A Global L^p-Gradient Estimate on Weak Solutions of Nonlinear Parabolic Systems under Mixed Boundary Conditions

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Abstract

In this paper, we prove the integrability of the spatial gradient Du to an exponent $p > 2$ near the boundary, u being a weak solution of a nonlinear parabolic system under mixed boundary conditions. Our method of proof relies on an adaption of a technique by Gehring-Giaquinta-Modica (higher integrability by reverse Hölder inequality) to cubes which possibly intersect a hyperplane. Combined with the interior estimate in [8] we obtain a global L^p-gradient estimate.

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1 Introduction. Statement of the Main Result

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$, which is assumed to be decomposed into two parts:

$$\partial\Omega = \Gamma_0 \cup \Gamma_1,\ \Gamma_0 \cap \Gamma_1 = \emptyset, \quad \Gamma_0 \ \textit{closed},\ \mathrm{int}(\Gamma_0) \neq \emptyset.$$

Set $Q = \Omega \times (0, T)$ $(0 < T < +\infty$ fixed).

We consider the following initial-boundary value problem:

$$(1.1) \qquad \frac{\partial u^i}{\partial t} - D_\alpha a_i^\alpha(x, t, u, Du) = f_i - D_\alpha g_i^\alpha \ ^{1)} \ \text{ in } Q \quad (i = 1, \ldots, N),$$

$$(1.2) \qquad \begin{cases} u = 0 \quad \text{on} \quad \Gamma_0 \times (0, T), \\ a_i^\alpha(x, t, u, Du)\nu_\alpha = 0 \ \text{ on } \ \Gamma_1 \times (0, T) \quad (i = 1, \ldots, N), \end{cases}$$

$$(1.3) \qquad u = 0 \quad \text{on} \quad \Omega \times \{0\},$$

[1)] Throughout a repeated Greek resp. Latin index implies summation over $1, \ldots, n$ resp. $1, \ldots, N$.

where:

$$\begin{aligned} u &= \{u^1, \dots, u^N\}, \\ Du &= \{D_\alpha u^i\} \quad \text{(matrix of first spatial derivatives)}, \\ \nu &= \{\nu_1, \dots, \nu_n\} \quad \text{(unit outward normal along } \partial\Omega); \\ f &= \{f^1, \dots, f^N\} \text{ and } g = \{g_i^\alpha\} \text{ are given functions on } Q. \end{aligned}$$

Define, for any matrix $a = \{a_i^\alpha\}$,

$$\|a\| = \left(\sum_{\alpha=1}^{N} \sum_{i=1}^{n} (a_i^\alpha)^2 \right)^{1/2}$$

Throughout the whole paper, we impose on the functions a_i^α in (1.1) the following conditions:

$$\text{(1.4)} \qquad \begin{cases} a_i^\alpha \textit{ is a Carathéodory function on } Q \times \mathbb{R}^N \times \mathbb{R}^{nN} \\ (\alpha = 1, \dots, n;\ i = 1, \dots, N) \end{cases}$$

$$\text{(1.5)} \qquad \begin{cases} \|a(x,t,u,\xi)\| \le a_0(1 + \|\xi\|) \\ \forall (x,t,u,\xi) \in Q \times \mathbb{R}^N \times \mathbb{R}^{nN} \ (a_0 = \text{const}); \end{cases}$$

$$\text{(1.6)} \qquad \begin{cases} a_i^\alpha(x,t,u,\xi)\xi_\alpha^i \ge \lambda_0 \|\xi\|^2 \\ \forall (x,t,u,\xi) \in Q \times \mathbb{R}^N \times \mathbb{R}^{nN} \ (\lambda_0 = \text{const} > 0). \end{cases}$$

Remark. An inspection of the proofs below shows that the main result of the present paper continues to hold if (1.5) is replaced by the more general growth condition

$$\begin{cases} \|a(x,t,u,\xi)\| \le a_0(1 + |u|^{(n+2)/n} + \|\xi\|) \\ \forall (x,t,u,\xi) \in Q \times \mathbb{R}^N \times \mathbb{R}^{nN} \end{cases}$$

(cf. [8]). ■

By $W_p^1(\Omega)$ we denote the usual Sobolev space over Ω. Next, we define

$$W_2^{1,1}(Q) = \left\{ v \in L^2(Q) : D_\alpha v,\ \frac{\partial v}{\partial t} \in L^2(Q) \ (\alpha = 1, \dots, n) \right\},$$

and

$$\begin{aligned} W_2^{1,0}(Q) &= \{v \in L^2(Q) : D_\alpha v \in L^2(Q) \ (\alpha = 1, \dots, n)\} \\ V_2^{1,0}(Q) &= \left\{ v \in W_2^{1,0}(Q) : \operatorname*{ess\,sup}_{(0,T)} \int_\Omega v^2(x,t) \mathrm{d}x < +\infty \right\}. \end{aligned}$$

$L^p(Q; \mathbb{R}^N)$, $W_2^{1,1}(Q; \mathbb{R}^N)$ etc. denote the space of vector valued functions $v = \{v^1, \dots, v^N\}$ whose components belong to $L^p(Q)$ resp. $W_2^{1,1}(Q)$ etc.

Let (1.4) und (1.5) be satisfied. We introduce the following

DEFINITION *Assume*

$$f \in L^{2(n+2)/(n+4)}(Q; \mathbb{R}^N), \quad g \in L^2(Q; \mathbb{R}^{nN}).$$

The function $u \in V_2^{1,0}(Q; \mathbb{R}^N)$ *is called a weak solution of* (1.1) – (1.3) *if*

$$(1.7)\quad \begin{cases} \int_\Omega u^i(x,t)\varphi^i(x,t)\mathrm{d}x - \int_0^t\int_\Omega u^i \frac{\partial \varphi^i}{\partial s}\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega a_i^\alpha(x,s,u,Du)D_\alpha\varphi^i\mathrm{d}x\mathrm{d}s = \\ = \int_0^t\int_\Omega (f^i\varphi^i + g_i^\alpha D_\alpha\varphi^i)\mathrm{d}x\mathrm{d}s \\ \text{for a.a. } t \in (0,T), \text{ and all } \varphi \in W_2^{1,1}(Q;\mathbb{R}^N) \text{ with} \\ \varphi = 0 \quad \text{a.e. on } \Gamma_0 \times (0,T), \end{cases}$$

$$(1.8)\qquad u = 0 \quad \text{a.e. on } \Gamma_0 \times (0,T). \qquad \blacksquare$$

