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On Colourings of Hypergraphs Without Monochromatic Fano Planes

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For Tom Trotter on his 65th birthday

For k-uniform hypergraphs F and H and an integer r, let $c_{r,F}(H)$ denote the number of r-colourings of the set of hyperedges of H with no monochromatic copy of F, and let $c_{r,F}(n) = \max_{H\in\mathcal{H}_n} c_{r,F}(H)$, where the maximum runs over all k-uniform hypergraphs on n vertices. Moreover, let $\mathrm{ex}(n, F)$ be the usual *extremal* or *Turán function*, *i.e.*, the maximum number of hyperedges of an n-vertex k-uniform hypergraph which contains no copy of F.

For complete graphs $F = K_\ell$ and $r = 2$, Erdős and Rothschild conjectured that $c_{2,K_\ell}(n) = 2^{\mathrm{ex}(n,K_\ell)}$. This conjecture was proved by Yuster for $\ell = 3$ and by Alon, Balogh, Keevash and Sudakov for arbitrary ℓ. In this paper, we consider the question for hypergraphs and show that, in the special case when F is the Fano plane and $r = 2$ or 3, then $c_{r,F}(n) = r^{\mathrm{ex}(n,F)}$, while $c_{r,F}(n) \gg r^{\mathrm{ex}(n,F)}$ for $r \geqslant 4$.

1. Introduction and results

We consider k-uniform hypergraphs $H = (V, E)$, where $E = E(H) \subseteq \binom{V}{k}$. For k-uniform hypergraphs F and H and an integer r, let $c_{r,F}(H)$ denote the number of r-colourings of the set of hyperedges of H with no monochromatic copy of F, and let $c_{r,F}(n) = \max_{H\in\mathcal{H}_n} c_{r,F}(H)$, where the maximum runs over all k-uniform hypergraphs on n vertices.

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Moreover, let $\mathrm{ex}(n, F)$ be the usual *extremal* or *Turán function*, *i.e.*, the maximum number of hyperedges of an n-vertex k-uniform hypergraph which contains no copy of F. We say a hypergraph H on n vertices is extremal for F if $e(H) = |E(H)| = \mathrm{ex}(n, F)$.

Clearly, every edge colouring of any extremal hypergraph H for F contains no monochromatic copy of F and, consequently,

$$c_{r,F}(n) \geqslant r^{\mathrm{ex}(n,F)}$$

for all $r \geqslant 2$. On the other hand, let $\mathrm{Forb}_F(n)$ denote the family of all labelled hypergraphs on n vertices which contain no copy of F. Since every 2-colouring of the set of hyperedges of a hypergraph H which contains no monochromatic copy of F gives rise to a member of $\mathrm{Forb}_F(n)$, *e.g.*, always consider the sub-hypergraph in one of the two colours, we have

$$c_{2,F}(n) \leqslant |\,\mathrm{Forb}_F(n)|.$$

The size of $\mathrm{Forb}_F(n)$ was first studied by Erdős, Kleitman and Rothschild [8] and Kolaitis, Prömel and Rothschild [13, 14] for graph cliques $F = K_\ell$ on ℓ vertices, and by Erdős, Frankl, and Rödl [7] for arbitrary graphs F, *i.e.*, $|\,\mathrm{Forb}_F(n)| \leqslant 2^{\mathrm{ex}(n,F)+o(n^2)}$ (see [4, 3] for recent improvements). Recently, the result from [7] was extended in [15, 16] to k-uniform hypergraphs F, *i.e.*,

$$|\,\mathrm{Forb}_F(n)| \leqslant 2^{\mathrm{ex}(n,F)+o(n^k)}$$

(see [17] for recent improvements when F is the hypergraph of the Fano plane). Returning to the maximum number of hyperedge colourings without a monochromatic copy of an arbitrary k-uniform hypergraph F, we have for two colours

$$2^{\mathrm{ex}(n,F)} \leqslant c_{2,F}(n) \leqslant 2^{\mathrm{ex}(n,F)+o(n^k)}. \tag{1.1}$$

In the graph case, when $F = K_\ell$ is a graph clique, Yuster [20] (for $\ell = 3$) and Alon, Balogh, Keevash and Sudakov [1] (for arbitrary ℓ) closed the gap in (1.1), and showed that the lower bound is the correct order of $c_{2,K_\ell}(n)$, *i.e.*, $c_{2,K_\ell}(n) = 2^{\mathrm{ex}(n,K_\ell)}$, which was conjectured by Erdős and Rothschild (see [6]). Moreover, Alon, Balogh, Keevash and Sudakov showed that $c_{3,K_\ell}(n) = 3^{\mathrm{ex}(n,K_\ell)}$, and in both cases, $r = 2, 3$, we have

$$c_{r,K_\ell}(H) = c_{r,K_\ell}(n) = r^{\mathrm{ex}(n,K_\ell)}$$

only when H is the $(\ell - 1)$-partite Turán graph. In fact, it was shown in [1] that the same result holds for ℓ-chromatic graphs that contain a colour-critical edge. Furthermore, it was observed in [1] that $c_{r,K_\ell}(n) \gg r^{\mathrm{ex}(n,K_\ell)}$ for $r \geqslant 4$.

In this paper, we determine $c_{r,F}(n)$ for $r = 2, 3$ and F being the 3-uniform hypergraph of the Fano plane, *i.e.*, the unique triple system with 7 hyperedges on 7 vertices in which every pair of vertices is contained in precisely one hyperedge. It was shown independently by Füredi and Simonovits [10] and Keevash and Sudakov [11] that, for n sufficiently large, the unique extremal Fano plane-free hypergraph on n vertices is the balanced, complete, bipartite hypergraph $B_n = (U \dot\cup W, E(B_n))$, where $|U| = \lfloor n/2 \rfloor$, $|W| = \lceil n/2 \rceil$ and $E(B_n)$ consists of all hyperedges with at least one vertex in U and one vertex in W.

Therefore, for the Fano plane F we have, for sufficiently large n,

$$\mathrm{ex}(n,F) = e(B_n) = |E(B_n)| = \binom{n}{3} - \binom{\lceil n/2 \rceil}{3} - \binom{\lfloor n/2 \rfloor}{3} \leqslant \frac{n^3}{8} - \frac{n^2}{4} \leqslant \frac{n^3}{8} \tag{1.2}$$

and

$$\delta_1(B_n) = e(B_n) - e(B_{n-1}) = \left(\left\lceil \frac{n}{2} \right\rceil - 1\right)\left\lfloor \frac{n}{2} \right\rfloor + \binom{\lfloor n/2 \rfloor}{2} \geqslant \frac{3}{8}n^2 - n, \tag{1.3}$$

where for a hypergraph $H = (V,E)$ we let $\delta_1(H)$ denote the minimum vertex degree, *i.e.*, $\delta_1(H) = \min_{u \in V} |\{\{v,w\} : \{u,v,w\} \in E\}|$.

Theorem 1.1. *Let F be the 3-uniform hypergraph of the Fano plane and let $r = 2$ or $r = 3$. There exists an integer n_r such that, for every 3-uniform hypergraph H on $n \geqslant n_r$ vertices, we have*

$$c_{r,F}(H) \leqslant r^{\mathrm{ex}(n,F)}.$$

Moreover, the only 3-uniform hypergraph H on n vertices with $c_{r,F}(H) = r^{\mathrm{ex}(n,F)}$ is the extremal hypergraph for F, i.e., H is isomorphic to B_n, the balanced, complete, bipartite hypergraph on n vertices.

The following result shows that, as in the case of graph cliques, Theorem 1.1 does not extend to more than 3 colours (see also (5.1)).

