Double Field Theory as the Double Copy of Yang-Mills Theory via Homotopy Algebras

Dissertation zur Erlangung des akademischen Grades

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Abstract

This thesis deals with a relation between Yang-Mills theory and gravity called the double copy, which was first formulated in the framework of scattering amplitudes. Yang-Mills scattering amplitudes carry kinematic information as well as information associated with a property called color. The double copy states that, provided that certain algebraic conditions are met, replacing the color information of Yang-Mills amplitudes with another copy of their kinematic information yields gravitational amplitudes. The algebraic conditions required by the double copy hint at the existence of an algebra underlying the kinematics of Yang-Mills called the kinematic algebra. In the last fifteen years, the double copy has revolutionized the way that scattering computations are performed in gravity, and, yet, a first principle understanding of the double copy and the kinematic algebra remains elusive. In this thesis we divert from scattering amplitudes and address this problem in the framework of homotopy algebras, which are the mathematical structures underlying perturbative field theories, including their off-shell structure. In particular, the algebra underlying Yang-Mills theory factorizes into color and kinematic algebraic structures. Using this factorization, we construct explicitly a kinematic algebra for Yang-Mills theory to quartic order in perturbation theory. Then, following the double copy, we replace the color algebra with a second copy of the kinematic algebra and we obtain gravity up to and including quartic interactions. Moreover, we explain how the kinematic algebra is responsible for the consistency of the resulting gravity theory. Our algebraic approach to the double copy is completely off-shell, gauge independent and local, and provides a novel perspective on the algebraic foundations and origins of the double copy.

Zusammenfassung

Diese Arbeit befasst sich mit der sogenannten Doppelkopie, welche erstmals im Rahmen von Streuamaplituden formuliert wurde und eine Beziehung zwischen Yang-Mills-Theorie und der Gravitation herstellt. Yang-Mills-Streuamplituden tragen sowohl kinematische Informationen als auch Informationen, die mit einer Eigenschaft namens Farbe verbunden sind. Die Doppelkopie besagt, dass die Ersetzung der Farbinformation in Yang-Mills-Amplituden durch eine andere Kopie ihrer kinematischen Information zu Gravitationsamplituden führt, sofern bestimmte algebraische Bedingungen erfüllt sind. Die algebraischen Bedingungen, die für die Doppelkopie erforderlich sind, deuten auf die Existenz einer Algebra hin, die der Kinematik der Yang-Mills-Theorie zugrunde liegt und die kinematische Algebra genannt wird. In den letzten fünfzehn Jahren hat die Doppelkopie die Art und Weise, wie Streuungsberechnungen in der Gravitation durchgeführt werden, revolutioniert, und dennoch bleibt ein grundsätzliches Verständnis der Doppelkopie und der kinematischen Algebra schwer zu fassen. In dieser Arbeit verlassen wir den Rahmen der Streuamplituden und behandeln dieses Problem mit Hilfe von Homotopie-Algebren, den mathematischen Strukturen, die perturbativen Feldtheorien, einschließlich ihrer Off-Shell Struktur, zugrunde liegen. Insbesondere die der Yang-Mills-Theorie zugrunde liegende Algebra lässt sich in algebraische Strukturen für Farbe und Kinematik faktorisieren. Unter Verwendung dieser Faktorisierung konstruieren wir explizit eine kinematische Algebra für die Yang-Mills Theorie bis zur quartischen Ordnung in der Störungstheorie. Dann ersetzen wir die Farbalgebra durch eine zweite Kopie der kinematischen Algebra und erhalten eine Gravitationstheorie bis hinzu und einschließlich der quartischen Wechselwirkungen. Außerdem erklären wir, inwiefern die kinematische Algebra die Struktur ist, die für die Konsistenz der resultierenden Gravitationstheorie verantwortlich ist. Unsere algebraische Herangehensweise an die Doppelkopie ist vollständig Off-Shell, eichunabhängig und lokal und bietet eine neue Perspektive auf die algebraischen Grundlagen und Ursprünge der Doppelkopie.

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List of publications

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Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß §7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 42/2018 am 11.07.2018 angegebenen Hilfsmittel angefertigt habe

Ort, Datum	Felipe Díaz-Jaramillo

I declare that I have completed the thesis independently using only the aids and tools specified. I have not applied for a doctor's degree in the doctoral subject elsewhere and do not hold a corresponding doctor's degree. I have taken due note of the Faculty of Mathematics and Natural Sciences PhD Regulations, published in the Official Gazette of Humboldt-Universität zu Berlin no. 42/2018 on 11/07/2018.

Place, date	Felipe Díaz-Jaramillo

Contents

Chapter 1

Introduction

One of the main challenges that theoretical physics faces today is to reconcile gravity, which governs the large scale structure of the universe, with the strong, weak and electromagnetic interactions, which constitute the standard model of particle physics and govern physics at microscopic scales. The three fundamental interactions of the standard model are mediated by particles called gauge bosons which are ripples of fundamental *gauge fields*. Mathematically, the dynamics of the gauge fields associated with each one of the three interactions is described by the same perturbative field theory called *Yang-Mills theory*. Gravity, on the other hand, is well and most commonly described by Einstein's General Theory of Relativity (GR) as the geometry of spacetime. However, in order to describe certain phenomena such as gravitational waves, it is convenient to think of gravity as being mediated by particles called gravitons which are ripples of a fundamental *graviton field*. In this sense, similarly to Yang-Mills, gravity is described as a perturbative field theory that governs the dynamics of the graviton field. In the following, we adopt the perturbative perspective on gravity.

As perturbative field theories, Yang-Mills and GR are vastly different. On the one hand, Yang-Mills theory has up to quartic self-interaction terms in the action. On the other hand, in sharp contrast, in the perturbative expansion of GR, the graviton field has an infinite number of self-interaction terms. Moreover, Yang-Mills theory is renormalizable in four dimensions, while gravity needs an infinite number of counterterms to deal with ultraviolet divergences rendering the theory non-renormalizable. Yet, despite these fundamental differences, scattering amplitudes unveil a remarkable relation between both theories known as *the double copy*. The double copy, roughly speaking, states that kinematic building blocks of Yang-Mills scattering amplitudes carry enough information to construct gravitational amplitudes. In this thesis, we extend this statement beyond scattering amplitudes. More precisely, we conjecture that the kinematic building blocks of the complete theory of Yang-Mills, including its off-shell structure and non-linear gauge redundancies, carry enough information to construct a complete theory of gravity. We prove this conjecture to quartic order in perturbation theory.

The purpose of this chapter is to give context to the rest of the thesis. To that end, in section 1.1 we introduce a variety of topics that will play an important role throughout the main text. In particular, we discuss the double copy and the frameworks and tools that we use in this thesis. The discussion on the different topics will be superficial and by no means rigorous. In later chapters, we will come back to many of these topics in order to introduce necessary technical details. After the survey of topics, in section 1.2 we state the main objective of this thesis and the most relevant results obtained with this work.

1.1 Survey of topics

In this section, we provide a brief survey of the different topics that will be relevant to the thesis. This survey intends to motivate the main objective and results of the thesis discussed in the following sections. Whenever necessary, the same topics will be explained in detail in the forthcoming chapters.

1.1.1 Yang-Mills theory

In the standard textbook approach to quantum field theory, Yang-Mills theory is presented as a perturbative field theory with an action principle with cubic and quartic self-interactions. The action, written in a Lorentz covariant form, describes the dynamics of a gauge field (or gluon field) A^a_μ, where μ is a spacetime index and a is an index associated to a property called *color*. Mathematically color is encoded in a Lie algebra $\mathfrak{g}$. In four-dimensional Minkowski spacetime, the gauge field A^a_μ has two physical propagating degrees of freedom, contrary to the four components that one would naively expect from a vector. Indeed, the theory is invariant under non-linear gauge transformations which eliminate two unphysical degrees of freedom when computing physical observables.

For instance, in the context of scattering amplitudes, the two physical degrees of freedom are called *gluon polarizations*, and they are usually encoded in constrained *polarization vectors* ϵ_μ. Scattering amplitudes in Yang-Mills theory split into *kinematic information*, in the form of polarization vectors and momenta of the external particles of the process, and *color information*, in the form of elements of the color Lie algebra $\mathfrak{g}$. The two sets of information are independent of each other.

1.1.2 Gravity

General relativity is most commonly described as a theory based on geometry where spacetime is a manifold with a metric $g_{\mu\nu}$. One then can write an action principle, the Einstein-Hilbert action, which determines the dynamics of the metric. Alternatively one can think of general relativity as a perturbative field theory on Minkowski spacetime. To that end, one writes the metric as the following sum: $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the background Minkowski metric, κ is the gravitational coupling constant, and $h_{\mu\nu}$ is the *graviton field*. This rewriting of the metric leads to a perturbative expansion in powers of κ (and $h_{\mu\nu}$) of the Einstein-Hilbert action which can then be interpreted as a field theory of the graviton field $h_{\mu\nu}$ propagating in Minkowski space with an infinite number of self-interactions.

The perturbative approach to gravity, similarly to Yang-Mills, has a non-linear gauge redundancy that follows from diffeomorphism invariance of the geometrical formulation. In four-dimensional Minkowski space, the gauge redundancy implies that the graviton field has two physical propagating degrees of freedom which in the context of scattering amplitudes we call *graviton polarizations*. These physical components are encoded in constrained *polarization tensors* $\epsilon_{\mu\nu}$. In sharp contrast to Yang-Mills theory, graviton scattering amplitudes only contain kinematic information.

In this thesis, we do not restrict our attention to general relativity and consider a gravity theory that contains more degrees of freedom. In the following, we consider so-called $N = 0$ supergravity (or NS-NS supergravity) which, in addition to the graviton field $h_{\mu\nu}$, consists of a two-form field called the B-field (or Kalb-Ramond field) $b_{\mu\nu}$, and a scalar field called the dilaton ϕ. In total, perturbative $N = 0$ supergravity in four-dimensional Minkowski spacetime carries four (the square of those in Yang-Mills) physical propagating degrees of freedom: two polarizations of the graviton, one (pseudo) scalar degree of freedom of the B-field and one scalar degree of freedom of the dilaton. It is worth mentioning that this theory formulated in twenty-six spacetime dimensions corresponds to the low energy effective action of closed bosonic string theory. Here, in general, we work with an arbitrary number of spacetime dimensions, unless otherwise stated.

1.1.3 Double copy

The double copy program originates from string theory. In the late eighties, Kawai, Lwellen and Tye (KLT) [8] found that closed string scattering amplitudes, which correspond to gravity in the field theory limit, can be written as "squares" of open string scattering amplitudes, which correspond to Yang-Mills in the field theory limit. More recently, Bern, Carrasco and Johansson (BCJ) [9] proposed a reformulation and extension of the results by KLT in the field theory limit in terms of a duality between color and kinematics. One of the main messages of the BCJ double copy is that replacing the color information of Yang-Mills scattering amplitudes with another copy of their kinematic information leads to amplitudes in ($N = 0$ super-) gravity. This reduces significantly the complexity of gravitational computations. For this reason, double copy techniques are widely used in the scattering amplitudes program as a tool to simplify computations in gravity that can easily become intractable.

The double copy is not an exclusive feature of pure Yang-Mills theory. Other theories with color degrees of freedom have been used in this framework to construct gravity theories beyond $N = 0$ supergravity. For instance, among others, the BCJ double copy has been successfully used in theories with matter, as well as in supersymmetric extensions of Yang-Mills theory, such as $N = 4$ super Yang-Mills. In particular, starting from $N = 4$ super Yang-Mills theory, the BCJ double copy has enabled the computation of high loop integrands in $N = 8$ supergravity, shedding some light on its ultraviolet behavior [10–17]. Furthermore, in view of phenomenological applications, the double copy has been used in computations to study the two-body problem using amplitude-based methods [18–33].

In addition to the BCJ double copy of scattering amplitudes, there exists the *classical double copy* approach to classical solutions. Pioneered by Monteiro, O'Connell and White [34], the classical double copy attempts to write solutions of gravity field equations as double copies of solutions of Yang-Mills equations, and it has been successful in the construction of non-perturbative and perturbative spacetimes, as well as in applications to compute gravitational observables using amplitude-based methods [35–50]. See [51] for a complete set of references.

1.1.4 Double field theory

Originally, double field theory (DFT) was constructed by Hull and Zwiebach in [52] as a massless subsector of closed string field theory on toroidal backgrounds that is invariant under T-duality (or $O(D, D)$ transformations). See also [53] for earlier work by Siegel. Nowadays the original construction by Hull and Zwiebach is called *weakly constrained* DFT. One of the main features of the theory is that the spacetime fields and gauge parameters incorporate the Kaluza-Klein modes of the closed string arising from toroidal compactifications, as well as their T-dual winding modes generated by the string wrapping around the torus. The inclusion of the winding modes is encoded in the fact that all fields and gauge parameters depend on *dual coordinates* $\bar{x}^{\bar{\mu}}$, which can be interpreted as the canonical conjugates to winding modes, similarly to the standard coordinates x^μ being the canonical conjugates to momenta.

The theory is defined on a double $2D$-dimensional space with D coordinates x^μ with associated spacetime indices μ, and D coordinates $\bar{x}^{\bar{\mu}}$ with associated indices $\bar{\mu}$. Hence weakly constrained DFT exhibits a manifest *index factorization* related to the left- and right-moving modes of the string which correspond to x^μ and $\bar{x}^{\bar{\mu}}$, respectively. The general consistency of the theory relies on the *weak constraint* which arises from the level-matching condition of closed string theory. The physical field content consists of a *tensor fluctuation* $e_{\mu\bar{\nu}}(x, \bar{x})$ and auxiliary and pure gauge fields. The field $e_{\mu\bar{\nu}}$ contains the graviton, B-field and dilaton, among their respective tower of Kaluza-Klein and winding modes. An explicit action for weakly constrained DFT was constructed up to cubic interactions in 2009 [52] and since then, the construction of the theory beyond that order in perturbation theory has been a major challenge that we overcome here.

In this thesis we also use the better known *strongly constrained* double field theory [54, 55], which is a reformulation of $N = 0$ supergravity. This theory is defined on a double $2D$-dimensional space and is manifestly invariant under the T-duality group $O(D, D)$ *prior* to any toroidal compactification and, as a consequence, perturbative strongly constrained DFT exhibits an index factorization to all orders in perturbation theory [56]. Commonly, strongly constrained DFT is presented as an exact gravity theory and its field content consists of a so-called *duality invariant dilaton* d and a so-called *generalized metric* $\mathcal{H}$, which combines the spacetime metric $g_{\mu\nu}$ and the B-field $b_{\mu\nu}$ into a single geometrical object.

The adjective *strongly constrained* refers to the fact that the weak constraint is replaced by a stronger section condition which eliminates the dependence on D coordinates of the fields. In the case of toroidal compactifications, due to the fields only effectively depending on D of the coordinates, the strongly constrained theory does not capture *all* the Kaluza-Klein and winding modes simultaneously. In this thesis, however, we do not consider toroidal compactifications in strongly constrained DFT and only use the extra coordinates as purely auxiliary to make $O(D, D)$ and hence index factorization manifest. For weakly constrained DFT, though, we do consider toroidal backgrounds in the second to last chapter of the thesis.

1.1.5 Homotopy algebras

An algebra is a vector space equipped with at least one bilinear map, also called *product* or *bracket*. One classifies algebras according to the type of relations that the maps obey. We call these relations *defining relations*. For example, an associative algebra is a vector space equipped with a bilinear product that obeys associativity. Similarly, a Lie algebra is a vector space equipped with a two-bracket that obeys the Jacobi identity. We call associative algebras and Lie algebras *strict* because their defining relations, associativity and the Jacobi identity, respectively, hold strictly. Homotopy algebras are generalizations of strict algebras in that the defining relations do not hold strictly, but rather *up to homotopy*. This then implies the existence of maps with higher multiplicity (also called *homotopies* or higher maps) that control the failure of the defining relations to hold strictly. Additionally, for the homotopy algebra to be consistent, all the higher multiplicity maps that one introduces have to obey additional compatibility conditions among themselves.

In order to understand the physical significance of homotopy algebras it will be convenient to first talk about symmetries. When we first learn about continuous infinitesimal symmetries in field theory (be they global or local), we are taught that there exist mathematical structures that encode the symmetries and their consistency. In most cases familiar to physicists, like in Yang-Mills theory and Einstein-Hilbert gravity, the continuous infinitesimal symmetries are encoded in Lie algebras. Similarly, the homotopy version of Lie algebras called strongly homotopy Lie algebras or L_∞-algebras, encode perturbative field theories and their consistency. For instance, there is an L_∞-algebra underlying Yang-Mills, where the action and gauge structure of the theory are encoded in the maps of the algebra while the consistency of the theory is encoded in the so-called L_∞ relations, which are the compatibility relations that all the maps obey.

The L_∞-algebra underlying Yang-Mills theory exhibits a manifest factorization between color and kinematics which resembles the fact that gluon amplitudes have independent color and kinematic information. More precisely, the L_∞-algebra of Yang-Mills theory is the tensor product $\mathcal{K}^{\rm YM}\otimes\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra encoding color, while $\mathcal{K}^{\rm YM}$ is an exotic homotopy algebra that encodes kinematics. Following the logic of the double copy, where one replaces color with another copy of kinematics, in this thesis we ask: to what extent is the tensor product $\mathcal{K}^{\rm YM}\otimes\bar{\mathcal{K}}^{\rm YM}$ an L_∞-algebra that encodes gravity?

1.1.6 Kinematic algebras

A necessary condition to implement the BCJ double copy is that *color-kinematics duality* be obeyed. This is the case if the kinematic building blocks of Yang-Mills amplitudes called *kine-*

matic numerators, obey the same algebraic relations as the color building blocks called *color factors*. Color-kinematics duality is required for the gravity amplitude to be invariant under linearized gauge transformations. The algebraic relations obeyed by the color factors are a consequence of the Jacobi identity of the color Lie algebra $\mathfrak{g}$. For this reason, it is believed that there should exist a *kinematic algebra*. Then, the algebraic relations obeyed by the kinematic numerators should be a consequence of the defining relation of the kinematic algebraic structure.

The first explicit realization of a kinematic algebra was found for self-dual Yang-Mills theory and a subsector of pure Yang-Mills theory by Monteiro and O'Connell [57]. More recently, Ben-Shahar and Johansson found a kinematic algebra for Chern-Simons theory [58] (see also [59] for related work on the pure spinor formulation of super Yang-Mills theory). In these two cases, the authors found that the kinematic algebras of these theories are Lie algebras. For scattering amplitudes of the complete theory of Yang-Mills, Reiterer constructed a kinematic homotopy algebra in [60] called $\mathrm{BV}_\infty^\square$-algebra, which we define in detail in the main text. Kinematic algebras of this type have been found in other theories, including self-dual Yang-Mills and Chern-Simons, and in these two particular cases, they contain as consistent subsectors the kinematic Lie algebras of Monteiro and O'Connell, and Ben-Shahar and Johansson [3, 6, 61, 62]. Furthermore, there are algebraic approaches to constructing kinematic numerators that obey color-kinematics duality. In these approaches, however, the relations obeyed by the kinematic numerators are not a consequence of the defining relations of the algebras [63–67].

As we mentioned above, the double copy has been used to construct gravity integrands for loop computations. However, a complete proof of color-kinematics duality (and BCJ double copy) at loop level remains elusive. One of the main issues with finding such a proof is that the kinematic numerators in loop-level amplitudes depend on off-shell loop momenta and color-kinematics duality is poorly understood off-shell. Similar issues arise in classical solutions where an off-shell algebraic kinematic understanding would allow one to systematically carry out the double copy of any classical solution in an efficient manner. Moreover, color-kinematics duality relies on a choice of gauge. Indeed, in general, one relies on so-called *generalized gauge transformations* to make the kinematic numerators obey the duality. These transformations may be non-local, which would alter the unitarity of the amplitudes. Hence, in this thesis we ask the following question: is it possible to find an off-shell, gauge independent and local approach to color-kinematics duality and the double copy? Answering this question can lead to a first principle understanding and systematic implementation of the double copy.

1.2 Main objective and results

In order to have a first principle understanding of the double copy, the main objective of this thesis is **to find an off-shell, gauge independent and local double copy prescription that allows one to construct off-shell and gauge invariant gravity using the kinematic building blocks of Yang-Mills theory**. Finding such a prescription allows one to find an off-shell and gauge independent extension of color-kinematics duality. To that end, we use homotopy algebras and double field theory. On the one hand, the use of homotopy algebras is motivated by the fact that these mathematical structures encode the dynamics, gauge structure and consistency of perturbative field theories. This makes these algebras the natural candidates to find a way to construct a complete and consistent gravity theory in an off-shell and gauge independent manner. On the other hand, double field theory provides a framework that encodes all the degrees of freedom that follow from the BCJ double copy in a way that an *index factorization* property is manifest to all orders in perturbation theory. This factorization property, similarly to the KLT relations, originates from string theory and it is an important property to render the double copy manifest at the level of an action.

In this thesis, we find such a double copy prescription up to and including quartic interactions in the action. In particular, one of our main results is the construction of an off-shell and gauge independent kinematic algebra for Yang-Mills theory to quartic order. This kinematic algebra is a homotopy algebra of the type that Reiterer introduced, namely a $\mathrm{BV}_\infty^\Box$-algebra. Moreover, using the kinematic $\mathrm{BV}_\infty^\Box$-algebra, we implement a perturbative double copy prescription up to quartic order in fields in the action. This double copy prescription consists of taking the tensor product of two copies of the kinematic algebra which, after imposing constraints, leads to an L_∞-algebra that encodes gravity in the form of strongly constrained double field theory. To check the consistency of our algebraic double copy prescription, we construct the quadratic and cubic double field theory actions, while to probe quartic interactions we compute four-point scattering amplitudes. Moreover, we find that at four points the algebraic relations between the kinematic numerators associated with color-kinematics duality is, in fact, a deformed Poisson identity which is one of the defining relations of the kinematic $\mathrm{BV}_\infty^\Box$-algebra. Furthermore, using our algebraic double copy prescription, we construct weakly constrained double field theory to quartic order. To present these results, the remainder of the thesis is organized as follows:

- In **Chapter 2** we give a brief introduction to various topics that play an important role in this thesis. First, we discuss color ordering of Yang-Mills scattering amplitudes. Then, after showing how these amplitudes exhibit a manifest factorization between color and kinematics, we turn to introducing color-kinematics duality and the BCJ double copy. Subsequently, we turn to double field theory and, finally, we close the chapter by implementing a double copy prescription at the level of the Lagrangian that, starting with Yang-Mills theory, leads to double field theory to cubic order.

- **Chapter 3** deals with the mathematical structures central to this thesis: homotopy algebras. We start with a brief introduction to the mathematical concepts behind L_∞-algebras and then turn to formulating classical perturbative field theories in this mathematical framework. As concrete examples, we formulate Chern-Simons theory and Yang-Mils theory in this algebraic framework and discuss a manifest factorization between color and kinematics that their corresponding algebras exhibit. This factorization allows us to construct so-called kinematic $\mathrm{BV}_\infty^\Box$-algebras, which we can later use for the double copy.

- In **Chapter 4** we use the kinematic algebra of Yang-Mills to construct strongly constrained double field theory up to and including quartic interactions. More precisely, similarly to the BCJ double copy of amplitudes, we exchange the color Lie algebra of Yang-Mills with another copy of the kinematic $\mathrm{BV}_\infty^\Box$-algebra.

- In **Chapter 5** we turn to weakly constrained double field theory. We start this chapter with some general comments on weakly constrained double field theory. Then, we introduce the notion of homotopy transfer which plays a central role in the algebraic construction of the theory to quartic order. Finally, using our algebraic approach to the double copy in combination with homotopy transfer, we construct weakly constrained DFT to quartic order.

- Finally, in **Chapter 6** we present a discussion on the results and potential future directions.

Chapter 2

Double copy and double field theory

In this chapter we will introduce a series of concepts upon which the rest of the thesis is based. We begin in section 2.1 with a review of color ordering, which relies on the observation that Yang-Mills amplitudes exhibit a manifest factorization of color and kinematics. Subsequently, in section 2.2, we give a brief introduction to the double copy of scattering amplitudes which allows one to construct gravity amplitudes using kinematic information of Yang-Mills theory. Moving forward, in section 2.3, we review basic concepts of double field theory and finally, in section 2.4 we propose a simple color-kinematic substitution at the level of the action to cubic order in fields that, starting from Yang-Mills theory, yields gauge invariant double field theory at linearized order, while at cubic order one obtains double field theory in so-called Siegel gauge.

Section 2.4 is largely based on [1], and certain computations in that section are taken from this reference.

2.1 Color ordered amplitudes

Let us start this section with some general remarks about tree-level[1] Yang-Mills (or gluon) scattering amplitudes[2] in D dimensional Minkowski space. We impose Feynman gauge so that the propagators are proportional to $\frac{1}{p^2}$. Gluon amplitudes contain two types of information: information about kinematics in the form of momenta of the external legs p_i and polarization vectors $\epsilon_{i\,\mu}(p_i)$, and information about color in the form of Lie algebra generators t_a or structure constants f^a_{bc} of the color Lie group under consideration. Indeed, in general a Yang-Mills scattering amplitude can be written in a form that manifests a factorization between color and kinematics as

$$\mathcal{A}^{(n)}(1,2,...,n) = g^{n-2} \sum_{\mathcal{P}(2,3,..,n)} \mathrm{Tr}\big\{t_{a_1} t_{a_2} ... t_{a_n}\big\}\, A^{(n)}(1,2,...,n)\ , \tag{2.1}$$

[1] In this thesis we only consider tree-level amplitudes.

[2] For a more complete review of topics in scattering amplitudes refer to [68, 69].

where the sum runs through all non-cyclic permutations of the particle labels and the $A^{(n)}$ are[3] $(n-1)!$ *color ordered* or *partial* amplitudes which depend only on kinematic information. These partial amplitudes obey properties such as cyclicity and reflection of the the particle labels. One says that the amplitude above is written in the *trace basis* because it contains a trace over the generators t_{a_i}. Alternatively, one could write the full color dressed amplitude in the so-called Del Duca, Dixon, Maltoni (DDM) basis [70] in terms of the structure constants f^a_{bc}, which reads

$$\mathcal{A}^{(n)\,a_1\ldots a_n}(1,2,...,n) = g^{n-2} \sum_{\sigma(2,...,n-1)} f^{a_1 a_2 e_1}\, f^{e_1 a_3 e_2}\, f^{e_2 a_4 e_3} ... f^{e_{n-3} a_{n-1} a_n}\, A^{(n)}(1,2,...,n) \;, \tag{2.2}$$

where the sum runs though the permutation σ of $(n-2)$ color indices and labels, and in this case there are $(n-2)!$ color ordered amplitudes due to the structure constants being antisymmetric in all indices and the Jacobi identity

$$f^{e[a_1 a_2}\, f^{a_3]a_4 e} = 0 \;. \tag{2.3}$$

Notice that it is always possible to go from one basis to the other by writing the structure constants in terms of commutators of the generators. Performing such a change of basis should not alter the amplitude. Thus, the fact that there is a different number of independent color ordered amplitudes in the two bases can only be explained by additional relations between color ordered amplitudes. Indeed, there exists a set of relations called *Kleiss-Kuijf* relations [71] which reduce the number of color ordered amplitudes from $(n-1)!$ to $(n-2)!$. The Kleiss-Kuijf relations read as follows:

$$A^{(n)}(1,\{\alpha\},n,\{\beta\}) = (-1)^{n_\beta} \sum_{\sigma\in\alpha\sqcup\beta} A^{(n)}(1,\sigma,n) \;. \tag{2.4}$$

In the above equation one considers two ordered sets, $\{\alpha\}$ and $\{\beta\}$, n_β is the number of elements of $\{\beta\}$, $\{\beta^T\}$ is a set with the same elements of $\{\beta\}$ but with a reversed ordering, and the sum runs over the *shuffles* or *ordered permutations* of $\{\alpha\}$ and $\{\beta^T\}$. In order to illustrate how to deal with shuffles, let us consider the sets $\{\alpha\} = \{1,2\}$ and $\{\beta\} = \{3,4\}$ so that $\{\beta^T\} = \{4,3\}$. Then, the sum over shuffles is

$$\{1,2\}\sqcup\{4,3\} = \{1,2,4,3\}+\{1,4,2,3\}+\{1,4,3,2\}+\{4,3,1,2\}+\{4,1,3,2\}+\{4,1,2,3\} \;. \tag{2.5}$$

Let us now investigate certain consequences of the Kleiss-Kuijf relation of a four-point amplitude with $\{\alpha\} = \{2\}$ and $\{\beta\} = \{3\} = \{\beta^T\}$:

$$A^{(4)}(1,2,4,3) = -A^{(4)}(1,2,3,4) - A^{(4)}(1,3,2,4) \;. \tag{2.6}$$

Using cyclicity and reflection of the partial amplitudes we can recast the above equation as

$$A^{(4)}(1,2,3,4) + A^{(4)}(1,3,4,2) + A^{(4)}(1,4,2,3) = 0 \;, \tag{2.7}$$

[3] This number of independent color ordered amplitudes is due to cyclicity of the trace.

which, in particular, is the so-called *photon decoupling identity* at four points. The partial amplitudes $A^{(4)}$ are fully determined by their pole structure and some objects called *kinematic numerators*. It is important to remark that color ordered amplitudes are computed using planar diagrams. As a consequence, four-point partial amplitudes only have two contributing color ordered Feynman diagrams, implying that they only have two simple poles. For this reason, one can choose a prametrization of the four-point partial amplitudes that solves the photon decoupling identity (2.7) which has the correct pole structure [9], namely

$$\begin{aligned} A^{(4)}(1,2,3,4) &= -\frac{n_s}{s} + \frac{n_t}{t} \ , \\ A^{(4)}(1,3,4,2) &= -\frac{n_u}{u} + \frac{n_s}{s} \ , \\ A^{(4)}(1,4,2,3) &= -\frac{n_t}{t} + \frac{n_u}{u} \ , \end{aligned} \tag{2.8}$$

where the kinematic numerators $n_{s_{ij}}$ are associated to the Mandelstam variables[4] $s_{ij} = (p_i+p_j)^2$. Notice that such a parametrization of the partial amplitudes is always possible if one *blows up* the quartic vertices by inserting a one in the guise of a propagator multiplied by an inverse propagator and hence the kinematic numerators have information from both the cubic and quartic vertices.

An alternative solution to the photon decoupling identity can be obtained by a different reasoning [9]. Due to Lorentz invariance, the only way that the three amplitudes in the photon decoupling identity (2.7) add up to zero is if they are proportional to the Mandelstam invariants s, t, u, whose sum vanishes:

$$A^{(4)}(1,2,3,4) + A^{(4)}(1,3,4,2) + A^{(4)}(1,4,2,3) = (s+t+u)\chi = 0 \ . \tag{2.9}$$

This relation implies that all amplitudes are proportional to each other. In order to find the explicit form of each amplitude in terms of s_{ij} and χ, one needs to assign the correct Mandelstam invariant to each amplitude. Take for instance $A^{(4)}(1,2,3,4)$. This partial amplitude is invariant under the exchange $s \leftrightarrow t$, or equivalently, the exchange of the legs $(1 \leftrightarrow 3)$:

$$A^{(4)}(1,2,3,4) \to A^{(4)}(3,2,1,4) = A^{(4)}(4,1,2,3) = A^{(4)}(1,2,3,4) \ , \tag{2.10}$$

where in the first equality we used reflection and in the second we used cyclicity. This means that the amplitude has to be proportional to $s_{13} \equiv u$, which is invariant under the exchange of the legs $(1 \leftrightarrow 3)$. Analogous arguments lead to the following solution of the photon decoupling identity:

$$A^{(4)}(1,2,3,4) = u\,\chi \ , \quad A^{(4)}(1,3,4,2) = t\,\chi \ , \quad A^{(4)}(1,4,2,3) = s\,\chi \ . \tag{2.11}$$

Eliminating χ leads to a new set of relations between the color ordered amplitudes, called the

[4]Note that $s_{12} \equiv s$, $s_{23} \equiv t$, $s_{31} \equiv u$.

Bern-Carrasco-Johansson (BCJ) relations [9]

$$\begin{aligned} t\, A^{(4)}(1,2,3,4) &= u\, A^{(4)}(1,3,4,2)\ , \\ s\, A^{(4)}(1,2,3,4) &= u\, A^{(4)}(1,4,2,3)\ , \\ t\, A^{(4)}(1,4,2,3) &= s\, A^{(4)}(1,3,4,2)\ , \end{aligned} \tag{2.12}$$

which can be extended to higher n-point amplitudes. For general n the BCJ relations reduce the number of independent partial amplitudes to $(n-3)!$.

The BCJ relations have far reaching implications. On the one hand, the BCJ relations reduce significantly the number of independent color ordered amplitudes and, as a consequence, computing loop amplitudes using unitarity methods becomes simpler. On the other hand, substituting the partial amplitudes written in terms of the kinematic numerators (2.8) into the BCJ relations leads to the following relation:

$$n_s + n_t + n_u = 0\ . \tag{2.13}$$

This equation is called "kinematic Jacobi identity", for reasons that will be apparent in the next section, and it plays a crucial role in this thesis because it is the basis of *color-kinematics* duality which is, in turn, the basis of the modern double copy program.

Let us close this section by discussing gauge invariance of the partial amplitudes. A full color dressed Yang-Mills amplitude $\mathcal{A}^{(n)}$ should be invariant under linearized gauge transformations which are generated by any shift of a polarization vector of the form

$$\delta\epsilon_{i\,\mu}(p_i) = p_{i\,\mu}\ . \tag{2.14}$$

Gauge invariance of the color dressed amplitude $\mathcal{A}^{(n)}$ implies that the partial amplitudes $A^{(n)}$ be gauge invariant. Indeed, if one expands $\mathcal{A}^{(n)}$ in the trace basis (2.1), all the color structures are independent and as a consequence all the color ordered amplitudes must be gauge invariant independently. Hence, due to gauge invariance, under the shift of any external leg, the kinematic numerators of the four-point partial amplitudes showed in (2.8) must transform as

$$\delta_k n_{s_{ij}} = \alpha_k\, s_{ij}\ , \tag{2.15}$$

where k is the label of the particle whose polarization vector is shifted and α_k is a common factor that all kinematic numerators share.

2.2 Double copy

In this section we give a brief introduction to the concepts necessary to understand the basics of the double copy construction of scattering amplitudes. We start by introducing color kinematics duality, and subsequently we construct gravity scattering amplitudes using the kinematic building blocks of Yang-Mills. The concepts developed in this section will serve as motivation to extend this double copy construction beyond scattering amplitudes in later chapters.

2.2.1 Color-kinematics duality

Having introduced color ordered amplitudes and the BCJ relations, we now turn to introducing color-kinematics duality at four points. To that end, consider a four-point Yang-Mills amplitude in the DDM basis:

$$\mathcal{A}^{(4)} = g^2\, T_{a_1} T_{a_2} T_{a_3} T_{a_4} \Big\{ f^{a_1 a_2 e}\, f^{e a_3 a_4}\, A^{(4)}(1,2,3,4) + f^{a_1 a_3 e}\, f^{e a_2 a_4}\, A^{(4)}(1,3,2,4) \Big\} \,, \tag{2.16}$$

where we saturate the color indices with *color polarizations* T_{a_i}. Using reflection and cyclicity of the partial amplitude in the second term as well as antisymmetry of the structure constant leads to

$$\mathcal{A}^{(4)} = g^2 \Big\{ - c_s\, A(1,2,3,4) + c_u\, A(1,4,2,3) \Big\} \,, \tag{2.17}$$

where the *color factors* c_{s_i} read

$$c_s := -T_{a_1} T_{a_2} T_{a_3} T_{a_4}\, f^{a_1 a_2 e}\, f^{e a_3 a_4} \,, \quad c_u := -T_{a_1} T_{a_2} T_{a_3} T_{a_4}\, f^{a_3 a_1 e}\, f^{e a_2 a_4} \,. \tag{2.18}$$

We can use the expressions for the partial amplitudes (2.8) that we found in the previous section to rewrite the full amplitude in terms of the kinematic numerators as

$$\mathcal{A}^{(4)} = g^2 \left\{ \frac{c_s\, n_s}{s} - (c_u + c_s)\, \frac{n_t}{t} + \frac{c_u\, n_u}{u} \right\} \,, \tag{2.19}$$

and the sum of color factors in the second term can be written using another color factor c_t defined as

$$c_t := -T_{a_1} T_{a_2} T_{a_3} T_{a_4}\, f^{a_2 a_3 e}\, f^{e a_1 a_4} \,, \tag{2.20}$$

which is related to the other two color factors via the Jacobi identity

$$c_s + c_t + c_u = 0 \,. \tag{2.21}$$

Thus, we can write the full four-point amplitude as

$$\mathcal{A}^{(4)} = g^2 \left\{ \frac{c_s\, n_s}{s} + \frac{c_t\, n_t}{t} + \frac{c_u\, n_u}{u} \right\} \,. \tag{2.22}$$

Notice that the color factors and the kinematic numerators appear on the same footing in the above form of the amplitude. The color factors obey the Jacobi identity (2.21), while the kinematic numerators, as shown in the previous section, obey the "kinematic Jacobi identity"

$$n_s + n_t + n_u = 0 \,, \tag{2.23}$$

whose name originates from the similarity between this equation and the Jacobi identity (2.21).

Color-kinematics duality, originally conjectured by Bern, Carrasco and Johansson in [9], is the statement that it is possible to organize the perturbative expansion of Yang-Mills in terms of diagrams with only cubic vertices[5] (trivalent diagrams) such that the kinematic numerators

[5] This can be achieved by blowing up the quartic vertices with ones in the guise of propagators multiplied by inverse propagators.

obey the same type of three-term relations as the color factors [72]. More precisely, one can write a generic n-point tree-level Yang-Mills scattering amplitude as a sum of diagrams with only cubic vertices as

$$\mathcal{A}^{(n)} = g^{n-2} \sum_{i \in \text{cubic}} \frac{c_i \, n_i}{D_i} \,, \tag{2.24}$$

where D_i are products of denominators of propagators. Color-kinematics duality states that if three color factors c_i, c_j and c_k are related by the Jacobi identity, then their kinematic counterparts are also related by the three-term kinematic relation, namely

$$c_i + c_j + c_k = 0 \;\leftrightarrow\; n_i + n_j + n_k = 0 \,. \tag{2.25}$$

Color-kinematics duality is manifest at four points because the color factors and the kinematic numerators obey the same algebraic relations. At higher points the duality is not manifest in general. However, one has the freedom to redefine the kinematic numerators in ways that leave the amplitude invariant by means of *generalized gauge transformations*. One can use these transformations to make the numerators obey the duality. In this thesis, however, since we do not consider higher than four-point amplitudes, we will not review these generalized redundancies. Color-kinematics duality in Yang-Mills has been proven at tree-level [73–75], and it extends to a plethora of theories with color degrees of freedom. See [72] for a complete set of theories with color-kinematics duality and a corresponding set of references. Moreover, there have been attempts to render color-kinematics duality manifest at the level of the Yang-Mills action [73, 76–79]. This would lead to Feynman rules that generate color-kinematics obeying numerators.

The fact that the kinematic numerators obey the three-term relation (2.23), which is similar to the Jacobi identity of the color factors, suggests that there may be an algebraic structure underlying the kinematics of Yang-Mills, dubbed the *kinematic algebra*, just like there is an algebraic structure underlying color. This has been investigated from different perspectives. For instance, Monteiro and O'Connell [57] found that in the self-dual sector of Yang-Mills, upon imposing a light-cone gauge condition, there is a kinematic Lie algebra of area preserving diffeomorphisms which can be extended to the scattering amplitudes of full Yang-Mills by choosing certain helicity configurations of the external particles (Maximally Helicity Violating, or MHV configurations). Similarly, more recently, Ben-Shahar and Johansson found a kinematic Lie algebra for off-shell correlators in Chern-Simons theory [58]. In order to compute the correlators they use Lorenz gauge, and they find the Lie algebra of volume preserving diffeomorphisms in three dimensions. In addition, there have been other approaches regarding homotopy algebras, which we describe in more detail in the following chapter. For a more complete account on the advances on the kinematic algebra and color kinematics duality see [80] and references therein.

Lastly, let us examine gauge invariance of the full four-point amplitude $\mathcal{A}^{(4)}$ in the form (2.22). Let us shift, for instance, the fourth external leg, i.e $\delta\epsilon_{4\,\mu} = p_{4\,\mu}$. It then follows that the variation of the amplitude is

$$\begin{aligned}\delta_4 \mathcal{A}^{(4)} &= g^2 \left\{ \frac{c_s\, \delta_4 n_s}{s} + \frac{c_t\, \delta_4 n_t}{t} + \frac{c_u\, \delta_4 n_u}{u} \right\} \\ &= g^2\, \alpha_4 \left\{ c_s + c_t + c_u \right\} = 0 \ ,\end{aligned} \tag{2.26}$$

where we used the transformations of the kinematic numerators (2.15) and the Jacobi identity (2.21). This shows that the Jacobi identity plays a crucial role in the gauge invariance of Yang-Mills scattering amplitudes.

2.2.2 Gravity amplitudes from Yang-Mills

The double copy originates from a set of relations between closed string and open string scattering amplitudes found by Kawai, Lwellen and Tye (KLT) [8]. These relations formulate closed string scattering amplitudes as "squares" of open string scattering amplitudes. In the field theory limit, closed string scattering amplitudes are associated to gravity amplitudes, while open string amplitudes are associated to color ordered Yang-Mills amplitudes. Hence the KLT relations imply that gravity amplitudes can be thought of as "squares" of Yang-Mills amplitudes. Here, the term "gravity amplitudes" includes the graviton, B-field and dilaton degrees of freedom. The KLT relations for three-, four- and five-point tree-level amplitudes read

$$\begin{aligned}\mathcal{M}^{(3)}(1,2,3) &= A^{(3)}(1,2,3)\, \bar{A}^{(3)}(1,2,3) \ , \\ \mathcal{M}^{(4)}(1,2,3,4) &= -s_{12}\, A^{(4)}(1,2,3,4)\, \bar{A}^{(4)}(1,2,4,3) \ , \\ \mathcal{M}^{(5)}(1,2,3,4,5) &= s_{12} s_{45}\, A^{(5)}(1,2,3,4,5)\, \bar{A}^{(5)}(1,3,5,4,2) \\ &\quad + s_{14} s_{25}\, A^{(5)}(1,4,2,5)\, \bar{A}^{(5)}(1,3,5,2,4) \ ,\end{aligned} \tag{2.27}$$

where $\mathcal{M}^{(n)}$ are n-point gravity amplitudes while $A^{(n)}$ and $\bar{A}^{(n)}$ are two copies of color ordered Yang-Mills n-point amplitudes. These relations have been extended to all orders in points [81,82], and their complexity grows fast when going to higher points.

The modern double copy program, the BCJ double copy first introduced in [9], is basically a reformulation of the above KLT relations based on color-kinematics duality. The main idea behind the BCJ double copy is that replacing color information by kinematic information in Yang-Mills scattering amplitudes leads to gravity amplitudes. More precisely, if one has a tree-level Yang-Mills amplitude of the form (omitting the coupling constant)

$$\mathcal{A}^{(n)} = \sum_{i \in \text{cubic}} \frac{c_i\, n_i}{D_i} \ , \tag{2.28}$$

such that color-kinematics duality is satisfied, i.e the kinematic numerators obey (2.23), then one can replace the color factors by a new copy of the kinematic numerators $\bar{n}_i$ that also obey the three-term relation (2.23) while leaving the denominators D_i untouched, to obtain

$$\mathcal{M}^{(n)}_{\text{DC}} = \sum_{i \in \text{cubic}} \frac{n_i\, \bar{n}_i}{D_i} \ . \tag{2.29}$$

If the new set of numerators is the same as the original ones, namely $n_i = \bar{n}_i$, the above object is a graviton amplitude[6]. In principle, however, the two sets of numerators may be different and then such a substitution leads to different gravity theories. Let us illustrate how this works at four points. We start with the full Yang-Mills amplitude in the form

$$\mathcal{A}^{(4)} = \frac{c_s\, n_s}{s} + \frac{c_t\, n_t}{t} + \frac{c_u\, n_u}{u} \, . \tag{2.30}$$

Color-kinematics duality is manifest at four points, as we argued previously, and hence, according to the BCJ double copy, we can exchange the color factors by new kinematic numerators leading to

$$\mathcal{M}^{(4)}_{\rm DC} = \frac{n_s\, \bar{n}_s}{s} + \frac{n_t\, \bar{n}_t}{t} + \frac{n_u\, \bar{n}_u}{u} \, , \tag{2.31}$$

where, in order to keep track of the factorization into two copies of kinematics of the amplitude, we keep the bar on the new set of numerators as well as on the polarization vectors upon which they depend $\bar{\epsilon}_\mu$. Importantly, however, here and in the remainder of the thesis both sets of numerators are associated to pure Yang-Mills amplitudes and hence their explicit expressions are identical, modulo putting a bar on the polarization vectors in the barred copy $\bar{n}_{s_{ij}}$. Doing the following straightforward computation one finds that $\mathcal{M}^{(4)}_{\rm DC}$ coincides with the gravity amplitude $\mathcal{M}^{(4)}$ of the KLT relations (2.27):

$$\begin{aligned} \mathcal{M}^{(4)}(1,2,3,4) &= -s\, A^{(4)}(1,2,3,4)\, \bar{A}^{(4)}(1,2,4,3) \\ &= -s\, A^{(4)}(1,2,3,4)\, \bar{A}^{(4)}(1,3,4,2) \, , \end{aligned} \tag{2.32}$$

where in the second line we used the reflection property and cyclicity of the barred partial amplitude. Substituting the color ordered amplitudes by their solution to the photon decoupling identity in (2.8), one obtains

$$\begin{aligned} \mathcal{M}^{(4)} &= -s\left(-\frac{n_s}{s} + \frac{n_t}{t}\right)\left(-\frac{\bar{n}_u}{u} + \frac{\bar{n}_s}{s}\right) \\ &= \frac{n_s\, \bar{n}_s}{s} - \frac{n_t\, (\bar{n}_s + \bar{n}_u)}{t} - \frac{(n_s + n_t)\, n_u}{u} \\ &= \frac{n_s\, \bar{n}_s}{s} + \frac{n_t\, \bar{n}_t}{t} + \frac{n_u\, \bar{n}_u}{u} \, , \end{aligned} \tag{2.33}$$

where to arrive to the second line we used $s + t + u = 0$, and for the final equality we used the three-term relation obeyed by the two sets of kinematic numerators. This proves that the object that follows from the BCJ double copy is indeed a gravity amplitude. Moreover, color-kinematics duality and the BCJ double copy are conjectured to work at loop-level [9, 10, 83], where a number of examples have ben explicitly worked out. However, it is more difficult to find kinematic numerators that obey color-kinematics duality because in loop amplitudes the numerators depend on off-shell loop momenta [80, 84].

Let us comment on gauge invariance of the gravity amplitude (2.31). The amplitude has two independent symmetries. On the one hand, $\mathcal{M}^{(4)}_{\rm DC}$ is invariant under gauge symmetries associated to the polarization vectors ϵ_μ. More precisely, varying the fourth unbarred polarization vector as $\delta\epsilon_{4\,\mu} = p_{4\,\mu}$, the unbarred kinematic numerators transform as $\delta_4 n_{s_{ij}} = s_{ij}\, \alpha_4$ and the variation

[6] In principle, if the polarization vectors do not obey $\epsilon\cdot\epsilon = 0$, then there are also contributions from the dilaton.

of the amplitude yields

$$\delta_4 \mathcal{M}^{(4)} = \alpha_4 \left(\bar{n}_s + \bar{n}_t + \bar{n}_u \right) = 0 \ . \tag{2.34}$$

The above variation vanishes due to the three-term kinematic relation (2.23) obeyed by the barred numerators. Similarly, varying the fourth barred polarization vector as $\bar{\delta}_4 \bar{\epsilon}_{4\,\mu} = p_{4\,\mu}$ induces the variation of the numerators $\bar{\delta}_4 \bar{n}_{s_{ij}} = \bar{\alpha}_4\, s_{ij}$, and the variation of the amplitude is

$$\bar{\delta}_4 \mathcal{M}^{(4)} = \bar{\alpha}_4 \left(n_s + n_t + n_u \right) = 0 \ , \tag{2.35}$$

which, again, vanishes by the same argument. From this analysis, one concludes that the "kinematic Jacobi identity" plays a crucial role in the gauge invariance of the gravity amplitudes, in complete analogy to the role played by the Jacobi identity of the color factors for the color dressed Yang-Mills amplitude (recall equation (2.26)). The two sets of polarization vectors, ϵ_μ and $\bar{\epsilon}_\mu$, that appear in each copy of the kinematic numerators can be thought of as coming from the factorization of polarization tensors[7] $\varepsilon_{\mu\nu}$:

$$\varepsilon_{i\,\mu\nu}(p_i) = \epsilon_{i\,\mu}(p_i)\,\bar{\epsilon}_{i\,\nu}(p_i) \ . \tag{2.36}$$

The polarization tensors can be decomposed into a symmetric traceless part, an antisymmetric part and a pure trace part, corresponding to external gravitons, B-fields and dilatons, respectively, namely

$$\varepsilon_{i\,\mu\nu} = \epsilon_{i\,(\mu\nu)} + \tilde{\epsilon}_{i\,[\mu\nu]} + \tfrac{1}{D}\eta_{\mu\nu}\epsilon_i \ , \tag{2.37}$$

with $\epsilon^\mu{}_\mu = 0$ and $\epsilon := \varepsilon^\mu{}_\mu$. Notice that the two independent gauge symmetries encode more redundancy than standard linearized diffeomorphisms. This is because the polarization tensor $\varepsilon_{\mu\nu}$ also contains the B-field, which has its own gauge redundancy. Nevertheless, one can *project out* the B-field and dilaton degrees of freedom. To that end, one can drop the bar on the second set polarization vectors, which amounts to taking the symmetric part of $\varepsilon_{\mu\nu}$, and one can impose $\epsilon_\mu \epsilon^\mu = 0$, which amounts to eliminating the dilaton. With such a projection, the amplitude is only invariant under linearized diffeomorphisms which act as

$$\delta\epsilon_{\mu\nu} = p_\mu \epsilon_\nu + p_\nu \epsilon_\mu \ . \tag{2.38}$$

In the remainder of the thesis we will always keep all the $N = 0$ supergravity degrees of freedom, in the form of double field theory fields.

2.3 Strongly constrained double field theory

In this section we introduce strongly constrained double field theory (DFT), while the weakly constrained theory will be discussed at length in chapter 5. Thus, in the following we will omit the adjective *strongly constrained* and simply write DFT unless absolutely necessary. As we alluded to in the introduction, DFT is a reformulation of $N = 0$ supergravity on double $2D$-dimensional spacetime with coordinates $(x^\mu, \bar{x}^{\bar{\mu}})$. The theory is invariant under global $O(D, D)$

[7] Polarization tensors can only be factorized in this way if one fully specifies the helicity of the associated external particle. For instance, for a tensor particle with helicity eigenvalue +2 one has $\varepsilon^{++}_{i\,\mu\nu}(p_i) = \epsilon^{+}_{i\,\mu}(p_i)\,\bar{\epsilon}^{+}_{i\,\nu}(p_i)$.

transformations and local gauge symmetries that encode diffeomorphisms and B-field gauge transformations.

