigarrow$	$(\mathrm{G}_0, -, 1)$

Once more, remark 2.3.5 addresses half of the table, and the remaining half is tackled similarly to, if not more easily than, its F_0 counterpart.

Possibility $(\mathrm{G}_0, +, 1)$**.** We may repeat $(\mathrm{F}_0, +, 1)$'s argument, but a simpler procedure is available in this case. Since E_λ is a genus 1 curve, it holds that

$$\left.\begin{aligned} h^0(\theta_{E_\lambda}) &= h^0(\theta_{E_\lambda} \otimes \eta_{E_\lambda}) &= 1 \\ \deg(\theta_{E_\lambda}) &= \deg(\theta_{E_\lambda} \otimes \eta_{E_\lambda}) &= 0 \end{aligned}\right\} \Rightarrow \begin{cases} \theta_{E_\lambda} = \theta_{E_\lambda} \otimes \eta_{E_\lambda} = \mathcal{O}_{E_\lambda} \Rightarrow \\ \qquad\qquad \Rightarrow \eta_{E_\lambda} = \mathcal{O}_{E_\lambda} \quad (!!) \end{cases}$$

in direct contradiction with the nontriviality of a Prym root.

Possibility $(\mathrm{G}_0, -, 1)$**.** Same contradiction as in $(\mathrm{F}_0, -, 1)$.

Hence both intersections are empty again and $G_0 \cdot \overline{\mathcal{P}}^+_{\text{null}} = G_0 \cdot \overline{\mathcal{P}}^-_{\text{null}} = 0$.

Finally, let us consider $H_0 \cap \overline{\mathcal{P}}^+_{\text{null}}$ (resp. $\overline{\mathcal{P}}^-_{\text{null}}$). If a stable Prym curve

$$H_{0,\lambda} = (C \cup_{p\sim e} E_\lambda, (\eta_C, \eta_{E_\lambda})) \in H_0$$

lies in $\overline{\mathcal{P}}^+_{\text{null}}$ (resp. $\overline{\mathcal{P}}^-_{\text{null}}$), we can produce a limit $\mathfrak{g}^1_{g-1}$ on $C \cup_{p\sim e} E_\lambda$ such that

$$\begin{cases} L_C &= \theta_C \otimes \eta_C\,(p) \\ L_{E_\lambda} &= \theta_{E_\lambda} \otimes \eta_{E_\lambda}\,((g-1)e) \end{cases}$$

with $h^0(\theta_C \otimes \eta_C) + h^0(\theta_{E_\lambda} \otimes \eta_{E_\lambda}) \equiv 0 \mod 2$, where θ_C and θ_{E_λ} have the same parity (resp. opposite parity). Then $\theta_C \otimes \eta_C$ and $\theta_{E_\lambda} \otimes \eta_{E_\lambda}$ have the same parity and we get the following possibilities:

	$h^0(\theta_C)$	$h^0(\theta_{E_\lambda})$	$h^0(\theta_C \otimes \eta_C)$	$h^0(\theta_{E_\lambda} \otimes \eta_{E_\lambda})$		
$\overline{\mathcal{P}}^+_{\text{null}}$	0	0	0	0	$\rightsquigarrow$	contradiction
	1	1	0	0	$\rightsquigarrow$	contradiction
	0	0	1	1	$\rightsquigarrow$	$(\mathrm{H}_0, +, 1, 0)$
	1	1	1	1	$\rightsquigarrow$	$(\mathrm{H}_0, +, 1, 1)$
$\overline{\mathcal{P}}^-_{\text{null}}$	0	1	0	0	$\rightsquigarrow$	contradiction
	1	0	0	0	$\rightsquigarrow$	contradiction
	0	1	1	1	$\rightsquigarrow$	$(\mathrm{H}_0, -, 1, 1)$
	1	0	1	1	$\rightsquigarrow$	$(\mathrm{H}_0, -, 1, 0)$

Even though there are more cases, they are all covered by already discussed arguments. We point to those outside of the scope of remark 2.3.5.

Possibilities $(\mathrm{H}_0, +, 1, 0)$, $(\mathrm{H}_0, -, 1, 0)$**.** Same contradiction as in $(\mathrm{F}_0, +, 1)$.

Possibilities $(\mathrm{H}_0, +, 1, 1)$, $(\mathrm{H}_0, -, 1, 1)$**.** Same contradiction as in $(\mathrm{G}_0, +, 1)$.

As a result, it is clear that $H_0 \cdot \overline{\mathcal{P}}^+_{\text{null}} = H_0 \cdot \overline{\mathcal{P}}^-_{\text{null}} = 0$ as well.

Remark 2.3.10. The intersection numbers above, coupled with example 2.1.25 and proposition 2.2.5, show that

$$F_0 \cdot \overline{\mathcal{P}}^+_{\text{null}} + F_0 \cdot \overline{\mathcal{P}}^-_{\text{null}} \quad = \quad 0 \quad = \quad F_0 \cdot \overline{\mathcal{P}}_{\text{null}}$$

(resp. G_0, H_0), as expected by remark 2.2.6.

Lemma 2.3.11. *In the setting of proposition 2.3.8, the families F_0, G_0 and H_0 provide three linearly independent linear relations*

$$F_0 \quad \rightsquigarrow \quad \begin{cases} \lambda^+ - 12\,\delta_0^{\mathrm{p},+} & = \quad -2^{g-3}(2^{g-2}-1) \\ \lambda^- - 12\,\delta_0^{\mathrm{p},-} & = \quad -2^{2g-5} \end{cases}$$

$$G_0 \quad \rightsquigarrow \quad \begin{cases} \lambda^+ - 4\,\delta_0^{\mathrm{t},+} - 4\,\delta_0^{\mathrm{b},+} & = \quad 0 \\ \lambda^- - 4\,\delta_0^{\mathrm{t},-} - 4\,\delta_0^{\mathrm{b},-} & = \quad -2^{g-3}(2^{g-1}-1) \end{cases}$$

$$H_0 \quad \rightsquigarrow \quad \begin{cases} \lambda^+ - 4\,\delta_0^{\mathrm{p},+} - 4\,\delta_0^{\mathrm{b},+} & = \quad -2^{2g-5} \\ \lambda^+ - 4\,\delta_0^{\mathrm{p},-} - 4\,\delta_0^{\mathrm{b},-} & = \quad -2^{g-3}(2^{g-2}-1) \end{cases}$$

between the coefficients of λ, δ_0^{t}, δ_0^{p}, $\delta_0^{\mathrm{b}} \in \mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$ in each expansion.

Proof. Follows from combining proposition 2.3.8 with the intersection of F_0 and the Prym-null divisors (resp. G_0, H_0). $\square$

We need one more relation, which will be supplied by the last family of test curves that was defined in section 2.1.

2.3.3 Over irreducible nodal curves

Recall the test curve from example 2.1.26, that is

$$\begin{aligned} Y_0 \quad &\equiv \quad \{(B_{py},\, \eta_y^{\mathrm{t}}) \;/\; \eta_y^{\mathrm{t}} \in \Delta_0^{\mathrm{t}}(B_{py})\}_{y\in B} \quad \subset \quad \Delta_0^{\mathrm{t}} \quad \subset \quad \overline{\mathcal{R}}_g \\ \text{with} \quad & (B,p) \in \mathcal{M}_{g-1,1} \;\; \text{general}, \\ & B_{py} = B/\{y \sim p\} \;\; \text{irreducible nodal curve for } y \neq p, \\ \text{and} \quad & B_{pp} \;\; \text{copy of } B \text{ with a pigtail attached to } p. \end{aligned}$$

As we will see below, its extended intersection table is:

	λ	δ_0^{t}	δ_0^{p}	δ_0^{b}	δ_1^{n}	δ_1^{t}	δ_1^{p}	$\delta_{(j\geq 2)}$	$\overline{\mathcal{P}}_{\mathrm{null}}^{+}$	$\overline{\mathcal{P}}_{\mathrm{null}}^{-}$
Y_0	0	$2-2g$	0	0	1	0	0	0	0	$2^{g-3}(2^{g-2}(g-3)+1)$

The reason that the Prym-null intersection gravitates entirely towards the odd side is the parity change explored in proposition 2.2.13. With the help of both this result and the notation used to prove it, we can easily determine $Y_0 \cap \overline{\mathcal{P}}_{\mathrm{null}}^{+}$ and $Y_0 \cap \overline{\mathcal{P}}_{\mathrm{null}}^{-}$.

On the one hand, if a stable Prym curve

$$Y_{0,\,y} \quad = \quad (B_{py},\, \eta_y^{\mathrm{t}}) \quad \in \quad Y_0$$

lies in the even Prym-null divisor $\overline{\mathcal{P}}_{\mathrm{null}}^{+}$, then by definition there exists a stable vanishing theta-null $\theta_y \in \overline{\Theta}_{\mathrm{null}} \subset \overline{\mathcal{S}}_g^{+}$ over B_{py} such that $\theta_y \otimes \eta_y^{\mathrm{t}} \in \overline{\mathcal{S}}_g^{+}$ is even. Since proposition 2.2.13 shows that θ_y and $\theta_y \otimes \eta_y^{\mathrm{t}}$ always have opposite parity, no such vanishing theta-null can exist, and therefore $Y_0 \cdot \overline{\mathcal{P}}_{\mathrm{null}}^{+} = 0$.

On the other hand, if a stable Prym curve

$$Y_{0,\,y} \quad = \quad (B_{py},\, \eta_y^{\mathrm{t}}) \quad \in \quad Y_0$$

lies in the odd Prym-null divisor $\overline{\mathcal{P}}_{\mathrm{null}}^{-}$ instead, then by definition there exists a stable vanishing theta-null $\theta_y \in \overline{\Theta}_{\mathrm{null}} \subset \overline{\mathcal{S}}_g^{+}$ over B_{py} such that $\theta_y \otimes \eta_y^{\mathrm{t}} \in \overline{\mathcal{S}}_g^{-}$ is odd. Proposition 2.2.13 now makes redundant the second part of the condition, so we just need to count how many B_{py} admit a vanishing theta-null. This can be done by taking advantage of the formula for the theta-null class

$$\overline{\vartheta}_{\text{null}} \;=\; \frac{1}{4}\,\lambda - \frac{1}{16}\,\delta_0^{\text{n}} - \sum_{i=1}^{\lfloor g/2 \rfloor} \delta_i^-$$

introduced in section 2.2 and originally given by [Far10]. If we consider the test curve Y_0^{n} obtained as the pullback of $\{B_{py}\}_{y\in B}$ by the divisor $\Delta_0^{\text{n}} \subset \overline{\mathcal{S}}_g^+$, i.e.

$$Y_0^{\text{n}} \;\equiv\; \{(B_{py},\, \theta_y) \;/\; \theta_y \in \Delta_0^{\text{n}}(B_{py})\}_{y\in B} \;\subset\; \Delta_0^{\text{n}} \;\subset\; \overline{\mathcal{S}}_g^+$$

then the previous discussion identifies the intersection $Y_0 \cap \overline{\mathcal{P}}^-_{\text{null}}$ with the intersection $Y_0^{\text{n}} \cap \overline{\Theta}_{\text{null}}$. The latter can easily be derived from the theta-null formula and the intersection table of Y_0^{n} with the generators of $\text{Pic}(\overline{\mathcal{S}}_g^+)_{\mathbb{Q}}$, which is:

	λ	δ_0^{n}	δ_0^{b}	δ_1^+	δ_1^-	$\delta_{(j\geq 2)}$
Y_0^{n}	0	$2^{2g-1}(1-g)$	0	$2^{g-3}(2^{g-1}+1)$	$2^{g-3}(2^{g-1}-1)$	0

Indeed, example 2.1.26 and $\deg(\Delta_0^{\text{n}}|\Delta_0) = 2^{2g-2}$ yield all coefficients except for the δ_1^+, δ_1^- ones, while over the special point $(B_{pp},\, \theta_p)$ we can see that

$$\theta_p \;=\; (\theta_B,\, \mathcal{O}_E(1),\, (\mathcal{O}_{\mathbb{P}^1}, \varphi)) \;\in\; \Delta_0^{\text{n}}(B_{pp})$$

for some $\theta_B \in S_{g-1}(B)$ and $\varphi \in \{\psi, -\psi\}$ such that $h^0(\theta_B) \equiv h^0(\mathcal{O}_{\mathbb{P}^1}, \varphi) \mod 2$. Since by construction of ψ and $-\psi$ we have

$$h^0(\mathcal{O}_{\mathbb{P}^1}, \varphi) \;=\; \begin{cases} 1 & \Leftrightarrow \quad \varphi = \psi \\ 0 & \Leftrightarrow \quad \varphi = -\psi \end{cases}$$

it follows that φ is determined by the parity of θ_B and thus

$$\begin{aligned} Y_0^{\text{n}} \cdot (2\,\delta_1^+) \;&=\; \#S_{g-1}^+(B) \;=\; 2^{g-2}(2^{g-1}+1) \\ Y_0^{\text{n}} \cdot (2\,\delta_1^-) \;&=\; \#S_{g-1}^-(B) \;=\; 2^{g-2}(2^{g-1}-1) \end{aligned}$$

accounting for the appropriate factor of 2. In conclusion, we get

$$\begin{aligned} Y_0 \cdot \overline{\mathcal{P}}^-_{\text{null}} \;=\; Y_0^{\text{n}} \cdot \overline{\Theta}_{\text{null}} \;&=\; -2^{-4}\, Y_0^{\text{n}} \cdot \delta_0^{\text{n}} - Y_0^{\text{n}} \cdot \delta_1^- \\ &=\; -2^{2g-5}(1-g) - 2^{g-3}(2^{g-1}-1) \\ &=\; 2^{g-3}(2^{g-2}(g-3)+1) \end{aligned}$$

as indicated earlier.

Remark 2.3.12. Again, these intersection numbers add up to

$$Y_0 \cdot \overline{\mathcal{P}}^+_{\mathrm{null}} + Y_0 \cdot \overline{\mathcal{P}}^-_{\mathrm{null}} \quad = \quad 2^{g-3}(2^{g-2}(g-3)+1) \quad = \quad Y_0 \cdot \overline{\mathcal{P}}_{\mathrm{null}}$$

in line with example 2.1.26, proposition 2.2.5 and remark 2.2.6.

Proposition 2.3.13. *In the setting of proposition 2.3.8, the generating classes* λ, δ_0^{t}, δ_0^{p}, $\delta_0^{\mathrm{b}} \in \mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$ *have coefficients*

$$\varrho^+_{\mathrm{null}} \left| \begin{array}{rclcl} \lambda^+ & = & 2^{g-3}(2^{g-1}+1) & = & 2^{g-3}(2^{g-1}+1) \\ \delta_0^{\mathrm{t},+} & = & 0 & = & 0 \\ \delta_0^{\mathrm{p},+} & = & 2^{2g-7} & = & 2^{g-3}\,2^{-2}\,2^{g-2} \\ \delta_0^{\mathrm{b},+} & = & 2^{g-5}(2^{g-1}+1) & = & 2^{g-3}\,2^{-2}(2^{g-1}+1) \end{array} \right.$$

$$\varrho^-_{\mathrm{null}} \left| \begin{array}{rclcl} \lambda^- & = & 2^{2g-4} & = & 2^{g-3}\,2^{g-1} \\ \delta_0^{\mathrm{t},-} & = & 2^{2g-6} & = & 2^{g-3}\,2^{-2}\,2^{g-1} \\ \delta_0^{\mathrm{p},-} & = & 2^{2g-7} & = & 2^{g-3}\,2^{-2}\,2^{g-2} \\ \delta_0^{\mathrm{b},-} & = & 2^{g-5}(2^{g-1}-1) & = & 2^{g-3}\,2^{-2}(2^{g-1}-1) \end{array} \right.$$

in the rational expansions of the Prym-null classes in genus g.

Proof. Since the δ_1^{n} coefficients have already been computed (proposition 2.3.8), it is straightforward to check that the linear relation provided by the family Y_0 in each case directly determines the corresponding δ_0^{t} coefficient:

$$Y_0 \quad \rightsquigarrow \quad \left\{ \begin{array}{rcl} \delta_0^{\mathrm{t},+} & = & 0 \\ \delta_0^{\mathrm{t},-} & = & 2^{2g-6} \end{array} \right.$$

Plugging these into lemma 2.3.11, we obtain the following linear systems:

$$\begin{pmatrix} 1 & -12 & 0 \\ 1 & 0 & -4 \\ 1 & -4 & -4 \end{pmatrix} \cdot \begin{pmatrix} \lambda^+ \\ \delta_0^{\mathrm{p},+} \\ \delta_0^{\mathrm{b},+} \end{pmatrix} = \begin{pmatrix} -2^{g-3}(2^{g-2}-1) \\ 0 \\ -2^{2g-5} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -12 & 0 \\ 1 & 0 & -4 \\ 1 & -4 & -4 \end{pmatrix} \cdot \begin{pmatrix} \lambda^- \\ \delta_0^{\mathrm{p},-} \\ \delta_0^{\mathrm{b},-} \end{pmatrix} = \begin{pmatrix} -2^{2g-5} \\ 2^{g-3} \\ -2^{g-3}(2^{g-2}-1) \end{pmatrix}$$

The two sets of solutions are precisely the expressions stated above. □

As far as coefficients go, proposition 2.3.13 completes the process started in proposition 2.3.8. The only thing left to do is to assemble the class expansions, and examine some of their interactions with other families of curves.

2.3.4 Class expansion and application to other families

For the first time, all of the rational coefficients introduced in remark 2.2.7 are known to us, by virtue of propositions 2.3.8 and 2.3.13. As a result, we are finally in a position to express the rational classes of $\overline{\mathcal{P}}^+_{\text{null}}$ and $\overline{\mathcal{P}}^-_{\text{null}}$ in terms of the generating classes of $\text{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$, which was our main goal.

Theorem 2.3.14. *For $g \geq 5$, the classes ϱ^+_{null}, $\varrho^-_{\text{null}} \in \text{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}$ are given by*

$$\begin{aligned}
\varrho^+_{\text{null}} \;=\; 2^{g-3} \Bigg(& (2^{g-1}+1)\,\lambda \;-\; \frac{1}{4}\left(2^{g-2}\,\delta_0^{\text{p}} + (2^{g-1}+1)\,\delta_0^{\text{b}}\right) \\
& - \sum_{i=1}^{k} \Big((2^{i-1}-1)(2^{g-i}-1)\,\delta_i^{\text{n}} + (2^i-1)(2^{g-i-1}-1)\,\delta_i^{\text{t}} + \\
& \qquad\qquad + (2^{g-1}-2^{i-1}-2^{g-i-1}+1)\,\delta_i^{\text{p}} \Big) \\
& - \psi(g)\cdot\left((2^{g/2-1}-1)(2^{g/2}-1)\,\delta_{g/2}^{\text{n}} + (2^{g-1}-2^{g/2}+1)\,\delta_{g/2}^{\text{p}} \right) \Bigg)
\end{aligned}$$

$$\begin{aligned}
\varrho^-_{\text{null}} \;=\; 2^{g-3} \Bigg(& 2^{g-1}\,\lambda \;-\; \frac{1}{4}\left(2^{g-1}\,\delta_0^{\text{t}} + 2^{g-2}\,\delta_0^{\text{p}} + (2^{g-1}-1)\,\delta_0^{\text{b}}\right) \\
& - \sum_{i=1}^{k} \Big(2^{i-1}\,(2^{g-i}-1)\,\delta_i^{\text{n}} + (2^i-1)\,2^{g-i-1}\,\delta_i^{\text{t}} + \\
& \qquad\qquad + (2^{g-1}-2^{i-1}-2^{g-i-1})\,\delta_i^{\text{p}} \Big) \\
& - \psi(g)\cdot\left(2^{g/2-1}\,(2^{g/2}-1)\,\delta_{g/2}^{\text{n}} + (2^{g-1}-2^{g/2})\,\delta_{g/2}^{\text{p}} \right) \Bigg)
\end{aligned}$$

where the upper bound k and the parity-checking function $\psi(g)$, defined as

$$k \;=\; \lceil g/2 \rceil - 1 \;=\; \begin{cases} \lfloor g/2 \rfloor & \text{if } g \text{ odd} \\ \lfloor g/2 \rfloor - 1 & \text{if } g \text{ even} \end{cases}$$

$$\psi(g) \;=\; \frac{1+(-1)^g}{2} \;=\; \begin{cases} 0 & \text{if } g \text{ odd} \\ 1 & \text{if } g \text{ even} \end{cases}$$

account for the slight variation that occurs when $g = 2i$.