In order to state the main result of our paper, we make more precise the conditions on the boundary $\partial\Omega$. Define

$$\begin{aligned} x &= \{x_1,\dots,x_n\} = \{x', x_n\}, \quad x' = \{x_1,\dots,x_{n-1}\}, \\ x &= \{x_1,\dots,x_n\} = \{x_1, x'', x_n\}, \quad x'' = \{x_2,\dots,x_{n-1}\}, \end{aligned}$$

and

$$\begin{aligned} B_r(x_0) &= \{x \in \mathbb{R}^n : |x - x_0| < r\}, \\ \Delta_\rho(x_0') &= \{x' \in \mathbb{R}^{n-1} : |x_i - x_{0i}| < \rho \ (i = 1,\dots,n-1)\}, \\ \Delta_\rho(x_0'') &= \{x'' \in \mathbb{R}^{n-2} : |x_i - x_{0i}| < \rho \ (i = 2,\dots,n-1)\}. \end{aligned}$$

Then the conditions on $\partial\Omega$ are as follows:

$$(1.9)\quad \begin{cases} \text{for every } x_0 \in \partial\Omega \text{ there exist a ball } B_{r_0} = B_{r_0}(x_0) \\ \text{and a cube } \Delta_{\rho_0} = \Delta_{\rho_0}(0') \subset \mathbb{R}^{n-1} \text{ such that} \\ \qquad \mathcal{T}_1(x_0) = 0, \\ \qquad \mathcal{T}_1(\Omega \cap B_{r_0}) = \{z \in \mathbb{R}^n : z' \in \Delta_{\rho_0},\, z_n > H(z')\}, \\ \qquad \mathcal{T}_1(\partial\Omega \cap B_{r_0}) = \{z \in \mathbb{R}^n : z' \in \Delta_{\rho_0},\, z_n = H(z')\}, \\ \text{where}: \\ \quad z = \mathcal{T}_1(x) = \mathcal{O}x + \zeta_0,\ \mathcal{O} \text{ orthogonal matrix, } \zeta_0 \in \mathbb{R}^n \text{ fixed,} \\ \quad H \text{ Lipschitz continuous on } \Delta_{\rho_0}. \end{cases}$$

Without any loss of generality, in the sequel we may assume that

$$\partial(\Omega \cap B_{r_0}) \quad \text{is Lipschitzian.}$$

Let $x_0 \in \Gamma_0 \cap \overline{\Gamma}_1$ be arbitrary. We complete (1.9) by the following conditions:

$$(1.10_{\mathrm{n}}) \quad \begin{cases} n \geq 3: \; \textit{there exists a cube } \Delta_{\rho_1} = \Delta_{\rho_1}(0'') \subset I\!R^{n-2} \textit{ such that} \\ \quad \mathcal{T}_1(\Gamma_0 \cap \overline{\Gamma}_1 \cap B_{r_0}) = \{z \in I\!R^n : z_n = H(z'),\, z_1 = h(z'') \\ \qquad\qquad\qquad\qquad\qquad (z' \in \Delta_{\rho_0},\, z'' \in \Delta_{\rho_1})\}, \\ \quad \mathcal{T}_1(\Gamma_0 \cap B_{r_0}) \qquad = \{z \in I\!R^n : z_n = H(z'),\, z_1 \leq h(z'') \\ \qquad\qquad\qquad\qquad\qquad (z' \in \Delta_{\rho_0},\, z'' \in \Delta_{\rho_1})\}, \\ \quad \mathcal{T}_1(\Gamma_1 \cap B_{r_0}) \qquad = \{z \in I\!R^n : z_n = H(z'),\, z_1 > h(z'') \\ \qquad\qquad\qquad\qquad\qquad (z' \in \Delta_{\rho_0},\, z'' \in \Delta_{\rho_1})\}, \\ \textit{where}: \\ \qquad H \textit{ according to } (1.9), \\ \qquad h \textit{ Lipschitz continuous on } \Delta_{\rho_1}; \end{cases}$$

$$(1.10_2) \quad \begin{cases} n = 2: \; \textit{there exists an interval } \Delta_{\rho_1} = (-\rho_1, \rho_1) \subset I\!R^1 \\ \textit{such that} \\ \qquad \mathcal{T}_1(\Gamma_0 \cap \overline{\Gamma}_1 \cap B_{r_0}) = \{0\}, \\ \qquad \mathcal{T}_1(\Gamma_0 \cap B_{r_0}) \qquad = \{z \in I\!R^2 : z_2 = H(z_1),\, -\rho_1 < z_1 \leq 0\}, \\ \qquad \mathcal{T}_1(\Gamma_1 \cap B_{r_0}) \qquad = \{z \in I\!R^2 : z_2 = H(z_1),\, 0 < z_1 < \rho_1\} \\ \qquad (H \textit{ according to } (1.9)). \end{cases}$$

Throughout the whole paper, conditions (1.9) and (1.10_{n}) $(n \geq 2)$ are assumed to hold.

Then our main result is the following

THEOREM *Assume*

$$f \in L^{q_1}(Q; I\!R^N) \left(q_1 > \frac{2(n+2)}{n+4} \right), \quad g \in L^{q_2}(Q; I\!R^{nN}) \; (q_2 > 2).$$

Let $u \in V_2^{1,0}(Q; I\!R^N)$ *be a weak solution of* (1.1) – (1.3). *Then there exists a* $p > 2$ *such that*

$$\int\limits_Q \|Du\|^p \mathrm{d}x\mathrm{d}t \leq$$

$$(1.11) \qquad \leq c \left\{ \left(\int\limits_Q \|Du\|^2 \mathrm{d}x\mathrm{d}t \right)^{p/2} + \int\limits_Q (1 + |f|^{q_1} + \|g\|^{q_2}) \mathrm{d}x\mathrm{d}t \right\},$$

where the constant c *depends only on* n, N, a_0, λ_0, q_1, q_2, $\|f\|_{L^{2(n+2)/(n+4)}(Q;I\!R^N)}$ *and geometric properties of* $\partial\Omega$ *and* $\Gamma_0 \cap \overline{\Gamma}_1$.

Our method of proof consists in establishing a reverse Hölder inequality on Du near the boundary $\partial\Omega$, from which the higher integrability of Du follows. Combined with the interior higher integrability from [8] we obtain the global estimate (1.11).