DFT has been constructed as an exact, non-perturbative gravity theory in [54, 55]. In the "textbook" approach to DFT, the theory is formulated using a *duality invariant dilaton* d and a *generalized metric* $\mathcal{H}$, which combines the spacetime metric $g_{\mu\nu}$ and the B-field $b_{\mu\nu}$ into a single geometrical object. Nevertheless, due to the perturbative nature of the BCJ double copy, in this thesis we will exclusively work in perturbation theory, where the physical field content is a *tensor fluctuation* $e_{\mu\bar{\nu}}(x,\bar{x})$ and the dilaton $d(x,\bar{x})$. These fields should not be confused with those of the weakly constrained theory because here none of the dimensions are compactified. Hence, the fields do not carry information about Kaluza-Klein nor winding modes, and the additional coordinates $\bar{x}^{\bar{\mu}}$ are just purely formal objects that allow the index factorization property to be manifest. Unless otherwise stated, we will consider perturbation theory around a double Minkowski background with metrics $(\eta_{\mu}, \bar{\eta}_{\bar{\mu}\bar{\nu}})$, which we use to contract the spacetime indices.

Gauge invariance relies on the *strong constraint* which, in this basis, acting on fields collectively denoted by $\Psi(x,\bar{x})$ reads

$$\big(\Box - \bar{\Box}\big)\Psi(x,\bar{x}) = 0 \;, \quad \partial_\mu \Psi_1(x,\bar{x})\partial^\mu \Psi_2(x,\bar{x}) = \bar{\partial}_{\bar{\mu}}\Psi_1(x,\bar{x})\bar{\partial}^{\bar{\mu}}\Psi_2(x,\bar{x}) \;, \tag{2.39}$$

and analogous for gauge parameters and products of fields and gauge parameters. The two sets of derivatives are $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ and $\bar{\partial}_{\bar{\mu}} \equiv \frac{\partial}{\partial \bar{x}^{\bar{\mu}}}$, and the d'Alembert operators are defined as $\Box \equiv \partial_\mu \partial^\mu$ and $\bar{\Box} \equiv \bar{\partial}_{\bar{\mu}}\bar{\partial}^{\bar{\mu}}$. This constraint eliminates the dependence of the fields and gauge parameters on D of the coordinates.

For the reader familiar with DFT, the formulation that we introduce in this section and construct throughout chapter 4 can be obtained from a perturbative expansion of the generalized metric formulation, in combination with a suitable vielbein formalism as well as field redefinitions which for the sake of compactness we do not develop here. In our field basis the free double field theory reads

$$S^{(2)}_{\text{DFT}} = \frac{1}{4}\int d^D x\, d^D \bar{x} \Big\{ e^{\mu\bar{\nu}}\Box e_{\mu\bar{\nu}} + \partial^\mu e_{\mu\bar{\nu}}\,\partial^\rho e_\rho{}^{\bar{\nu}} + \bar{\partial}^{\bar{\nu}} e_{\mu\bar{\nu}}\,\bar{\partial}^{\bar{\sigma}} e^\mu{}_{\bar{\sigma}} - d\Box d + 2d\partial^\mu\bar{\partial}^{\bar{\nu}} e_{\mu\bar{\nu}} \Big\} \;, \tag{2.40}$$

which is invariant under global double Lorentz transformations. The theory is invariant under a set of gauge transformations generated by two gauge parameters λ_μ and $\bar{\lambda}_{\bar{\mu}}$, under which the fields transform as

$$\begin{aligned} \delta e_{\mu\bar{\nu}} &= \partial_\mu \bar{\lambda}_{\bar{\nu}} + \bar{\partial}_{\bar{\nu}}\lambda_\mu \;, \\ \delta d &= \partial\cdot\lambda + \bar{\partial}\cdot\bar{\lambda} \;. \end{aligned} \tag{2.41}$$

Going to higher orders in perturbation theory these transformations acquire non-linear terms and, as we show below, they encode diffeomorphisms and B-field gauge transformations.

We now turn to making contact with standard perturbative gravity. As stated above, strongly constrained DFT is a reformulation of $N = 0$ supergravity. Indeed, upon choosing a solution to the strong constraint called the *supergravity solution*, which amounts to identifying the two sets of coordinates and indices $x^\mu = \bar{x}^{\bar{\mu}}$, one has the fields $e_{\mu\nu}(x)$ and $d(x)$. At linearized level the tensor fluctuation in the supergravity solution $e_{\mu\nu}$ contains a symmetric part corresponding to the graviton field $h_{\mu\nu}$, while its antisymmetric part contains the B-field $b_{\mu\nu}$. Similarly, the duality invariant dilaton d contains the string frame dilaton ϕ and the trace of the graviton h. More precisely, in the supergravity solution at linearized level[8] the fields decompose as

$$\begin{aligned} e_{\mu\nu} &= h_{\mu\nu} + b_{\mu\nu} \ , \\ d &= h - 4\phi \ , \end{aligned} \tag{2.42}$$

and the action in terms of the supergravity fields becomes

$$\begin{aligned} S^{(2)}_{N=0} = \frac{1}{4}\int d^Dx \Big\{ & h_{\mu\nu}\Box h^{\mu\nu} + 2\,\partial^\mu h_{\mu\nu}\,\partial_\rho h^{\rho\nu} + 2\,h\,\partial^\mu\partial^\nu h_{\mu\nu} - h\Box h \\ & + b_{\mu\nu}\Box b^{\mu\nu} + 2\,\partial^\mu b_{\mu\nu}\,\partial_\rho b^{\rho\nu} \\ & -16\,\phi\Box\phi - 8\,\phi\,\partial^\mu\partial^\nu h_{\mu\nu} + 8\,\phi\Box h \Big\} \ , \end{aligned} \tag{2.43}$$

where the first line is the Fierz-Pauli action, the second line is the B-field action and the last line is the dilaton action. The above free theory is obtained by expanding to quadratic order in fields the full string frame action

$$S_{N=0} = \int d^Dx\sqrt{-g}\,e^{-2\phi}\Big\{R - \tfrac{1}{12}H^2 - 4\phi\Box\phi\Big\} \ , \tag{2.44}$$

with R the Ricci scalar of the metric $g_{\mu\nu}$ and H the field strength of the B-field. The linearized action (2.43) is invariant under the following linearized gauge transformations:

$$\begin{aligned} \delta h_{\mu\nu} &= 2\,\partial_{(\mu}\lambda_{\nu)} + 2\,\partial_{(\nu}\bar\lambda_{\mu)} \ , \\ \delta b_{\mu\nu} &= 2\,\partial_{[\mu}\lambda_{\nu]} + 2\,\partial_{[\nu}\bar\lambda_{\mu]} \ , \\ \delta h &= \partial\cdot\lambda + \partial\cdot\bar\lambda \ , \\ \delta\phi &= 0 \ , \end{aligned} \tag{2.45}$$

and one can immediately identify that taking linear combinations of λ_μ and $\bar\lambda_\mu$ to define new gauge parameters $\xi_\mu = \lambda_\mu + \bar\lambda_\mu$ and $\zeta_\mu = \lambda_\mu - \bar\lambda_\mu$ leads to the more familiar linearized diffeomorphisms and B-field gauge transformations

$$\begin{aligned} \delta h_{\mu\nu} &= \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \ , \\ \delta b_{\mu\nu} &= \partial_\mu\zeta_\nu - \partial_\nu\zeta_\mu \ . \end{aligned} \tag{2.46}$$

Following similar steps one can start from an all-order formulation of DFT and find that solving the strong constraint with the supergravity solution leads to the non-perturbative $N=0$ supergravity action (2.44). Analogous remarks follow for the DFT gauge transformations which are called generalized diffeomorphisms. In standard formulations of the theory they form an algebraic structure called Courant algebroid [85]. In the field and parameter basis that we use in this thesis for DFT, the gauge algebra is a larger L_∞-algebra, as we will review in subsequent chapters. Nevertheless, performing field and parameter redefinitions one should recover

[8] To describe the theory to higher orders in perturbation theory, this decompositions of the fields may acquire non-linear terms that mix all the supergravity fields.

the standard Courant algebroid structure.

Having explained how DFT is a reformulation of $N = 0$ supergravity let us now turn to a rewriting of DFT that allows us to make contact with the original work by Hull and Zwiebach, and it will also help us identify the spectrum that follows from our off-shell and gauge invariant double copy prescription in chapter 4. To that end, we enlarge the field content of DFT and introduce two auxiliary fields f_μ and $\bar{f}_{\bar{\mu}}$, as well as two scalars e and $\bar{e}$ with the dilaton corresponding to the linear combination $d = 2\,(\bar{e} - e)$. We then write a new DFT action equivalent to (2.40) that reads

$$\begin{aligned} S^{(2)}_{\text{H.Z.}} = \int d^D x\, d^D \bar{x} \Big\{ \tfrac{1}{4}\, e^{\mu\bar{\nu}} \Box e_{\mu\bar{\nu}} + 2\,\bar{e}\,\Box e - f^\mu f_\mu - \bar{f}^{\bar{\mu}} \bar{f}_{\bar{\mu}} - f^\mu \Big(\bar{\partial}^{\bar{\nu}} e_{\mu\bar{\nu}} - 2\,\partial_\mu \bar{e} \Big) \\ + \bar{f}^{\bar{\nu}} \Big(\partial^\mu e_{\mu\bar{\nu}} + 2\,\bar{\partial}_{\bar{\nu}} e \Big) \Big\} \,. \end{aligned} \tag{2.47}$$

As it can be easily checked, the field equations of the auxiliary fields are

$$f_\mu = \partial_\mu \bar{e} - \tfrac{1}{2}\,\bar{\partial}^{\bar{\rho}} e_{\mu\bar{\rho}} \,, \qquad \bar{f}_{\bar{\mu}} = \bar{\partial}_{\bar{\mu}} e + \tfrac{1}{2}\,\partial^\rho e_{\rho\bar{\mu}} \,. \tag{2.48}$$

We can use these field equations to integrate out the auxiliary fields, which leads to $S^{(2)}_{\text{DFT}}$ in (2.40). The action $S^{(2)}_{\text{H.Z.}}$ in (2.47) is invariant under gauge transformations generated by λ_μ and $\bar{\lambda}_{\bar{\mu}}$, and the inclusion of the two scalars e and $\bar{e}$ also generates an additional Stückelberg symmetry. In total, the action (2.47) is invariant under the following gauge transformations:

$$\begin{aligned} \delta e_{\mu\bar{\nu}} &= \partial_\mu \bar{\lambda}_{\bar{\nu}} + \bar{\partial}_{\bar{\nu}} \lambda_\mu \,, \\ \delta e &= -\tfrac{1}{2} \partial \cdot \lambda + \eta \,, & \delta \bar{e} &= \tfrac{1}{2} \bar{\partial} \cdot \bar{\lambda} + \eta \,, \\ \delta f_\mu &= -\tfrac{1}{2}\, \Box \lambda_\mu + \partial_\mu \eta \,, & \delta \bar{f}_{\bar{\mu}} &= \tfrac{1}{2}\, \Box \bar{\lambda}_{\bar{\mu}} + \bar{\partial}_{\bar{\mu}} \eta \,. \end{aligned} \tag{2.49}$$

The DFT dilaton $d = 2\,(\bar{e} - e)$ is invariant under the η symmetry, while the other linear combination $\tilde{d} = 2\,(\bar{e} + e)$ is pure gauge and drops out of the action after integrating out the auxiliary fields f_μ and $\bar{f}_{\bar{\mu}}$. In this field basis the cubic action of DFT reads

$$\begin{aligned} S^{(3)}_{\text{H.Z.}} = \int d^D x d^D \bar{x} \, \big[\tfrac{1}{8} e_{\mu\bar{\nu}} \big(2\,\partial^\mu e_{\rho\bar{\sigma}} \bar{\partial}^{\bar{\nu}} e^{\rho\bar{\sigma}} - 2\,\partial^\mu e_{\rho\bar{\sigma}} \bar{\partial}^{\bar{\sigma}} e^{\rho\bar{\nu}} - 2\,\partial^\rho e^{\mu\bar{\sigma}} \bar{\partial}^{\bar{\nu}} e_{\rho\bar{\sigma}} + 2\,\partial_\rho e^{\rho\bar{\nu}} \bar{\partial}_{\bar{\rho}} e^{\mu\bar{\rho}} + \partial_\rho e^{\rho\bar{\sigma}} \bar{\partial}_{\bar{\sigma}} e^{\mu\bar{\nu}} \big) \\ + \tfrac{1}{2} e_{\mu\bar{\nu}} f^\mu \bar{f}^{\bar{\nu}} - \tfrac{1}{2} f_\mu f^\mu \bar{e} + \tfrac{1}{2} \bar{f}_{\bar{\nu}} \bar{f}^{\bar{\nu}} e - \tfrac{1}{4} f^\mu \left(e_{\mu\bar{\nu}} \bar{\partial}^{\bar{\nu}} \bar{e} + \bar{\partial}^{\bar{\nu}} \left(e_{\mu\bar{\nu}} \bar{e} \right) \right) \\ - \tfrac{1}{4} \bar{f}^{\bar{\nu}} \left(e_{\mu\bar{\nu}} \partial^\mu e + \partial^\mu \left(e_{\mu\bar{\nu}} e \right) \right) + \tfrac{1}{4} f^\mu \left(\bar{e} \partial_\mu e - e \partial_\mu \bar{e} \right) + \tfrac{1}{4} \bar{f}^{\bar{\nu}} \left(\bar{e} \bar{\partial}_{\bar{\nu}} e - e \bar{\partial}_{\bar{\nu}} \bar{e} \right) \\ - \tfrac{1}{8} e_{\mu\bar{\nu}} \left(\bar{e} \partial^\mu \bar{\partial}^{\bar{\nu}} e + e \partial^\mu \bar{\partial}^{\bar{\nu}} \bar{e} - \partial^\mu e \bar{\partial}^{\bar{\nu}} \bar{e} - \bar{\partial}^{\bar{\nu}} e \partial^\mu \bar{e} \right) \big] \,. \end{aligned} \tag{2.50}$$

This cubic action combined with the free theory is invariant under a non-linear extension of the gauge transformations (2.49), which we construct explicitly in section 4. Expanding, for instance, the generalized metric formulation of DFT to cubic order in fields, one finds a cubic action that is equivalent to the above after integrating out the auxiliary fields and performing field redefinitions.

The free and cubic actions (2.47) and (2.50) coincide with the quadratic and cubic actions that Hull and Zwiebach constructed in [52] using string field theory methods for weakly constrained

DFT. It is important to stress that, even though strongly and weakly constrained DFT are two different theories, up to cubic order one can write the same action for both theories. The physical interpretation of the fields to this order, however, depends on the type of constraint. Moreover, the real technical differences between the strongly and weakly constrained theories start to arise beyond cubic order in perturbation theory. Indeed, as we will see in chapter 5, the two quartic theories are vastly different. This is due to fact that in weakly constraint DFT one has to introduce projectors to impose the weak constraint appropriately.

2.4 Double field theory as the double copy of Yang-Mills

As reviewed in section 2.2, double copy techniques are mostly used in the scattering amplitudes program, which is largely based on on-shell techniques that eliminate the need for an action principle and gauge redundancies. However, an off-shell approach to the double copy can be beneficial to develop an off-shell understanding of color-kinematics duality, which can elucidate aspects of the loop-level double copy and the classical double copy. As a result, even though it is far from obvious that an off-shell double copy prescription should work, numerous attempts to implement double copy prescriptions at the Lagrangian level have been made [73, 77, 86–90].

In order to successfully implement a Lagrangian double copy prescription, one usually needs to organize the perturbative expansion of gravity in a way that an index factorization property is manifest in order to have Feynman rules that can be factorized into two copies. This requires elaborate field redefinitions of the gravity fields [86, 91, 92]. This issue is immediately overcome in the framework of double field theory because its perturbative expansion exhibits a manifest index factorization property to all orders in perturbation theory inherited from string theory. (See [38, 93, 94] for double copy prescriptions of classical solutions in double field theory.)

This section is based on [1], where we show how a simple double copy prescription starting from the Yang-Mills action leads to double field theory. Given that the double copy instructs us to replace color factors with a second set of kinematic factors which come with their own momenta, this naturally leads to a double field theory with doubled momenta or, in position space, a doubled set of coordinates. Our approach is perturbative and here we will implement a color-kinematic substitution in the Yang-Mills action up to cubic order inspired by the BCJ double copy of amplitudes. We will see that at quadratic order our prescription leads to gauge invariant double field theory, while at cubic order this match requires a gauge choice, called Siegel gauge.

2.4.1 Quadratic theory

We start from the action for Yang-Mills theory in D dimensions,

$$S_{\rm YM} = -\frac{1}{4}\int d^Dx\, \kappa_{ab}\, F^{\mu\nu\, a} F_{\mu\nu}{}^{b}\ , \tag{2.51}$$

with the field strength for the gauge bosons ${A_\mu}^a$,

$$ {F_{\mu\nu}}^a = \partial_\mu {A_\nu}^a - \partial_\nu {A_\mu}^a + f^a{}_{bc} {A_\mu}^b {A_\nu}^c \ . \tag{2.52} $$

Here, $f^a{}_{bc}$ denote the structure constants of the color gauge group, with adjoint indices $a, b, \ldots$, and invariant Cartan-Killing form κ_{ab} that lowers adjoint indices, so that $f_{abc} \equiv \kappa_{ad} f^d{}_{bc}$ is totally antisymmetric. Expanding the Yang-Mills action to quadratic order in fields and integrating by parts one obtains

$$ S^{(2)}_{\rm YM} = \frac{1}{2} \int d^D x \, \kappa_{ab} \, A^{\mu a} \big(\Box {A_\mu}^b - \partial_\mu \partial^\nu {A_\nu}^b \big) \ . \tag{2.53} $$

In order to motivate the double copy prescription it is convenient to pass over to momentum space. Defining ${A_\mu}^a(k) \equiv \frac{1}{(2\pi)^{D/2}} \int d^D x {A_\mu}^a(x) e^{ikx}$ the quadratic action reads

$$ S^{(2)}_{\rm YM} = -\frac{1}{2} \int_k \kappa_{ab} \, k^2 \, \Pi^{\mu\nu}(k) {A_\mu}^a(-k) {A_\nu}^b(k) \ , \tag{2.54} $$

where $\int_k := \int d^D k$, and we have scaled out k^2, in order to define the projector in terms of the Minkowski metric $\eta_{\mu\nu}$

$$ \Pi^{\mu\nu}(k) \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \ , \tag{2.55} $$

which obeys the identities

$$ \Pi^{\mu\nu}(k) k_\nu \equiv 0 \ , \qquad \Pi^{\mu\nu} \Pi_{\nu\rho} \equiv \Pi^\mu{}_\rho \ . \tag{2.56} $$

The first identity implies gauge invariance under

$$ \delta {A_\mu}^a(k) = k_\mu \lambda^a(k) \ , \tag{2.57} $$

where the gauge parameter $\lambda^a(k)$ is an arbitrary function.

Let us now turn to the double copy construction of a gravity theory. We take the double copy prescription to lead to a double field theory: Replace the color indices a by a *second* set of spacetime indices denoted by a bar, $a \rightarrow \bar{\mu}$, corresponding to a *second* set of spacetime momenta $\bar{k}^{\bar{\mu}}$:

$$ {A_\mu}^a(k) \ \rightarrow \ e_{\mu\bar{\mu}}(k, \bar{k}) \ . \tag{2.58} $$

To complete the double copy prescription for the quadratic theory we need to define a substitution rule for the Cartan-Killing metric κ_{ab}. We will see that the following replacement does the job:

$$ \kappa_{ab} \ \rightarrow \ \tfrac{1}{2} \, \bar{\Pi}^{\bar{\mu}\bar{\nu}}(\bar{k}) \, , \tag{2.59} $$

where $\bar{\Pi}^{\bar{\mu}\bar{\nu}}$ is defined as in (2.55), but with all momenta replaced by barred momenta and all indices replaced by barred indices. This prescription is motivated from the double copy rule at the level of amplitudes: for a gauge theory amplitude $\mathcal{A} = \sum_i \frac{n_i c_i}{D_i}$, where n_i are kinematic factors, c_i are color factors, and the D_i are the inverse propagators, the double copy amounts to replacing c_i by kinematic factors n_i, while the $D_i \sim k^2$ are untouched. Thus, it is natural to scale out k^2 from the kinetic operator and to double only the resulting projector $\Pi^{\mu\nu}$, which also guarantees that a two-derivative theory is mapped to a two-derivative theory.

The quadratic gravity action following from this double copy (DC) prescription then reads

$$S^{(2)}_{\rm DC} = -\frac{1}{4}\int_{k,\bar{k}} k^2\,\Pi^{\mu\nu}(k)\bar{\Pi}^{\bar{\mu}\bar{\nu}}(\bar{k})e_{\mu\bar{\mu}}(-k,-\bar{k})e_{\nu\bar{\nu}}(k,\bar{k})\,. \tag{2.60}$$

We emphasize that the field $e_{\mu\bar{\mu}}$ now depends on doubled momenta $K \equiv (k,\bar{k})$. Note that the momenta k and $\bar{k}$ enter the action on the same footing, except that we have chosen the factor in front to be k^2 rather than $\bar{k}^2$, but in double field theory this asymmetry is resolved due to the strong constraint which in momentum space reads

$$k^2 = \bar{k}^2\ . \tag{2.61}$$

In order to match with double field theory, and also to lead to a local action, we thus have to assume that the doubled momenta are subject to this constraint (which does have more general solutions than the trivial $k=\bar{k}$ for which the theory reduces to a standard linearized gravity theory). We also note that, owing to the first identity in (2.56), the action is manifestly gauge invariant under

$$\delta e_{\mu\bar{\nu}} = k_\mu\bar{\lambda}_{\bar{\nu}} + \bar{k}_{\bar{\nu}}\lambda_\mu\ , \tag{2.62}$$

with two independent gauge parameters λ_μ and $\bar{\lambda}_{\bar{\mu}}$ that depend on doubled momenta $K\equiv(k,\bar{k})$, subject to (2.61).

We will now show that (2.60) is indeed equivalent to (quadratic) double field theory. Writing out the projectors with (2.55) and using the level-matching constraint (2.61) the action reads

$$S^{(2)}_{\rm DC} = -\frac{1}{4}\int_{k,\bar{k}}\Big(k^2 e^{\mu\bar{\nu}}e_{\mu\bar{\nu}} - k^\mu k^\rho e_{\mu\bar{\nu}}e_\rho{}^{\bar{\nu}} - \bar{k}^{\bar{\nu}}\bar{k}^{\bar{\sigma}}e_{\mu\bar{\nu}}e^\mu{}_{\bar{\sigma}} + \frac{1}{k^2}k^\mu k^\rho\bar{k}^{\bar{\nu}}\bar{k}^{\bar{\sigma}}e_{\mu\bar{\nu}}e_{\rho\bar{\sigma}}\Big)\,.$$

In order to compare with the standard double field theory action we have to Fourier transform to (doubled) position space. This is straightforward except for the last term in (2.63), which due to the factor $\frac{1}{k^2}$ would yield a non-local term. This problem is resolved by introducing an auxiliary scalar field $d(k,\bar{k})$ (the dilaton):

$$S^{(2)}_{\rm DC} = -\frac{1}{4}\int_{k,\bar{k}}\Big(k^2 e^{\mu\bar{\nu}}e_{\mu\bar{\nu}} - k^\mu k^\rho e_{\mu\bar{\nu}}e_\rho{}^{\bar{\nu}} - \bar{k}^{\bar{\nu}}\bar{k}^{\bar{\sigma}}e_{\mu\bar{\nu}}e^\mu{}_{\bar{\sigma}} - k^2d^2 + 2\,d\,k^\mu\bar{k}^{\bar{\nu}}e_{\mu\bar{\nu}}\Big)\,.$$

Integrating out d by solving its own field equations,

$$d = \frac{1}{k^2}k^\mu\bar{k}^{\bar{\nu}}e_{\mu\bar{\nu}}\ , \tag{2.63}$$

and back-substituting into the action we recover the non-local (2.63). Alternatively, without integrating out fields, one may redefine the dilaton as $d\to d' = d - \frac{1}{k^2}k^\mu\bar{k}^{\bar{\nu}}e_{\mu\bar{\nu}}$, which decouples d' from $e_{\mu\bar{\nu}}$. The action (2.63) is of course still gauge invariant, with a gauge transformation for d that is determined by the variation of (2.63):

$$\delta d = k_\mu\lambda^\mu + \bar{k}_{\bar{\mu}}\bar{\lambda}^{\bar{\mu}}\ , \tag{2.64}$$

where we used (2.61). With the action in the form (2.63) it is then straightforward to Fourier transform to a local action in doubled position space:

$$S^{(2)}_{\rm DC} = \frac{1}{4}\int d^Dx\, d^D\bar{x}\Big(e^{\mu\bar{\nu}}\Box e_{\mu\bar{\nu}} + \partial^\mu e_{\mu\bar{\nu}}\,\partial^\rho e_\rho{}^{\bar{\nu}} + \bar{\partial}^{\bar{\nu}} e_{\mu\bar{\nu}}\,\bar{\partial}^{\bar{\sigma}} e^\mu{}_{\bar{\sigma}} - d\Box d + 2d\partial^\mu\bar{\partial}^{\bar{\nu}} e_{\mu\bar{\nu}}\Big), \tag{2.65}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\bar{\partial}_{\bar{\mu}} = \frac{\partial}{\partial \bar{x}^{\bar{\mu}}}$ are the partial derivatives corresponding to the coordinates that are dual to k^μ and $\bar{k}^{\bar{\mu}}$ and hence by (2.61) subject to the constraint

$$\Box \equiv \partial^\mu\partial_\mu = \bar{\partial}^{\bar{\mu}}\bar{\partial}_{\bar{\mu}}\ . \tag{2.66}$$

The gauge transformations (2.62) and (2.64) translate in doubled position space to

$$\begin{aligned} \delta e_{\mu\bar{\nu}} &= \partial_\mu\bar{\lambda}_{\bar{\nu}} + \bar{\partial}_{\bar{\nu}}\lambda_\mu\ , \\ \delta d &= \partial_\mu\lambda^\mu + \bar{\partial}_{\bar{\mu}}\bar{\lambda}^{\bar{\mu}}\ , \end{aligned} \tag{2.67}$$

under which (2.65) is invariant, upon imposing the constraint (2.66). The action (2.65) coincides precisely with the standard quadratic double field theory action (2.40), which upon setting $x = \bar{x}$ is equivalent, up to field redefinitions, to the familiar free action for gravity, antisymmetric tensor and dilaton, as we proved in the previous section.

2.4.2 Cubic theory

We now turn to the cubic vertex of Yang-Mills theory and extend the double copy construction to the cubic action of double field theory. The cubic part of the Yang-Mills action (2.51) reads

$$S^{(3)}_{\rm YM} = -\int d^Dx\, f_{abc}\,\partial^\mu A^{\nu a}\, A_\mu{}^b\, A_\nu{}^c\ . \tag{2.68}$$

Upon Fourier transforming to momentum space this becomes

$$S^{(3)}_{\rm YM} = \frac{i}{(2\pi)^{D/2}}\int_{k_1,k_2,k_3}\delta(k_1+k_2+k_3)\, f_{abc}\, k_1^\mu A_1^\nu{}^a A_{2\mu}{}^b A_{3\nu}{}^c\,, \tag{2.69}$$

where we use the short-hand notation $A_i \equiv A(k_i)$, and we performed the x-integration, introducing the delta function. It is convenient to write this more symmetrically as

$$S^{(3)}_{\rm YM} = -\frac{i}{6(2\pi)^{D/2}}\int_{k_1,k_2,k_3}\delta(k_1+k_2+k_3)\, f_{abc}\,\Pi^{\mu\nu\rho}(k_1,k_2,k_3)\, A_{1\mu}{}^a A_{2\nu}{}^b A_{3\rho}{}^c\,, \tag{2.70}$$

where we defined

$$\Pi^{\mu\nu\rho}(k_1,k_2,k_3) \equiv \eta^{\mu\nu}k_{12}^\rho + \eta^{\nu\rho}k_{23}^\mu + \eta^{\rho\mu}k_{31}^\nu\,, \tag{2.71}$$

with $k_{ij} \equiv k_i - k_j$. Note that this tensor has the anti-symmetry properties required by the structure it multiplies, e.g., $\Pi^{\mu\nu\rho}(k_1,k_2,k_3) = -\Pi^{\nu\mu\rho}(k_2,k_1,k_3)$.

Our task now is to give the double copy prescription that extends (2.58), (2.59) to the cubic theory. The natural substitution rule is

$$f_{abc} \;\to\; \tfrac{i}{4}\,\bar{\Pi}^{\bar{\mu}\bar{\nu}\bar{\rho}}(\bar{k}_1,\bar{k}_2,\bar{k}_3)\;, \tag{2.72}$$

where the factor of i is needed since we relate a theory with one derivative to a theory with two derivatives. This gives the cubic action

$$S^{(3)}_{\rm DC} = \tfrac{1}{48(2\pi)^{D/2}}\int dK_1 dK_2 dK_3\,\delta(K_1+K_2+K_3)\,\bar{\Pi}^{\bar{\mu}\bar{\nu}\bar{\rho}}(\bar{k}_1,\bar{k}_2,\bar{k}_3)\,\Pi^{\mu\nu\rho}(k_1,k_2,k_3)\,e_{1\,\mu\bar{\mu}}\,e_{2\,\nu\bar{\nu}}\,e_{3\,\rho\bar{\rho}}\,,$$

where we use the short-hand notation $e_{i\,\mu\bar{\mu}} \equiv e_{\mu\bar{\mu}}(K_i)$, with $K \equiv (k,\bar{k})$ for doubled momenta, and $dK \equiv d^{2D}K$. Writing out $\Pi^{\mu\nu\rho}$ and $\bar{\Pi}^{\bar{\mu}\bar{\nu}\bar{\rho}}$ yields nine terms which, upon relabeling momentum variables and indices, reduce to two terms, and then writing out $k_{ij} = k_i - k_j$ the action becomes

$$\begin{aligned} S^{(3)}_{\rm DC} = \tfrac{1}{8(2\pi)^{D/2}} &\int dK_1 dK_2 dK_3\,\delta(K_1+K_2+K_3) \\ &\times\, e_{1\,\mu\bar{\mu}}\Big[-k_2^{\mu}\,e_{2\,\rho\bar{\rho}}\,\bar{k}_3^{\bar{\mu}}\,e_3^{\rho\bar{\rho}} + k_2^{\mu}\,e_{2\,\nu\bar{\rho}}\,\bar{k}_3^{\bar{\rho}}\,e_3^{\nu\bar{\mu}} + k_2^{\rho}\,e_2^{\mu\bar{\rho}}\,\bar{k}_3^{\bar{\mu}}\,e_{3\,\rho\bar{\rho}} + k_2^{\mu}\,\bar{k}_2^{\bar{\mu}}\,e_{2\,\rho\bar{\rho}}\,e_3^{\rho\bar{\rho}} \\ &- k_{2\,\rho}\,e_2^{\mu\bar{\rho}}\,\bar{k}_{3\,\bar{\rho}}\,e_3^{\rho\bar{\mu}} - k_2^{\rho}\,\bar{k}_2^{\bar{\mu}}\,e_2^{\mu\bar{\rho}}\,e_{3\rho\bar{\rho}}\Big]\,. \end{aligned}$$

Fourier transforming to position space and integrating by parts, we finally obtain

$$\begin{aligned} S^{(3)}_{\rm DC} = \frac{1}{8}\int d^Dx\,d^D\bar{x}\;e_{\mu\bar{\mu}}\Big[&2\partial^{\mu}e_{\rho\bar{\rho}}\,\bar{\partial}^{\bar{\mu}}e^{\rho\bar{\rho}} - 2\partial^{\mu}e_{\nu\bar{\rho}}\,\bar{\partial}^{\bar{\rho}}e^{\nu\bar{\mu}} - 2\partial^{\rho}e^{\mu\bar{\rho}}\,\bar{\partial}^{\bar{\mu}}e_{\rho\bar{\rho}} \\ &+ \partial^{\rho}e_{\rho\bar{\rho}}\,\bar{\partial}^{\bar{\rho}}e^{\mu\bar{\mu}} + \bar{\partial}_{\bar{\rho}}e^{\mu\bar{\rho}}\,\partial_{\rho}e^{\rho\bar{\mu}}\Big]\,, \end{aligned} \tag{2.73}$$

which is precisely the original action by Hull and Zwiebach (2.50) introduced in the previous section, after setting the auxiliary fields f_{μ} and $\bar{f}_{\bar{\mu}}$ to zero. This corresponds to a gauge condition called Siegel gauge which will play an important role when we turn to scattering amplitudes in chapter 4. Notice, however, that our double copy prescription does not yield any interactions where the scalars e and $\bar{e}$ appear. Indeed, in Siegel gauge, the original action of Hull and Zwiebach to cubic order reads

$$S_{\rm H.Z.} = \int d^Dx\,d^D\bar{x}\,\Big[\frac{1}{4}e_{\mu\bar{\mu}}\Box e^{\mu\bar{\mu}} + 2\bar{e}\Box e + \mathcal{L}^{(3)}_{\rm DC} - \frac{1}{8}e_{\mu\bar{\nu}}\big(\bar{e}\partial^{\mu}\bar{\partial}^{\bar{\nu}}e + e\partial^{\mu}\bar{\partial}^{\bar{\nu}}\bar{e} - \partial^{\mu}e\bar{\partial}^{\bar{\nu}}\bar{e} - \bar{\partial}^{\bar{\nu}}e\partial^{\mu}\bar{e}\big)\Big]. \tag{2.74}$$

For the quadratic theory we had to integrate out the dilaton $d \equiv 2\,(e-\bar{e})$ from double field theory in order to show that it equals the double copy of Yang-Mills. A subtlety at this stage is that, after picking Siegel gauge, it is no longer true that only the combination $d \equiv 2\,(e-\bar{e})$ enters the action. Thus, we have to integrate out the pair of fields $(e,\bar{e})$. Since these enter the action (2.74) only quadratically, at tree-level integrating them out just amounts to setting $e = \bar{e} = 0$.[9] Thus, cubic double field theory in Siegel gauge precisely coincides with the double copy of Yang-Mills theory upon integrating out the scalar fields.

The procedure outlined in this section is arguably the most direct double copy prescription

[9]Indeed, there are no tree-level diagrams for only external $e_{\mu\bar{\mu}}$ states that involve these fields and hence setting them to zero is the correct procedure of integrating them out at tree-level.

for the Yang-Mills action to cubic order. Albeit encouraging, the cubic theory that follows from this prescription requires a gauge choice, thus rendering the result not general enough for our purposes. For this reason we have to resort to more sophisticated methods. In the remainder of the thesis we will use the framework of homotopy algebras which will allow us to find a local double copy prescription that does not require a gauge condition. In the next chapter we will introduce these algebraic structures and state their relation to the double copy.

Chapter 3

Homotopy algebras and field theories

In this chapter we introduce the mathematical structures central to this thesis: homotopy algebras. These algebras are generalizations of more familiar algebras (at least to theoretical physicists), such as Lie algebras and commutative algebras. We start this chapter with the definition of L_∞-algebras in section 3.1 [95–97]. In section 3.2 we explain that these algebras are the mathematical structures underlying perturbative field theories in that they encode their gauge structure, together with dynamics and consistent interactions. Then, as concrete examples relevant for the double copy program, in sections 3.3 and 3.4 we formulate, respectively, Chern-Simons theory and Yang-Mills theory in this algebraic framework. In addition, we show how the L_∞-algebras of these two theories factorize into a color part and a kinematic part. In both theories, color is encoded in a finite dimensional Lie algebra $\mathfrak{g}$ while kinematics is encoded in a so-called $\mathrm{BV}_\infty^\square$-algebra. These exotic algebras were first constructed by Reiterer in [60] and, at least up to and including quartic interactions, they are the mathematical structures responsible for the consistency of the double copy, as we will show in the following chapter.

This chapter is largely based on [2,3] and the upcoming paper [5] and parts of some sections are taken from these references.

3.1 L_∞-algebras

L_∞-algebras, also known as strongly homotopy Lie algebras, are generalizations of Lie algebras that encode the information of perturbative field theories by means of a (possibly infinite) set of multilinear maps acting on a graded vector space that obey a (possibly infinite) set of relations that generalize the Jacobi identity. In this section we introduce L_∞-algebras from a mathematical perspective. This will set the stage to formulate field theories in this framework in the following section.

Given that Lie algebras are a particular case of L_∞-algebras, let us start our discussion with the definition of a Lie algebra: A Lie algebra is a vector space V equipped with an antisymmetric bilinear map (or Lie bracket) $[\cdot,\cdot] : V \times V \to V$ that obeys the Jacobi identity

$$[[u,v],w] + [[v,w],u] + [[w,u],v] = 0 \ , \text{with } u,v,w \in V \ . \tag{3.1}$$

From this definition one can conclude that a Lie algebra has three main constituents:

- A vector space (V).
- An antisymmetric map ($[\cdot,\cdot]$).
- A relation obeyed by the map (Jacobi identity).

Similarly, L_∞-algebras, albeit in a more general sense, have the same number of constituents. More precisely, an L_∞-algebra is an integer graded vector space $\mathcal{X} = \bigoplus_i X_i$ equipped with a set of multilinear graded symmetric maps (or brackets) $B_n : \mathcal{X}^{\otimes n} \to \mathcal{X}$ that obey the so-called generalized Jacobi identities. In the following we explain in detail all the concepts introduced in this definition.

- **Graded vector space**

An integer graded vector space $\mathcal{X}$ is a vector space that is constructed as the direct sum of smaller vector spaces X_i, i.e

$$\mathcal{X} = \bigoplus_i X_i \ , \quad i \in \mathbb{Z} \ . \tag{3.2}$$

The integer number i that labels each vector space is called the *degree* of the vector space. This degree is inherited by the elements of each vector space $x_i \in X_i$ and we denote it by $|x_i| = i$. We say that $x_i \in X_i$ has degree i.

- **Graded symmetric maps B_n**

In an L_∞-algebra, the vector space $\mathcal{X}$ is equipped with (possibly infinitely many) multilinear maps $B_n : \mathcal{X}^{\otimes n} \to \mathcal{X}$ for $n > 0$. These take n elements of $\mathcal{X}$ and give back one element of $\mathcal{X}$. We call these maps n-brackets, where n is the number of inputs that they take. In a Lie algebra the Lie bracket $[\cdot,\cdot]$ is antisymmetric. The maps B_n, on the other hand, are graded symmetric. This means that given two adjacent inputs $x_i \in X_i$ and $x_j \in X_j$ of B_n, we can exchange their order inside of the bracket at the cost of a sign determined by their degree, i.e

$$B_n(..., x_i, x_j, ...) = (-1)^{|x_i||x_j|} \, B_n(..., x_j, x_i, ...) \ . \tag{3.3}$$

In order to simplify our notation, from now on when we refer to the degrees of the inputs in the exponent of phase factors we will omit the absolute value sign and simply write

$$B_n(..., x_i, x_j, ...) = (-1)^{x_i x_j} \, B_n(..., x_j, x_i, ...) \ . \tag{3.4}$$

In addition, we assign a degree to the multilinear maps. In our conventions they have degree one, i.e $|B_n| = 1$. In order to find the degree of the output of the B_n, one simply sums the degrees of all the inputs plus the degree of B_n as

$$|B_n(x_1, x_2, ..., x_n)| = \sum_{i=1}^{n} |x_i| + |B_n| = \sum_{i=1}^{n} |x_i| + 1 \ . \tag{3.5}$$

- L_∞ **relations**

The final constituent of the definition of an L_∞-algebra is the set of relations between the maps B_n that generalize the Jacobi identity. These (possibly infinite) relations are called generalized Jacobi identities or L_∞ relations and they read

$$\sum_{i+j=l+1} \sum_{\sigma} \epsilon(\sigma;x) B_j(B_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), x_{\sigma(i+1)}, ..., x_{\sigma(l)}) = 0 \ , x_i \in \mathcal{X} \ , \tag{3.6}$$

where we restrict the sum over the permutations σ to *unshuffles*, meaning that the arguments of the brackets are partially ordered as

$$\sigma(1) < ... < \sigma(i) \ , \quad \sigma(i+1) < ... < \sigma(l) \ , \tag{3.7}$$

and $\epsilon(\sigma;x)$ is the Koszul sign which depends on the degrees of the inputs x_i. The Koszul sign is the sign that one obtains upon permuting adjacent elements of an ordered list

$$(x_1, x_2, ...x_n) = \epsilon(\sigma;x)\,(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(l)}) \ . \tag{3.8}$$

As an example, consider a list of three elements (x_1, x_2, x_3) of $\mathcal{X}$. One of the permutations of this list is (x_1, x_3, x_2). The Koszul sign associated to such a permutation is determined by the relation

$$(x_1, x_2, x_3) = (-1)^{x_2 x_3}\,(x_1, x_3, x_2) \ . \tag{3.9}$$

There is an L_∞ relation for every value of l. The first three read

$$\begin{aligned}
l = 1: \quad & B_1(B_1(x)) = 0 \ , \\
l = 2: \quad & B_1 B_2(x_1, x_2) + B_2(B_1(x_1), x_2) + (-1)^{x_1 x_2} B_2(B_1(x_2), x_1) = 0 \ , \\
l = 3: \quad & B_2(B_2(x_1, x_2), x_3) + (-1)^{x_2 x_3} B_2(B_2(x_1, x_3), x_2) + (-1)^{x_1(x_2+x_3)} B_2(B_2(x_2, x_3), x_1) \\
& + B_1 B_3(x_1, x_2, x_3) + B_3(B_1(x_1), x_2, x_3) + (-1)^{x_1 x_2} B_3(B_1(x_2), x_1, x_3) \\
& + (-1)^{x_3(x_1+x_2)} B_3(B_1(x_3), x_1, x_2) = 0 \ .
\end{aligned} \tag{3.10}$$

The following relation ($l = 4$) reads, schematically, $B_3 B_2 + B_3 B_2 + B_1 B_4 + B_4 B_1 = 0$ and so on, and the number of non-trivial relations depends on the number of non-trivial brackets.

The first relation in (3.10) ($l = 1$) states that the map B_1, which we call the *differential*, is nilpotent. The second relation ($l = 2$) is the Leibniz rule of the differential B_1 with respect to the two-bracket B_2. Finally, and perhaps most interestingly, the third relation ($l = 3$) is the so-called homotopy Jacobi identity. This relation quantifies the failure of the two-bracket B_2 to be a (graded) Lie bracket, and tells us that this failure is governed by the differential and the three-bracket B_3. In the language of homological algebra, B_3 is called a *homotopy*, and one then says that the Jacobi identity holds up to homotopy. For this reason, these algebras are named strongly homotopy Lie algebra.

These algebras are the mathematical structures underlying perturbative field theories and gauge symmetries. As we will see, each of the three constituents of an L_∞-algebra plays an important role in field theories. For instance, we assign a vector space to each ingredient of our theory: we say that the fields live in the vector space X_0, while the gauge parameters live in the space X_{-1}. The n-brackets represent the interactions between the fields and the gauge transformations, and finally, the L_∞ relations encode the consistency of the theory, namely gauge covariance of the field equations, Noether identities, closure of the gauge algebra and so on.

3.2 Field theories in the language of L_∞-algebras

We now turn to formulating field theories in the language of L_∞-algebras. We start this section with a brief review of general Lagrangian perturbative field theories and their gauge structure. Subsequently we interpret all the elements and consistency conditions of the field theory in the framework of L_∞-algebras. In order to illustrate how the consistency of the theory is encoded in the L_∞ relations, we will perform some explicit computations up to trilinear order which involve the three L_∞ relations shown in equation (3.10). Nevertheless, it is important to emphasize that one can extend the analysis that we will show here to all orders in perturbation theory (see for instance [97, 98]).

We consider a Lagrangian perturbative field theory with fields collectively denoted by ψ^i and action functional $S[\psi]$. Using the variational principle we find the field equations $\mathcal{E}_i$ that describe the dynamics of ψ^i by taking a functional derivative of the action as

$$\mathcal{E}_i := \frac{\delta S[\psi]}{\delta \psi^i} \ . \tag{3.11}$$

The theory is gauge invariant if the action functional $S[\psi]$ is invariant under local infinitesimal transformations generated by gauge parameters Λ, i.e for a particular parameter Λ_1 we have

$$0 = \delta_{\Lambda_1} S[\psi] = \int dx \, \mathcal{E}_i \, \delta_{\Lambda_1} \psi^i \ , \tag{3.12}$$

which translates into gauge covariance of the field equations. Gauge invariance of the action implies the existence of relations between the field equations called Noether (or Bianchi) identities $\mathcal{N}$, which are a manifestation of the redundancy that follows from gauge symmetry.

In physics, whenever a theory is invariant under a gauge transformation, there is an underlying gauge algebra that encodes the symmetry and its consistency. We now turn to examining the gauge structure of general field theories. In order to probe the gauge algebra we act with a second gauge transformation with parameter Λ_2 on the variation of the action (3.12) leading to

$$0 = \delta_{\Lambda_2}\delta_{\Lambda_1} S[\psi] = \int dx \left\{ \frac{\delta^2 S}{\delta\psi^j \delta\psi^i} \, \delta_{\Lambda_2}\psi^j \, \delta_{\Lambda_1}\psi^i + \mathcal{E}_i \, \delta_{\Lambda_2}\delta_{\Lambda_1}\psi^i \right\} . \tag{3.13}$$

From this equation it is obvious to conclude that two consecutive infinitesimal gauge transformations do not constitute another gauge transformation because of the first term of the last equality which has two functional derivatives acting on the action. Nevertheless, we can antisymmetrize equation (3.13) in the labels of the gauge parameters, thus eliminating the first term and yielding the commutator of gauge variations

$$0 = \left\{\delta_{\Lambda_2}\delta_{\Lambda_1} - \delta_{\Lambda_1}\delta_{\Lambda_2}\right\} S[\psi] = \int dx\, \mathcal{E}_i\, [\delta_{\Lambda_2}, \delta_{\Lambda_1}]\psi^i \ . \tag{3.14}$$

The commutator gives rise to a bracket $[\cdot, \cdot]$ that combines Λ_1 and Λ_2, and possibly the fields, into a single gauge parameter $\Lambda_{12} \equiv -[\Lambda_1, \Lambda_2]$. This bracket defines the *gauge algebra* of the theory, and consistency requires the *closure* condition

$$\int dx\, \mathcal{E}_i\, [\delta_{\Lambda_2}, \delta_{\Lambda_1}]\psi^i = \int dx\, \mathcal{E}_i\, \delta_{-[\Lambda_1,\Lambda_2]}\psi^i \ . \tag{3.15}$$

Alternatively we can write the unintegrated version

$$[\delta_{\Lambda_2}, \delta_{\Lambda_1}]\psi^i = \delta_{-[\Lambda_1,\Lambda_2]}\psi^i + \mu^{ij}\, \mathcal{E}_j \ , \tag{3.16}$$

where μ^{ij} is antisymmetric. The last term in the right hand side accounts for equation of motion symmetries.

Since the commutator obeys the Jacobi identity, the *Jacobiator* of the gauge variations has to vanish when acting on fields, namely

$$\mathrm{Jac}(\delta_{\Lambda_1}, \delta_{\Lambda_2}, \delta_{\Lambda_3})\, \psi^i = \left\{[[\delta_{\Lambda_1}, \delta_{\Lambda_2}], \delta_{\Lambda_3}] + [[\delta_{\Lambda_2}, \delta_{\Lambda_3}], \delta_{\Lambda_1}] + [[\delta_{\Lambda_3}, \delta_{\Lambda_1}], \delta_{\Lambda_2}]\right\} \psi^i = 0 \ . \tag{3.17}$$

Thus, one would naively think that the bracket between gauge parameters $[\Lambda_1, \Lambda_2]$ also obeys the Jacobi identity. Nevertheless, there is a loophole in arriving to this conclusion. There may exist choices of gauge parameters that leave the fields invariant. This reflects a redundancy in the gauge parameters sometimes referred to as a *gauge-for-gauge* symmetry generated by a *gauge-for-gague parameter* χ, which in turn implies *Noether-for-Noether identities* $\mathcal{B}$. We call these symmetries *reducible*. Reducibility means that there exist *trivial* gauge parameters $\Lambda(\chi)$ that do not generate a gauge variation of the fields, namely

$$\delta_{\Lambda(\chi)}\psi^i = 0 \ . \tag{3.18}$$

Then, in order to account for reducibility, the bracket between gauge parameters obeys the following relation:

$$[[\Lambda_1, \Lambda_2], \Lambda_3] + [[\Lambda_2, \Lambda_3], \Lambda_1] + [[\Lambda_3, \Lambda_1], \Lambda_2] + \Lambda(\chi) = 0 \ , \tag{3.19}$$

This is the defining relation of the gauge algebra and it is a deformation of the Jacobi identity of the bracket $[\cdot, \cdot]$ by an additional term $\Lambda(\chi)$ which, as we will see in the following, has an L_∞ interpretation. In most theories familiar to theoretical physicists the gauge algebra is a Lie algebra, i.e there is no additional term $\Lambda(\chi)$ and thus $[\cdot, \cdot]$ obeys the Jacobi identity. For example, in Yang-Mills theory and pure Einstein-Hilbert gravity, the closure of the gauge symmetries determines a field independent Lie bracket which obeys the Jacobi identity. For other theories, such as double field theory, the closure condition (3.16) and the Jacobi identity

(3.19) imply the existence of higher (than two-) brackets which form an L_∞-algebra.

The mathematical structure underlying a field theory with the above characteristics is an L_∞-algebra. The first step to formulate a field theory in L_∞-form is to consider a graded vector space $\mathcal{X}$, where we assign a vector space $X_i \subset \mathcal{X}$ to each one of the ingredients of the theory. We can illustrate this using a *chain complex*

$$\begin{array}{ccccccccccccccc} ... & \xrightarrow{B_1} & X_{-2} & \xrightarrow{B_1} & X_{-1} & \xrightarrow{B_1} & X_0 & \xrightarrow{B_1} & X_1 & \xrightarrow{B_1} & X_2 & \xrightarrow{B_1} & X_3 & \xrightarrow{B_1} & ... \\ & & \chi & & \Lambda & & \psi & & \mathcal{E} & & \mathcal{N} & & \mathcal{B} & & \end{array} \quad . \tag{3.20}$$

We say that the gauge parameters have degree one, the fields degree zero, etc., and the ellipses next to the first and last arrows indicate that there may be higher order reducibility, indicating the existence of gauge-for-gauge-for-gague parameters and so on. In this thesis, however, we will only consider theories that have up to first order reducibility which means that we truncate the chain complex so that $\mathcal{X} = \bigoplus_{i=-2}^{3} X_i$ for DFT, and $\mathcal{X} = \bigoplus_{i=-1}^{2} X_i$ for Yang-Mills. Moreover, in the chain complex above we introduced a differential B_1 which contains information of the free theory and has to be nilpotent according to the L_∞ relations.

The next task in the formulation of field theories in the framework of L_∞-algebras is to read off the action of the differential B_1 and all other higher brackets on the different elements of $\mathcal{X}$. To that end, we assume that the field equations, gauge transformations and trivial gauge parameters can be written perturbatively as

$$\begin{aligned} \mathcal{E}(\psi) :&= B_1(\psi) + \tfrac{1}{2!}B_2(\psi,\psi) + \tfrac{1}{3!}B_3(\psi,\psi,\psi) + ... = 0 \ \in X_1 \ , \\ \delta_\Lambda \psi &= B_1(\Lambda) + B_2(\Lambda,\psi) + \tfrac{1}{2}\, B_3(\Lambda,\psi,\psi) + ... \ \in X_0 \ , \\ \Lambda_{\text{Trivial}}(\chi) &= B_1(\chi) + B_2(\chi,\psi) + B_3(\chi,\psi,\psi) ... \ \in X_{-1} \ , \end{aligned} \tag{3.21}$$

where the ellipses denote possible higher terms in perturbation theory, and we explicitly wrote the ψ dependence in the field equation $\mathcal{E}(\psi)$ to specify that we are referring to the field equations of ψ and not an arbitrary element of X_1. Then, by simple comparison with the field equations and gauge transformations of the theory, one can read off the action of the differential and higher brackets on fields, and on one gauge parameter and multiple fields, and so on. In Yang-Mills theory, for example, the field equations have up to trilinear terms and hence there exists only a B_3 that acts on three fields. In addition, the only non-linear contribution to the Yang-Mills gauge transformations is $B_2(\Lambda, \psi)$. In pure perturbative gravity, in contrast, while the gauge transformations also have up to bilinear terms, the field equations have an infinite number of higher order corrections, meaning that there exists an infinite number of brackets between fields.