Remark 2.3.15. Indeed, the check $\varrho^+_{\text{null}} + \varrho^-_{\text{null}} = \varrho_{\text{null}}$ of remark 2.2.6 holds, as already evidenced by earlier remarks.

With the class expansions of $\varrho^+_{\mathrm{null}}$ and $\varrho^-_{\mathrm{null}}$ under our belt, it becomes easier to study the intersection of the Prym-null divisors with other families of curves. We give a couple of interesting examples next.

Example 2.3.16 (quartic tails)**.** We fix a general curve $(C,p) \in \mathcal{M}_{g-3,1}$ and a general pencil $\gamma\colon \mathrm{Bl}_{16}(\mathbb{P}^2) \to \mathbb{P}^1$ of plane quartics, with fibers

$$\{Q_\lambda = \gamma^{-1}(\lambda)\}_{\lambda\in\mathbb{P}^1} \quad\subset\quad \overline{\mathcal{M}}_3$$

together with a section $\zeta\colon \mathbb{P}^1 \to \mathrm{Bl}_{16}(\mathbb{P}^2)$ induced by one of the basepoints. We may then glue the curve (C,p) to the pencil γ along ζ, thus producing a pencil of stable curves

$$\mathcal{Q} = (C \times \mathbb{P}_1) \cup_{\{p\}\times\mathbb{P}^1 \sim \zeta(\mathbb{P}^1)} \mathrm{Bl}_{16}(\mathbb{P}^2) \longrightarrow \mathbb{P}^1$$

which corresponds to

$$\mathcal{Q} \quad\equiv\quad \{C \cup_{p\sim\zeta(\lambda)} Q_\lambda\}_{\lambda\in\mathbb{P}^1} \quad\subset\quad \Delta_3 \quad\subset\quad \overline{\mathcal{M}}_g$$

Standard techniques show its intersection table to be:

	λ	δ_0	δ_1	δ_2	δ_3	$\delta_{(j\geq 4)}$
$\mathcal{Q}$	3	27	0	0	-1	0

We now fix a Prym root $\eta_C \in R_{g-3}(C)$ and lift $\mathcal{Q}$ to a test curve R, as follows:

$$R \quad\equiv\quad \{(C \cup_{p\sim\zeta(\lambda)} Q_\lambda,\, (\eta_C, \mathcal{O}_{Q_\lambda}))\}_{\lambda\in\mathbb{P}^1} \quad\subset\quad \Delta_3^{\mathrm{t}} \quad\subset\quad \overline{\mathcal{R}}_g$$

Observe that $\pi_*(R) = \mathcal{Q}$. Then $R\cdot\lambda = \mathcal{Q}\cdot\lambda = 3$ and

$$R\cdot\delta_3^{\mathrm{t}} \quad=\quad \mathcal{Q}\cdot\delta_3 \quad=\quad -1$$

If we look at the 27 points $\lambda_\infty \in \mathbb{P}^1$ corresponding to singular quartics of γ and blow up the node of the component $Q_{\lambda_\infty} \in \Delta_0$, we can see that the pullback of $\eta_{\lambda_\infty} = (\eta_C, \mathcal{O}_{Q_{\lambda_\infty}})$ is $(\eta_C, \mathcal{O}_{\mathbb{P}^1})$, which is nontrivial. Hence $R_{\lambda_\infty} \in \Delta_0^{\mathrm{p}}$ and

$$R\cdot\delta_0^{\mathrm{p}} \quad=\quad \mathcal{Q}\cdot\delta_0 \quad=\quad 27$$

All other intersection numbers are 0, so we get a table:

	λ	δ_0^{t}	δ_0^{p}	δ_0^{b}	δ_3^{n}	δ_3^{t}	δ_3^{p}	$\delta_{(j\neq 0,3)}$
R	3	0	27	0	0	-1	0	0

Note the similarities between R and the family F_0 from example 2.1.25.

Remark 2.3.17. Applying theorem 2.3.14 to example 2.3.16, we get

$$\begin{aligned} R\cdot\overline{\mathcal{P}}^{+}_{\mathrm{null}} &= 3\,\lambda^{+} - 27\,\delta_0^{\mathrm{p},+} + \delta_3^{\mathrm{t},+} &&= 2^{g-1}(2^{g-4}-1) \\ R\cdot\overline{\mathcal{P}}^{-}_{\mathrm{null}} &= 3\,\lambda^{-} - 27\,\delta_0^{\mathrm{p},-} + \delta_3^{\mathrm{t},-} &&= 2^{2g-5} \end{aligned}$$

These intersection numbers may in fact be interpreted, as the limit linear series techniques introduced in earlier cases are also quite useful here. In this setting, the problem happens to turn into a beautiful bitangent count.

Again, we want to consider $R\cap\overline{\mathcal{P}}^{+}_{\mathrm{null}}$ (resp. $\overline{\mathcal{P}}^{-}_{\mathrm{null}}$). If a stable Prym curve

$$R_\lambda \;=\; (C\cup_{p\sim z} Q_\lambda,\,(\eta_C,\mathcal{O}_{Q_\lambda})) \;\in\; R$$

lies in $\overline{\mathcal{P}}^{+}_{\mathrm{null}}$ (resp. $\overline{\mathcal{P}}^{-}_{\mathrm{null}}$), we can produce a limit $\mathfrak{g}^1_{g-1}$ on $C\cup_{p\sim z} Q_\lambda$ such that

$$\begin{cases} L_C &= \theta_C\otimes\eta_C\,(3p) \\ L_{Q_\lambda} &= \theta_{Q_\lambda}\,((g-3)z) \end{cases}$$

with $h^0(\theta_C\otimes\eta_C)+h^0(\theta_{Q_\lambda})\equiv 0 \mod 2$, where θ_C and θ_{Q_λ} have the same parity (resp. opposite parity). Then $\theta_C\otimes\eta_C$ and θ_{Q_λ} have the same parity and we get the following possibilities:

	$h^0(\theta_C)$	$h^0(\theta_{Q_\lambda})$	$h^0(\theta_C\otimes\eta_C)$		
$\overline{\mathcal{P}}^{+}_{\mathrm{null}}$	0	0	0	$\rightsquigarrow$	contradiction
	1	1	1	$\rightsquigarrow$	$(\mathrm{R},+,1)$
$\overline{\mathcal{P}}^{-}_{\mathrm{null}}$	1	0	0	$\rightsquigarrow$	contradiction
	0	1	1	$\rightsquigarrow$	$(\mathrm{R},-,1)$

In order to deal with the cases not covered by remark 2.3.5, we first need to understand how theta characteristics on canonical genus 3 curves look like. Let us quickly elaborate on this.

Given a nonhyperelliptic arbitrary curve $X\in\mathcal{M}_3$, its canonical embedding realises it as a plane quartic $Q\hookrightarrow\mathbb{P}^2$, with the canonical series manifesting as the restriction of the hyperplane series to the curve.

Take a theta characteristic θ on X, with $\theta^{\otimes 2}\cong\omega_X\in W^2_4(X)$. Then we have $\deg(\theta)=2$, and θ is of type $\mathfrak{g}^r_2$ on X whenever $h^0(\theta)=r+1>0$. But X is not hyperelliptic, so it does not admit any $\mathfrak{g}^1_2$ and thus $r\le 0 \Rightarrow h^0(\theta)=r+1\le 1$. Therefore, the 36 even theta characteristics of X have $h^0(\theta)=0$ and the 28 odd ones have $h^0(\theta)=1$. In particular, X has no vanishing theta-nulls.

Let θ be odd. Then $|\theta| = \{D\}$ with $2D \sim K_X$, that is, $D = x + y$ and

$$2D \;=\; 2x + 2y \;=\; H \cap Q$$

for some hyperplane $H \hookrightarrow \mathbb{P}^2$. If moreover $x \neq y$ (for example, for X general), we get a one-to-one correspondence between odd theta characteristics of X and bitangents to its canonical model Q. Note that, if X is special enough for Q to have any hyperflexes, then the tangent lines at such points must be included in the correspondence too.

Remark 2.3.18. If X is hyperelliptic, the $\mathfrak{g}^1_2$ is a vanishing theta-null, and the $28 = \binom{8}{2}$ odd theta characteristics correspond to pairs of ramification points.

Keeping this description in mind, we tackle the remaining possibilities:

Possibilities $(\mathrm{R}, +, 1)$, $(\mathrm{R}, -, 1)$**.** In both of these cases, we have

$$\begin{aligned} h^0(\theta_C \otimes \eta_C) = 1 \;\;&\Rightarrow\;\; a_0^{\ell_C}(p) < 3 \leq a_1^{\ell_C}(p) \\ h^0(\theta_{Q_\lambda}) = 1 \;\;&\Rightarrow\;\; a_0^{\ell_{Q_\lambda}}(z) < g-3 \leq a_1^{\ell_{Q_\lambda}}(z) \end{aligned}$$

Now (C, p) is general, so we may assume that $p \notin \mathrm{supp}(\theta_C \otimes \eta_C)$. Therefore

$$\begin{aligned} h^0(\theta_C \otimes \eta_C\,(-p)) = h^0(\theta_C \otimes \eta_C) - 1 = 0 \;\;&\Rightarrow\;\; a_1^{\ell_C}(p) < 4 \\ h^0(\theta_C \otimes \eta_C\,(p)) = h^0(\theta_C \otimes \eta_C) = 1 \;\;&\Rightarrow\;\; a_0^{\ell_C}(p) < 2 \leq a_1^{\ell_C}(p) \end{aligned}$$

Hence $a_1^{\ell_C}(p) = 3$ and, via the limit $\mathfrak{g}^1_{g-1}$ condition, $a_0^{\ell_{Q_\lambda}}(z) = g - 4$. Moreover, we may assume that the basepoint z is not a hyperflex of Q_λ, as the pencil γ is also general. Consequently, $\mathrm{supp}(\theta_{Q_\lambda})$ does not consist of z twice, that is,

$$\mathrm{div}(\theta_{Q_\lambda}) \neq 2z \;\;\Rightarrow\;\; a_1^{\ell_{Q_\lambda}}(z) \leq g-2$$

which combined with the condition $a_0^{\ell_C}(p) + a_1^{\ell_{Q_\lambda}}(z) \geq g - 1$ yields $a_0^{\ell_C}(p) = 1$ and $a_1^{\ell_{Q_\lambda}}(z) = g - 2$. In turn, this means that $z \in \mathrm{supp}(\theta_{Q_\lambda})$, and that ℓ is a refined limit $\mathfrak{g}^1_{g-1}$ of the form

$$\begin{aligned} \ell_C \;&=\; |\theta_C \otimes \eta_C\,(2p)| + p \;\;\in\;\; G^1_{g-1}(C) \\ \ell_{Q_\lambda} \;&=\; |\theta_{Q_\lambda}\,(z)| + (g-4)z \;\;\in\;\; G^1_{g-1}(Q_\lambda) \end{aligned}$$

with vanishing sequences $(1,\,3)$ and $(g-4,\,g-2)$.

In conclusion, for each pair $(Q_\lambda,\,\theta_{Q_\lambda})$ consisting of a plane quartic Q_λ of γ equipped with an odd theta characteristic θ_{Q_λ} such that $z = \zeta(\lambda) \in \mathrm{supp}(\theta_{Q_\lambda})$, then every $\theta_C \in S^-_{g-3}(C)$ with $\theta_C \otimes \eta_C \in S^-_{g-3}(C)$ yields a limit $\mathfrak{g}^1_{g-1}$ as above, and these limit linear series are the only ones contributing to the intersection

$R \cap \overline{\mathcal{P}}^+_{\mathrm{null}}$ (resp. $\theta_C \in S^+_{g-3}(C)$ with $\theta_C \otimes \eta_C \in S^-_{g-3}(C)$, $R \cap \overline{\mathcal{P}}^-_{\mathrm{null}}$). The natural question then arises as to how many such pairs $(Q_\lambda,\, \theta_{Q_\lambda})$ there are.

We have discussed that odd theta characteristics θ_{Q_λ} of the plane curve Q_λ correspond to bitangents to the quartic. Under this identification, the condition $z = \zeta(\lambda) \in \mathrm{supp}(\theta_{Q_\lambda})$ corresponds to the bitangent having the basepoint z as one of its contact points. In particular, for each Q_λ we get only one candidate: the tangent line $T_z(Q_\lambda) \subset \mathbb{P}^2$, which will intersect Q_λ in two additional points. If we find out for how many values of λ these two points coincide, we will have found the pairs $(Q_\lambda,\, \theta_{Q_\lambda}) \equiv (Q_\lambda, T_z(Q_\lambda))$ we are trying to count.

Now, we can study the pencil γ by taking two general polynomials

$$F(\mathsf{x}) = \textstyle\sum_{i+j+k=4} a_{ijk}\, x_0^i x_1^j x_2^k \,, \quad G(\mathsf{x}) = \textstyle\sum_{i+j+k=4} b_{ijk}\, x_0^i x_1^j x_2^k \;\; \in \;\; \mathbb{C}[x_0, x_1, x_2]_4$$

and considering the family $\{Q_\lambda\}$ described by $H(\mathsf{x}, \lambda) = \lambda_0\, F(\mathsf{x}) + \lambda_1\, G(\mathsf{x}) = 0$, with basepoints $\{(\mathsf{x}, \lambda) \;/\; F(\mathsf{x}) = G(\mathsf{x}) = 0\} \ni \zeta(\lambda) = z$. By a suitable change of coordinates, we may assume $z = (1:0:0) \in \mathbb{P}^2$. If we write $H(\mathsf{x}, \lambda)$ as

$$H(\mathsf{x}, \lambda) \;\; = \;\; H_\lambda(\mathsf{x}) \;\; = \;\; \textstyle\sum_{i+j+k=4} c_{ijk}(\lambda)\, x_0^i x_1^j x_2^k \;\; \in \;\; \mathbb{C}[x_0, x_1, x_2]_4$$

with $c_{ijk}(\lambda) = \lambda_0\, a_{ijk} + \lambda_1\, b_{ijk} \in \mathbb{C}[\lambda]_1$, this means that $c_{400}(\lambda) = 0$.

Moreover, the tangent line $T_z(Q_\lambda)$ is given by

$$\tfrac{\partial H_\lambda(\mathsf{x})}{\partial x_0}(z)\, x_0 + \tfrac{\partial H_\lambda(\mathsf{x})}{\partial x_1}(z)\, x_1 + \tfrac{\partial H_\lambda(\mathsf{x})}{\partial x_2}(z)\, x_2 \;\; = \;\; c_{310}(\lambda)\, x_1 + c_{301}(\lambda)\, x_2 \;\; = \;\; 0$$

Switching to coordinates u, v on $T_z(Q_\lambda) = \mathbb{P}^1_{u,v}$, that is,

$$x_0 = u \;, \quad x_1 = -c_{301}(\lambda)\, v \;, \quad x_2 = c_{310}(\lambda)\, v$$

we see that $z = \{v = 0\} = (1:0) \in \mathbb{P}^1_{u,v}$ and that the intersection $Q_\lambda \cap T_z(Q_\lambda)$ is given by

$$\overline{H}_\lambda(u, v) \;\; = \;\; \textstyle\sum_{i+j+k=4} (-1)^j\, c_{301}(\lambda)^j\, c_{310}(\lambda)^k\, c_{ijk}(\lambda)\, u^i v^{j+k} \;\; = \;\; 0$$

Since the intersection contains z twice, this polynomial has no v^0, v^1 terms:

$$\begin{aligned} j+k=0 \;\; &\Rightarrow \;\; i=4 \;\; \rightsquigarrow & c_{400}(\lambda) &= 0 \\ j+k=1 \;\; &\Rightarrow \;\; i=3 \;\; \rightsquigarrow & -c_{301}(\lambda)\, c_{310}(\lambda) + c_{310}(\lambda)\, c_{301}(\lambda) &= 0 \end{aligned}$$

Factoring out v^2, we get a quadric

$$\begin{aligned} q_\lambda(u,v) &= \textstyle\sum_{\substack{0\leq i\leq 2\\ j+k=4-i}} (-1)^j\, c_{301}(\lambda)^j\, c_{310}(\lambda)^k\, c_{ijk}(\lambda)\, u^i v^{2-i} \\ &= c_3^{uu}(\lambda)\, u^2 + c_4^{uv}(\lambda)\, uv + c_5^{vv}(\lambda)\, v^2 \end{aligned}$$

whose roots correspond to the two additional points lying in $Q_\lambda \cap T_z(Q_\lambda)$. The degree of each summand $(-1)^j\, c_{301}(\lambda)^j\, c_{310}(\lambda)^k\, c_{ijk}(\lambda)$ is $j+k+1$, so we have

$$\begin{aligned} c_3^{uu}(\lambda) &= \textstyle\sum_{j+k=2} (-1)^j\, c_{301}(\lambda)^j\, c_{310}(\lambda)^k\, c_{2jk}(\lambda) \;\in\; \mathbb{C}[\lambda_0,\lambda_1]_3 \\ c_4^{uv}(\lambda) &= \textstyle\sum_{j+k=3} (-1)^j\, c_{301}(\lambda)^j\, c_{310}(\lambda)^k\, c_{1jk}(\lambda) \;\in\; \mathbb{C}[\lambda_0,\lambda_1]_4 \\ c_5^{vv}(\lambda) &= \textstyle\sum_{j+k=4} (-1)^j\, c_{301}(\lambda)^j\, c_{310}(\lambda)^k\, c_{0jk}(\lambda) \;\in\; \mathbb{C}[\lambda_0,\lambda_1]_5 \end{aligned}$$

Finally, the values of λ for which the roots of $q_\lambda(u,v)$ coincide are determined by the roots of the discriminant

$$\Delta(\lambda) \;=\; \Delta(q_\lambda(u,v)) \;=\; c_4^{uv}(\lambda)^2 - 4\, c_3^{uu}(\lambda)\, c_5^{vv}(\lambda) \;\in\; \mathbb{C}[\lambda_0,\lambda_1]_8$$

which is an octic polynomial. Therefore we obtain

$$\#\{(Q_\lambda, \theta_{Q_\lambda}) \;/\; z \in \operatorname{supp}(\theta_{Q_\lambda})\} \;=\; \#\{\lambda \in \mathbb{P}^1 \;/\; \Delta(\lambda) = 0\} \;=\; 8 \;=\; 2^3$$

and we can finally observe the appearance of the intersection numbers provided by theorem 2.3.14 and remark 2.3.17, as the count becomes:

$$\begin{aligned} &\#\{(Q_\lambda, \theta_{Q_\lambda}) \;/\; z \in \operatorname{supp}(\theta_{Q_\lambda})\} \cdot \\ &\cdot \#\{\theta_C \in S^-_{g-3}(C) \;/\; \theta_C \otimes \eta_C \in S^-_{g-3}(C)\} \;=\; 2^3 \cdot \mathrm{N}^-_{g-3} \;=\; 2^{g-1}(2^{g-4}-1) \end{aligned}$$

$$\begin{aligned} &\#\{(Q_\lambda, \theta_{Q_\lambda}) \;/\; z \in \operatorname{supp}(\theta_{Q_\lambda})\} \cdot \\ &\cdot \#\{\theta_C \in S^+_{g-3}(C) \;/\; \theta_C \otimes \eta_C \in S^-_{g-3}(C)\} \;=\; 2^3 \cdot \mathrm{N}^\pm_{g-3} \;=\; 2^{2g-5} \end{aligned}$$

In particular, this reveals the lack of contribution from singular fibers.