By using the same method, analogous results have been proved in [1] – [4] for Neumann boundary conditions along the whole boundary $\partial\Omega$. Under more restrictive assumptions on the coefficients a_i^α, the higher integrability of Du has been obtained in [6], [7] by an entirely different method. ■

2 Extension onto $\Omega \times (-T, 0)$

We extend the data in (1.1) from $Q = \Omega \times (0,T)$ onto $\Omega \times (-T, 0)$ as follows:

$$\hat{a}_i^\alpha(x,t,u,\xi) = \begin{cases} a_i^\alpha(x,t,u,\xi) & \textit{for a.a. } (x,t) \in \Omega \times (0,T), \\ a_i^\alpha(x,-t,0,0) & \textit{for a.a. } (x,t) \in \Omega \times (-T,0) \\ \forall\, (u,\xi) \in I\!R^N \times I\!R^{nN}; & \end{cases}$$

$$\hat{f}^i(x,t) = \begin{cases} f^i(x,t) & \textit{for a.a. } (x,t) \in \Omega \times (0,T), \\ 0 & \textit{for a.a. } (x,t) \in \Omega \times (-T,0); \end{cases}$$

$$\hat{g}_i^\alpha(x,t) = \begin{cases} g_i^\alpha(x,t) & \textit{for a.a. } (x,t) \in \Omega \times (0,T), \\ a_i^\alpha(x,-t,0,0) & \textit{for a.a. } (x,t) \in \Omega \times (-T,0). \end{cases}$$

Clearly, the functions $\hat{a}_i^\alpha$ need not be continuous at $u = 0$, $\xi = 0$. However, if $u = u(x,t)\,(\in I\!R^N)$ and $v = v(x,t)\,(\in I\!R^{nN})$ are measurable functions on $\Omega \times (-T,T)$ then $(x,t) \mapsto \hat{a}_i^\alpha(x,t,u(x,t),v(x,t))$ is measurable on $\Omega \times (-T,T)$.

Next, let $u \in V_2^{1,0}(Q; I\!R^N)$ be a weak solution of (1.1) – (1.3). Define

$$\hat{u}(x,t) = \begin{cases} u(x,t) & \textit{for a.a. } (x,t) \in \Omega \times (0,T), \\ 0 & \textit{for a.a. } (x,t) \in \Omega \times (-T,0). \end{cases}$$

It follows that $\hat{u} \in V_2^{1,0}(\Omega \times (-T,T); I\!R^N)$, and (1.7), (1.8) imply

$$\text{(2.1)} \quad \begin{cases} \displaystyle\int_\Omega \hat{u}^i(x,t)\varphi^i(x,t)\mathrm{d}x - \int_{-T}^{t}\int_\Omega \hat{u}^i \frac{\partial \varphi^i}{\partial s}\mathrm{d}x\mathrm{d}s + \\ \displaystyle\quad + \int_{-T}^{t}\int_\Omega \hat{a}_i^\alpha(x,s,\hat{u},D\hat{u})D_\alpha\varphi^i\mathrm{d}x\mathrm{d}s = \int_{-T}^{t}\int_\Omega (\hat{f}^i\varphi^i + \hat{g}_i^\alpha D_\alpha\varphi^i)\mathrm{d}x\mathrm{d}s \\ \textit{for a.a. } t \in (-T,T), \textit{ and all } \varphi \in W_2^{1,1}(\Omega \times (-T,T); I\!R^N) \\ \textit{with } \varphi = 0 \textit{ a.e. on } \Gamma_0 \times (-T,T), \end{cases}$$

$$\text{(2.2)} \qquad \hat{u} = 0 \quad \textit{a.e. on } \Gamma_0 \times (-T,T). \qquad \blacksquare$$

3 Regularization and localization

Fix any $t_0 \in (-T,T)$. The *Steklov mean* of $f \in L^1(\Omega \times (-T,T))$ is:

$$f_k(x,t) = k\int_t^{t+\frac{1}{k}} f(x,s)\mathrm{d}s \textit{ for a.a. } (x,t) \in \Omega \times (-T,t_0) \left(k > \frac{1}{T - t_0}\right).$$

We refer to [8] for integrability, differentiability and convergence properties (if $k \to +\infty$) of f_k. $\blacksquare$

We regularize any given weak solution u of (1.1) – (1.3) by introducing its Steklov mean. To this end, define

$$W^1_{2,\Gamma_0}(\Omega) = \{v \in W^1_2(\Omega) : v = 0 \text{ a.e. on } \Gamma_0\}.$$

The space $W^1_{2,\Gamma_0}(\Omega)$ being separable, an analogous (even slightly simplified) proof as that of [8, Th. 2.1] gives:

Let $u \in V_2^{1,0}(Q; \mathbb{R}^N)$ be a weak solution of (1.1) – (1.3). *Let $\hat{u}_k$ be the Steklov mean of $\hat{u}$ ($\hat{u}$ according to Section* 2). *Then:*

$$(3.1)\qquad \begin{cases} \displaystyle\int_\Omega \frac{\partial \hat{u}^i_k}{\partial t}(x,t)\psi^i(x)\mathrm{d}x + \int_\Omega (\hat{a}^\alpha_i)_k(x,t,\hat{u},D\hat{u})D_\alpha\psi^i(x)\mathrm{d}x = \\ \displaystyle = \int_\Omega [(\hat{f}^i)_k(x,t)\psi^i(x) + (\hat{g}^\alpha_i)_k(x,t)D_\alpha\psi^i(x)]\mathrm{d}x \\ \textit{for a.a. } t \in (-T,t_0),\ \forall\, \psi \in W^1_{2,\Gamma_0}(\Omega;\mathbb{R}^N); \end{cases}$$

$$(3.2)\qquad \hat{u}_k = 0 \quad \textit{a.e. on } \Gamma_0 \times (-T,t_0)$$

for all integers $k > \dfrac{1}{T-t_0}$. ■

Let $x_0 \in \Gamma_0 \cap \overline{\Gamma}_1$. Let $B_{r_0} = B_{r_0}(x_0)$ be a ball according to (1.9). We localize (3.1) by using test functions ψ vanishing a.e. in $\Omega \setminus (\Omega \cap B_{r_0})$. Thus, we obtain

$$(3.3)\qquad \begin{cases} \displaystyle\int_{\Omega\cap B_{r_0}} \frac{\partial \hat{u}^i_k}{\partial t}\psi^i\mathrm{d}x + \int_{\Omega\cap B_{r_0}} (\hat{a}^\alpha_i)_k D_\alpha\psi^i\mathrm{d}x = \\ \displaystyle = \int_{\Omega\cap B_{r_0}} [(\hat{f}^i)_k\psi^i + (\hat{g}^\alpha_i)_k D_\alpha\psi^i]\mathrm{d}x \\ \textit{for a.a. } t \in (-T,t_0),\ \forall\, \psi \in W^1_2(\Omega\cap B_{r_0};\mathbb{R}^N) \textit{ with} \\ \psi = 0 \textit{ a.e. on } (\Gamma_0 \cap B_{r_0}) \cup (\Omega \cap \partial B_{r_0}) \end{cases}$$

and for all integers $k > \dfrac{1}{T-t_0}$.