We now turn to showing the relationship between the L_∞ relations and the consistency of the field theory. First, consistency requires that the field equations be covariant under gauge transformations. Let us explicitly prove this statement perturbatively up to trilinear order using the L_∞ relations. Performing a gauge transformation of the field equations leads to

$$\delta_\Lambda \mathcal{E}(\psi) = B_1(\delta_\Lambda \psi) + B_2(\delta_\Lambda \psi, \psi) + \tfrac{1}{2}\, B_3(\delta_\Lambda \psi, \psi, \psi) \ . \tag{3.22}$$

Substituting the gauge variation $\delta_\Lambda\psi$ by its expression in terms of the brackets displayed in equation (3.21) yields to trilinear order

$$\begin{aligned}\delta_\Lambda\mathcal{E}(\psi) = {} & B_1(B_1(\Lambda)) \\ & + B_1B_2(\Lambda,\psi) + B_2(B_1(\Lambda),\psi) \\ & + B_2(B_2(\Lambda,\psi),\psi) + \tfrac{1}{2}\,B_1B_3(\Lambda,\psi,\psi) + \tfrac{1}{2}B_3(B_1(\Lambda),\psi,\psi)\ .\end{aligned} \tag{3.23}$$

Assuming that the brackets that define our theory obey the L_∞ relations, we can use those relations to reorganize the above expression. The first line vanishes due to the nilpotency of the differential. In the second and third lines we can use, respectively, the Leibniz relation and the homotopy Jacobi relation

$$\begin{aligned}B_1B_2(\Lambda,\psi) + B_2(B_1(\Lambda),\psi) - B_2(\Lambda,B_1(\psi)) = 0\ ,\\ B_2(B_2(\psi,\psi),\Lambda) + 2\,B_2(B_2(\Lambda,\psi),\psi) + B_1B_3(\Lambda,\psi,\psi)\\ + B_3(B_1(\Lambda),\psi,\psi) - 2\,B_3(\Lambda,\psi,B_1(\psi)) = 0\ ,\end{aligned} \tag{3.24}$$

leading to

$$\begin{aligned}\delta_\Lambda\mathcal{E}(\psi) &= B_2(\Lambda,B_1(\psi)) + \tfrac{1}{2}\,B_2(\Lambda,B_2(\psi,\psi)) + B_3(\Lambda,\psi,B_1(\psi))\\ &= B_2(\Lambda,\mathcal{E}(\psi)) + \tfrac{1}{2}\,B_3(\Lambda,\mathcal{E}(\psi),\psi) + \tfrac{1}{2}\,B_3(\Lambda,\psi,\mathcal{E}(\psi))\ .\end{aligned} \tag{3.25}$$

This equation, besides proving that the field equations are covariant under gauge transformations, determines the brackets between field equations with gauge parameters, and fields with field equations and gauge parameters. This computation shows that the generalized Jacobi identities are needed to prove gauge covariance of the field equations.

We now turn to the Noether identities. Given the brackets in the field equations (3.21), one can introduce a pairing $\langle\cdot,\cdot\rangle : X_i \times X_{-i+1} \to \mathbb{R}$ that allows one to construct an action as

$$S[\psi] = \tfrac{1}{2!}\,\langle\psi,B_1(\psi)\rangle + \tfrac{1}{3!}\,\langle\psi,B_2(\psi,\psi)\rangle + \tfrac{1}{4!}\,\langle\psi,B_3(\psi,\psi,\psi)\rangle + \ldots\ . \tag{3.26}$$

The pairing $\langle\cdot,\cdot\rangle$ has the symmetry properties

$$\langle x_1,x_2\rangle = (-1)^{(x_1+1)(x_2+1)}\,\langle x_2,x_1\rangle\ ,\quad x_1,x_2\in\mathcal{X}\ , \tag{3.27}$$

and

$$\langle x_i,B_n(\ldots,x_j,\ldots)\rangle = (-1)^{x_ix_j}\,\langle x_j,B_n(\ldots,x_i,\ldots)\rangle\ . \tag{3.28}$$

An L_∞-algebra equipped with a pairing $\langle\cdot,\cdot\rangle$ obeying the above properties is called *cyclic.* A gauge transformation of the action yields

$$\begin{aligned}0 = \delta_\Lambda S[\psi] &= \langle\delta_\Lambda\psi,\mathcal{E}(\psi)\rangle\\ &= \langle B_1(\Lambda) + B_2(\Lambda,\psi) + \tfrac{1}{2}\,B_3(\Lambda,\psi,\psi) + \ldots,\mathcal{E}(\psi)\rangle\ .\end{aligned} \tag{3.29}$$

We can isolate the gauge parameter using the property (3.28) to obtain

$$0 = \delta_\Lambda S[\psi] = -\langle\Lambda, B_1(\mathcal{E}(\psi)) + B_2(\mathcal{E}(\psi),\psi) + \tfrac{1}{2}B_3(\mathcal{E}(\psi),\psi,\psi) + \ldots\rangle\ , \tag{3.30}$$

which implies the Noether identity

$$\mathcal{N}(\psi) := B_1(\mathcal{E}(\psi)) + B_2(\mathcal{E}(\psi),\psi) + \tfrac{1}{2}B_3(\mathcal{E}(\psi),\psi,\psi) + ... \equiv 0 \ , \tag{3.31}$$

where the dependence of the Noether identity on the fields $\mathcal{N}(\psi)$ indicates that this relation holds identically and that we are not dealing with an arbitrary element of X_2. Perturbatively, the Noether identity is encoded in the L_∞ relations when all the inputs of the brackets are fields. In order to see this more clearly, let us substitute $\mathcal{E}(\psi)$ by its expression in terms of the brackets and expand the Noether identity $\mathcal{N}(\psi)$ to trilinear order. Doing so leads to

$$\begin{aligned}\mathcal{N}(\psi) :&= B_1(B_1(\psi))\\ &+ \tfrac{1}{2}\left\{B_1B_2(\psi,\psi) + 2\,B_2(B_1(\psi),\psi)\right\}\\ &+ \tfrac{1}{6}\left\{3\,B_2(B_2(\psi,\psi),\psi) + B_1B_3(\psi,\psi,\psi) + 3\,B_3(B_1(\psi),\psi,\psi)\right\} \equiv 0 \ .\end{aligned} \tag{3.32}$$

The first line vanishes due to the nilpotency of the differential, the second line due to the Leibniz rule, and the third line vanishes due to the homotopy Jacobi identity.

An additional consistency condition is that the Noether identities be preserved under gauge transformations. We shall prove this to trilinear order. A gauge variation of the Noether identity (3.31) yields

$$\begin{aligned}\delta_\Lambda\mathcal{N} &= B_1B_2(\Lambda,\mathcal{E}) + B_2(\mathcal{E},B_1(\Lambda))\\ &\quad + B_2(B_2(\Lambda,\mathcal{E}),\psi) + B_2(B_2(\Lambda,\psi),\mathcal{E}) + B_1B_3(\Lambda,\mathcal{E},\psi) + B_3(\mathcal{E},\psi,B_1(\Lambda)) \ ,\end{aligned} \tag{3.33}$$

where we used the covariance of the field equations. Using the Leibniz rule and homotopy Jacobi identity we can bring the above equation to

$$\delta_\Lambda\mathcal{N} = B_2(\Lambda,\mathcal{N}) + B_3(\Lambda,\psi,\mathcal{N}) \ , \tag{3.34}$$

showing that the Noether identity is preserved. Additionally, from this computation it is possible to read off the brackets between Noether identities and a gauge parameter, and between Noether identities, gauge parameters and multiple fields.

Let us now look at the gauge algebra. We start with the closure condition (3.16), and evaluate explicitly the commutator of two gauge transformations using the L_∞ brackets up to trilinear order, i.e

$$\begin{aligned}[\delta_{\Lambda_2},\delta_{\Lambda_1}]\psi =&\ \delta_{\Lambda_2}\left\{B_1(\Lambda_1) + B_2(\Lambda_1,\psi) + \tfrac{1}{2}\,B_3(\Lambda_1,\psi,\psi)\right\} - (1\leftrightarrow 2)\\ \overset{[12]}{=}&\ 2\,B_2(\Lambda_1,B_1(\Lambda_2)) + 2\,B_2(\Lambda_1,B_2(\Lambda_2,\psi)) + 2\,B_3(\Lambda_1,B_1(\Lambda_2),\psi) \ ,\end{aligned} \tag{3.35}$$

where [12] over the equal sign in the second line denotes implicit antisymmetrization in the labels 1 and 2 of the gauge parameters[1] . In order to proceed and find a composite gauge parameter Λ_{12}, we need to use the Leibniz rule and the homotopy Jacobi identity which, with implicit

[1]Recall our conventions $a_{[1}a_{2]} = \frac{1}{2}\left\{a_1a_2 - a_2a_1\right\}$.

antisymmetrization of the labels 1 and 2, respectively read

$$
\begin{aligned}
B_1B_2(\Lambda_1,\Lambda_2)+2\,B_2(B_1(\Lambda_1),\Lambda_2)=0\ ,&\\
B_2(B_2(\Lambda_1,\Lambda_2),\psi)+2\,B_2(B_2(\psi,\Lambda_1),\Lambda_2)+B_1B_3(\Lambda_1,\Lambda_2,\psi)&\\
+\,2\,B_3(B_1(\Lambda_1),\Lambda_2,\psi)+B_3(\Lambda_1,\Lambda_2,B_1(\psi))=0\ .&
\end{aligned}
\tag{3.36}
$$

Using these relations in (3.35) leads to

$$
\begin{aligned}
[\delta_{\Lambda_2},\delta_{\Lambda_1}]\psi &= B_1\left\{B_2(\Lambda_1,\Lambda_2)+B_3(\Lambda_1,\Lambda_2,\psi)\right\}+B_2(\psi,B_2(\Lambda_1,\Lambda_2))+B_3(\Lambda_1,\Lambda_2,B_1(\psi))\\
&= B_1(\Lambda_{12})+B_2(\Lambda_{12},\psi)+B_3(\Lambda_1,\Lambda_2,\mathcal{E}(\psi))\ .
\end{aligned}
\tag{3.37}
$$

To trilinear order one can identify

$$
\Lambda_{12}=B_2(\Lambda_1,\Lambda_2)+B_3(\Lambda_1,\Lambda_2,\psi)\ ,\quad \mu^{ij}\mathcal{E}_j\sim B_3(\Lambda_1,\Lambda_2,\mathcal{E}(\psi))\ , \tag{3.38}
$$

showing, once again, how the consistency of the theory is encoded in the generalized Jacobi relations. In theories such as Yang-Mills and pure gravity one has $B_3(\Lambda_1,\Lambda_2,\psi)=B_3(\Lambda_1,\Lambda_2,\mathcal{E}(\psi))=0$. Hence, from the closure of the gauge algebra one only finds a two-bracket that forms a Lie algebra. More generally, however, this is not the case. In order to see this from the perspective of L_∞-algebras, we turn to the Jacobiator of the commutator of gauge variations acting on a field

$$
0\equiv \mathrm{Jac}(\delta_{\Lambda_3},\delta_{\Lambda_2},\delta_{\Lambda_1})\,\psi \overset{[123]}{=} 3\,[[\delta_{\Lambda_1},\delta_{\Lambda_2}],\delta_{\Lambda_3}]\,\psi\ . \tag{3.39}
$$

Evaluating the gauge variations explicitly to trilinear order yields

$$
\mathrm{Jac}(\delta_{\Lambda_3},\delta_{\Lambda_2},\delta_{\Lambda_1})\,\psi \overset{[123]}{=} 3\,B_1\big\{B_2(B_2(\Lambda_1,\Lambda_2),\Lambda_3)+B_3(B_1(\Lambda_1),\Lambda_2,\Lambda_3)\big\}\ . \tag{3.40}
$$

This equation must vanish. In order to see this, we use the homotopy Jacobi relation (*gauge algebra*)

$$
3\,B_2(B_2(\Lambda_{[1},\Lambda_2),\Lambda_{3]})+3\,B_3(B_1(\Lambda_{[1}),\Lambda_2,\Lambda_{3]})+B_1B_3(\Lambda_1,\Lambda_2,\Lambda_3)=0\ . \tag{3.41}
$$

Using this in equation (3.40) leads to

$$
\mathrm{Jac}(\delta_{\Lambda_3},\delta_{\Lambda_2},\delta_{\Lambda_1})\,\psi=-B_1\big\{B_1B_3(\Lambda_1,\Lambda_2,\Lambda_3)\big\}=0\ , \tag{3.42}
$$

which vanishes because $B_1^2=0$. This may be understood from the perspective of reducibility because, in this particular case, the three bracket $B_3(\Lambda_1,\Lambda_2,\Lambda_3)$ can be seen as a gauge-for-gauge parameter namely

$$
\Lambda^{123}_{\rm Trivial}=B_1(\chi_{123})=B_1B_3(\Lambda_1,\Lambda_2,\Lambda_3)\ , \tag{3.43}
$$

and due to the nilpotency of the differential, to trilinear order we have

$$
\delta_{\Lambda^{123}_{\rm Trivial}}\psi=B_1B_1B_3(\Lambda_1,\Lambda_2,\Lambda_3)=0\ . \tag{3.44}
$$

The presence of higher brackets and the need to use the L_∞ relations beyond the standard Jacobi identity implies that, in general, the gauge algebra of a perturbative field theory is an L_∞-algebra. From this analysis we can also find brackets between gauge parameters, brackets between multiple gauge parameters and multiple fields and so on. There exist more relations that involve other elements of $\mathcal{X}$ such as the Noether-for-Noether identities etc., but for the sake of brevity we will not discuss them here.

Finally, in order to close this introduction to L_∞-algebras and field theories, let us make an important remark regarding gauge fixing. The field equations of a field theory, in general, can be written compactly as

$$B_1(\psi) = \mathcal{J} \ , \tag{3.45}$$

where we denote all the interactions and possible external currents with $\mathcal{J}$. This equation, in general, cannot be solved perturbatively because the theory can have a gauge symmetry and as a consequence B_1 is not invertible. Thus, in order to solve the field equations, there exists a linear operator $b : X_i \to X_{i-1}$ of degree $|b| = -1$ that allows us to impose gauge fixing conditions. In addition, we require b to obey

$$[B_1, b] := B_1 b + b B_1 = \Box \ , \tag{3.46}$$

where the square brackets denote the (graded) commutator of linear maps. This commutation relation is required in order to obtain propagators G that go as $G \sim \frac{1}{\Box}$. Indeed, if we impose *Siegel* gauge conditions, which are given by

$$b(\psi) = 0 \ , \tag{3.47}$$

then, we can act with b on the field equation (3.45) yielding

$$b B_1(\psi) = b\mathcal{J} \ . \tag{3.48}$$

We can use the commutation relation (3.46) to write the above equation as

$$- B_1(b(\psi)) + \Box\psi = b\mathcal{J} \ , \tag{3.49}$$

and, due to the gauge condition (3.47), the first term vanishes and we obtain

$$\Box\psi = b\mathcal{J} \ . \tag{3.50}$$

Assuming that $\mathcal{J}$ does not belong to the kernel of $\Box$, one can solve this equations as

$$\psi = \frac{b}{\Box}\mathcal{J} \ , \tag{3.51}$$

and we conclude that the propagator is $G = \frac{b}{\Box}$. The operator b will play a crucial role in sections 3.3.2 and 3.4.2. It is important to stress that in this thesis we will only impose gauge fixing conditions whenever we analyze scattering amplitudes in chapter 4 section 4.5. The content of the thesis prior to that chapter will be completely gauge independent.

3.3 Chern-Simons theory

After having discussed how to formulate theories using L_∞-algebras, we now turn to a concrete example: non-abelian Chern-Simons theory on a three dimensional manifold M_3. The L_∞-algebra formulation of Chern-Simons will be illustrative to later proceed with pure Yang-Mills theory, whose algebraic structure is more intricate. In addition, in this section we take advantage of a manifest factorization of the L_∞-algebra of Chern-Simons into color and kinematics to construct a *kinematic algebra*.

3.3.1 L_∞-algebra formulation of Chern-Simons theory

Chern-Simons theory contains one field, A, which is a one-form in M_3 that takes values in a color Lie algebra $\mathfrak{g}$ with basis elements $t_a \in \mathfrak{g}$. Thus for the field A we write

$$A = A^a\, t_a = A_\mu{}^a dx^\mu \otimes t_a \; . \tag{3.52}$$

The action of Chern-Simons theory can be expressed as

$$S_{\rm CS}[A] = \tfrac{1}{2}\langle A, dA\rangle + \tfrac{1}{3!}\langle A, [A,A]\rangle \; , \tag{3.53}$$

where d is the de Rham differential, and the bracket $[\cdot,\cdot]$ is defined in terms of the structure constants $f_{bc}{}^a$ of $\mathfrak{g}$ and the wedge product of p-forms $\wedge$. For a p-form B and a q-form C that are $\mathfrak{g}$-valued $[\cdot,\cdot]$ is defined as

$$[B, C] = B^b \wedge C^c\, f_{bc}{}^a\, t_a \; . \tag{3.54}$$

This bracket inherits the graded symmetry property of the wedge product and the antisymmetry of the structure constants, implying the graded antisymmetry

$$[B, C] = (-1)^{pq+1}[C, B] \; . \tag{3.55}$$

The de Rham differential obeys the Leibniz rule with respect to the wedge product. This means that for a p-form B and a q-form C (not necessarily $\mathfrak{g}$-valued), d obeys

$$d(B \wedge C) = dB \wedge C + (-1)^p B \wedge dC \; , \tag{3.56}$$

implying

$$d[B, C] = [dB, C] + (-1)^p[B, dC] \; . \tag{3.57}$$

Given a $\mathfrak{g}$-valued one-form A and a $\mathfrak{g}$-valued two-form E, the inner product $\langle\cdot,\cdot\rangle$ is defined by the Cartan-Killing form κ_{ab} of $\mathfrak{g}$ and integration over M_3 as

$$\langle A, E\rangle := \kappa_{ab} \int_{M_3} A^a \wedge E^b \; . \tag{3.58}$$

Varying the action (3.53) produces the two-form field equation

$$F(A) := dA + \tfrac{1}{2}[A, A] = 0 \; , \tag{3.59}$$

where $F(A)$ is the field strength of A. The action (3.53) is invariant under the gauge transformations generated by the $\mathfrak{g}$-valued zero-form λ

$$\delta A = d\lambda + [A, \lambda] \; , \tag{3.60}$$

and gauge invariance of the action implies, in turn, the Noether identity

$$dF(A) + [A, F(A)] \equiv 0 \; , \tag{3.61}$$

which is a three-form. Our task now is to formulate the theory in the language of L_∞-algebras. Given that the gauge symmetry is irreducible, the underlying vector space of the theory is $\mathcal{X}^{CS} = \bigoplus_{i=-1}^{2} X_i^{\text{CS}}$, which together with a differential B_1 generates the chain complex

$$\begin{array}{ccccccc} X_{-1}^{\text{CS}} & \xrightarrow{B_1} & X_0^{\text{CS}} & \xrightarrow{B_1} & X_1^{\text{CS}} & \xrightarrow{B_1} & X_2^{\text{CS}} \\ \lambda & & A & & E & & \mathcal{N} \end{array} \; . \tag{3.62}$$

Chern-Simons theory has a simple algebraic structure because it does not have brackets that take more than two inputs. More precisely, the algebraic structure underlying Chern-Simons theory is a so-called differential graded Lie algebra or dgLa for short. In such algebras, due to the lack of non-vanishing higher brackets, the only non-trivial L_∞ relations are the nilpotency of the differential, the (graded) Leibniz rule, and the (graded) homotopy Jacobi identity which reduces to a (graded) Jacobi identity. In the following we will identify the action of the differential B_1 and the two-bracket B_2 on all the elements of $\mathcal{X}^{\text{CS}}$, and we will show how the different relations that ensure the consistency of the theory are encoded in the language of L_∞-algebras.

Free theory and the differential B_1

We can read off the action of the differential B_1 on gauge parameters from the linear piece of equation (3.60), while we can read off the action of the differential on fields and field equations from the linear pieces of the field equation (3.59) and the Noether identity (3.61), respectively. From simple inspection we conclude that

$$B_1(\lambda) = d\lambda \in X_0^{CS} \; , \quad B_1(A) = dA \in X_1^{CS} \; , \qquad B_1(E) = dE \in X_2^{CS} \; . \tag{3.63}$$

The vector space $\mathcal{X}^{\text{CS}}$ and the differential B_1 define the free theory, where the only relevant L_∞ relation is the nilpotency of the differential $B_1^2 = 0$ which encodes linearized gauge invariance and the linearized Noether identity. Let us first look at the linearized gauge invariance of the free field equations:

$$\delta^{(0)} B_1(A) = B_1(\delta^{(0)} A) = B_1(B_1(\lambda)) = d(d\lambda) = 0 \; , \tag{3.64}$$

which follows because the de Rham differential is nilpotent, and the superscript (0) denotes that we performed a linearized gauge transformation. Similarly, to check that the Noether identity is fulfilled, we act with B_1 on the linearized field equation. Given that B_1 acts also as d on field equations, we obtain a vanishing result. This proves that the differential B_1 is nilpotent and hence that the free theory is consistent.

Interacting theory and the bracket B_2

In order to find all the non-vanishing two-brackets we need to work slightly harder. Let us first look at the two-brackets that can be read off in a straightforward manner from the equations that define the theory. Comparing the field equation (3.59) to (3.21), we conclude that the two-bracket between two fields is

$$B_2(A_1, A_2) = [A_1, A_2] \in X_1^{\mathrm{CS}} , \tag{3.65}$$

where we used the polarization identity

$$2\, B_2(A_1, A_2) = B_2(A_1 + A_2, A_1 + A_2) - B_2(A_1, A_1) - B_2(A_2, A_2) , \tag{3.66}$$

in order to find the action of the two-bracket on two distinct elements of X_0^{CS}. This is necessary because from the field equations one can only directly read off the action of the two-bracket on two copies of the same field.

Similarly, we can read off the two-bracket between a gauge parameter and a field from the gauge transformation (3.60)

$$B_2(\lambda, A) = [A, \lambda] \in X_0^{\mathrm{CS}} , \tag{3.67}$$

and finally, we can read off the two-bracket between a field equation and a field from the Noether identity (3.61)

$$B_2(A, E) = [A, E] \in X_2^{\mathrm{CS}} . \tag{3.68}$$

Notice that the symmetry properties of B_2 are not necessarily the same as the symmetry properties of $[\cdot, \cdot]$. This is the case in general, and one has to be careful when moving inputs around in calculations.

The brackets above are not the only non-vanishing two-brackets of Chern-Simons. In addition, from degree considerations we expect, in principle, non-vanishing brackets between two gauge parameters, a gauge parameter and an equation, and a gauge parameter and a Noether identity. In order to find them we have to look at the gauge covariance of the field equations, compatibility of the Noether identity with gauge symmetry, and the closure of the gauge algebra.

First, in the interest of finding the two-bracket between gauge parameters and field equations, we perform a gauge variation of the field equation (3.59) to check its gauge covariance. Doing so yields

$$\delta F(A) = -[\lambda, F(A)] , \tag{3.69}$$

and we can read off the two-bracket

$$B_2(\lambda, E) = -[\lambda, E] \ \in X_1^{\mathrm{CS}} . \tag{3.70}$$

Next, in order to obtain the two-bracket between gauge parameters and Noether identities, we have to look at the compatibility of the Noether identity with the gauge symmetry. This can be checked by performing a gauge variation of equation (3.61)

$$\delta\big\{dF + [A, F]\big\} = [dF + [A, F], \lambda] \ , \tag{3.71}$$

from which we infer

$$B_2(\lambda, \mathcal{N}) = -[\lambda, \mathcal{N}] \ \in X_2^{\mathrm{CS}} \ . \tag{3.72}$$

Finally, for the gauge algebra, consider the closure condition which for Chern-Simons simply reads

$$[\delta_{\lambda_2}, \delta_{\lambda_1}]A = \delta_{\lambda_{12}} A \ , \tag{3.73}$$

because there are no equation-of-motion symmetries. Recall that the bracket in the left hand side of the equation is the commutator of operators, which should not be confused with the bracket $[\cdot, \cdot]$ that acts on elements of $\mathcal{X}^{\mathrm{CS}}$. Expanding the commutator in the left hand side of equation (3.73) yields

$$\begin{aligned} [\delta_{\lambda_2}, \delta_{\lambda_1}]A &= -d[\lambda_1, \lambda_2] - [A, [\lambda_1, \lambda_2]] \\ &= \delta_{-[\lambda_1, \lambda_2]} A \ , \end{aligned} \tag{3.74}$$

implying the two-bracket

$$B_2(\lambda_1, \lambda_2) = -[\lambda_1, \lambda_2] \ \in X_{-1}^{\mathrm{CS}} \ , \tag{3.75}$$

and there are no higher brackets contributing to the gauge algebra.

After identifying the differential and the two-bracket acting on all the elements of $\mathcal{X}^{\mathrm{CS}}$, one then has to check that the L_∞ relations hold. The nilpotency of the differential is obvious as we saw when discussing the free theory. In the interest of illustrating how to perform explicit computations, let us look at some examples. Take for instance the Leibniz rule

$$B_1 B_2(\lambda, A) + B_2(B_1(\lambda), A) - B_2(\lambda, B_1(A)) = 0 \ . \tag{3.76}$$

This relation is associated to the gauge covariance of the field equations. In order to see how to write this equation in terms of the explicit action of the maps on the gauge parameter and field, notice that in the first term $B_2(\lambda, A) \in X_0^{\mathrm{CS}}$. This means that B_1 acts on $B_2(\lambda, A)$ as if it were acting on a field and thus as the de Rham differential. In the second term, given that $B_1(\lambda) \in X_0^{\mathrm{CS}}$, the two-bracket B_2 acts as if it had two field inputs. In the last term, due to $B_1(A) \in X_1^{\mathrm{CS}}$, we have to consider the two bracket between a gauge parameter and a field equation. Thus we have

$$d[A, \lambda] + [d\lambda, A] + [\lambda, dA] \equiv 0 \ , \tag{3.77}$$

whose vanishing follows because d obeys the Leibniz rule (3.57) and because of the graded antisymmetry of $[\cdot,\cdot]$.

As an additional example consider the Jacobi relation

$$B_2(B_2(\lambda_1,\lambda_2),A) - B_2(B_2(\lambda_2,A),\lambda_1) + B_2(B_2(A,\lambda_1),\lambda_2) = 0\ . \tag{3.78}$$

In terms of $[\cdot,\cdot]$ the graded Jacobi relation is

$$[A,[\lambda_1,\lambda_2]] - [[A,\lambda_2],\lambda_1] + [[A,\lambda_1],\lambda_2] \equiv 0\ , \tag{3.79}$$

which vanishes because of the graded antisymmetry of $[\cdot,\cdot]$ and the Jacobi identity obeyed by the structure constants $f_{bc}{}^{a}$. Similarly, one can check all the relations of the maps B_1 and B_2 acting on all possible elements of $\mathcal{X}^{\rm CS}$, and one finds that indeed they obey the L_∞ relations. In addition, due to the lack of higher brackets, there are no more L_∞ relations beyond the Leibniz rule and graded Jacobi identity to be checked. It is worth stressing that in Chern-Simons theory the differential and the two-bracket act in the same way on all the elements of $\mathcal{X}^{\rm CS}$: B_1 acts as the de Rham differential and B_2 always involves $[\cdot,\cdot]$. This is not the case in general.

Lastly, let us turn to identifying the map b that allows us to fix a gauge and solve the field equations. In Chern-Simons theory a natural choice is the adjoint operator $b = -d^\dagger$, which in our conventions is defined in terms of the Hodge star[2] $\star : X_i^{\rm CS} \to X_{3-i}^{\rm CS}$ as $d^\dagger(u) = (-1)^u \star d \star (u)$. This map has degree -1 because it lowers the form degree by one and it obeys the required commutation relation

$$[B_1, b] = -[d, d^\dagger] = -dd^\dagger - d^\dagger d = \Box\ . \tag{3.80}$$

Moreover, it squares to zero, i.e $d^{\dagger\,2} = 0$, and if we were to fix a gauge, imposing the condition $b(A) = 0$ would correspond to Lorenz gauge in components i.e $\partial_\mu A^\mu = 0$. Nevertheless, we refrain from doing so in the following.

3.3.2 Kinematic algebra of Chern-Simons theory

The L_∞-algebra of Chern-Simons theory exhibits a manifest factorization of color and kinematics. More precisely, the vector space $\mathcal{X}^{\rm CS}$ factorizes as

$$\mathcal{X}^{\rm CS} = \Omega_\bullet \otimes \mathfrak{g}\ , \tag{3.81}$$

where the kinematic space $\Omega_\bullet = \bigoplus_{p=0}^{3} \Omega_p$ is the space of p-forms on M_3, and $\mathfrak{g}$ is a finite dimensional Lie algebra that encodes color. The factorization is obvious because gauge parameters are $\mathfrak{g}$-valued zero-forms, the fields are one-forms, the field equations are two-forms and the Noether identities are three-forms. In general, we can write any element $y(x) \in \mathcal{X}^{\rm CS}$ as

$$y(x) = u^a(x) \otimes t_a\ , \text{with } u^a(x) \in \Omega_\bullet\ ,\ t_a \in \mathfrak{g}\ . \tag{3.82}$$

The p-forms $u^a(x)$ should be thought of as *color-stripped* quantities with only kinematic information. Thus we can ignore the color index and simply write $u(x) \in \Omega_\bullet$. Moreover, the L_∞

[2] We use the convention $\star^2 = -1$.

maps B_n also factorize into kinematic products denoted by m_n, and the structure constants of the color Lie group ${f_{ab}}^c$. The differential is *color-blind* as it acts as the de Rham differential, namely

$$B_1(y(x)) = m_1(u^a(x)) \otimes t_a = du^a(x) \otimes t_a \ . \tag{3.83}$$

The two-bracket, on the other hand, factorizes as

$$B_2(y_1, y_2) = (-1)^{y_1} m_2(u_1^a, u_2^b) \otimes {f_{ab}}^c t_c \ , \tag{3.84}$$

where m_2 is a kinematic two-product. By inspection of the definition of B_2 in terms of $[\cdot,\cdot]$, one concludes that m_2 is the wedge product of p-forms, i.e

$$m_2(u_1, u_2) = u_1 \wedge u_2 \ , \tag{3.85}$$

for all $u_i \in \Omega_\bullet$. In our conventions, the color generators t_a have degree $|t_a| = -1$, while the Lie bracket of $\mathfrak{g}$ associated to the structure constants ${f_{ab}}^c$ has degree one. For this reason, the elements and maps of the kinematic space carry *kinematic degree*

$$|u(x)| = |y(x)|_{\mathcal{X}^{\text{CS}}} + 1 \ , \quad |m_1| = |B_1|_{\mathcal{X}^{\text{CS}}} \ , \quad |m_2| = |B_2|_{\mathcal{X}^{\text{CS}}} - 1 = 0 \ , \tag{3.86}$$

which coincide with the form degree of the u_i, the degree of the form degree de Rham differential, and the form degree of the wedge product.

The space $\Omega_\bullet$ equipped with the de Rham differential and the wedge product of p-forms make up a so-called *differential graded commutative algebra* or dgca for short. These algebras consist of a graded vector space equipped with a nilpotent differential m_1 ($m_1^2 = 0$) and a bilinear graded commutative product m_2 that obey

$$\begin{aligned} m_1 m_2(u_1, u_2) - m_2(m_1(u_1, u_2)) - (-1)^{u_1} m_2(u_1, m_1(u_2)) = 0 \ , \\ m_2(m_2(u_1, u_2), u_3) - m_2(u_1, m_2(u_2, u_3)) = 0 \ , \end{aligned} \tag{3.87}$$

where the first relation is the Leibniz rule of m_1 with respect to m_2 and the second relation is associativity of the two-product. Indeed, the de Rham differential is nilpotent and obeys the Leibniz rule with respect to the wedge product, and the wedge product is associative. Nevertheless, as explained in the previous chapter, one expects a Lie-type algebra to be the kinematic algebraic structure responsible for the double copy, not an associative commutative algebra. In the interest of constructing a Lie-type algebra with the kinematic structures identified thus far, it will be illustrative to study the compatibility of the operator $b = -d^\dagger$ with the associative structure. In contrast to the differential m_1, the operator b does not obey the Leibniz rule with respect to m_2. One can parametrize the failure of Leibniz rule of b by defining a new bilinear map, or generalized Poisson (or so-called Gerstenhaber) bracket[3], $b_2 : \Omega_\bullet \times \Omega_\bullet \to \Omega_\bullet$

$$\begin{aligned} b_2(u_1, u_2) :&= b m_2(u_1, u_2) - m_2(b u_1, u_2) - (-1)^{u_1} m_2(u_1, b u_2) \\ &= d^\dagger(u_1 \wedge u_2) - d^\dagger u_1 \wedge u_2 - (-1)^{u_1} u_1 \wedge d^\dagger u_2 \end{aligned} \tag{3.88}$$

[3]Here, the term Poisson bracket refers to the fact that b_2 obeys a Poisson compatibility relation (see equation (3.90)). The terminology does not refer to the standard Poisson bracket of classical mechanics.

The operator b obeys the Leibniz rule with respect to b_2, but the differential m_1 fails to obey it *modulo box*. More precisely,

$$\begin{aligned} m_1 b_2(u_1,u_2) - b_2(m_1(u_1),u_2) - (-1)^{u_2} b_2(u_1, m_1(u_2)) &= [\Box, m_2](u_1,u_2) \\ &\equiv \Box m_2(u_1,u_2) - m_2(\Box u_1, u_2) - m_2(u_1, \Box u_2) \\ &= 2\, m_2(\partial_\mu u_1, \partial^\mu u_2) \ . \end{aligned} \tag{3.89}$$

This can be proved by using the definition of b_2 in terms of b and m_2 (3.88), and the commutation relation $[m_1, b] = \Box$. Moreover, the two-bracket b_2 and the two-product m_2 obey the compatibility condition

$$b_2(m_2(u_1,u_2),u_3) - (-1)^{u_1(u_2+u_3)} m_2(b_2(u_2,u_3),u_1) - (-1)^{u_3(u_1+u_2)} m_2(b_2(u_3,u_1),u_2) = 0 \ , \tag{3.90}$$

which, in turn, implies that b_2 obeys the graded Jacobi identity

$$b_2(b_2(u_1,u_2),u_3) + (-1)^{u_1(u_2+u_3)} b_2(b_2(u_2,u_3)) + (-1)^{u_3(u_1+u_2)} b_2(b_2(u_3,u_1),u_2) = 0 \ . \tag{3.91}$$

This is proven by acting with b on the relation (3.143), which turns all the products m_2 into brackets b_2 and converts (3.90) into the Jacobi identity of b_2 (the details of the proof can be found in [3]).

The above is an example of a *Batalin–Vilkovisky algebra* (or BV algebra for short): This is a graded vector space with a degree-(-1) differential b obeying $b^2 = 0$ (a chain complex) equipped with a graded commutative and associative product m_2, and a differential graded Lie algebra structure with differential b and Lie bracket b_2 satisfying the compatibility condition (3.90) between m_2 and b_2. [Upon ignoring the differential, a BV algebra is known as Gerstenhaber algebra, which is a generalization of the Poisson algebra of functions on phase space. Here the product is just the ordinary product of functions and the Lie bracket is the Poisson bracket, which indeed satisfies the compatibility condition (Poisson identity).]

This definition, as well as the explicit check of the two relations (3.91), (3.90), can be simplified by noting that in a BV algebra the differential is of 'second order'. To explain this notion for our special case note that while $d^\dagger$ is defined in terms of a first-order differential operator it does not act via the Leibniz rule on the wedge product, as noted above, and in this sense is of higher order. It is actually of second order in that it acts like the Laplacian on a product of functions.[4] One can then define a BV algebra as a graded commutative associative algebra equipped with a differential of second order. The graded Lie bracket is then a derived notion, defined as in (3.89) as the failure of the differential to obey the Leibniz rule with respect to the graded commutative product. Both the Jacobi identity and the compatibility condition are consequences of the differential being second order.

[4]More precisely, $d^\dagger$ being of second order means

$$d^\dagger(uvw) = -(d^\dagger u)vw - (-1)^u u(d^\dagger v)w - (-1)^{u+v} uvd^\dagger w + d^\dagger(uv)w + (-1)^u ud^\dagger(vw) + (-1)^{(u+1)v} vd^\dagger(uw) \ , \tag{3.92}$$

where we left the wedge product implicit. The second order character of $d^\dagger$ is clear in the equivalent space of polyvectors, see (3.99) below.

After this abstract discussion let us return to the example at hand, which actually has the following simple geometric interpretation. Given the metric we can identify differential forms with polyvectors (completely anti-symmetric contravariant tensors) by raising indices. The inner product (3.58) on forms then gives rise to the natural pairing between a p-form and a rank-p polyvector. This pairing does not depend on the full metric but only on the volume form, whose corresponding density we denote by $\rho = \sqrt{|g|}$. The adjoint operator $d^\dagger$ is transported to the covariant divergence on polyvectors, which we denote by Δ, and which indeed decreases the rank by one. On a rank-p polyvector $u^{\mu_1\cdots\mu_p}$ we have

$$(\Delta u)^{\mu_1\cdots\mu_{p-1}} := \rho^{-1}\partial_\nu\big(\rho\, u^{\nu\mu_1\cdots\mu_{p-1}}\big)\ . \tag{3.93}$$

This is a differential in that $\Delta^2 = 0$ but it does not act via the Leibniz rule on the wedge product of polyvectors. Rather, the failure defines the so-called Schouten–Nijenhuis bracket on polyvectors, which for vector fields reduces to the familiar Lie bracket generating infinitesimal diffeomorphisms. Indeed, setting

$$[u_1, u_2] := \Delta(u_1 \wedge u_2) - \Delta u_1 \wedge u_2 - (-1)^{u_1} u_1 \wedge \Delta u_2\ , \tag{3.94}$$

and specializing to vector fields u_1, u_2 one finds

$$\begin{aligned}[u_1, u_2]^\mu &= \big(\rho^{-1}\partial_\nu\big(\rho\, 2u_1^{[\nu}u_2^{\mu]}\big) - \rho^{-1}\partial_\nu(\rho u_1^\nu)u_2^\mu + u_1^\mu\rho^{-1}\partial_\nu(\rho u_2^\nu)\big)\\ &= u_1^\nu\partial_\nu u_2^\mu - u_2^\nu\partial_\nu u_1^\mu\ ,\end{aligned} \tag{3.95}$$

which is the diffeomorphism covariant Lie bracket of vector fields (in which the volume factors have cancelled). This bracket, and the Schouten–Nijenhuis bracket more generally, of course satisfy the Jacobi identity. Moreover, the compatibility condition (3.90) has a simple geometric interpretation: it means that the wedge product of polyvectors is covariant under infinitesimal diffeomorphisms. Thus, the polyvectors equipped with the wedge product and the second order differential Δ form a BV algebra.

As an aside, let us note that in this picture of polyvectors there is a particularly intuitive way to understand that Δ is of second order and hence defines a BV algebra. Following [99] we start by viewing the de Rham complex as functions of even coordinates x^μ and odd anti-commuting coordinates θ^μ playing the role of dx^μ. The expansion of a function $f(x,\theta)$ reads

$$f(x,\theta) = \sum_p \tfrac{1}{p!}\, f_{\mu_1\ldots\mu_p}(x)\,\theta^{\mu_1}\cdots\,\theta^{\mu_p}\ , \tag{3.96}$$

and thus this space of functions is equivalent to the de Rham complex of differential forms. Moreover, the pointwise product $f \cdot g$ of functions encodes the wedge product of differential forms. The de Rham differential is now realized as

$$d = \theta^\mu \frac{\partial}{\partial x^\mu}\ , \tag{3.97}$$

and thus, taking the form of a vector field, acts as a derivation on the product. Turning then to the chain complex of polyvector fields, these can be realized as functions of x^μ and new odd variables ϑ_μ,

$$F(x,\vartheta) = \sum_p \tfrac{1}{p!}\, F^{\mu_1\ldots\mu_p}(x)\, \vartheta_{\mu_1} \cdots \vartheta_{\mu_p}\,, \tag{3.98}$$

for which the pointwise product yields the wedge product of polyvectors. The differential given by the above divergence operator Δ is then realized, say for trivial volume measure $\rho = 1$, as

$$\Delta = \frac{\partial^2}{\partial x^\mu \partial \vartheta_\mu}\,. \tag{3.99}$$

This makes it manifest that Δ is of second order with respect to the wedge product of polyvectors and hence, in the isomorphic space of differential forms, that $d^\dagger$ is second order.

After this aside, we finally turn to the kinematic Lie algebra of Chern-Simons theory, which has recently been identified by Ben-Shahar and Johansson [58] and turns out to be a small subalgebra of the above BV algebra. To see this we specialize to the fields of Chern-Simons theory and impose the condition

$$bA = 0\,, \tag{3.100}$$

which means that the divergence of the corresponding vector field vanishes. This is just a standard gauge fixing condition (as one needs to impose for any quantum computations). The Lie bracket is closed on divergence-free vector fields for which it is known as the algebra of volume preserving diffeomorphisms, which was identified in [58] as the kinematic Lie algebra of Chern-Simons theory. The operator b is perfectly suited to impose a gauge fixing condition, but we see here that there is a rich algebraic structure whether one imposes $bA = 0$ or not.

We close this section by pointing out that there is actually more structure than a BV algebra, because the de Rham differential d plays no role in the latter. A BV algebra equipped with a second differential (of opposite degree to the first) that acts as a derivation on the product is known as a differential graded BV algebra provided both differentials anti-commute. Here, however, they anticommute to the d'Alembert operator $\Box$. Following Reiterer we will refer to such a structure as a $\mathrm{BV}^\Box$-algebra.[5] While for the considerations in [58] all this extra structure was not needed, this changes for genuine Yang-Mills theory in arbitrary dimensions. At least in its known local formulations, in order to double copy Yang-Mills theory the full $\mathrm{BV}^\Box$-algebra in its homotopy version, denoted $\mathrm{BV}^\Box_\infty$ in the following, is needed. We call these exotic algebras *kinematic algebras* because, as we will see in the following chapter, they are the mathematical structures responsible for the double copy.

3.4 Yang-Mills theory

Following the approach taken above for Chern-Simons theory, in this section we formulate Yang-Mills theory in D-dimensional Minkowski space $\mathcal{M}_D$ with metric $\eta_{\mu\nu}$ in L_∞ form. The L_∞-algebra encoding Yang-Mills, similarly to the one of Chern-Simons, factorizes into a color part and a kinematic part. This factorization allows for the construction of a kinematic algebra. More precisely, the kinematic algebra of Yang-Mills is a homotopy version of a $\mathrm{BV}^\Box$-algebra

[5] It was also noted in [61] that the $\mathrm{BV}^\Box$ algebra of Reiterer is present in Chern-Simons theory.

namely a $\mathrm{BV}_\infty^\Box$-algebra, which we will define below.

3.4.1 L_∞-algebra formulation of Yang-Mills theory

In this section and in the remainder of the thesis we will use a version of the Yang-Mills action presented in [2] that has an auxiliary field φ in the kinetic terms which reads

$$S_{\mathrm{YM}} = \int d^D x \, \mathrm{Tr} \left\{ \tfrac{1}{2} A^\mu \Box A_\mu - \tfrac{1}{2} \varphi^2 + \varphi \, \partial_\mu A^\mu - \partial_\mu A_\nu \, [A^\mu, A^\nu] - \tfrac{1}{4} [A^\mu, A^\nu] \, [A_\mu, A_\nu] \right\} . \quad (3.101)$$

As before, we consider fields taking values in a color Lie algebra $\mathfrak{g}$, $A_\mu = A_\mu^a \, t_a$ and $\varphi = \varphi^a \, t_a$, and, in contrast to Chern-Simons theory, the bracket $[\cdot, \cdot]$ in the Yang-Mills action (4.69) is the Lie bracket of $\mathfrak{g}$ defined by the structure constants ${f_{bc}}^a$ as

$$[t_a, t_b] = {f_{ab}}^c \, t_c \, . \quad (3.102)$$

Recall that for Chern-Simons we combined the structure constants with the wedge product of p-forms. In this section $[\cdot, \cdot]$ is solely defined by the structure constants ${f_{bc}}^a$ and hence is antisymmetric and obeys the Jacobi identity.

The field equations of the gauge and auxiliary fields are, respectively,

$$\begin{aligned} E^\mu(A, \varphi) &:= \Box A^\mu - \partial^\mu \varphi + \partial_\nu [A^\nu, A^\mu] + [f^{\mu\nu}, A_\nu] + [A_\nu, [A^\nu, A^\mu]] = 0 \, , \\ E(A, \varphi) &:= \partial \cdot A - \varphi = 0 \, , \end{aligned} \quad (3.103)$$

where $f_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ is the abelian field strength of A_μ. Notice that if we use the on-shell value of φ in the equation of motion of A_μ, one recovers the standard Yang-Mills equation. The field equations (3.103) are covariant under the non-abelian gauge transformations

$$\delta A_\mu = \partial_\mu \lambda + [A_\mu, \lambda] \, , \quad \delta \varphi = \Box \lambda + \partial_\mu \, [A^\mu, \lambda] \, , \quad (3.104)$$

which imply the Noether identity

$$N(A, \varphi) := -\partial_\mu E^\mu(A, \varphi) + \Box E(A, \varphi) - [A_\mu, E^\mu(A, \varphi)] + [A_\mu, \partial^\mu E(A, \varphi)] \equiv 0 \, . \quad (3.105)$$

Following the procedure that we executed for Chern-Simons, we now turn to the formulation of Yang-Mills theory in the language of L_∞-algebras. The vector space underlying Yang-Mills theory is $\mathcal{X}^{\mathrm{YM}} = \bigoplus_{i=-1}^{2} X_i^{\mathrm{YM}}$, and equipped with a differential B_1, we have the following chain complex:

$$\begin{array}{ccccccc} X_{-1}^{\mathrm{YM}} & \xrightarrow{B_1} & X_0^{\mathrm{YM}} & \xrightarrow{B_1} & X_1^{\mathrm{YM}} & \xrightarrow{B_1} & X_2^{\mathrm{YM}} \\ \Lambda & & \mathcal{A} & & \mathcal{E} & & \mathcal{N} \end{array} \, . \quad (3.106)$$

Given that in our formulation of Yang-Mills we have two fields, the elements of $\mathcal{X}^{\mathrm{YM}}$ are doublets with components

$$\Lambda = \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \ , \quad \mathcal{A} = \begin{pmatrix} A_\mu \\ \varphi \end{pmatrix} \ , \quad \mathcal{E} = \begin{pmatrix} E \\ E_\mu \end{pmatrix} \ , \quad \mathcal{N} = \begin{pmatrix} 0 \\ N \end{pmatrix} \ . \tag{3.107}$$

Alternatively, we can depict the chain complex in terms of the components as

$$\begin{array}{ccccccc} X_{-1}^{\rm YM} & \xrightarrow{B_1} & X_0^{\rm YM} & \xrightarrow{B_1} & X_1^{\rm YM} & \xrightarrow{B_1} & X_2^{\rm YM} \\ \lambda & & A_\mu & & E & & \\ & & \varphi & & E_\mu & & N \end{array} \ , \tag{3.108}$$

which will be more convenient later on in the thesis due to the manifest $\mathbb{Z}_2$ split between the upper and lower lines of the complex. It is important to stress, however, that the elements upon which the L_∞ maps act are the doublets in equation (3.107).

Free theory and the differential B_1

It is straightforward to read off the action of the differential on the different elements of $\mathcal{X}^{\rm YM}$. From the field equations (3.103), the gauge transformations (3.104) and the Noether identity (3.105), we see that the the differential B_1 acts as

$$\begin{gathered} B_1(\Lambda) = \begin{pmatrix} \partial_\mu \lambda \\ \Box \lambda \end{pmatrix} \in X_0^{\rm YM} \ , \quad B_1(\mathcal{A}) = \begin{pmatrix} \partial \cdot A - \varphi \\ \Box A_\mu - \partial_\mu \varphi \end{pmatrix} \in X_1^{\rm YM}, \\ B_1(\mathcal{E}) = \begin{pmatrix} 0 \\ -\partial_\mu E^\mu + \Box E \end{pmatrix} \in X_2^{\rm YM} \ . \end{gathered} \tag{3.109}$$

As advertised at the end of section 3.3, B_1 in Yang-Mills theory acts differently on objects with different degrees, in contrast to Chern-Simons. This differential defines the free theory and we can now check linearized gauge invariance of the field equations, which in the L_∞ language translates to nilpotency of B_1. In components we have

$$\delta^{(0)} B_1(\mathcal{A}) = B_1(B_1(\Lambda)) = \begin{pmatrix} \partial^\mu(\partial_\mu \lambda) - \Box \lambda \\ \Box \partial_\mu \lambda - \partial_\mu \Box \lambda \end{pmatrix} \equiv 0 \ . \tag{3.110}$$

Performing a similar computation for the Noether identity, $B_1(B_1(\mathcal{A})) = 0$, one proves that B_1 is indeed nilpotent acting on all elements of $\mathcal{X}^{\rm YM}$, and hence the free theory is consistent.