Theorem 1.2. *For the Fano plane F and $r > 3$ we have $c_{r,F}(n) \gg r^{\mathrm{ex}(n,F)}$ for sufficiently large n.*

Theorem 1.1 and Theorem 1.2 are a first extension of the results from [1] to hypergraphs. In fact, our proof proceeds along similar lines, and is based on the stability result for the Fano plane due to Keevash and Sudakov [11] and Füredi and Simonovits [10] and the weak hypergraph regularity lemma.

2. Tools

Throughout this paper we study 3-uniform hypergraphs and from now on by a *hypergraph* we always mean a 3-uniform hypergraph. For a hypergraph $H = (V,E)$ and a subset $U \subseteq V$ of the vertex set V we write $E_H(U)$, or simply $E(U)$ if the hypergraph under consideration is obvious, for the hyperedges of H that are completely contained in U, *i.e.*, $E_H(U) = E \cap \binom{U}{3}$. We define the cardinality of $E_H(U)$ by $e_H(U)$ or simply $e(U)$. Similarly, for two disjoint subsets U and W we write

$$E(U,W) = \{e \in E : e \subseteq U \cup W, e \cap U \neq \emptyset, e \cap W \neq \emptyset\} = E(U \cup W) \setminus (E(U) \cup E(W))$$

and $e(U,W) = |E(U,W)|$. Analogously, we define $E(W_1,W_2,W_3)$ and $e(W_1,W_2,W_3)$ for triples of pairwise disjoint subsets.

The following stability result for Fano plane-free hypergraphs was proved by Füredi and Simonovits [10] and Keevash and Sudakov [11].

Theorem 2.1 (Stability theorem for Fano plane-free hypergraphs). *For every $\delta > 0$ there exist $\varepsilon > 0$ and n_0 such that every Fano plane-free hypergraph H on $n \geqslant n_0$ vertices with at least $(\frac{1}{8} - \varepsilon)n^3$ hyperedges admits a partition $V(H) = X \dot{\cup} Y$ with $e(X) + e(Y) < \delta n^3$.*

Another tool we use in this paper is the so-called *weak hypergraph regularity lemma*. This result is a straightforward extension of Szemerédi's regularity lemma [19] for graphs. We only state the version for 3-uniform hypergraphs here. Let $H = (V, E)$ be a hypergraph and let W_1, W_2, W_3 be mutually disjoint non-empty subsets of V. We denote by $d_H(W_1, W_2, W_3) = d(W_1, W_2, W_3)$ the *density* of the 3-partite induced sub-hypergraph $H[W_1, W_2, W_3]$ of H, defined by

$$d_H(W_1, W_2, W_3) = \frac{e_H(W_1, W_2, W_3)}{|W_1||W_2||W_3|}.$$

We say the triple (V_1, V_2, V_3) of mutually disjoint subsets $V_1, V_2, V_3 \subseteq V$ is (ε, d)-*regular*, for positive constants ε and d, if

$$|d_H(W_1, W_2, W_3) - d| \leqslant \varepsilon$$

for all triples of subsets $W_1 \subseteq V_1, W_2 \subseteq V_2, W_3 \subseteq V_3$ with $|W_1||W_2||W_3| \geqslant \varepsilon |V_1||V_2||V_3|$. We say the triple (V_1, V_2, V_3) is ε-*regular* if it is (ε, d)-regular for some $d \geqslant 0$.

An ε-regular partition of a vertex set $V(H)$ has the following properties:

(i) $V = V_1 \dot{\cup} \cdots \dot{\cup} V_t$,
(ii) $||V_i| - |V_j|| \leqslant 1$ for all i, j,
(iii) for all but at most $\varepsilon\binom{t}{3}$ sets $\{i_1, i_2, i_3\} \subseteq [t] = \{1, \ldots, t\}$, the triple $(V_{i_1}, V_{i_2}, V_{i_3})$ is ε-regular.

The coloured version of the weak regularity lemma (see, *e.g.*, [5, 9, 18]) states the following.

Theorem 2.2. *For all integers $r \geqslant 1$ and $t_0 \geqslant 1$, and every $\varepsilon > 0$, there exist $T_0 = T_0(r, t_0, \varepsilon)$ and $N_0 = N_0(r, t_0, \varepsilon)$ such that, for every hypergraph $H = (V, E)$ on $n \geqslant N_0$ vertices, whose hyperedges are r-coloured $E(H) = E_1 \dot{\cup} \cdots \dot{\cup} E_r$, there exists a partition $V = V_1 \dot{\cup} \cdots \dot{\cup} V_t$, with $t_0 \leqslant t \leqslant T_0$, which is ε-regular simultaneously with respect to all sub-hypergraphs $H_i = (V, E_i)$ for $1 \leqslant i \leqslant r$.*

For a hypergraph H and a regular partition of its vertex set, we use the concept of a cluster-hypergraph.

Definition. For a hypergraph $H = (V, E)$ and an ε-regular partition $V = V_1 \dot{\cup} \cdots \dot{\cup} V_t$ of its vertex set and a number $\gamma > 0$, let $H(\gamma) = (V^*, E^*)$ be the cluster-hypergraph with vertex set $V^* = [t] = \{1, \ldots, t\}$ and edge set E^*, where for $1 \leqslant i < j < k \leqslant t$ it is $\{i, j, k\} \in E^*$ if and only if the triple (V_i, V_j, V_k) is ε-regular and the density satisfies $d_H(V_i, V_j, V_k) \geqslant \gamma$.

In [12] a counting lemma for linear hypergraphs in the context of the weak hypergraph regularity lemma was proved, where a hypergraph is said to be linear if no two of its hyperedges intersect in more than one vertex. Since the Fano plane is a linear hypergraph, we obtain the following lemma.

Lemma 2.3. *For all $\gamma > 0$ there exists $\varepsilon = \varepsilon(\gamma) > 0$ and an integer $m_0 = m_0(\gamma)$ such that for every positive integer t the following holds. Let $H = (V, E)$ be a hypergraph with an ε-regular partition $V = V_1 \dot\cup \cdots \dot\cup V_t$ such that $|V_i| \geqslant m_0$ for every $i \in [t]$. If the cluster-hypergraph $H(\gamma)$ contains a copy of the Fano plane, then the hypergraph H contains a Fano plane too.*

3. Structure of hypergraphs with many edge colourings

For the proof of Theorem 1.1 we first analyse the structure of those hypergraphs which admit 'many' Fano plane-free colourings.

Lemma 3.1 (Main lemma). *Let $r = 2$ or $r = 3$ and let F be the hypergraph of the Fano plane. Then, for every $\delta > 0$ there exists $n_0 = n_0(r, \delta)$ such that every hypergraph $H = (V, E)$ on $n \geqslant n_0$ vertices with $c_{r,F}(H) \geqslant r^{e(B_n)}$ admits a partition $V = X \dot\cup Y$ of its vertex set with $e(X) + e(Y) < \delta n^3$.*

Proof. We prove the lemma only for $r = 3$, as the proof for $r = 2$ is very similar. Let $\delta > 0$ be given. Let $h(x) := -x \log x - (1 - x) \log(1 - x)$ for $0 < x < 1$ be the entropy function. Fix γ sufficiently small with $0 < \gamma < 1$ such that

$$133\gamma + 66h(6\gamma) < \frac{\delta}{2} \quad \text{and} \quad 44\gamma + 22h(6\gamma) < \varepsilon'(\delta/2), \tag{3.1}$$

where $\varepsilon'(\delta/2)$ is given by Theorem 2.1. Note that such a γ exists, since $h(6\gamma) \to 0$ as $\gamma \to 0$. Let $\varepsilon = \varepsilon(\gamma) > 0$ with $\varepsilon < \gamma/2$ be such that Lemma 2.3 is satisfied. Moreover, let $t_0 = \max\{1/\varepsilon, t'\}$, where t' is sufficiently large, so that (1.2) holds, *i.e.*, $\mathrm{ex}(t, F) = e(B_t)$ for every $t \geqslant t'$, and so that Theorem 2.1 holds for $\delta/2$ for all hypergraphs on at least t' vertices.