The localized weak formulation (3.3) will be the point of departure to prove $Du \in L^p((\Omega \cap B_{r_1}) \times (0,T); \mathbb{R}^{nN})$ $(p > 2)$ for an appropriate $0 < r_1 < r_0$. It is readily seen that our discussion continues to hold (with simplified reasonings) when $x_0 \in \mathrm{int}(\Gamma_0)$ or $x_0 \in \Gamma_1$. By a compactness argument, we extend this result to a boundary strip of Ω. ■

4 Change of variables

We pass from the variables $x \in \Omega \cap B_{r_0}$ to new rectangular coordinates

$$y = \mathcal{T}x,$$

where

$$\begin{aligned} \mathcal{T} &= \mathcal{T}_3 \circ \mathcal{T}_2 \circ \mathcal{T}_1 \qquad if \quad n \geq 3, \\ \mathcal{T} &= \mathcal{T}_2 \circ \mathcal{T}_1 \qquad\qquad if \quad n = 2, \end{aligned}$$

and

$$\begin{aligned} &\mathcal{T}_1 \qquad\quad \textit{according to } \; (1.9), (1.10_{\mathrm{n}}), \\ &\mathcal{T}_2(z) \;=\; w = \{z', z_n - H(z')\}, \\ &\mathcal{T}_3(w) \;=\; y = \{-w_1 + h(w''), w_2, \ldots, w_n\}. \end{aligned}$$

We have

$$\begin{aligned} \mathcal{T}'(x) &= \mathcal{T}_3'(\mathcal{T}_2 \circ \mathcal{T}_1(x)) \circ \mathcal{T}_2'(\mathcal{T}_1(x)) \circ \mathcal{O}x \qquad \text{if} \quad n \geq 3, \\ \mathcal{T}'(x) &= \mathcal{T}_2'(\mathcal{T}_1(x)) \circ \mathcal{O}x \qquad\qquad\qquad\qquad\quad \text{if} \quad n = 2 \end{aligned}$$

for a.a. $x \in \Omega \cap B_{r_0}$, where

$$\mathcal{T}_2'(z) = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -\dfrac{\partial H}{\partial z_1} & -\dfrac{\partial H}{\partial z_2} & \ldots & -\dfrac{\partial H}{\partial z_{n-1}} & 1 \end{pmatrix}$$

$$\mathcal{T}_3'(w) = \begin{pmatrix} -1 & \dfrac{\partial h}{\partial w_2} & \ldots & \dfrac{\partial h}{\partial w_{n-1}} & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

It follows that

$$(4.1) \qquad \begin{cases} \mathcal{T}, \mathcal{T}^{-1} \quad \textit{are Lipschitz mappings,} \\ |\det \mathcal{T}'(x)| = 1 \quad \textit{for a.a.} \quad x \in \Omega \cap B_{r_0}. \end{cases} \qquad \blacksquare$$

For the subsequent discussion, we introduce the following notations:

$$\begin{aligned} C_r(y_0) &= \{y \in \mathbb{R}^n : |y_i - y_{0i}| < r \; (i = 1, \ldots, n)\}, \\ C_r^+(y_0) &= \{y \in C_r(y_0) : y_n > 0\}, \\ C_r^-(y_0) &= \{y \in C_r(y_0) : y_n < 0\}. \end{aligned}$$

Let $x_0 \in \Gamma_0 \cap \overline{\Gamma}_1$. Let $B_{r_0} = B_{r_0}(x_0)$ be a ball according to (1.9). From (1.9), (1.10_{n}) we obtain the existence of a $\rho_0 > 0$, such that

$$\mathcal{T}^{-1} : \overline{C_{\rho_0}^+(0)} \longrightarrow \overline{\Omega} \cap B_{r_0},$$

where

$$
(4.2) \qquad \begin{cases} \mathcal{T}^{-1}(C^+_{\rho_0}(0)) \subset \Omega \cap B_{r_0}, \\ \mathcal{T}^{-1}(\{y \in \overline{C^+_{\rho_0}(0)} : y_1 \le 0,\ y_n = 0\}) \subset \Gamma_0 \cap B_{r_0}, \\ \mathcal{T}^{-1}(\{y \in \overline{C^+_{\rho_0}(0)} : y_1 > 0,\ y_n = 0\}) \subset \Gamma_1 \cap B_{r_0}, \end{cases}
$$

We note that both r_0 and ρ_0 possibly depend on x_0. ■

5 Transformation of (3.3)

We begin by transforming the functions $\hat{a}_i^\alpha$ and $\hat{f}^i$, $\hat{g}_i^\alpha$ (cf. Section 2):

$$
\begin{aligned}
A_i^\alpha(y,t,v,\eta) &= \hat{a}_i^\beta(x,t,v,\eta\circ\mathcal{T}'(x))\frac{\partial(\mathcal{T}x)_\alpha}{\partial x_\beta} \\
F^i(y,t) &= \hat{f}^i(x,t), \\
G_i^\alpha(y,t) &= \hat{g}_i^\beta(x,t)\frac{\partial(\mathcal{T}x)_\alpha}{\partial x_\beta}
\end{aligned}
$$

for a.a. $(y,t) \in C^+_{\rho_0}(0)\times(-T,T)$ $(y = \mathcal{T}x)$ and all $(v,\eta) \in \mathbb{R}^N \times \mathbb{R}^{nN}$ $(\alpha = 1,\dots,n;$ $i = 1,\dots,N)$. A_i^α are Carathéodory functions on $C^+_{\rho_0}(0) \times (-T,T) \times \mathbb{R}^N \times \mathbb{R}^{nN}$ which satisfy an analogous growth condition as a_i^α do (cf. (1.5)).