Interacting theory and the brackets B_2 and B_3

From the bilinear parts of the field equations (3.103), gauge transformations (3.104) and the

Noether identity (3.105), we can read off the two-brackets B_2

$$
\begin{aligned}
B_2(\mathcal{A}_1, \mathcal{A}_2) &= \begin{pmatrix} 0 \\ B_2^\mu(A_1, A_2) \end{pmatrix} = 2 \begin{pmatrix} 0 \\ \partial_\nu \left[A^\nu_{(1}, A^\mu_{2)}\right] + \left[f^{\mu\nu}_{(1}, A_{2)\nu}\right] \end{pmatrix} \in X_1^{\mathrm{YM}} , \\
B_2(\Lambda, \mathcal{A}) &= \begin{pmatrix} B_2^\mu(\lambda, A) \\ B_2^\varphi(\lambda, A) \end{pmatrix} = \begin{pmatrix} [A^\mu, \lambda] \\ \partial_\mu [A^\mu, \lambda] \end{pmatrix} \in X_0^{\mathrm{YM}} , \\
B_2(\mathcal{A}, \mathcal{E}) &= \begin{pmatrix} 0 \\ B_2(A, \mathcal{E}) \end{pmatrix} = \begin{pmatrix} 0 \\ [A_\mu, \partial^\mu E - E^\mu] \end{pmatrix} \in X_2^{\mathrm{YM}} .
\end{aligned} \tag{3.111}
$$

Moreover, the field equations have a cubic term which implies the existence of a three-bracket B_3 between fields:

$$
B_3(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) = \begin{pmatrix} 0 \\ B_3^\mu(A_1, A_2, A_3) \end{pmatrix} = 6 \begin{pmatrix} 0 \\ \left[A_{\nu(1}, \left[A_2^\nu, A^\mu_{3)}\right]\right] \end{pmatrix} \in X_1^{\mathrm{YM}} . \tag{3.112}
$$

Notice that the auxiliary field φ does not appear in any of the two-brackets nor the three-bracket, and similarly to the differential, the action of B_2 on the elements of $\mathcal{X}^{\mathrm{YM}}$ depends on their degree. With the above brackets and differential, we can write the Yang-Mills action in terms of the L_∞ maps as

$$
S_{\mathrm{YM}}[\mathcal{A}] = \tfrac{1}{2!} \langle \mathcal{A}, B_1(\mathcal{A}) \rangle + \tfrac{1}{3!} \langle \mathcal{A}, B_2(\mathcal{A}, \mathcal{A}) \rangle + \tfrac{1}{4!} \langle \mathcal{A}, B_3(\mathcal{A}, \mathcal{A}, \mathcal{A}) \rangle . \tag{3.113}
$$

In order to find the remaining two-brackets (there are no other three-brackets), we need to look at the same computations that we examined in Chern-Simons theory. First, we turn to the gauge covariance of the field equations. Performing a gauge variation of the field equations yields

$$
\begin{aligned}
\delta E_\mu(A, \varphi) &= -[\lambda, E_\mu(A, \varphi) - \partial_\mu E(A, \varphi)] , \\
\delta E(A, \varphi) &= 0 .
\end{aligned} \tag{3.114}
$$

This result determines the action of the two-bracket between a gauge parameter and a field equation

$$
B_2(\Lambda, \mathcal{E}) = \begin{pmatrix} 0 \\ B_2^\mu(\lambda, \mathcal{E}) \end{pmatrix} = - \begin{pmatrix} 0 \\ [\lambda, E^\mu - \partial^\mu E] \end{pmatrix} \in X_1^{\mathrm{YM}} . \tag{3.115}
$$

To find the two-bracket between gauge parameters and Noether identities, we turn to the compatibility of the Noether identity (3.105) with gauge symmetry. To that end we perform a gauge transformation of the Noether identity which yields

$$
\delta N(A, \varphi) = -[\lambda, N(A, \varphi)] \equiv 0 , \tag{3.116}
$$

implying

$$
B_2(\Lambda, \mathcal{N}) = - \begin{pmatrix} 0 \\ [\lambda, N] \end{pmatrix} \in X_2^{\mathrm{YM}} . \tag{3.117}
$$

Finally, we now turn to the gauge algebra. The closure conditions for the fields in our formulation of Yang-Mills read

$$\begin{aligned}[][\delta_{\Lambda_2},\delta_{\Lambda_1}]A_\mu &= -\partial_\mu[\lambda_1,\lambda_2]-[A_\mu,[\lambda_1,\lambda_2]]\ ,\\ [\delta_{\Lambda_2},\delta_{\Lambda_1}]\varphi &= \Box[\lambda_1,\lambda_2]+\partial^\mu[A_\mu,[\lambda_1,\lambda_2]]\ ,\end{aligned} \tag{3.118}$$

implying

$$B_2(\Lambda_1,\Lambda_2)=\begin{pmatrix}-[\lambda_1,\lambda_2]\\ 0\end{pmatrix}\ \in X_1^{\mathrm{YM}}\ . \tag{3.119}$$

Similarly to Chern-Simons, the gauge algebra of our formulation of Yang-Mills is a Lie algebra since there are no higher brackets nor equation-of-motion symetries.

As an example of the type of relations that one has to check in Yang-Mills theory consider the Leibniz rule associated to gauge covariance of the field equations

$$B_1B_2(\Lambda,\mathcal{A})+B_2(B_1(\Lambda),\mathcal{A})-B_2(\Lambda,B_1(\mathcal{A}))=0\ . \tag{3.120}$$

In components the relation reads

$$\begin{aligned}&\begin{pmatrix}\partial^\mu[A_\mu,\lambda]-\partial_\nu[A^\nu,\lambda]\\ \Box[A_\mu,\lambda]-\partial_\mu\,\partial^\nu[A_\nu,\lambda]\end{pmatrix}+\begin{pmatrix}0\\ \partial_\nu[\partial^\nu\lambda,A_\mu]+\partial_\nu[A^\nu,\partial_\mu\lambda]+[f_{\mu\nu},\partial^\nu\lambda]\end{pmatrix}\\ &\qquad\qquad+\begin{pmatrix}0\\ [\lambda,\Box A_\mu]-[\lambda,\partial_\mu\partial^\nu A_\nu]\end{pmatrix}\equiv 0\ ,\end{aligned} \tag{3.121}$$

whose vanishing relies on the symmetry properties of the Lie bracket $[\cdot,\cdot]$ and the Leibniz rule of derivatives with respect to the pointwise product of functions.

In addition to the Leibniz rule, in Yang-Mills we have to check homotopy Jacobi relations. For instance when computing the Noether identity one encounters

$$3\,B_2(B_2(\mathcal{A}_{(1},\mathcal{A}_2),\mathcal{A}_{3)})+B_1B_3(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)=0\ . \tag{3.122}$$

Moreover, one also encounters higher relations between the three-bracket and the two-bracket as, for example, when looking at the gauge covariance of the field equations which involves the relation

$$3B_3(B_2(\Lambda,\mathcal{A}),\mathcal{A},\mathcal{A})+B_2(B_3(\mathcal{A},\mathcal{A},\mathcal{A}),\Lambda)=0\ , \tag{3.123}$$

among others. For the sake of brevity we will not perform these computations explicitly nor list all the remaining L_∞ relations in Yang-Mills theory.

Finally, there also exists a map $b:X_i^{\mathrm{YM}}\to X_{i-1}^{\mathrm{YM}}$ that allows one to impose gauge conditions and determine a propagator. In our formulation of Yang-Mills theory the action of b is given by

$$b\Lambda=0\ ,\ b\mathcal{A}=\begin{pmatrix}\varphi\\ 0\end{pmatrix}\in X_{-1}^{\mathrm{YM}}\ ,\ b\mathcal{E}=\begin{pmatrix}E_\mu\\ 0\end{pmatrix}\in X_0^{\mathrm{YM}}\ ,\ b\mathcal{N}=\begin{pmatrix}N\\ 0\end{pmatrix}\in X_1^{\mathrm{YM}}\ , \tag{3.124}$$

or diagramatically

$$
\begin{array}{ccccccc}
X_{-1}^{\rm YM} & \xrightarrow{B_1} & X_0^{\rm YM} & \xrightarrow{B_1} & X_1^{\rm YM} & \xrightarrow{B_1} & X_2^{\rm YM} \\
\lambda & \overset{b}{\nwarrow} & A_\mu & \overset{b}{\nwarrow} & E & \overset{b}{\nwarrow} & \\
 & & \varphi & & E_\mu & & N
\end{array} \quad . \tag{3.125}
$$

In order to construct the kinematic algebra, one needs to prove that b obeys $[B_1, b] = \Box$ and that it is nilpotent. Nilpotency can be checked straightforwardly. Now we turn to proving $[B_1, b] = \Box$. To that end, let us consider the the commutator $[B_1, b]$ acting on a field $\mathcal{A}$, i.e

$$
\begin{aligned}
[B_1, b]\mathcal{A} &= B_1(b(\mathcal{A})) + b(B_1(\mathcal{A})) \\
&= B_1 \begin{pmatrix} \varphi \\ 0 \end{pmatrix} + b \begin{pmatrix} \partial \cdot A - \varphi \\ \Box A_\mu - \partial_\mu \varphi \end{pmatrix} \\
&= \begin{pmatrix} \partial_\mu \varphi \\ \Box \varphi \end{pmatrix} + \begin{pmatrix} \Box A_\mu - \partial_\mu \varphi \\ 0 \end{pmatrix} \\
&= \Box \begin{pmatrix} A_\mu \\ \varphi \end{pmatrix} \\
&= \Box \mathcal{A} \; .
\end{aligned} \tag{3.126}
$$

Doing the same computation for the other elements of $\mathcal{X}^{\rm YM}$ one finds that, indeed, b as defined in (3.124) obeys $[B_1, b] = \Box$. In this version of Yang-Mills, the gauge choice $b\mathcal{A} = 0$ sets φ to zero and the scalar field equation reduces to the condition $\partial \cdot A = 0$. We will, however, not impose such a gauge fixing in the present chapter.

3.4.2 Kinematic algebra of Yang-Mills theory

We now turn to constructing the kinematic algebra of Yang-Mills theory. Similarly to Chern-Simons theory, Yang-Mills theory exhibits a factorization between color and kinematics (or *off-shell color ordering*), in complete analogy to the scattering amplitudes as we discussed in the previous chapter. The vector space factorizes as

$$
\mathcal{X}^{\rm YM} = \mathcal{K}^{\rm YM} \otimes \mathfrak{g} \; , \tag{3.127}
$$

where $\mathcal{K}^{\rm YM}$ is an infnite dimensional graded space and, exactly as in Chern-Simons theory, the elements of the L_∞-algebra, differential and two-bracket factorize as before, namely

$$
\begin{aligned}
y(x) = u^a(x) \otimes t_a \; , \quad & u^a(x) \in \mathcal{K}^{\rm YM} \; , \quad t_a \in \mathfrak{g} \; , \\
B_1(y(x)) = m_1(u^a(x)) \otimes t_a \; , \quad & B_2(y_1, y_2) = (-1)^{y_1} m_2(u_1^a, u_2^b) \otimes f_{ab}{}^c t_c \; ,
\end{aligned} \tag{3.128}
$$

with kinematic degrees

$$
|u(x)| = |y(x)|_{\mathcal{X}^{\rm YM}} + 1 \; , \quad |m_1| = |B_1|_{\mathcal{X}^{\rm YM}} \; , \quad |m_2| = |B_2|_{\mathcal{X}^{\rm YM}} - 1 = 0 \; . \tag{3.129}
$$

In contrast to Chern-Simons, however, in Yang-Mills there exists a three-bracket between three fields which factorizes as

$$B_3(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) = 2\, m_3(\mathcal{A}^a_{(1}, \mathcal{A}^b_2, \mathcal{A}^c_{3)}) \otimes f_{be}{}^a f_{cd}{}^e t_d \;, \tag{3.130}$$

where the kinematic three-product has degree $|m_3| = -1$ because B_3 contains two Lie brackets of $\mathfrak{g}$, each with degree one. The kinematic space of Yang-Mills theory $\mathcal{K}^{\rm YM}$ equipped with the differential m_1, and the products m_2 and m_3 is a so-called C_∞-algebra [100]. More precisely, a C_∞-algebra is a graded vector space equipped with a (possibly infinite) set of n-products m_n that obey a (possibly infinite) set of relations, called C_∞ relations. In Yang-Mills, nevertheless, the only non-trivial maps are m_1, m_2 and m_3. The symmetry property of C_∞ products m_n is determined by requiring that they vanish on shuffles. With our degree conventions this property reads

$$\begin{aligned} m_2(u_1, u_2) - (-)^{u_1 u_2} m_2(u_2, u_1) = 0 \;, \\ m_3(u_1, u_2, u_3) - (-)^{u_1 u_2} m_3(u_2, u_1, u_3) + (-)^{u_1(u_2+u_3)} m_3(u_2, u_3, u_1) = 0 \;, \end{aligned} \tag{3.131}$$

which for m_2 is the same as being graded symmetric and in principle, for general C_∞-algebras, there may exist higher brackets with more complicated symmetry properties. In Yang-Mills, the nontrivial C_∞ relations amount to nilpotency of m_1, the Leibniz property of m_1 with respect to m_2 (m_1 is a derivation for m_2), and associativity of m_2 up to homotopy:

$$\begin{aligned} m_1^2(u) &= 0 \;, \\ m_1(m_2(u_1, u_2)) &= m_2(m_1(u_1), u_2) + (-)^{u_1} m_2(u_1, m_1(u_2)) \;, \\ m_2\big(m_2(u_1, u_2), u_3\big) - m_2\big(u_1, m_2(u_2, u_3)\big) &= m_1(m_3(u_1, u_2, u_3)) + m_3(m_1(u_1), u_2, u_3) \\ &\quad + (-)^{u_1} m_3(u_1, m_1(u_2), u_3) + (-)^{u_1+u_2} m_3(u_1, u_2, m_1(u_3)) \;. \end{aligned} \tag{3.132}$$

Notice that the last relation, in the same spirit of L_∞-algebras, parametrizes the failure of the two-product m_2 to be associative.

The kinematic maps m_n acting on all the color stripped elements of $\mathcal{K}^{\rm YM}$ are the following:

Differential m_1

$$m_1(\Lambda) = \begin{pmatrix} \partial_\mu \lambda \\ \Box \lambda \end{pmatrix} \in K_1 \,, \quad m_1(\mathcal{A}) = \begin{pmatrix} \partial \cdot A - \varphi \\ \Box A_\mu - \partial_\mu \varphi \end{pmatrix} \in K_2 \,, \quad m_1(\mathcal{E}) = \begin{pmatrix} 0 \\ \Box E - \partial_\mu E^\mu \end{pmatrix} \in K_3 \,, \tag{3.133}$$

Two-product m_2

$$
m_2\big(\Lambda_1,\Lambda_2\big) = \begin{pmatrix} \lambda_1\lambda_2 \\ 0 \end{pmatrix} \in K_0 \ , \qquad m_2\big(\Lambda,\mathcal{A}\big) = \begin{pmatrix} \lambda\, A_\mu \\ \partial^\nu(\lambda A_\nu) \end{pmatrix} \in K_1 \ ,
$$
$$
m_2\big(\mathcal{A}_1,\mathcal{A}_2\big) = \begin{pmatrix} 0 \\ \big(A_1 \bullet A_2\big)_\mu \end{pmatrix} \in \ K_2 \ , \qquad m_2\big(\Lambda,\mathcal{E}\big) = \begin{pmatrix} 0 \\ \lambda(E_\mu - \partial_\mu E) \end{pmatrix} \in \ K_2 \ , \tag{3.134}
$$
$$
m_2\big(\mathcal{A},\mathcal{E}\big) = \begin{pmatrix} 0 \\ -A_\mu(E^\mu - \partial^\mu E) \end{pmatrix} \in \ K_3 \ , \quad m_2\big(\Lambda,\mathcal{N}\big) = \begin{pmatrix} 0 \\ \lambda\, N \end{pmatrix} \in \ K_3 \ ,
$$

where the antisymmetric product $\bullet$ is given by

$$
\big(V \bullet W\big)^\mu = V^\nu \partial_\nu W^\mu + \big(\partial^\mu V_\nu - \partial_\nu V^\mu\big) W^\nu + \big(\partial_\nu V^\nu\big) W^\mu - \big(V \leftrightarrow W\big) \ . \tag{3.135}
$$

Three-product m_3

$$
m_3(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3) = \begin{pmatrix} 0 \\ A_1 \cdot A_2\, A_{3\,\mu} + A_3 \cdot A_2\, A_{1\,\mu} - 2\, A_1 \cdot A_3\, A_{2\,\mu} \end{pmatrix} \in K_2 \ . \tag{3.136}
$$

Alternatively, one can redefine m_3 as

$$
m_{3h}(\mathcal{A}_1,\mathcal{A}_2|\mathcal{A}_3) = \begin{pmatrix} 0 \\ A_{1\,\mu} A_2 \cdot A_3 - A_{2\,\mu} A_1 \cdot A_3 \end{pmatrix} \ . \tag{3.137}
$$

This redefined product is a graded hook in the labels, meaning that it is graded symmetric in the first two inputs (which we highlight by the vertical bar) and vanishes upon total graded symmetrization.

Having found the C_∞-algebra above, we now turn to the construction of the kinematic algebra of Yang-Mills theory. Following the construction of the kinematic algebra of Chern-Simons in the previous section, now that we have an associative-commutative-type algebra, we can look at the algebraic behaviour of the kinematic products m_n with respect to the map b. As we will see momentarily, the kinematic algebra of Yang-Mills is a generalization of $\mathrm{BV}^\square$-algebras, named $\mathrm{BV}^\square_\infty$-algebras.

The action of b on $\mathcal{K}^{\mathrm{YM}}$ is:

$$
b\Lambda = 0 \ , \ b\mathcal{A} = \begin{pmatrix} \varphi \\ 0 \end{pmatrix} \in K_0^{\mathrm{YM}} \ , \ b\mathcal{E} = \begin{pmatrix} E_\mu \\ 0 \end{pmatrix} \in K_1^{\mathrm{YM}} \ , \ b\mathcal{N} = \begin{pmatrix} N \\ 0 \end{pmatrix} \in K_2^{\mathrm{YM}} \ , \tag{3.138}
$$

or diagramatically

$$
\begin{array}{ccccccc}
K_0^{\mathrm{YM}} & \xrightarrow{m_1} & K_1^{\mathrm{YM}} & \xrightarrow{m_1} & K_2^{\mathrm{YM}} & \xrightarrow{m_1} & K_3^{\mathrm{YM}} \\
\lambda & & A_\mu & & E & & \\
& \overset{b}{\nwarrow} & & \overset{b}{\nwarrow} & & \overset{b}{\nwarrow} & \\
& & \varphi & & E_\mu & & \mathcal{N}
\end{array} \ , \tag{3.139}
$$

where all the elements of the chain complex are color-stripped objects. Notice that b, like the differential, has the same explicit realization on $\mathcal{X}^{\mathrm{YM}}$ and $\mathcal{K}^{\mathrm{YM}}$ and thus the commutation relation

$$[B_1, b] \equiv [m_1, b] = \square \; , \tag{3.140}$$

still holds on the kinematic space $\mathcal{K}^{\mathrm{YM}}$. Similarly to Chern-Simons, b in Yang-Mills does not obey the Leibniz rule with respect to the kinematic two-product m_2. Hence we can parametrize its failure by defining a new bracket $b_2 : \mathcal{K}^{\mathrm{YM}} \times \mathcal{K}^{\mathrm{YM}} \to \mathcal{K}^{\mathrm{YM}}$ as

$$b_2(u_1, u_2) := bm_2(u_1, u_2) - m_2(bu_1, u_2) - (-1)^{u_1} m_2(u_1, bu_2) \; . \tag{3.141}$$

The differential m_1 obeys the Leibniz rule with respect to b_2 modulo box, i.e

$$\begin{aligned} m_1 b_2(u_1, u_2) - b_2(m_1(u_1), u_2) - (-1)^{u_2} b_2(u_1, m_1(u_2)) &= [\square, m_2](u_1, u_2) \\ &\equiv \square m_2(u_1, u_2) - m_2(\square u_1, u_2) - m_2(u_1, \square u_2) \\ &= 2\, m_2(\partial_\mu u_1, \partial^\mu u_2) \; , \end{aligned} \tag{3.142}$$

which follows from the commutation relation (3.140).

Recall that one of the defining relations of $\mathrm{BV}^{\square}$-algebras is the Poisson compatibility condition

$$\mathrm{Poiss}(u_1, u_2, u_3) = 0 \; , \tag{3.143}$$

where for the sake of future notation we introduced the *Poissonator*

$$\begin{aligned} \mathrm{Poiss}(u_1, u_2, u_3) := b_2(m_2(u_1, u_2), u_3) &- (-1)^{u_1(u_2+u_3)} m_2(b_2(u_2, u_3), u_1) \\ &- (-1)^{u_3(u_1+u_2)} m_2(b_2(u_3, u_1), u_2) \; . \end{aligned} \tag{3.144}$$

The kinematic algebra of Chern-Simons is *strict* in the sense that the Poissonator vanishes strictly, as is the case when m_2 is the wedge product of forms and b_2 is related to the Schouten-Nijenhuis bracket by the metric. From the vanishing of the Poissonator it then follows that b_2 obeys the Jacobi identity, and hence a consistent subsector of the kinematic algebra of Chern-Simons is a genuine Lie algebra. This is *not* the case in Yang-Mills theory as we will illustrate with examples below. Rather, the above compatibility relation is deformed in a controlled manner and the resulting algebraic structure is then a $\mathrm{BV}^{\square}_{\infty}$-algebra. Explicitly, the compatibility relation (3.143) gets deformed as:

$$\begin{aligned} \mathrm{Poiss}(u_1, u_2, u_3) =& [m_1, \theta_3](u_1, u_2, u_3) + m_{3h} d_{\square}(u_1, u_2, u_3) - m_{3h} d_s(u_1, u_2, u_3) \\ & - (-1)^{u_1(u_2+u_3)} m_{3h} d_s(u_2, u_3, u_1) - (-1)^{u_3(u_1+u_2)} m_{3h} d_s(u_3, u_1, u_2) \; , \end{aligned} \tag{3.145}$$

where θ_3 is a degree -2 trilinear map, the commutator $[m_1, \theta_3]$ is defined as

$$[m_1,\theta_3](u_1,u_2,u_3) := m_1\theta_3(u_1,u_2,u_3) - \theta_3(m_1(u_1),u_2,u_3) - (-1)^{u_1}\theta_3(u_1,m_1(u_2),u_3) - (-1)^{u_1+u_2}\theta_3(u_1,u_2,m_1(u_3)) \ , \tag{3.146}$$

and the d_s and $d_\Box$ are derivative operators that act on the inputs as

$$\begin{aligned} d_s(u_1,u_2,u_3) &:= 2\,(\partial^\mu u_1, \partial_\mu u_2, u_3) \ , \\ d_t(u_1,u_2,u_3) &:= 2\,(u_1, \partial^\mu u_2, \partial_\mu u_3) \ , \\ d_u(u_1,u_2,u_3) &:= 2\,(\partial^\mu u_1, u_2, \partial_\mu u_3) \ , \\ d_\Box &:= d_s + d_t + d_u \ . \end{aligned} \tag{3.147}$$

The compatibility relation (3.145) parametrizes the failure of b_2 to be a generalized Poisson (or Gerstenhaber) backet. This failure is controlled by a homotopy θ_3, and box failures generated by the operator d_s. Let us emphasize that our present construction is only up to trilinear order. However, in principle there may exist higher homotopies (e.g. θ_4) that obey higher relations. Recall that in Chern-Simons the compatibility relation (3.143) leads to the Jacobi identity upon acting with the b operator. Acting with b on the deformed compatibility relation (3.145), instead, leads to a more exotic relation, where the Jacobi identity fails on the one hand by an homotopy b_3, and on the other hand by box failures that contain θ_3. More explicitly, the Jacobi-like relation obeyed by b_2 is the following:

$$\mathrm{Jac}(u_1,u_2,u_3) + [m_1,b_3](u_1,u_2,u_3) + \theta_3 d_s(u_1,u_2,u_3) + (-1)^{u_1(u_2+u_3)}\,\theta_3 d_s(u_2,u_3,u_1) + (-1)^{u_3(u_1+u_2)}\,\theta_3(u_3,u_1,u_2) = 0 \ . \tag{3.148}$$

The Jacobiator $\mathrm{Jac}(u_1,u_2,u_3)$ is defined as

$$\mathrm{Jac}(u_1,u_2,u_3) := b_2(b_2(u_1,u_2),u_3) + (-1)^{u_1(u_2+u_3)}\,b_2(b_2(u_2,u_3),u_1) + (-1)^{u_3(u_1+u_2)}\,b_2(b_2(u_3,u_1),u_2) \ , \tag{3.149}$$

and the three-bracket b_3 is given by the commutator

$$b_3(u_1,u_2,u_3) = -[b,\theta_{3\,s}](u_1,u_2,u_3) \ , \tag{3.150}$$

where $\theta_{3\,s}$ is the part of θ_3 that is (graded) symmetric in the arguments. In addition, θ_3 has a hook component

$$\theta_{3\,h}(u_1,u_2|u_3) = -[b,m_{3\,h}](u_1,u_2|u_3) \ , \tag{3.151}$$

which is graded symmetric in the first two inputs and vanishes upon summing over a graded symmetric combination of inputs, i.e

$$\theta_{3\,h}(u_1,u_2|u_3) + (-1)^{u_1(u_2+u_3)}\theta_{3\,h}(u_2,u_3|u_1) + (-1)^{u_3(u_1+u_2)}\theta_{3\,h}(u_3,u_1|u_2) = 0 \ . \tag{3.152}$$

To summarize, the vector space $\mathcal{K}^{\mathrm{YM}}$ endowed with the kinematic products m_n, the operator

b that allows us to construct b_2, the d'Alambert operator $\Box$ and θ_3, forms a $\mathrm{BV}_\infty^\Box$-algebra to trilinear order provided that the following holds:

- The kinematics maps m_n obey the C_∞ relations (3.132).
- The operator b is nilpotent and obeys the commutation relation with the differential $[m_1, b] = \Box$.
- The differential m_1 obeys the Leibniz rule with respect to b_2 modulo box as in equation (3.142).
- The Poisson compatibility condition holds up to a homotopy θ_3 and up to box failures as in equation (3.145) and acting with b leads to the Jacobi-like relation (3.148).

Let us stress that, even though $\mathrm{BV}_\infty^\Box$-algebras do not contain Lie (or L_∞-) algebra as a consistent subsector, the maps of these exotic algebras obey Jacobi-like relations. For this reason, one can think of $\mathrm{BV}_\infty^\Box$ as Lie-type algebras with controlled failures to obey the Jacobi identity.

In order to prove that the kinematic algebra of Yang-Mills theory is indeed a $\mathrm{BV}_\infty^\Box$-algebra to trilinear order, it is necessary to first explicitly compute the Poissonator and the terms involving $m_{3\,h}$ with all possible combinations of inputs that are non-trivial by degree. Then, from the resulting expressions, one should be able to identify the differential acting on various elements and determine θ_3. In order to illustrate the procedure, let us compute as an example the Poissonator of two gauge parameters and one field $\mathrm{Poiss}(\Lambda_1, \Lambda_2, \mathcal{A})$, and let us ignore the terms containing $m_{3\,h}$ as they are trivial with this combination of inputs. The deformed compatibility condition (3.145) for our choice of inputs reads

$$\mathrm{Poiss}(\Lambda_1, \Lambda_2, \mathcal{A}) = m_1\theta_3(\Lambda_1, \Lambda_2, \mathcal{A}) - \theta_3(m_1(\Lambda_1), \Lambda_2, \mathcal{A}) - \theta_3(\Lambda_1, m_1(\Lambda_2), \mathcal{A}) \ . \tag{3.153}$$

Our task now is to compute the Poissonator and then read off θ_3. To that end, we have to find the two-bracket b_2 between gauge parameters and between a gauge parameter and an equation. From the definition of b_2 (3.141), one finds

$$\begin{aligned} b_2(\Lambda_1, \Lambda_2) &:= bm_2(\Lambda_1, \Lambda_2) - m_2(b\Lambda_1, \Lambda_2) - m_2(\Lambda_1, b\Lambda_2) = 0 \ , \\ b_2(\Lambda, \mathcal{A}) &:= bm_2(\Lambda, \mathcal{A}) - m_2(b\Lambda, \mathcal{A}) - m_2(\Lambda, b\mathcal{A}) = \partial^\mu\big(\lambda\, A_\mu\big) - \lambda\,\varphi \in K_0 \ . \end{aligned} \tag{3.154}$$

where we used the explicit expression for the action of m_2 on the elements of $\mathcal{K}^{\mathrm{YM}}$ displayed in equation (3.134), as well as $b\Lambda = 0$. Using the above b_2, the Poissonator $\mathrm{Poiss}(\Lambda_1, \Lambda_2, \mathcal{A})$ is

$$\mathrm{Poiss}(\Lambda_1, \Lambda_2, \mathcal{A}) = -\lambda_1\,\lambda_2\,\big(\partial\cdot A - \varphi\big) \stackrel{!}{=} [m_1, \theta_3](\Lambda_1, \Lambda_2, \mathcal{A}) \ , \tag{3.155}$$

from which we can immediately identify the scalar component of action of the differential on the field, corresponding to $\theta_3(\Lambda_1, \Lambda_2, m_1(\mathcal{A}))$, and thus we conclude:

$$\begin{aligned} \theta_3(\Lambda_1, \Lambda_2, \mathcal{A}) &= 0 \in K_0 \ , \\ \theta_3(\Lambda, \mathcal{A}_1, \mathcal{A}_2) &= 0 \in K_0 \ , \\ \theta_3(\Lambda_1, \Lambda_2, \mathcal{E}) &= \lambda_1\,\lambda_2\,E \in K_0 \ . \end{aligned} \tag{3.156}$$

Performing analogous computations one finds the remaining θ_3. Here we list the non-trivial ones:

$$\begin{aligned}
\theta_3(\mathcal{E},\lambda_1,\lambda_2) &= \lambda_1\lambda_2\, E\in K_0\ ,\\
\theta_3(\mathcal{N},\lambda_1,\lambda_2) &= -\begin{pmatrix}0\\ \lambda_1\lambda_2\,\mathcal{N}\end{pmatrix}\in K_1\ ,\\
\theta_3(\mathcal{E},\mathcal{A},\lambda) &= \begin{pmatrix}A_\mu\lambda E\\ A^\nu E_\nu\lambda + A^\nu\partial_\nu\lambda E+\varphi\lambda E\end{pmatrix}\in K_1\ ,\\
\theta_3(\lambda,\mathcal{E}_1,\mathcal{E}_2) &= \begin{pmatrix}\lambda E_1E_2\\ \lambda(E_1^\mu E_2+E_2^\mu E_1)\end{pmatrix}\in K_2\ ,\\
\theta_3(\lambda,\mathcal{E},\mathcal{N}) &= \lambda E\mathcal{N}\in K_3\ ,\\
\theta_3(\mathcal{A},\mathcal{E}_1,\mathcal{E}_2) &= \varphi E_1E_2+2\,A^\mu\partial_\mu(E_1E_2)-A_\mu(E_1^\mu E_2+E_2^\mu E_1)\in K_3\ .
\end{aligned}\tag{3.157}$$

The last two θ_3 maps have both a graded symmetric and a hook part, which we give explicitly:

$$\begin{aligned}
&\theta_3(\mathcal{A}_1,\mathcal{A}_2|\mathcal{A}_3) = \theta_{3s}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)+\theta_{3h}(\mathcal{A}_1,\mathcal{A}_2|\mathcal{A}_3)\in K_1\ ,\quad \text{where}\\
&\theta_{3s}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3) \overset{[123]}{=} 6\begin{pmatrix}0\\ A_1\cdot\partial A_2\cdot A_3\end{pmatrix}\ ,\quad \theta_{3h}(\mathcal{A}_1,\mathcal{A}_2|\mathcal{A}_3) \overset{[12]}{=} -2\begin{pmatrix}A_1^\mu A_2\cdot A_3\\ 0\end{pmatrix}\ ,
\end{aligned}\tag{3.158}$$

and

$$\begin{aligned}
&\theta_3(\mathcal{A}_1,\mathcal{A}_2|\mathcal{E}) = \theta_{3s}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{E})+\theta_{3h}(\mathcal{A}_1,\mathcal{A}_2|\mathcal{E})\in K_2\ ,\\
&\theta_3(\mathcal{E},\mathcal{A}_1|\mathcal{A}_2) = \theta_{3s}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{E})+\theta_{3h}(\mathcal{E},\mathcal{A}_1|\mathcal{A}_2)\in K_2\ ,\quad \text{where}\\
&\theta_{3s}(\mathcal{E},\mathcal{A}_1,\mathcal{A}_2) \overset{[12]}{=} 2\begin{pmatrix}0\\ A_2^\mu A_1^\nu\big(E_\nu+\partial_\nu E\big)+E\,\big(2\,A_1\cdot\partial A_2^\mu+\partial^\mu A_1\cdot A_2\big)+A_2^\mu\varphi_1 E\end{pmatrix}\ ,\\
&\theta_{3h}(\mathcal{A}_1,\mathcal{A}_2|\mathcal{E}) \overset{[12]}{=} -2\begin{pmatrix}0\\ A_1^\mu A_2\cdot E\end{pmatrix}\ ,\quad \theta_{3h}(\mathcal{E},\mathcal{A}_1|\mathcal{A}_2) = \begin{pmatrix}0\\ A_1^\mu A_2\cdot E - E^\mu A_1\cdot A_2\end{pmatrix}\ .
\end{aligned}\tag{3.159}$$

We do not give the 3-brackets b_3, since they can be computed directly from $b_3=-[b,\theta_{3s}]$.

3.5 Constructing a basis for the kinematic algebra

In order to close this chapter, in this section we develop a technology that will allow us to implement a precise and well defined double copy prescription in the following chapter. To that end, let us describe in more detail the structure of the kinematic space $\mathcal{K}^{\rm YM}\equiv\mathcal{K}$, where we drop the superscript YM for the sake of brevity. To begin with, recall that $\mathcal{K}$ is a graded vector space given by the direct sum of subspaces K_i of homogenous degree: $\mathcal{K}=\bigoplus_{i=0}^3 K_i$. Elements in each K_i are identified as gauge parameters Λ, fields $\mathcal{A}$, field equations $\mathcal{E}$ and Noether identities $\mathcal{N}$

according to the following diagram:

$$\begin{array}{cccc} K_0 & K_1 & K_2 & K_3 \\ \Lambda & \mathcal{A} & \mathcal{E} & \mathcal{N} \end{array} . \tag{3.160}$$

In order to display explicitly its degree structure, we find useful to view $\mathcal{K}$ as the tensor product of a finite dimensional graded vector space $\mathcal{Z} = \bigoplus_{i=0}^{3} \mathcal{Z}_i$ with spacetime smooth functions of degree zero: $\mathcal{K} = \mathcal{Z} \otimes C^\infty(\mathcal{M})$, where $\mathcal{M}$ is flat $D-$dimensional Minkowski spacetime. To construct a basis for $\mathcal{Z}$, let us introduce $d+2$ graded vectors $\theta_M = (\theta_+, \theta_\mu, \theta_-)$, where $\mu = 0, 1, \dots D-1$ is a Lorentz vector index. We assign degrees $|\theta_M| = 1 - M$, meaning

$$|\theta_+| = 0 \ , \quad |\theta_\mu| = 1 \ , \quad |\theta_-| = 2 \ . \tag{3.161}$$

We now take a second copy of these vectors with degrees shifted by one, which we denote by $c\,\theta_M$, with $|c\,\theta_M| = 2 - M$, or

$$|c\,\theta_+| = 1 \ , \quad |c\,\theta_\mu| = 2 \ , \quad |c\,\theta_-| = 3 \ . \tag{3.162}$$

A basis of $\mathcal{Z}$ is given by $Z_A = (\theta_M, c\,\theta_M)$, which shows a natural $\mathbb{Z}_2$ symmetry. The isomorphism between θ_M and $c\,\theta_M$ can be implemented by nilpotent operators b and c defined by their action on the basis Z_A:

$$\begin{aligned} & c\,(\theta_M) := c\,\theta_M \ , \quad c\,(c\,\theta_M) := 0 \ , \\ & b\,(\theta_M) := 0 \ , \qquad b\,(c\,\theta_M) := \theta_M \ . \end{aligned} \tag{3.163}$$

The degrees of b and c are thus fixed to be $|c| = +1$ and $|b| = -1$, and from their definition one can see that they obey the algebra

$$c^2 = 0 \ , \quad b^2 = 0 \ , \quad bc + cb = 1 \ . \tag{3.164}$$

The basis elements of $\mathcal{Z}$ can be displayed according to their degree in a way that emphasizes the $\mathbb{Z}_2$ symmetry:

$$\begin{array}{ccccccc} \mathcal{Z}_0 & & \mathcal{Z}_1 & & \mathcal{Z}_2 & & \mathcal{Z}_3 \\ \theta_+ & & \theta_\mu & & \theta_- & & \\ & \overset{b}{\nwarrow} & & \overset{b}{\nwarrow} & & \overset{b}{\nwarrow} & \\ & & c\,\theta_+ & & c\,\theta_\mu & & c\,\theta_- \end{array} \ , \tag{3.165}$$

where we have indicated the action of b (c acts by reversing the arrows).

Upon tensoring $\mathcal{Z}$ with smooth functions, we obtain the kinematic space $\mathcal{K}$ of Yang-Mills theory. The degree in $\mathcal{K}$ coincides with the one in $\mathcal{Z}$, meaning that for an homogeneous element $\psi = Z\,f(x)$ one has $|\psi| = |Z|$. An arbitrary element in $\mathcal{K}$ can thus be expanded as

$$\psi = Z_A\,\psi^A(x) \ . \tag{3.166}$$

Comparing the degree structure (3.160) of $\mathcal{K}$ with (3.165), one can see that the Yang-Mills fields, parameters and so on are given by the following vectors in $\mathcal{K}$ with homogeneous degrees:

$$
\begin{aligned}
\lambda &= \theta_+ \, \lambda(x) \, \in \mathcal{K}_0 \, , \\
\mathcal{A} &= \theta_\mu \, A^\mu(x) + c \, \theta_+ \, \varphi(x) \, \in \mathcal{K}_1 \, , \\
\mathcal{E} &= \theta_- \, E(x) + c \, \theta_\mu \, E^\mu(x) \, \in \mathcal{K}_2 \, , \\
\mathcal{N} &= c \, \theta_- \, \mathcal{N}(x) \, \in \mathcal{K}_3 \, .
\end{aligned}
\tag{3.167}
$$

The $\mathbb{Z}_2$ structure and the action of b and c are inherited from $\mathcal{Z}$. One can indeed draw the same diagram (3.165) in $\mathcal{K}$ to display this:

$$
\begin{array}{cccccccc}
K_0 & & K_1 & & K_2 & & K_3 & \\
\lambda & & A^\mu & & E & & & , \\
& \overset{b}{\nwarrow} & & \overset{b}{\nwarrow} & & \overset{b}{\nwarrow} & & \\
& & \varphi & & E^\mu & & \mathcal{N} &
\end{array}
\tag{3.168}
$$

where we omitted the Z_A and only wrote the component fields. Notice that the operator b introduced here is the same operator that we used to construct the kinematic $\mathrm{BV}_\infty^\square$-algebra of Yang-Mills in the previous section.

We now turn to defining the action of the kinematic maps on the basis elements Z_A. Given the tensor product structure of $\mathcal{K} = \mathcal{Z} \otimes C^\infty(\mathcal{M})$ and the expansion (3.166) of arbitrary vectors, the kinematic products m_n act on elements of $\mathcal{K}$ as follows:

$$
\begin{aligned}
m_1(\psi) &= \hat{m}_1(Z_A) \, \psi^A(x) \, , \\
m_2(\psi_1, \psi_2) &= \mu \left[\hat{m}_2(Z_A, Z_B) \left(\psi_1^A(x) \otimes \psi_2^B(x) \right) \right] \, , \\
m_{3h}(\psi_1, \psi_2, \psi_3) &= \mu \left[\hat{m}_{3h}(Z_A, Z_B, Z_C) \left(\psi_1^A(x) \otimes \psi_2^B(x) \otimes \psi_3^C(x) \right) \right] \, .
\end{aligned}
\tag{3.169}
$$

The operators $\hat{m}_n(Z_{A_1}, \ldots, Z_{A_n})$ are $\mathcal{Z}-$valued multi-differential operators acting on the component fields:

$$
\begin{aligned}
\hat{m}_2(Z_1, Z_2) : C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M}) \; &\to \; \mathcal{Z} \otimes \Big(C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M}) \Big) \, , \\
\hat{m}_{3h}(Z_1, Z_2, Z_3) : C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M}) \; &\to \; \mathcal{Z} \otimes \Big(C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M}) \otimes C^\infty(\mathcal{M}) \Big) \, ,
\end{aligned}
\tag{3.170}
$$

and μ denotes the local pointwise product:

$$
\mu \left[f_1(x) \otimes \ldots \otimes f_n(x) \right] = f_1(x) \ldots f_n(x) \, . \tag{3.171}
$$

To clarify this point, let us give some explicit examples. Acting on the basis vectors of $\mathcal{Z}_1$ (corresponding to fields), we have

$$
\begin{aligned}
\hat{m}_1(\theta_\mu) &= c \, \theta_\mu \, \square + \theta_- \, \partial_\mu \, , \\
\hat{m}_1(c \, \theta_+) &= -c \, \theta_\mu \, \partial^\mu - \theta_- \, .
\end{aligned}
\tag{3.172}
$$

Using (3.169) one can see that the differential m_1 acts on a field $\mathcal{A}$ as

$$\begin{aligned} m_1(\mathcal{A}) &= \hat{m}_1(\theta_\mu)\, A^\mu + \hat{m}_1(c\,\theta_+)\,\varphi \\ &= c\,\theta_\mu \Big(\Box A^\mu - \partial^\mu \varphi\Big) + \theta_- \Big(\partial\cdot A - \varphi\Big)\,. \end{aligned} \tag{3.173}$$

For the next example, the non-vanishing part of m_2 between fields $\mathcal{A}_1$ and $\mathcal{A}_2$ is encoded in the bi-differential operator

$$\begin{aligned} \hat{m}_2(\theta_\mu,\theta_\nu) = c\,\theta_\nu &\Big[\big(\partial_\mu\otimes\mathbb{1}\big) + 2\,\big(\mathbb{1}\otimes\partial_\mu\big)\Big] - c\,\theta_\mu\Big[\big(\mathbb{1}\otimes\partial_\nu\big) + 2\,\big(\partial_\nu\otimes\mathbb{1}\big)\Big] \\ &+ c\,\theta_\rho\,\eta_{\mu\nu}\Big[\big(\partial^\rho\otimes\mathbb{1}\big) - \big(\mathbb{1}\otimes\partial^\rho\big)\Big]\,. \end{aligned} \tag{3.174}$$

This acts on $\big(A_1^\mu\otimes A_2^\nu\big)$ as

$$\begin{aligned} \hat{m}_2(\theta_\mu,\theta_\nu)\,\big(A_1^\mu\otimes A_2^\nu\big) &= c\,\theta_\nu\Big[\big(\partial\cdot A_1\otimes A_2^\nu\big) + 2\,\big(A_1^\mu\otimes\partial_\mu A_2^\nu\big)\Big] \\ &\quad - c\,\theta_\mu\Big[\big(A_1^\mu\otimes\partial\cdot A_2\big) + 2\,\big(\partial_\nu A_1^\mu\otimes A_2^\nu\big)\Big] \\ &\quad + c\,\theta_\rho\,\Big[\big(\partial^\rho A_1^\mu\otimes A_{2\mu}\big) - \big(A_1^\mu\otimes\partial^\rho A_{2\mu}\big)\Big]\,, \end{aligned} \tag{3.175}$$

and the pointwise evaluation with μ finally yields (with a dot denoting contraction of Lorentz indices)

$$\begin{aligned} m_2(\mathcal{A}_1,\mathcal{A}_2) &= \mu\left[\hat{m}_2(\theta_\mu,\theta_\nu)\,\big(A_1^\mu\otimes A_2^\nu\big)\right] \\ &= c\,\theta_\mu\,\Big(\partial\cdot A_1\,A_2^\mu + 2\,A_1\cdot\partial A_2^\mu + \partial^\mu A_1\cdot A_2 - (1\leftrightarrow 2)\Big) \\ &\equiv c\,\theta_\mu\,\big(A_1\bullet A_2\big)^\mu\,, \end{aligned} \tag{3.176}$$

which corresponds to the color-stripped cubic vertex of Yang-Mills. Similarly, the only non-vanishing component of m_{3h} comes from the operator

$$\hat{m}_{3h}(\theta_\mu,\theta_\nu,\theta_\rho) = \Big(c\,\theta_\mu\,\eta_{\nu\rho} - c\,\theta_\nu\,\eta_{\mu\rho}\Big)\,\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big)\,, \tag{3.177}$$

yielding

$$\begin{aligned} m_{3h}(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3) &= \mu\left[\hat{m}_{3h}(\theta_\mu,\theta_\nu,\theta_\rho)\,\big(A_1^\mu\otimes A_2^\nu\otimes A_3^\rho\big)\right] \\ &= c\,\theta_\mu\,\Big(A_1^\mu\,A_2\cdot A_3 - (1\leftrightarrow 2)\Big)\,, \end{aligned} \tag{3.178}$$

corresponding to the color-stripped quartic vertex. Similarly, for θ_3 we have the operator $\hat{\theta}_3$ that acts on this basis and leads to

$$\theta_3(\psi_1,\psi_2,\psi_3) = \mu\left[\hat{\theta}_3(Z_A,Z_B,Z_C)\,\Big(\psi_1^A(x)\otimes\psi_2^B(x)\otimes\psi_3^C(x)\Big)\right]\,. \tag{3.179}$$

The same logic can be applied to find $\hat{b}_2$ and $\hat{b}_3$. A complete list of the operators $\hat{m}_n$ and θ_3 can be found in appendix A.

3.5.1 Intrinsic input-free notation

Having now an explicit basis for our kinematic algebra, in this section we develop a convenient input-free notation developed in the upcoming paper [5] which will significantly simplify computations when implementing our double copy prescription in the forthcoming chapter. Here we will denote by $\mathcal{O}$ any linear operator in $\mathcal{K}$, of degree $|\mathcal{O}|$. Generic bilinear and trilinear maps will be denoted by $\mathcal{M}$ and $\mathcal{T}$, respectively, with arbitrary intrinsic degrees $|\mathcal{M}|$ and $|\mathcal{T}|$. Similarly to (3.169), these generic maps act on elements of $\mathcal{K}$ as

$$\begin{aligned} \mathcal{O}(\psi) &= \hat{\mathcal{O}}(Z_A)\,\psi^A(x)\;, \\ \mathcal{M}(\psi_1,\psi_2) &= \mu\left[\hat{\mathcal{M}}(Z_A,Z_B)\left(\psi_1^A(x)\otimes\psi_2^B(x)\right)\right]\;, \\ \mathcal{T}(\psi_1,\psi_2,\psi_3) &= \mu\left[\hat{\mathcal{T}}(Z_A,Z_B,Z_C)\left(\psi_1^A(x)\otimes\psi_2^B(x)\otimes\psi_3^C(x)\right)\right]\;, \end{aligned} \tag{3.180}$$

where $\hat{\mathcal{O}}$, $\hat{\mathcal{M}}$ and $\hat{\mathcal{T}}$ are $\mathcal{Z}-$valued multidifferential operators acting on the component functions. We define the graded commutator of operators $\mathcal{O}_1$, $\mathcal{O}_2$ by

$$[\mathcal{O}_1,\mathcal{O}_2](\psi) := \mathcal{O}_1\big(\mathcal{O}_2\psi\big) - (-1)^{\mathcal{O}_1\mathcal{O}_2}\mathcal{O}_2\big(\mathcal{O}_1\psi\big)\;, \tag{3.181}$$

where every symbol in exponents refers to the degree of a map or element. The commutators of an operator $\mathcal{O}$ with bilinear and trilinear maps $\mathcal{M}$ and $\mathcal{T}$ are the bilinear map $[\mathcal{O},\mathcal{M}]$ and trilinear map $[\mathcal{O},\mathcal{T}]$ given by

$$\begin{aligned} [\mathcal{O},\mathcal{M}](\psi_1,\psi_2) &:= \mathcal{O}\mathcal{M}(\psi_1,\psi_2) - (-1)^{\mathcal{O}\mathcal{M}}\Big[\mathcal{M}(\mathcal{O}\psi_1,\psi_2) + (-1)^{\psi_1\mathcal{O}}\mathcal{M}(\psi_1,\mathcal{O}\psi_2)\Big]\;, \\ [\mathcal{O},\mathcal{T}](\psi_1,\psi_2,\psi_3) &:= \mathcal{O}\mathcal{T}(\psi_1,\psi_2,\psi_3) - (-1)^{\mathcal{O}\mathcal{T}}\Big[\mathcal{T}(\mathcal{O}\psi_1,\psi_2,\psi_3) \\ &\qquad + (-1)^{\mathcal{O}\psi_1}\mathcal{T}(\psi_1,\mathcal{O}\psi_2,\psi_3) + (-1)^{\mathcal{O}(\psi_1+\psi_2)}\mathcal{T}(\psi_1,\psi_2,\mathcal{O}\psi_3)\Big]\;. \end{aligned} \tag{3.182}$$

The action of an operator $\mathcal{O}$ on a map (be it another operator, a bilinear or trilinear map) gives a map of the same kind, *e.g.*

$$\big(\mathcal{O}\mathcal{M}\big)(\psi_1,\psi_2) := \mathcal{O}\big(\mathcal{M}(\psi_1,\psi_2)\big)\;. \tag{3.183}$$

Finally, composition of bilinear maps is defined from the left and denoted by juxtaposition:

$$\mathcal{M}_1\mathcal{M}_2(\psi_1,\psi_2,\psi_3) := \mathcal{M}_1\big(\mathcal{M}_2(\psi_1,\psi_2),\psi_3\big)\;. \tag{3.184}$$

This is sufficient for our purposes, since all bilinear maps involved are graded symmetric. With this notation one can check that $[\mathcal{O},-]$ is a derivation on commutators and compositions, in the sense that it obeys

$$\begin{aligned} [\mathcal{O}_1,[\mathcal{O}_2,\mathcal{M}]] &= [[\mathcal{O}_1,\mathcal{O}_2],\mathcal{M}] + (-1)^{\mathcal{O}_1\mathcal{O}_2}[\mathcal{O}_2,[\mathcal{O}_1,\mathcal{M}]]\;, \\ [\mathcal{O}_1,\mathcal{O}_2\mathcal{M}] &= [\mathcal{O}_1,\mathcal{O}_2]\mathcal{M} + (-1)^{\mathcal{O}_1\mathcal{O}_2}\mathcal{O}_2[\mathcal{O}_1,\mathcal{M}]\;, \\ [\mathcal{O},\mathcal{M}_1\mathcal{M}_2] &= [\mathcal{O},\mathcal{M}_1]\mathcal{M}_2 + (-1)^{\mathcal{O}\mathcal{M}_1}\mathcal{M}_1[\mathcal{O},\mathcal{M}_2]\;. \end{aligned} \tag{3.185}$$

We now turn to discuss the symmetry properties of trilinear maps $\mathcal{T}$. Since they are all graded symmetric in the first two arguments (this is the reason we chose to work with $m_{3\text{h}}$ rather than m_3), they can be decomposed into a totally graded symmmetric part, which we denote by $\mathcal{T}_\text{s}$, and a graded hook part $\mathcal{T}_\text{h} := \mathcal{T} - \mathcal{T}_\text{s}$. In terms of $\mathcal{T}$, the symmetrized map $\mathcal{T}_\text{s}$ acts on three inputs as

$$\begin{aligned}\mathcal{T}_\text{s}(\psi_1,\psi_2,\psi_3) = \frac{1}{3}\Big\{&\mathcal{T}(\psi_1,\psi_2,\psi_3) + (-1)^{\psi_1(\psi_2+\psi_3)}\mathcal{T}(\psi_2,\psi_3,\psi_1) \\ &+ (-1)^{\psi_3(\psi_1+\psi_2)}\mathcal{T}(\psi_3,\psi_1,\psi_2)\Big\} \,.\end{aligned} \tag{3.186}$$

According to (3.180), we want to associate a multidifferential operator $\hat{\mathcal{T}}_\text{s}$ to $\mathcal{T}_\text{s}$, such that

$$\mathcal{T}_\text{s}(\psi_1,\psi_2,\psi_3) = \mu\left[\hat{\mathcal{T}}_\text{s}(Z_A,Z_B,Z_C)\left(\psi_1^A\otimes\psi_2^B\otimes\psi_3^C\right)\right] \,. \tag{3.187}$$

To do so, we start by introducing a permutation operator Σ, which acts on trilinear operators as

$$\big(\mathcal{O}_1\otimes\mathcal{O}_2\otimes\mathcal{O}_3\big)\Sigma := \big(\mathcal{O}_3\otimes\mathcal{O}_1\otimes\mathcal{O}_2\big) \,, \tag{3.188}$$

and thus obeys

$$\begin{aligned}\mu\Big[\hat{\mathcal{T}}(Z_A,Z_B,Z_C)\Sigma\left(f_1\otimes f_2\otimes f_3\right)\Big] &= \mu\Big[\hat{\mathcal{T}}(Z_A,Z_B,Z_C)\big(f_2\otimes f_3\otimes f_1\big)\Big] \,, \\ \mu\Big[\hat{\mathcal{T}}(Z_A,Z_B,Z_C)\Sigma^2\left(f_1\otimes f_2\otimes f_3\right)\Big] &= \mu\Big[\hat{\mathcal{T}}(Z_A,Z_B,Z_C)\big(f_3\otimes f_1\otimes f_2\big)\Big] \,,\end{aligned} \tag{3.189}$$

and $\Sigma^3 = 1$. We then use Σ to define a projector π, obeying $\pi^2 = \pi$, so that the symmetrized and hook operators $\hat{\mathcal{T}}_\text{s}$ and $\hat{\mathcal{T}}_\text{h}$ are defined via

$$\hat{\mathcal{T}}_\text{s} := \hat{\mathcal{T}}\pi \,, \quad \hat{\mathcal{T}}_\text{h} := \hat{\mathcal{T}}(1-\pi) \,, \quad \hat{\mathcal{T}} = \hat{\mathcal{T}}_\text{s} + \hat{\mathcal{T}}_\text{h} \,. \tag{3.190}$$

In terms of the permutation operator Σ, π is explicitly given by

$$\begin{aligned}\hat{\mathcal{T}}\pi(Z_A,Z_B,Z_C) = \frac{1}{3}\Big\{&\hat{\mathcal{T}}(Z_A,Z_B,Z_C) + (-1)^{Z_A(Z_B+Z_C)}\hat{\mathcal{T}}(Z_B,Z_C,Z_A)\,\Sigma \\ &+ (-1)^{Z_C(Z_A+Z_B)}\hat{\mathcal{T}}(Z_C,Z_A,Z_B)\,\Sigma^2\Big\} \,,\end{aligned} \tag{3.191}$$

which reproduces the expression (3.186) for the map $\mathcal{T}_\text{s}$ upon using (3.187). We will then use interchangeably the notation $\mathcal{T}_\text{s} \equiv \mathcal{T}\pi$ for the symmetrized map as well. The operator $\hat{\mathcal{T}}_\text{s}$ obeys the graded symmetry property

$$\hat{\mathcal{T}}_\text{s}(Z_A,Z_B,Z_C) = (-1)^{Z_A(Z_B+Z_C)}\hat{\mathcal{T}}_\text{s}(Z_B,Z_C,Z_A)\Sigma = (-1)^{Z_C(Z_A+Z_B)}\hat{\mathcal{T}}_\text{s}(Z_C,Z_A,Z_B)\Sigma^2 \,, \tag{3.192}$$

which implies the standard graded symmetry of the map $\mathcal{T}_\text{s}(\psi_1,\psi_2,\psi_3)$ upon permutations of the inputs.