Example 2.3.19 (more irreducible nodal curves)**.** If we recall the family

$$\mathcal{Y} \;\equiv\; \{B_{py}\}_{y\in B} \;\subset\; \Delta_0 \;\subset\; \overline{\mathcal{M}}_g$$

from example 2.1.26, which was lifted to a test curve $Y_0 \subset \Delta_0^{\mathrm{t}} \subset \overline{\mathcal{R}}_g$, then there are two more standard lifts Z_0 and T_0 in $\overline{\mathcal{R}}_g$, which arise when $\mathcal{Y}$ is pulled back by the maps $\Delta_0^{\mathrm{p}} \to \Delta_0$ and $\Delta_0^{\mathrm{b}} \to \Delta_0$ respectively:

$$\begin{aligned} Z_0 &\equiv \{(B_{py},\, \eta_y^{\mathrm{p}}) \;/\; \eta_y^{\mathrm{p}} \in \Delta_0^{\mathrm{p}}(B_{py})\}_{y\in B} &&\subset \Delta_0^{\mathrm{p}} \subset \overline{\mathcal{R}}_g \\ T_0 &\equiv \{(B \cup_{p\sim 0,\, y\sim\infty} E,\, \eta_y^{\mathrm{b}}) \;/\; \eta_y^{\mathrm{b}} \in \Delta_0^{\mathrm{b}}(B_{py})\}_{y\in B} &&\subset \Delta_0^{\mathrm{b}} \subset \overline{\mathcal{R}}_g \end{aligned}$$

If we set $k = \#R_{g-1}(B) = 2^{2g-2} - 1$, we can see that their intersection table is:

	λ	δ_0^{t}	δ_0^{p}	δ_0^{b}	δ_1^{n}	δ_1^{t}	δ_1^{p}	$\delta_{(j\geq 2)}$
Y_0	0	$2-2g$	0	0	1	0	0	0
Z_0	0	0	$4k(1-g)$	0	0	k	k	0
T_0	0	0	0	$2^{2g-2}(1-g)$	1	0	k	0

Note that $\deg(\Delta_0^{\mathrm{p}}|\Delta_0) = 2k$ and $\deg(\Delta_0^{\mathrm{b}}|\Delta_0) = 2^{2g-2} = k+1$.

Remark 2.3.20. Applying theorem 2.3.14 to example 2.3.19, it follows that

$$\begin{aligned} Z_0 \cdot \overline{\mathcal{P}}^{+}_{\mathrm{null}} &= (2^{2g-2}-1)\,\mu &&= \#R_{g-1}(B)\cdot\mu \\ Z_0 \cdot \overline{\mathcal{P}}^{-}_{\mathrm{null}} &= (2^{2g-2}-1)\,\mu &&= \#R_{g-1}(B)\cdot\mu \\ T_0 \cdot \overline{\mathcal{P}}^{+}_{\mathrm{null}} &= 2^{g-2}(2^{g-1}+1)\,\mu &&= \#S^{+}_{g-1}(B)\cdot\mu \\ T_0 \cdot \overline{\mathcal{P}}^{-}_{\mathrm{null}} &= 2^{g-2}(2^{g-1}-1)\,\mu &&= \#S^{-}_{g-1}(B)\cdot\mu \end{aligned}$$

with the factor $\mu = Y_0^{\mathrm{n}} \cdot \overline{\Theta}_{\mathrm{null}} = 2^{g-3}(2^{g-2}(g-3)+1)$ indicating the number of nodal curves B_{py} in $\mathcal{Y}$ that admit a vanishing theta-null $\theta_y \in \overline{\Theta}_{\mathrm{null}}(B_{py})$, which we computed in the argument preceding preposition 2.3.13. Once more, it may be interesting to provide an interpretation of these results.

According to example 2.1.17, any Prym root $\eta_B \in R_{g-1}(B)$ gives rise to two elements $\eta_y^{\mathrm{p},+}$, $\eta_y^{\mathrm{p},-} \in \Delta_0^{\mathrm{p}}(B_{py})$, depending on which of the two possible gluings $\eta_B|_p \cong \eta_B|_y$ is chosen. In particular, for each pair $(B_{py},\theta_y) \in \overline{\Theta}_{\mathrm{null}}$, tensoring θ_y by either $\eta_y^{\mathrm{p},+}$ or $\eta_y^{\mathrm{p},-}$ produces stable spin curves of opposite parity, so that

$$\begin{aligned} (B_{py},\,\eta_y^{\mathrm{p},+}) &\in Z_0 \cap \overline{\mathcal{P}}^{+}_{\mathrm{null}} \\ (B_{py},\,\eta_y^{\mathrm{p},-}) &\in Z_0 \cap \overline{\mathcal{P}}^{-}_{\mathrm{null}} \end{aligned}$$

which explains the emergence of the factors

$$\begin{aligned} k &= \#\{\eta_B \in R_{g-1}(B)\} \\ \mu &= \#\{(B_{py},\,\theta_y) \in \overline{\Theta}_{\mathrm{null}}\} \end{aligned}$$

in the intersection numbers $Z_0 \cdot \overline{\mathcal{P}}^{+}_{\mathrm{null}}$ and $Z_0 \cdot \overline{\mathcal{P}}^{-}_{\mathrm{null}}$.

Similarly, for each pair $(B_{py},\theta_y) \in \overline{\Theta}_{\mathrm{null}}$, the root $\theta_y|_B \in \sqrt{\omega_B(p+q)}$ can be subtracted from any theta characteristic $\theta_B \in S_{g-1}(B) = \sqrt{\omega_B}$ so as to create a root $\eta_B \in \sqrt{\mathcal{O}_B(-p-q)}$. This in turn yields a unique stable Prym curve

$$(X,\eta_y^{\mathrm{b}}) \;=\; (B \cup_{p\sim 0,\, y\sim\infty} E,\; \eta_y^{\mathrm{b}}) \;\in\; T_0 \cap \overline{\mathcal{P}}_{\mathrm{null}}$$

such that η_y^{b} restricts to $(\eta_B, \mathcal{O}_E(1))$ on $\mathrm{Pic}(B) \oplus \mathrm{Pic}(E)$. Furthermore, (X, η_y^{b}) lies in $\overline{\mathcal{P}}_{\mathrm{null}}^{+}$ (resp. $\overline{\mathcal{P}}_{\mathrm{null}}^{-}$) whenever θ_B is even (resp. odd) by construction of the Prym-null divisors, which brings to light the connection between

$$\#S_{g-1}^{+}(B), \quad \#S_{g-1}^{-}(B), \quad \mu$$

and the intersection numbers $T_0 \cdot \overline{\mathcal{P}}_{\mathrm{null}}^{+}$ and $T_0 \cdot \overline{\mathcal{P}}_{\mathrm{null}}^{-}$.

2.4 Moduli spaces of multiple roots

Initially, section 2.3 was meant to be the end of our journey. Nonetheless, it soon became clear that $\overline{\mathcal{R}}_g$ is not really the most natural environment in which to discuss the Prym-null divisors, in the same manner that the divisor $\overline{\mathcal{M}}_g^{\mathrm{null}}$ of $\overline{\mathcal{M}}_g$ finds a more intrinsic expression in $\overline{\mathcal{S}}_g^{+}$ thanks to its theta-null counterpart. Instead of $\overline{\mathcal{R}}_g$, we would rather work in a moduli space of curves equipped with both a Prym root and a theta characteristic, or *Prym-spin curves*.

Unfortunately, compactifications of spaces of curves carrying multiple roots of different bundles have not been constructed in the literature yet. The closest silver lining that we get is offered by Sertöz, who in [Ser17] and [Ser19] focuses on the geometry of multiple spin curves and builds compactifications of moduli spaces of curves carrying multiple roots of the same bundle.

In this section, we review the original construction $\mathcal{S}_{\mathrm{lim}}^{2}(\mathcal{N})$ given by Sertöz and propose an extension $\mathcal{S}_{\mathrm{lim}}^{2}(\mathcal{N}_1, \mathcal{N}_2)$ for roots of different bundles, as well as a compactification $\overline{\mathcal{RS}}_g$ of the space of *Prym-spin curves* of genus g. Next, we examine the consequences that the existence of these spaces would have for the Prym-null divisors, and comment on the intriguing possibility of compactifying the standard tensor product of square roots.

2.4.1 Multiple limit roots

We want to study the space $\mathcal{RS}_g \to \mathcal{M}_g$ parametrizing points of the form

$$(\mathcal{RS}_g)_{\mathbb{C}} \;\; = \;\; \{(C, \eta, \theta) \;/\; C \in (\mathcal{M}_g)_{\mathbb{C}},\, \eta \in R_g(C),\, \theta \in S_g(C)\}$$

or rather, to find a good compactification of this space. More generally, we can take the universal curve $\phi\colon \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$, together with two line bundles $\mathcal{N}_1$, $\mathcal{N}_2$ on

$\overline{\mathcal{C}}_g$ of relative even degree, and attempt to build a space

$$\overline{\mathcal{S}}^2(\mathcal{N}_1, \mathcal{N}_2) \to \overline{\mathcal{M}}_g$$

of curves with a *double root* of $(\mathcal{N}_1, \mathcal{N}_2)$, that is, curves carrying both a square root of $\mathcal{N}_1$ and a square root of $\mathcal{N}_2$ at the same time. Observe that

$$\overline{\mathcal{RS}}_g \quad = \quad \overline{\mathcal{S}}^2(\mathcal{O}_{\overline{\mathcal{C}}_g}, \omega_\phi) - \overline{\mathcal{S}}_g$$

should these moduli spaces exist.

Remark 2.4.1. Since $\mathcal{RS}_g = \mathcal{R}_g \times_{\mathcal{M}_g} \mathcal{S}_g \to \mathcal{M}_g$ is the fiber product of each of the moduli spaces $\mathcal{R}_g$ and $\mathcal{S}_g$ over $\mathcal{M}_g$, it may seem like the best candidate for $\overline{\mathcal{RS}}_g$ is the fiber product of the corresponding compactifications, that is,

$$\overline{\mathcal{R}}_g \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g$$

which is indeed compact. However, the modular interpretation of $\overline{\mathcal{R}}_g \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{S}}_g$ is not quite natural, as the roots η, θ parametrized by a point

$$((X_1, \eta), (X_2, \theta)) \quad \in \quad (\overline{\mathcal{R}}_g \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{S}}_g)_{\mathbb{C}}$$

are defined over quasistable curves X_1, X_2 that have the same stable model, but whose exceptional components can possibly differ. For instance, consider

$$\left.\begin{array}{ccccc} (X_1, \eta) & \in & \Delta_0^{\mathrm{b}} & \subset & \overline{\mathcal{R}}_g \\ (X_2, \theta) & \in & \Delta_0^{\mathrm{n}} & \subset & \overline{\mathcal{S}}_g^+ \end{array}\right\} \Rightarrow \left\{\begin{array}{ccccc} \mathrm{st}(X_1) & = & \mathrm{st}(X_2) & \in & \overline{\mathcal{M}}_g \\ X_1 & \neq & X_2 & & \end{array}\right.$$

The potential discrepancy $X_1 \neq X_2$ is somewhat cumbersome, and prevents the fiber product from being normal, as shown in [Ser17] Prop. 5.56.

In both [Ser17] and [Ser19], the partial identification issue posed by remark 2.4.1 is tackled through the introduction of a master curve X dominating both X_1 and X_2, which provides a common ground for η and θ to be pulled back to and studied over. Although the referenced work focuses only on multiple roots of the same bundle, its ideas can be carried over to the more general setting of multiple roots of several (possibly different) bundles.

We first describe the original treatment of the case $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}$, and then explore the natural extension to $\mathcal{N}_1 \neq \mathcal{N}_2$.

To begin with, let Y be a stable curve (of genus g), and N be a line bundle on Y of even degree d. In more general terms, let $\mathcal{Y} \to S$ be a stable curve (of genus g) over a base S, and $\mathcal{N}$ be a line bundle on $\mathcal{Y}$ of relative even degree d. Recall that we are interested in $\phi\colon \mathcal{Y} = \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ and $\mathcal{N} \in \{\mathcal{O}_{\overline{\mathcal{C}}_g}, \omega_\phi\}$.

Definition 2.4.2. Given a subset $Z \subset \mathrm{Sing}(Y)$ of nodes of Y with ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_Y$, the projection

$$\pi\colon X = \mathbb{P}(\mathcal{I}_Z) = \underline{\mathrm{Proj}}(\mathrm{Sym}_{\mathcal{O}_Y} \mathcal{I}_Z) \longrightarrow Y$$

is referred to as an *exceptional blow-up* of Y at Z. Note that each node $z \in Z$ is replaced by an exceptional component $\pi^{-1}(z) \cong \mathbb{P}^1$, so that X is quasistable. Similarly, a morphism $\mathcal{X} \to \mathcal{Y}$ over S is said to be an *exceptional blow-up* of $\mathcal{Y}$ if every geometric fiber $\mathcal{X}_s \to \mathcal{Y}_s$ is an exceptional blow-up of $\mathcal{Y}_s$.

The prime example of an exceptional blow-up is the stabilisation

$$X \longrightarrow Y = \mathrm{st}(X)$$

of a quasistable curve X, in which case Z corresponds to the contraction of the exceptional components of X.

Definition 2.4.3. We say that (π, L, α) *is a limit root of* N (resp. *stabilizes to a limit root of* N) if:

(i) $\pi\colon X \to Y$ is an exceptional blow-up of Y.

(ii) $L \in \mathrm{Pic}^{d/2}(X)$ is a line bundle such that $\deg(L|_E) = 1$ (resp. $\in \{0, 1\}$) for every exceptional component E of X.

(iii) $\alpha\colon L^{\otimes 2} \to \pi^* N$ is a sheaf homomorphism such that the restriction $\alpha|_A$ is generically non-zero for every non-exceptional component A of X.

In relative fashion, we say that $(\pi, \mathcal{L}, \alpha)$ *is a limit root of* $\mathcal{N}$ (resp. *stabilizes to a limit root of* $\mathcal{N}$) if:

(i) $\pi\colon \mathcal{X} \to \mathcal{Y}$ is an exceptional blow-up of $\mathcal{Y}$.

(ii) $\mathcal{L} \in \mathrm{Pic}^{d/2}(\mathcal{X})$ is a line bundle on $\mathcal{X}$.

(iii) $\alpha\colon \mathcal{L}^{\otimes 2} \to \pi^* \mathcal{N}$ is a sheaf homomorphism.

(iv) Every geometric fiber $(\pi_s,\ \mathcal{L}_s,\ \alpha_s)$ is a limit root of $\mathcal{N}_s$ (resp. stabilizes to a limit root of $\mathcal{N}_s$).

An isomorphism $(\mathcal{X} \to \mathcal{Y},\ \mathcal{L},\ \alpha) \cong (\mathcal{X}' \to \mathcal{Y},\ \mathcal{L}',\ \alpha')$ is a pair (φ, ψ) where:

(i) $\varphi\colon \mathcal{X} \cong \mathcal{X}'$ is an isomorphism over $\mathcal{Y}$.

(ii) $\psi\colon \varphi^*(\mathcal{L}') \cong \mathcal{L}$ is a sheaf isomorphism such that $\varphi^*(\alpha') = \alpha \circ \psi^{\otimes 2}$.

The moduli space of limit roots of $\mathcal{N}$ is denoted $\mathcal{S}_{\mathrm{lim}}(\mathcal{N}) \to S$.

Comparing these notions with definitions 2.1.8 and 2.1.9, we see that

$$\begin{aligned}\overline{\mathcal{R}}_g &= \mathcal{S}_{\lim}(\mathcal{O}_{\overline{\mathcal{C}}_g}) - \overline{\mathcal{M}}_g \\ \overline{\mathcal{S}}_g &= \mathcal{S}_{\lim}(\omega_\phi)\end{aligned}$$

so stable Prym and spin curves are, as expected, particular cases of limit roots. On the other hand, observe that if (π, L, α) stabilizes to a limit root of N, then contracting the exceptional components where L is trivial yields a limit root of N, which not only justifies the name chosen for this concept but also connects it with the master curve idea mentioned earlier. Indeed, given two limit roots

$$(\pi_1\colon X_1 \to Y,\, L_1,\, \alpha_1),\quad (\pi_2\colon X_2 \to Y,\, L_2,\, \alpha_2) \quad \in \quad \mathcal{S}_{\lim}(\mathcal{N})_{\mathbb{C}}$$

then $X = X_1 \times_Y X_2$ is a quasistable curve fitting into the diagram

$$\begin{tikzcd} & X_1 \arrow[dr, "\pi_1"] & \\ X \arrow[ur, "\rho_1"] \arrow[rr, "\pi"] \arrow[dr, "\rho_2"'] & & Y \\ & X_2 \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

so that $(\pi\colon X \to Y,\, \rho_i^* L_i,\, \rho_i^* \alpha_i)$ stabilizes to (π_i, L_i, α_i) for each $i \in \{1, 2\}$. This addresses the partial identification issue, but a new problem arises if we require the moduli space of double roots of $\mathcal{N}$ to have finite fibers over $\overline{\mathcal{M}}_g$.

As already mentioned in the case of stable Prym curves (see remark 2.1.18), the collection of all limit roots having an exceptional component over a certain, fixed node gives rise to a finite number of isomorphism classes, since individual limit roots are considered up to the isomorphisms from definition 2.4.3 and not simply as isolated line bundles. In particular, this means that the moduli space of (single) limit roots of $\mathcal{N}$ does have finite fibers over $\overline{\mathcal{M}}_g$.

Unfortunately, when trying to pair two limit roots that are exceptional over the same node, the corresponding isomorphisms now modify both bundles simultaneously. As a result, we end up with infinitely many nonisomorphic ways of lifting one such pair, most of which need to be filtered out in order to obtain a workable space. This can be achieved through the introduction of an additional piece of information that synchronizes the pullbacks of both roots.

Definition 2.4.4. Let $\{(\pi_i\colon \mathcal{X}_i \to \mathcal{Y},\, \mathcal{L}'_i,\, \alpha'_i)\}_{i=1,2}$ be a pair of limit roots of $\mathcal{N}$, and $\{\pi\colon \mathcal{X} \to \mathcal{Y},\, \mathcal{L}_i,\, \alpha_i\}_{i=1,2}$ be such that $(\pi, \mathcal{L}_i, \alpha_i)$ stabilizes to $(\pi_i, \mathcal{L}'_i, \alpha'_i)$ for each $i \in \{1, 2\}$. Then we have partial stabilization maps $\{\rho_i\colon \mathcal{X} \to \mathcal{X}_i\}_{i=1,2}$ and a commutative diagram:

$$\begin{tikzcd} & \mathcal{X}_1 \arrow[dr, "\pi_1"] & \\ \mathcal{X} \arrow[ur, "\rho_1"] \arrow[rr, "\pi"] \arrow[dr, "\rho_2"'] & & \mathcal{Y} \\ & \mathcal{X}_2 \arrow[ur, "\pi_2"'] & \end{tikzcd} \qquad \left\{ \begin{array}{lclcl} \rho_i^*\mathcal{L}'_i & = & \mathcal{L}_i & \in & \mathrm{Pic}(\mathcal{X}) \\ \rho_i^*\alpha'_i & = & \alpha_i : & \mathcal{L}_i^{\otimes 2} & \to \pi^*\mathcal{N} \end{array} \right.$$

We say that a *synchronization data* of $\{\pi, \mathcal{L}_i, \alpha_i\}_{i=1,2}$ is a collection $\{\mathcal{F}, \chi_i\}_{i=1,2}$ of a line bundle $\mathcal{F}$ on $\mathcal{X}$ and sheaf homomorphisms $\chi_i \colon \mathcal{F} \to \mathcal{L}_i^{\otimes 2}$ such that:

(i) Each χ_i restricts to an isomorphism over the largest open set on which ρ_i is an isomorphism, which we denote by $V_i = V(\mathcal{L}_i) \subset \mathcal{X}$.