Next, define

$$
U(y,t) = \hat{u}(x,t) \quad (y = \mathcal{T}x)
$$

for a.a. $(y,t) \in C^+_{\rho_0}(0) \times (-T,T)$. Then $U \in V_2^{1,0}(C^+_{\rho_0}(0)\times(-T,T);\mathbb{R}^N)$ and

$$
\begin{aligned}
U_k(y,t) &= k\int_t^{t+\frac{1}{k}} \hat{u}(\mathcal{T}^{-1}y,s)\mathrm{d}s = \hat{u}_k(\mathcal{T}^{-1}y,t), \\
\frac{\partial U_k}{\partial t}(y,t) &= \frac{\partial \hat{u}_k}{\partial t}(\mathcal{T}^{-1}y,t)
\end{aligned}
$$

for a.a. $(y,t) \in C^+_{\rho_0}(0) \times (-T,t_0)$ $\left(t_0 \in (0,T),\ k > \dfrac{1}{T-t_0}\right)$. By (1.6),

$$
\begin{aligned}
A_i^\alpha(y,t,U,DU)D_\alpha U^i &= \hat{a}_i^\beta(x,t,U,(DU)\circ\mathcal{T}'(x))(D_\alpha U^i)\frac{\partial(\mathcal{T}x)_\alpha}{\partial x_\beta} \\
&\ge \lambda_0\|(DU)\circ\mathcal{T}'(x)\|^2.
\end{aligned}
$$

On the other hand, (4.1) implies the existence of a constant $\mu_0 > 0$ such that

$$
\|\xi\circ\mathcal{T}'(x)\| \ge \mu_0\|\xi\| \quad \forall\, \xi \in \mathbb{R}^{nN},\ \textit{for a.a. } x \in \Omega\cap B_{r_0}.
$$

Thus,

$$
(5.1) \qquad \begin{cases} A_i^\alpha(y,t,U,DU)D_\alpha U^i \ge \lambda_0\mu_0^2\|DU\|^2 \\ \textit{for a.a. } (y,t) \in C^+_{\rho_0}(0)\times(-T,T). \end{cases}
$$

From (3.3) it follows that

$$
(5.2)\qquad \begin{cases} \displaystyle\int\limits_{C^+_{\rho_0}(0)} \frac{\partial U^i_k}{\partial t}\chi^i \mathrm{d}y + \int\limits_{C^+_{\rho_0}(0)} (A^\alpha_i(y,t,U,DU))_k D_\alpha \chi^i \mathrm{d}y = \\ \displaystyle = \int\limits_{C^+_{\rho_0}(0)} (F^i_k \chi^i + G^\alpha_{ik} D_\alpha \chi^i)\mathrm{d}y \\ \textit{for a.a. } t \in (-T, t_0), \textit{ and all } \chi \in W^1_2(C^+_{\rho_0}(0); I\!R^N) \textit{ with} \\ \chi = 0 \textit{ a.e. on } \partial C^+_{\rho_0}(0) \cap (\{y : y_1 \le 0, y_n = 0\} \cup \{y : y_n > 0\}) \end{cases}
$$

and all integers $k > \dfrac{1}{T - t_0}$. Indeed, define $\psi(x) = \chi(\mathcal{T}x)$ for a.a. $x \in \mathcal{T}^{-1}(C^+_{\rho_0}(0))$. We extend ψ by zero onto $(\Omega \cap B_{r_0}) \setminus \mathcal{T}^{-1}(C^+_{\rho_0}(0))$ and obtain $\psi \in W^1_2(\Omega \cap B_{r_0}; I\!R^N)$ with $\psi = 0$ a.e. on $(\Gamma_0 \cap B_{r_0}) \cup (\Omega \cap \partial B_{r_0})$. Thus, (5.2) follows from (3.3).

Finally, (3.2) implies

$$
(5.3)\qquad U_k = 0 \ \text{ a.e. on } \ \partial C^+_{\rho_0}(0) \cap \{y : y_1 \le 0,\ y_n = 0\}. \qquad \blacksquare
$$

6 Caccioppoli and Poincaré inequalities

Throughout this section, we assume

$$
y_0 \in C^+_{\rho_0/2}(0),\ t_0 \in \left(-\frac{T}{2}, T\right),\ 0 < r < \min\left\{\frac{1}{12}\sqrt{\frac{T}{2}}, \frac{\rho_0}{24}\right\}.
$$

Let $\zeta_{2r} \in C^\infty(I\!R^n)$ and $\tau_{2r} \in C^\infty(I\!R)$ be cut-off functions as follows:

$$
\begin{cases} \zeta_{2r} \equiv 1 \text{ on } C_r(y_0),\ \zeta_{2r} \equiv 0 \text{ in } I\!R^n \setminus C_{2r}(y_0), \\ 0 \le \zeta_{2r} \le 1,\ |D\zeta_{2r}| \le \dfrac{\sigma_1}{r} \text{ in } I\!R^n; \end{cases}
$$

$$
\begin{cases} \tau_{2r} \equiv 1 \text{ on } (t_0 - r^2, +\infty),\ \tau_{2r} \equiv 0 \text{ on } (-\infty, t_0 - 4r^2), \\ 0 \le \tau_{2r} \le 1,\ 0 \le \tau'_{2r} \le \dfrac{\sigma_2}{r^2} \text{ on } I\!R \end{cases}
$$

($\sigma_1, \sigma_2 = \text{const} > 0$ independent of r).

Set $C^+_{2r} = C^+_{2r}(y_0)$. Define

$$
\tilde{v}_{C^+_{2r}} = \left(\int\limits_{C^+_{2r}} \zeta^2_{2r} \mathrm{d}z\right)^{-1} \int\limits_{C^+_{2r}} v \zeta^2_{2r} \mathrm{d}y,
$$

and

$$
\begin{aligned} Q_r &= Q_r(y_0, t_0) = C_r(y_0) \times (t_0 - r^2, t_0), \\ Q^+_r &= Q^+_r(y_0, t_0) = C^+_r(y_0) \times (t_0 - r^2, t_0). \end{aligned}
$$

We distinguish two cases concerning the position of the cube $C_{4r}(y_0)$:

(I) $C_{4r}(y_0) \cap \{y : y_1 \le 0,\ y_n = 0\} = \emptyset$,

(II) $C_{4r}(y_0) \cap \{y : y_1 \le 0,\ y_n = 0\} \neq \emptyset$.

LEMMA 1 (Caccioppoli *inequalities*).
Let $U \in V_2^{1,0}(C^+_{\rho_0}(0) \times (-T,T); I\!R^N)$ *satisfy* (5.2), (5.3).
1. *Assume* (I). *Then*:

$$\operatorname*{ess\,sup}_{(t_0-r^2,t_0)} \int_{C_r^+} |U(y,t) - \tilde{U}_{C_r^+}(t)|^2 \mathrm{d}y + \int_{Q_r^+} \|DU\|^2 \mathrm{d}y\mathrm{d}t \leq$$

$$\leq c_1 \left\{ \frac{1}{r^2} \int_{Q_{2r}^+} |U - \tilde{U}_{C_{2r}^+}|^2 \mathrm{d}y\mathrm{d}t + \int_{Q_{2r}^+} (1 + \|G\|^2) \mathrm{d}y\mathrm{d}t + \left(\int_{Q_{2r}^+} |F|^{2(n+2)/(n+4)} \mathrm{d}y\mathrm{d}t \right)^{(n+4)/(n+2)} \right\} \tag{6.1}$$

($c_1 = \text{const} > 0$ *independent of* r).
2. *Assume* (II). *Then*:

$$\operatorname*{ess\,sup}_{(t_0-r^2,t_0)} \int_{C_r^+} |U(y,t)|^2 \mathrm{d}y + \int_{Q_r^+} \|DU\|^2 \mathrm{d}y\mathrm{d}t \leq$$

$$\leq c_2 \left\{ \frac{1}{r^2} \int_{Q_{2r}^+} |U|^2 \mathrm{d}y\mathrm{d}t + \int_{Q_{2r}^+} (1 + \|G\|^2) \mathrm{d}y\mathrm{d}t + \left(\int_{Q_{2r}^+} |F|^{2(n+2)/(n+4)} \mathrm{d}y\mathrm{d}t \right)^{(n+4)/(n+2)} \right\} \tag{6.2}$$

($c_2 = \text{const} > 0$ *independent of* r).