Let us illustrate the action of π with a concrete example. We consider the trilinear map $\mathcal{T}$ associated with the operator

$$\hat{\mathcal{T}}(\theta_\mu,\theta_\nu,\theta_\rho) = c\,\theta_-\,\eta_{\mu\nu}\left[\big(\mathbb{1}\otimes\partial_\rho\otimes\mathbb{1}\big) - \big(\partial_\rho\otimes\mathbb{1}\otimes\mathbb{1}\big)\right] \,, \tag{3.193}$$

which is part of the actual map $m_2 m_2(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$. Acting with (3.193) on $(A_1^\mu \otimes A_2^\nu \otimes A_3^\rho)$ and evaluating the pointwise product one obtains

$$\mathcal{T}(A_1, A_2, A_3) = c\,\theta_-\left(A_{1\mu}\partial_\rho A_2^\mu A_3^\rho - A_{2\mu}\partial_\rho A_1^\mu A_3^\rho\right)\,, \tag{3.194}$$

where we abbreviated $A_i = \theta_\mu A_i^\mu$. According to the definitions (3.191), the symmetrized operator $\hat{\mathcal{T}}\pi$ is given by

$$\begin{aligned}\hat{\mathcal{T}}\pi(\theta_\mu, \theta_\nu, \theta_\rho) = \frac{1}{3}\,c\,\theta_-\Big(&\eta_{\mu\nu}\left(\mathbb{1}\otimes\partial_\rho\otimes\mathbb{1}\right) - \eta_{\mu\nu}\left(\partial_\rho\otimes\mathbb{1}\otimes\mathbb{1}\right) + \eta_{\nu\rho}\left(\mathbb{1}\otimes\mathbb{1}\otimes\partial_\mu\right)\\ &- \eta_{\nu\rho}\left(\mathbb{1}\otimes\partial_\mu\otimes\mathbb{1}\right) + \eta_{\mu\rho}\left(\partial_\nu\otimes\mathbb{1}\otimes\mathbb{1}\right) - \eta_{\mu\rho}\left(\mathbb{1}\otimes\mathbb{1}\otimes\partial_\nu\right)\Big)\,,\end{aligned} \tag{3.195}$$

yielding the symmetrized map

$$\begin{aligned}\mathcal{T}\pi(A_1, A_2, A_3) = \frac{1}{3}\,c\,\theta_-\Big(&A_{1\mu}\partial_\rho A_2^\mu A_3^\rho - A_{2\mu}\partial_\rho A_1^\mu A_3^\rho + A_{2\mu}\partial_\rho A_3^\mu A_1^\rho - A_{3\mu}\partial_\rho A_2^\mu A_1^\rho\\ &+ A_{3\mu}\partial_\rho A_1^\mu A_2^\rho - A_{1\mu}\partial_\rho A_3^\mu A_2^\rho\Big)\,.\end{aligned} \tag{3.196}$$

From the definition (3.182) of the commutator $[\mathcal{O}, \mathcal{T}]$, one can check that the action of $\mathcal{O}$ preserves the symmetry property of the map $\mathcal{T}$, meaning that

$$[\mathcal{O}, \mathcal{T}]\pi = [\mathcal{O}, \mathcal{T}\pi]\,. \tag{3.197}$$

We conclude formulating this input-free notation by focusing on the possible $\Box$ obstructions. Since we are working on flat spacetime, the d'Alambert operator $\Box$ commutes with all the multidifferential operators $\hat{\mathcal{O}}$, $\hat{\mathcal{M}}$ and $\hat{\mathcal{T}}$ in (3.180). Its commutators with the maps $\mathcal{O}$, $\mathcal{M}$ and $\mathcal{T}$ are thus entirely determined by the commutator of $\Box$ on the pointwise product of functions. We thus define the following operators, acting on three local functions:

$$\begin{aligned}&d_s(f_1\otimes f_2\otimes f_3) := 2\left(\partial^\mu f_1\otimes\partial_\mu f_2\otimes f_3\right)\,,\\ &d_\Box(f_1\otimes f_2\otimes f_3) := 2\left(\partial^\mu f_1\otimes\partial_\mu f_2\otimes f_3\right) + 2\left(f_1\otimes\partial^\mu f_2\otimes\partial_\mu f_3\right) + 2\left(\partial^\mu f_1\otimes f_2\otimes\partial_\mu f_3\right)\,.\end{aligned} \tag{3.198}$$

The subscript in d_s alludes to the Mandelstam variable s, and should not be confused with the symmetrization $\mathcal{T}_\text{s}$. One can compose a $\mathcal{Z}-$valued tri-differential operator $\hat{\mathcal{T}}$ with d_s and $d_\Box$, which we denote by juxtaposition, yielding

$$\begin{aligned}&\hat{\mathcal{T}}d_s := 2\,\hat{\mathcal{T}}\circ\left(\partial^\mu\otimes\partial_\mu\otimes\mathbb{1}\right)\,,\\ &\hat{\mathcal{T}}d_\Box := 2\,\hat{\mathcal{T}}\circ\Big\{\left(\partial^\mu\otimes\partial_\mu\otimes\mathbb{1}\right) + \left(\mathbb{1}\otimes\partial^\mu\otimes\partial_\mu\right) + \left(\partial^\mu\otimes\mathbb{1}\otimes\partial_\mu\right)\Big\}\,,\end{aligned} \tag{3.199}$$

where $\circ$ denotes the composition of operators. These are also $\mathcal{Z}-$valued tri-differential operators which generate the corresponding maps $\mathcal{T}d_s$ and $\mathcal{T}d_\Box$. For instance, one has $\mathcal{T}d_s(\psi_1, \psi_2, \psi_3) = 2\,\mathcal{T}(\partial^\mu\psi_1, \partial_\mu\psi_2, \psi_3)$ and so on. Under projection by π, d_s and $d_\Box$ obey

$$\mathcal{T}d_\Box\,\pi = \mathcal{T}\pi\,d_\Box = 3\,\mathcal{T}\pi\,d_s\,\pi\,. \tag{3.200}$$

The $d_\Box$ operator is always related to a total commutator with $\Box$, in the sense that

$$[\Box, \mathcal{T}] = \mathcal{T}\, d_\Box \ , \tag{3.201}$$

while $\mathcal{T}\, d_s$ is not. Lastly, from the definition of $[\mathcal{O}, \mathcal{T}]$ it follows that d_s and $d_\Box$ commute with linear operators $\mathcal{O}$:

$$[\mathcal{O}, \mathcal{T}]d_s = [\mathcal{O}, \mathcal{T} d_s] \ , \quad [\mathcal{O}, \mathcal{T}]d_\Box = [\mathcal{O}, \mathcal{T} d_\Box] \ . \tag{3.202}$$

With this notation at hand, we can summarize all the relevant $\mathrm{BV}_\infty^\Box$ relations up to trilinear maps [3]:

$$\begin{aligned}
&m_1^2 = 0 \ , \quad b^2 = 0 \ , \quad [m_1, b] = \Box \ , && \text{differentials and } \Box \text{ obstruction,} \\
&[m_1, m_2] = 0 \ , \quad m_2 m_2 (1 - \pi) = [m_1, m_{3\mathrm{h}}] \ , && C_\infty \text{ structure,} \\
&b_2 = [b, m_2] \ , \quad [m_1, b_2] = [\Box, m_2] \ , && \text{two-bracket and deformed Leibniz,} \\
&b_2 m_2 + m_2 b_2 \, (1 - 3\,\pi) = [m_1, \theta_3] + m_{3\mathrm{h}}(d_\Box - 3\, d_s\, \pi) \ , && \text{deformed homotopy Poisson,} \\
&3\, b_2 b_2 \pi + [m_1, b_3] + 3\, \theta_3 d_s \pi = 0 \ , && \text{deformed homotopy Jacobi,} \\
&\theta_{3\mathrm{h}} + [b, m_{3\mathrm{h}}] = 0 \ , \quad b_3 + [b, \theta_{3\mathrm{s}}] = 0 \ , && \text{compatibility of homotopies.}
\end{aligned} \tag{3.203}$$

For example, using this notation, the Jacobiator and the Poissonator can be written as

$$\mathrm{Jac} = 3\, b_2 b_2 \pi \ , \quad \mathrm{Poiss} = b_2 m_2 + m_2 b_2 \, (1 - 3\,\pi) \ . \tag{3.204}$$

Chapter 4

Off-shell and gauge invariant double copy

This chapter deals with the construction of strongly constrained double field theory up to quartic interactions, using the kinematic building blocks of Yang-Mills theory in an off-shell, gauge independent and local manner. We start with the factorization of the L_∞-algebra underlying Yang-Mills theory, $\mathcal{K}^{\rm YM} \otimes \mathfrak{g}$, and ask the following question: is it possible to obtain a gravity theory by replacing $\mathfrak{g}$ by another copy of the kinematic algebra $\bar{\mathcal{K}}^{\rm YM}$? The answer, as we will prove below, is affirmative. In order to prove this, one first has to identify a vector space that encodes the elements of a gravity theory, as well as a set of brackets that obey the L_∞ relations. Indeed, at least a subsector of the tensor product of kinematic algebras $\mathcal{K}^{\rm YM} \otimes \bar{\mathcal{K}}^{\rm YM}$ is the L_∞-algebra that underlies $N = 0$ supergravity in the form of double field theory.

In section 4.1 we start by replacing color by another copy of kinematics at the level of vector spaces. In order to obtain a vector space that contains the correct DFT degrees of freedom, one has to impose some constraints on the elements of the tensor product $\mathcal{K}^{\rm YM} \otimes \bar{\mathcal{K}}^{\rm YM}$. In section 4.2 we build a differential B_1 which gives the free double field theory action, and in sections 4.3 and 4.4 we construct, respectively, a two-bracket B_2 and a three-bracket B_3 which encode cubic and quartic interactions in the action. Finally, in section 4.5 we compute four-point scattering amplitudes for Yang-Mills and DFT using our algebraic double copy prescription. We find that the so-called "kinematic Jacobi identity" is, in fact, not a strict Jacobi identity, but instead one of the defining relations of the kinematic $\mathrm{BV}^{\square}_{\infty}$-algebra. Moreover, we explain how our algebraic double copy is consistent with the BCJ double copy of scattering amplitudes, at least at four-points.

The ideas regarding homotopy BV-algebras and their tensor products in Yang-Mills and gravity were first proposed by Zeitlin in the context of string theory in [100–103]. For related approaches to the double copy using homotopy algebras see [78, 104].

This chapter is largely based on [2, 3] and the upcoming paper [5], and fragments of some sections are taken from these references.

4.1 Tensor product of kinematic spaces and notation

In this section we start the construction of strongly constrained double field theory using the kinematic building blocks of Yang-Mills theory. First, we construct the vector space that encodes double field theory, and afterwards we extend the input-free notation of section 3.5.1 to the DFT vector space up to trilinear order.

Let us start with the vector space. Following BCJ we replace the color Lie algebra $\mathfrak{g}$ of Yang-Mills by a second copy of the kinematic space $\bar{\mathcal{K}}^{\text{YM}}$. Recall that the L_∞-algebra of Yang-Mills factorizes as

$$\mathcal{X}^{\text{YM}} = \mathcal{K}^{\text{YM}} \otimes \mathfrak{g} \; . \tag{4.1}$$

Replacing color by kinematics as $\mathfrak{g} \to \bar{\mathcal{K}}^{\text{YM}}$ then leads to the following space:

$$\mathcal{V} = \mathcal{K}^{\text{YM}} \otimes \bar{\mathcal{K}}^{\text{YM}} \; . \tag{4.2}$$

For the sake of simplicity, let us drop the superscript in the kinematic space $\mathcal{K}^{\text{YM}}$ and simply write $\mathcal{K}$. Analogously to $\mathcal{K}$, the second copy of the kinematic space decomposes as $\bar{\mathcal{K}} = \bar{\mathcal{Z}} \otimes C^\infty(\bar{\mathcal{M}})$ (in complete analogy to $\mathcal{K}$ as explained in section 3.5), with $\bar{\mathcal{M}}$ a second copy of $D-$dimensional Minkowski space, and it comes with its own set of coordinates and spacetime indices $\bar{x}^{\bar{\mu}}$. A generic element of this space can be written as

$$\bar{\psi}(\bar{x}) = \bar{Z}_{\bar{A}}\, \bar{\psi}^{\bar{A}}(\bar{x}) \; , \tag{4.3}$$

where the $\bar{Z}_{\bar{A}} = (\bar{\theta}_{\bar{M}}, \bar{c}\,\bar{\theta}_{\bar{M}})$ form the basis of the space $\bar{\mathcal{Z}}$ and $\bar{\psi}^{\bar{A}}(\bar{x})$ are functions on $C^\infty(\bar{\mathcal{M}})$.

With the above decomposition of $\bar{\mathcal{K}}$, the tensor product of the two copies of the kinematic space yields

$$\begin{aligned} \mathcal{V} = \mathcal{K} \otimes \bar{\mathcal{K}} &= \big(\mathcal{Z} \otimes C^\infty(\mathcal{M})\big) \otimes \big(\bar{\mathcal{Z}} \otimes C^\infty(\bar{\mathcal{M}})\big) \\ &= \big(\mathcal{Z} \otimes \bar{\mathcal{Z}}\big) \otimes C^\infty(\mathcal{M} \times \bar{\mathcal{M}}) \; , \end{aligned} \tag{4.4}$$

where we used that, under certain topological assumptions, $C^\infty(\mathcal{M}) \otimes C^\infty(\bar{\mathcal{M}}) \simeq C^\infty(\mathcal{M} \times \bar{\mathcal{M}})$. A generic element of the tensor product space $\mathcal{V}$ can thus be expanded as

$$\Xi(x,\bar{x}) = Z_A \otimes \bar{Z}_{\bar{B}}\, \Xi^{A\bar{B}}(x,\bar{x}) \equiv Z_A \bar{Z}_{\bar{B}}\, \Xi^{A\bar{B}}(x,\bar{x}) \; , \tag{4.5}$$

where Ξ has degree $|\Xi| = |Z_A + \bar{Z}_{\bar{B}}|$, in the last equality we omitted the tensor product symbol in the basis elements to simplify our notation, i.e $Z_A \otimes \bar{Z}_{\bar{B}} \equiv Z_A \bar{Z}_{\bar{B}}$, and $\Xi^{A\bar{B}}(x,\bar{x})$ are functions on $C^\infty(\mathcal{M} \times \bar{\mathcal{M}})$. Nevertheless, since we are interested in constructing strongly constrained DFT, we impose the strong constraint

$$\Delta := \tfrac{1}{2}\big(\Box - \bar{\Box}\big) \equiv 0 \; , \tag{4.6}$$

and as a consequence, the elements of the physical space only depend on D of the coordinates.

Let us now turn to the physical interpretation of the tensor product space. The space $\mathcal{V}$ is too big and contains twice as many elements as the ones present in double field theory. For this reason, it is necessary to constrain the space in order to correctly reproduce the field spectrum of

DFT. Inspired by the string field theory formulation of DFT, we restrict our space by imposing the condition

$$b^-\Psi(x,\bar{x}) = 0\;, \tag{4.7}$$

where the operator $b^- : \mathcal{V} \to \mathcal{V}$ is defined as the difference of the operator $b : \mathcal{K} \to \mathcal{K}$ and $\bar{b} : \bar{\mathcal{K}} \to \bar{\mathcal{K}}$, namely

$$b^- := \tfrac{1}{2}\big(b\otimes 1 - 1\otimes \bar{b}\big)\;. \tag{4.8}$$

Hence, the vector space of interest is the following:

$$\mathcal{V}^{\rm DFT} = \Big\{\Psi(x,\bar{x}) \in \mathcal{V}|\ \Delta \equiv 0\;,\ b^-\Psi(x,\bar{x}) = 0\Big\}\;. \tag{4.9}$$

Before writing out all the elements Ψ of $\mathcal{V}_{\rm DFT}$ explicitly, let us illustrate how b^- constrains the spectrum of the theory. First, it will be convenient to write out a general element of $\mathcal{V}$, prior to any constraint, in terms of the θ_M, $c\theta_M$ and their barred counterparts as

$$\Xi = Z_A\bar{Z}_{\bar{B}}\Xi^{A\bar{B}} = \theta_M\bar{\theta}_{\bar{N}}\,\varphi^{M\bar{N}} + c\,\theta_M\bar{\theta}_{\bar{N}}\,\Sigma^{M\bar{N}} + \theta_M\bar{c}\,\bar{\theta}_{\bar{N}}\,\tilde{\Sigma}^{M\bar{N}} + c\theta_M\bar{c}\,\bar{\theta}_{\bar{N}}\,\Omega^{M\bar{N}}\;. \tag{4.10}$$

We can change the above basis by defining $c^\pm$ as

$$c^\pm = c\otimes 1 \pm 1\otimes \bar{c}\;, \tag{4.11}$$

and we can expand a generic element Ξ of $\mathcal{V}$ as

$$\Xi = Z_A\bar{Z}_{\bar{B}}\Xi^{A\bar{B}} = \theta_M\bar{\theta}_{\bar{N}}\,\varphi^{M\bar{N}} + c^+\,\theta_M\bar{\theta}_{\bar{N}}\,\sigma^{M\bar{N}} + c^-\,\theta_M\bar{\theta}_{\bar{N}}\,\tilde{\sigma}^{M\bar{N}} + c^+c^-\,\theta_M\bar{\theta}_{\bar{N}}\,\omega^{M\bar{N}}\;, \tag{4.12}$$

where $\sigma^{M\bar{N}}$ and $\tilde{\sigma}^{M\bar{N}}$ are linear combinations of $\Sigma^{M\bar{N}}$ and $\tilde{\Sigma}^{M\bar{N}}$, and $\omega^{M\bar{N}}$ is equivalent to $\Omega^{M\bar{N}}$ up to a phase factor. The operators $b^\pm$ and $c^\pm$ obey the following commutation relations:

$$[b^\pm, c^\pm] := b^\pm c^\pm + c^\pm b^\pm = 1\;,\ \ [b^\pm, c^\mp] := b^\pm c^\mp + c^\mp b^\pm = 0\;, \tag{4.13}$$

which follows from $[b,c] = 1 = [\bar{b},\bar{c}]$, and as a consequence $b^\pm$ acts on functions $F(c^\pm)$ of $c^\pm$ as

$$b^\pm F(c^\pm) = \frac{\partial}{\partial c^\pm}\,F(c^\pm)\;. \tag{4.14}$$

Hence the condition $b^-\Psi = 0$ sets the components $\tilde{\sigma}^{M\bar{N}}$ and $\omega^{M\bar{N}}$ to zero, and the elements Ψ of the physical space $\mathcal{V}^{\rm DFT}$ can be expanded, in general, as

$$\Psi = Z_A\bar{Z}_{\bar{B}}\Psi^{A\bar{B}} = \theta_M\bar{\theta}_{\bar{N}}\,\varphi^{M\bar{N}} + c^+\,\theta_M\bar{\theta}_{\bar{N}}\,\sigma^{M\bar{N}}\;,\ \ \Psi \in \mathcal{V}_{\rm DFT}\;. \tag{4.15}$$

With this truncated space and the following degree shift all the elements of $\mathcal{V}^{\rm DFT}$ by minus two:

$$|\Psi| = |Z_A + \bar{Z}_{\bar{B}} - 2|\;, \tag{4.16}$$

which we implement in order to have conventional degrees in the L_∞ formulation of field theories, the vector subspace of $\mathcal{V}^{\rm DFT}$ of lowest degree has degree -2, while the one with the highest degree has degree +3, i.e

$$\mathcal{V}^{\mathrm{DFT}} = \bigoplus_{i=-2}^{3} V_i^{\mathrm{DFT}} \ , \tag{4.17}$$

and the components of the elements of all degrees are:

$$\begin{aligned}
&\chi = \theta_+\bar{\theta}_+ \, \chi \in V_{-2}^{\mathrm{DFT}} \ , \\
&\Lambda = \theta_+\bar{\theta}_{\bar{\mu}} \, \bar{\lambda}^{\bar{\mu}} - \theta_\mu\bar{\theta}_+ \, \lambda^\mu - 2\, c^+ \, \theta_+\bar{\theta}_+ \, \eta \in V_{-1}^{\mathrm{DFT}} \ , \\
&\psi = \theta_\mu\bar{\theta}_{\bar{\nu}} \, e^{\mu\bar{\nu}} + 2\, \theta_+\bar{\theta}_- \, \bar{e} + 2\, \theta_-\bar{\theta}_+ \, e + 2\, c^+ \, \theta_+\bar{\theta}_{\bar{\mu}} \, \bar{f}^{\bar{\mu}} + 2\, c^+ \, \theta_\mu\bar{\theta}_+ \, f^\mu \in V_0^{\mathrm{DFT}} \ , \\
&\mathcal{F} = c^+ \, \theta_\mu\bar{\theta}_{\bar{\nu}} \, F^{\mu\bar{\nu}} + c^+ \, \theta_+\bar{\theta}_- \, \bar{F} + c^+ \, \theta_-\bar{\theta}_+ \, F + \theta_\mu\bar{\theta}_- \, F^\mu + \theta_-\bar{\theta}_{\bar{\mu}} \, \bar{F}^{\bar{\mu}} \in V_1^{\mathrm{DFT}} \ , \\
&\mathcal{N} = 2\, c^+ \, \theta_-\bar{\theta}_{\bar{\mu}} \, \bar{N}^{\bar{\mu}} - 2\, c^+ \, \theta_\mu\bar{\theta}_- \, N^\mu - \theta_-\bar{\theta}_- \, N \in V_2^{\mathrm{DFT}} \ , \\
&\mathcal{R} = -c^+ \, \theta_-\bar{\theta}_- \, \mathcal{R} \in V_3^{\mathrm{DFT}} \ .
\end{aligned} \tag{4.18}$$

The elements χ of V_{-2}^{DFT} are gauge for gauge parameters, the elements Λ of V_{-1}^{DFT} are gauge parameters, the elements ψ of V_0^{DFT} are the fields with field equations $\mathcal{F} \in V_1^{\mathrm{DFT}}$, and $\mathcal{N} \in V_2^{\mathrm{DFT}}$ and $\mathcal{R} \in V_3^{\mathrm{DFT}}$ are the spaces of Noether and Noether-for-Noether identities, respectively. This spectrum coincides with the fields, gauge parameters and so on that we introduced for DFT in chapter 2.

In the remainder of this section, we extend the input-free notation of section 3.5.1 to the DFT space $\mathcal{V}^{\mathrm{DFT}}$. This technology will significantly simplify the computations needed to check the consistency of our algebraic double copy prescription. Given the definition of the linear, bilinear and trilinear *single copy* maps acting on $\mathcal{K}$ (and $\bar{\mathcal{K}}$)

$$\begin{aligned}
\mathcal{O}(\psi) &= \hat{\mathcal{O}}(Z_A) \, \psi^A(x) \ , \\
\mathcal{M}(\psi_1, \psi_2) &= \mu \left[\hat{\mathcal{M}}(Z_A, Z_B) \left(\psi_1^A(x) \otimes \psi_2^B(x) \right) \right] \ , \\
\mathcal{T}(\psi_1, \psi_2, \psi_3) &= \mu \left[\hat{\mathcal{T}}(Z_A, Z_B, Z_C) \left(\psi_1^A(x) \otimes \psi_2^B(x) \otimes \psi_3^C(x) \right) \right] \ ,
\end{aligned} \tag{4.19}$$

and the expansion (4.15), we now proceed to lift the action of operators $\mathcal{O} : \mathcal{K} \to \mathcal{K}$ and $\bar{\mathcal{O}} : \bar{\mathcal{K}} \to \bar{\mathcal{K}}$ to $\mathcal{V}^{\mathrm{DFT}}$ by defining

$$\mathcal{O}(\Psi) := \hat{\mathcal{O}}(Z_A) \bar{Z}_{\bar{B}} \, \Psi^{A\bar{B}}(x, \bar{x}) \ , \quad \bar{\mathcal{O}}(\Psi) := (-1)^{Z_A \bar{\mathcal{O}}} Z_A \, \hat{\bar{\mathcal{O}}}(\bar{Z}_{\bar{B}}) \, \Psi^{A\bar{B}}(x, \bar{x}) \ , \tag{4.20}$$

Where the differential operators $\hat{\mathcal{O}}$ and $\hat{\bar{\mathcal{O}}}$ act on functions of x and $\bar{x}$ by taking $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\bar{\partial}_{\bar{\mu}} = \frac{\partial}{\partial \bar{x}^{\bar{\mu}}}$ derivatives, respectively. This allows us to sum operators from $\mathcal{K}$ and $\bar{\mathcal{K}}$ and yield well-defined operators on $\mathcal{V}^{\mathrm{DFT}}$, such as $\mathcal{O} + \bar{\mathcal{O}}$. Similarly, tensor products of bilinear maps $\mathcal{M}$ and $\bar{\mathcal{M}}$ are defined to act on elements of $\mathcal{V}^{\mathrm{DFT}}$ as follows:

$$\begin{aligned}
&\big(\mathcal{M} \otimes \bar{\mathcal{M}}\big)(\Psi_1, \Psi_2) = \big(\mathcal{M} \otimes \bar{\mathcal{M}}\big)\big(Z_A \bar{Z}_{\bar{B}} \, \Psi_1^{A\bar{B}}, Z_C \bar{Z}_{\bar{D}} \, \Psi_2^{C\bar{D}}\big) \\
&:= (-1)^{Z_C \bar{Z}_{\bar{B}} + \bar{\mathcal{M}}(Z_A + Z_C)} \mu \left[\hat{\mathcal{M}}(Z_A, Z_C) \, \hat{\bar{\mathcal{M}}}(\bar{Z}_{\bar{B}}, \bar{Z}_{\bar{D}}) \left(\Psi_1^{A\bar{B}}(x, \bar{x}) \otimes \Psi_2^{C\bar{D}}(x, \bar{x}) \right) \right] \ ,
\end{aligned} \tag{4.21}$$

with a completely analogous expression for $\big(\mathcal{T} \otimes \bar{\mathcal{T}}\big)(\Psi_1, \Psi_2, \Psi_3)$. With these definitions we can extend the input-free notation of section 3.5.1 to $\mathcal{V}^{\mathrm{DFT}}$. It turns out that operators $\mathcal{O}$ and $\bar{\mathcal{O}}$

commute (in the graded sense). To show this we compute

$$\begin{aligned}\mathcal{O}(\bar{\mathcal{O}}\,\Psi) &= (-1)^{Z_A\bar{\mathcal{O}}}\mathcal{O}\big(Z_A\,\hat{\bar{\mathcal{O}}}(\bar{Z}_{\bar{B}})\,\Psi^{A\bar{B}}\big)\\ &= (-1)^{Z_A\bar{\mathcal{O}}}\hat{\mathcal{O}}(Z_A)\,\hat{\bar{\mathcal{O}}}(\bar{Z}_{\bar{B}})\,\Psi^{A\bar{B}}(x,\bar{x})\\ &= (-1)^{Z_A\bar{\mathcal{O}}+\bar{\mathcal{O}}(Z_A+\mathcal{O})}\bar{\mathcal{O}}\big(\hat{\mathcal{O}}(Z_A)\,\bar{Z}_{\bar{B}}\,\Psi^{A\bar{B}}\big)\\ &= (-1)^{\mathcal{O}\bar{\mathcal{O}}}\bar{\mathcal{O}}(\mathcal{O}\,\Psi)\ ,\end{aligned} \tag{4.22}$$

where we omitted the explicit dependence on $(x,\bar{x})$ in intermediate steps. This can be written as the input-free relation $[\mathcal{O},\bar{\mathcal{O}}]=0$, where the graded commutator $[\mathcal{O},\bar{\mathcal{O}}]:\mathcal{V}^{\rm DFT}\to\mathcal{V}^{\rm DFT}$ is defined as

$$\big[\mathcal{O},\bar{\mathcal{O}}\big] := \mathcal{O}\bar{\mathcal{O}}-(-1)^{\mathcal{O}\bar{\mathcal{O}}}\bar{\mathcal{O}}\mathcal{O}\ . \tag{4.23}$$

A similar computation using the definition (4.21) shows that operators of $\mathcal{K}$ commute with bilinear and trilinear maps of $\bar{\mathcal{K}}$ and viceversa, in the sense

$$[\mathcal{O},\mathcal{M}\otimes\bar{\mathcal{M}}]=[\mathcal{O},\mathcal{M}]\otimes\bar{\mathcal{M}}\ ,\quad [\bar{\mathcal{O}},\mathcal{M}\otimes\bar{\mathcal{M}}]=(-1)^{\mathcal{M}\bar{\mathcal{O}}}\mathcal{M}\otimes[\bar{\mathcal{O}},\bar{\mathcal{M}}]\ , \tag{4.24}$$

with analogous formulas for commutators with $\mathcal{T}\otimes\bar{\mathcal{T}}$. Nesting of bilinear maps can also be extended naturally by defining

$$\big(\mathcal{M}_1\otimes\bar{\mathcal{M}}_1\big)\big(\mathcal{M}_2\otimes\bar{\mathcal{M}}_2\big) := (-1)^{\bar{\mathcal{M}}_1\mathcal{M}_2}\mathcal{M}_1\mathcal{M}_2\otimes\bar{\mathcal{M}}_1\bar{\mathcal{M}}_2\ , \tag{4.25}$$

where we remind the reader that the composition of single copy maps $\mathcal{M}_1\mathcal{M}_2$ (and $\bar{\mathcal{M}}_1\bar{\mathcal{M}}_2$) is defined by

$$\mathcal{M}_1\mathcal{M}_2(\psi_1,\psi_2,\psi_3) := \mathcal{M}_1\big(\mathcal{M}_2(\psi_1,\psi_2),\psi_3\big)\ . \tag{4.26}$$

Finally, one can introduce on $\mathcal{V}^{\rm DFT}$ a symmetric projector Π, obeying $\Pi^2=\Pi$, via

$$\begin{aligned}&\big(\mathcal{T}\otimes\bar{\mathcal{T}}\big)\Pi\,\big(\Psi_1,\Psi_2,\Psi_3\big) = \frac{1}{3}\,(-1)^{\mathcal{E}}\mu\Big[\Big(\hat{\mathcal{T}}(Z_A,Z_B,Z_C)\,\hat{\bar{\mathcal{T}}}(\bar{Z}_{\bar{A}},\bar{Z}_{\bar{B}},\bar{Z}_{\bar{C}})\\ &+(-1)^{Z_A(Z_B+Z_C)+\bar{Z}_{\bar{A}}(\bar{Z}_{\bar{B}}+\bar{Z}_{\bar{C}})}\hat{\mathcal{T}}(Z_B,Z_C,Z_A)\Sigma\,\hat{\bar{\mathcal{T}}}(\bar{Z}_{\bar{B}},\bar{Z}_{\bar{C}},\bar{Z}_{\bar{A}})\bar{\Sigma}\\ &+(-1)^{Z_C(Z_A+Z_B)+\bar{Z}_{\bar{C}}(\bar{Z}_{\bar{A}}+\bar{Z}_{\bar{B}})}\hat{\mathcal{T}}(Z_C,Z_A,Z_B)\Sigma^2\,\hat{\bar{\mathcal{T}}}(\bar{Z}_{\bar{C}},\bar{Z}_{\bar{A}},\bar{Z}_{\bar{B}})\bar{\Sigma}^2\Big)\Big(\Psi_1^{A\bar{A}}\otimes\Psi_2^{B\bar{B}}\otimes\Psi_3^{C\bar{C}}\Big)\Big]\ ,\end{aligned} \tag{4.27}$$

where the global phase is $\mathcal{E}=Z_B\bar{Z}_{\bar{A}}+Z_C(\bar{Z}_{\bar{A}}+\bar{Z}_{\bar{B}})+(Z_A+Z_B+Z_C)\bar{\mathcal{T}}$. In terms of the map $\mathcal{T}\otimes\bar{\mathcal{T}}$, this results in

$$\begin{aligned}\big(\mathcal{T}\otimes\bar{\mathcal{T}}\big)\Pi(\Psi_1,\Psi_2,\Psi_3) := \frac{1}{3}\Big\{&\big(\mathcal{T}\otimes\bar{\mathcal{T}}\big)(\Psi_1,\Psi_2,\Psi_3)+(-1)^{\Psi_1(\Psi_2+\Psi_3)}\big(\mathcal{T}\otimes\bar{\mathcal{T}}\big)(\Psi_2,\Psi_3,\Psi_1)\\ &+(-1)^{\Psi_3(\Psi_1+\Psi_2)}\big(\mathcal{T}\otimes\bar{\mathcal{T}}\big)(\Psi_3,\Psi_1,\Psi_2)\Big\}\ ,\end{aligned} \tag{4.28}$$

which makes the graded symmetry manifest. One can lift the definition of the single copy π or $\bar{\pi}$ to a trilinear map $\mathcal{T}\otimes\bar{\mathcal{T}}$ on $\mathcal{V}^{\rm DFT}$ by

$$(\mathcal{T}\otimes\bar{\mathcal{T}})\pi := (\mathcal{T}\pi)\otimes\bar{\mathcal{T}}\ ,\quad (\mathcal{T}\otimes\bar{\mathcal{T}})\bar{\pi} := \mathcal{T}\otimes(\bar{\mathcal{T}}\bar{\pi})\ , \tag{4.29}$$

and using (3.187), (3.191) for the single copy symmetrized maps. From this it follows that $\pi\Pi = \bar{\pi}\Pi = \pi\bar{\pi}$, which further implies the decomposition

$$\begin{aligned} \Pi &= \Big[\pi\bar{\pi} + (1-\pi)(1-\bar{\pi})\Big]\Pi \ , \\ 1-\Pi &= \Big[\pi(1-\bar{\pi}) + (1-\pi)\bar{\pi} + (1-\pi)(1-\bar{\pi})\Big](1-\Pi) \ . \end{aligned} \tag{4.30}$$

Finally, let us close this section with the strong constraint in this notation. Recall that the action of the *box operators* in the single copy space $\mathcal{K}$

$$\begin{aligned} d_s(f_1 \otimes f_2 \otimes f_3) &:= 2\,(\partial^\mu f_1 \otimes \partial_\mu f_2 \otimes f_3) \ , \\ d_t(f_1 \otimes f_2 \otimes f_3) &:= 2\,(f_1 \otimes \partial_\mu f_2 \otimes \partial^\mu f_3) \ , \\ d_u(f_1 \otimes f_2 \otimes f_3) &:= 2\,(\partial^\mu f_1 \otimes f_2 \otimes \partial_\mu f_3) \ , \\ d_\square(f_1 \otimes f_2 \otimes f_3) &:= \big(d_s + d_t + d_u\big)(f_1 \otimes f_2 \otimes f_3) \ . \end{aligned} \tag{4.31}$$

Then, on elements Ψ of $\mathcal{V}^{\mathrm{DFT}}$, the strong constraint amounts to setting

$$\begin{gathered} d_s \equiv \bar{d}_{\bar{s}} \ , \ \ d_t \equiv \bar{d}_{\bar{t}} \ , \ \ d_u \equiv \bar{d}_{\bar{u}} \ , \\ d_\square \equiv \bar{d}_{\bar{\square}} \ . \end{gathered} \tag{4.32}$$

4.2 Free theory

Having identified the relevant vector space $\mathcal{V}^{\mathrm{DFT}}$ and the physical interpretation of its elements, we now turn to constructing a differential B_1 that encodes linearized double field theory[1]. To that end, we need to find linear operators that act on $\mathcal{K}$ and $\bar{\mathcal{K}}$ whose combination is another linear operator that acts on $\mathcal{V}^{\mathrm{DFT}}$ and is nilpotent. The natural candidate for such a combination is

$$B_1 = m_1 + \bar{m}_1 \ . \tag{4.33}$$

Indeed, this operator acts linearly on $\mathcal{V}^{\mathrm{DFT}}$ and is nilpotent, which is proven by the following computation:

$$\begin{aligned} B_1^2 &= \big(m_1 + \bar{m}_1\big)\big(m_1 + \bar{m}_1\big) \\ &= m_1^2 + m_1\bar{m}_1 - m_1\bar{m}_1 + \bar{m}_1^2 \\ &= 0 \ , \end{aligned} \tag{4.34}$$

where we used that both m_1 and $\bar{m}_1$ are nilpotent (as we proved in the previous chapter), and the minus sign in the third term of the second line follows from (4.22), namely m_1 and $\bar{m}_1$ have both odd degree, and $\bar{m}_1$ *jumps* m_1 when computing $\bar{m}_1 m_1 = -m_1\bar{m}_1$.

The vector space $\mathcal{V}^{\mathrm{DFT}}$ equipped with this differential forms the DFT chain complex

$$\begin{matrix} V_{-1} & \xrightarrow{B_1} & V_{-1} & \xrightarrow{B_1} & V_0 & \xrightarrow{B_1} & V_1 & \xrightarrow{B_1} & V_2 & \xrightarrow{B_1} & V_3 \\ \chi & & \Lambda & & \psi & & \mathcal{F} & & \mathcal{N} & & \mathcal{R} \end{matrix} \ , \tag{4.35}$$

[1]Not to be confused with the differential of the L_∞-algebra of Yang-Mills or Chern-Simons.

and the differential defines a consistent free field theory because, by construction, it is nilpotent. Nevertheless, in order to confirm that the consistent free field theory that arises from our double copy prescription is indeed linearized DFT, let us explicitly analyze the equations that arise from this construction. We start by explicitly finding the lowest order gauge transformations of the fields using the new differential. To that end, we have to act with B_1 on a gauge parameter Λ, read off the resulting components and identify the gauge transformations of the various fields of the theory. In order to do so, we need to use the notation developed in the previous section (see equation (4.20)). We would like to explicitly evaluate the following expression:

$$\delta\psi = B_1(\Lambda) = \big(m_1 + \bar{m}_1\big)(\theta_+\bar{\theta}_{\bar{\nu}}\,\bar{\lambda}^{\bar{\nu}} - \theta_\mu\bar{\theta}_+\,\lambda^\mu - 2\,c^+\,\theta_+\bar{\theta}_+\eta)\;, \tag{4.36}$$

where we expanded Λ in components as in equation (4.18). For the sake of brevity, let us focus only on the $\theta_+\bar{\theta}_{\bar{\nu}}\bar{\lambda}^{\bar{\nu}}$ component, namely

$$\begin{aligned}\delta_{\bar{\lambda}}\psi &= \big(m_1 + \bar{m}_1\big)(\theta_+\bar{\theta}_{\bar{\nu}}\,\bar{\lambda}^{\bar{\nu}})\\ &= \hat{m}_1(\theta_+)\bar{\theta}_{\bar{\nu}}\,\bar{\lambda}^{\bar{\nu}} + \theta_+\hat{\bar{m}}_1(\bar{\theta}_{\bar{\nu}})\,\bar{\lambda}^{\bar{\nu}}\\ &= \theta_\mu\bar{\theta}_{\bar{\nu}}\partial^\mu\bar{\lambda}^{\bar{\nu}} + c\,\theta_+\bar{\theta}_{\bar{\nu}}\,\Box\bar{\lambda}^{\bar{\nu}} + \bar{c}\,\theta_+\bar{\theta}_{\bar{\nu}}\,\bar{\Box}\bar{\lambda}^{\bar{\nu}} + \theta_+\bar{\theta}_-\,\bar{\partial}_{\bar{\nu}}\bar{\lambda}^{\bar{\nu}}\\ &= \theta_\mu\bar{\theta}_{\bar{\nu}}\partial^\mu\bar{\lambda}^{\bar{\nu}} + \theta_+\bar{\theta}_-\,\bar{\partial}_{\bar{\nu}}\bar{\lambda}^{\bar{\nu}} + c^+\,\theta_+\bar{\theta}_{\bar{\nu}}\,\Box\bar{\lambda}^{\bar{\nu}}\;,\end{aligned} \tag{4.37}$$

where we used the explicit expressions for $\hat{m}_1(\theta_+)$ and $\hat{\bar{m}}_1(\bar{\theta}_{\bar{\nu}})$ (see appendix A), as well as the strong constraint $\Box \equiv \bar{\Box}$. Notice that the basis element $\theta_\mu\bar{\theta}_{\bar{\nu}}$ of the first term of the last equality in equation (4.37) coincides with the basis element of the tensor fluctuation $e_{\mu\bar{\nu}}$ in equation (4.18), the basis element $\theta_+\bar{\theta}_-$ of the second term coincides with the basis element of $\bar{e}$ and the basis element $c^+\,\theta_\mu\bar{\theta}_+$ of the last term coincides with the basis element of the auxiliary field f_μ. Thus, one can infer that the linearized gauge transformation of $e_{\mu\bar{\nu}}$ with respect to $\bar{\lambda}_{\bar{\nu}}$ is

$$\delta_{\bar{\lambda}}e_{\mu\bar{\nu}} = \partial_\mu\bar{\lambda}_{\bar{\nu}}\;, \tag{4.38}$$

and similar for the other fields. Doing analogous computations for the other components leads to the complete set of linearized gauge transformations:

$$\begin{aligned}\delta e_{\mu\bar{\nu}} &= \partial_\mu\bar{\lambda}_{\bar{\nu}} + \bar{\partial}_{\bar{\nu}}\lambda_\mu\;,\\ \delta f_\mu &= -\tfrac{1}{2}\,\Box\lambda_\mu + \partial_\mu\eta\;, \qquad \delta\bar{f}_{\bar{\mu}} = \tfrac{1}{2}\,\Box\bar{\lambda}_{\bar{\mu}} + \bar{\partial}_{\bar{\mu}}\eta\;,\\ \delta e &= -\tfrac{1}{2}\,\partial\cdot\lambda + \eta\;, \qquad \delta\bar{e} = \tfrac{1}{2}\,\bar{\partial}\cdot\bar{\lambda} + \eta\;.\end{aligned} \tag{4.39}$$

Similarly, we can find the trivial gauge parameters (or linearized reducibility) by acting with B_1 on a gauge-for-gauge parameter as $\Lambda = B_1(\chi)$, i.e.,

$$\lambda_\mu = -\partial_\mu\chi\;, \qquad \bar{\lambda}_{\bar{\mu}} = \bar{\partial}_{\bar{\mu}}\chi\;, \qquad \eta = -\tfrac{1}{2}\,\Box\chi\;, \tag{4.40}$$

which generate no gauge transformations at this order.

For the dynamics, the free field equations are encoded on the differential acting on the fields as $B_1(\psi) = 0$, and the various components read

$$
\begin{aligned}
&\Box\, e_{\mu\bar{\nu}} + 2\,\bar{\partial}_{\bar{\nu}} f_\mu - 2\,\partial_\mu \bar{f}_{\bar{\nu}} = 0\ ,\\
&\Box\, e - \partial \cdot f = 0\ , \qquad \Box\, \bar{e} - \bar{\partial} \cdot \bar{f} = 0\ ,\\
&\partial_\mu \bar{e} - \tfrac{1}{2}\,\bar{\partial}^{\bar{\rho}} e_{\mu\bar{\rho}} - f_\mu = 0\ , \qquad \bar{\partial}_{\bar{\mu}} e + \tfrac{1}{2}\,\partial^\rho e_{\rho\bar{\mu}} - \bar{f}_{\bar{\mu}} = 0\ ,
\end{aligned}
\tag{4.41}
$$

These field equations can be found by varying an action $S_{\rm DFT}$. Indeed, we can define a non-degenerate pairing $\langle \cdot, \cdot \rangle_{\rm DFT}$ to construct an action as

$$
S^{(2)}_{\rm DFT} = \tfrac{1}{2}\,\big\langle \psi, B_1(\psi) \big\rangle_{\rm DFT}\ , \tag{4.42}
$$

which upon expanding all components yields

$$
S^{(2)}_{\rm DFT} = \int dx d\bar{x}\,\Big\{ \tfrac{1}{4} e_{\mu\bar{\mu}} \Box e^{\mu\bar{\mu}} + 2\,\bar{e}\Box e - f_\mu f^\mu - \bar{f}_{\bar{\mu}} \bar{f}^{\bar{\mu}} - f^\mu \left(\bar{\partial}^{\bar{\nu}} e_{\mu\bar{\nu}} - \partial_\mu \bar{e} \right) + \bar{f}^{\bar{\nu}} \left(\partial^\mu e_{\mu\bar{\nu}} + 2\,\bar{\partial}_{\bar{\nu}} e \right) \Big\}\ , \tag{4.43}
$$

which coincides exactly with quadratic part of the action $S^{\rm H.Z}_{\rm DFT}$ in equation (2.47).

Finally, given the field equations $\mathcal{F}$, the Noether identities are obtained by $\mathcal{N} = B_1(\mathcal{F})$ and, analogously, $\mathcal{R} = B_1(\mathcal{N})$ is the Noether-for-Noether identity.

4.3 Cubic theory

We now proceed to extend our double copy prescription to include interactions. In this section we construct double field theory up to cubic interactions in the action. Algebraically, this corresponds to finding a DFT two-bracket B_2 that encodes the bilinear terms in the field equations. To that end, we need a bilinear operator acting on $\mathcal{V}^{\rm DFT}$ built with kinematic Yang-Mills operators acting on $\mathcal{K}$ and $\bar{\mathcal{K}}$.

Given that double field theory is a consistent perturbative field theory, B_1 and B_2 have to obey the L_∞ relations. Up to cubic order, in addition to nilpotency of B_1, the consistency of the theory relies on the Leibniz rule $[B_1, B_2] = 0$. Moreover, in view of going to higher order than cubic in perturbation theory, B_2 should obey the homotopy Jacobi relation. Thus, at least a subset of the kinematic Yang-Mills constituents that make up B_2 should be of Lie-type. Furthermore, the elements Ψ of $\mathcal{V}^{\rm DFT}$ obey $b^-\Psi = 0$, and hence, in order to preserve the algebra structure, the two-bracket should act on the kernel of b^- as $B_2 : \ker b^- \times \ker b^- \to \ker b^-$. A natural candidate to preserve the b^- constraint is[2]

$$
B_2 = b^-\, m_2 \otimes \bar{m}_2\ , \tag{4.44}
$$

where $\bar{m}_2 : \bar{\mathcal{K}} \times \bar{\mathcal{K}} \to \bar{\mathcal{K}}$ is the C_∞ two-product of the second copy of thekinematic algebra. Notice that the output of B_2 lives in the image of b^-, $\mathrm{im\,b}^-$, and hence it is annihilated by b^-, preserving the algebra structure. Moreover, given that all the inputs obey $b^-\Psi = 0$, B_2 admits

[2]This definition of B_2 differs from the definition of B_2 in [2] and [3] by a factor of $-\frac{1}{2}$.

the following alternative rewriting:

$$B_2 = [b^-, m_2 \otimes \bar{m}_2] = \tfrac{1}{2}\left(b_2 \otimes \bar{m}_2 - m_2 \otimes \bar{b}_2\right) , \tag{4.45}$$

where $\bar{b}_2 = [\bar{b}, \bar{m}_2]$ is the generalized Poisson bracket of the second copy of kinematic $\mathrm{BV}_\infty^\square$-algebra $\bar{\mathcal{K}}$. As discussed in section 3.4.2, b_2 and $\bar{b}_2$ are Lie-type brackets that obey deformed Jacobi relations, and hence makes (4.45) an appropriate two-bracket.

Let us prove that B_2 obeys the Leibniz rule $[B_1, B_2] = 0$ and as a consequence defines a consistent field theory to cubic order. To do so, we will use the intrinsic input-free notation developed in the previous section. In terms of the single copy maps, the Leibniz relation reads

$$\begin{aligned}
[B_1, B_2] &= \tfrac{1}{2}\,[m_1 + \bar{m}_1, b_2 \otimes \bar{m}_2 - m_2 \otimes \bar{b}_2] \\
&= \tfrac{1}{2}\Big([m_1, b_2] \otimes \bar{m}_2 - m_2 \otimes [\bar{m}_1, \bar{b}_2]\Big) \\
&= \tfrac{1}{2}\,[\square - \bar{\square}, m_2 \otimes \bar{m}_2] \\
&= [\Delta, m_2 \otimes \bar{m}_2] \equiv 0 ,
\end{aligned} \tag{4.46}$$

where in the last equality we used the strong constraint $\Delta \equiv 0$. This proves that $\mathcal{V}^{\mathrm{DFT}}$ equipped with B_1 and B_2 defines an L_∞-algebra to bilinear order, and hence it encodes a consistent theory to cubic order.