(ii) It holds that $\alpha_1 \circ \chi_1 = \alpha_2 \circ \chi_2$, that is, the diagram

$$\begin{tikzcd} & \mathcal{L}_1^{\otimes 2} \arrow[dr, "\alpha_1"] & \\ \mathcal{F} \arrow[ur, "\chi_1"] \arrow[dr, "\chi_2"'] \arrow[rr, phantom, "\circlearrowleft"] & & \pi^*\mathcal{N} \\ & \mathcal{L}_2^{\otimes 2} \arrow[ur, "\alpha_2"'] & \end{tikzcd}$$

of line bundles on $\mathcal{X}$ is also commutative.

The combined collection $\{\pi, \mathcal{L}_i, \alpha_i, \mathcal{F}, \chi_i\}_{i=1,2}$ is called a *double limit root* of $\mathcal{N}$, and isomorphisms of double limit roots are pairs of isomorphisms of limit roots which are compatible with the synchronization data. Finally, the moduli space of double limit roots of $\mathcal{N}$ is denoted $\mathcal{S}^2_{\lim}(\mathcal{N}) \to S$.

The previous discussion may be clarified by looking at a basic example that encapsulates both the lifting issue and its solution.

Example 2.4.5. Let us see how adding the synchronization data yields a finite number of lifts for any given pair of limit roots. Using the notation of example 2.1.17, consider the exceptional blow-up

$$\pi\colon \quad X = B \cup_{p\sim 0,\, q\sim\infty} E \quad\longrightarrow\quad \mathrm{st}(X) = Y = B_{pq}$$

of an irreducible 1-nodal curve Y, together with two limit roots

$$(\pi\colon X \to Y,\, L_1,\, \alpha_1), \quad (\pi\colon X \to Y,\, L_2,\, \alpha_2) \quad\in\quad \mathcal{S}_{\lim}(N)$$

of some line bundle $N \in \mathrm{Pic}(Y)$. Then the Picard groups of Y and X sit in the middle of interlocking exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C}^* & \longrightarrow & \mathrm{Pic}(Y) & \xrightarrow{\nu^*} & \mathrm{Pic}(B) & \longrightarrow & 0 \\
0 & \longrightarrow & \mathbb{C}^* & \xrightarrow{\zeta} & \mathrm{Pic}(X) & \xrightarrow{\xi} & \mathrm{Pic}(B) \oplus \mathrm{Pic}(E) & \longrightarrow & 0
\end{array}
$$

from which we can deduce that, for each $i \in \{1, 2\}$, the line bundle $L_i \in \mathrm{Pic}(X)$ is determined up to a constant by its restriction to B. Indeed, if we take

$$
\begin{cases}
\varphi_\lambda \colon X \to X \quad \text{such that} \quad \varphi_\lambda|_B = \mathrm{Id}_B, \quad \varphi_\lambda|_E \colon e \mapsto \lambda \cdot e, \quad \text{and} \\
M \in \mathrm{Pic}(X) \quad \text{such that} \quad M|_E = \mathcal{O}_E(k),
\end{cases}
$$

with $\lambda \in \mathbb{C}^*$ and $k \in \mathbb{Z}^+$, then $(\varphi_\lambda)^* M = M \otimes \zeta(\lambda^k)$, so in particular every line bundle of the form

$$
L_{i,\lambda_i} \;=\; (\varphi_{\lambda_i})^* L_i \;=\; L_i \otimes \zeta(\lambda_i) \;\in\; \xi^{-1}(L_i|_B, \mathcal{O}_E(1)) \;\cong\; \mathbb{C}^*
$$

gives rise to an isomorphism of (single) limit roots

$$
(\varphi_{\lambda_i}, \mathrm{Id})\colon \quad (\pi,\, L_{i,\lambda_i},\, \alpha_{i,\lambda_i}) \;\cong\; (\pi,\, L_i,\, \alpha_i) \;\in\; \mathcal{S}_{\mathrm{lim}}(N)
$$

However, when taken as a pair of limit roots, two sets of choices

$$
(L_{1,\lambda_1},\, L_{2,\lambda_2}),\ (L_{1,\mu_1},\, L_{2,\mu_2}) \;\in\; \xi^{-1}(L_1|_B, \mathcal{O}_E(1)) \times \xi^{-1}(L_2|_B, \mathcal{O}_E(1))
$$

only give rise to isomorphic (unsynchronized) double roots whenever

$$
\lambda_1/\mu_1 = \lambda_2/\mu_2 \;\Leftrightarrow\; \lambda_1/\lambda_2 = \mu_1/\mu_2 \;\Leftrightarrow\; (\lambda_1 : \lambda_2) = (\mu_1 : \mu_2)
$$

since the scaling now happens uniformly. In other words, each of the infinitely many points

$$
(\lambda_1 : \lambda_2) \;\in\; \mathbb{P}(\mathbb{C}^2) - \{(1:0), (0:1)\} \;\cong\; \mathbb{C}^*
$$

corresponds to a different (unsynchronized) lift of $\{(\pi, L_i, \alpha_i)\}_{i=1,2}$. This can be avoided with the inclusion of synchronization data, which restricts the selection of representatives L_1 and L_2 to those having isomorphic squares:

$$
\begin{tikzcd}
 & L_1^{\otimes 2} \arrow[dr, "\alpha_1"] & \\
F \arrow[ur, "\cong"] \arrow[dr, "\cong"'] \arrow[rr, phantom, "\circlearrowleft"] & & \pi^* N \\
 & L_2^{\otimes 2} \arrow[ur, "\alpha_2"'] &
\end{tikzcd}
$$

As a result, any other choice is limited by a diagram

$$\begin{tikzcd}[column sep=small] & L_{1,\lambda_1}^{\otimes 2} = L_1^{\otimes 2} \otimes \zeta(\lambda_1^2) \arrow[dr] \arrow[dd, phantom, "\circlearrowleft"] & \\ F' \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & & \pi^* N \\ & L_{2,\lambda_2}^{\otimes 2} = L_2^{\otimes 2} \otimes \zeta(\lambda_2^2) \arrow[ur] & \end{tikzcd}$$

with $L_1^{\otimes 2} = F = L_2^{\otimes 2} \in \mathrm{Pic}(X)$, so that $\lambda_1^2 = \lambda_2^2 \in \mathbb{C}^*$ and thus

$$\lambda_1^2 = \lambda_2^2 \quad \Leftrightarrow \quad (\lambda_1/\lambda_2)^2 = 1 \quad \Leftrightarrow \quad (\lambda_1 : \lambda_2) \in \{(1:1),\, (1:-1)\}$$

In conclusion, we are left with only two possible nonisomorphic lifts

$$\{\pi, (L_{1,1}, L_{2,1}), F\}, \quad \{\pi, (L_{1,1}, L_{2,-1}), F\} \quad \in \quad \mathcal{S}^2_{\mathrm{lim}}(N)$$

of the pair $\{(\pi, L_i, \alpha_i)\}_{i=1,2} \in \mathcal{S}_{\mathrm{lim}}(N) \times \mathcal{S}_{\mathrm{lim}}(N)$, instead of infinitely many.

After some work, the case $\mathcal{N}_1 = \mathcal{N}_2$ has been successfully resolved, since the moduli space $\mathcal{S}^2_{\mathrm{lim}}(\mathcal{N})$ provides the compactification we were aiming for:

Theorem 2.4.6. *Let $\phi\colon \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ be the universal curve, and $\mathcal{N} \in \{\mathcal{O}_{\overline{\mathcal{C}}_g}, \omega_\phi\}$. Then the moduli space $\mathcal{S}^2_{\mathrm{lim}}(\mathcal{N})$ of double limit roots is proper and contains the moduli space of smooth double roots as an open subset. Moreover, the forgetful map $\mathcal{S}^2_{\mathrm{lim}}(\mathcal{N}) \to \overline{\mathcal{M}}_g$ is quasi-finite.*

Proof. This is the main result of [Ser17] Part I, or [Ser19] Th. 6.8. □

It is clear that in order to build $\mathcal{S}^2_{\mathrm{lim}}(\mathcal{N}_1, \mathcal{N}_2)$ we need to extend the notion of synchronizing to a setting where $\mathcal{N}_1$ and $\mathcal{N}_2$ may not be the same. The main apparent obstacle is the fact that the morphisms α_i do not have the same target anymore. We can tackle this by observing that synchronization data serves the purpose of controlling how the squares of the synchronized limit roots actually look like, and how they relate to one another. Thus, if we want to take roots of two different line bundles $\mathcal{N}_1$ and $\mathcal{N}_2$, we just need to keep track of the difference between these bundles:

$$\mathcal{D} \quad = \quad \mathcal{D}(\mathcal{N}_1, \mathcal{N}_2) \quad = \quad \mathcal{N}_1 \otimes \mathcal{N}_2^\vee \quad \in \quad \mathrm{Pic}(\mathcal{Y})$$

With this in mind, definition 2.4.4 can be readily generalised.

Definition 2.4.7. Let $\{(\pi_i\colon \mathcal{X}_i \to \mathcal{Y},\, \mathcal{L}'_i,\, \alpha'_i)\}_{i=1,2}$ be a pair of limit roots of $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively, and again $\{\pi\colon \mathcal{X} \to \mathcal{Y},\, \mathcal{L}_i,\, \alpha_i\}_{i=1,2}$ be such that $(\pi, \mathcal{L}_i, \alpha_i)$

stabilizes to $(\pi_i, \mathcal{L}'_i, \alpha'_i)$ for each $i \in \{1,2\}$, with a commutative diagram:

$$\begin{tikzcd} & \mathcal{X}_1 \arrow[dr, "\pi_1"] & \\ \mathcal{X} \arrow[ur, "\rho_1"] \arrow[rr, "\pi"] \arrow[dr, "\rho_2"'] & & \mathcal{Y} \\ & \mathcal{X}_2 \arrow[ur, "\pi_2"'] & \end{tikzcd} \qquad \begin{cases} \rho_i^* \mathcal{L}'_i & = \quad \mathcal{L}_i \quad \in \quad \mathrm{Pic}(\mathcal{X}) \\ \rho_i^* \alpha'_i & = \quad \alpha_i \;:\; \mathcal{L}_i^{\otimes 2} \;\to\; \pi^* \mathcal{N}_i \end{cases}$$

We say that a *synchronization data* of $\{\pi, \mathcal{L}_i, \alpha_i\}_{i=1,2}$ is a collection $\{\mathcal{F}, \chi_i\}_{i=1,2}$ of a line bundle $\mathcal{F}$ on $\mathcal{X}$ and sheaf homomorphisms

$$\begin{array}{llll} \chi_1 \colon & \mathcal{F} \otimes \pi^* \mathcal{D} & \longrightarrow & \mathcal{L}_1^{\otimes 2} \\ \chi_2 \colon & \mathcal{F} & \longrightarrow & \mathcal{L}_2^{\otimes 2} \end{array}$$

such that:

(i) Each χ_i restricts to an isomorphism over $V_i = V(\mathcal{L}_i) \subset \mathcal{X}$.

(ii) The sequence $\mathcal{F} \to \mathcal{L}_2^{\otimes 2} \to \pi^* \mathcal{N}_2$ induces a commutative diagram

$$\begin{tikzcd} & \mathcal{L}_1^{\otimes 2} \arrow[dr, "\alpha_1"] & \\ \mathcal{F} \otimes \pi^* \mathcal{D} \arrow[ur, "\chi_1"] \arrow[dr, "\chi_2 \otimes \mathrm{Id}"'] & \circlearrowleft & \pi^* \mathcal{N}_1 \\ & \mathcal{L}_2^{\otimes 2} \otimes \pi^* \mathcal{D} \arrow[ur, "\alpha_2 \otimes \mathrm{Id}"'] & \end{tikzcd}$$

of line bundles on $\mathcal{X}$.

Accordingly, we say that $\{\pi, \mathcal{L}_i, \alpha_i, \mathcal{F}, \chi_i\}_{i=1,2}$ is a *double limit root* of $(\mathcal{N}_1, \mathcal{N}_2)$, and denote their corresponding moduli space as $\mathcal{S}^2_{\mathrm{lim}}(\mathcal{N}_1, \mathcal{N}_2) \to S$.

Remark 2.4.8. Observe that:

(i) If we set $\mathcal{N}_1 = \mathcal{N}_2 = \mathcal{N}$, then $\mathcal{D} = \mathcal{O}_{\mathcal{Y}}$ is trivial and we get

$$\mathcal{S}^2_{\mathrm{lim}}(\mathcal{N}, \mathcal{N}) \;\cong\; \mathcal{S}^2_{\mathrm{lim}}(\mathcal{N})$$

since definition 2.4.4 is recovered.

(ii) If we interchange $\mathcal{N}_1$ and $\mathcal{N}_2$, then the new difference bundle

$$\mathcal{D}(\mathcal{N}_2, \mathcal{N}_1) \;=\; \mathcal{N}_2 \otimes \mathcal{N}_1^{\vee} \;=\; \mathcal{D}(\mathcal{N}_1, \mathcal{N}_2)^{\vee} \;=\; \mathcal{D}^{\vee}$$

is the dual of the previous one, which yields an identification

$$\mathcal{S}^2_{\mathrm{lim}}(\mathcal{N}_1, \mathcal{N}_2) \;\cong\; \mathcal{S}^2_{\mathrm{lim}}(\mathcal{N}_2, \mathcal{N}_1)$$

where $(\mathcal{F} \otimes \pi^* \mathcal{D}, \chi_2, \chi_1)$ synchronizes $(\mathcal{L}_2, \mathcal{L}_1)$.

(iii) If we take $n \in \mathbb{N}$ and a sequence $\{\mathcal{N}_i\}_{i=1}^n$ of line bundles on $\mathcal{Y}$, then we can consider all of their associated difference bundles

$$\mathcal{D}_{ij} \quad = \quad \mathcal{D}(\mathcal{N}_i, \mathcal{N}_j) \quad = \quad \mathcal{N}_i \otimes \mathcal{N}_j^\vee \quad \in \quad \mathrm{Pic}(\mathcal{Y})$$

and define $\mathcal{S}_{\lim}^n(\mathcal{N}_i)_{i=1}^n$ by means of the synchronization data

$$\mathcal{F}' \in \mathrm{Pic}(\mathcal{X}), \qquad \chi_i\colon \mathcal{F}' \otimes \pi^*\mathcal{N}_i \to \mathcal{L}_i^{\otimes 2}$$

such that each χ_i restricts to an isomorphism over V_i and the diagrams

$$\begin{tikzcd}
 & \mathcal{L}_i^{\otimes 2} \arrow[dr, "\alpha_i"] & \\
\mathcal{F}' \otimes \pi^*\mathcal{N}_i \arrow[ur, "\chi_i"] \arrow[dr, "\chi_j \otimes \mathrm{Id}"'] \arrow[rr, phantom, "\circlearrowleft"] & & \pi^*\mathcal{N}_i \\
 & \mathcal{L}_j^{\otimes 2} \otimes \pi^*\mathcal{D}_{ij} \arrow[ur, "\alpha_j \otimes \mathrm{Id}"'] &
\end{tikzcd}$$

all commute. Note that the (purely stylistic) choice of $\mathcal{F}' = \mathcal{F} \otimes \pi^*\mathcal{N}_2^\vee$ corresponds to the convention used in definition 2.4.7.

(iv) For $\lambda = (a_1, \ldots, a_\ell)$ with $a_i \in \mathbb{N}$ and $\sum a_i = n$, we can write

$$\mathcal{S}_{\lim}^\lambda(\mathcal{N}_i)_{i=1}^\ell \quad = \quad \mathcal{S}_{\lim}^n(\mathcal{N}_1, \overset{(a_1)}{\ldots}, \mathcal{N}_1, \ldots, \mathcal{N}_\ell, \overset{(a_\ell)}{\ldots}, \mathcal{N}_\ell)$$

which is consistent with the previous notation and [Ser19].

Remark 2.4.9. Whenever $\mathcal{N}_1 \neq \mathcal{N}_2$, we do not have a statement equivalent to theorem 2.4.6, although we believe it is very likely to exist. At first sight, there is no reason for the proof of such result not to work in the more general setting, but a proper claim of this nature would require a thorough check of the theory developed by Sertöz and its adaptability to the case $\mathcal{N}_1 \neq \mathcal{N}_2$, which is outside of the scope of this thesis. Therefore, from here on we adopt the assumption of existence and appropriate behaviour for the spaces $\mathcal{S}_{\lim}^2(\mathcal{N}_1, \mathcal{N}_2)$, in the interest of studying some of their potential uses.

By virtue of definition 2.4.7, we can start thinking about a compactification of the Prym-spin moduli space, since we can set $(\mathcal{N}_1, \mathcal{N}_2) = (\mathcal{O}_{\overline{\mathcal{C}}_g}, \omega_\phi)$.

2.4.2 Prym-spin curves with a vanishing theta-null

We wanted to compactify the moduli space

$$\mathcal{RS}_g \quad = \quad \mathcal{R}_g \times_{\mathcal{M}_g} \mathcal{S}_g \quad \to \quad \mathcal{M}_g$$

of *Prym-spin curves*, i.e. of triplets (C, η, θ) such that

$$C \in \mathcal{M}_g, \qquad \eta \in R_g(C) = J_2(C) - \{\mathcal{O}_C\}, \qquad \theta \in S_g(C) = \sqrt{\omega_C}$$

Clearly, the compactification $\overline{\mathcal{RS}}_g$ sits within the moduli space of double limit roots of $(\mathcal{O}_{\overline{\mathcal{C}}_g}, \omega_\phi)$, where $\phi\colon \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ is as usual the universal curve over $\overline{\mathcal{M}}_g$. In particular, we need to remove the component of trivial roots of $\mathcal{O}_{\overline{\mathcal{C}}_g}$, that is, the closure of the locus of double roots of the form

$$\{(C, \mathcal{O}_C, \theta) \ / \ \theta \in S_g(C)\} \quad \cong \quad \{(C, \theta) \ / \ \theta \in S_g(C)\} \quad = \quad (\mathcal{S}_g)_{\mathbb{C}}$$

Since this component is isomorphic to $\overline{\mathcal{S}}_g$, we can state the following:

Definition 2.4.10. We refer to the space

$$\overline{\mathcal{RS}}_g \quad = \quad \mathcal{S}^2_{\lim}(\mathcal{O}_{\overline{\mathcal{C}}_g}, \omega_\phi) - \overline{\mathcal{S}}_g$$

as the *moduli space of stable Prym-spin curves* of genus g.