Inequalities (6.1) *and* (6.2) *continue to hold with* $2r$ *in place of* r.

Proof. – 1. We have

$$\int_{C_{2r}^+} [U_k^i(y,t) - (\widetilde{U_k^i})_{C_{2r}^+}(t)] \zeta_{2r}^2(y) \mathrm{d}y = 0 \quad (i = 1, \ldots, N)$$

for all $t \in (t_0 - 4r^2, t_0)$ $\left(k > \dfrac{1}{T - t_0} \text{ integer} \right)$. Hence

$$\int_{t_0-4r^2}^{t} \int_{C_{2r}^+} \frac{\partial U_k^i}{\partial s} (U_k^i - (\widetilde{U_k^i})_{C_{2r}^+}) \zeta_{2r}^2 \tau_{2r}^2 \mathrm{d}y\mathrm{d}s =$$

$$= \frac{1}{2} \int_{C_{2r}^+} |U_k^i(y,t) - (\widetilde{U_k^i})_{C_{2r}^+}(t)|^2 \zeta_{2r}^2(y) \mathrm{d}y \, \tau_{2r}^2(t)$$

$$- \int_{t_0-4r^2}^{t} \int_{C_{2r}^+} |U_k - (\widetilde{U}_k)_{C_{2r}^+}|^2 \zeta_{2r}^2 \tau_{2r} \tau_{2r}' \mathrm{d}y\mathrm{d}t$$

for all $t \in (t_0 - 4r^2, t_0)$.

By (I), the function

$$\chi = (U_k(\cdot,t) - (\widetilde{U_k})_{C^+_{2r}}(t))\zeta^2_{2r}\tau^2_{2r}(t), \quad t \in (t_0 - 4r^2, t_0)$$

[as well as $\chi = (U_k(\cdot,t) - (\widetilde{U_k})_{C^+_{4r}}(t))\zeta^2_{4r}\tau^2_{4r}(t)$, $t \in (t_0 - 16r^2, t_0)$] is admissible in (5.2). We obtain

$$\frac{1}{2}\int\limits_{C^+_{2r}} |U_k(y,t) - (\widetilde{U_k})_{C^+_{2r}}(t)|^2 \zeta^2_{2r}(y)\mathrm{d}y\,\tau^2_{2r}(t)$$

$$+ \int\limits_{t_0-4r^2}^{t} \int\limits_{C^+_{2r}} (A^\alpha_i)_k (D_\alpha U^i_k)\zeta^2_{2r}\tau^2_{2r}\mathrm{d}y\mathrm{d}s =$$

$$= -2\int\limits_{t_0-4r^2}^{t} \int\limits_{C^+_{2r}} (A^\alpha_i)_k (U^i_k - (\widetilde{U^i_k})_{C^+_{2r}})\zeta_{2r}(D_\alpha\zeta_{2r})\tau^2_{2r}\mathrm{d}y\mathrm{d}s$$

$$+ \int\limits_{t_0-4r^2}^{t} \int\limits_{C^+_{2r}} |U_k - (\widetilde{U_k})_{C^+_{2r}}|^2 \zeta^2_{2r}\tau_{2r}\tau'_{2r}\mathrm{d}y\mathrm{d}s$$

$$+ \int\limits_{t_0-4r^2}^{t} \int\limits_{C^+_{2r}} F^i_k (U^i_k - (\widetilde{U^i_k})_{C^+_{2r}})\zeta^2_{2r}\tau^2_{2r}\mathrm{d}y\mathrm{d}s$$

$$+ \int\limits_{t_0-4r^2}^{t} \int\limits_{C^+_{2r}} G^\alpha_{ik}[(D_\alpha U^i_k)\zeta^2_{2r} + 2(U^i_k - (\widetilde{U^i_k})_{C^+_{2r}})\zeta_{2r}(D_\alpha\zeta_{2r})]\tau^2_{2r}\mathrm{d}y\mathrm{d}s$$

for all $t \in (t_0 - 4r^2, t_0)$. Letting tend $k \to +\infty$ (cf. [8]) and making use of (5.1) one finds by routine calculations

$$\operatorname*{ess\,sup}_{(t_0-4r^2,t_0)} \int\limits_{C^+_{2r}} |U(y,t) - \tilde{U}_{C^+_{2r}}(t)|^2 \zeta^2_{2r}\tau^2_{2r}\mathrm{d}y + \int\limits_{Q^+_{2r}} \|DU\|^2 \zeta^2_{2r}\tau^2_{2r}\mathrm{d}y\mathrm{d}t \leq$$

$$\leq c\Bigg\{\frac{1}{r^2}\int\limits_{Q^+_{2r}} |U - \tilde{U}_{C^+_{2r}}|^2\mathrm{d}y\mathrm{d}t + \int\limits_{Q^+_{2r}} (1 + \|G\|^2)\mathrm{d}y\mathrm{d}t$$

$$+ \int\limits_{Q^+_{2r}} |F|\,|U - \tilde{U}_{C^+_{2r}}|\zeta^2_{2r}\tau^2_{2r}\mathrm{d}y\mathrm{d}t\Bigg\}.$$