We now turn to explicitly evaluating the action of B_2 on various elements of $\mathcal{V}^{\mathrm{DFT}}$ in order to compare our results with the original formulation by Hull and Zwiebach. In the interest of illustrating how to do these calculations, let us consider the two-bracket between two gauge parameters which defines the gauge algebra of DFT: $\Lambda_{12} = B_2(\Lambda_1, \Lambda_2)$. In principle, for generic gauge parameters, one would have to evaluate six terms corresponding to all possible combinations of components. In order to simplify our example, let us only consider one of the diagonal terms $B_2(\theta_\mu\bar{\theta}_+\,\lambda_1^\mu, \theta_\nu\bar{\theta}_+\,\lambda_2^\nu)$ giving

$$\begin{aligned}
B_2(\theta_\mu\bar{\theta}_+\,\lambda_1^\mu, \theta_\nu\bar{\theta}_+\,\lambda_2^\nu) &= b^-\, m_2 \otimes \bar{m}_2\big(\theta_\mu\bar{\theta}_+\,\lambda_1^\mu, \theta_\nu\bar{\theta}_+\,\lambda_2^\nu\big) \\
&= b^-\,\mu\Big[\hat{m}_2(\theta_\mu, \theta_\nu)\,\hat{\bar{m}}_2(\theta_+, \theta_+)\,(\lambda_1^\mu \otimes \lambda_2^\nu)\Big] \\
&= b^-\,\mu\Bigg\{\Big[c\,\theta_\nu\Big[\big(\partial_\mu \otimes \mathbb{1}\big) + 2\,\big(\mathbb{1} \otimes \partial_\mu\big)\Big] - c\,\theta_\mu\Big[\big(\mathbb{1} \otimes \partial_\nu\big) + 2\,\big(\partial_\nu \otimes \mathbb{1}\big)\Big] \\
&\quad + c\,\theta_\rho\,\eta_{\mu\nu}\,\Big[\big(\partial^\rho \otimes \mathbb{1}\big) - \big(\mathbb{1} \otimes \partial^\rho\big)\Big]\Big]\;\bar{\theta}_+\,\big(\mathbb{1} \otimes \mathbb{1}\big)\,\big](\lambda_1^\mu \otimes \lambda_2^\nu)\Bigg\} \\
&= b^- c\,\theta_\rho\,\bar{\theta}_+\,\Big(\partial \cdot \lambda_1\,\lambda_2^\rho + 2\,\lambda_1 \cdot \partial\lambda_2^\rho + \partial^\rho\lambda_1 \cdot \lambda_2 - (1 \leftrightarrow 2)\Big) \\
&= \tfrac{1}{2}\,\theta_\rho\,\bar{\theta}_+\,\Big(\partial \cdot \lambda_1\,\lambda_2^\rho + 2\,\lambda_1 \cdot \partial\lambda_2^\rho + \partial^\rho\lambda_1 \cdot \lambda_2 - (1 \leftrightarrow 2)\Big) \\
&\equiv \tfrac{1}{2}\,\theta_\rho\bar{\theta}_+\,\big(\lambda_1 \bullet \lambda_2\big)^\rho \;\in \mathcal{V}_{-1} ,
\end{aligned} \tag{4.47}$$

where $\bullet$ is the operator that determines the kinematic part of the cubic vertex of Yang-Mills. Hence, the gauge structure of double field theory follows from Yang-Mills theory. Doing analogous computations for the other parameters, one finds the components

$$\begin{aligned} B_2^\mu(\Lambda_1,\Lambda_2) &= \lambda_{12}^\mu = -\tfrac{1}{2}\,(\lambda_1\bullet\lambda_2)^\mu + \tfrac{1}{2}\,\bar\partial_{\bar\nu}\big(\lambda_1^\mu\,\bar\lambda_2^{\bar\nu}\big) - \tfrac{1}{2}\,\bar\partial_{\bar\nu}\big(\lambda_2^\mu\,\bar\lambda_1^{\bar\nu}\big)\ , \\ B_2^{\bar\mu}(\Lambda_1,\Lambda_2) &= \bar\lambda_{12}^{\bar\mu} = -\tfrac{1}{2}\,(\bar\lambda_1\bullet\bar\lambda_2)^{\bar\mu} + \tfrac{1}{2}\,\partial_\nu\big(\bar\lambda_1^{\bar\mu}\lambda_2^\nu\big) - \tfrac{1}{2}\,\partial_\nu\big(\bar\lambda_2^{\bar\mu}\lambda_1^\nu\big)\ , \\ B_2^\eta(\Lambda_1,\Lambda_2) &= \eta_{12} = \tfrac{1}{4}\,\partial_\mu\bar\partial_{\bar\nu}\big(\lambda_1^\mu\bar\lambda_2^{\bar\nu} - \lambda_2^\mu\bar\lambda_1^{\bar\nu}\big)\ . \end{aligned} \tag{4.48}$$

Evaluating $B_2(\Lambda,\psi)$ gives the gauge transformations (including the linearized ones found in the previous section)

$$\begin{aligned} \delta e_{\mu\bar\nu} &= \partial_\mu\bar\lambda_{\bar\nu} + \bar\partial_{\bar\nu}\lambda_\mu - \tfrac{1}{2}\,(\lambda\bullet e_{\bar\nu})_\mu - \tfrac{1}{2}\,(\bar\lambda\bullet e_\mu)_{\bar\nu} - \lambda_\mu\,\big(\bar f_{\bar\nu} - \bar\partial_{\bar\nu}\bar e\big) + \bar\lambda_{\bar\nu}\,\big(f_\mu - \partial_\mu e\big)\ , \\ \delta e &= -\tfrac{1}{2}\,\partial\cdot\lambda + \eta + \tfrac{1}{2}\,\lambda^\mu\big(f_\mu - \partial_\mu e\big)\ , \\ \delta\bar e &= \tfrac{1}{2}\,\bar\partial\cdot\bar\lambda + \eta + \tfrac{1}{2}\,\bar\lambda^{\bar\mu}\big(\bar f_{\bar\mu} - \bar\partial_{\bar\mu}\bar e\big)\ , \\ \delta f_\mu &= -\tfrac{1}{2}\,\Box\lambda_\mu + \partial_\mu\eta + \tfrac{1}{4}\,\bar\partial^{\bar\nu}(\lambda\bullet e_{\bar\nu})_\mu - \tfrac{1}{2}\,\bar\partial^{\bar\nu}\big[\bar\lambda_{\bar\nu}\,(f_\mu - \partial_\mu e)\big]\ , \\ \delta\bar f_{\bar\mu} &= \tfrac{1}{2}\,\Box\bar\lambda_{\bar\mu} + \bar\partial_{\bar\mu}\eta - \tfrac{1}{4}\,\partial^\nu(\bar\lambda\bullet e_\nu)_{\bar\mu} - \tfrac{1}{2}\,\partial^\nu\big[\lambda_\nu\,(\bar f_{\bar\mu} - \bar\partial_{\bar\mu}\bar e)\big]\ . \end{aligned} \tag{4.49}$$

Finally, the action at cubic order in fields is expressed in L_∞ form as

$$S^{(3)}_{\rm DFT} = \tfrac{1}{6}\big\langle\psi, B_2(\psi,\psi)\big\rangle\ , \tag{4.50}$$

and is explicitly is given by

$$\begin{aligned} S^{(3)}_{\rm DFT} = \int dx d\bar x\,\Big[&\tfrac{1}{8}\,e^{\mu\bar\nu}\Big(\bar\partial^{\bar\lambda}e_{\mu\bar\lambda}\,\partial^\rho e_{\rho\bar\nu} + \partial^\lambda e_{\lambda\bar\rho}\,\bar\partial^{\bar\rho}e_{\mu\bar\nu} + 2\,\partial_\mu e_{\lambda\bar\rho}\,\bar\partial_{\bar\nu}e^{\lambda\bar\rho} - 2\,\partial_\mu e^{\lambda\bar\rho}\,\bar\partial_{\bar\rho}e_{\lambda\bar\nu} - 2\,\bar\partial_{\bar\nu}e^{\lambda\bar\rho}\,\partial_\lambda e_{\mu\bar\rho}\Big) \\ &+ \tfrac{1}{2}\,e^{\mu\bar\nu}\Big(f_\mu - \partial_\mu e\Big)\Big(\bar f_{\bar\nu} - \bar\partial_{\bar\nu}\bar e\Big)\Big]\ . \end{aligned} \tag{4.51}$$

This action has one additional term with respect to the action that we found with the naive color-kinematic substitution in section 2.4, namely the term in the last line. This additional term makes the action invariant to cubic order under the gauge transformations (4.49). The full DFT action that arises from our prescription (the cubic action (4.51) in combination with the quadratic action (4.43)) is significantly simpler than the original formulation by Hull and Zwiebach $S^{\rm H.Z}_{\rm DFT}$ in (2.50). Both actions are related by local non-linear field and parameter redefinitions, which were worked out in [2].

4.4 Quartic theory

In this section we construct strongly constrained double field theory up to quartic interactions which, algebraically, corresponds to building a DFT three-bracket B_3 using the maps from the two copies of the kinematic algebra of Yang-Mills. To this order in perturbation theory, the L_∞ relation that encodes consistency of the theory is the homotopy Jacobi relation

$$\begin{aligned} B_2(B_2(\Psi_1,\Psi_2),\Psi_3) + (-1)^{\Psi_1(\Psi_2+\Psi_3)}B_2(B_2(\Psi_2,\Psi_3),\Psi_1) + (-1)^{\Psi_3(\Psi_1+\Psi_2)}B_2(B_2(\Psi_3,\Psi_1),\Psi_2) \\ + [B_1,B_3](\Psi_1,\Psi_2,\Psi_3) = 0\ . \end{aligned} \tag{4.52}$$

The strategy to construct B_3 is to compute the graded Jacobiator of B_2

$$\begin{aligned}\mathrm{Jac}_{B_2}(\psi_1,\psi_2,\psi_3) := B_2(B_2(\Psi_1,\Psi_2),\Psi_3) + (-1)^{\Psi_1(\Psi_2+\Psi_3)} B_2(B_2(\Psi_2,\Psi_3),\Psi_1) \\ + (-1)^{\Psi_3(\Psi_1+\Psi_2)} B_2(B_2(\Psi_3,\Psi_1),\Psi_2)\end{aligned} \tag{4.53}$$

in terms of the single copy maps m_2 and b_2. The $\mathrm{BV}_\infty^\square$ relations (3.203) of the two copies will then allow us to prove that Jac_{B_2} is a $(m_1+\bar{m}_1)-$commutator and this, in turn, allows us to identify B_3. We will do this computation using input-free notation, where the jacobiator can be written as

$$\mathrm{Jac}_{B_2} := 3\, B_2 B_2 \Pi\ , \tag{4.54}$$

and the homotopy Jacobi relation (4.52) becomes

$$\mathrm{Jac}_{B_2} + [B_1, B_3] = 0\ . \tag{4.55}$$

Using the single copy maps, the Jacobiator is

$$\begin{aligned}\mathrm{Jac}_{B_2} &= \tfrac{3}{2}\, b^-\, m_2 \otimes \bar{m}_2 \big\{ b_2 \otimes \bar{m}_2 - m_2 \otimes \bar{b}_2 \big\}\, \Pi \\ &= \tfrac{3}{2}\, b^- \big\{ m_2 b_2 \otimes \bar{m}_2 \bar{m}_2 - m_2 m_2 \otimes \bar{m}_2 \bar{b}_2 \big\}\, \Pi\ .\end{aligned} \tag{4.56}$$

Notice that in the above Jacobiator we used the two equivalent versions of B_2: the outermost bracket is of the form (4.44), while the innermost bracket is of the form (4.45). This makes the computation more economical. The next step is to split the maps into symmetric and hook components as

$$\begin{aligned}\mathrm{Jac}_{B_2} = \tfrac{3}{2}\, b^-\, \big\{ m_2 b_2 \pi \otimes \bar{m}_2 \bar{m}_2 \bar{\pi} - m_2 m_2 \pi \otimes \bar{m}_2 \bar{b}_2 \bar{\pi} \big\} \\ + \tfrac{3}{2}\, b^- \big\{ m_2 b_2 (1-\pi) \otimes \bar{m}_2 \bar{m}_2 (1-\bar{\pi}) - m_2 m_2 (1-\pi) \otimes \bar{m}_2 \bar{b}_2 (1-\bar{\pi}) \big\}\, \Pi\ ,\end{aligned} \tag{4.57}$$

where we used $\pi\Pi = \bar{\pi}\Pi = \pi\bar{\pi}$ in the first line. Now, in order to be able to identify a differential $m_1 + \bar{m}_1$, we have to use the $\mathrm{BV}_\infty^\square$ relations of the single copy maps. In the first line we use the symmetric projection of the homotopy Poisson compatibility condition

$$[b, m_2 m_2 \pi] - 3\, m_2 b_2 \pi = [m_1, \theta_{3\mathrm{s}}] - 3\, m_{3\mathrm{h}} d_s \pi\ , \tag{4.58}$$

while in the second line we use the homotopy associativity relation

$$m_2 m_2 (1-\pi) = [m_1, m_{3\mathrm{h}}]\ , \tag{4.59}$$

which leads to

$$\begin{aligned}\mathrm{Jac}_{B_2} = \tfrac{1}{2}\, b^-\, \big\{ - [m_1, \theta_{3\mathrm{s}}] \otimes \bar{m}_2 \bar{m}_2 \bar{\pi} + m_2 m_2 \pi \otimes [\bar{m}_1, \bar{\theta}_{3\mathrm{s}}] \big\} \\ + \tfrac{3}{2}\, b^- \big\{ m_2 b_2 (1-\pi) \otimes [\bar{m}_1, \bar{m}_{3\mathrm{h}}] - [m_1, m_{3\mathrm{h}}] \otimes \bar{m}_2 \bar{b}_2 (1-\bar{\pi}) \big\}\, \Pi \\ + \tfrac{3}{2}\, b^- \big\{ m_{3\mathrm{h}} d_s \pi \otimes \bar{m}_2 \bar{m}_2 \pi - m_2 m_2 \pi \otimes \bar{m}_{3\mathrm{h}} \bar{d}_{\bar{s}} \bar{\pi} \big\}\ ,\end{aligned} \tag{4.60}$$

where the terms involving $[b, m_2 m_2]$ (and their barred counterpart) in the symmetric projection of the homotopy Poisson relation (4.58) vanish due to the b^- constraint.

From the Jacobiator in the form (4.60) it is not possible to immediately extract a DFT differential $B_1 = m_1 + \bar{m}_1$ and obtain an explicit expression for B_3. To that end, we can insert a zero in the guise of Leibniz relations as

$$\begin{aligned} 0 = \tfrac{1}{2}\, b^- \big\{ [m_1, m_2 m_2 \pi] \otimes \bar{\theta}_{3\text{s}} - \theta_{3\text{s}} \otimes [\bar{m}_1, \bar{m}_2 \bar{m}_2 \bar{\pi}] \big\} \\ + \tfrac{3}{2}\, b^- \big\{ m_{3\text{h}} \otimes [\bar{m}_1, \bar{m}_2 \bar{b}_2] - [m_1, m_2 b_2] \otimes \bar{m}_{3\text{h}} \big\}\, \Pi \\ + \tfrac{3}{2}\, b^- \big\{ m_2 m_2 d_s \otimes \bar{m}_{3\text{h}} - m_{3\text{h}} \otimes \bar{m}_2 \bar{m}_2 \bar{d}_{\bar{s}} \big\}\, \Pi\,. \end{aligned} \tag{4.61}$$

The first line is zero due to the Leibniz relation of the homotopy associative algebra, whereas the second and third lines are zero by virtue of the Leibniz rule modulo box of the bracket b_2. Notice, very importantly, that the terms added as the Leibniz rule modulo box of b_2 are not projected, so they contain both their symmetric and hook components. Adding the above zero to the Jacobiator leads to

$$\begin{aligned} \text{Jac}_{B_2} = \tfrac{1}{2}\, b^- \Big\{ \big[m_1 + \bar{m}_1, m_2 m_2 \pi \otimes \bar{\theta}_{3\text{s}} \big] - \big[m_1 + \bar{m}_1, \theta_{3\text{s}} \otimes \bar{m}_2 \bar{m}_2 \bar{\pi} \big] \Big\} \\ - \tfrac{3}{2}\, b^- \Big\{ \big[m_1 + \bar{m}_1, m_2 b_2 (1-\pi) \otimes \bar{m}_{3\text{h}} \big] + \big[m_1 + \bar{m}_1, m_{3\text{h}} \otimes \bar{m}_2 \bar{b}_2 (1-\bar{\pi}) \big] \Big\}\, \Pi \\ + \tfrac{3}{2}\, b^- \Big\{ m_{3\text{h}} d_s \pi \otimes \bar{m}_2 \bar{m}_2 \bar{\pi} - m_{3\text{h}} \otimes \bar{m}_2 \bar{m}_2 \bar{d}_{\bar{s}}\, \Pi \\ + m_2 m_2 d_s \otimes \bar{m}_{3\text{h}}\, \Pi - m_2 m_2 \pi \otimes \bar{m}_{3\text{h}} \bar{d}_{\bar{s}} \bar{\pi} \Big\}\,, \end{aligned} \tag{4.62}$$

where to arrive at the terms in the second line we used

$$\big(1\otimes 1 - \pi \otimes 1\big)\Pi = \big(1 \otimes 1 - 1 \otimes \bar{\pi}\big)\Pi = \big(1 \otimes 1 - 1 \otimes \bar{\pi}\big)\big(1 \otimes 1 - \pi \otimes 1\big)\Pi\,, \tag{4.63}$$

in order to project some of the terms involving $m_2 b_2$ using the fact that $m_{3\text{h}}$ contains a $(1-\pi)$ projector. Even though it is straightforward to extract the DFT differential from the first two lines of (4.62), it is not obvious how to do so in the last two lines. To this end, one has to use the strong constraint $d_s \equiv \bar{d}_{\bar{s}}$. Let us deal explicitly with the last line of (4.62):

$$\begin{aligned} 3\, m_2 m_2 d_s \otimes \bar{m}_{3\text{h}}\, \Pi - 3\, m_2 m_2 \pi \otimes \bar{m}_{3\text{h}} \bar{d}_{\bar{s}} \bar{\pi} &= 3 \big\{ m_2 m_2 \otimes \bar{m}_{3\text{h}} \bar{d}_{\bar{s}} - m_2 m_2 \pi \otimes \bar{m}_{3\text{h}} \bar{d}_{\bar{s}} \big\}\, \Pi \\ &= 3\, m_2 m_2 (1-\pi) \otimes \bar{m}_{3\text{h}} \bar{d}_{\bar{s}}\, \Pi \\ &= 3\, [m_1, m_{3\text{h}}] \otimes \bar{m}_{3\text{h}} \bar{d}_{\bar{s}}\, \Pi\,. \end{aligned} \tag{4.64}$$

In the first equality we used $\pi\Pi = \pi\bar{\pi}\,\Pi = \pi\bar{\pi}$, and to obtain the final line we used the homotopy associativity relation (4.59). Repeating the same procedure for the other term in the Jacobiator yields

$$\begin{aligned} \text{Jac}_{B_2} = \tfrac{1}{2}\, b^- \Big\{ \big[m_1 + \bar{m}_1, m_2 m_2 \pi \otimes \bar{\theta}_{3\text{s}} \big] - \big[m_1 + \bar{m}_1, \theta_{3\text{s}} \otimes \bar{m}_2 \bar{m}_2 \bar{\pi} \big] \Big\} \\ - \tfrac{3}{2}\, b^- \Big\{ \big[m_1 + \bar{m}_1, m_2 b_2 (1-\pi) \otimes \bar{m}_{3\text{h}} \big] + \big[m_1 + \bar{m}_1, m_{3\text{h}} \otimes \bar{m}_2 \bar{b}_2 (1-\bar{\pi}) \big] \Big\}\, \Pi \\ + \tfrac{3}{2}\, b^- \Big\{ \big[m_1 + \bar{m}_1, m_{3\text{h}} \otimes \bar{m}_{3\text{h}} \bar{d}_{\bar{s}} \big] \Big\}\, \Pi\,. \end{aligned} \tag{4.65}$$

From this form of the Jacobiator it is possible to read-off B_3, which is given by

$$B_3 = -\tfrac{1}{2}\, b^- \Big\{ \theta_{3\text{s}} \otimes \bar{m}_2 \bar{m}_2 \bar{\pi} - m_2 m_2 \pi \otimes \bar{\theta}_{3\text{s}} \\ + 3\left[m_2 b_2 (1-\pi) \otimes \bar{m}_{3\text{h}} + m_{3\text{h}} \otimes \bar{m}_2 \bar{b}_2 (1-\bar{\pi}) \right] \Pi \\ - 3\, m_{3\text{h}} \otimes \bar{m}_{3\text{h}} \bar{d}_{\bar{s}}\, \Pi \Big\} \,. \tag{4.66}$$

The above expression can be made simpler by noticing that since $\theta_{3\text{s}}$ and $\bar{\theta}_{3\text{s}}$, and $m_{3\text{h}}$ and $\bar{m}_{3\text{h}}$, are projected onto their symmetric and hook parts, respectively, one can drop the explicit projectors of the maps that multiply them in the tensor product. This yields

$$B_3 = -\tfrac{1}{2}\, b^- \Big\{ \theta_{3\text{s}} \otimes \bar{m}_2 \bar{m}_2 - m_2 m_2 \otimes \bar{\theta}_{3\text{s}} + 3\, m_2 b_2 \otimes \bar{m}_{3\text{h}} + 3\, m_{3\text{h}} \otimes \bar{m}_2 \bar{b}_2 \\ - 3\, m_{3\text{h}} \otimes \bar{m}_{3\text{h}} \bar{d}_{\bar{s}} \Big\}\, \Pi \,, \tag{4.67}$$

where we factored out the symmetric operator Π. This map determines gauge invariant gravity in the form of double field theory up to quartic order in the action, because by construction B_1, B_2 and B_3 obey the L_∞ relations to trilinear order. Indeed, we could use the above three-bracket to construct a quartic action of the form

$$S^{(4)}_{\text{DFT}} = \tfrac{1}{4!} \big\langle \psi, B_3(\psi,\psi,\psi) \big\rangle \,, \tag{4.68}$$

which is local, since all the single copy maps that we used to build the DFT brackets are local. It is worth stressing that the fact that the DFT maps B_1, B_2 and B_3 obey the L_∞ relations and hence encode a consistent theory to quartic order, is due to the $\text{BV}^{\square}_{\infty}$ relations that the single copy maps obey. For this reason, the algebraic structure responsible for the consistency of our double copy prescription is the kinematic $\text{BV}^{\square}_{\infty}$-algebra. In this section, in contrast to to the previous two, we will not find the quartic DFT action explicitly. Instead, in the next section we compute four-point scattering amplitudes.

4.5 Four-point scattering amplitudes

In this section we compute four-point DFT scattering amplitudes as a consistency check of our double copy prescription. This will allow us to give an algebraic interpretation of known results in the double copy literature. We start this section with a brief introduction on how to formulate scattering amplitudes in terms of the algebraic maps that we have discussed so far. In order to illustrate this, we will start with Yang-Mills theory and subsequently proceed to DFT. Additionally, in this section we state the relation between the standard BCJ double copy prescription of scattering amplitudes and our algebraic approach, and we give an interpretation of the "kinematic Jacobi identity" as an algebraic relation of the kinematic $\text{BV}^{\square}_{\infty}$-algebra of Yang-Mills.

4.5.1 Yang-Mills scattering amplitudes

Recall the version of Yang-Mills theory that we use in this thesis:

$$S_{\rm YM} = \int d^Dx \,{\rm Tr}\left\{\tfrac{1}{2}\,A^\mu\Box A_\mu - \tfrac{1}{2}\,\varphi^2 + \varphi\,\partial_\mu A^\mu - \partial_\mu A_\nu\,[A^\mu, A^\nu] - \tfrac{1}{4}\,[A^\mu, A^\nu]\,[A_\mu, A_\nu]\right\} . \quad (4.69)$$

Our task now is to compute scattering amplitudes of Yang-Mills using the L_∞ data of the above action. The first step is to impose a gauge condition. As discussed at the end of section 3.2 the b operator, which plays a central role in the kinematic algebra, can be used to impose gauge conditions. At this stage we work with all the fields taking values in $\mathfrak{g}$, and we impose Siegel gauge on fields:

$$b(\mathcal{A}) \equiv \begin{pmatrix}\varphi\\0\end{pmatrix} = 0 \,, \quad (4.70)$$

where use the notation of section 3.4 and the operator b acts as in the diagram (3.139) and obeys $[B_1, b] = \Box$. The field equation of the scalar φ, $\partial\cdot A - \varphi = 0$, thus implies after imposing the gauge condition that the fields A_μ have to be transverse, i.e

$$\partial\cdot A = 0 \,. \quad (4.71)$$

The external states of scattering amplitudes are considered to be asymptotically free, and as a consequence only the free theory is relevant to prepare the external scattered particles. The above gauge choice implies that there is a residual gauge symmetry in the free theory where all gauge parameters are harmonic, i.e $\Box\lambda = 0$.

In our algebraic formalism, the on-shell conditions reads

$$B_1(\mathcal{A}) = 0 \,, \quad (4.72)$$

which together with $b(\mathcal{A}) = 0$ is the familiar massless wave equation

$$\Box A_\mu = 0 \,. \quad (4.73)$$

The solutions to the free field equations can then be expressed as

$$A_\mu(x) = \epsilon^a_\mu(x)\otimes t_a \,, \quad (4.74)$$

where we have factorized the gauge field into a kinematic part or polarization vector $\epsilon^a_\mu(x)$, and a color part $t_a\in\mathfrak{g}$. We can think of the polarization vector as a color stripped object and we drop the color index a to write it as

$$\epsilon_\mu(x) = \epsilon_\mu(p)\,e^{ipx} \,. \quad (4.75)$$

Following the discussion of section 3.2 , the propagator for our version of Yang-Mills in Siegel gauge in momentum space is given by

$$G(p) = \frac{b}{p^2} \,, \quad (4.76)$$

where the Feynman $i\varepsilon$-prescription is implied. When computing amplitudes we assign a label i, a Lie algebra element t_i, and a polarization vector $\epsilon_{i\,\mu}(p_i)$ to each external particle. In momentum space, the gauge and on-shell conditions for all external particles read

$$p_i \cdot \epsilon_i = 0 \;, \quad p_i^2 = 0 \;. \tag{4.77}$$

A four-point tree-level Yang-Mills scattering amplitude can be written in terms of the L_∞ maps as

$$\mathcal{A}^{(4)}_{\text{Tree}} = \, -g^2_{\text{YM}} \, \big\langle A_4, B_2(G_{12}\, B_2(A_1, A_2), A_3)\big\rangle + g^2_{\text{YM}}\big\langle A_4, B_3(A_1, A_2, A_3)\big\rangle + \text{cyclic} \;, \tag{4.78}$$

where the pairing $\langle\cdot,\cdot\rangle$ is the same pairing used to construct the action in the framework of homotopy algebras, in the arguments of the brackets we write A instead of $\mathcal{A}$ to emphasize that we have gauge fixed and on-shell fields and the cyclic sum is with respect to the labels (123) while keeping the label 4 fixed. From now on we work in momentum space. The first term in the amplitude is the exchange contribution. This can be easily inferred because of the presence of the propagator[3] $G_{12} = -\frac{b}{s_{12}}$ and the two two-brackets B_2, associated to the cubic vertex of Yang-Mills. The second term, on the other hand, is the contact contribution to the amplitude, as it consists only of the bracket B_3 which defines the quartic vertex of Yang-Mills. In order to state the relation between the BCJ double copy and our algebraic approach, it will be useful to factorize color and kinematics in the amplitude as

$$\mathcal{A}^{(4)}_{\text{Tree}} = -g^2_{\text{YM}}\Big(\epsilon_4, \big\{m_2(G_{12}\, m_2(\epsilon_1, \epsilon_2), \epsilon_3) - m_{3h}(\epsilon_1, \epsilon_2|\epsilon_3)\big\}\Big)\, \text{Tr}\big(t_4[[t_1, t_2], t_3]\big) + \text{cyclic} \;, \tag{4.79}$$

where the arguments of the kinematic products m_n are color stripped polarization vectors

$$\epsilon_i = \theta_\mu\, \epsilon_i^\mu(p_i)\, e^{ip_i x} \;, \tag{4.80}$$

t_i are the generators of $\mathfrak{g}$ associated to each external particle, and the kinematic pairing $(\cdot,\cdot)$ is given by

$$\big(\epsilon_i, J_j\big) = \epsilon_{i\,\mu}(p_i)\, J_j^\mu(p_j)\, \delta(p_i + p_j) \;, \tag{4.81}$$

where J^μ denotes a current built from external particles data, belonging to the space K_2 of field equations. In the following we will not write the delta functions explicitly.

In the double copy literature (see also section 2.2) Yang-Mills scattering amplitudes are usually expressed in terms of so-called kinematic numerators n_{sij}, which depend on polarization vectors and momenta, and color factors which are color-traces of generators of the gauge group. Explicitly, the four-gluon amplitude can be written as

$$\mathcal{A}^{(4)}_{\text{Tree}} = g^2_{\text{YM}}\Big\{\frac{n_s\, c_s}{s} + \frac{n_t\, c_t}{t} + \frac{n_u\, c_u}{u}\Big\} \;. \tag{4.82}$$

Comparing equations (4.82) and (4.79), one can read-off the algebraic form of the kinematic

[3]In our conventions, for massless particles we use: $s_{12} = s = 2\, p_1 \cdot p_2$, $s_{23} = t = 2\, p_2 \cdot p_3$, $s_{13} = u = 2\, p_1 \cdot p_3$.

numerators and the color factors

$$\begin{aligned} n_s &:= \big(\epsilon_4, \mathfrak{n}_s\big) \ , \\ c_s &:= \mathrm{Tr}\big(t_4\,[[t_1,t_2],t_3]\big) \ , \end{aligned} \tag{4.83}$$

where we defined the current

$$\mathfrak{n}_s := m_2(b_2(\epsilon_1,\epsilon_2),\epsilon_3) + s\, m_{3h}(\epsilon_1,\epsilon_2|\epsilon_3) \, \in K_2 \ . \tag{4.84}$$

The expressions for the other channels can be found by relabeling the external fields. Note that here bm_2 equals $b_2 = [b, m_2]$ when acting on fields, since the inputs are annihilated by b due to gauge fixing.

Let us close the discussion on Yang-Mills amplitudes with some remarks on color-kinematics duality. As mentioned in section 2.2, the color factors of the four-point amplitude obey

$$c_s + c_t + c_u = 0 \ . \tag{4.85}$$

Indeed, taking the explicit definition of the c_{s_i} in terms of the Lie brackets yields

$$\mathrm{Tr}\Big\{t_4\,[[t_1,t_2],t_3] + t_4\,[[t_2,t_3],t_1] + t_4\,[[t_3,t_1],t_2]\Big\} = 0 \ , \tag{4.86}$$

whose vanishing is due to the Jacobi identity of $[\cdot,\cdot]$. Similarly, the BCJ double copy requires the kinematic numerators to obey the so-called "*kinematic Jacobi identity*". Now we explain how this relation follows in a straightforward manner from the homotopy Poisson relation of the kinematic $\mathrm{BV}_\infty^\square$-algebra. Let us recall the homotopy Poisson relation in an input free form:

$$[b, m_2 m_2] - 3m_2 b_2 \pi - [m_1, \theta_3] - m_{3h}(d_\square - 3\, d_s\, \pi) = 0 \ . \tag{4.87}$$

Since we want to relate this equation to the kinematic numerators, all the inputs that we will consider are polarization vectors obeying the gauge and on-shell conditions. Additionally, in order to recover the kinematic numerators from this equation, it is necessary to take the inner product of the Poisson relation with a polarization vector $\epsilon_{4\,\mu}$. Doing so the first term vanishes because all the polarization vectors are annihilated by b due to gauge fixing. The third and fourth term vanish because all the the polarization vectors are on-shell, and hence are annihilated by m_1 and $\square$. Notice that the second term in combination with the last term is the cyclic sum of the currents $\mathfrak{n}_{s_{ij}}$. Thus, upon taking the inner product with a polarization vector ϵ_4 we obtain the "kinematic Jacobi idenity"

$$n_s + n_t + n_u = 0 \ . \tag{4.88}$$

This observation provides an alternative understanding of color-kinematics duality, where the kinematic algebra does not necessarily have to be a strict Lie algebra, but rather it can be a Lie-type algebra with controlled failures. Recall that the consistency of the off-shell and gauge invariant quartic DFT construction relied heavily on the $\mathrm{BV}_\infty^\square$ relations of the two copies of the kinematic algebra, and thus it is no mystery to find that the consistency of the amplitudes relies on the same type of relations.

4.5.2 Double field theory scattering amplitudes

Let us now turn to computing double field theory scattering amplitudes. Similarly to Yang-Mills, the first step in computing DFT amplitudes is imposing gauge and on-shell conditions on the external states. In DFT, as discussed in section 2.4, Siegel gauge is imposed as

$$b^+(\psi) \equiv \begin{pmatrix} f_\mu \\ \bar{f}_{\bar{\nu}} \end{pmatrix} = 0 \ , \tag{4.89}$$

where $b^+ := \frac{1}{2}\big(b \otimes 1 + 1 \otimes \bar{b}\big)$. This gauge choice in combination with the free field equations of the auxiliary fields shown in equation (4.41) imply the de Donder-type condition for the free external states

$$\partial_\mu \bar{e} - \tfrac{1}{2}\,\bar{\partial}^{\bar{\rho}} e_{\mu\bar{\rho}} = 0 \ , \qquad \bar{\partial}_{\bar{\mu}} e + \tfrac{1}{2}\,\partial^\rho e_{\rho\bar{\mu}} = 0 \ . \tag{4.90}$$

In this gauge the free theory has a residual gauge symmetry consisting of harmonic gauge parameters $\Box\lambda_\mu = \Box\bar{\lambda}_{\bar{\nu}} = 0$. We can use further gauge redundancy to fix the scalars e and $\bar{e}$ to zero on-shell. Let us stress that setting e and $\bar{e}$ to zero can only be achieved at the linearized level, and it amounts to discarding all the external e and $\bar{e}$ states. However, they are present in the interaction vertices. In this thesis we only consider tensors as external particles and at four-points and tree level, for such a choice, there are no internal e nor $\bar{e}$ particles. Setting the scalars to zero, the residual gauge symmetry consists of harmonic and transverse gauge parameters. With this choice the condition on the tensor fluctuation is reduced to

$$\bar{\partial}^{\bar{\rho}} e_{\mu\bar{\rho}} = 0 \ , \ \ \partial^\rho e_{\rho\bar{\mu}} = 0 \ . \tag{4.91}$$

The free theory field equations in the algebraic form are

$$B_1(\psi) = 0 \ , \tag{4.92}$$

and with the above gauge choices, the tensor component of the free field equations read

$$\Box e_{\mu\bar{\nu}} = 0 \ , \tag{4.93}$$

which we impose on external states, and thus we consider particles with double plane wave solutions of the form

$$e_{\mu\bar{\nu}}(x,\bar{x}) = \varepsilon_{\mu\bar{\nu}}(p,\bar{p})\, e^{i(px+\bar{p}\bar{x})} \ . \tag{4.94}$$

The four-point DFT amplitude can be written in terms of the algebraic maps as

$$\mathcal{M}^{(4)}_{\text{Tree}} = -2\kappa^2 \Big\langle e_4, [B_2(\mathfrak{h}_{12}\, B_2(e_1,e_2),e_3) + \text{cyclic}] - B_3(e_1,e_2,e_3) \Big\rangle_{\text{DFT}} \ , \tag{4.95}$$

where κ is the gravitational coupling constant, the propagator is $\mathfrak{h} = -\frac{b^+}{s_{ij}}$, and the pairing $\langle \cdot,\cdot \rangle_{\text{DFT}}$ will be defined explicitly below. We wrote e_i instead of ψ_i in the arguments of the above maps to emphasize that we are dealing with external fields of the form

$$e_i = \theta_\mu \bar{\theta}_{\bar{\nu}}\, \varepsilon_i^{\mu\bar{\nu}}(p_i,\bar{p}_i)\, e^{i(p_i x+\bar{p}_i\bar{x})} \ , \tag{4.96}$$

subject to the gauge and on-shell conditions

$$p_i^{\mu}\varepsilon_{i\,\mu\bar{\nu}}(p_i,\bar{p}_i)=0\ ,\ \ \bar{p}_i^{\bar{\nu}}\varepsilon_{i\,\mu\bar{\nu}}(p_i,\bar{p}_i)=0\ ,\ \ p_i^2=\bar{p}_i^2=0\ . \tag{4.97}$$

Our task now is to compute the DFT amplitude (4.95) explicitly using the single copy Yang-Mills maps, because doing so will make direct contact with the BCJ double copy of scattering amplitudes. Before performing the explicit computation, it will be useful to first resort to the input-free notation and organize conveniently the maps that contribute to the amplitude. If we write out the exchange and contact contributions to the amplitude in terms of the single copy maps we obtain

$$\begin{aligned}B_2\mathfrak{h}B_2\Pi-B_3=\tfrac{1}{2}\,b^-\Big\{&\frac{1}{s_{ij}}m_2b_2\otimes\bar{m}_2\bar{b}_2+\tfrac{1}{3}\,\theta_{3\mathrm{s}}\otimes\bar{m}_2\bar{m}_2-\tfrac{1}{3}\,m_2m_2\otimes\bar{\theta}_{3\mathrm{s}}\\&+m_2b_2\otimes\bar{m}_{3\mathrm{h}}+m_{3\mathrm{h}}\otimes\bar{m}_2\bar{b}_2-m_{3\mathrm{h}}\otimes\bar{m}_{3\mathrm{h}}\bar{d}_{\bar{s}}\Big\}\,\Pi\ ,\end{aligned} \tag{4.98}$$

where the symmetric operator Π encodes the cyclic sum over label permutations, the first term in the right hand side is the exchange contribution, and the remaining terms come from the expression that we found for B_3 in the section4.4. To arrive at the above form of the exchange term we used the explicit expression for the propagator $\mathfrak{h}_{ij}=-\frac{b^+}{s_{ij}}$ with ij denoting the particle labels of the first two inputs, and $B_2b^+B_2=-\frac{1}{2}\,b^-m_2b_2\otimes\bar{m}_2\bar{b}_2$ together with $b(e_i)=\bar{b}(e_i)=0$.

Not all the terms in (4.98) contribute to the amplitude. For instance, the two terms with $\theta_{3\mathrm{s}}$ and $\bar{\theta}_{3\mathrm{s}}$, with three tensor inputs as considered here, only contribute to amplitudes with at least an external scalar field. This can be inferred from the fact the output of $\hat{\theta}_{3\mathrm{s}}$ is the $c\theta_+$ component of a field equation or current. Given that we are only interested in all-tensor amplitudes, we can ignore those two terms, and the amplitude in terms of the single copy maps becomes

$$\begin{aligned}\mathcal{M}^{(4)}_{\mathrm{Tree}}=-\kappa^2\Big\langle e_4,\frac{1}{s_{ij}}\,b^-\,\big[&m_2b_2\otimes\bar{m}_2\bar{b}_2+s_{ij}\,m_2b_2\otimes\bar{m}_{3\mathrm{h}}+s_{ij}\,m_{3\mathrm{h}}\otimes\bar{m}_2\bar{b}_2\\&+s_{ij}^2\,m_{3\mathrm{h}}\otimes\bar{m}_{3\mathrm{h}}\big]\Pi(e_1,e_2,e_3)\Big\rangle_{\mathrm{DFT}}\ ,\end{aligned} \tag{4.99}$$

where we emphasize that all the maps are in the image of b^- (as we factored it out from the square brackets) and we used that d_s acts in momentum space as

$$\begin{aligned}\bar{d}_s(\varepsilon_i\otimes\varepsilon_j\otimes\varepsilon_k)e^{ix(p_i+p_j+p_k)_i\bar{x}(\bar{p}_i+\bar{p}_j+\bar{p}_k)}&=-2p_i\cdot p_j\,(\varepsilon_i\otimes\varepsilon_j\otimes\varepsilon_k)e^{ix(p_i+p_j+p_k)_i\bar{x}(\bar{p}_i+\bar{p}_j+\bar{p}_k)}\\&=-s_{ij}\,(\varepsilon_i\otimes\varepsilon_j\otimes\varepsilon_k)e^{ix(p_i+p_j+p_k)_i\bar{x}(\bar{p}_i+\bar{p}_j+\bar{p}_k)}\ ,\end{aligned} \tag{4.100}$$

as well as the strong constraint $s_{ij}\equiv\bar{s}_{ij}$.

Now we illustrate how to perform explicit computations using the DFT brackets in terms of the single copy maps in momentum space. To that end, let us remark that the polarization tensors $\varepsilon_{\mu\bar{\nu}}(p,\bar{p})$ can be parametrized as

$$\begin{aligned}e_i&=\theta_\mu\bar{\theta}_{\bar{\nu}}\,\epsilon_i^{\mu}(p_i)\,e^{ip_ix}\,\bar{\epsilon}_i^{\bar{\nu}}(\bar{p}_i)\,e^{i\bar{p}_i\bar{x}}\\&=\epsilon_i\otimes\bar{\epsilon}_i\ .\end{aligned} \tag{4.101}$$

As a concrete example, let us compute the tensor product map $m_{3\text{h}} \otimes \bar{m}_{3\text{h}}$ that appears in the last term of the DFT amplitude, using the above factorization of the polarization tensors:

$$\begin{aligned} m_{3\text{h}} \otimes \bar{m}_{3\text{h}}(e_1, e_2, e_3) &= \mu \Big[\hat{m}_{3\text{h}}(\theta_\mu, \theta_\nu, \theta_\rho)\, \hat{\bar{m}}_{3\text{h}}(\bar{\theta}_{\bar{\mu}}, \bar{\theta}_{\bar{\nu}}, \bar{\theta}_{\bar{\rho}}) \big(\epsilon_1^\mu \, \bar{\epsilon}_1^{\bar{\mu}} \otimes \epsilon_2^\nu \, \bar{\epsilon}_2^{\bar{\nu}} \otimes \epsilon_3^\rho \, \bar{\epsilon}_3^{\bar{\rho}} \big) \Big] \\ &= \bar{c}\, c\, \theta_\mu \bar{\theta}_{\bar{\nu}} \Big\{ \big(\epsilon_1^\mu \, \epsilon_2 \cdot \epsilon_3 - \epsilon_2^\mu \, \epsilon_1 \cdot \epsilon_3 \big) \big(\bar{\epsilon}_1^{\bar{\nu}} \, \bar{\epsilon}_2 \cdot \bar{\epsilon}_3 - \bar{\epsilon}_2^{\bar{\nu}} \, \bar{\epsilon}_1 \cdot \bar{\epsilon}_3 \big) \Big\} \\ &= \tfrac{1}{2}\, c^+ \, c^- \, \theta_\mu \bar{\theta}_{\bar{\nu}} \Big\{ \big(\epsilon_1^\mu \, \epsilon_2 \cdot \epsilon_3 - \epsilon_2^\mu \, \epsilon_1 \cdot \epsilon_3 \big) \big(\bar{\epsilon}_1^{\bar{\nu}} \, \bar{\epsilon}_2 \cdot \bar{\epsilon}_3 - \bar{\epsilon}_2^{\bar{\nu}} \, \bar{\epsilon}_1 \cdot \bar{\epsilon}_3 \big) \Big\} \, . \end{aligned} \tag{4.102}$$

In the above computation we ignored all phase factors for simplicity and we used the expressions for $\hat{m}_{3\text{h}}$ and $\hat{\bar{m}}_{3\text{h}}$ from appendix A. Next, in order to compute the amplitude, we also have to define the pairing or inner product $\langle \cdot, \cdot \rangle_{\text{DFT}}$ between on-shell fields and tensor currents in momentum space $\mathcal{J} = c^+ \, \theta_\mu \bar{\theta}_{\bar{\nu}} \mathcal{J}^{\mu\bar{\nu}}(p_i, \bar{p}_i) e^{ip_i x + i\bar{x}\bar{p}_i}$ in the following way:

$$\big\langle e, \mathcal{J} \big\rangle_{\text{DFT}} = \varepsilon_{\mu\bar{\nu}}(p_i, \bar{p}_i)\, \mathcal{J}^{\mu\bar{\nu}}(p_j, \bar{p}_j) \delta(p_i + p_j)\, \bar{\delta}(\bar{p}_i + \bar{p}_j) \, . \tag{4.103}$$

This inner product exhibits a factorization property associated to the factorization of polarization tensors. Indeed, given that all the maps that make up the currents are tensor products of two single copy maps in the image of b^-, i.e $\mathcal{J} = -b^- \, J \otimes \bar{J}$, and given that the on-shell fields factorize as in (4.101), the inner product factorizes as

$$- \big\langle e, b^- \, \mathcal{J} \big\rangle_{\text{DFT}} = (\epsilon, J)\, (\bar{\epsilon}, \bar{J}) \, , \tag{4.104}$$

where the pairings $(\cdot, \cdot)$ in the right hand side are the kinematic Yang-Mills pairings of the two kinematic spaces defined as in (4.81).

Using the factorization property of the polarization tensors, currents and the inner product, upon computing all terms in (4.99) as we did in the example (4.102), one obtains the amplitude

$$\begin{aligned} \mathcal{M}^{(4)}_{\text{Tree}} &= \kappa^2 \left\{ \frac{\big(\epsilon_4, \mathfrak{n}_s\big)\, \big(\bar{\epsilon}_4, \bar{\mathfrak{n}}_s\big)}{s} \right\} + \text{cyclic} \\ &= \kappa^2 \left\{ \frac{n_s \, \bar{n}_s}{s} + \frac{n_t \, \bar{n}_t}{t} + \frac{n_u \, \bar{n}_u}{u} \right\} \, , \end{aligned} \tag{4.105}$$

where we used the definition of the kinematic numerators in terms of the currents $n_{s_{ij}} = (\epsilon_4, \mathfrak{n}_{s_{ij}})$. This amplitude agrees with the expectation from the BCJ double copy. Indeed, the above amplitude can be obtained by exchanging color and kinematics à la BCJ, namely exchanging $c_{s_{ij}} \to \bar{n}_{s_{ij}}$ and $g_{\text{YM}} \to \kappa$ in the Yang-Mills amplitude (4.82). Moreover, if we solve the strong constraint by setting $p_\mu = \bar{p}_{\bar{\mu}}$, one recovers the four-point amplitude of $N = 0$ supergravity. This shows that our double copy prescription is consistent with the BCJ double copy and reproduces the correct four-point scattering amplitudes.

Chapter 5

Homotopy transfer and weakly constrained double field theory

In this chapter we construct weakly constrained double field theory to quartic order. To that end, we start with a brief review of weakly constrained DFT in section 5.1. We discuss the field content of the theory and its interpretation as well as some technical details that are required by the background geometry that we use. Afterwards, in section 5.2 we introduce the notion of homotopy transfer. This notion allows one to transfer algebraic structures from one space to another and plays a central role in the construction of the quartic theory. Subsequently, in section 5.3 we construct a homotopy algebra called $\mathrm{BV}_\infty^\Delta$-algebra which corresponds to the tensor product of the two copies of the Yang-Mills kinematic algebra prior to imposing any conditions, and then, in section 5.4, we abstractly construct the three-bracket of weakly constrained DFT which determines the quartic theory. Finally, as a concrete example, in section 5.5 we compute a consistent subsector of the gauge algebra of weakly constrained DFT to trilinear order.

This section is based on [4] and the upcoming paper [5].

5.1 Generalities of weakly constrained double field theory

As we stated in the introduction, weakly constrained double field is a massless subsector[1] of closed string field theory on toroidal backgrounds originally constructed by Hull and Zwiebach to cubic order in the action in [52]. The field content and gauge parameters that arise from the string field theory construction coincide with the field content that follows from the double copy prescription described in the previous chapter for strongly constrained double field theory, namely

$$\begin{aligned}\Psi(x,\bar{x}) &= \Big\{e_{\mu\bar{\nu}}(x,\bar{x}), e(x,\bar{x}), \bar{e}(x,\bar{x}), f_\mu(x,\bar{x}), \bar{f}_{\bar{\mu}}(x,\bar{x})\Big\} \\ \Lambda(x,\bar{x}) &= \Big\{\lambda_\mu(x,\bar{x}), \bar{\lambda}_{\bar{\mu}}(x,\bar{x}), \eta(x,\bar{x})\Big\}\ ,\end{aligned} \tag{5.1}$$

[1] Massless here refers to the fields in the higher dimensional theory prior to compactification.

Nevertheless, in the weakly constrained theory these fields and parameters have different interpretations than in the strongly constrained case because of the background geometry and the type of constraint that is imposed. In particular, for weakly constrained DFT we consider backgrounds with some coordinates compactified on tori, in contrast to the double Minkowski background that we considered in the previous chapters for the strongly constrained theory. Moreover, here we impose the *weak constraint* (or level matching condition)

$$(\square - \bar{\square})\Psi(x, \bar{x}) = 0 \; , \quad (\square - \bar{\square})\Lambda(x, \bar{x}) = 0 \; , \tag{5.2}$$

which, contrary to the strong constraint, does not eliminate any coordinate dependance. Thus, weakly constrained DFT is truly doubled and as a consequence the theory contains information about all the Kaluza-Klein and winding modes of the string simultaneously, encoded in the x- and $\bar{x}$-dependence, respectively. Hence, the tensor fluctuation $e_{\mu\bar{\nu}}$, in addition to the massless graviton and B-field, contains massive Kaluza-Klein modes, as well as their respective T-dual winding modes. Additionally, the string field theory construction requires the following constraint:

$$b_0^- \Psi(x, \bar{x}) \; , \quad b_0^- \Lambda(x, \bar{x}) = 0 \; , \quad b_0^- := \tfrac{1}{2}\big(b_0 - \bar{b}_0\big) \; , \tag{5.3}$$

where b_0 and $\bar{b}_0$ are right and left-moving zero modes of the closed string ghosts. This constraint is the string theory analogue of the algebraic b^- constraint that we imposed in the previous chapter to truncate the tensor product space to the physical one.