Let $T_0 = T_0(3, t_0, \varepsilon)$ and $N_0 = N_0(3, t_0, \varepsilon)$ be according to Theorem 2.2 and let $m_0 = m_0(\gamma)$ be according to Lemma 2.3. Finally, set $n_0 := \max\{N_0, T_0 \cdot m_0\}$.

Let $H = (V, E)$ be a hypergraph on $n \geqslant n_0$ vertices, which admits at least $3^{e(B_n)}$ Fano plane-free 3-colourings of the set of hyperedges. Let us denote the colours by red, blue and green.

Consider any fixed Fano plane-free 3-colouring of the set of hyperedges of H. By Theorem 2.2, for $r = 3$ there exists a positive integer $T_0 = T_0(3, t_0, \varepsilon)$ and there exists a partition $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$ of the vertex set $V(H)$, $t_0 \leqslant t \leqslant T_0$, which is ε-regular with respect to each colour class, where $|V_i| \leqslant \lceil n/t \rceil$, $1 \leqslant i \leqslant t$. To simplify the calculations, we assume in the following that $|V_i| = n/t \in \mathbb{N}$, $1 \leqslant i \leqslant t$. This does not change our asymptotic analysis.

Let $H_{\mathrm{red}}(\gamma), H_{\mathrm{blue}}(\gamma)$ and $H_{\mathrm{green}}(\gamma)$ be the corresponding cluster-hypergraphs on the vertex set $[t] = \{1, \ldots, t\}$, *i.e.*, $H_{\mathrm{col}}(\gamma)$ corresponds to all those hyperedges with colour $\mathrm{col} \in \{\mathrm{red}, \mathrm{blue}, \mathrm{green}\}$ which are contained in ε-regular triples of density at least γ. By our assumption and by Lemma 2.3, each hypergraph $H_{\mathrm{col}}(\gamma)$ is Fano plane-free, and hence each contains at most $\mathrm{ex}(t, F) = e(B_t)$ hyperedges.

We count the number of 3-colourings of the set of hyperedges which yield the partition $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$ of the vertex set and the cluster-hypergraphs $H_{\mathrm{red}}(\gamma)$, $H_{\mathrm{blue}}(\gamma)$, and

$H_{\mathrm{green}}(\gamma)$. To do so, we first bound from above the number of hyperedges $e \in E(H)$ which intersect some set V_i, $1 \leqslant i \leqslant t$, in at least two vertices, or are contained in a triple (V_i, V_j, V_k) which is not ε-regular, or for one colour class are contained in a triple (V_i, V_j, V_k) of edge-density less than γ, $1 \leqslant i < j < k \leqslant t$.

The number of hyperedges $e \in E(H)$ that intersect one of the sets $V_1, \ldots, V_t$ in at least two vertices is at most

$$t\binom{n/t}{2}n < \frac{1}{2t}n^3. \tag{3.2}$$

The number of hyperedges $e \in E(H)$ that are contained in one of the at most $3\varepsilon\binom{t}{3}$ ε-irregular triples (V_i, V_j, V_k), $1 \leqslant i < j < k \leqslant t$, is at most

$$3\varepsilon\binom{t}{3}\left(\frac{n}{t}\right)^3 < \frac{\varepsilon}{2}n^3. \tag{3.3}$$

The number of hyperedges $e \in E(H)$ which, for one of the three colour classes, are contained in triples (V_i, V_j, V_k) of density less than γ, $1 \leqslant i < j < k$, is at most

$$3\binom{t}{3}\gamma\left(\frac{n}{t}\right)^3 < \frac{\gamma}{2}n^3. \tag{3.4}$$

With $t \geqslant t_0 \geqslant 1/\varepsilon$ and $\varepsilon < \gamma/2$, the total number of all these hyperedges is by (3.2)–(3.4) less than

$$\gamma n^3. \tag{3.5}$$

These hyperedges can be chosen in at most

$$\binom{\binom{n}{3}}{\gamma n^3} < \binom{n^3/6}{\gamma n^3} \leqslant 2^{h(6\gamma)n^3/6} \tag{3.6}$$

ways – here we used $\binom{n}{\alpha n} \leqslant 2^{h(\alpha)n}$ for $0 < \alpha < 1$ – and can be coloured red, blue or green in at most

$$3^{\gamma n^3} \tag{3.7}$$

ways.

Next we consider the set of all remaining hyperedges in H, *i.e.*, those contained in ε-regular triples (V_i, V_j, V_k) of density at least γ for every colour class, $1 \leqslant i < j < k$. If $\{i, j, k\}$ is a hyperedge in exactly s, $1 \leqslant s \leqslant 3$, of the cluster-hypergraphs $H_{\mathrm{red}}(\gamma), H_{\mathrm{blue}}(\gamma), H_{\mathrm{green}}(\gamma)$, then in the hypergraph H every remaining hyperedge in the ε-regular triple (V_i, V_j, V_k) is coloured by one of s possible colours. As $e(V_i, V_j, V_k) \leqslant (n/t)^3$, we can colour these hyperedges in at most

$$s^{(n/t)^3} \tag{3.8}$$

ways. Let e_s be the number of triples $\{i, j, k\}$, $1 \leqslant i < j < k \leqslant t$, which are hyperedges in exactly s cluster-hypergraphs. Hence, the number of 3-colourings that yield the partition $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$ of the vertex set $V(H)$ and the cluster-hypergraphs $H_{\mathrm{red}}(\gamma), H_{\mathrm{blue}}(\gamma)$,

$H_{\mathrm{green}}(\gamma)$ is, by (3.6)–(3.8), with $e(V_i, V_j, V_k) \leqslant (n/t)^3$, $1 \leqslant i < j < k \leqslant t$, at most

$$2^{h(6\gamma)n^3/6} \cdot 3^{\gamma n^3} \cdot \left(1^{e_1} 2^{e_2} 3^{e_3}\right)^{(n/t)^3} = 2^{h(6\gamma)n^3/6} \cdot 3^{\gamma n^3} \cdot \left(2^{e_2} 3^{e_3}\right)^{(n/t)^3}. \tag{3.9}$$

None of the cluster-hypergraphs contain a Fano plane, and hence they have at most $e(B_t)$ hyperedges, *i.e.*, $e(H_{\mathrm{col}}(\gamma)) \leqslant e(B_t) \leqslant t^3/8$ for col $\in \{\mathrm{red}, \mathrm{blue}, \mathrm{green}\}$. Observe that

$$\begin{aligned} 2e_2 + 3e_3 \leqslant e_1 + 2e_2 + 3e_3 &= e(H_{\mathrm{red}}(\gamma)) + e(H_{\mathrm{blue}}(\gamma)) + e(H_{\mathrm{green}}(\gamma)) \\ &\leqslant 3e(B_t) \leqslant \frac{3t^3}{8}, \end{aligned} \tag{3.10}$$

thus

$$e_2 \leqslant \frac{3t^3}{16} - \frac{3e_3}{2}, \tag{3.11}$$

and we infer by using $2 < 3^{7/11}$ that

$$2^{e_2} \cdot 3^{e_3} \overset{(3.11)}{\leqslant} 2^{3t^3/16 - 3e_3/2} \cdot 3^{e_3} < 3^{(7/11)(3t^3/16 - 3e_3/2)} \cdot 3^{e_3} \leqslant 3^{21t^3/176 + e_3/22}. \tag{3.12}$$