To estimate the last integral on the right, we make use of (A3) [Appendix] with $2r$ in place of r. We obtain, for any $\delta > 0$,

$$\int\limits_{Q^+_{2r}} |F|\,|U - \tilde{U}_{C^+_{2r}}|\zeta^2_{2r}\tau^2_{2r}\mathrm{d}y\mathrm{d}t \leq$$

$$\leq \delta\Bigg(\int\limits_{Q^+_{2r}} |(U - \tilde{U}_{C^+_{2r}})\zeta_{2r}\tau_{2r}|^{2(n+2)/n}\mathrm{d}y\mathrm{d}t\Bigg)^{n/(n+2)} +$$

$$+c(\delta)\bigg(\int\limits_{Q^+_{2r}}|F|^{2(n+2)/(n+4)}\mathrm{d}y\mathrm{d}t\bigg)^{(n+4)/(n+2)}$$

$$\le \delta\gamma_1^{n/(n+2)}\bigg\{\operatorname*{ess\,sup}_{(t_0-4r^2,t_0)}\int\limits_{C^+_{2r}}|(U-\tilde U_{C^+_{2r}})\zeta_{2r}\tau_{2r}|^2\mathrm{d}y$$

$$+\int\limits_{Q^+_{2r}}\|D((U-\tilde U_{C^+_{2r}})\zeta_{2r}\tau_{2r})\|^2\mathrm{d}y\mathrm{d}t\bigg\}$$

$$+c(\delta)\bigg(\int\limits_{Q^+_{2r}}|F|^{2(n+2)/(n+4)}\mathrm{d}y\mathrm{d}t\bigg)^{(n+4)/(n+2)}.$$

Thus, choosing $\delta>0$ appropriately and observing that

$$\int\limits_{C^+_r}(v-\tilde v_{C^+_r})^2\mathrm{d}y\le 2(1+2^{n+3})\int\limits_{C^+_r}(v-\tilde v_{C^+_{2r}})^2\mathrm{d}y\quad \forall\, v\in L^2(C^+_{2r})$$

(cf. p. 27), (6.1) follows.

2. Let (II) hold. Then, regardless of the position of the cube C_{2r} (resp. C_{4r}), the function

$$\chi=U_k(\cdot,t)\zeta^2_{2r}\tau^2_{2r}(t),\quad t\in(t_0-4r^2,t_0)$$

[resp. $\chi=U_k(\cdot,t)\zeta^2_{4r}\tau^2_{4r}(t)$, $t\in(t_0-16r^2,t_0)$] is admissible in (5.2). Repeating word by word the preceding arguments we obtain (6.2). ■

LEMMA 2 (Poincaré *inequalities*).
Let $U\in V_2^{1,0}(C^+_{\rho_0}(0)\times(-T,T);\mathbb{R}^N)$ *satisfy* (5.2), (5.3).
1. *Assume* (I). *Then*:

$$\operatorname*{ess\,sup}_{(t_0-4r^2,t_0)}\int\limits_{C^+_{2r}}|U(y,t)-\tilde U_{C^+_{2r}}(t)|^2\mathrm{d}y\le$$

$$\le c_3\bigg\{\int\limits_{Q^+_{8r}}(1+\|DU\|^2+\|G\|^2)\mathrm{d}y\mathrm{d}t+ \tag{6.3}$$

$$+\bigg(\int\limits_{Q^+_{8r}}\|F\|^{2(n+2)/(n+4)}\mathrm{d}y\mathrm{d}t\bigg)^{(n+4)/(n+2)}\bigg\}$$

($c_3=\mathrm{const}>0$ *independent of* r).
2. *Assume* (II). *Then*:

$$\operatorname*{ess\,sup}_{(t_0-4r^2,t_0)}\int\limits_{C^+_{2r}}|U(y,t)|^2\mathrm{d}y\le$$

$$\le c_4\bigg\{\int\limits_{Q^+_{12r}}(1+\|DU\|^2+\|G\|^2)\mathrm{d}y\mathrm{d}t+ \tag{6.4}$$

$$+\left(\int\limits_{Q^+_{12r}} |F|^{2(n+2)/(n+4)}\mathrm{d}y\mathrm{d}t\right)^{(n+4)/(n+2)}\Bigg\}$$

($c_4 = \text{const} > 0$ *independent of* r).

Proof. – 1. By (A2) [Appendix],

$$\int\limits_{Q^+_{4r}} |U - \tilde{U}_{C^+_{4r}}|^2\mathrm{d}y\mathrm{d}t \le \gamma_0 r^2 \int\limits_{Q^+_{8r}} \|DU\|^2\mathrm{d}y\mathrm{d}t.$$

Inserting this into (6.1) [with $2r$ in place of r] gives (6.3).

2. To prove (6.4) we observe that

$$U(\cdot,t) = 0 \quad \text{a.e. on } C^+_{12r}(y_0) \cap \{y : y_1 \le 0,\ y_n = 0\}.$$

Thus, by (A5) [Appendix] [with $4r$ in place of r; $s = q = 2$],

$$\int\limits_{Q^+_{4r}} |U|^2\mathrm{d}y\mathrm{d}t \le \gamma_2^2 r^2 \int\limits_{Q^+_{12r}} \|DU\|^2\mathrm{d}y\mathrm{d}t.$$

Then (6.4) follows from (6.2). ■

7 Proof of the Theorem

7.1 *Preliminaries.* Let C_r be any cube of side length $2r$. The following SOBOLEV-POINCARÉ inequality is well-known:

$$\left(\int\limits_{C_r} |v - \tilde{v}_{C_r}|^s\mathrm{d}y\right)^{1/s} \le \sigma_0 r^{1+n/s-n/q}\left(\int\limits_{C_r} |Dv|^q\mathrm{d}y\right)^{1/q} \quad \forall\, v \in W^1_q(C_r), \tag{7.1}$$

where:

$$\begin{aligned} 1 \le s \le \frac{nq}{n-q} \qquad & \textit{if } 1 \le n < q, \\ 1 \le s < +\infty \qquad & \textit{if } n = q, \\ \sigma_0 = \text{const} > 0 \qquad & \textit{independent of } r. \end{aligned}$$

The structure of the constant $\sigma_0 r^{1+n/s-n/q}$ in (7.1) can be established by a standard homothetical argument.

Let $v \in W^1_2(C^+_{\rho_0}(0))$. We extend v onto $C^-_{\rho_0}(0)$ by reflection with respect to $\{y \mid y_n = 0\}$:

$$\bar{v}(y) := \begin{cases} v(y', y_n) & \text{for a.a. } \ y \in C^+_{\rho_0}(0), \\ v(y', -y_n) & \text{for a.a. } \ y \in C^-_{\rho_0}(0). \end{cases} \tag{7.2}$$

Let

$$y_0 \in C_{\rho_0/2}(0), \quad 0 < r < \frac{\rho_0}{2}.$$

If $y_{0n} \ge 0$ we have

$$\bar{v} \in W^1_2(C_r(y_0)), \quad \int\limits_{C_r(y_0)} |D\bar{v}|^2\mathrm{d}y \le 2 \int\limits_{C^+_r(y_0)} |Dv|^2\mathrm{d}y. \tag{7.3}$$

Assume $y_{0n} < 0$. Set $y_0^* = \{y_0', -y_{0n}\}$. It follows that

$$(7.4) \qquad \bar{v} \in W_2^1(C_r(y_0^*)), \qquad \int\limits_{C_r(y_0)} |D\bar{v}|^2 \mathrm{d}y \le 2 \int\limits_{C_r^+(y_0^*)} |Dv|^2 \mathrm{d}y. \qquad \blacksquare$$