In this thesis we choose the background geometry to be a double Euclidean torus, where all fields and gauge parameters can be written as a double Fourier series:

$$f(x, \bar{x}) = \sum_{k, \bar{k}} \tilde{f}(k, \bar{k}) \, e^{ik\cdot x + i\bar{k}\cdot\bar{x}} \; , \tag{5.4}$$

with integer momenta $(k^\mu, \bar{k}^{\bar{\mu}}) \in \mathbb{Z}^{2d}$. Let us define the operator $\Delta := \frac{1}{2}(\square - \bar{\square})$. In Fourier space, this operator acts on functions on the double torus as

$$\Delta \, f(x, \bar{x}) = -\frac{1}{2} \sum_{k, \bar{k}} (k^2 - \bar{k}^2) \, \tilde{f}(k, \bar{k}) \, e^{ik\cdot x + i\bar{k}\cdot\bar{x}} \; . \tag{5.5}$$

The weak constraint then requires the fields and gauge parameters to be in the kernel of Δ. For our particular choice of space, we can define a projector $\mathcal{P}_\Delta$ to the kernel of Δ in terms of the Kronecker delta symbol acting as

$$\Big(\mathcal{P}_\Delta f\Big)(x, \bar{x}) = \sum_{k, \bar{k}} \delta_{k^2, \bar{k}^2} \tilde{f}(k, \bar{k}) \, e^{ik\cdot x + i\bar{k}\cdot\bar{x}} = \sum_{k^2 = \bar{k}^2} \tilde{f}(k, \bar{k}) \, e^{ik\cdot x + i\bar{k}\cdot\bar{x}} \; . \tag{5.6}$$

This operator clearly projects to the kernel of Δ, since

$$\Delta \, \mathcal{P}_\Delta = \mathcal{P}_\Delta \, \Delta = 0 \; , \tag{5.7}$$

and squares to itself: $\mathcal{P}_\Delta^2 = \mathcal{P}_\Delta$. We should think of all the fields Ψ and gauge parameters Λ to be projected objects, even if we do not write the projector explicitly. It is important to stress that, in general, the product of two projected functions does not belong to the kernel of Δ, just like the product of two harmonic functions is not harmonic. For this reason, all the products of fields and gauge parameters in the theory should be projected with $\mathcal{P}_\Delta$, i.e

$$\mathcal{P}_\Delta(f(x,\bar{x})g(x,\bar{x})) = \sum_{k,\bar{k}} \sum_{l,\bar{l}} \delta_{(k+l)^2,(\bar{k}+\bar{l})^2} \tilde{f}(k,\bar{k})\, \tilde{g}(l,\bar{l})\, e^{i(k+l)\cdot x + i(\bar{k}+\bar{l})\cdot\bar{x}} \;, \tag{5.8}$$

and similar for higher order products. The quadratic action of weakly constrained DFT reads

$$\begin{aligned} S^{(2)}_{\text{H.Z.}} = \int d^D x\, d^D \bar{x} \Big\{ \tfrac{1}{4}\, e^{\mu\bar{\nu}} \Box e_{\mu\bar{\nu}} + 2\, \bar{e}\, \Box e - f^\mu f_\mu - \bar{f}^{\bar{\mu}} \bar{f}_{\bar{\mu}} - f^\mu \Big(\bar{\partial}^{\bar{\nu}} e_{\mu\bar{\nu}} - 2\, \partial_\mu \bar{e} \Big) \\ + \bar{f}^{\bar{\nu}} \Big(\partial^\mu e_{\mu\bar{\nu}} + 2\, \bar{\partial}_{\bar{\nu}} e \Big) \Big\} \;, \end{aligned} \tag{5.9}$$

while the cubic action is given by

$$\begin{aligned} S^{(3)}_{\text{H.Z.}} = \int d^D x d^D \bar{x}\, \Big[& \tfrac{1}{8} e_{\mu\bar{\nu}} \left(2\, \partial^\mu e_{\rho\bar{\sigma}} \bar{\partial}^{\bar{\nu}} e^{\rho\bar{\sigma}} - 2\, \partial^\mu e_{\rho\bar{\sigma}} \bar{\partial}^{\bar{\sigma}} e^{\rho\bar{\nu}} - 2\, \partial^\rho e^{\mu\bar{\sigma}} \bar{\partial}^{\bar{\nu}} e_{\rho\bar{\sigma}} + 2\, \partial_\rho e^{\rho\bar{\nu}} \bar{\partial}_{\bar{\rho}} e^{\mu\bar{\rho}} + \partial_\rho e^{\rho\bar{\sigma}} \bar{\partial}_{\bar{\sigma}} e^{\mu\bar{\nu}} \right) \\ & + \tfrac{1}{2} e_{\mu\bar{\nu}} f^\mu \bar{f}^{\bar{\nu}} - \tfrac{1}{2} f_\mu f^\mu \bar{e} + \tfrac{1}{2} \bar{f}_{\bar{\nu}} \bar{f}^{\bar{\nu}} e - \tfrac{1}{4} f^\mu \left(e_{\mu\bar{\nu}} \bar{\partial}^{\bar{\nu}} \bar{e} + \bar{\partial}^{\bar{\nu}} \left(e_{\mu\bar{\nu}} \bar{e} \right) \right) \\ & - \tfrac{1}{4} \bar{f}^{\bar{\nu}} \left(e_{\mu\bar{\nu}} \partial^\mu e + \partial^\mu \left(e_{\mu\bar{\nu}} e \right) \right) + \tfrac{1}{4} f^\mu \left(\bar{e} \partial_\mu e - e \partial_\mu \bar{e} \right) + \tfrac{1}{4} \bar{f}^{\bar{\nu}} \left(\bar{e} \bar{\partial}_{\bar{\nu}} e - e \bar{\partial}_{\bar{\nu}} \bar{e} \right) \\ & - \tfrac{1}{8} e_{\mu\bar{\nu}} \left(\bar{e} \partial^\mu \bar{\partial}^{\bar{\nu}} e + e \partial^\mu \bar{\partial}^{\bar{\nu}} \bar{e} - \partial^\mu e \bar{\partial}^{\bar{\nu}} \bar{e} - \bar{\partial}^{\bar{\nu}} e \partial^\mu \bar{e} \right) \Big] \;. \end{aligned} \tag{5.10}$$

All the products of field in both actions should be thought of as being projected by $\mathcal{P}_\Delta$. We will only write the projector explicitly when it is necessary from now on. Let us emphasize that, structurally, the above actions are the same as the ones we introduced for strongly constrained DFT in chapter 2. The interpretation of the actions, however, depends on the background geometry and the type of constraint imposed.

Constructing weakly constrained DFT beyond cubic order in the action has been a major challenge since the theory was first constructed in 2009. In the following we sketch why this has been the case. From a Noetherian perspective, an action to cubic order in fields and a set of non-linear gauge transformations is, in principle, enough information to construct an action to quartic order in fields. Indeed, this is the approach that we took to construct the quartic theory of strongly constrained DFT in the previous chapter. We had full knowledge of the cubic theory, so we took advantage of our algebraic understanding of the theory and constructed the quartic interactions. More precisely, we had an explicit expression for the two-bracket B_2, which determines the cubic theory, and then computing its Jacobiator we managed to extract the three-bracket B_3, which determines the quartic theory. For the weakly constrained theory one could naively think that one should proceed in a similar fashion. However, the projector complicates this perspective on the construction of perturbative field theories. Algebraically, the two-bracket in weakly constrained DFT can be written in terms of the two-bracket of the strongly constrained theory as

$$\mathcal{B}_2 = \mathcal{P}_\Delta\, B_2 \;, \tag{5.11}$$

due to both theories being the same to cubic order with the exception of the inclusion of the projector. The Jacobiator of the above bracket in terms of B_2 reads

$$3\,\mathcal{B}_2\mathcal{B}_2\Pi = 3\,\mathcal{P}_\Delta B_2 \mathcal{P}_\Delta B_2\Pi \ . \tag{5.12}$$

From the strongly constrained theory we know exactly what the Jacobiator of B_2 is, namely $3\,B_2B_2\Pi = -[B_1, B_3]$. Here, however, we cannot use this information due to the projector in the middle of the two B_2's not commuting with them. Hence, the naive Noetherian approach is difficult to implement in order to extract a three-bracket $\mathcal{B}_3$. As we will see, we can use the framework of homotopy algebras and homotopy transfer (which we introduce in the following section) to overcome this problem and construct weakly constrained DFT to quartic order.

5.2 Homotopy transfer

In this section we introduce the concept of *homotopy transfer*. This will be a crucial tool that will allow us to construct weakly constrained double field theory up to quartic interactions later on. The main idea behind homotopy transfer is to transport the algebraic structure defined on a given vector space to a different vector space in a consistent way. In this section we will give a very brief introduction to the topic with only the minimum required details. The reader interested in a rigorous and complete introduction to the topic can refer to [98]. In order to illustrate how homotopy transfer works, let us consider a differential commutative associative algebra (dgca). That is a graded vector space $\mathcal{V}$ equipped with a differential Q and a graded symmetric associative product μ_2 obeying the following relations

$$Q^2 = 0 \ , \quad [Q, \mu_2] = 0 \ , \quad \mu_2\mu_2(1-\pi) = 0 \ , \tag{5.13}$$

where we used the input-free notation of section 3.5.1. We use the degrees $|Q| = 1$, $|\mu_2| = 0$. Our task now is to consider another graded vector space $\bar{\mathcal{V}}$ that is a subspace of $\mathcal{V}$, i.e $\overline{\mathcal{V}} \subset \mathcal{V}$, and we ask the following question: How is the dgca algebraic structure on $\mathcal{V}$ transported to the subspace $\overline{\mathcal{V}}$?

To answer this question, the first step is to find a map that takes us from the full space $\mathcal{V}$, where we know the algebraic structure, to the subspace $\overline{\mathcal{V}}$. We call such a map a *projection* $P : \mathcal{V} \to \overline{\mathcal{V}}$. Moreover, we can define a map that goes in the other direction called the *inclusion map* $\iota : \overline{\mathcal{V}} \to \mathcal{V}$ which tells us what the elements of $\overline{\mathcal{V}}$ "look like" in the bigger space $\mathcal{V}$. For our purposes a trivial inclusion will suffice. Such an inclusion map is defined as follows: given an element $\bar{x}$ of $\overline{\mathcal{V}}$, the trivial inclusion map acts as

$$\iota(\bar{x}) = \bar{x} \in \mathcal{V} \ . \tag{5.14}$$

Notice that the element $\bar{x}$ remains untouched. The only difference is that after acting with ι we think of $\bar{x}$ as belonging to the bigger space $\mathcal{V}$. The projection and inclusion obey trivially $P \circ \iota = 1$, where $\circ$ denotes the composition of operators and 1 here is the identity map. Let us now turn to the linear structure on both spaces. The vector $\mathcal{V}$ is equipped with the differential Q, making it into a chain complex. Similarly, in order to have a chain complex structure on

$\overline{\mathcal{V}}$, we need a differential acting on this space which we denote by $\bar{Q}$. Then, we require the projection and inclusion to be *chain maps*, i.e

$$PQ = \bar{Q}P \ , \quad \iota\bar{Q} = Q\iota \ . \tag{5.15}$$

Additionally, in order to consistently transport the algebraic structure from $\mathcal{V}$ to $\overline{\mathcal{V}}$, homotopy transfer relies on the following crucial *homotopy transfer relation*

$$[Q, h] = 1 - \iota \circ P \ , \tag{5.16}$$

where h is a degree minus one map called *homotopy map.* One can interpret the above equation as the statement that ι is the inverse of P *up to homotopy.* Moreover, we further require the following *side conditions*

$$h^2 = h\iota = Ph = 0 \ . \tag{5.17}$$

In order to have a cleaner notation, let us define the composite operator $\mathcal{P} := \iota \circ P$. With this operator, all the above relations and conditions can be written as

$$\mathcal{P}Q = Q\mathcal{P} \ , \qquad \text{Chain map condition,} \tag{5.18}$$

$$[Q, h] = 1 - \mathcal{P} \ , \qquad \text{Homotopy relation,} \tag{5.19}$$

$$h^2 = \mathcal{P}h = h\mathcal{P} = 0 \ , \qquad \text{Side conditions.} \tag{5.20}$$

Given that here we work with a trivial inclusion, we can abuse terminology and refer to either $\mathcal{P}$ or P as the projector. Hence we will not make any formal distinction between the two.

We now turn to transporting the non-linear algebraic structure, namely the two-product μ_2. Naturally, there is a two-product $\bar{\mu}_2$ that acts on the subspace $\overline{\mathcal{V}}$ defined in terms of the two-product of the bigger space $\mathcal{V}$ as

$$\bar{\mu}_2 = \mathcal{P}\mu_2 \ , \tag{5.21}$$

where the inputs of μ_2 are acted upon by the inclusion map ι. This projected product, however, is not obviously associative as μ_2 is. In fact, we claim that $\bar{\mu}_2$ is associative up to homotopy. Let us prove this claim: First we look at the *associator* $\bar{\mu}_2\bar{\mu}_2(1-\pi)$. Writing this associator in terms of the brackets that act on $\mathcal{V}$ we have

$$\bar{\mu}_2\bar{\mu}_2(1-\pi) = \mathcal{P}\mu_2\,\mathcal{P}\mu_2(1-\pi) \tag{5.22}$$

In order to further manipulate this expression[2] using the known associativity relations obeyed by μ_2, we can use the homotopy relation to rewrite the innermost projector as $\mathcal{P} = 1 - [Q, h]$. This leads to

$$\bar{\mu}_2\bar{\mu}_2(1-\pi) = \mathcal{P}\,\mu_2\mu_2(1-\pi) - \mathcal{P}\mu_2\,[Q, h]\,\mu_2(1-\pi) \ . \tag{5.23}$$

The first term in the above equation vanishes due to μ_2 being associative, namely $\mu_2\mu_2(1-\pi) = 0$,

[2]Notice that this is precisely the situation that we described when showing the difficulties that arise in constructing weakly constrained DFT to quartic order in equation (5.12).

and we can write out explicitly the commutator $[Q, h]$ to obtain

$$\bar{\mu}_2\bar{\mu}_2(1-\pi) = -\mathcal{P}\mu_2\, Qh\, \mu_2(1-\pi) - \mathcal{P}\mu_2\, hQ\, \mu_2(1-\pi) \; . \tag{5.24}$$

Using the Leibniz rule $[Q, \mu_2] = 0$ and the chain map condition $\mathcal{P}Q = Q\mathcal{P}$, we can recast the above equation as

$$\bar{\mu}_2\bar{\mu}_2(1-\pi) = [Q, \bar{\mu}_3] \; , \tag{5.25}$$

where the new three-product $\bar{\mu}_3$ is given by

$$\bar{\mu}_3 = -\mathcal{P}\mu_2\, h\, \mu_2(1-\pi) \; . \tag{5.26}$$

Notice that this relation is the homotopy associativity relation that we introduced when we first discussed C_∞-algebras in section 3.4.2. This means that the dgca structure on $\mathcal{V}$ is transported to a C_∞ strcuture on the smaller space $\overline{\mathcal{V}}$, provided that the homotopy map h is non-vanishing. Otherwise, one obtains a dgca structure on $\overline{\mathcal{V}}$. Homotopy transfer can also be performed in other types of algebras as for example Lie or L_∞-algebras and, as we will show shortly, BV-type algebras.

5.3 $\mathrm{BV}^{\Delta}_{\infty}$-algebra

In this section we construct a $\mathrm{BV}^{\Delta}_{\infty}$-algebra on the tensor product $\mathcal{V} = \mathcal{K} \otimes \bar{\mathcal{K}}$, where $\mathcal{K}$ and $\bar{\mathcal{K}}$ are two copies of the kinematic algebras of Yang-Mills, prior to imposing any constraint. In this algebra, the operator $\Delta := \frac{1}{2}(\Box - \bar{\Box})$ plays the same role that the Laplace operator plays in the kinematic algebra of Yang-Mills (since here we are in Euclidean $\Box$ is a Lapacian). This larger algebra will give us a starting point from which we can construct both strongly and weakly constrained DFT. Obtaining the strongly constrained theory from the $\mathrm{BV}^{\Delta}_{\infty}$ is straightforward. The construction of the weakly constrained theory, however, presents subtleties as we will see in the next section.

5.3.1 C_∞-algebra on $\mathcal{V}$

Our starting point is the vector space $\mathcal{V} = \mathcal{K} \otimes \bar{\mathcal{K}}[-2]$, where $[-2]$ signifies a degree-shift by minus two[3]. Recall that each copy of the kinematic algebra of Yang-Mills contains, as a consistent subsector, a C_∞-algebra. For the unbarred copy we have products (m_1, m_2, m_3) acting on $\mathcal{K}$, while for the barred copy we have $(\bar{m}_1, \bar{m}_2, \bar{m}_3)$ acting on $\bar{\mathcal{K}}$. We now turn to proving that one can define a C_∞-algebra on $\mathcal{V}$ to trilinear order, using the kinematic C_∞ products of both copies of the kinematic algebra of Yang-Mills. To that end, using the input-free notation, let us define a differential and a two-bracket as

$$M_1 := m_1 + \bar{m}_1 \; , \quad M_2 := m_2 \otimes \bar{m}_2 \; . \tag{5.27}$$

[3]This is the same degree shift that we implemented for strongly constrained DFT in the previous chapter. See equation (4.16).

Recall that the first two C_∞ relations, corresponding to relations with one and two inputs, are: nilpotency of the differential and the Leibniz rule. One can easily check that these two relations are obeyed. Nilpotency of the differential is obvious because, independently, the single copy differentials m_1 and $\bar{m}_1$ are nilpotent and they anticommute with each other. For the Leibniz rule, on the other hand, we have the compute the following commutator $[M_1, M_2] = 0$. In terms of the single copy maps we have

$$\begin{aligned} [M_1, M_2] =&[m_1, m_2 \otimes \bar{m}_2] + [\bar{m}_1, m_2 \otimes \bar{m}_2] \\ =&[m_1, m_2] \otimes m_2 + m_2 \otimes [\bar{m}_1, \bar{m}_2] \\ =&0 \ , \end{aligned} \tag{5.28}$$

proving that the Leibniz relation is obeyed and we recall that the degrees of the single copy maps are $|m_1| = 1$, $|m_2| = 0$ and similar for the barred maps. The degrees of the double copy maps are $|M_1| = 1$, $|M_2| = 2$. The degree of M_2 follows from the degree shift by two that we performed in the tensor product.

We next study the associativity of M_2 by computing its associator. Due to graded symmetry of M_2, the latter is equivalent to the hook projection $M_2M_2(1-\Pi)$. Using the definition (5.28), the property (4.30) of projectors and the homotopy associativity of m_2 and $\bar{m}_2$ we can compute

$$\begin{aligned} M_2M_2(1-\Pi) &= m_2m_2 \otimes \bar{m}_2\bar{m}_2 \Big(\pi(1-\bar{\pi}) + (1-\pi)\bar{\pi} + (1-\pi)(1-\bar{\pi})\Big)(1-\Pi) \\ &= \Big\{m_2m_2\pi \otimes [\bar{m}_1, \bar{m}_{3\text{h}}] + [m_1, m_{3\text{h}}] \otimes \bar{m}_2\bar{m}_2\bar{\pi} + [m_1, m_{3\text{h}}] \otimes [\bar{m}_1, \bar{m}_{3\text{h}}]\Big\}(1-\Pi) \\ &= \big[M_1, m_2m_2\pi \otimes \bar{m}_{3\text{h}} + m_{3\text{h}} \otimes \bar{m}_2\bar{m}_2\bar{\pi}\big](1-\Pi) + [m_1, m_{3\text{h}}] \otimes [\bar{m}_1, \bar{m}_{3\text{h}}](1-\Pi) \ , \end{aligned} \tag{5.29}$$

where in the last step we used $[m_1, m_2m_2] = 0$ (and the barred relation) to extract a total differential M_1. At this stage an ambiguity arises on how to treat the last term, since

$$[m_1, m_{3\text{h}}] \otimes [\bar{m}_1, \bar{m}_{3\text{h}}] = \big[M_1, (\tfrac{1}{2} - \xi)\, m_{3\text{h}} \otimes [\bar{m}_1, \bar{m}_{3\text{h}}] + (\tfrac{1}{2} + \xi)\, [m_1, m_{3\text{h}}] \otimes \bar{m}_{3\text{h}}\big] \tag{5.30}$$

for arbitrary ξ. Keeping ξ arbitrary leads to a one-parameter family of three-products, differing by an M_1-exact term (which for maps means a total M_1 commutator). This is expected, since in a homotopy associative algebra the three-product is defined only up to an M_1-closed quantity. For simplicity we choose $\xi = 0$ and obtain

$$M_{3\text{h}} = \frac{1}{2}\Big\{m_{3\text{h}} \otimes \bar{m}_2\bar{m}_2(1+\bar{\pi}) + m_2m_2(1+\pi) \otimes \bar{m}_{3\text{h}}\Big\}(1-\Pi) \ , \tag{5.31}$$

obeying homotopy associativity in the form $M_2M_2(1-\Pi) = [M_1, M_{3\text{h}}]$. This proves that, at least to trilinear order, one can define a C_∞-algebra structure on $\mathcal{V}$. It is important to remark that even though the original C_∞-algebras on $\mathcal{K}$ and $\bar{\mathcal{K}}$ have no higher products than $m_{3\text{h}}$, the tensor algebra $\mathcal{V}$ may have infinitely many higher M_n, which we will not explore further.

5.3.2 The b^- operator and Δ

Having found a C_∞-algebra on the tensor product space using the two copies of the Yang-Mills kinematic maps, we now turn to constructing a $\mathrm{BV}_\infty^\Delta$-algebra. To that end, we follow the same steps as in section 3.4.2, where we first constructed the kinematic $\mathrm{BV}_\infty^\Box$-algebra. Our task now is to find the analog of the b operator in the kinematic algebra of Yang-Mills. The two natural candidates for such an operator on the tensor product are

$$b^\pm := \tfrac{1}{2}\big(b \pm \bar{b}\big) \ . \tag{5.32}$$

Both $b^\pm$ are nilpotent and (anti)commute with each other. Their commutators with M_1, which determine the obstruction Δ, are given by

$$[M_1, b^\pm] = \tfrac{1}{2}\left(\Box \pm \bar{\Box}\right) \ . \tag{5.33}$$

For establishing a $\mathrm{BV}_\infty^\Delta$ algebra on $\mathcal{V}$ both choices $b^\pm$ are equivalent. Our choice is dictated by the goal of constructing double field theory on a suitable subspace of $\mathcal{V}$ which, in particular, requires an unobstructed L_∞ algebra. In view of the fact that the we wish to obtain weakly constrained DFT, where one imposes the weak constraint $\big(\Box - \bar{\Box}\big)\Psi = 0$, the natural choice for the analogue of the b operator is b^-, yielding

$$[M_1, b^-] = \Delta \ , \quad \Delta := \tfrac{1}{2}\left(\Box - \bar{\Box}\right) \ . \tag{5.34}$$

With this choice, one can construct a two-bracket B_2 as

$$B_2 := [b^-, M_2] \ , \tag{5.35}$$

which is the analogue of the generalized Poisson bracket of the kinematic algebra $b_2 := [b, m_2]$. Notice that, after imposing the b^- constraint, i.e $b^-\Psi = 0$, the above two-bracket reduces to the two-bracket that we used to construct the L_∞-algebra of strongly constrained DFT, namely $B_2 = b^-\, m_2 \otimes \bar{m}_2$. However, for the moment, let us *not* impose it and keep the full double space $\mathcal{V}$.

The differential M_1 does not obey the Leibniz rule with respect to the two-bracket B_2. Instead, one finds that the Leibniz rule holds *modulo* Δ, i.e

$$\begin{aligned} [M_1, B_2] &= [M_1, [b^-, M_2]] \\ &= [[M_1, b^-], M_2] - [b^-, [M_1, M_2]] \\ &= [\Delta, M_2] \ , \end{aligned} \tag{5.36}$$

where we used $[M_1, M_2] = 0$. The above is analogue of the Leibniz rule modulo box of the kinematic algebra of Yang-Mills. In view of going to trilinear relations, for the $\Delta-$obstructions on trilinear maps we define $D_\Delta := \frac{1}{2}\left(d_\Box - \bar{d}_{\bar{\Box}}\right)$ and $D_s := \frac{1}{2}\left(d_s - \bar{d}_s\right)$ where d_s and $d_\Box$ and their barred counterparts are defined in (4.31). The Δ obstruction operators act on three functions

$F_i(x,\bar{x})$ as

$$\begin{aligned} D_s(F_1\otimes F_2\otimes F_3) &= (\partial^\mu F_1\otimes\partial_\mu F_2\otimes F_3) - (\bar\partial^{\bar\mu} F_1\otimes\bar\partial_{\bar\mu} F_2\otimes F_3)\ , \\ D_\Delta(F_1\otimes F_2\otimes F_3) &= (\partial^\mu F_1\otimes\partial_\mu F_2\otimes F_3) - (\bar\partial^{\bar\mu} F_1\otimes\bar\partial_{\bar\mu} F_2\otimes F_3) \\ &\quad + (F_1\otimes\partial^\mu F_2\otimes\partial_\mu F_3) - (F_1\otimes\bar\partial^{\bar\mu} F_2\otimes\bar\partial_{\bar\mu} F_3) \\ &\quad + (\partial^\mu F_1\otimes F_2\otimes\partial_\mu F_3) - (\bar\partial^{\bar\mu} F_1\otimes F_2\otimes\bar\partial_{\bar\mu} F_3)\ . \end{aligned} \tag{5.37}$$

Given the definition of D_s and D_Δ and the projector (4.27), they obey

$$\big(\mathcal{T}\otimes\bar{\mathcal{T}}\big)\,D_\Delta\,\Pi = \big(\mathcal{T}\otimes\bar{\mathcal{T}}\big)\,\Pi\,D_\Delta = 3\,\big(\mathcal{T}\otimes\bar{\mathcal{T}}\big)\,\Pi\,D_s\,\Pi\ , \tag{5.38}$$

similarly to (3.200) for d_s and $d_\Box$.

We now turn to examining the Poissonator of M_2 and B_2 (see equation (3.204) and replace $m_2\to M_2$ and $b_2\to B_2$) which reads

$$\mathrm{Poiss} = B_2M_2 + M_2B_2(1-3\,\Pi)\ . \tag{5.39}$$

If one computes the Poissonator using the single copy maps and the $\mathrm{BV}_\infty^\Box$ relations of both copies of the kinematic algebra, one finds that the result is non-vanishing. In particulat, one finds the following deformed Poisson relation associated to a $\mathrm{BV}_\infty^\Delta$-algebra:

$$\mathrm{Poiss} = \big[M_1,\Theta_3\big] + M_{3\mathrm{h}}\big(D_\Delta - 3\,D_s\,\Pi\big)\ . \tag{5.40}$$

This is a somewhat cumbersome computation and in the interest of simplicity we do not display it here explicitly. The reader interested in the details may refer to the upcoming paper [5]. Similarly to θ_3 in the kinematic algebra, Θ_3 splits into a totally symmetric part $\Theta_{3\mathrm{s}}$ and hook part $\Theta_{3\mathrm{h}}$. The symmetric part, in terms of the single copy maps, is given by

$$\begin{aligned} \Theta_{3\mathrm{s}} = \frac{1}{2}\Big\{&\theta_{3\mathrm{s}}\otimes\bar m_2\bar m_2 - m_2m_2\otimes\bar\theta_{3\mathrm{s}} + 3\,m_2b_2\otimes\bar m_{3\mathrm{h}} + 3\,m_{3\mathrm{h}}\otimes\bar m_2\bar b_2 - \tfrac{3}{2}\,m_{3\mathrm{h}}\otimes\bar m_{3\mathrm{h}}\,(d_s+\bar d_s) \\ &- \big[b^-, m_{3\mathrm{h}}\otimes\bar m_2\bar m_2 + m_2m_2\otimes\bar m_{3\mathrm{h}}\big]\Big\}\,\Pi\ , \end{aligned} \tag{5.41}$$

while the hook part is determined by the C_∞ map $M_{3\mathrm{h}}$ in equation (5.31) as

$$\Theta_{3\mathrm{h}} = -[b^-, M_{3\mathrm{h}}]\ . \tag{5.42}$$

Moreover, in complete analogy to the kinematic algebra of Yang-Mills, one can take a b^- commutator of the deformed Poisson relation (5.40) to obtain a deformed homotopy Jacobi relation with a three-bracket $B_3 = -[b^-,\Theta_{3\mathrm{s}}]$. This deformed homotopy Jacobi identity exhausts the relations of the $\mathrm{BV}_\infty^\Delta$-algebra up to trilinear maps. We can summarize them in the following table, which takes the same form as (3.203):

$$\begin{aligned}
&M_1^2 = 0\ , \quad (b^-)^2 = 0\ , \quad [M_1, b^-] = \Delta\ , && \text{differentials and central obstruction,}\\
&[M_1, M_2] = 0\ , \quad M_2M_2(1-\Pi) = [M_1, M_{3\text{h}}]\ , && C_\infty \text{ structure,}\\
&B_2 = [b^-, M_2]\ , \quad [M_1, B_2] = [\Delta, M_2]\ , && \text{two-bracket and deformed Leibniz,}\\
&B_2M_2 + M_2B_2\,(1-3\,\Pi) = [M_1, \Theta_3] + M_{3\text{h}}(D_\Delta - 3\,D_s\,\Pi)\ , && \text{deformed homotopy Poisson,}\\
&3\,B_2B_2\Pi + [M_1, B_3] + 3\,\Theta_3\,D_s\,\Pi = 0\ , && \text{deformed homotopy Jacobi,}\\
&\Theta_{3\text{h}} + [b^-, M_{3\text{h}}] = 0\ , \quad B_3 + [b^-, \Theta_{3\text{s}}] = 0\ , && \text{compatibility of homotopies.}
\end{aligned} \tag{5.43}$$

In particular, notice that the brackets of the L_∞ sector (albeit obstructed) are all determined in terms of other structures as $B_1 \equiv M_1$, $B_2 = [b^-, M_2]$, $B_3 = -[b^-, \Theta_{3\text{s}}]$.

5.3.3 Strongly constrained double field theory as a subsector of the BV_∞^Δ-algebra

In this subsection we show how one can recover strongly constrained double field theory, starting from the BV_∞^Δ-algebra to trilinear order constructed above. Strongly constrained DFT requires the strong constraint $\Delta \equiv 0$, which implies

$$D_\Delta \equiv 0\ , \quad D_s \equiv D_t \equiv D_u \equiv 0\ , \tag{5.44}$$

where the vanishing of $D_{s_{ij}}$ follows from $d_{s_{ij}} \equiv \bar{d}_{s_{ij}}$. Thus, upon imposing the strong constraint, the BV_∞^Δ relations turn into the relations of a BV_∞-algebra to trilinear order, namely

$$\begin{aligned}
&M_1^2 = 0\ , \quad (b^-)^2 = 0\ , \quad [M_1, b^-] = 0\ , && \text{differentials,}\\
&[M_1, M_2] = 0\ , \quad M_2M_2(1-\Pi) = [M_1, M_{3\text{h}}]\ , && C_\infty \text{ structure,}\\
&B_2 = [b^-, M_2]\ , \quad [M_1, B_2] = 0\ , && \text{Leibniz,}\\
&B_2M_2 + M_2B_2\,(1-3\,\Pi) = [M_1, \Theta_3]\ , && \text{homotopy Poisson,}\\
&3\,B_2B_2\Pi + [M_1, B_3] = 0\ , && \text{homotopy Jacobi,}\\
&\Theta_{3\text{h}} + [b^-, M_{3\text{h}}] = 0\ , \quad B_3 + [b^-, \Theta_{3\text{s}}] = 0\ , && \text{compatibility of homotopies.}
\end{aligned} \tag{5.45}$$

In particular, BV_∞-algebras contain L_∞-algebras as consistent subsectors. Indeed, the maps $M_1 \equiv B_1$, B_2 and B_3 form an L_∞-algebra on $\mathcal{V}$ to trilinear order. This L_∞-algebra contained in the BV_∞-algebra, however, is too large and does *not* encode strongly constrained double field theory. Recall that in order to obtain the correct spectrum of DFT we need to impose the b^- constraint so that the relevant space is

$$\mathcal{V}_\text{s}^\text{DFT} = \Big\{\Psi(x,\bar{x}) \in \mathcal{V} |\ \Delta \equiv 0\ ,\ b^-\Psi(x,\bar{x}) = 0\Big\}\ . \tag{5.46}$$

In this subspace of $\mathcal{V}$, there exists an L_∞-algebra to trilinear order with the following maps:

$$
\begin{aligned}
B_1 &= m_1 + \bar{m}_1 \\
B_2 &= b^- \, m_2 \otimes \bar{m}_2 \, , \\
B_3 &= -\frac{1}{2}\, b^- \Big\{ \theta_{3\mathrm{s}} \otimes \bar{m}_2 \bar{m}_2 - m_2 m_2 \otimes \bar{\theta}_{3\mathrm{s}} + 3\, m_2 b_2 \otimes \bar{m}_{3\mathrm{h}} + 3\, m_{3\mathrm{h}} \otimes \bar{m}_2 \bar{b}_2 \\
&\qquad - 3\, m_{3\mathrm{h}} \otimes \bar{m}_{3\mathrm{h}}\, d_s \Big\} \Pi \, .
\end{aligned}
\tag{5.47}
$$

These maps acting on the vector space $\mathcal{V}_{\mathrm{s}}^{\mathrm{DFT}}$ form the L_∞-algebra underlying strongly constrained DFT to quartic order.

5.4 Weakly constrained double field theory and homotopy transfer

We now turn to constructing the L_∞-algebra of weakly constrained DFT on a Euclidean double torus starting from the $\mathrm{BV}_\infty^\Delta$-algebra to trilinear order that we constructed above. To that end, we proceed in two steps: we will first transport the $\mathrm{BV}_\infty^\Delta$structure of $\mathcal{V}$ to the subspace $\overline{\mathcal{V}} := \mathrm{ker}\Delta$ via homotopy transfer, to be discussed momentarily, and in the second step we will restrict the maps to act on $\mathrm{ker} b^-$. In this last step the $\mathrm{BV}_\infty^\Delta$ structure will be lost, leaving an unobstructed L_∞ algebra that encodes weakly constrained double field theory. Let us first perform homotopy transfer.

Given that we are projecting onto a subspace $\overline{\mathcal{V}} := \mathrm{ker}\Delta$ of $\mathcal{V}$, we choose a trivial inclusion map and thus the projector is the projector onto the kernel of Δ, $\mathcal{P}_\Delta$. The differential is M_1 and we require the chain map condition

$$
M_1 \mathcal{P}_\Delta = \mathcal{P}_\Delta M_1 \, , \tag{5.48}
$$

together with the homotopy relation

$$
[M_1, h] = 1 - \mathcal{P}_\Delta \, , \tag{5.49}
$$

and the side conditions

$$
\mathcal{P}_\Delta h = h \mathcal{P}_\Delta = h^2 = 0 \, . \tag{5.50}
$$

With these side conditions and the homotopy equation, we can find a homotopy map h such that the homotopy equation is satisfied. In that interest, we shall first introduce the "propagator" G. Given a function orthogonal to $\mathrm{ker}\Delta$, meaning it obeys $\mathcal{P}_\Delta f = 0$, one can invert Δ by means of the propagator G, defined as

$$
\big(Gf\big)(x, \bar{x}) = -\sum_{k,\bar{k}} \frac{2}{k^2 - \bar{k}^2}\, \tilde{f}(k, \bar{k})\, e^{ik\cdot x + i\bar{k}\cdot\bar{x}} \, , \quad \tilde{f}(k, \bar{k}) \equiv 0 \;\; \forall \;\; k^2 = \bar{k}^2 \, . \tag{5.51}
$$

Such operator is clearly not defined on $\ker\Delta$, and obeys

$$G\,\Delta = 1 - \mathcal{P}_\Delta\ , \quad \Delta\,G\,(1-\mathcal{P}_\Delta) = 1 - \mathcal{P}_\Delta\ . \tag{5.52}$$

Since $\Delta = [M_1, b^-]$, it is now straightforward to find the homotopy:

$$h := b^- G\,(1-\mathcal{P}_\Delta)\ , \tag{5.53}$$

which obeys the fundamental relation (5.49) and the side conditions (5.50), since b^- is nilpotent and commutes with $\mathcal{P}_\Delta$.

With the projector and homotopy map at our disposal, we now turn to transporting the $\mathrm{BV}_\infty^\Delta$ algebraic structure to $\overline{\mathcal{V}}$. We will thus proceed step by step in a constructive way. From now on, we will denote all transferred maps in $\overline{\mathcal{V}}$ with an overline, which should not be confused with the second copy $\bar{\mathcal{K}}$, since the underlying Yang-Mills maps will not play a role anymore. Here all inputs are intended as $\overline{\Psi}_i$, living in $\ker\Delta$. Since we do not display any input, we shall denote by $\mathcal{M}|_{\overline{\mathcal{V}}}$ the restriction of any $k-$linear map $\mathcal{M}$ (including operators) to act on elements of the subspace $\overline{\mathcal{V}} \subset \mathcal{V}$. First of all, since $[M_1, \mathcal{P}_\Delta] = 0$, the differential is unchanged:

$$\overline{M}_1 := M_1|_{\overline{\mathcal{V}}}\ . \tag{5.54}$$

We proceed with the transferred M_2, which is given, as usual, just by projection:

$$\overline{M}_2 := \mathcal{P}_\Delta M_2|_{\overline{\mathcal{V}}}\ , \quad \Delta\overline{M}_2 = 0\ . \tag{5.55}$$

Similarly, $\overline{B}_2$ is given by projection and retains its BV relation with $\overline{M}_2$:

$$\overline{B}_2 := \mathcal{P}_\Delta\, B_2|_{\overline{\mathcal{V}}} = \mathcal{P}_\Delta\,[b^-, M_2]|_{\overline{\mathcal{V}}} = [b^-, \mathcal{P}_\Delta\, M_2|_{\overline{\mathcal{V}}}] = [b^-, \overline{M}_2]\ . \tag{5.56}$$

The original failure of M_1 to be a derivation of B_2 is now cured for $\overline{B}_2$:

$$[\overline{M}_1, \overline{B}_2] = [M_1, [b^-, \overline{M}_2]] = [\Delta, \overline{M}_2] = 0\ , \tag{5.57}$$

where we used the fact that acting on elements of $\ker\Delta$ one has $[\Delta, \overline{M}_2] = \Delta\overline{M}_2 = 0$. We move on to the first trilinear map, which requires the homotopy operator. In order to find $\overline{M}_{3\mathrm{h}}$, we compute the associator of $\overline{M}_2$ (we leave the restriction $(\cdots)|_{\overline{\mathcal{V}}}$ implicit):

$$\begin{aligned} \overline{M}_2\overline{M}_2\,(1-\Pi) &= \mathcal{P}_\Delta M_2\,\mathcal{P}_\Delta M_2\,(1-\Pi) = \mathcal{P}_\Delta\Big(M_2M_2 - M_2[M_1,h]M_2\Big)\,(1-\Pi) \\ &= \mathcal{P}_\Delta\Big([M_1, M_{3\mathrm{h}}] - [M_1, M_2hM_2]\Big)\,(1-\Pi) \\ &= [\overline{M}_1, \overline{M}_{3\mathrm{h}}]\ , \end{aligned} \tag{5.58}$$

with the transferred three-product

$$\overline{M}_{3\mathrm{h}} = \mathcal{P}_\Delta\Big(M_{3\mathrm{h}} - M_2hM_2\,(1-\Pi)\Big)\Big|_{\overline{\mathcal{V}}}\ . \tag{5.59}$$

Notice that we used $[M_1, M_2] = 0$ to compute

$$[M_1, M_2 h M_2] = M_2[M_1, hM_2] = M_2[M_1, h]M_2 \, . \tag{5.60}$$

We now define the Poisson compatibility of the transferred $\overline{M}_2$ and $\overline{B}_2$ as the one in $\mathcal{V}$ by

$$\overline{B}_2\overline{M}_2 + \overline{M}_2\overline{B}_2\,(1 - 3\,\Pi) \equiv [b^-, \overline{M}_2\overline{M}_2] - 3\,\overline{M}_2\overline{B}_2\,\Pi \, . \tag{5.61}$$

Since now both $\overline{M}_2$ and $\overline{B}_2$ commute with M_1, one immediately has that the expression (5.61) is M_1-closed. Its hook part is also exact, since

$$\begin{aligned}\Big\{[b^-, \overline{M}_2\overline{M}_2] - 3\,\overline{M}_2\overline{B}_2\,\Pi\Big\}\,(1-\Pi) &= [b^-, \overline{M}_2\overline{M}_2\,(1-\Pi)] = [b^-, [M_1, \overline{M}_{3\mathrm{h}}]] \\ &= -[M_1, [b^-, \overline{M}_{3\mathrm{h}}]] + [\Delta, \overline{M}_{3\mathrm{h}}] \\ &= [M_1, \overline{\Theta}_{3\mathrm{h}}] \, ,\end{aligned} \tag{5.62}$$

for $\overline{\Theta}_{3\mathrm{h}} = -[b^-, \overline{M}_{3\mathrm{h}}]$. The $\Delta-$obstruction above vanishes, since $[\Delta, \overline{M}_{3\mathrm{h}}] = \Delta\overline{M}_{3\mathrm{h}} = 0$ when acting on inputs in $\overline{\mathcal{V}}$. Computing the symmetric part of (5.61) is, as usual, more complicated. We obtain

$$\begin{aligned}&\Big\{[b^-, \overline{M}_2\overline{M}_2] - 3\,\overline{M}_2\overline{B}_2\,\Pi\Big\}\,\Pi = \mathcal{P}_\Delta\,\Big([b^-, M_2\mathcal{P}_\Delta M_2] - 3\,M_2\mathcal{P}_\Delta B_2\Big)\,\Pi \\ &= \mathcal{P}_\Delta\,\Big([b^-, M_2M_2] - 3\,M_2B_2 - \big[b^-, M_2[M_1, h]M_2\big] + 3\,M_2[M_1, h]B_2\Big)\,\Pi \, .\end{aligned} \tag{5.63}$$

Now we can use

$$\begin{aligned}M_2[M_1, h]M_2 &= [M_1, M_2hM_2] \, , \\ [M_1, hB_2] &= [M_1, h]B_2 - h[M_1, B_2] = [M_1, h]B_2 - h[\Delta, M_2]\end{aligned} \tag{5.64}$$

to pull out some M_1-commutators:

$$\begin{aligned}(5.63) &= \mathcal{P}_\Delta\,\Big([b^-, M_2M_2] - 3\,M_2B_2 - [b^-, [M_1, M_2hM_2]] + 3\,[M_1, M_2hB_2] + 3\,M_2h[\Delta, M_2]\Big)\Pi \\ &= \mathcal{P}_\Delta\,\Big([b^-, M_2M_2] - 3\,M_2B_2 + \big[M_1, [b^-, M_2hM_2] + 3\,M_2hB_2\big] + 3\,M_2h[\Delta, M_2]\Big)\Pi \, ,\end{aligned} \tag{5.65}$$

where we used $\mathcal{P}_\Delta\,[\Delta, M_2hM_2]|_{\overline{\mathcal{V}}} = \mathcal{P}_\Delta\,\Delta M_2hM_2|_{\overline{\mathcal{V}}} = 0$. The last term above can be further manipulated as follows:

$$\begin{aligned}3\,\mathcal{P}_\Delta\,M_2h[\Delta, M_2]\Pi &= 3\,\mathcal{P}_\Delta\,M_2hM_2D_s\Pi = 3\,p\,M_2hM_2\big(\Pi + 1 - \Pi\big)D_s\Pi \\ &= \mathcal{P}_\Delta\,M_2hM_2D_\Delta\Pi + 3\,\mathcal{P}_\Delta\,M_2hM_2(1-\Pi)D_s\Pi \\ &= \mathcal{P}_\Delta\,[\Delta, M_2hM_2]\Pi + 3\,\mathcal{P}_\Delta\,M_2hM_2(1-\Pi)D_s\Pi \\ &= 3\,\mathcal{P}_\Delta\,M_2hM_2(1-\Pi)D_s\Pi \, ,\end{aligned} \tag{5.66}$$

where we used (5.38) and the vanishing of total $\Delta-$commutators under $\mathcal{P}_\Delta$. We can now use the homotopy Poisson relation (5.40) on $\mathcal{V}$ to rewrite (5.65) as

$$\begin{aligned}(5.65) &= \mathcal{P}_\Delta\Big([M_1,\Theta_{3\text{s}}] - 3\,M_{3\text{h}}D_s + \big[M_1,[b^-,M_2hM_2] + 3\,M_2hB_2\big] + 3\,M_2hM_2(1-\Pi)D_s\Big)\Pi \\ &= \Big[M_1,\mathcal{P}_\Delta\left(\Theta_{3\text{s}} + [b^-,M_2hM_2] + 3\,M_2hB_2\right)\Pi\Big] - 3\,\mathcal{P}_\Delta\left(M_{3\text{h}} - M_2hM_2\,(1-\Pi)\right)D_s\Pi \\ &= [\overline{M}_1,\overline{\Theta}_{3\text{s}}] - 3\,\overline{M}_{3\text{h}}D_s\Pi\;,\end{aligned} \tag{5.67}$$

with the transferred $\overline{\Theta}_3$ given by

$$\overline{\Theta}_3 = \mathcal{P}_\Delta\left(\Theta_3 + [b^-,M_2hM_2] + 3\,M_2hB_2\,\Pi\right)\Big|_{\overline{\mathcal{V}}}\,. \tag{5.68}$$

The full homotopy Poisson relation thus reads

$$[b^-,\overline{M}_2\overline{M}_2] - 3\,\overline{M}_2\overline{B}_2\,\Pi = [\overline{M}_1,\overline{\Theta}_3] - 3\,\overline{M}_{3\text{h}}D_s\Pi\;. \tag{5.69}$$

This is telling us that the BV_∞^Δ algebra is well-behaved under homotopy transfer, but this is not enough to completely remove the obstructions. The reason is that Δ is not zero as an operator on $\overline{\mathcal{V}}$. Rather, only total $\Delta-$commutators are transferred to zero. The main difference of (5.69) compared to the identity (5.40) is that both sides of (5.69) are M_1-closed. This is obvious for the left-hand side, since $\overline{M}_1$ commutes with both $\overline{M}_2$ and $\overline{B}_2$, while checking it for the right-hand side requires some computation:

$$\begin{aligned}\Big[\overline{M}_1,\overline{M}_{3\text{h}}D_s\Pi\Big] &= [\overline{M}_1,\overline{M}_{3\text{h}}]D_s\Pi \\ &= \overline{M}_2\overline{M}_2(1-\Pi)D_s\Pi \\ &= -\tfrac{1}{3}\,\overline{M}_2\overline{M}_2D_\Delta\Pi + \overline{M}_2\overline{M}_2D_s\Pi \\ &= -\tfrac{1}{3}\,[\Delta,\overline{M}_2\overline{M}_2]\Pi + \overline{M}_2[\Delta,\overline{M}_2]\Pi = 0\;.\end{aligned} \tag{5.70}$$

We thus see that the $\Delta-$obstruction in (5.69) is M_1-closed. If it were exact, one could shift $\overline{\Theta}_3$ to obtain a genuine homotopy Poisson identity. If this is not the case, on the other hand, $\overline{M}_{3\text{h}}D_s\Pi$ would be a genuine cohomological obstruction.

For the last step, let us determine the homotopy Jacobi relation. Due to $\overline{B}_2 = [b^-,\overline{M}_2]$, the Jacobiator is given by taking a b^-−commutator of the left-hand side of (5.69), leading to the Jacobi-like relation

$$3\,\overline{B}_2\overline{B}_2\Pi + [\overline{M}_1,\overline{B}_3] + 3\,\overline{\Theta}_3D_s\Pi = 0\;, \tag{5.71}$$

including the deformation. It is interesting to note that only the hook part of $\overline{\Theta}_3$ (and thus $\overline{M}_{3\text{h}}$) contributes to (5.71), since

$$\overline{\Theta}_{3\text{s}}D_s\Pi = \overline{\Theta}_3\Pi D_s\Pi = \tfrac{1}{3}\,\overline{\Theta}_3D_\Delta\Pi = 0\;. \tag{5.72}$$

The transferred three-bracket is given by $\overline{B}_3 = -[b^-,\overline{\Theta}_{3\text{s}}]$, as dictated by BV_∞^Δ. Upon using the expression (5.68), one finds that $\overline{B}_3$ is given by

$$\overline{B}_3 = \mathcal{P}_\Delta\left(B_3 - 3\,B_2hB_2\,\Pi\right)\Big|_{\overline{\mathcal{V}}}\,. \tag{5.73}$$

In the following we will study the obstruction (5.71) to the homotopy Jacobi identity upon restricting the inputs to $\ker b^-$. Only the L_∞ brackets $\overline{B}_n$ will restrict to $\mathcal{V}_{\rm DFT}$, and we will show that a well-defined albeit non-local deformation of $\overline{B}_3$ can be used to absorb the obstruction.

5.4.1 Dealing with the obstruction

We now turn to constructing a well-defined three-bracket that defines quartic weakly constrained DFT by dealing with the obstruction found above. From now on we will focus on the (obstructed) L_∞ sector on $\overline{\mathcal{V}}$, with brackets $\overline{B}_n$ given by

$$\overline{B}_1 = \overline{M}_1 \ , \quad \overline{B}_2 = [b^-, \overline{M}_2] \ , \quad \overline{B}_3 = -[b^-, \overline{\Theta}_{3{\rm s}}] \ , \tag{5.74}$$

obeying the following quadratic relations:

$$\begin{aligned} \overline{B}_1^2 &= 0 \ , \\ \big[\overline{B}_1, \overline{B}_2\big] &= 0 \ , \\ 3\,\overline{B}_2\overline{B}_2\,\Pi + [\overline{B}_1, \overline{B}_3] &= -3\,\overline{\Theta}_{3{\rm h}}\, D_s\, \Pi \ . \end{aligned} \tag{5.75}$$

In the interest of studying the obstruction of the homotopy Jacobi identity, we further restrict the space by considering the kernel of b^-, $\ker b^-$, which gives rise to the weakly constrained DFT space $\mathcal{V}_{\rm w}^{\rm DFT}$

$$\mathcal{V}_{\rm w}^{\rm DFT} = \Big\{\Psi(x,\bar{x}) \in \mathcal{V} |\ \Delta\Psi = 0\ ,\ b^-\Psi(x,\bar{x}) = 0\Big\}\ . \tag{5.76}$$

This vector space contains the correct spectrum of weakly constraint double field theory. In particular, this vector space has the same number and type of elements as the space of strongly constrained DFT $\mathcal{V}_{\rm s}^{\rm DFT}$. More precisely, the elements of $\mathcal{V}_{\rm w}^{\rm DFT}$ of all degree are

$$\begin{aligned} &\chi = \theta_+\bar{\theta}_+\,\chi \in V_{{\rm w}\,-2}^{\rm DFT}\ , \\ &\Lambda = \theta_+\bar{\theta}_{\bar{\mu}}\,\bar{\lambda}^{\bar{\mu}} - \theta_\mu\bar{\theta}_+\,\lambda^\mu - 2\,c^+\,\theta_+\bar{\theta}_+\,\eta \in V_{{\rm w}\,-1}^{\rm DFT}\ , \\ &\psi = \theta_\mu\bar{\theta}_{\bar{\nu}}\,e^{\mu\bar{\nu}} + 2\,\theta_+\bar{\theta}_-\,\bar{e} + 2\,\theta_-\bar{\theta}_+\,e + 2\,c^+\,\theta_+\bar{\theta}_{\bar{\mu}}\,\bar{f}^{\bar{\mu}} + 2\,c^+\,\theta_\mu\bar{\theta}_+\,f^\mu \in V_{{\rm w}\,0}^{\rm DFT}\ , \\ &\mathcal{F} = c^+\,\theta_\mu\bar{\theta}_{\bar{\nu}}\,F^{\mu\bar{\nu}} + c^+\,\theta_+\bar{\theta}_-\,\bar{F} + c^+\,\theta_-\bar{\theta}_+\,F + \theta_\mu\bar{\theta}_-\,F^\mu + \theta_-\bar{\theta}_{\bar{\mu}}\,\bar{F}^{\bar{\mu}} \in V_{{\rm w}\,1}^{\rm DFT}\ , \\ &\mathcal{N} = 2\,c^+\,\theta_-\bar{\theta}_{\bar{\mu}}\,\bar{N}^{\bar{\mu}} - 2\,c^+\,\theta_\mu\bar{\theta}_-\,N^\mu - \theta_-\bar{\theta}_-\,N \in V_{{\rm w}\,2}^{\rm DFT}\ , \\ &\mathcal{R} = -c^+\,\theta_-\bar{\theta}_-\,\mathcal{R} \in V_{{\rm w}\,3}^{\rm DFT}\ . \end{aligned} \tag{5.77}$$

All the elements above are to be thought of as being acted upon by the projector $\mathcal{P}_\Delta$. We can define the following differential and two-bracket on $\mathcal{V}_{\rm w}^{\rm DFT}$:

$$\begin{aligned} \mathcal{B}_1 &:= \overline{B}_1|_{\ker b^-}\ , \\ \mathcal{B}_2 &:= \overline{B}_2|_{\ker b^-} = \big[b^-, \overline{M}_2\big]|_{\ker b^-} = b^-\overline{M}_2|_{\ker b^-}\ . \end{aligned} \tag{5.78}$$

which are well-defined on $\ker b^-$ because they obey $b^-\mathcal{B}_i = 0$. While it is trivial to see this for $\mathcal{B}_2$ from (5.78) because ${b^-}^2 = 0$, for the differential it can be shown by computing

$$\begin{aligned} b^-\mathcal{B}_1 &= b^-\overline{B}_1|_{\ker b^-} = \big[b^-, \overline{B}_1\big]|_{\ker b^-} \\ &= \big[b^-, M_1\big]|_{\ker\Delta}|_{\ker b^-} = \Delta|_{\ker\Delta}|_{\ker b^-} = 0 \ . \end{aligned} \tag{5.79}$$

This confirms that $\mathcal{B}_1 : \mathcal{V}_{\rm w}^{\rm DFT} \to \mathcal{V}_{\rm w}^{\rm DFT}$ and $\mathcal{B}_2 : \mathcal{V}_{\rm w}^{\rm DFT} \times \mathcal{V}_{\rm w}^{\rm DFT} \to \mathcal{V}_{\rm w}^{\rm DFT}$ restrict correctly and obey nilpotency and the Leibniz relation, thus defining a consistent double field theory to cubic order.