Assume that for every choice of a Fano plane-free colouring of the set of hyperedges of H we obtain

$$e_3 < \frac{t^3}{8} - 44\gamma t^3 - 22h(6\gamma)t^3.$$

Then, we have

$$2^{e_2} \cdot 3^{e_3} \overset{(3.12)}{<} 3^{t^3/8 - 2\gamma t^3 - h(6\gamma)t^3}. \tag{3.13}$$

Recalling that there are at most n^{T_0} partitions of the vertex set V into at most T_0 classes, and at most $2^{3\binom{T_0}{3}} < 2^{T_0^3}$ choices for the cluster-hypergraphs $H_{\mathrm{red}}(\gamma), H_{\mathrm{blue}}(\gamma), H_{\mathrm{green}}(\gamma)$, we infer from (3.9) and (3.13) that the total number of such 3-colourings of H is at most

$$\begin{aligned} n^{T_0} \cdot 2^{T_0^3} &\cdot 2^{h(6\gamma)n^3/6} \cdot 3^{\gamma n^3} \cdot \left(3^{t^3/8 - 2\gamma t^3 - h(6\gamma)t^3}\right)^{(n/t)^3} \\ &= n^{T_0} \cdot 2^{T_0^3} \cdot 2^{h(6\gamma)n^3/6} \cdot 3^{\gamma n^3} \cdot 3^{n^3/8 - 2\gamma n^3 - h(6\gamma)n^3} \\ &< n^{T_0} \cdot 2^{T_0^3} \cdot 3^{n^3/8 - \gamma n^3 - 5h(6\gamma)n^3/6} < 3^{e(B_n)} \end{aligned}$$

for sufficiently large n, which contradicts our assumption.

Hence, there exists a Fano plane-free 3-colouring of H which yields a partition $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$, $t \leqslant T_0$, and cluster-hypergraphs $H_{\mathrm{red}}(\gamma)$, $H_{\mathrm{blue}}(\gamma)$, $H_{\mathrm{green}}(\gamma)$ such that

$$e_3 \geqslant \frac{t^3}{8} - 44\gamma t^3 - 22h(6\gamma)t^3. \tag{3.14}$$

We infer

$$e_1 + e_2 \leqslant e_1 + 2e_2 \overset{(3.10),(3.14)}{\leqslant} 132\gamma t^3 + 66h(6\gamma)t^3. \tag{3.15}$$

Let H_3 be that hypergraph on the vertex set $[t]$ which consists of all hyperedges that are contained in all three cluster-hypergraphs. Let H' be the sub-hypergraph of H which contains all those hyperedges from H corresponding to the hyperedges in H_3, *i.e.*, $\{i, j, k\} \in E(H_3)$ if and only if $E(V_i, V_j, V_k) \subseteq E(H')$.

Due to (3.14) and (3.1), by Theorem 2.1 there exists a partition $[t] = A\dot{\cup}B$ such that

$$e_{H_3}(A) + e_{H_3}(B) < \frac{\delta}{2}t^3. \tag{3.16}$$

Set $X = \bigcup_{j\in A} V_j$ and $Y = \bigcup_{j\in B} V_j$. Then, we have

$$\begin{aligned} e_H(X) + e_H(Y) &\overset{(3.5)}{\leqslant} \gamma n^3 + (n/t)^3(e_{H_3}(A) + e_{H_3}(B) + e_1 + e_2) \\ &\overset{(3.15),(3.16)}{\leqslant} \gamma n^3 + (n/t)^3(\delta t^3/2 + 132\gamma t^3 + 66h(6\gamma)t^3) \\ &\leqslant \gamma n^3 + \delta n^3/2 + 132\gamma n^3 + 66h(6\gamma)n^3 \\ &\overset{(3.1)}{<} \delta n^3, \end{aligned}$$

which yields the desired partition $V(H) = X\dot{\cup}Y$. □

4. Proof of main result

Proof of Theorem 1.1. We prove only the case $r = 3$, as the proof for two colours is similar. We first fix all constants needed for the proof. Let ξ, ϱ, and ζ be defined by the following equations:

$$3^6 - 1 = 3^{6-\xi}, \quad 3^4 - 1 = 3^{4-\varrho} \quad \text{and} \quad (3h(2\zeta) + 1)(1 + 8\zeta)\log_3(2) = 1 - \zeta, \tag{4.1}$$

where $h(x) := -x\log x - (1-x)\log(1-x)$ is the entropy function. Recall that $h(x) \to 0$ as $x \to 0$ and, since $\log_3(2) < 1$, there exists such a $\zeta > 0$ satisfying the above such that $(3h(2\gamma) + 1)(1 + 8\gamma)\log_3(2) < 1 - \gamma$ for all $0 < \gamma < \zeta$. We set

$$\gamma := \min\left\{\frac{\xi}{2000}, \frac{\zeta}{2}\right\} \leqslant \frac{1}{25} \quad \text{and} \quad \delta := \frac{\varrho\gamma^3}{1000}. \tag{4.2}$$

For the main steps of the proof it is sufficient to keep in mind that

$$0 < \delta \ll \gamma \ll \varrho, \xi, \zeta.$$

Let $n_0 = n_0(3, \delta)$ be given by Lemma 3.1 and set $n_r = n_3 \geqslant n_0 + 3\binom{n_0}{3}$ sufficiently large.

The proof is similar to that in [1] and proceeds by contradiction. Assume that we are given a hypergraph H on $n > n_3$ vertices with $c_{3,F}(H) \geqslant 3^{e(B_n)+m}$ for some $m \geqslant 0$. We show the following claim.

Claim 4.1. *If $c_{3,F}(H) \geqslant 3^{e(B_n)+m}$ for some $m \geqslant 0$, and H is not the balanced, complete, bipartite hypergraph B_n, then there exists an induced sub-hypergraph H' on n' vertices with $n' \geqslant n - 3$ and $c_{3,F}(H') \geqslant 3^{e(B_{n'})+m+1}$.*

Inductively, we arrive at some sub-hypergraph H_0 with at least n_0 vertices that admits at least $3^{e(B_{n_0})+\binom{n_0}{3}+1}$ Fano plane-free 3-colourings of the set of hyperedges, which is impossible and yields the desired contradiction, and it is left to verify Claim 4.1. □

Proof of Claim 4.1. Let H be a hypergraph on n vertices, $H \neq B_n$ and $c_{3,F}(H) \geqslant 3^{e(B_n)+m}$ with $m \geqslant 0$. Clearly, this implies $e(H) \geqslant e(B_n)$. Without loss of generality we may assume

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that $\delta_1(H) \geqslant \delta_1(B_n)$. Otherwise let v be a vertex of minimum degree in H and consider $H' := H - v$. Since $e(B_{n-1}) = e(B_n) - \delta_1(B_n) \leqslant e(B_n) - \delta_1(H) - 1$, we have

$$c_{3,F}(H') \geqslant \frac{c_{3,F}(H)}{3^{\delta_1(H)}} = 3^{e(B_n)-\delta_1(H)+m} \geqslant 3^{e(B_{n-1})+m+1}.$$

In view of (1.3), from now on we may assume $\delta_1(H) \geqslant \delta_1(B_n) \geqslant 3n^2/8 - n$. Consider a partition of $V(H) = X \dot{\cup} Y$ which minimizes $e(X) + e(Y)$. Because of Lemma 3.1 we know that $e(X) + e(Y) < \delta n^3$, and hence

$$e(H) < e(B_n) + \delta n^3,$$

and it follows from $e(H) \geqslant e(B_n)$ that

$$e(X, Y) \geqslant e(B_n) - \delta n^3 \geqslant n^3/8 - n^2/4 - \delta n^3,$$

which in turn implies

$$n/2 - 2\sqrt{\delta}n \leqslant |X|, |Y| \leqslant n/2 + 2\sqrt{\delta}n. \tag{4.3}$$

Our argument splits into two cases depending on the link (graph). For a vertex v of H, define its link $L(v) := \{\{u, w\} : \{v, u, w\} \in E(H)\}$, which is a graph on $V(H)$. First (in Case 1) we will assume that there exists a vertex v with at least γn^2 link edges in its 'own' partition class.