The two partial stabilizations of a double limit root yield projections

$$\begin{array}{lllllll} \rho_{\mathcal{R}}\colon & \overline{\mathcal{RS}}_g & \to & \overline{\mathcal{R}}_g, & (\pi, \eta, \theta, \beta, \alpha, \mathcal{F}, \chi_1, \chi_2) & \mapsto & (X, \eta', \beta') \\ \rho_{\mathcal{S}}\colon & \overline{\mathcal{RS}}_g & \to & \overline{\mathcal{S}}_g, & (\pi, \eta, \theta, \beta, \alpha, \mathcal{F}, \chi_1, \chi_2) & \mapsto & (X, \theta', \alpha') \end{array}$$

and consequently a commutative diagram

$$\begin{tikzcd} & \overline{\mathcal{R}}_g \arrow[dr, "\pi_{\mathcal{R}}"] & \\ \overline{\mathcal{RS}}_g \arrow[ur, "\rho_{\mathcal{R}}"] \arrow[rr, "\pi"] \arrow[dr, "\rho_{\mathcal{S}}"'] & & \overline{\mathcal{M}}_g \\ & \overline{\mathcal{S}}_g \arrow[ur, "\pi_{\mathcal{S}}"'] & \end{tikzcd}$$

where all maps are finite and ramified over the boundary. We may then wonder about how the connected components of $\overline{\mathcal{RS}}_g$ look like.

Given a Prym pair (C, η) of genus g, definition 2.2.8 provides subsets

$$S^{\mathrm{x,y}}_\eta(C) \quad = \quad \{\theta \in S^{\mathrm{x}}_g(C) \ / \ \theta \otimes \eta \in S^{\mathrm{y}}_g(C)\} \quad \subset \quad S^{\mathrm{x}}_g(C)$$

for $\mathrm{x}, \mathrm{y} \in \{+, -\}$, and in doing so refines the standard, parity-based decomposition $S_g(C) = S^+_g(C) \sqcup S^-_g(C)$ into a four-piece decomposition

$$S_g(C) \quad = \quad \underbrace{S^{+,+}_\eta(C) \sqcup S^{+,-}_\eta(C)}_{S^+_g(C)} \sqcup \underbrace{S^{-,+}_\eta(C) \sqcup S^{-,-}_\eta(C)}_{S^-_g(C)}$$

which depends critically on the Prym root η. As a result, the space $\overline{\mathcal{RS}}_g$ splits into four connected components $\overline{\mathcal{RS}}_g^{\mathrm{xy}}$ corresponding to the closure of

$$RS_g^{\mathrm{xy}} \;=\; \{(C,\eta,\theta) \;/\; \eta \in R_g(C),\, \theta \in S_\eta^{\mathrm{x,y}}(C) \subset S_g^{\mathrm{x}}(C)\} \;\subset\; (\mathcal{RS}_g)_{\mathbb{C}}$$

for $\mathrm{x},\mathrm{y} \in \{+,-\}$, so that we have:

$$\overline{\mathcal{RS}}_g \;=\; \underbrace{\overline{\mathcal{RS}}_g^{++} \sqcup \overline{\mathcal{RS}}_g^{+-}}_{\overline{\mathcal{RS}}_g^{+}} \sqcup \underbrace{\overline{\mathcal{RS}}_g^{-+} \sqcup \overline{\mathcal{RS}}_g^{--}}_{\overline{\mathcal{RS}}_g^{-}} \;\subset\; \mathcal{S}^2_{\mathrm{lim}}(\mathcal{O}_{\overline{\mathcal{C}}_g},\omega_\phi)$$

The spaces $\overline{\mathcal{RS}}_g^{++}$ and $\overline{\mathcal{RS}}_g^{+-}$, together with the commutative diagrams

$$\begin{tikzcd} & \overline{\mathcal{R}}_g \arrow[dr, "\pi_{\mathcal{R}}"] & \\ \overline{\mathcal{RS}}_g^{++} \arrow[ur, "\rho^+_{\mathcal{R}}"] \arrow[rr, "\pi^+"] \arrow[dr, "\rho^+_{\mathcal{S}}"'] & & \overline{\mathcal{M}}_g \\ & \overline{\mathcal{S}}_g^{+} \arrow[ur, "\pi_{\mathcal{S}}"'] & \end{tikzcd} \quad \Big| \quad \begin{tikzcd} & \overline{\mathcal{R}}_g \arrow[dr, "\pi_{\mathcal{R}}"] & \\ \overline{\mathcal{RS}}_g^{+-} \arrow[ur, "\rho^-_{\mathcal{R}}"] \arrow[rr, "\pi^-"] \arrow[dr, "\rho^-_{\mathcal{S}}"'] & & \overline{\mathcal{M}}_g \\ & \overline{\mathcal{S}}_g^{+} \arrow[ur, "\pi_{\mathcal{S}}"'] & \end{tikzcd}$$

induced around them by the previous diagram, are of special interest to us. On each of these connected components we find a very familiar divisor, namely the pullback of the theta-null divisor $\overline{\Theta}_{\mathrm{null}} \subset \overline{\mathcal{S}}_g^+$, or equivalently the closure of:

$$\begin{aligned} \Theta^+_{\mathrm{null}} &= \{(C,\eta,\theta) \;/\; \eta \in R_g(C),\, \theta \in \Theta_{\mathrm{null}}(C),\, \theta\otimes\eta \in S_g^+(C)\} \;\subset\; RS_g^{++} \\ \Theta^-_{\mathrm{null}} &= \{(C,\eta,\theta) \;/\; \eta \in R_g(C),\, \theta \in \Theta_{\mathrm{null}}(C),\, \theta\otimes\eta \in S_g^-(C)\} \;\subset\; RS_g^{+-} \end{aligned}$$

As evidenced by definition 2.2.2, the divisors $\overline{\Theta}^+_{\mathrm{null}}$ and $\overline{\Theta}^-_{\mathrm{null}}$ can be respectively pushed forward to our even and odd Prym-null divisors:

$$\begin{aligned} (\rho^+_{\mathcal{S}})^*\,\overline{\Theta}_{\mathrm{null}} &= \overline{\Theta}^+_{\mathrm{null}} \subset \overline{\mathcal{RS}}_g^{++} \;\rightsquigarrow\; (\rho^+_{\mathcal{R}})_*\,\overline{\Theta}^+_{\mathrm{null}} = \overline{\mathcal{P}}^+_{\mathrm{null}} \subset \overline{\mathcal{R}}_g \\ (\rho^-_{\mathcal{S}})^*\,\overline{\Theta}_{\mathrm{null}} &= \overline{\Theta}^-_{\mathrm{null}} \subset \overline{\mathcal{RS}}_g^{+-} \;\rightsquigarrow\; (\rho^-_{\mathcal{R}})_*\,\overline{\Theta}^-_{\mathrm{null}} = \overline{\mathcal{P}}^-_{\mathrm{null}} \subset \overline{\mathcal{R}}_g \end{aligned}$$

thus suggesting an alternative method of computing their class expansion. This new approach requires an understanding of the boundary divisors of $\overline{\mathcal{RS}}_g^{++}$ and $\overline{\mathcal{RS}}_g^{+-}$ over both $\overline{\mathcal{R}}_g$ and $\overline{\mathcal{S}}_g^+$, which we concisely provide next.

Remark 2.4.11 (boundary table legends)**.** Since the boundaries of $\overline{\mathcal{RS}}_g^{++}$ and $\overline{\mathcal{RS}}_g^{+-}$ contain a very large number of irreducible components, it seems sensible to summarize their description in some accessible manner. We aim to do so by collecting the different boundary divisors in tables 2.4.12 and 2.4.13, which are structured according to the following criteria:

(i) First column ($\overline{\mathcal{RS}}_g^{+\mathrm{x}}$): name of the boundary divisor Δ_{RS} of $\overline{\mathcal{RS}}_g^{+\mathrm{x}}$.

- ($\nexists$) A certain divisor might be expected to exist, but in fact not be present in either $\overline{\mathcal{RS}}_g^{++}$ or $\overline{\mathcal{RS}}_g^{+-}$.

(ii) Second column (trait): defining feature of the general point

$$(\pi : X \to Y,\ \eta,\ \theta,\ \beta,\ \alpha,\ \mathcal{F},\ \chi_1,\ \chi_2) \quad \in \quad \Delta_{\mathrm{RS}}$$

whose partial stabilizations are distinguished by symbols ($\times$, ↘, ⊖):

($\times$) As in examples 2.1.16 and 2.1.19, here we have

$$X = C \cup_{p\sim 0} E \cup_{q\sim\infty} D \longrightarrow Y = C \cup_{p\sim q} D,$$
$$\eta = (\eta_C, \mathcal{O}_E, \eta_D), \qquad \theta = (\theta_C, \mathcal{O}_E(1), \theta_D)$$

The notation $(\eta_C, \mathcal{O}_D)$ *et al.* indicates that

$$\eta_C \in R_g(C), \qquad \eta_D = \mathcal{O}_D$$

while the notation $(\theta_C^{\mathrm{u}}, \theta_D^{\mathrm{v}})_{\mathrm{yz}}$ indicates that

$$\begin{cases} \theta_C \ \in \ S_i^{\mathrm{u}}(C) \\ \theta_D \ \in \ S_{g-i}^{\mathrm{v}}(D) \end{cases} \qquad \begin{cases} \theta_C \otimes \eta_C \ \in \ S_i^{\mathrm{y}}(C) \\ \theta_D \otimes \eta_D \ \in \ S_{g-i}^{\mathrm{z}}(D) \end{cases}$$

for $\mathrm{u}, \mathrm{v}, \mathrm{y}, \mathrm{z} \in \{+, -\}$.

(↘) As in examples 2.1.17 and 2.1.20, here we have η stabilizing to

$$(\mathcal{O}_B, -1) \in \Delta_0^{\mathrm{t}}(B_{pq}) \quad \text{or} \quad (\eta_B, \varphi_\eta) \in \Delta_0^{\mathrm{p}}(B_{pq})$$

with $\eta_B \in R_{g-1}(B)$ and $\varphi_\eta \colon \eta_B|_p \cong \eta_B|_q$, or θ stabilizing to

$$(\theta_B^{\sim}, \varphi_\theta)_{\mathrm{wx}} \in \Delta_0^{\mathrm{n}}(B_{pq})$$

with $\theta_B^{\sim} \in \sqrt{\omega_B(p+q)}$ and $\varphi_\theta \colon \theta_B|_p \cong \theta_B|_q$. Now $\mathrm{w}, \mathrm{x} \in \{+, -\}$ indicate the parity of θ and $\theta \otimes \eta$, respectively.

(⊖) As in examples 2.1.17 and 2.1.20, here we have η stabilizing to

$$(\eta_B^{\sim}) \quad \in \quad \xi^{-1}(\eta_B^{\sim}, \mathcal{O}_E(1)) \quad \subset \quad \mathrm{Pic}(B \cup_{p\sim 0,\, q\sim\infty} E)$$

with $\eta_B^{\sim} \in \sqrt{\mathcal{O}_B(-p-q)}$, or θ stabilizing to

$$(\theta_B^{\mathrm{w}})_{\mathrm{x}} \quad \in \quad \xi^{-1}(\theta_B^{\mathrm{w}}, \mathcal{O}_E(1)) \quad \subset \quad \mathrm{Pic}(B \cup_{p\sim 0,\, q\sim\infty} E)$$

with $\theta_B^{\mathrm{w}} \in S_{g-1}^{\mathrm{w}}(B)$. Once again, $\mathrm{x} \in \{+, -\}$ refers to the parity of the combination of θ and η.

(iii) Third column ($\overline{\mathcal{R}}_g$): boundary divisor Δ_{R} of $\overline{\mathcal{R}}_g$ over which Δ_{RS} lies.

- ($*$) A single asterisk under Δ_{R} indicates that Δ_{RS} is a ramification divisor of the morphism $\overline{\mathcal{RS}}_g^{+\mathrm{x}} \to \overline{\mathcal{R}}_g$ of moduli stacks, but not of the corresponding coarse moduli map $\overline{RS}_g^{+\mathrm{x}} \to \overline{R}_g$.
- ($**$) A double asterisk under Δ_{R} indicates that Δ_{RS} is a ramification divisor of both $\overline{\mathcal{RS}}_g^{+\mathrm{x}} \to \overline{\mathcal{R}}_g$ and $\overline{RS}_g^{+\mathrm{x}} \to \overline{R}_g$.

(iv) Fourth column ($\overline{\mathcal{S}}_g^+$): boundary divisor Δ_{S} of $\overline{\mathcal{S}}_g^+$ over which Δ_{RS} lies.

- ($**$) A double asterisk under Δ_{S} indicates that Δ_{RS} is a ramification divisor of both $\overline{\mathcal{RS}}_g^{+\mathrm{x}} \to \overline{\mathcal{S}}_g^+$ and $\overline{RS}_g^{+\mathrm{x}} \to \overline{S}_g^+$.

(v) Final columns (deg): degree of $\Delta_{\mathrm{RS}} \to \Delta_{\mathrm{R}}$ and $\Delta_{\mathrm{RS}} \to \Delta_{\mathrm{S}}$ considered as restrictions of the respective coarse moduli maps. We write

$$\begin{cases} \mathrm{s}_g = \#S_g(C) = 2^{2g} \\ \mathrm{s}_g^+ = \#S_g^+(C) = 2^{g-1}(2^g+1) \\ \mathrm{s}_g^- = \#S_g^-(C) = 2^{g-1}(2^g-1) \end{cases} \qquad \begin{cases} \mathrm{N}_g^\pm = 2^{2g-2} \\ \mathrm{N}_g^+ = 2^{g-1}(2^{g-1}+1) \\ \mathrm{N}_g^- = 2^{g-1}(2^{g-1}-1) \end{cases}$$

as in proposition 2.2.12, so that $\#\sqrt{N} = \mathrm{s}_g$ and $\#R_g(C) = \mathrm{s}_g - 1$.

Table 2.4.12 (boundary of $\overline{\mathcal{RS}}_g^{++}$)**.** With the previous notation, the boundary divisors of $\overline{\mathcal{RS}}_g^{++} \to \overline{\mathcal{R}}_g$, $\overline{\mathcal{S}}_g^{+}$ can be described as follows:

$\overline{\mathcal{RS}}_g^{++}$	Trait	$\overline{\mathcal{R}}_g$	$\overline{\mathcal{S}}_g^{+}$	$\deg\lvert_{\overline{R}_g}$	$\deg\lvert_{\overline{S}_g^{+}}$
$\Delta_i^{\mathrm{n}+}$	$\left[\begin{array}{l} \times - (\eta_C, \mathcal{O}_D) \\ \times - (\theta_C^+, \theta_D^+)_{++} \end{array}\right.$	$\underset{*}{\Delta_i^{\mathrm{n}}}$	Δ_i^{+}	$\mathrm{N}_i^+\, \mathrm{s}_{g-i}^+$	$\mathrm{s}_i^+ - 1$
$\Delta_i^{\mathrm{n}-}$	$\left[\begin{array}{l} \times - (\eta_C, \mathcal{O}_D) \\ \times - (\theta_C^-, \theta_D^-)_{--} \end{array}\right.$	$\underset{*}{\Delta_i^{\mathrm{n}}}$	Δ_i^{-}	$\mathrm{N}_i^-\, \mathrm{s}_{g-i}^-$	$\mathrm{s}_i^- - 1$
$\Delta_i^{\mathrm{t}+}$	$\left[\begin{array}{l} \times - (\mathcal{O}_C, \eta_D) \\ \times - (\theta_C^+, \theta_D^+)_{++} \end{array}\right.$	$\underset{*}{\Delta_i^{\mathrm{t}}}$	Δ_i^{+}	$\mathrm{s}_i^+\, \mathrm{N}_{g-i}^+$	$\mathrm{s}_{g-i}^+ - 1$
$\Delta_i^{\mathrm{t}-}$	$\left[\begin{array}{l} \times - (\mathcal{O}_C, \eta_D) \\ \times - (\theta_C^-, \theta_D^-)_{--} \end{array}\right.$	$\underset{*}{\Delta_i^{\mathrm{t}}}$	Δ_i^{-}	$\mathrm{s}_i^-\, \mathrm{N}_{g-i}^-$	$\mathrm{s}_{g-i}^- - 1$
$\Delta_i^{\mathrm{p}++}$	$\left[\begin{array}{l} \times - (\eta_C, \eta_D) \\ \times - (\theta_C^+, \theta_D^+)_{++} \end{array}\right.$	$\underset{*}{\Delta_i^{\mathrm{p}}}$	Δ_i^{+}	$\mathrm{N}_i^+\, \mathrm{N}_{g-i}^+$	$(\mathrm{s}_i^+ - 1)\,(\mathrm{s}_{g-i}^+ - 1)$
$\Delta_i^{\mathrm{p}+-}$	$\left[\begin{array}{l} \times - (\eta_C, \eta_D) \\ \times - (\theta_C^+, \theta_D^+)_{--} \end{array}\right.$	$\underset{*}{\Delta_i^{\mathrm{p}}}$	Δ_i^{+}	$\mathrm{N}_i^\pm\, \mathrm{N}_{g-i}^\pm$	$\mathrm{s}_i^-\, \mathrm{s}_{g-i}^-$
$\Delta_i^{\mathrm{p}-+}$	$\left[\begin{array}{l} \times - (\eta_C, \eta_D) \\ \times - (\theta_C^-, \theta_D^-)_{++} \end{array}\right.$	$\underset{*}{\Delta_i^{\mathrm{p}}}$	Δ_i^{-}	$\mathrm{N}_i^\pm\, \mathrm{N}_{g-i}^\pm$	$\mathrm{s}_i^+\, \mathrm{s}_{g-i}^+$
$\Delta_i^{\mathrm{p}--}$	$\left[\begin{array}{l} \times - (\eta_C, \eta_D) \\ \times - (\theta_C^-, \theta_D^-)_{--} \end{array}\right.$	$\underset{*}{\Delta_i^{\mathrm{p}}}$	Δ_i^{-}	$\mathrm{N}_i^-\, \mathrm{N}_{g-i}^-$	$(\mathrm{s}_i^- - 1)\,(\mathrm{s}_{g-i}^- - 1)$
$\nexists\, \Delta_0^{\mathrm{tn}}$	$\left[\begin{array}{l} \propto - (\mathcal{O}_B, -1) \\ \propto - (\theta_B^\sim, \varphi_\theta)_{++} \end{array}\right.$	Not present: parity must change			
Δ_0^{pn}	$\left[\begin{array}{l} \propto - (\eta_B, \varphi_\eta) \\ \propto - (\theta_B^\sim, \varphi_\theta)_{++} \end{array}\right.$	Δ_0^{p}	Δ_0^{n}	$\mathrm{s}_{g-1}/2$	$\mathrm{s}_{g-1} - 1$
Δ_0^{tb}	$\left[\begin{array}{l} \propto - (\mathcal{O}_B, -1) \\ \ominus - (\theta_B^+)_+ \end{array}\right.$	$\underset{**}{\Delta_0^{\mathrm{t}}}$	Δ_0^{b}	s_{g-1}^+	1
Δ_0^{pb}	$\left[\begin{array}{l} \propto - (\eta_B, \varphi_\eta) \\ \ominus - (\theta_B^+)_+ \end{array}\right.$	$\underset{**}{\Delta_0^{\mathrm{p}}}$	Δ_0^{b}	N_{g-1}^+	$2\,(\mathrm{s}_{g-1}^+ - 1)$
Δ_0^{bn}	$\left[\begin{array}{l} \ominus - (\eta_B^\sim) \\ \propto - (\theta_B^\sim, \varphi_\theta)_{++} \end{array}\right.$	Δ_0^{b}	$\underset{**}{\Delta_0^{\mathrm{n}}}$	s_{g-1}^+	s_{g-1}^+
Δ_0^{bb}	$\left[\begin{array}{l} \ominus - (\eta_B^\sim) \\ \ominus - (\theta_B^+)_+ \end{array}\right.$	Δ_0^{b}	Δ_0^{b}	s_{g-1}^+	s_{g-1}

Note that all degrees are given with respect to the coarse moduli maps, for the sake of simplicity and pushforwards. With respect to their stacky counterparts, an additional factor of 2 appears over $\overline{\mathcal{R}}_g$ whenever $i \neq 0$, as in example 2.1.19.