We now move on to the Jacobi identity of $\mathcal{B}_2$, which is given by restriction of the corresponding relation (5.75):

$$3\,\mathcal{B}_2\mathcal{B}_2\,\Pi + \big[\mathcal{B}_1, \overline{B}_3|_{\ker b^-}\big] = \mathcal{O} \ , \quad \mathcal{O} := -3\,\overline{\Theta}_{3\rm h}\,D_s\,\Pi|_{\ker b^-} \ , \tag{5.80}$$

where we denoted the obstruction by $\mathcal{O}$. Taking into account the restriction to $\ker b^-$, one can express the obstruction as

$$\begin{aligned} \mathcal{O} &= -3\,\overline{\Theta}_{3\rm h}\,D_s\,\Pi|_{\ker b^-} = 3\,[b^-, \overline{M}_{3\rm h}]\,D_s\,\Pi|_{\ker b^-} \\ &= 3\,b^-\overline{M}_{3\rm h}\,D_s\,\Pi|_{\ker b^-} \ , \end{aligned} \tag{5.81}$$

where we used that $[b^-, \mathcal{T}]|_{\ker b^-} = b^-\mathcal{T}|_{\ker b^-}$. This expression can be further manipulated by using the definition (5.59) of the transferred $\overline{M}_{3\rm h}$:

$$\begin{aligned} \mathcal{O} &= 3\,b^-\overline{M}_{3\rm h}\,D_s\,\Pi|_{\ker b^-} \\ &= 3\,\mathcal{P}_\Delta\,b^-\Big(M_{3\rm h} - M_2 h M_2\,(1-\Pi))\Big)\,D_s\,\Pi\Big|_{\ker b^- \cap \ker\Delta} \\ &= 3\,\mathcal{P}_\Delta\,b^-\Big(M_{3\rm h}\,D_s\,\Pi - M_2\Delta h M_2\,\Pi\Big)\Big|_{\ker b^- \cap \ker\Delta} \ , \end{aligned} \tag{5.82}$$

where we used $M_2\,hM_2\,D_s|_{\ker\Delta} = M_2\,[\Delta, hM_2]|_{\ker\Delta} = M_2\,\Delta h\,M_2|_{\ker\Delta}$. The homotopy (5.53) obeys $\Delta h = b^-(1-\mathcal{P}_\Delta)$ and, for inputs in $\ker b^-$, we can also write

$$b^-M_2\,b^-M_2|_{\ker b^-} = [b^-, M_2]\,b^-M_2|_{\ker b^-} = B_2\,b^-M_2|_{\ker b^-} = B_2 B_2|_{\ker b^-} \ , \tag{5.83}$$

yielding a simpler expression for the obstruction:

$$\mathcal{O} = 3\,\mathcal{P}_\Delta\,\Big(b^-M_{3\rm h}\,D_s - B_2\,(1-\mathcal{P}_\Delta)\,B_2\Big)\,\Pi\Big|_{\ker b^- \cap \ker\Delta} \ , \tag{5.84}$$

which we will use to determine whether it can be removed. First of all, the obstruction is closed: $[\mathcal{B}_1, \mathcal{O}] = 0$, as it can be seen by taking a $\mathcal{B}_1-$commutator of (5.80). However, $\mathcal{O}$ is given by projection $\mathcal{P}_\Delta$ of an otherwise not closed quantity:

$$\mathcal{O} = \mathcal{P}_\Delta\,\widetilde{\mathcal{O}} \ , \quad \big[\mathcal{B}_1, \widetilde{\mathcal{O}}\big] = \Delta\,\mathcal{W} \quad \Rightarrow \quad \big[\mathcal{B}_1, \mathcal{O}\big] = 0 \ , \tag{5.85}$$

with explicit $\widetilde{\mathcal{O}}$ and $\mathcal{W}$ given by

$$\begin{aligned}\widetilde{\mathcal{O}} &= 3\Big(b^- M_{3\text{h}}\, D_s - B_2\,(1-\mathcal{P}_\Delta)\, B_2\Big)\Pi\Big|_{\text{ker}b^-\cap\,\text{ker}\Delta}\ ,\\ \mathcal{W} &= \Big(3\, M_{3\text{h}}\, D_s + b^- M_2 M_2 - 3\, M_2\,(1-\mathcal{P}_\Delta)\, B_2\Big)\,\Pi\Big|_{\text{ker}b^-\cap\,\text{ker}\Delta}\ .\end{aligned} \tag{5.86}$$

Since $\widetilde{\mathcal{O}}$ is not closed, it certainly cannot be exact. It is thus hard to expect that one can extract a $\mathcal{B}_1-$commutator from $\mathcal{O}$ in a simple way.

In order to prove that $\mathcal{O}$ is, in fact, exact, we shall consider the "dynamical" Laplacian

$$\Delta_+ := \frac{1}{2}\left(\partial^\mu\partial_\mu + \bar\partial^{\bar\mu}\bar\partial_{\bar\mu}\right)\,,\quad \big[\mathcal{B}_1, b^+\big] = \Delta_+\,, \tag{5.87}$$

which acts on the Fourier expansion (5.4) as

$$\Delta_+\, f(x,\bar x) = -\frac{1}{2}\sum_{k,\bar k}(k^2+\bar k^2)\,\tilde f(k,\bar k)\, e^{ik\cdot x + i\bar k\cdot\bar x}\ . \tag{5.88}$$

Since the metric in both k^2 and $\bar k^2$ is Euclidean, Δ_+ is almost invertible. The only solution to $\Delta_+ f(x,\bar x) = 0$ is the doubled zero mode $\tilde f(0,0)$, which is allowed on the doubled torus due to its nontrivial topology. We can thus associate a zero mode projector $\mathcal{P}_0$ to $\text{ker}\Delta_+$, with a corresponding homotopy

$$h_0 := b^+ \frac{1}{\Delta_+}\,(1-\mathcal{P}_0)\,,\quad \big[\mathcal{B}_1, h_0\big] = 1-\mathcal{P}_0\ . \tag{5.89}$$

At this stage, it is important to notice that nonlinear combinations of fields of the form

$$\Big((1-\mathcal{P}_\Delta) f\,\mathcal{P}_\Delta g\Big)(x,\bar x)\sum_{k^2\neq\bar k^2\ ,\ l^2=\bar l^2}\tilde f(k,\bar k)\,\tilde g(l,\bar l)\, e^{i(k+l)\cdot x + i(\bar k+\bar l)\cdot\bar x}\ , \tag{5.90}$$

do not contain zero modes. This is easily seen from the fact that in the sum above $(k^\mu,\bar k^{\bar\mu})\neq -(l^\mu,\bar l^{\bar\mu})$, given that $k^2\neq\bar k^2$, while $l^2=\bar l^2$. The total momentum above is thus $(k^\mu+l^\mu,\bar k^{\bar\mu}+\bar l^{\bar\mu})\neq(0,0)$. On such combinations one has $(1-\mathcal{P}_0)=1$, yielding

$$(1-\mathcal{P}_\Delta) f\,\mathcal{P}_\Delta g = \big[\mathcal{B}_1, h_0\big]\Big((1-\mathcal{P}_\Delta) f\,\mathcal{P}_\Delta g\Big)\ . \tag{5.91}$$

In order to see that we can apply this argument to our obstruction, let us act with $\mathcal{O}$ (5.84) on three arbitrary inputs (Ψ_1,Ψ_2,Ψ_3) in $\mathcal{V}_{\text{DFT}}$:

$$\begin{aligned}\mathcal{O}(\Psi_1,\Psi_2,\Psi_3) &\overset{(123)}{=} 3\,\mathcal{P}_\Delta\, b^-\Big(M_{3\text{h}}\big(\partial^\mu\Psi_1,\partial_\mu\Psi_2,\Psi_3\big) - M_{3\text{h}}\big(\bar\partial^{\bar\mu}\Psi_1,\bar\partial_{\bar\mu}\Psi_2,\Psi_3\big)\Big)\\ &\quad - 3\,\mathcal{P}_\Delta\, B_2\big((1-\mathcal{P}_\Delta)B_2(\Psi_1,\Psi_2),\Psi_3\big)\,,\end{aligned} \tag{5.92}$$

where by (123) we denote graded symmetrization in the labels. The B_2B_2 term above has momenta of the form (5.90), given the explicit projector $(1-\mathcal{P}_\Delta)$ and that $\mathcal{P}_\Delta\Psi_i=\Psi_i$. The

$M_{3\text{h}}$ term falls in the same category, since $\mathcal{P}_\Delta$ only acts on input functions, and one has

$$\begin{aligned}\mu\Big[D_s(F_1\otimes F_2\otimes F_3)\Big] &= \mu\Big[(\partial^\mu F_1\otimes\partial_\mu F_2\otimes F_3)-(\bar\partial^{\bar\mu}F_1\otimes\bar\partial_{\bar\mu}F_2\otimes F_3)\Big]\\ &= \partial^\mu F_1\,\partial_\mu F_2\,F_3-\bar\partial^{\bar\mu}F_1\,\bar\partial_{\bar\mu}F_2\,F_3=\Delta(F_1F_2)\,F_3\\ &= (1-\mathcal{P}_\Delta)\big[\Delta(F_1F_2)\big]\,\mathcal{P}_\Delta F_3\ ,\end{aligned} \tag{5.93}$$

for weakly constrained functions $F_i(x,\bar x)$ obeying $\Delta F_i=0$.

Having shown that Δ_+ is invertible on $\mathcal{O}$, we can prove that it is exact:

$$\begin{aligned}\mathcal{O} &= \Big[\mathcal{B}_1,\frac{b^+}{\Delta_+}\Big]\mathcal{O}=\Big[\mathcal{B}_1,\frac{b^+}{\Delta_+}\mathcal{O}\Big]+\frac{b^+}{\Delta_+}\Big[\mathcal{B}_1,\mathcal{O}\Big]\\ &= \Big[\mathcal{B}_1,\frac{b^+}{\Delta_+}\mathcal{O}\Big]\ ,\end{aligned} \tag{5.94}$$

where we used $[\mathcal{B}_1,\mathcal{O}]=0$.

Since we have shown that the obstruction is exact, we can shift the original $\overline{B}_3|_{\text{ker}b^-}$ appearing in (5.80) by $\frac{b^+}{\Delta_+}\mathcal{O}$ and obtain a genuine L_∞ relation on $\mathcal{V}_{\text{DFT}}$:

$$3\,\mathcal{B}_2\mathcal{B}_2\,\Pi+[\mathcal{B}_1,\mathcal{B}_3]=0\ , \tag{5.95}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are given by (5.78), and the final three-bracket reads

$$\mathcal{B}_3=\mathcal{P}_\Delta\Big(B_3-3\,\frac{b^+}{\Delta_+}\big(b^-M_{3\text{h}}\,D_s-B_2\,(1-\mathcal{P}_\Delta)\,B_2\big)\,\Pi\Big)\Big|_{\text{ker}b^-\cap\,\text{ker}\Delta}\ . \tag{5.96}$$

Notice that the standard homotopy part B_2hB_2 in the definition (5.73) of the transported $\overline{B}_3$ drops on kerb^-, due to $h\propto b^-$ and $B_2\propto b^-$. We have thus succeeded in constructing the three-bracket of weakly constrained DFT on a purely spatial torus. Since the whole construction is fairly abstract and intricate, in the next section we will provide an explicit check of the above results.

5.5 Gauge algebra

As a concrete example, we now turn to computing the gauge algebra of weakly constrained DFT to trilinear order with L_∞ bracket $\mathcal{B}_3$ that we found in the previous section. We will only focus on a subsector of the gauge algebra in order to keep things simple. The gauge algebra is encoded in the homotopy Jacobi relation

$$\text{Jac}(\Lambda_1,\Lambda_2,\Lambda_2)+[\mathcal{B}_1,\mathcal{B}_3](\Lambda_1,\Lambda_2,\Lambda_3)=0\ , \tag{5.97}$$

with the Jacobiator $\text{Jac}(\Lambda_1,\Lambda_2,\Lambda_3)$ defined as

$$\text{Jac}(\Lambda_1,\Lambda_2,\Lambda_3):=3\,\mathcal{B}_2(\mathcal{B}_2(\Lambda_{[1},\Lambda_2),\Lambda_{3]})\overset{[123]}{=}3\,\mathcal{B}_2(\mathcal{B}_2(\Lambda_1,\Lambda_2),\Lambda_3)\ . \tag{5.98}$$

The input labels inside of the square brackets [123] on top of the the last equal sign denote antisymmetrization of the labels, and in the above equation and in the remainder of the paper we use the convention

$$3\,\mathcal{B}_2(\mathcal{B}_2(\Lambda_{[1},\Lambda_2),\Lambda_{3]}) = \mathcal{B}_2(\mathcal{B}_2(\Lambda_1,\Lambda_2),\Lambda_3) + \mathcal{B}_2(\mathcal{B}_2(\Lambda_2,\Lambda_3),\Lambda_1) + \mathcal{B}_2(\mathcal{B}_2(\Lambda_3,\Lambda_1),\Lambda_2)\ , \quad (5.99)$$

where we used the antisymmetry of $\mathcal{B}_2$ when acting on gauge parameters: $\mathcal{B}_2(\Lambda_1,\Lambda_2) = -\mathcal{B}_2(\Lambda_2,\Lambda_1)$.

In order to check the identity (5.97), it will be convenient to rewrite the individual terms of the Jacobiator in a different but equivalent way. One can rewrite the inner projector of the nested brackets in the Jacobiator as $\mathcal{P}_\Delta = 1-(1-\mathcal{P}_\Delta)$ while keeping the external projector untouched. Doing so yields

$$\begin{aligned}\mathrm{Jac}(\Lambda_1,\Lambda_2,\Lambda_3) &\overset{[123]}{=} 3\,\mathcal{P}_\Delta\, B_2(B_2(\Lambda_1,\Lambda_2),\Lambda_3) - 3\,\mathcal{P}_\Delta\, B_2(\,(1-\mathcal{P}_\Delta)\,B_2(\Lambda_1,\Lambda_2),\Lambda_3)\\ &\overset{[123]}{=} 3\,\mathcal{P}_\Delta\, B_2(B_2(\Lambda_1,\Lambda_2),\Lambda_3) - 3\,\mathcal{P}_\Delta\, B_2\big(\big\{B_2(\Lambda_1,\Lambda_2)\big\}_\perp,\Lambda_3\big)\ ,\end{aligned} \quad (5.100)$$

where here and in what follows it is understood that all the maps are acting on $\ker b^- \cap \ker\Delta$ and in order to simplify our notation we introduced the perpendicular projector $\big\{B_2\big\}_\perp$ in the second term in the last line which denotes $(1-\mathcal{P}_\Delta)\,B_2$. The above split will be useful once we compute the part of the homotopy Jacobi relation that contains the three-bracket $\mathcal{B}_3$ because it contains a term with $\mathcal{P}_\Delta\, B_2(1-\mathcal{P}_\Delta)B_2$.

Recall that a generic gauge parameter in double field theory has three components: two vector components and one scalar component. However, in order to simplify the computation we will restrict our attention to vanishing $\bar\lambda^{\bar\nu}$ and η while only keeping λ^μ. For this reason from now on we consider the consistent subsector of the gauge algebra with parameters of the form (see equation (5.77))

$$\Lambda = -\theta_\mu\bar\theta_+\,\lambda^\mu + \theta_+\bar\theta_{\bar\nu}\,0 + c^+\,\theta_+\bar\theta_+\,0\ . \quad (5.101)$$

The homotopy Jacobi relation (5.97) takes values in the space of gauge parameters, and hence consists of three components. For this reason we will check the gauge algebra explicitly displaying the basis elements of the DFT space $Z_A\bar Z_{\bar B}$. This will allow us to keep track of the different components of the relation.

We now turn to finding the two-bracket between gauge parameters using the technology developed in the previous chapter. For gauge parameters defined as in (5.101), we have

$$
\begin{aligned}
\mathcal{B}_2(\Lambda_1,\Lambda_2) &= \mathcal{P}_\Delta\, b^-\, m_2\otimes \bar{m}_2\big(\theta_\mu\bar\theta_+\,\lambda_1^\mu,\theta_\nu\bar\theta_+\,\lambda_2^\nu\big)\\
&= \mathcal{P}_\Delta\, b^-\,\mu\Big[\hat m_2(\theta_\mu,\theta_\nu)\,\hat{\bar m}_2(\theta_+,\theta_+)\,(\lambda_1^\mu\otimes\lambda_2^\nu)\Big]\\
&= \mathcal{P}_\Delta\, b^-\,\mu\Bigg\{\Big[c\,\theta_\nu\Big[(\partial_\mu\otimes\mathbb{1})+2\,(\mathbb{1}\otimes\partial_\mu)\Big]-c\,\theta_\mu\Big[(\mathbb{1}\otimes\partial_\nu)+2\,(\partial_\nu\otimes\mathbb{1})\Big]\\
&\quad+c\,\theta_\rho\,\eta_{\mu\nu}\Big[(\partial^\rho\otimes\mathbb{1})-(\mathbb{1}\otimes\partial^\rho)\Big]\Big]\,\bar\theta_+\,(\mathbb{1}\otimes\mathbb{1})\,]\big(\lambda_1^\mu\otimes\lambda_2^\nu\big)\Bigg\}\\
&= \mathcal{P}_\Delta\, b^- c\,\theta_\rho\,\bar\theta_+\Big(\partial\cdot\lambda_1\,\lambda_2^\rho+2\,\lambda_1\cdot\partial\lambda_2^\rho+\partial^\rho\lambda_1\cdot\lambda_2-(1\leftrightarrow 2)\Big)\\
&= \tfrac{1}{2}\,\mathcal{P}_\Delta\,\theta_\rho\,\bar\theta_+\Big(\partial\cdot\lambda_1\,\lambda_2^\rho+2\,\lambda_1\cdot\partial\lambda_2^\rho+\partial^\rho\lambda_1\cdot\lambda_2-(1\leftrightarrow 2)\Big)\\
&\equiv \tfrac{1}{2}\,\mathcal{P}_\Delta\,\theta_\rho\bar\theta_+\,\big(\lambda_1\bullet\lambda_2\big)^\rho\ \in\mathcal{V}_{-1}\,,
\end{aligned}
\tag{5.102}
$$

and we used the component form of $\hat m_2(\theta_\mu,\theta_\nu)$ and $\hat{\bar m}_2(\bar\theta_+,\bar\theta_+)$, which can be found in the appendix in equation (A.4). Using the above expression for the two-bracket B_2 we obtain the following Jacobiator:

$$
\begin{aligned}
\mathrm{Jac}(\Lambda_1,\Lambda_2,\Lambda_3) \overset{[123]}{=} &-\tfrac{3}{2}\,\mathcal{P}_\Delta\,\theta_\mu\bar\theta_+\,\Big[\partial^\mu(\lambda_{1\,\rho}\,\partial^\rho\lambda_{2\,\nu}\,\lambda_3^\nu)+2\,\partial_\rho\lambda_{1\,\nu}\,\lambda_2^\nu\,\partial^\rho\lambda_3^\mu+\Delta_+\lambda_{1\,\rho}\,\lambda_2^\rho\,\lambda_3^\mu\\
&\qquad+2\,\partial_\rho\lambda_1^\rho\,\lambda_2^\nu\,\partial_\nu\lambda_3^\mu+\partial_\rho\lambda_1^\rho\,\partial^\mu\lambda_{2\,\nu}\,\lambda_3^\nu+\lambda_2^\nu\,\partial_\nu\partial_\rho\lambda_1^\rho\,\lambda_3^\mu\Big]\\
&+\tfrac{3}{4}\,\mathcal{P}_\Delta\,\theta_\mu\bar\theta_+\,\Big[\big\{\lambda_1\bullet\lambda_2\big\}_\perp\bullet\lambda_3\Big]^\mu\,,
\end{aligned}
\tag{5.103}
$$

where we use $\mathcal{P}_\Delta\,\Box=\mathcal{P}_\Delta\,\Delta_+$.

Having the Jacobiator (5.103) at our disposal, in order to verify the homotopy Jacobi relation (5.97) we need the following components of $\mathcal{B}_3$: first, $\mathcal{B}_3$ on three gauge parameters, whose only non-trivial part can be found with the following computation:

$$
\begin{aligned}
\mathcal{B}_3(\Lambda_1,\Lambda_2,\Lambda_3) &= \mathcal{P}_\Delta\,B_3(\Lambda_1,\Lambda_2,\Lambda_2)\\
&= -\tfrac{1}{2}\,\mathcal{P}_\Delta\,b^-\,\theta_{3\mathrm{s}}\otimes\bar m_2\bar m_2\,\Pi(\Lambda_1,\Lambda_2,\Lambda_3)\\
&\overset{[123]}{=} \tfrac{1}{2}\,\mathcal{P}_\Delta\,b^-\,\theta_{3\mathrm{s}}\otimes\bar m_2\bar m_2\,(\theta_\mu\bar\theta_+\,\lambda_1^\mu,\theta_\nu\bar\theta_+\,\lambda_2^\nu,\theta_\rho\bar\theta_+\,\lambda_3^\rho)\\
&\overset{[123]}{=} \tfrac{1}{2}\,\mathcal{P}_\Delta\,b^-\,\mu\Big[\hat\theta_{3\mathrm{s}}(\theta_\mu,\theta_\nu,\theta_\rho)\,\bar\theta_+\,\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big)\,\big(\lambda_1^\mu\otimes\lambda_2^\nu\otimes\lambda_3^\rho\big)\Big]\\
&\overset{[123]}{=} \tfrac{1}{2}\,\mathcal{P}_\Delta\,b^-\,\mu\Bigg\{c\,\theta_+\Big[\eta_{\mu\nu}\,\big(\partial_\rho\otimes\mathbb{1}\otimes\mathbb{1}\big)-\eta_{\mu\nu}\,\big(\mathbb{1}\otimes\partial_\rho\otimes\mathbb{1}\big)+\eta_{\nu\rho}\,\big(\mathbb{1}\otimes\partial_\mu\otimes\mathbb{1}\big)\\
&\quad-\eta_{\nu\rho}\,\big(\mathbb{1}\otimes\mathbb{1}\otimes\partial_\mu\big)+\eta_{\mu\rho}\,\big(\mathbb{1}\otimes\mathbb{1}\otimes\partial_\nu\big)-\eta_{\mu\rho}\,\big(\partial_\nu\otimes\mathbb{1}\otimes\mathbb{1}\big)\Big]\,\bar\theta_+\,\big(\lambda_1^\mu\otimes\lambda_2^\nu\otimes\lambda_3^\rho\big)\Bigg\}\\
&\overset{[123]}{=} 3\,\mathcal{P}_\Delta\,b^-\,c\,\theta_+\bar\theta_+\,\big\{\lambda_{1\,\rho}\,\partial^\rho\lambda_{2\,\nu}\,\lambda_3^\nu\big\}\\
&\overset{[123]}{=} \tfrac{3}{2}\,\mathcal{P}_\Delta\,\theta_+\bar\theta_+\,\big\{\lambda_{1\,\rho}\,\partial^\rho\lambda_{2\,\nu}\,\lambda_3^\nu\big\}\in\mathcal{V}_{-2}\,,
\end{aligned}
\tag{5.104}
$$

where we used the component form $\hat\theta_{3\mathrm{s}}(\theta_\mu,\theta_\nu,\theta_\rho)$ shown in equation in appendix A.

Second, we need to find $\mathcal{B}_3(\Lambda_1,\Lambda_2,\Psi)$. From a computation analogous to the above, we find

$$
\begin{aligned}
\mathcal{B}_3(\Lambda_1,\Lambda_2,\Psi) \overset{[12]}{=} & -\tfrac{1}{2}\,\mathcal{P}_\Delta\,\theta_\mu\bar\theta_+\Big[2\,f_\rho\,\lambda_1^\rho\,\lambda_2^\mu + 4e\,\lambda_1^\nu\,\partial_\nu\lambda_2^\mu + 2\lambda_1^\nu\,\partial_\nu e\,\lambda_2^\mu + 2e\,\partial^\mu\lambda_{1\,\nu}\lambda_2^\nu \\
& -\;e^{\mu\bar\nu}\,\bar\partial_{\bar\nu}\lambda_{1\,\rho}\,\lambda_2^\rho + \bar\partial_{\bar\nu}\lambda_1^\mu\,e^{\rho\bar\nu}\,\lambda_{2\,\rho} - \big(2\,f^\rho + \bar\partial_{\bar\nu}e^{\rho\bar\nu}\big)\,\lambda_{1\,\rho}\,\lambda_2^\mu\Big] \\
& + \mathcal{P}_\Delta\,\tfrac{1}{\Delta_+}\,\theta_\mu\bar\theta_+\,\bar\partial^{\bar\nu}\Big[\partial_\rho\lambda_1^\mu\,\partial^\rho\lambda_2^\nu\,e_{\nu\bar\nu} + \lambda_1^\nu\,\partial_\rho\lambda_2^\mu\,\partial^\rho e_{\nu\bar\nu} + \partial_\rho e^\mu{}_{\bar\nu}\,\partial^\rho\lambda_{1\,\nu}\,\lambda_2^\nu \\
& - \bar\partial_{\bar\rho}\lambda_1^\mu\,\bar\partial^{\bar\rho}\lambda_2^\nu\,e_{\nu\bar\nu} - \lambda_1^\nu\,\bar\partial_{\bar\rho}\lambda_2^\mu\,\bar\partial^{\bar\rho}e_{\nu\bar\nu} - \bar\partial_{\bar\rho}e^\mu{}_{\bar\nu}\,\bar\partial^{\bar\rho}\lambda_{1\,\nu}\,\lambda_2^\nu\Big] \\
& - \tfrac{1}{4}\,\mathcal{P}_\Delta\,\tfrac{1}{\Delta_+}\,\theta_\mu\bar\theta_+\,\bar\partial^{\bar\nu}\Big[\big\{\lambda_1\bullet\lambda_2\big\}_\perp\bullet e_{\bar\nu} + 2\,\lambda_2\bullet\big\{\lambda_1\bullet e_{\bar\nu}\big\}_\perp\Big]^\mu \\
& - \tfrac{1}{2}\,\mathcal{P}_\Delta\,\theta_+\bar\theta_{\bar\nu}\Big[\lambda_1^\rho\,\partial_\rho\lambda_{2\,\nu}\,e^{\nu\bar\nu} + \lambda_2^\rho\,\partial_\rho e^{\nu\bar\nu}\,\lambda_{1\,\nu} + e^{\nu\bar\nu}\,\partial_\nu\lambda_{1\,\rho}\,\lambda_2^\rho\Big] \\
& - \tfrac{1}{2}\,\mathcal{P}_\Delta\,c^+\theta_+\bar\theta_+\,\bar\partial_{\bar\nu}\Big[\lambda_1^\rho\,\partial_\rho\lambda_{2\,\nu}\,e^{\nu\bar\nu} + \lambda_2^\rho\,\partial_\rho e^{\nu\bar\nu}\,\lambda_{1\,\nu} + e^{\nu\bar\nu}\,\partial_\nu\lambda_{1\,\rho}\,\lambda_2^\rho\Big] \;\in \mathcal{V}_{-1}\;.
\end{aligned}
\tag{5.105}
$$

From this expression one infers by inspection of the third to fifth line that the non-locality inherent in $\frac{1}{\Delta_+}$ is unavoidable: there is no overall Δ_+ that can be factored out to cancel it, as $\bar\partial^{\bar\nu}$ is contracted with $e_{\mu\bar\nu}$ and not with a derivative. This changes after replacing the field in (5.105) by $\mathcal{B}_1(\Lambda)$, which is the next step in order to verify the homotopy Jacobi relation. For instance, in the last line in equation (5.105) one obtains

$$
\begin{aligned}
&\mathcal{P}_\Delta\,\tfrac{1}{\Delta_+}\bar\partial^{\bar\nu}\big\{\big\{\lambda_{[1}\bullet\lambda_2\big\}_\perp\bullet\bar\partial_{\bar\nu}\lambda_{3]} + 2\,\lambda_{[2}\bullet\big\{\lambda_1\bullet\bar\partial_{\bar\nu}\lambda_{3]}\big\}_\perp\big\}^\mu \\
&= \;\mathcal{P}_\Delta\,\tfrac{1}{\Delta_+}\,\bar\partial^{\bar\nu}\bar\partial_{\bar\nu}\big\{\big\{\lambda_{[1}\bullet\lambda_2\big\}_\perp\bullet\lambda_{3]}\big\}^\mu\;,
\end{aligned}
\tag{5.106}
$$

where the equality follows using the Leibniz rule and the antisymmetry of the labels. Under the projector $\mathcal{P}_\Delta$ we can then use the weak constraint $\bar\partial_{\bar\nu}\bar\partial^{\bar\nu}\equiv\bar\Box=\Box$ together with $\mathcal{P}_\Delta\Box=\mathcal{P}_\Delta\Delta_+$ to cancel $\frac{1}{\Delta_+}$. Doing so for the other terms and appropriate antisymmetrizations of the inputs leads to

$$
\begin{aligned}
3\,\mathcal{B}_3(\Lambda_{[1},\Lambda_2,\mathcal{B}_1(\Lambda_{3]})) \overset{[123]}{=} & \tfrac{3}{2}\,\mathcal{P}_\Delta\,\theta_\mu\bar\theta_+\Big\{\Delta_+\lambda_{3\,\rho}\,\lambda_1^\rho\,\lambda_2^\mu + 2\,\partial_\rho\lambda_3^\rho\,\lambda_1^\nu\,\partial_\nu\lambda_2^\mu + \lambda_1^\nu\,\partial_\nu\partial_\rho\lambda_3^\rho\,\lambda_2^\mu \\
& +\partial_\rho\lambda_3^\rho\,\partial^\mu\lambda_{1\,\nu}\,\lambda_2^\nu + 2\,\partial_\rho\lambda_1^\nu\,\lambda_{2\,\nu}\,\partial^\rho\lambda_3^\mu\Big\} \\
& -\tfrac{3}{4}\,\mathcal{P}_\Delta\,\theta_\mu\bar\theta_+\,\big\{\big\{\lambda_1\bullet\lambda_2\big\}_\perp\bullet\lambda_3\big\}^\mu \\
& -\tfrac{3}{2}\,\mathcal{P}_\Delta\,\theta_+\bar\theta_{\bar\nu}\,\bar\partial^{\bar\nu}\,\Big\{\lambda_{1\,\rho}\,\partial^\rho\lambda_{2\,\nu}\,\lambda_3^\nu\Big\} \\
& -\tfrac{3}{2}\,\mathcal{P}_\Delta\,c^+\,\theta_+\bar\theta_+\,\Delta_+\Big\{\lambda_{1\,\rho}\,\partial^\rho\lambda_{2\,\nu}\,\lambda_3^\nu\Big\}\;,
\end{aligned}
\tag{5.107}
$$

which has no non-localities. Next, we act with the differential on $\mathcal{B}_3(\Lambda_1,\Lambda_2,\Lambda_3)$, which yields

$$
\begin{aligned}
\mathcal{B}_1\mathcal{B}_3(\Lambda_1,\Lambda_2,\Lambda_3) \overset{[123]}{=} & \tfrac{3}{2}\,\mathcal{P}_\Delta\,\theta_\mu\bar\theta_+\,\partial^\mu\big(\lambda_{1\,\rho}\,\partial^\rho\lambda_{2\,\nu}\,\lambda_3^\nu\big) \\
& +\tfrac{3}{2}\,\mathcal{P}_\Delta\,\theta_+\bar\theta_{\bar\nu}\,\bar\partial^{\bar\nu}\big(\lambda_{1\,\rho}\,\partial^\rho\lambda_{2\,\nu}\,\lambda_3^\nu\big) \\
& +\tfrac{3}{2}\,\mathcal{P}_\Delta\,c^+\theta_+\bar\theta_+\,\Delta_+\big(\lambda_{1\,\rho}\,\partial^\rho\lambda_{2\,\nu}\,\lambda_3^\nu\big)\;.
\end{aligned}
\tag{5.108}
$$

Finally, adding up (5.103), (5.108) and (5.107) one verifies the homotopy Jacobi relation (5.97).

Chapter 6

Summary, discussion and outlook

In this thesis, we developed a gauge independent, off-shell and local approach to the double copy to quartic order in interactions. We used our double copy prescription to construct gravity in the form of strongly constrained double field theory, as well as weakly constrained double field theory. Our approach is based on the L_∞ description of perturbative field theories. In this algebraic framework the L_∞-algebra of Yang-Mills theory is the tensor product $\mathcal{K} \otimes \mathfrak{g}$, where $\mathcal{K}$ is a kinematic $\mathrm{BV}_\infty^\square$-algebra and $\mathfrak{g}$ is a color Lie algebra. In complete analogy to the BCJ double copy, our double copy prescription consists of exchanging the color information encoded in $\mathfrak{g}$ by another copy of kinematic information encoded in a second copy $\bar{\mathcal{K}}$ of the kinematic algebra. This prescription leads to the tensor product $\mathcal{K} \otimes \bar{\mathcal{K}}$ which, after imposing appropriate constraints, gives rise to the L_∞ description of either strongly constrained or weakly constrained double field theory, depending on the constraints imposed on the tensor product.

For the strongly constrained theory, we constructed the abstract L_∞ maps B_1, B_2 and B_3 which determine the off-shell and gauge invariant theory to quartic order. During the construction of the abstract off-shell brackets in terms of the single copy maps, we learned that the consistency of the theory to quartic order, encoded in the L_∞ relations, relies on the relations that define the kinematic $\mathrm{BV}_\infty^\square$-algebra of Yang-Mills. In order to perform explicit checks of our prescription, we built the DFT action explicitly to cubic order using B_1 and B_2. For the quartic interactions, on the other hand, we computed double field theory four-point scattering amplitudes. We found that the celebrated "kinematic Jacobi identity" of scattering amplitudes is, in fact, the on-shell version of the deformed Poisson compatibility condition of the kinematic $\mathrm{BV}_\infty^\square$-algebra.

In order to construct the weakly constrained double field theory we had to deal with issues arising from the projector $\mathcal{P}_\Delta$. This projector imposes the weak constraint (or level-matching condition) on fields and gauge parameters and their products. We dealt with these issues using homotopy transfer. First, we transferred a $\mathrm{BV}_\infty^\Delta$ structure on the tensor product $\mathcal{K} \otimes \bar{\mathcal{K}}$ to the subspace of level-matched elements, where we found an obstructed L_∞-algebra. Then, we further restricted our space with the b^- constraint, where we could get rid of the obstruction by the inclusion of a non-locality which is well-defined due to the fact that on the double Euclidean torus one can invert the Laplacian. Finally, as a concrete example, we computed the gauge algebra of weakly constrained DFT to trilinear order.

Our approach to the double copy offers a novel insight into color-kinematics duality. With our results, we have proved that the consistency of the off-shell theory to quartic order relies on all the relations of a $BV_{\infty}^{\square}$-algebra, not on the Jacobi identity of a Lie algebra. This observation extends to the amplitudes, where we proved that, even on-shell, the three-term relation obeyed by the kinematic numerators is not a Jacobi identity, but rather a deformed Poisson compatibility relation. This observation, however, is limited to the quartic theory and four-point scattering amplitudes.

The double copy prescription developed in this thesis is, to our knowledge, the only off-shell prescription that does not require a choice of gauge and that, in addition to giving the full gravitational dynamics, also allows one to reconstruct the non-linear gauge structure of gravity in terms of the kinematic building blocks of Yang-Mills theory. Moreover, the double copy in the framework of homotopy algebras has proved useful in constructing weakly constrained double field theory to quartic order, which was not known before. This approach, however, due to the obstruction that requires to invert the Laplacian, is limited to Euclidean spaces. For this reason, it is not possible to further test our results with, for example, scattering amplitudes.

Future directions

In the following, we list a series of interesting potential future directions to improve and extend our double copy prescription:

- **Tensor product of $BV_{\infty}^{\square}$-algebras beyond trilinear maps**: The main shortcoming of our current prescription is that, so far, we have constructed the kinematic algebra of our version of Yang-Mills to trilinear order. However, an all-order construction of gravity requires an all-order kinematic algebra. Then, a prescription to take the tensor product of two $BV_{\infty}^{\square}$-algebras to all orders should be developed.
- **On and off-shell color-kinematics duality at higher points**: We have seen that at four-points the three-term relation that the kinematic numerators obey is an on-shell version of the deformed Poisson relation of the kinematic algebra. However, it is not clear how to extend this beyond four points. Understanding the connection between the (higher than trilinear) off-shell $BV_{\infty}^{\square}$ relations and the three-term relation of the numerators in amplitudes can, potentially, lead to a proof of color-kinematics duality and the BCJ double copy at loop level.
- **Classical solutions and the classical double copy**: An interesting application of our double copy construction would be in the context of general classical solutions. To that end, one would have to understand, from an algebraic perspective, how color-stripping and the construction of the kinematic algebra work when considering solutions to the equations of motion written in terms of L_{∞} maps.

- **A more general construction**: Our double copy prescription works well for a particular version of Yang-Mills with an auxiliary field. However, it would be beneficial to understand how to implement an algebraic double copy prescription using kinematic $\mathrm{BV}_\infty^\square$-algebras with other versions of Yang-Mills or more general theories, such as gauge theories with matter fields or supersymmetric extensions of Yang-Mills.
- **Beyond the double Euclidean torus**: The construction of quartic weakly constrained double field theory, though novel, is only consistent if the background geometry has Euclidean signature. As a consequence, one cannot compute interesting objects like scattering amplitudes. Thus, it would be interesting to see whether there are alternative ways to construct the quartic theory without relying on Euclidean backgrounds. A potential alternative would be to take advantage of the on-shell methods used in the scattering amplitudes program, such as BCFW recursion, where one can construct higher-point amplitudes starting from three-point amplitudes. Cubic weakly constrained double field theory is well-defined on Lorentzian backgrounds, and hence one can compute three-point scattering amplitudes. If these scattering amplitudes are consistent and have the correct analytical properties, one could use them to construct four- and higher-point amplitudes and prove the existence of the theory beyond cubic order in Minkowski space with some coordinates compactified on a (double) torus.

Appendix A

Yang-Mills maps and operators

This appendix is taken from the upcoming paper [5]. In this appendix we collect all the relevant operators associated to the maps of the $\mathrm{BV}_\infty^\square$ algebra of $\mathcal{K}$. We start from the operators $\hat{m}_1, \hat{m}_2$ and $\hat{m}_{3h}$ (3.169) corresponding to the C_∞ subalgebra. The differential m_1 is related to $\hat{m}_1$ via

$$m_1(\psi) = \hat{m}_1(Z_A)\,\psi^A(x)\;, \tag{A.1}$$

where the complete list of operators $\hat{m}_1(Z_A)$ is given by

$$\begin{aligned} \hat{m}_1(\theta_+) &= \theta_\mu\,\partial^\mu + c\,\theta_+\,\Box\;, & \hat{m}_1(c\,\theta_+) &= -c\,\theta_\mu\,\partial^\mu - \theta_-\;,\\ \hat{m}_1(\theta_\mu) &= c\,\theta_\mu\,\Box + \theta_-\,\partial_\mu\;, & \hat{m}_1(c\,\theta_\mu) &= -c\,\theta_-\,\partial_\mu\;,\\ \hat{m}_1(\theta_-) &= c\,\theta_-\,\Box\;, & \hat{m}_1(c\,\theta_-) &= 0\;. \end{aligned} \tag{A.2}$$

We continue with the two-product m_2, which acts as

$$m_2(\psi_1,\psi_2) = \mu\left[\hat{m}_2(Z_A,Z_B)\left(\psi_1^A(x)\otimes\psi_2^B(x)\right)\right]\;, \tag{A.3}$$

and the non-vanishing bidifferential operators $\hat{m}_2(Z_A,Z_B)$ read

$$\begin{aligned} \hat{m}_2(\theta_+,\theta_+) &= \theta_+\big(\mathbb{1}\otimes\mathbb{1}\big)\;,\\ \hat{m}_2(\theta_\mu,\theta_\nu) &= c\,\theta_\nu\Big[\big(\partial_\mu\otimes\mathbb{1}\big) + 2\,\big(\mathbb{1}\otimes\partial_\mu\big)\Big] - c\,\theta_\mu\Big[\big(\mathbb{1}\otimes\partial_\nu\big) + 2\,\big(\partial_\nu\otimes\mathbb{1}\big)\Big]\\ &\quad + c\,\theta_\rho\,\eta_{\mu\nu}\Big[\big(\partial^\rho\otimes\mathbb{1}\big) - \big(\mathbb{1}\otimes\partial^\rho\big)\Big]\;, \end{aligned} \tag{A.4}$$

for "diagonal" (Z_A, Z_B), while for "off-diagonal" ones we give both orderings explicitly:

$$\begin{aligned} \hat{m}_2(\theta_\mu,\theta_+) &= \theta_\mu\big(\mathbb{1}\otimes\mathbb{1}\big) + c\,\theta_+\big(\partial_\mu\otimes\mathbb{1} + \mathbb{1}\otimes\partial_\mu\big)\;, & \hat{m}_2(\theta_+,\theta_\mu) &= \hat{m}_2(\theta_\mu,\theta_+)\;,\\ \hat{m}_2(\theta_+,c\,\theta_\mu) &= c\,\theta_\mu\big(\mathbb{1}\otimes\mathbb{1}\big)\;, & \hat{m}_2(c\,\theta_\mu,\theta_+) &= \hat{m}_2(\theta_+,c\,\theta_\mu)\;,\\ \hat{m}_2(\theta_+,\theta_-) &= -c\,\theta_\mu\big(\mathbb{1}\otimes\partial^\mu\big)\;, & \hat{m}_2(\theta_-,\theta_+) &= -c\,\theta_\mu\big(\partial^\mu\otimes\mathbb{1}\big)\;,\\ \hat{m}_2(\theta_\mu,c\,\theta_\nu) &= -c\,\theta_-\eta_{\mu\nu}\big(\mathbb{1}\otimes\mathbb{1}\big)\;, & \hat{m}_2(c\,\theta_\nu,\theta_\mu) &= \hat{m}_2(\theta_\mu,c\,\theta_\nu)\;,\\ \hat{m}_2(\theta_\mu,\theta_-) &= c\,\theta_-\big(\mathbb{1}\otimes\partial_\mu\big)\;, & \hat{m}_2(\theta_-,\theta_\mu) &= c\,\theta_-\big(\partial_\mu\otimes\mathbb{1}\big)\;,\\ \hat{m}_2(\theta_+,c\,\theta_-) &= c\,\theta_-\big(\mathbb{1}\otimes\mathbb{1}\big)\;, & \hat{m}_2(c\,\theta_-,\theta_+) &= \hat{m}_2(\theta_+,c\,\theta_-)\;, \end{aligned} \tag{A.5}$$

which enforce graded symmetry of the map m_2. The C_∞ maps are exhausted with the three-product

$$m_{3h}(\psi_1,\psi_2,\psi_3) = \mu\left[\hat{m}_{3h}(Z_A,Z_B,Z_C)\left(\psi_1^A(x)\otimes\psi_2^B(x)\otimes\psi_3^C(x)\right)\right] , \tag{A.6}$$

whose only non-vanishing component is associated to the operator

$$\hat{m}_{3h}(\theta_\mu,\theta_\nu,\theta_\rho) = \left(c\,\theta_\mu\,\eta_{\nu\rho} - c\,\theta_\nu\,\eta_{\mu\rho}\right)\left(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\right) . \tag{A.7}$$

Coming to the $\mathrm{BV}_\infty^\square$ structure, the two-bracket b_2 is associated to a bidifferential operator $\hat{b}_2$ exactly as in (A.4):

$$b_2(\psi_1,\psi_2) = \mu\left[\hat{b}_2(Z_A,Z_B)\left(\psi_1^A(x)\otimes\psi_2^B(x)\right)\right] , \tag{A.8}$$

but we do not give the explicit form of the operators $\hat{b}_2(Z_A,Z_B)$, since they can be straightforwardly derived from $b_2 = [b, m_2]$. The homotopy Poisson map θ_3 is related to tridifferential operators $\hat{\theta}_3(Z_A,Z_B,Z_C)$ by

$$\theta_3(\psi_1,\psi_2,\psi_3) = \mu\left[\hat{\theta}_3(Z_A,Z_B,Z_C)\left(\psi_1^A(x)\otimes\psi_2^B(x)\otimes\psi_3^C(x)\right)\right] . \tag{A.9}$$

The following operators correspond to totally graded symmetric maps:

$$\begin{aligned}
\hat{\theta}_3(\theta_+,\theta_+,\theta_-) &= \theta_+\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big) ,\\
\hat{\theta}_3(\theta_+,\theta_+,c\,\theta_-) &= -c\,\theta_+\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big) ,\\
\hat{\theta}_3(\theta_+,\theta_\mu,c\,\theta_\nu) &= c\,\theta_+\eta_{\mu\nu}\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big) ,\\
\hat{\theta}_3(\theta_+,\theta_\mu,\theta_-) &= \theta_\mu\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big) + c\,\theta_+\big(\partial_\mu\otimes\mathbb{1}\otimes\mathbb{1}\big) ,\\
\hat{\theta}_3(\theta_+,c\,\theta_+,\theta_-) &= c\,\theta_+\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big) ,\\
\hat{\theta}_3(\theta_+,\theta_-,\theta_-) &= \theta_-\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big) ,\\
\hat{\theta}_3(\theta_+,\theta_-,c\,\theta_\mu) &= c\,\theta_\mu\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big) ,\\
\hat{\theta}_3(\theta_+,\theta_-,c\,\theta_-) &= c\,\theta_-\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big) ,\\
\hat{\theta}_3(\theta_\mu,\theta_-,\theta_-) &= 2\,c\,\theta_-\big(\mathbb{1}\otimes\partial_\mu\otimes\mathbb{1}+\mathbb{1}\otimes\mathbb{1}\otimes\partial_\mu\big) ,\\
\hat{\theta}_3(\theta_\mu,c\,\theta_\nu,\theta_-) &= -c\,\theta_-\eta_{\mu\nu}\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big) ,\\
\hat{\theta}_3(c\,\theta_+,\theta_-,\theta_-) &= c\,\theta_-\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big) ,
\end{aligned} \tag{A.10}$$

Given the ordering of (Z_A,Z_B,Z_C) above, the operator corresponding to the exchange of the first two Z's, *i.e.* $\hat{\theta}_3(Z_B,Z_A,Z_C)$, is obtained by just exchanging the first two factors in $\big(\mathcal{O}_1\otimes\mathcal{O}_2\otimes\mathcal{O}_3\big)$, since the sign $(-1)^{Z_AZ_B}$ in all these cases is $+1$. For instance, given the above expression for $\hat{\theta}_3(\theta_+,\theta_\mu,\theta_-)$, one has

$$\hat{\theta}_3(\theta_\mu,\theta_+,\theta_-) = \theta_\mu\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big) + c\,\theta_+\big(\mathbb{1}\otimes\partial_\mu\otimes\mathbb{1}\big) . \tag{A.11}$$

The next $\hat{\theta}_3$ operators have both a totally graded symmetric part and a hook part, which we give separately:

$$\begin{aligned}
\hat{\theta}_{3s}(\theta_\mu,\theta_\nu,\theta_\rho) &= c\,\theta_+\Big[\eta_{\mu\nu}\left(\partial_\rho\otimes\mathbb{1}\otimes\mathbb{1}\right) - \eta_{\mu\nu}\left(\mathbb{1}\otimes\partial_\rho\otimes\mathbb{1}\right) + \eta_{\nu\rho}\left(\mathbb{1}\otimes\partial_\mu\otimes\mathbb{1}\right) \\
&\qquad - \eta_{\nu\rho}\left(\mathbb{1}\otimes\mathbb{1}\otimes\partial_\mu\right) + \eta_{\mu\rho}\left(\mathbb{1}\otimes\mathbb{1}\otimes\partial_\nu\right) - \eta_{\mu\rho}\left(\partial_\nu\otimes\mathbb{1}\otimes\mathbb{1}\right)\Big]\,, \qquad \text{(A.12)} \\
\hat{\theta}_{3h}(\theta_\mu,\theta_\nu,\theta_\rho) &= \Big(\theta_\nu\,\eta_{\mu\rho} - \theta_\mu\,\eta_{\nu\rho}\Big)\left(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\right)\,,
\end{aligned}$$

and one can see that they are antisymmetric in the simultaneous exchange of $\mu \leftrightarrow \nu$ and $\mathcal{O}_1 \leftrightarrow \mathcal{O}_2$ in the factors $\mathcal{O}_1\otimes\mathcal{O}_2\otimes\mathcal{O}_3$. The last group of non-vanishing $\hat{\theta}_3$ also has totally graded symmetric and hook components, given by

$$\begin{aligned}
\hat{\theta}_{3s}(\theta_\mu,\theta_\nu,c\,\theta_\rho) &= \Big(c\,\theta_\nu\,\eta_{\mu\rho} - c\,\theta_\mu\,\eta_{\nu\rho}\Big)\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big)\,, \\
\hat{\theta}_{3s}(\theta_\mu,\theta_\nu,\theta_-) &= c\,\theta_\nu\big(\mathbb{1}\otimes\mathbb{1}\otimes\partial_\mu\big) - c\,\theta_\mu\big(\mathbb{1}\otimes\mathbb{1}\otimes\partial_\nu\big) + 2\,c\,\theta_\nu\big(\mathbb{1}\otimes\partial_\mu\otimes\mathbb{1}\big) - 2\,c\,\theta_\mu\big(\partial_\nu\otimes\mathbb{1}\otimes\mathbb{1}\big) \\
&\qquad + c\,\theta_\rho\,\eta_{\mu\nu}\Big[\big(\partial^\rho\otimes\mathbb{1}\otimes\mathbb{1}\big) - \big(\mathbb{1}\otimes\partial^\rho\otimes\mathbb{1}\big)\Big]\,, \\
\hat{\theta}_{3s}(\theta_\mu,c\,\theta_+,\theta_-) &= -\hat{\theta}_{3s}(c\,\theta_+,\theta_\mu,\theta_-) = -c\,\theta_\mu\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big)\,, \\
\hat{\theta}_{3h}(\theta_\mu,\theta_\nu,c\,\theta_\rho) &= \Big(c\,\theta_\nu\,\eta_{\mu\rho} - c\,\theta_\mu\,\eta_{\nu\rho}\Big)\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big)\,, \\
\hat{\theta}_{3h}(c\,\theta_\rho,\theta_\mu,\theta_\nu) &= \hat{\theta}_{3h}(\theta_\mu,c\,\theta_\rho,\theta_\nu) = \Big(c\,\theta_\mu\,\eta_{\nu\rho} - c\,\theta_\rho\,\eta_{\mu\nu}\Big)\big(\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\big)\,.
\end{aligned} \qquad \text{(A.13)}$$

As for the two-bracket b_2, we do not give the explicit form of the operators $\hat{b}_3$ corresponding to the three-bracket, since they can be derived from $b_3 = -[b,\theta_{3s}]$.

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