Case 1. *H has the property that* $\exists Z \in \{X, Y\}$ *and* $\exists v \in Z : |L(v) \cap \binom{Z}{2}| \geqslant \gamma n^2$.

Without loss of generality we may assume $v \in Y$ with $|L(v) \cap \binom{Y}{2}| \geqslant \gamma n^2$. The minimality of $e(X) + e(Y)$ implies that $|L(v) \cap \binom{X}{2}| \geqslant \gamma n^2$, as otherwise we could move v to X decreasing $e(X) + e(Y)$.

We split the Fano plane-free colourings of H into two classes, $\mathcal{C}_1$ and $\mathcal{C}_2 = \overline{\mathcal{C}_1}$. Let $\mathcal{C}_1$ be the set of those colourings for which there exist $L'_Y \subset L(v) \cap \binom{Y}{2}$ and $L'_X \subset L(v) \cap \binom{X}{2}$, of size at least $\gamma n^2/4$ each, and all hyperedges of the form $\{v\} \cup f$ with $f \in L'_X \cup L'_Y$ have the same colour.

For a fixed colouring from $\mathcal{C}_1$ there exist matchings $M_X \subset L'_X$ and $M_Y \subset L'_Y$, such that $\min\{|M_X|, |M_Y|\} \geqslant \gamma n/5$. For three link edges f_1, f_2, f_3 with $f_1 \in M_Y$ and $f_2, f_3 \in M_X$ let $t_1, t_2, t_3, t_4 \in \binom{V}{3}$ be four triples (not necessarily hyperedges of H) such that $\{\{v\} \cup f_i : i = 1, 2, 3\} \cup \{t_1, \ldots, t_4\}$ forms a Fano plane. Note that each of the triples t_1, t_2, t_3, t_4 contains precisely one vertex from $f_1 \subset Y$ and precisely one vertex from each of f_2 and $f_3 \subset X$. (In fact, there are two different sets of four triples $t_1, \ldots, t_4$ for any given f_1, f_2, f_3 and we just fix one of those two sets.) Since $\{v\} \cup f_i$ are of the same colour, either one of the triples t_j must be missing in H or there are only $3^4 - 1$ ways to colour t_1, t_2, t_3, t_4. Since $|M_X|, |M_Y| \geqslant \gamma n/5$ there are at least $\frac{\gamma n}{5}\binom{\gamma n/5}{2}$ possible choices for f_1, f_2, f_3 and since there are at most $\delta n^3 \leqslant \gamma^3 n^3/1000$ hyperedges absent between X and Y, there are at least $\gamma^3 n^3/500$ such Fano planes present in H for a fixed colouring in $\mathcal{C}_1$. Furthermore, note that for two different choices of f_1, f_2, f_3 and f'_1, f'_2, f'_3 the corresponding sets $\{t_1, \ldots, t_4\}$

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and $\{t'_1, \ldots, t'_4\}$ are disjoint. Hence we obtain the following estimate on $|\mathcal{C}_1|$:

$$|\mathcal{C}_1| \leqslant 3\binom{\binom{|X|}{2}}{\gamma n^2/4}\binom{\binom{|Y|}{2}}{\gamma n^2/4}\frac{3^{e(H)}}{3^{4\gamma^3 n^3/500}}(3^4-1)^{\gamma^3 n^3/500}$$
$$\overset{(4.1)}{\leqslant} 3 \cdot 2^{n^2} \cdot 3^{e(B_n)+\delta n^3 - 4\gamma^3 n^3/500 + (4-\varrho)\gamma^3 n^3/500}$$
$$\overset{(4.2)}{=} 3 \cdot 2^{n^2} \cdot 3^{e(B_n)-\delta n^3}.$$

Consequently, for large enough n we have

$$|\mathcal{C}_1| \leqslant 3^{e(B_n)-1}.$$

Let $\mathcal{C}_2$ be the Fano plane-free edge colourings of H which do not belong to $\mathcal{C}_1$, *i.e.*, the family of those colourings for which there does not exist $L'_Y \subset L(v) \cap \binom{Y}{2}$ and $L'_X \subset L(v) \cap \binom{X}{2}$, of size at least $\gamma n^2/4$ each, and such that all hyperedges of the form $\{v\} \cup f$ with $f \in L'_X \cup L'_Y$ have the same colour. We have just shown that

$$\mathcal{C}_2 \geqslant 3^{e(B_n)+m-1}.$$

Next we estimate the number of colourings of the set of hyperedges incident to v that can be extended to a colouring in $\mathcal{C}_2$. For a set $W \subseteq V(H)$ we say $e \in E(H)$ *is a hyperedge from* v *to* W if $v \in e$ and $(e \setminus \{v\}) \subset W$.

For any colouring from $\mathcal{C}_2$, by definition, for every col $\in \{\text{red}, \text{blue}, \text{green}\}$ there is a vertex class $V_{\text{col}} \in \{X, Y\}$ such that there are at most $\gamma n^2/4$ hyperedges from v to V_{col}, since otherwise the colouring would belong to $\mathcal{C}_1$. Note that because of (4.2) and (4.3) the size of $\binom{V_{\text{col}}}{2}$ is at most $n^2/8 + \gamma n^2$, and consequently there are at most

$$\binom{n^2/8+\gamma n^2}{\gamma n^2/4} \leqslant 2^{h(\frac{2\gamma}{1+8\gamma})(1+8\gamma)n^2/8} \overset{(4.2)}{\leqslant} 2^{h(2\gamma)(1+8\gamma)n^2/8}$$

ways to choose the hyperedges of colour col between v and V_{col}.

Since $|L(v) \cap \binom{X}{2}|, |L(v) \cap \binom{Y}{2}| \geqslant \gamma n^2$ it is impossible that $V_{\text{red}} = V_{\text{blue}} = V_{\text{green}}$. Hence, for two colours, say red and blue, there will be at most $\gamma n^2/4$ hyperedges from v to, say, $X = V_{\text{red}} = V_{\text{blue}}$ (the case $Y = V_{\text{red}} = V_{\text{blue}}$ is symmetric here and the analysis is independent from the earlier assumption $v \in Y$). Then, for the remaining third colour there will be at most $\gamma n^2/4$ hyperedges of colour green from v to $Y = V_{\text{green}}$. Now we can colour the remaining hyperedges from v to X only green, and we can colour the remaining hyperedges (there are at most $n^2/8 + \gamma n^2$) from v to Y with two colours, red and blue. We also had only 6 different possibilities for choosing $V_{\text{red}}, V_{\text{blue}}, V_{\text{green}} \in \{X, Y\}$ in such a way.