Table 2.4.13 (boundary of $\overline{\mathcal{RS}}_g^{+-}$)**.** With the previous notation, the boundary divisors of $\overline{\mathcal{RS}}_g^{+-} \to \overline{\mathcal{R}}_g$, $\overline{\mathcal{S}}_g^+$ can be described as follows:

$\overline{\mathcal{RS}}_g^{+-}$	Trait	$\overline{\mathcal{R}}_g$	$\overline{\mathcal{S}}_g^+$	$\deg\vert_{\overline{R}_g}$	$\deg\vert_{\overline{S}_g^+}$
$\Delta_i^{\mathrm{n}+}$	$\times - (\eta_C, \mathcal{O}_D)$ $\times - (\theta_C^+, \theta_D^+)_{-+}$	$\underset{*}{\Delta_i^{\mathrm{n}}}$	Δ_i^+	$\mathrm{N}_i^{\pm}\,\mathrm{s}_{g-i}^+$	s_i^-
$\Delta_i^{\mathrm{n}-}$	$\times - (\eta_C, \mathcal{O}_D)$ $\times - (\theta_C^-, \theta_D^-)_{+-}$	$\underset{*}{\Delta_i^{\mathrm{n}}}$	Δ_i^-	$\mathrm{N}_i^{\pm}\,\mathrm{s}_{g-i}^-$	s_i^+
$\Delta_i^{\mathrm{t}+}$	$\times - (\mathcal{O}_C, \eta_D)$ $\times - (\theta_C^+, \theta_D^+)_{+-}$	$\underset{*}{\Delta_i^{\mathrm{t}}}$	Δ_i^+	$\mathrm{s}_i^+\,\mathrm{N}_{g-i}^{\pm}$	s_{g-i}^-
$\Delta_i^{\mathrm{t}-}$	$\times - (\mathcal{O}_C, \eta_D)$ $\times - (\theta_C^-, \theta_D^-)_{-+}$	$\underset{*}{\Delta_i^{\mathrm{t}}}$	Δ_i^-	$\mathrm{s}_i^-\,\mathrm{N}_{g-i}^{\pm}$	s_{g-i}^+
$\Delta_i^{\mathrm{p}++}$	$\times - (\eta_C, \eta_D)$ $\times - (\theta_C^+, \theta_D^+)_{+-}$	$\underset{*}{\Delta_i^{\mathrm{p}}}$	Δ_i^+	$\mathrm{N}_i^+\,\mathrm{N}_{g-i}^{\pm}$	$(\mathrm{s}_i^+ - 1)\,\mathrm{s}_{g-i}^-$
$\Delta_i^{\mathrm{p}+-}$	$\times - (\eta_C, \eta_D)$ $\times - (\theta_C^+, \theta_D^+)_{-+}$	$\underset{*}{\Delta_i^{\mathrm{p}}}$	Δ_i^+	$\mathrm{N}_i^{\pm}\,\mathrm{N}_{g-i}^+$	$\mathrm{s}_i^-\,(\mathrm{s}_{g-i}^+ - 1)$
$\Delta_i^{\mathrm{p}-+}$	$\times - (\eta_C, \eta_D)$ $\times - (\theta_C^-, \theta_D^-)_{+-}$	$\underset{*}{\Delta_i^{\mathrm{p}}}$	Δ_i^-	$\mathrm{N}_i^{\pm}\,\mathrm{N}_{g-i}^-$	$\mathrm{s}_i^+\,(\mathrm{s}_{g-i}^- - 1)$
$\Delta_i^{\mathrm{p}--}$	$\times - (\eta_C, \eta_D)$ $\times - (\theta_C^-, \theta_D^-)_{-+}$	$\underset{*}{\Delta_i^{\mathrm{p}}}$	Δ_i^-	$\mathrm{N}_i^-\,\mathrm{N}_{g-i}^{\pm}$	$(\mathrm{s}_i^- - 1)\,\mathrm{s}_{g-i}^+$
Δ_0^{tn}	$- (\mathcal{O}_B, -1)$ $- (\theta_B^{\sim}, \varphi_\theta)_{+-}$	Δ_0^{t}	Δ_0^{n}	s_{g-1}	1
Δ_0^{pn}	$- (\eta_B, \varphi_\eta)$ $- (\theta_B^{\sim}, \varphi_\theta)_{+-}$	Δ_0^{p}	Δ_0^{n}	$\mathrm{s}_{g-1}/2$	$\mathrm{s}_{g-1} - 1$
$\nexists\,\Delta_0^{\mathrm{tb}}$	$- (\mathcal{O}_B, -1)$ $- (\theta_B^+)_-$	Not present: parity cannot change			
Δ_0^{pb}	$- (\eta_B, \varphi_\eta)$ $- (\theta_B^+)_-$	$\underset{**}{\Delta_0^{\mathrm{p}}}$	Δ_0^{b}	$\mathrm{N}_{g-1}^{\pm}$	$2\,\mathrm{s}_{g-1}^-$
Δ_0^{bn}	$- (\eta_B^{\sim})$ $- (\theta_B^{\sim}, \varphi_\theta)_{+-}$	Δ_0^{b}	$\underset{**}{\Delta_0^{\mathrm{n}}}$	s_{g-1}^-	s_{g-1}^-
Δ_0^{bb}	$- (\eta_B^{\sim})$ $- (\theta_B^+)_-$	Δ_0^{b}	Δ_0^{b}	s_{g-1}^+	s_{g-1}

Note that all degrees are given with respect to the coarse moduli maps, for the sake of simplicity and pushforwards. With respect to their stacky counterparts, an additional factor of 2 appears over $\overline{\mathcal{R}}_g$ whenever $i \neq 0$, as in example 2.1.19.

Thanks to tables 2.4.12 and 2.4.13, we can write down the pullbacks of the boundary classes of $\overline{\mathcal{R}}_g$ and $\overline{\mathcal{S}}_g^+$ in terms of the boundary classes of $\overline{\mathcal{RS}}_g^{++}$ and $\overline{\mathcal{RS}}_g^{+-}$. As usual, we apply the convention $\delta = \mathcal{O}(\Delta)$ to refer to divisor classes in stacks, and obtain:

$$\begin{array}{c} \rho_{\mathcal{R}}^+ \colon \overline{\mathcal{RS}}_g^{++} \to \overline{\mathcal{R}}_g \\ (\text{deg. } \mathrm{N}_g^+) \end{array} \quad \begin{cases} (\rho_{\mathcal{R}}^+)^*(\delta_i^{\mathrm{n}}) &=& 2\,(\delta_i^{\mathrm{n}+} + \delta_i^{\mathrm{n}-}) \\ (\rho_{\mathcal{R}}^+)^*(\delta_i^{\mathrm{t}}) &=& 2\,(\delta_i^{\mathrm{t}+} + \delta_i^{\mathrm{t}-}) \\ (\rho_{\mathcal{R}}^+)^*(\delta_i^{\mathrm{p}}) &=& 2\,(\delta_i^{\mathrm{p}++} + \delta_i^{\mathrm{p}+-} + \delta_i^{\mathrm{p}-+} + \delta_i^{\mathrm{p}--}) \\ (\rho_{\mathcal{R}}^+)^*(\delta_0^{\mathrm{t}}) &=& 2\,\delta_0^{\mathrm{tb}} \\ (\rho_{\mathcal{R}}^+)^*(\delta_0^{\mathrm{p}}) &=& \delta_0^{\mathrm{pn}} + 2\,\delta_0^{\mathrm{pb}} \\ (\rho_{\mathcal{R}}^+)^*(\delta_0^{\mathrm{b}}) &=& \delta_0^{\mathrm{bn}} + \delta_0^{\mathrm{bb}} \end{cases}$$

$$\begin{array}{c} \rho_{\mathcal{S}}^+ \colon \overline{\mathcal{RS}}_g^{++} \to \overline{\mathcal{S}}_g^+ \\ (\text{deg. } \mathrm{s}_g^+ - 1) \end{array} \quad \begin{cases} (\rho_{\mathcal{S}}^+)^*(\delta_i^+) &=& \delta_i^{\mathrm{n}+} + \delta_i^{\mathrm{t}+} + \delta_i^{\mathrm{p}++} + \delta_i^{\mathrm{p}+-} \\ (\rho_{\mathcal{S}}^+)^*(\delta_i^-) &=& \delta_i^{\mathrm{n}-} + \delta_i^{\mathrm{t}-} + \delta_i^{\mathrm{p}-+} + \delta_i^{\mathrm{p}--} \\ (\rho_{\mathcal{S}}^+)^*(\delta_0^{\mathrm{n}}) &=& \delta_0^{\mathrm{pn}} + 2\,\delta_0^{\mathrm{bn}} \\ (\rho_{\mathcal{S}}^+)^*(\delta_0^{\mathrm{b}}) &=& \delta_0^{\mathrm{tb}} + \delta_0^{\mathrm{pb}} + \delta_0^{\mathrm{bb}} \end{cases}$$

$$\begin{array}{c} \rho_{\mathcal{R}}^- \colon \overline{\mathcal{RS}}_g^{+-} \to \overline{\mathcal{R}}_g \\ (\text{deg. } \mathrm{N}_g^\pm) \end{array} \quad \begin{cases} (\rho_{\mathcal{R}}^-)^*(\delta_i^{\mathrm{n}}) &=& 2\,(\delta_i^{\mathrm{n}+} + \delta_i^{\mathrm{n}-}) \\ (\rho_{\mathcal{R}}^-)^*(\delta_i^{\mathrm{t}}) &=& 2\,(\delta_i^{\mathrm{t}+} + \delta_i^{\mathrm{t}-}) \\ (\rho_{\mathcal{R}}^-)^*(\delta_i^{\mathrm{p}}) &=& 2\,(\delta_i^{\mathrm{p}++} + \delta_i^{\mathrm{p}+-} + \delta_i^{\mathrm{p}-+} + \delta_i^{\mathrm{p}--}) \\ (\rho_{\mathcal{R}}^-)^*(\delta_0^{\mathrm{t}}) &=& \delta_0^{\mathrm{tn}} \\ (\rho_{\mathcal{R}}^-)^*(\delta_0^{\mathrm{p}}) &=& \delta_0^{\mathrm{pn}} + 2\,\delta_0^{\mathrm{pb}} \\ (\rho_{\mathcal{R}}^-)^*(\delta_0^{\mathrm{b}}) &=& \delta_0^{\mathrm{bn}} + \delta_0^{\mathrm{bb}} \end{cases}$$

$$\begin{array}{c} \rho_{\mathcal{S}}^- \colon \overline{\mathcal{RS}}_g^{+-} \to \overline{\mathcal{S}}_g^+ \\ (\text{deg. } \mathrm{s}_g^-) \end{array} \quad \begin{cases} (\rho_{\mathcal{S}}^-)^*(\delta_i^+) &=& \delta_i^{\mathrm{n}+} + \delta_i^{\mathrm{t}+} + \delta_i^{\mathrm{p}++} + \delta_i^{\mathrm{p}+-} \\ (\rho_{\mathcal{S}}^-)^*(\delta_i^-) &=& \delta_i^{\mathrm{n}-} + \delta_i^{\mathrm{t}-} + \delta_i^{\mathrm{p}-+} + \delta_i^{\mathrm{p}--} \\ (\rho_{\mathcal{S}}^-)^*(\delta_0^{\mathrm{n}}) &=& \delta_0^{\mathrm{tn}} + \delta_0^{\mathrm{pn}} + 2\,\delta_0^{\mathrm{bn}} \\ (\rho_{\mathcal{S}}^-)^*(\delta_0^{\mathrm{b}}) &=& \delta_0^{\mathrm{pb}} + \delta_0^{\mathrm{bb}} \end{cases}$$

We now have all of the information needed to extract the Prym-null classes $\varrho_{\mathrm{null}}^+$ and $\varrho_{\mathrm{null}}^-$ from the theta-null class $\overline{\vartheta}_{\mathrm{null}}$, and a road map given by

$$\begin{array}{ccccccccc} (\rho_{\mathcal{S}}^+)^*\,\overline{\Theta}_{\mathrm{null}} &=& \overline{\Theta}_{\mathrm{null}}^+ &\subset& \overline{\mathcal{RS}}_g^{++} &\rightsquigarrow& (\rho_{\mathcal{R}}^+)_*\,\overline{\Theta}_{\mathrm{null}}^+ &=& \overline{\mathcal{P}}_{\mathrm{null}}^+ &\subset& \overline{\mathcal{R}}_g \\ (\rho_{\mathcal{S}}^-)^*\,\overline{\Theta}_{\mathrm{null}} &=& \overline{\Theta}_{\mathrm{null}}^- &\subset& \overline{\mathcal{RS}}_g^{+-} &\rightsquigarrow& (\rho_{\mathcal{R}}^-)_*\,\overline{\Theta}_{\mathrm{null}}^- &=& \overline{\mathcal{P}}_{\mathrm{null}}^- &\subset& \overline{\mathcal{R}}_g \end{array}$$

The aforementioned starting class, computed in [Far10], is of the form:

$$\overline{\vartheta}_{\mathrm{null}} \;=\; \frac{1}{4}\,\lambda - \frac{1}{16}\,\delta_0^{\mathrm{n}} - \sum_{i=1}^{\lfloor g/2 \rfloor} \delta_i^{-} \;\in\; \mathrm{Pic}(\overline{\mathcal{S}}_g^{+})_{\mathbb{Q}}$$

as stated in section 2.2. Pulling back by $\rho_{\mathcal{S}}^{+}$ and $\rho_{\mathcal{S}}^{-}$, we then deduce that:

$$\begin{aligned}
\overline{\vartheta}_{\mathrm{null}}^{+} &= (\rho_{\mathcal{S}}^{+})^{*}\,\overline{\vartheta}_{\mathrm{null}} \\
&= \frac{1}{4}\,\lambda - \frac{1}{16}\,\delta_0^{\mathrm{pn}} - \frac{1}{8}\,\delta_0^{\mathrm{bn}} - \sum_{i=1}^{\lfloor g/2 \rfloor} (\delta_i^{\mathrm{n}-} + \delta_i^{\mathrm{t}-} + \delta_i^{\mathrm{p}-+} + \delta_i^{\mathrm{p}--}) \\
&\in \mathrm{Pic}(\overline{\mathcal{RS}}_g^{++})_{\mathbb{Q}}
\end{aligned}$$

$$\begin{aligned}
\overline{\vartheta}_{\mathrm{null}}^{-} &= (\rho_{\mathcal{S}}^{-})^{*}\,\overline{\vartheta}_{\mathrm{null}} \\
&= \frac{1}{4}\,\lambda - \frac{1}{16}\,\delta_0^{\mathrm{tn}} - \frac{1}{16}\,\delta_0^{\mathrm{pn}} - \frac{1}{8}\,\delta_0^{\mathrm{bn}} - \sum_{i=1}^{\lfloor g/2 \rfloor} (\delta_i^{\mathrm{n}-} + \delta_i^{\mathrm{t}-} + \delta_i^{\mathrm{p}-+} + \delta_i^{\mathrm{p}--}) \\
&\in \mathrm{Pic}(\overline{\mathcal{RS}}_g^{+-})_{\mathbb{Q}}
\end{aligned}$$

with $\overline{\vartheta}_{\mathrm{null}}^{+} = \mathcal{O}_{\overline{\mathcal{RS}}_g^{++}}(\overline{\Theta}_{\mathrm{null}}^{+}) \in \mathrm{Pic}(\overline{\mathcal{RS}}_g^{++})$ and $\overline{\vartheta}_{\mathrm{null}}^{-} = \mathcal{O}_{\overline{\mathcal{RS}}_g^{+-}}(\overline{\Theta}_{\mathrm{null}}^{-}) \in \mathrm{Pic}(\overline{\mathcal{RS}}_g^{+-})$. Finally, we can use tables 2.4.12 and 2.4.13 to push these classes forward by $\rho_{\mathcal{R}}^{+}$ and $\rho_{\mathcal{R}}^{-}$, which brings us back to $\varrho_{\mathrm{null}}^{+}$, $\varrho_{\mathrm{null}}^{-} \in \mathrm{Pic}(\overline{\mathcal{R}}_g)$ and results in:

$$\begin{aligned}
\varrho_{\mathrm{null}}^{+} &= (\rho_{\mathcal{R}}^{+})_{*}\,\overline{\vartheta}_{\mathrm{null}}^{+} \\
&= \frac{1}{4}\,\mathrm{N}_g^{+}\,\lambda - \frac{1}{16}\,\frac{1}{2}\,\mathrm{s}_{g-1}\,\delta_0^{\mathrm{p}} - \frac{1}{8}\,\mathrm{s}_{g-1}^{+}\,\delta_0^{\mathrm{b}} \\
&\quad - \frac{1}{2}\sum_{i=1}^{\lfloor g/2 \rfloor} \Big(\mathrm{N}_i^{-}\,\mathrm{s}_{g-i}^{-}\,\delta_i^{\mathrm{n}} + \mathrm{s}_i^{-}\,\mathrm{N}_{g-i}^{-}\,\delta_i^{\mathrm{t}} + (\mathrm{N}_i^{\pm}\,\mathrm{N}_{g-i}^{\pm} + \mathrm{N}_i^{-}\,\mathrm{N}_{g-i}^{-})\,\delta_i^{\mathrm{p}}\Big) \\
&\in \mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}
\end{aligned}$$

$$\begin{aligned}
\varrho_{\mathrm{null}}^{-} &= (\rho_{\mathcal{R}}^{-})_{*}\,\overline{\vartheta}_{\mathrm{null}}^{-} \\
&= \frac{1}{4}\,\mathrm{N}_g^{\pm}\,\lambda - \frac{1}{16}\,\mathrm{s}_{g-1}\,\delta_0^{\mathrm{t}} - \frac{1}{16}\,\frac{1}{2}\,\mathrm{s}_{g-1}\,\delta_0^{\mathrm{p}} - \frac{1}{8}\,\mathrm{s}_{g-1}^{-}\,\delta_0^{\mathrm{b}} \\
&\quad - \frac{1}{2}\sum_{i=1}^{\lfloor g/2 \rfloor} \Big(\mathrm{N}_i^{\pm}\,\mathrm{s}_{g-i}^{-}\,\delta_i^{\mathrm{n}} + \mathrm{s}_i^{-}\,\mathrm{N}_{g-i}^{\pm}\,\delta_i^{\mathrm{t}} + (\mathrm{N}_i^{\pm}\,\mathrm{N}_{g-i}^{-} + \mathrm{N}_i^{-}\,\mathrm{N}_{g-i}^{\pm})\,\delta_i^{\mathrm{p}}\Big) \\
&\in \mathrm{Pic}(\overline{\mathcal{R}}_g)_{\mathbb{Q}}
\end{aligned}$$

A quick computation shows that these expansions agree with propositions 2.3.8

and 2.3.13, and thus with theorem 2.3.14. Indeed, the coefficients are:

$$\varrho_{\mathrm{null}}^{+}\left|\begin{array}{rcll}
\lambda^{+} & = & 2^{-2}\,\mathrm{N}_g^{+} & = \; 2^{g-3}(2^{g-1}+1)\\
\delta_0^{\mathrm{t},+} & = & 0 & = \; 0\\
\delta_0^{\mathrm{p},+} & = & 2^{-5}\,\mathrm{s}_{g-1} & = \; 2^{g-3}\,2^{-2}\,2^{g-2}\\
\delta_0^{\mathrm{b},+} & = & 2^{-3}\,\mathrm{s}_{g-1}^{+} & = \; 2^{g-3}\,2^{-2}(2^{g-1}+1)\\
\delta_i^{\mathrm{n},+} & = & 2^{-1}\,\mathrm{N}_i^{-}\,\mathrm{s}_{g-i}^{-} & = \; 2^{g-3}(2^{i-1}-1)(2^{g-i}-1)\\
\delta_i^{\mathrm{t},+} & = & 2^{-1}\,\mathrm{s}_i^{-}\,\mathrm{N}_{g-i}^{-} & = \; 2^{g-3}(2^{i}-1)(2^{g-i-1}-1)\\
\delta_i^{\mathrm{p},+} & = & 2^{-1}\,(\mathrm{N}_i^{\pm}\,\mathrm{N}_{g-i}^{\pm}+\mathrm{N}_i^{-}\,\mathrm{N}_{g-i}^{-}) & = \; 2^{g-3}(2^{g-1}-2^{i-1}-2^{g-i-1}+1)
\end{array}\right.$$

$$\varrho_{\mathrm{null}}^{-}\left|\begin{array}{rcll}
\lambda^{-} & = & 2^{-2}\,\mathrm{N}_g^{\pm} & = \; 2^{g-3}\,2^{g-1}\\
\delta_0^{\mathrm{t},-} & = & 2^{-4}\,\mathrm{s}_{g-1} & = \; 2^{g-3}\,2^{-2}\,2^{g-1}\\
\delta_0^{\mathrm{p},-} & = & 2^{-5}\,\mathrm{s}_{g-1} & = \; 2^{g-3}\,2^{-2}\,2^{g-2}\\
\delta_0^{\mathrm{b},-} & = & 2^{-3}\,\mathrm{s}_{g-1}^{-} & = \; 2^{g-3}\,2^{-2}(2^{g-1}-1)\\
\delta_i^{\mathrm{n},-} & = & 2^{-1}\,\mathrm{N}_i^{\pm}\,\mathrm{s}_{g-i}^{-} & = \; 2^{g-3}\,2^{i-1}(2^{g-i}-1)\\
\delta_i^{\mathrm{t},-} & = & 2^{-1}\,\mathrm{s}_i^{-}\,\mathrm{N}_{g-i}^{\pm} & = \; 2^{g-3}(2^{i}-1)\,2^{g-i-1}\\
\delta_i^{\mathrm{p},-} & = & 2^{-1}\,(\mathrm{N}_i^{\pm}\,\mathrm{N}_{g-i}^{-}+\mathrm{N}_i^{-}\,\mathrm{N}_{g-i}^{\pm}) & = \; 2^{g-3}(2^{g-1}-2^{i-1}-2^{g-i-1})
\end{array}\right.$$

as expected. This means that, at least under the assumption from remark 2.4.9, our analysis of the Prym-null divisors has truly come full circle.