Finally, there are at most $n^2/4$ hyperedges that contain v and intersect both X and Y, and they can be coloured arbitrarily, so in total in at most $3^{n^2/4}$ ways. Summarizing the above, we can estimate the number of possible colourings of the hyperedges incident with v (which extend to a colouring in $\mathcal{C}_2$) from above by

$$6 \cdot 2^{3h(2\gamma)(1+8\gamma)n^2/8} \cdot 2^{(1+8\gamma)n^2/8} \cdot 3^{n^2/4} = 6 \cdot 3^{(3h(2\gamma)+1)(1+8\gamma)\log_3(2)n^2/8+n^2/4}$$
$$\overset{(4.1)}{\leqslant} 3^{2+(1-\gamma)n^2/8+n^2/4} = 3^{3n^2/8-\gamma n^2/8+2} \overset{(1.3)}{\leqslant} 3^{\delta_1(B_n)-2}.$$

Setting $H' := H - v$, we obtain

$$c_{3,F}(H') \geqslant \frac{|\mathcal{C}_2|}{3^{\delta_1(B_n)-2}} \geqslant \frac{3^{e(B_n)+m-1}}{3^{\delta_1(B_n)-2}} = 3^{e(B_{n-1})+m+1},$$

which proves Claim 4.1 for hypergraphs H satisfying the assumptions of Case 1.

Next we consider the case that every vertex v has at most γn^2 link edges in its own partition class.

Case 2. H has the property that $\forall Z \in \{X, Y\}$ and $\forall v \in Z : |L(v) \cap \binom{Z}{2}| \leqslant \gamma n^2$.

Since $H \neq B_n$, there exists (without loss of generality) a hyperedge $e = \{v_1, v_2, v_3\} \subset Y$. Let $L := \bigcap_{i=1}^{3} L(v_i) \cap \binom{X}{2}$. From $\delta_1(H) \geqslant \delta_1(B_n) \geqslant 3n^2/8 - n$ it follows that

$$|L| \geqslant (1 - 4\gamma)\binom{|X|}{2} > (2/3 + 1/6)\binom{|X|}{2}$$

(see (4.2)). By Turán's theorem and (4.3) we find at least $\frac{1}{36}\binom{|X|}{2} \geqslant \frac{1}{360}n^2$ edge-disjoint K_4s in L. Denote them by $K^1, \ldots, K^q$, where

$$q \geqslant \frac{1}{360}n^2. \tag{4.4}$$

Since $K^j \subset L$ for every $j = 1, \ldots, q$, every such K^j forms together with the hyperedge e a Fano plane. Fixing a colour for e, we can colour the 6 hyperedges that correspond to the edges of every K^j in only $3^6 - 1$ instead of 3^6 different ways.

Set $H' := H - \{v_1, v_2, v_3\}$. Let E_e denote the set of hyperedges of H which contain at least one vertex from $e = \{v_1, v_2, v_3\}$. Obviously, $|E_e| \leqslant 3\gamma n^2 + 3\binom{|X|}{2} + 3|X||Y|$. It follows from the choice of $\delta \ll \gamma$ (see (4.2)), $e(X) + e(Y) < \delta n^3$, and $e(H) \geqslant e(B_n)$, that

$$\begin{aligned}|E_e| \overset{(4.3)}{\leqslant} \frac{9}{8}n^2 + 4\gamma n^2 &\overset{(1.3)}{\leqslant} \delta_1(B_n) + \delta_1(B_{n-1}) + \delta_1(B_{n-2}) + 5\gamma n^2\\ &= e(B_n) - e(B_{n-3}) + 5\gamma n^2.\end{aligned}$$

We can colour the set of hyperedges of E_e in at most

$$\frac{3^{|E_e|}}{3^{6q}}(3^6 - 1)^q \overset{(4.1)}{=} 3^{|E_e| - \xi q}$$

ways. Consequently,

$$c_{3,F}(H') \geqslant 3^{e(B_n)+m-|E_e|+\xi q} \geqslant 3^{e(B_{n-3})+m-5\gamma n^2+\xi q} \overset{(4.2),(4.4)}{\geqslant} 3^{e(B_{n-3})+m+1},$$

which concludes Case 2 and finishes the proof of Claim 4.1. □

5. Fano plane-free r-colourings ($r \geqslant 4$)

Proof of Theorem 1.2. Let $H = (V, E)$ be the complete 4-partite hypergraph with the vertex partition $V = V_1 \dot\cup V_2 \dot\cup V_3 \dot\cup V_4$ of almost equal size: $||V_i| - |V_j|| \leqslant 1$ for $1 \leqslant i < j \leqslant 4$. We colour its hyperedges with colours from $[r]$ as follows. The hyperedges from $E(V_1 \cup V_3, V_2 \cup V_4)$ can be coloured with colours from $\{1, \ldots, r-2\}$, from $E(V_1 \cup V_2, V_3 \cup V_4)$

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with colour $r-1$ and from $E(V_1 \cup V_4, V_2 \cup V_3)$ with colour r. Obviously, there are no monochromatic Fano planes, as all monochromatic induced sub-hypergraphs are bipartite. It remains to verify a lower bound on the number of possible colourings (we now assume for simplicity that 4 divides n), as follows.

- The hyperedges that intersect 3 of the possible 4 partition classes can be coloured arbitrarily (*i.e.*, by r colours), which gives
$$r^{4(\frac{n}{4})^3}$$
colourings for those hyperedges.
- The hyperedges from $E(V_1, V_2)$, $E(V_1, V_4)$, $E(V_2, V_3)$ or $E(V_3, V_4)$ can be coloured with $r-1$ colours, and since $e(V_i, V_j) = 2\binom{n/4}{2}\frac{n}{4}$ we obtain
$$(r-1)^{4\cdot 2\binom{n/4}{2}\frac{n}{4}}$$
colourings for these hyperedges.
- The hyperedges from $E(V_1, V_3)$ or $E(V_2, V_4)$ can be coloured with 2 colours in
$$2^{2\cdot 2\binom{n/4}{2}\frac{n}{4}}$$
many ways.

Consequently,
$$c_{r,F}(n) \geqslant r^{4(\frac{n}{4})^3}(r-1)^{4\cdot 2\binom{n/4}{2}\frac{n}{4}}2^{2\cdot 2\binom{n/4}{2}\frac{n}{4}} \geqslant \left(\sqrt{\sqrt{2}r(r-1)}\right)^{n^3/8-O(n^2)} \geqslant (r+\varepsilon)^{e(B_n)}$$
for any $r \geqslant 4$ and for some $\varepsilon > 0$ and sufficiently large n. □

We note that this lower bound on the number of Fano plane-free r-colourings can be easily improved. For example, if one distributes the available colours for the three bipartitions as evenly as possible, then one obtains the following for $r \geqslant 4$:
$$c_{r,F}(n) \geqslant f_r^{n^3/8-O(n^2)}, \quad \text{with} \quad f_r = \begin{cases} \left(\frac{2}{3}\right)^{3/4} r^{5/4} & \text{if } r = 0 \mod 3, \\ r^{1/2}\left\lceil\frac{2}{3}r\right\rceil^{1/2}\left\lfloor\frac{2}{3}r\right\rfloor^{1/4} & \text{if } r = 1 \mod 3, \\ r^{1/2}\left\lceil\frac{2}{3}r\right\rceil^{1/4}\left\lfloor\frac{2}{3}r\right\rfloor^{1/2} & \text{if } r = 2 \mod 3. \end{cases} \tag{5.1}$$

The next result gives an upper bound on $c_{r,F}(n)$ for any fixed integer $r \geqslant 4$.

Theorem 5.1. *For the Fano plane F and integers $r \geqslant 4$ we have*
$$c_{r,F}(n) \leqslant \left((3r/4)^{4/3}\right)^{n^3/8+o(n^3)}.$$

Proof. The arguments are similar to those used in the proof of Lemma 3.1. Let $\gamma > 0$ be arbitrary and set $\varepsilon = \varepsilon(\gamma) > 0$ with $\varepsilon < \gamma/2$ such that Lemma 2.3 is satisfied. Moreover, let $t_0 = \max\{1/\varepsilon, t'\}$, where t' is sufficiently large, so that (1.2) holds, *i.e.*, so that $\text{ex}(t, F) = e(B_t)$ for every $t \geqslant t'$. Let $T_0 = T_0(r, t_0, \varepsilon)$ and $N_0 = N_0(r, t_0, \varepsilon)$ be given by Theorem 2.2 and let $m_0 = m_0(\gamma)$ be given by Lemma 2.3. Set $n_0 := \max\{N_0, T_0 \cdot m_0\}$ and let $H = (V, E)$ be a hypergraph on $n \geqslant n_0$ vertices.

Consider any fixed r-colouring of the set of hyperedges of H without a monochromatic Fano plane F. By Theorem 2.2 there exists a partition $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$ of the vertex set $V(H)$, $t_0 \leqslant t \leqslant T_0$, which is ε-regular with respect to each colour class, where w.l.o.g. $|V_i| = n/t$, $1 \leqslant i \leqslant t$.

For $\gamma > 0$ and $\mathrm{col} \in [r]$ let $H_{\mathrm{col}}(\gamma)$ be the corresponding cluster-hypergraphs on the vertex set $[t] = \{1, \ldots, t\}$, *i.e.*, $H_{\mathrm{col}}(\gamma)$ corresponds to all hyperedges of colour $\mathrm{col} \in \{1, \ldots, r\}$ contained in ε-regular triples of density at least γ. Furthermore, for $s \in [r]$ let e_s be the number of triples $\{i, j, k\}$, $1 \leqslant i < j < k \leqslant t$, which are hyperedges in exactly s of the cluster-hypergraphs $H_{\mathrm{col}}(\gamma)$ with $\mathrm{col} \in [r]$. By our assumption and by Lemma 2.3 each hypergraph $H_{\mathrm{col}}(\gamma)$ is Fano plane-free, and hence contains at most $e(B_t)$ hyperedges:

$$\sum_{s=1}^{r} s e_s \leqslant r \cdot \mathrm{ex}(t, F) \leqslant r \cdot \frac{t^3}{8}. \tag{5.2}$$

As in (3.2)–(3.9), the number of r-colourings of the set of hyperedges of H which yield the vertex partition $V = V_1 \dot\cup \cdots \dot\cup V_t$ and the cluster-hypergraphs $H_1(\gamma), \ldots, H_r(\gamma)$ can be bounded from above by

$$\binom{\binom{n}{3}}{r\gamma n^3} \cdot r^{r\gamma n^3} \cdot \left(\prod_{s=1}^{r} s^{e_s}\right)^{(\frac{n}{t})^3} \leqslant 2^{h(6r\gamma)n^3/6} \cdot r^{r\gamma n^3} \cdot \left(\prod_{s=1}^{r} s^{e_s}\right)^{(\frac{n}{t})^3}. \tag{5.3}$$

Since

$$\sum_{s=1}^{r} e_s \leqslant \binom{t}{3} \leqslant \frac{t^3}{6},$$

we may view $\prod_{s=1}^{r} s^{e_s}$ as a product of at most $t^3/6$ factors. The sum of those factors equals $\sum_{s=1}^{r} s e_s$, which is due to (5.2) bounded from above by $rt^3/8$. Since a product of positive reals with bounded sum of the factors is maximized when all factors are equal, one can show that

$$\prod_{s=1}^{r} s^{e_s} \leqslant \left(\frac{rt^3/8}{t^3/6}\right)^{t^3/6} = \left(\frac{3r}{4}\right)^{t^3/6} \tag{5.4}$$

(see, *e.g.*, [1, Lemma 4.3]).

The number t of partition classes is at most T_0, hence there are at most n^{T_0} partitions of the vertex set V into at most T_0 classes. Given such a partition, we have at most $2^{r\binom{T_0}{3}} < 2^{rT_0^3}$ choices for the cluster-hypergraphs $H_1(\gamma), \ldots, H_r(\gamma)$. With (5.3) and (5.4) we obtain

$$\begin{aligned} c_{r,F}(n) &\leqslant n^{T_0} \cdot 2^{rT_0^3} \cdot 2^{h(6r\gamma)n^3/6} \cdot r^{r\gamma n^3} \cdot \left((3r/4)^{t^3/6}\right)^{(n/t)^3} \\ &\leqslant n^{T_0} \cdot 2^{rT_0^3} \cdot 2^{h(6r\gamma)n^3/6} \cdot r^{r\gamma n^3} \cdot \left((3r/4)^{4/3}\right)^{n^3/8} \\ &\leqslant \left((3r/4)^{4/3}\right)^{n^3/8 + o(n^3)}, \end{aligned} \tag{5.5}$$

as $\gamma > 0$ can be chosen to be arbitrarily small and the entropy $h(\gamma) \to 0$ as $\gamma \to 0$. □

Remark. The upper bound in Theorem 5.1 can be slightly improved. A more careful analysis of (5.4), which uses the fact that every factor of $\prod_{s=1}^{r} s^{e_s}$ is an integer, yields

$\prod_{s=1}^{r} s^{e_s} \leqslant \lfloor 3r/4 \rfloor^a \lceil 3r/4 \rceil^b$, where $a+b=t^3/6$ and $a=(\lceil 3r/4 \rceil - 3r/4)t^3/6$. This gives

$$c_{r,F}(n) \leqslant \left(\lfloor 3r/4 \rfloor^{a/3} \lceil 3r/4 \rceil^{b/3}\right)^{n^3/8+o(n^3)},$$

where $a+b=4$ and $a=4\lceil 3r/4 \rceil - 3r$.

6. Concluding remarks

The following generalization of the function $c_{2,K_\ell}(n)$ for graphs was studied by Balogh [2]. For a fixed k-uniform hypergraph F, an integer r, and an r-colouring χ of the hyperedges of F, which uses all r colours, we denote for a k-uniform hypergraph H by $c_{r,\chi,F}(H)$ the number of colourings of the set of hyperedges H with r colours which do not contain a copy of F that is identical to χ up to permutation of the colour classes. We call such colourings of H (χ, F)-free. Similarly, as before we set $c_{r,\chi,F}(n) = \max c_{r,\chi,F}(H)$, where the maximum runs over all k-uniform hypergraphs on n vertices.

6.1. Forbidden 2-colourings of the Fano plane

In [2] Balogh studied $c_{2,\chi,K_\ell}(n)$ and showed that $c_{2,\chi,K_\ell}(n) = 2^{\mathrm{ex}(n,K_\ell)}$. On the other hand, for three colours ($r=3$), it is easy to see that $c_{3,\chi,K_3}(n) \geqslant 2^{\binom{n}{2}} \gg 3^{n^2/4}$, since trivially no 2-colouring of K_n admits a triangle with 3 colours. We can prove a similar result for 2-colourings in the special case when F is the Fano plane.