2.4.3 Building a product of limit roots

The construction of $\mathcal{S}^2_{\mathrm{lim}}(\mathcal{N}_1,\mathcal{N}_2)$, or more generally of $\mathcal{S}^n_{\mathrm{lim}}(\mathcal{N}_i)_{i=1}^n$ for $n\in\mathbb{N}$, raises the question of whether it is possible to extend a basic operation of roots to limit roots: we refer, of course, to the *tensor product*.

Remark 2.4.14. On a smooth curve, the product $L_1\otimes L_2$ of two square roots $L_i\in\sqrt{N_i}$ is again a square root (of $N_1\otimes N_2$). However, when working over

$$\mathcal{S}^1_{\mathrm{lim}}(\mathcal{N}_1)\times_{\overline{\mathcal{M}}_g}\mathcal{S}^1_{\mathrm{lim}}(\mathcal{N}_2)\longrightarrow\overline{\mathcal{M}}_g$$

the tensor product is not so well-behaved. There are several problems:

(i) *Each limit root may be defined over a different quasistable curve.*
This is easy to solve: we can take the product of the lifts to a common exceptional blow-up.

(ii) *Different lifts would produce different products.*
This is quite inconvenient, but perhaps a specific lift can be selected.

(iii) *Even if we choose a specific lift, the tensor product of two pullbacks of limit roots is not always a limit root itself.*
This is harder, and much less intuitive to tackle.

In view of the third point, the question becomes: can we modify the product in any consistent way in order to end up with a limit root? To answer this, a shift in perspective is required: if we switch over to $\mathcal{S}^2_{\mathrm{lim}}(\mathcal{N}_1, \mathcal{N}_2)$, we not only bypass the second problem, but also have at our disposal a series of new data that will prove integral to dealing with the third one.

According to remark 2.4.14, we would like to build a map

$$\mathcal{S}^2_{\mathrm{lim}}(\mathcal{N}_1, \mathcal{N}_2) \longrightarrow \mathcal{S}^3_{\mathrm{lim}}(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_1 \otimes \mathcal{N}_2)$$

to act as a sort of product of limit roots, or "limit product". We will propose a candidate for this, but only at the level of points. First, we need to classify the nodes of the stable model of a double limit root in terms of the behaviour that each of the individual limit roots displays over them.

Given a stable curve $Y \in \overline{\mathcal{M}}_g$, any pair of exceptional blow-ups

$$\pi_1 \colon X_1 \to Y, \qquad \pi_2 \colon X_2 \to Y$$

induces an arrangement of the nodes of Y into four different sets.

Definition 2.4.15. Let (π_1, π_2) be a pair of exceptional blow-ups of Y. We say that a node $z \in \mathrm{Sing}(Y)$ is *of (π_1, π_2)-exceptional type* (a_1, a_2) when

$$a_i \;=\; a_i(z) \;=\; \dim\left(\pi_i^{-1}(z)\right) \;=\; \begin{cases} 0 & \text{if } \pi_i^{-1}(z) \cong \mathbb{P}^0 \\ 1 & \text{if } \pi_i^{-1}(z) \cong \mathbb{P}^1 \end{cases}$$

for each i. Furthermore, the set of (a_1, a_2)-nodes of Y is denoted

$$Z_{(a_1,a_2)} \;=\; Z_{(a_1,a_2)}(\pi_1, \pi_2) \;\subset\; \mathrm{Sing}(Y)$$

with $(a_1, a_2) \in \mathbb{F}_2^2$, which leads to a decomposition

$$\mathrm{Sing}(Y) \;=\; Z_{(0,0)} \sqcup Z_{(1,0)} \sqcup Z_{(0,1)} \sqcup Z_{(1,1)}$$

Note that, if we take $(L_1, L_2)_{\mathrm{sync}}$ to be a double limit root

$$(L_1, L_2)_{\mathrm{sync}} \;=\; \{\pi \colon X \to Y,\, L_i,\, \alpha_i, F, \chi_i\}_{i=1,2} \;\in\; \mathcal{S}^2_{\mathrm{lim}}(N_1, N_2)$$

partially stabilizing to the pair $\{(\pi_i\colon X_i \to Y,\, L_i',\, \alpha_i')\}_{i=1,2}$, then $z \in \mathrm{Sing}(Y)$ is a (π_1, π_2)-exceptional (a_1, a_2)-node when

$$a_i \;=\; a_i(z) \;=\; \deg(L_i|_{E_z}) \;=\; \begin{cases} 0 & \text{if } \; L_i|_{E_z} = \mathcal{O}_{E_z} \\ 1 & \text{if } \; L_i|_{E_z} = \mathcal{O}_{E_z}(1) \end{cases}$$

In this case, we have subsets

$$\begin{array}{lllllll} E_{(a_1,a_2)} & = & E_{(a_1,a_2)}(\pi_1,\pi_2) & = & \pi^{-1}(Z_{(a_1,a_2)}) & \subset & X \\ S_{(a_1,a_2)} & = & S_{(a_1,a_2)}(\pi_1,\pi_2) & = & E_{(a_1,a_2)} \cap \mathrm{Sing}(X) & \subset & \mathrm{Sing}(X) \end{array}$$

separating the exceptional components of X, as well as its nodes:

$$\mathrm{Sing}(X) \;=\; S_{(0,0)} \sqcup S_{(1,0)} \sqcup S_{(0,1)} \sqcup S_{(1,1)}$$

Generally, the pair (π_1, π_2) will be clear from the context, and thus omitted.

The obstruction to a naive extension of the tensor product is created in the nodes of exceptional type $(1,1)$. If we can compensate for this obstruction, it is not out of the question that a suitable product arises.

Proposition 2.4.16. *For any stable curve Y of genus g, and any pair (N_1, N_2) of line bundles on Y of even degree, there is a "limit product" map*

$$\begin{array}{rlll} \circledast\colon & \mathcal{S}^2_{\lim}(N_1, N_2) & \to & \mathcal{S}^3_{\lim}(N_1, N_2, N_1 \otimes N_2) \\ & (L_1, L_2)_{\mathrm{sync}} & \mapsto & (L_1, L_2, L_1 \circledast L_2)_{\mathrm{sync}} \end{array}$$

that corresponds to the standard tensor product of roots when Y is smooth.

Proof. Given a double limit root $(L_1, L_2)_{\mathrm{sync}}$ as in definition 2.4.15, we can find an open cover of X by separating its three types of exceptional components

$$E(X) \;=\; E_{(1,0)} \sqcup E_{(0,1)} \sqcup E_{(1,1)} \;\subset\; X$$

and considering adequate open neighbourhoods for each type:

$$\begin{array}{lllllllll} U_{10} & = & U_{10}(\pi_1,\pi_2) & = & X - (E_{(0,1)} \cup E_{(1,1)}) & \subset & V_1 & = & X - E_{(0,1)} \\ U_{01} & = & U_{01}(\pi_1,\pi_2) & = & X - (E_{(1,0)} \cup E_{(1,1)}) & \subset & V_2 & = & X - E_{(1,0)} \\ U_{11} & = & U_{11}(\pi_1,\pi_2) & = & X - (E_{(1,0)} \cup E_{(0,1)}) & = & V_1 \cap V_2 & & \end{array}$$

In keeping with definitions 2.4.4 and 2.4.7, we write $V_i = V(L_i) \subset X$. It is easy to check that these open sets are related by the equalities

$$\begin{array}{rcl} & & U_{10} \cup U_{01} \cup U_{11} \;=\; X \\ U_{10} \cap U_{01} &=& U_{10} \cap U_{11} \;=\; U_{01} \cap U_{11} \;=\; X - E(X) \end{array}$$

and that, using the notation from remark 2.4.8(iii), we have

$$\begin{array}{llcllcl} \chi_1 \colon & F' \otimes \pi^* N_1 & \to & L_1^{\otimes 2} & \text{isomorphism over} & V_1 & = U_{10} \cup U_{11} \\ \chi_2 \colon & F' \otimes \pi^* N_2 & \to & L_2^{\otimes 2} & \text{isomorphism over} & V_2 & = U_{01} \cup U_{11} \\ \alpha_1 \colon & L_1^{\otimes 2} & \to & \pi^* N_1 & \text{isomorphism over} & U_{01} & \\ \alpha_2 \colon & L_2^{\otimes 2} & \to & \pi^* N_2 & \text{isomorphism over} & U_{10} & \end{array}$$

In particular, the differences $\pi^* N_i \otimes (L_i^\vee)^{\otimes 2}$ can be locally described:

	$(\pi^* N_1 \otimes (L_1^\vee)^{\otimes 2})\|_U$		$(\pi^* N_2 \otimes (L_2^\vee)^{\otimes 2})\|_U$
$U = U_{10}$	$(F')^\vee\|_{U_{10}}$	$\ncong$	$\mathcal{O}_{U_{10}}$
$U = U_{01}$	$\mathcal{O}_{U_{01}}$	$\ncong$	$(F')^\vee\|_{U_{01}}$
$U = U_{11}$	$(F')^\vee\|_{U_{11}}$	$\cong$	$(F')^\vee\|_{U_{11}}$

The key observation here is that the standard square of a limit root can locally differ from its radicand, but by no more than F'. Consequently, the excess

$$(L_1^{\otimes 2} \otimes L_2^{\otimes 2})|_{U_{11}} \quad = \quad (\pi^* N_1 \otimes \pi^* N_2 \otimes \underbrace{(F')^{\otimes 2}})|_{U_{11}}$$

over $U_{11} \subset X$ prevents $L_1 \otimes L_2$ from being a limit root of $N_1 \otimes N_2$. However, it also reveals where the problem lies, and how to work around it through the use of a correction term R. Indeed, consider the line bundle R on X defined as the gluing, or *recollement*, of the following local data:

$$\{(U_{10}, \mathcal{O}_{U_{10}}), (U_{01}, \mathcal{O}_{U_{01}}), (U_{11}, (F')^\vee|_{U_{11}})\} \;\rightsquigarrow\; R \in \operatorname{Pic}(X)$$
$$\text{with} \quad R|_{U_{10}} = \mathcal{O}_{U_{10}}, \quad R|_{U_{01}} = \mathcal{O}_{U_{01}}, \quad R|_{U_{11}} = (F')^\vee|_{U_{11}}$$

The inclusion of the correction term R offers a natural candidate for the role of “limit product” of $(L_1, L_2)_{\text{sync}}$, starting with a line bundle

$$L_1 \circledast L_2 \quad = \quad L_1 \otimes L_2 \otimes R \quad \in \quad \operatorname{Pic}(X)$$

over X, whose local description is:

$$\begin{array}{lrcl} \text{over } U_{10}\colon & (L_1 \circledast L_2)|_{U_{10}} & \cong & (L_1 \otimes L_2)|_{U_{10}} \\ \text{over } U_{01}\colon & (L_1 \circledast L_2)|_{U_{01}} & \cong & (L_1 \otimes L_2)|_{U_{01}} \\ \text{over } U_{11}\colon & (L_1 \circledast L_2)|_{U_{11}} & \cong & (\pi^* N_1 \otimes L_2 \otimes L_1^\vee)|_{U_{11}} \\ & & \cong & (\pi^* N_2 \otimes L_1 \otimes L_2^\vee)|_{U_{11}} \end{array}$$

Moreover, $L_1 \circledast L_2$ is equipped with a sheaf homomorphism

$$\alpha_1 \circledast \alpha_2\colon \quad (L_1 \circledast L_2)^{\otimes 2} \;\; \to \;\; \pi^*(N_1 \otimes N_2)$$

obtained by gluing the following local data:

$$\{(U_{10},\, \alpha_1 \otimes \alpha_2),\, (U_{01},\, \alpha_1 \otimes \alpha_2),\, (U_{11},\, \mathrm{Id})\} \;\; \rightsquigarrow \;\; \alpha_1 \circledast \alpha_2$$

In other words, $\alpha_1 \circledast \alpha_2$ is locally given by:

$$\begin{array}{lrcl} \text{over } U_{10}\colon & (L_1 \circledast L_2)^{\otimes 2}|_{U_{10}} & \cong & (L_1^{\otimes 2} \otimes L_2^{\otimes 2})|_{U_{10}} \stackrel{\alpha_1 \otimes \alpha_2}{\longrightarrow} (\pi^* N_1 \otimes \pi^* N_2)|_{U_{10}} \\ \text{over } U_{01}\colon & (L_1 \circledast L_2)^{\otimes 2}|_{U_{01}} & \cong & (L_1^{\otimes 2} \otimes L_2^{\otimes 2})|_{U_{01}} \stackrel{\alpha_1 \otimes \alpha_2}{\longrightarrow} (\pi^* N_1 \otimes \pi^* N_2)|_{U_{01}} \\ \text{over } U_{11}\colon & (L_1 \circledast L_2)^{\otimes 2}|_{U_{11}} & \cong & (L_1^{\otimes 2} \otimes L_2^{\otimes 2} \otimes ((F')^\vee)^{\otimes 2})|_{U_{11}} \;\; \cong \\ & & & \qquad \cong \;\; (\pi^* N_1 \otimes \pi^* N_2)|_{U_{11}} \end{array}$$

Now, since $(\alpha_1 \circledast \alpha_2)|_{U_{11}}$ is an isomorphism, pulling back to $E_{(1,1)} \subset U_{11}$ we see that $L_1 \circledast L_2$ is trivial over $E_{(1,1)}$, and so the triplet

$$(X,\, L_1 \circledast L_2,\, \alpha_1 \circledast \alpha_2)$$

stabilizes to a limit root $(\pi_3\colon X_3 \to Y,\, L_3',\, \alpha_3')$ of $N_1 \otimes N_2$ which we denote

$$(\pi_3, L_3', \alpha_3') \;\; = \;\; (\pi_1, L_1', \alpha_1') \circledast_{\mathrm{sync}} (\pi_2, L_2', \alpha_2') \;\; \in \;\; \mathcal{S}_{\lim}(N_1 \otimes N_2)$$

where X_3 corresponds to contracting $E_{(1,1)} \subset X$. As it turns out, this new limit root does not exist in isolation; on the contrary, a sheaf homomorphism

$$\chi_1 \circledast \chi_2\colon \quad F' \otimes \pi^*(N_1 \otimes N_2) \;\; \to \;\; (L_1 \circledast L_2)^{\otimes 2}$$

is readily associated to it, namely by *recollement* of the following local data:

$$\{(U_{10},\, \chi_1 \otimes \alpha_2^{-1}),\, (U_{01},\, \alpha_1^{-1} \otimes \chi_2),\, (U_{11},\, \alpha_1 \circ \chi_1 \equiv \alpha_2 \circ \chi_2)\} \;\; \rightsquigarrow \;\; \chi_1 \circledast \chi_2$$

Observe that this process works due to the synchronizing conditions, yielding:

$$\begin{array}{lll}
\text{over } U_{10}: & ((F'\otimes\pi^*N_1)\otimes\pi^*N_2)|_{U_{10}} \overset{\chi_1\otimes\alpha_2^{-1}}{\cong} (L_1^{\otimes 2}\otimes L_2^{\otimes 2})|_{U_{10}} \\
\text{over } U_{01}: & (\pi^*N_1\otimes(F'\otimes\pi^*N_2))|_{U_{01}} \overset{\alpha_1^{-1}\otimes\chi_2}{\cong} (L_1^{\otimes 2}\otimes L_2^{\otimes 2})|_{U_{01}} \\
\text{over } U_{11}: & ((F'\otimes\pi^*N_1)\otimes\pi^*N_2)|_{U_{11}} \xrightarrow{\alpha_1\circ\chi_1} (\pi^*N_1\otimes\pi^*N_2)|_{U_{11}} \\
(\text{or equiv.} & (\pi^*N_1\otimes(F'\otimes\pi^*N_2))|_{U_{11}} \xrightarrow{\alpha_2\circ\chi_2} (\pi^*N_1\otimes\pi^*N_2)|_{U_{11}}\ )
\end{array}$$