Theorem 6.1. *For every 2-colouring χ of the hyperedges of the Fano plane F which uses both colours, there exists an n_0 such that for all $n \geqslant n_0$ we have $c_{2,\chi,F}(n) = 2^{\mathrm{ex}(n,F)}$, and the only 3-uniform hypergraph H on n vertices with $c_{2,\chi,F}(H) = 2^{\mathrm{ex}(n,F)}$ is B_n.*

The proof of Theorem 6.1 follows the lines of the proof of Theorem 1.1, and we discuss the required adjustments below.

Proof of Theorem 6.1 (sketch). First an analogous extension of Lemma 3.1 is proved. Again we use the weak hypergraph regularity lemma and obtain the cluster-hypergraphs H_{red} and H_{blue}. Lemma 2.3 implies that for every 2-colouring which does not contain a χ-coloured copy of F, the number $e(H_2)$ of hyperedges that appear in both cluster-hypergraphs satisfies $e(H_2) = |E(H_{\mathrm{red}}) \cap E(H_{\mathrm{blue}})| \leqslant e(B_t)$, where t is the number of vertex classes of the regular partition. Now a simple calculation, similar to (3.9)–(3.13), shows that if $e(H_2) < (1-o(1))e(B_t)$ for every (χ, F)-free colouring of H, then this contradicts the assumption that $c_{2,\chi,F}(H) \geqslant 2^{e(B_n)}$. Thus there must be a (χ, F)-free colouring of H with $e(H_2) \geqslant (1-o(1))e(B_t)$. Now the stability theorem for Fano plane-free hypergraphs yields a partition $A \dot\cup B = [t]$ with $|E_{H_2}(A) \cup E_{H_2}(B)| = o(t^3)$. However, we still have to bound the number of hyperedges of $H_1 = ([t], E(H_{\mathrm{red}}) \triangle E(H_{\mathrm{blue}}))$ which are completely contained in A or B. For that we note that $E(H_1) \cup E(H_2)$ cannot contain a copy of F with precisely one hyperedge in $E(H_1)$. Since then Lemma 2.3 again yields a copy of F that has the same colouring as χ. (Here we use the assumption that χ is indeed not a monochromatic colouring of F.) But since $e_{H_2}(A,B) \geqslant (1-o(1))e(B_t)$, this implies $e_{H_1}(A) + e_{H_1}(B) \leqslant o(t^3)$ by a simple counting argument, which gives the appropriate extension of Lemma 3.1.

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In the second part, we follow the arguments from Section 4. Again the proof goes by induction, and we show that if $c_{2,\chi,F}(H) \geqslant 2^{e(B_n)+m}$ and $H \neq B_n$ then there exists a sub-hypergraph H' on $n' \geqslant n-3$ vertices such that $c_{2,\chi,F}(H') \geqslant 2^{e(B_{n'})+m+1}$. The proof follows the lines of Section 4 (adjusted for the case $r=2$). We only have to change the definition of the set $\mathcal{C}_1$ in Case 1. Here we let $\mathcal{C}_1$ be those (χ, F)-free colourings of H such that the link graph L'_Y of v contains many $(\gamma n^2/3)$ blue edges and L'_X contains many red edges, or *vice versa*. With this adjustment the proof is verbatim. □

6.2. Forbidden 3- and 4-colourings of the Fano plane

We close this note with the observation that Theorem 1.2 can also be extended to this setting. More precisely, $c_{r,\chi,F} \gg r^{e(B_n)}$ for $r=4$. In fact, as in the example of Balogh for K_3 above, we have $c_{r,\chi,F}(n) \geqslant (r-1)^{\binom{n}{3}} \gg r^{e(B_n)}$ for $r \geqslant 4$.

This leaves the case $r=3$ open. However, a similar question is also open for graphs F with more than 3 edges: *e.g.*, to our knowledge it is not known whether $c_{3,\chi,K_4}(n) \gg 3^{2n^3/3}$ or if equality holds.

References

[1] Alon, N., Balogh, J., Keevash, P. and Sudakov, B. (2004) The number of edge colorings with no monochromatic cliques. *J. London Math. Soc.* (2) **70** 273–288.

[2] Balogh, J. (2006) A remark on the number of edge colorings of graphs. *Europ. J. Combin.* **27** 565–573.

[3] Balogh, J., Bollobás, B. and Simonovits, M. (2004) The number of graphs without forbidden subgraphs. *J. Combin. Theory Ser. B* **91** 1–24.

[4] Balogh, J., Bollobás, B. and Simonovits, M. (2009) The typical structure of graphs without given excluded subgraphs. *Random Struct. Alg.* **34** 305–318.

[5] Chung, F. R. K. (1991) Regularity lemmas for hypergraphs and quasi-randomness. *Random Struct. Alg.* **2** 241–252.

[6] Erdős, P. (1974) Some new applications of probability methods to combinatorial analysis and graph theory. In *Proc. Fifth Southeastern Conference on Combinatorics, Graph Theory and Computing* (Florida Atlantic University, Boca Raton 1974), pp. 39–51.

[7] Erdős, P., Frankl, P. and Rödl, V. (1986) The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent. *Graphs Combin.* **2** 113–121.

[8] Erdős, P., Kleitman, D. J. and Rothschild, B. L. (1976) Asymptotic enumeration of K_n-free graphs. In *Colloquio Internazionale sulle Teorie Combinatorie* (Rome 1973), Vol. II, Accad. Naz. Lincei, Rome, pp. 19–27.

[9] Frankl, P. and Rödl, V. (1992) The uniformity lemma for hypergraphs. *Graphs Combin.* **8** 309–312.

[10] Füredi, Z. and Simonovits, M. (2005) Triple systems not containing a Fano configuration. *Combin. Probab. Comput.* **14** 467–484.

[11] Keevash, P. and Sudakov, B. (2005) The Turán number of the Fano plane. *Combinatorica* **25** 561–574.

[12] Kohayakawa, Y., Nagle, B., Rödl, V. and Schacht, M. Weak hypergraph regularity and linear hypergraphs. *J. Combin. Theory, Sec. B*, to appear.

[13] Kolaitis, P. G., Prömel, H. J. and Rothschild, B. L. (1985) Asymptotic enumeration and a 0–1 law for *m*-clique free graphs. *Bull. Amer. Math. Soc.* (N.S.) **13** 160–162.

[14] Kolaitis, P. G., Prömel, H. J. and Rothschild, B. L. (1987) K_{l+1}-free graphs: Asymptotic structure and a 0–1 law. *Trans. Amer. Math. Soc.* **303** 637–671.

[15] Nagle, B. and Rödl, V. (2001) The asymptotic number of triple systems not containing a fixed one. *Discrete Math.* **235** 271–290.

[16] Nagle, B., Rödl, V. and Schacht, M. (2006) Extremal hypergraph problems and the regularity method. In *Topics in Discrete Mathematics, Algorithms Combin.*, Vol. 26, Springer, Berlin, pp. 247–278.

[17] Person, Y. and Schacht, M. (2009) Almost all hypergraphs without Fano planes are bipartite. In *Proc. Twentieth Annual ACM–SIAM Symposium on Discrete Algorithms* (C. Mathieu, ed.), ACM, pp. 217–226.

[18] Steger, A. (1990) Die Kleitman–Rothschild Methode. PhD thesis, Forschungsinstitut für Diskrete Mathematik, Rheinische Friedrichs–Wilhelms–Universität Bonn.

[19] Szemerédi, E. (1978) Regular partitions of graphs. In *Problèmes Combinatoires et Théorie des Graphes* (Colloq. Internat. CNRS, Orsay 1976), CNRS, Paris, pp. 399–401.

[20] Yuster, R. (1996) The number of edge colorings with no monochromatic triangle. *J. Graph Theory* **21** 441–452.