Not only does $\chi_1 \circledast \chi_2$ restrict to an isomorphism over $V(L_1 \circledast L_2) = U_{10} \cup U_{01}$, but it also gives rise to a commutative diagram

$$\begin{tikzcd}
 & L_1^{\otimes 2}\otimes\pi^*N_2 \arrow[dr, "\alpha_1\otimes\mathrm{Id}"] & \\
F'\otimes\pi^*(N_1\otimes N_2) \arrow[ur, "\chi_1\otimes\mathrm{Id}"] \arrow[r, "\chi_1\circledast\chi_2"] \arrow[dr, "\chi_2\otimes\mathrm{Id}"'] & (L_1\circledast L_2)^{\otimes 2} \arrow[r, "\alpha_1\circledast\alpha_2"] & \pi^*(N_1\otimes N_2) \\
 & L_2^{\otimes 2}\otimes\pi^*N_1 \arrow[ur, "\alpha_2\otimes\mathrm{Id}"'] &
\end{tikzcd}$$

thus making $\{F', \chi_1, \chi_2, \chi_1 \circledast \chi_2\}$ into synchronization data of $(L_1, L_2, L_1 \circledast L_2)$, in the sense of remark 2.4.8(iii). That is, we have built a triple limit root

$$\left\{\begin{array}{llll} X, & L_1, & L_2, & L_1\circledast L_2, \\ & \alpha_1, & \alpha_2, & \alpha_1\circledast\alpha_2, \\ F', & \chi_1, & \chi_2, & \chi_1\circledast\chi_2 \end{array}\right\} \in \mathcal{S}^3_{\mathrm{lim}}(N_1, N_2, N_1\otimes N_2)$$

of $(N_1, N_2, N_1 \otimes N_2)$, which we denote $(L_1, L_2, L_1 \circledast L_2)_{\mathrm{sync}}$, and so a map

$$\begin{array}{rccc} \circledast: & \mathcal{S}^2_{\mathrm{lim}}(N_1,N_2) & \to & \mathcal{S}^3_{\mathrm{lim}}(N_1,N_2,N_1\otimes N_2) \\ & (L_1,L_2)_{\mathrm{sync}} & \mapsto & (L_1,L_2,L_1\circledast L_2)_{\mathrm{sync}} \end{array}$$

fitted with a forgetful retraction $(L_1, L_2, L_3)_{\mathrm{sync}} \mapsto (L_1, L_2)_{\mathrm{sync}}$. $\square$

Recalling definition 2.4.15, note that the exceptional type of the nodes of Y with respect to the triplet (π_1, π_2, π_3) involved in the limit product

$$(\pi_3, L'_3, \alpha'_3) \;=\; (\pi_1, L'_1, \alpha'_1) \circledast_{\mathrm{sync}} (\pi_2, L'_2, \alpha'_2)$$

directly depends on their (π_1, π_2)-exceptional type. Specifically, we have

$$Z_{(a_1,a_2)}(\pi_1,\pi_2) \;=\; Z_{(a_1,a_2,a_3)}(\pi_1,\pi_2,\pi_3) \qquad \text{for} \qquad a_3 \;\equiv\; a_1 + a_2 \mod 2$$

hence the open subsets $\{U_{ab}\,(\pi_3,\pi_2)\}$ and $\{U_{ab}\,(\pi_1,\pi_2)\}$ are related by

$$\begin{cases} U'_{ab} \;=\; U_{ab}\,(\pi_3,\pi_2) \;=\; U_{(a+b \bmod 2)\,b}\,(\pi_1,\pi_2) \;=\; U_{(a+b \bmod 2)\,b} \\ U'_{10} \;=\; U_{10}, \quad U'_{01} \;=\; U_{11}, \quad U'_{11} \;=\; U_{01} \end{cases}$$

This indicates that an inverse operation can also be constructed, using

$$\{(U_{10},\mathcal{O}_{U_{10}}),\,(U_{01},F'|_{U_{01}} \cong (L_2^{\otimes 2}\otimes \pi^* N_2^{\vee})|_{U_{01}}),\,(U_{11},\mathcal{O}_{U_{11}})\} \;\rightsquigarrow\; R' \in \operatorname{Pic}(X)$$

as a correction term to deal with the now problematic $(0,1)$-nodes. If modified accordingly, the above proof results in a "limit division" map

$$\begin{array}{rrcl} \circledast^{\vee}: & \mathcal{S}^2_{\lim}(N_1,N_2) & \to & \mathcal{S}^3_{\lim}(N_1,N_2,N_1\otimes N_2^{\vee}) \\ & (L_1,L_2)_{\mathrm{sync}} & \mapsto & (L_1,L_2,L_1\otimes L_2^{\vee}\otimes R')_{\mathrm{sync}} \end{array}$$

which is supported by the commutative diagram

$$\begin{tikzcd} \mathcal{S}^2_{\lim}(N_1,N_2) \arrow[r, "\circledast"] & \mathcal{S}^3_{\lim}(N_1,N_2,N_1\otimes N_2) \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\ \mathcal{S}^3_{\lim}(N_2,N_1\otimes N_2,N_1) \arrow[u] & \mathcal{S}^2_{\lim}(N_2,N_1\otimes N_2) \arrow[l, "\circledast^{\vee}"'] \end{tikzcd}$$

$$\begin{tikzcd} (L_1,L_2)_{\mathrm{sync}} \arrow[r, mapsto, "\circledast"] & (L_1,L_2,L_1\circledast L_2)_{\mathrm{sync}} \arrow[d, mapsto] \\ (L_2,L_1\circledast L_2,L_1)_{\mathrm{sync}} \arrow[u, mapsto] & (L_2,L_1\circledast L_2)_{\mathrm{sync}} \arrow[l, mapsto, "\circledast^{\vee}"'] \end{tikzcd}$$

since $R' = R'(L_2,L_1\circledast L_2)_{\mathrm{sync}} = (R(L_1,L_2)_{\mathrm{sync}})^{\vee} = R^{\vee}$ implies that

$$(L_1\circledast L_2)\circledast^{\vee} L_2 \;=\; (L_1\otimes L_2\otimes R)\otimes L_2^{\vee}\otimes R' \;=\; L_1$$

and similarly $(\alpha_1\circledast\alpha_2)\circledast^{\vee}\alpha_2 = \alpha_1$, $(\chi_1\circledast\chi_2)\circledast^{\vee}\chi_2 = \chi_1$.

Once again, we attempt to clarify the situation with some examples.

Example 2.4.17. Let $B = B_X \subset X$ be the closure of $X - E(X)$. The bundles R and $L_1 \circledast L_2$ defined in proposition 2.4.16 restrict to:

$$\begin{array}{ccccccccc} \operatorname{Pic}(X) & \longrightarrow & \operatorname{Pic}(B) & \oplus & \operatorname{Pic}(E_{(1,0)}) & \oplus & \operatorname{Pic}(E_{(0,1)}) & \oplus & \operatorname{Pic}(E_{(1,1)}) \\ R & \longmapsto & \big(\mathcal{O}_B(S_{(1,1)}) & , & \mathcal{O}_{E_{(1,0)}} & , & \mathcal{O}_{E_{(0,1)}} & , & \mathcal{O}_{E_{(1,1)}}(-2)\big) \\ L_1\circledast L_2 & \longmapsto & \big(L_1|_B\otimes L_2|_B\otimes \mathcal{O}_B(S_{(1,1)}) & , & & & & & \\ & & & & \mathcal{O}_{E_{(1,0)}}(1) & , & \mathcal{O}_{E_{(0,1)}}(1) & , & \mathcal{O}_{E_{(1,1)}} \quad\big) \end{array}$$

since R is trivial outside of the $(1,1)$-nodes, and

$$R|_{U_{11}} \quad = \quad (F')^{\vee}|_{U_{11}} \quad = \quad (\pi^* N_1 \otimes (L_1^{\vee})^{\otimes 2})|_{U_{11}} \quad = \quad (\pi^* N_2 \otimes (L_2^{\vee})^{\otimes 2})|_{U_{11}}$$

otherwise; recall the sheaf isomorphisms given after definition 2.1.8.

Example 2.4.18. In the case of Prym-spin curves over Δ_0, the correction term of $(X_1, \theta, \alpha) \circledast_{\text{sync}} (X_2, \eta, \beta)$ amounts to

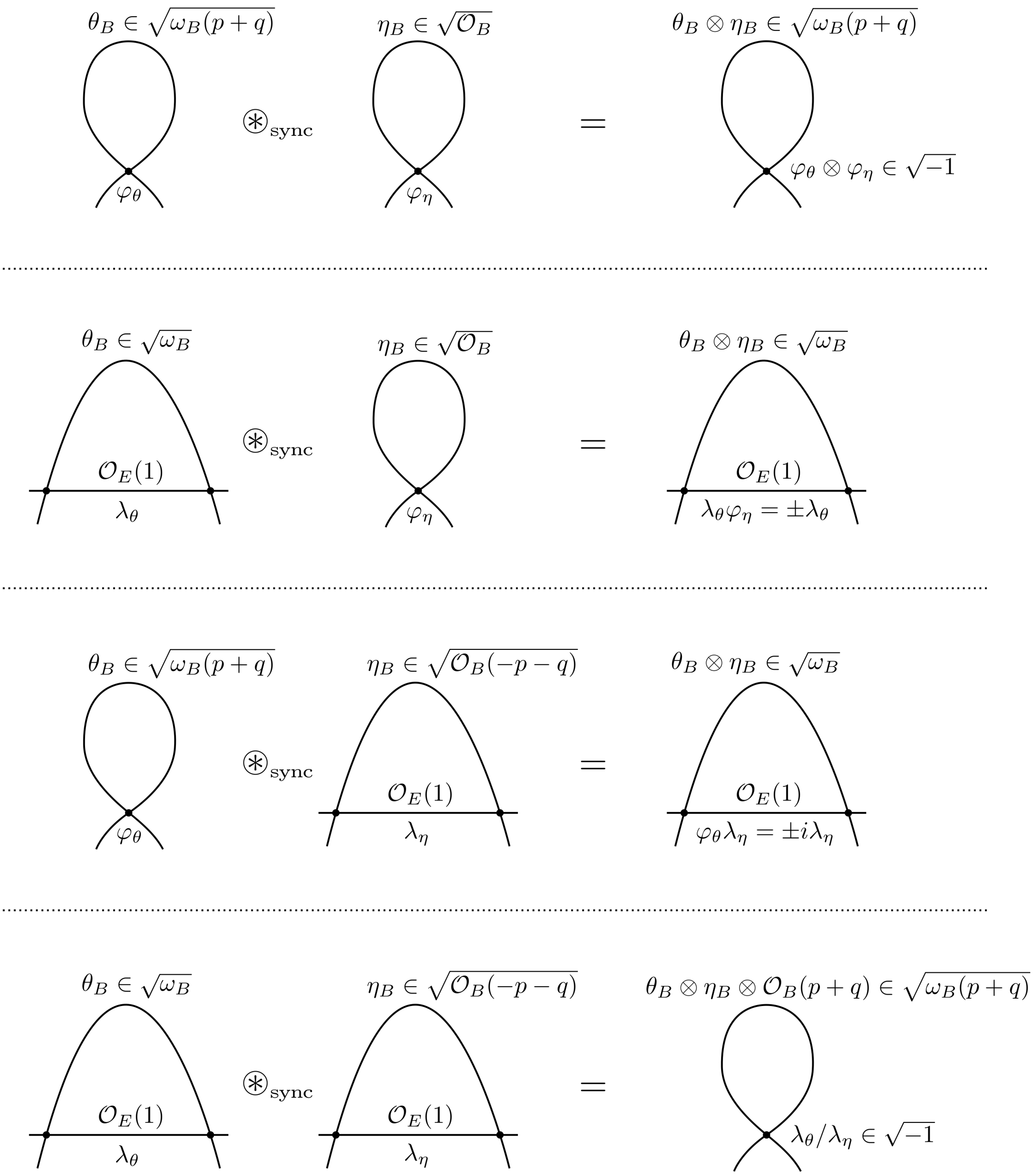
$$(\pi^* \omega_Y \otimes (\theta^{\vee})^{\otimes 2})|_{U_{11}} \quad = \quad (\pi^* \mathcal{O}_Y \otimes (\eta^{\vee})^{\otimes 2})|_{U_{11}} \quad = \quad (\eta^{\vee})^{\otimes 2}|_{U_{11}}$$

over U_{11}, so that the limit product $\theta \circledast \eta$ is of the form $\theta \otimes \eta^{\vee}$ over $(1,1)$-nodes and $\theta \otimes \eta$ elsewhere. With the notation from remark 2.4.11 and the discussion prior to proposition 2.2.13, the four basic combinations of limit roots and their respective products are depicted on the back of the page.

If the limit product $\circledast$ from proposition 2.4.16 could be defined over families of double limit roots, it would be of great value for the study of the spaces

$$\mathcal{S}^2_{\text{lim}}(\mathcal{N}_1, \mathcal{N}_2) \to \overline{\mathcal{M}}_g$$

that have been introduced in this section. We thus conclude this work with the hope that these notions will, sooner or later, receive a proper treatment.



[Example 2.4.18]

Bibliography

[AC87] E. Arbarello and M. Cornalba. "The Picard groups of the moduli spaces of curves". *Topology*, 26(2):153–171, 1987.

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Volume I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[Ati56] M. Atiyah. "On the Krull-Schmidt theorem with application to sheaves". *Bull. Soc. Math. France*, 84:307–317, 1956.

[BCF04] D. Ballico, C. Casagrande, and C. Fontanari. "Moduli of Prym curves". *Doc. Math.*, 9:265–281, 2004.

[Bea77] A. Beauville. "Prym varieties and the Schottky problem". *Invent. Math.*, 41(2):149–196, 1977.

[Bea13] A. Beauville. "Vanishing thetanulls on curves with involutions". *Rend. Circ. Mat. Palermo (2)*, 62(1):61–66, 2013.

[Cor89] M. Cornalba. "Moduli of curves and theta-characteristics". In *Lectures on Riemann surfaces (Trieste, 1987)*, pages 560–589. World Sci. Publ., Teaneck, NJ, 1989.

[dGP05] J. d'Almeida, L. Gruson, and N. Perrin. "Courbes de genre 5 munies d'une involution sans point fixe". *J. London Math. Soc. (2)*, 72(3):545–570, 2005.

[DM69] P. Deligne and D. Mumford. "The irreducibility of the space of curves of given genus". *Inst. Hautes Études Sci. Publ. Math.*, 36:75–109, 1969.

[Dol08] I. V. Dolgachev. "Rationality of $\mathcal{R}_2$ and $\mathcal{R}_3$". *Pure Appl. Math. Q.*, 4(2, Special Issue: In honor of Fedor Bogomolov. Part 1):501–508, 2008.

[Don92] R. Donagi. "The fibers of the Prym map". In *Curves, Jacobians, and abelian varieties (Amherst, MA, 1990)*, volume 136 of *Contemp. Math.*, pages 55–125. Amer. Math. Soc., Providence, RI, 1992.

[DS81] R. Donagi and R. C. Smith. "The structure of the Prym map". *Acta Math.*, 146(1-2):25–102, 1981.

[EH86] D. Eisenbud and J. Harris. "Limit linear series: basic theory". *Invent. Math.*, 85(2):337–371, 1986.

[EH87] D. Eisenbud and J. Harris. "The Kodaira dimension of the moduli space of curves of genus $\geqq 23$". *Invent. Math.*, 90(2):359–387, 1987.

[Far76] H. M. Farkas. "Unramified double coverings of hyperelliptic surfaces". *J. Analyse Math.*, 30:150–155, 1976.

[Far10] G. Farkas. "The birational type of the moduli space of even spin curves". *Adv. Math.*, 223(2):433–443, 2010.

[Far12] G. Farkas. "Prym varieties and their moduli". In *Contributions to algebraic geometry*, EMS Ser. Congr. Rep., pages 215–255. Eur. Math. Soc., Zürich, 2012.

[FL10] G. Farkas and K. Ludwig. "The Kodaira dimension of the moduli space of Prym varieties". *J. Eur. Math. Soc. (JEMS)*, 12(3):755–795, 2010.

[FS82] R. Friedman and R. Smith. "The generic Torelli theorem for the Prym map". *Invent. Math.*, 67(3):473–490, 1982.

[FV12] G. Farkas and A. Verra. "Moduli of theta-characteristics via Nikulin surfaces". *Math. Ann.*, 354(2):465–496, 2012.

[FV14] G. Farkas and A. Verra. "The geometry of the moduli space of odd spin curves". *Ann. of Math. (2)*, 180(3):927–970, 2014.

[FV16] G. Farkas and A. Verra. "The universal abelian variety over $\mathcal{A}_5$". *Ann. Sci. Éc. Norm. Supér. (4)*, 49(3):521–542, 2016.

[Har77] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Har83] J. Harer. "The second homology group of the mapping class group of an orientable surface". *Invent. Math.*, 72(2):221–239, 1983.

[Har92] J. Harris. *Algebraic geometry. A first course*, volume 133 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1992.

[Har93] J. Harer. “The rational Picard group of the moduli space of Riemann surfaces with spin structure”. In *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*, volume 150 of *Contemp. Math.*, pages 107–136. Amer. Math. Soc., Providence, RI, 1993.

[HM82] J. Harris and D. Mumford. “On the Kodaira dimension of the moduli space of curves”. *Invent. Math.*, 67(1):23–86, 1982. With an appendix by W. Fulton.

[HM98] J. Harris and I. Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1998.

[Huy16] D. Huybrechts. *Lectures on K3 surfaces*, volume 158 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2016.

[Laz04] R. Lazarsfeld. *Positivity in algebraic geometry I. Classical setting: line bundles and linear series*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, 2004.

[LZ21] Y. Len and D. Zakharov. “Kirchhoff’s theorem for Prym varieties”, 2021. With an appendix by S. Casalaina-Martin. arXiv: 2012.15235.

[Mas76] L. Masiewicki. “Universal properties of Prym varieties with an application to algebraic curves of genus five”. *Trans. Amer. Math. Soc.*, 222:221–240, 1976.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)].* Springer-Verlag, Berlin, third edition, 1994.

[Mum71] D. Mumford. “Theta characteristics of an algebraic curve”. *Ann. Sci. École Norm. Sup. (4)*, 4(2):181–192, 1971.

[Mum74] D. Mumford. “Prym varieties I”. In *Contributions to analysis (a collection of papers dedicated to Lipman Bers)*, pages 325–350. Academic Press, New York, 1974.

[Mum08] D. Mumford. *Abelian varieties*, volume 5 of *Tata Institute of Fundamental Research Studies in Mathematics.* Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book

Agency, New Delhi, 2008. With appendices by C. P. Ramanujam and Y. Manin. Corrected reprint of the second (1974) edition.

[Ols16] M. Olsson. *Algebraic spaces and stacks*, volume 62 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2016.

[Put12a] A. Putman. “The Picard group of the moduli space of curves with level structures”. *Duke Math. J.*, 161(4):623–674, 2012.

[Put12b] A. Putman. “The second rational homology group of the moduli space of curves with level structures”. *Adv. Math.*, 229(2):1205–1234, 2012.

[Rie57] B. Riemann. “Theorie der Abel'schen Functionen”. *J. Reine Angew. Math.*, 54:115–155, 1857.

[Roj21] A. Rojas. “The divisors of Prym semicanonical pencils”, 2021. arXiv: 2103.01687.

[Ser17] E. C. Sertöz. *Enumerative geometry of double spin curves*. PhD thesis, Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät, 2017.

[Ser19] E. C. Sertöz. “A compactification of the moduli space of multiple-spin curves”, 2019. arXiv: 1701.02303.

[TiB87] M. Teixidor i Bigas. “Half-canonical series on algebraic curves”. *Trans. Amer. Math. Soc.*, 302(1):99–115, 1987.

[TiB88] M. Teixidor i Bigas. “The divisor of curves with a vanishing theta-null”. *Compositio Math.*, 66(1):15–22, 1988.

[Ver13] A. Verra. “Rational parametrizations of moduli spaces of curves”. In *Handbook of moduli. Vol. III*, volume 26 of *Adv. Lect. Math. (ALM)*, pages 431–506. Int. Press, Somerville, MA, 2013.

[vGS07] B. van Geemen and A. Sarti. “Nikulin involutions on K3 surfaces”. *Math. Z.*, 255(4):731–753, 2007.

Declaration of independent work

I declare that I have completed the thesis independently using only the aids and tools specified. I have not applied for a doctor's degree in the doctoral subject elsewhere and do not hold a corresponding doctor's degree. I have taken due note of the Faculty of Mathematics and Natural Sciences PhD Regulations, published in the Official Gazette of Humboldt-Universität zu Berlin no. 42/2018 on 11.07.2018.

Berlin, 30.03.2021

Carlos Maestro Pérez