7.2 *Gradient estimate over* Q_r^+. Let

$$y_0 \in C^+_{\rho_0/2}(0), \quad t_0 \in \left(-\frac{T}{2}, T\right), \quad 0 < r < \min\left\{\frac{1}{12}\sqrt{\frac{T}{2}}, \frac{\rho_0}{24}\right\}.$$

Let $U \in V_2^{1,0}(Q \times (-T,T); I\!R^N)$ *satisfy* (5.2), (5.3). *Then*:

$$\int\limits_{Q_r^+} \|DU\|^2 \mathrm{d}y\mathrm{d}t \le$$

$$(7.5) \qquad \le c(\varepsilon) r^{-2(n+2)/n} \left(\int\limits_{Q_{12r}^+} \|DU\|^{2n/(n+2)} \mathrm{d}y\mathrm{d}t \right)^{(n+2)/n}$$

$$+\varepsilon \int\limits_{Q_{12r}^+} \|DU\|^2 \mathrm{d}y\mathrm{d}t + c \int\limits_{Q_{12r}^+} (1 + |F|^{2(n+2)/(n+4)} + \|G\|^2) \mathrm{d}y\mathrm{d}t$$

for all $0 < \varepsilon \le 1$ ($c(\varepsilon)$, $c = \mathrm{const} > 0$ *independent of* r; $Q_r^+ = C_r^+(y_0) \times (t_0 - r^2, t_0)$).

Proof. – 1. Assume (I). Let $n \ge 3$. We combine (6.3), (A1) [Appendix] and (7.1) $\left(\text{with } v = \overline{U} \text{ (cf. (7.2)) and } s = \dfrac{2n}{n-2},\ q = 2 \text{ resp. } s = q = \dfrac{2n}{n+2}\right)$ to obtain

$$\int\limits_{Q_{2r}^+} |U - \tilde{U}_{C_{2r}^+}|^2 \mathrm{d}y\mathrm{d}t \le$$

$$\le \left(\operatorname*{ess\,sup}_{(t_0-4r^2, t_0)} \int\limits_{C_{2r}^+} |U(y,s) - \tilde{U}_{C_{2r}^+}(s)|^2 \mathrm{d}y \right)^{1/2} \int\limits_{t_0-4r^2}^{t_0} \left(\int\limits_{C_{2r}^+} |U - \tilde{U}_{C_{2r}^+}|^2 \mathrm{d}y \right)^{1/2} \mathrm{d}t$$

$$\le c \left\{ \int\limits_{Q_{8r}^+} (1 + \|DU\|^2 + \|G\|^2) \mathrm{d}y\mathrm{d}t + \right.$$

$$\left. + \left(\int\limits_{Q_{8r}^+} |F|^{2(n+2)/(n+4)} \mathrm{d}y\mathrm{d}t \right)^{(n+4)/(n+2)} \right\}^{1/2} \times$$

$$\times \int\limits_{t_0-4r^2}^{t_0} \left(\int\limits_{C_{4r}} |\overline{U} - \tilde{\overline{U}}_{C_{4r}}|^{2n/(n-2)} \mathrm{d}y \right)^{(n-2)/4n} \times$$

$$\times \left(\int\limits_{C_{4r}} |\overline{U} - \tilde{\overline{U}}_{C_{4r}}|^{2n/(n+2)} \mathrm{d}y \right)^{(n+2)/4n} \mathrm{d}t$$

$$\leq c r^{1/2}\Bigg\{\int\limits_{Q^+_{8r}}(1+\|DU\|^2+|F|^{2(n+2)/(n+4)}+\|G\|^2)\mathrm{d}y\mathrm{d}t\Bigg\}^{1/2}\times$$

$$\times\int\limits_{t_0-4r^2}^{t_0}\Bigg(\int\limits_{C^+_{4r}}\|DU\|^2\mathrm{d}y\Bigg)^{1/4}\Bigg(\int\limits_{C^+_{4r}}\|DU\|^{2n/(n+2)}\mathrm{d}y\Bigg)^{(n+2)/4n}\mathrm{d}t$$

$$\leq c r^{3/2-1/n}\Bigg\{\int\limits_{Q^+_{8r}}(1+\|DU\|^2+|F|^{2(n+2)/(n+4)}+\|G\|^2)\mathrm{d}y\mathrm{d}t\Bigg\}^{3/4}\times$$

$$\times\Bigg\{\int\limits_{Q^+_{8r}}\|DU\|^{2n/(n+2)}\mathrm{d}y\mathrm{d}t\Bigg\}^{(n+2)/4n}.$$

Inserting this estimate into the right-hand side of (6.1) gives, for any $\varepsilon>0$,

$$\int\limits_{Q^+_r}\|DU\|^2\mathrm{d}y\mathrm{d}t\leq$$

$$\leq c(\varepsilon)r^{-2(n+2)/n}\Bigg(\int\limits_{Q^+_{8r}}\|DU\|^{2n/(n+2)}\mathrm{d}y\mathrm{d}t\Bigg)^{(n+2)/n}$$

$$+\varepsilon\int\limits_{Q^+_{8r}}(1+\|DU\|^2+|F|^{2(n+2)/(n+4)}+\|G\|^2)\mathrm{d}y\mathrm{d}t$$

$$+c\int\limits_{Q^+_{8r}}(1+|F|^{2(n+2)/(n+4)}+\|G\|^2)\mathrm{d}y\mathrm{d}t.$$

Whence (7.5).

Let $n=2$. Again combining (6.3), (A1) [Appendix] and (7.1) (with $s=2$, $q=1$) gives

$$\int\limits_{Q^+_{2r}}|U-\tilde{U}_{C^+_{2r}}|^2\mathrm{d}y\mathrm{d}t\leq$$

$$\leq c\Bigg\{\int\limits_{Q^+_{8r}}(1+\|DU\|^2+|F|^{4/3}+\|G\|^2)\mathrm{d}y\mathrm{d}t\Bigg\}^{1/2}\int\limits_{Q^+_{8r}}\|DU\|\mathrm{d}y\mathrm{d}t.$$

Thus, by (6.1),

$$\int\limits_{Q^+_r}\|DU\|^2\mathrm{d}y\mathrm{d}t\leq$$

$$\leq c(\varepsilon)r^{-4}\Bigg(\int\limits_{Q^+_{8r}}\|DU\|\mathrm{d}y\mathrm{d}t\Bigg)^2+\varepsilon\int\limits_{Q^+_{8r}}(1+\|DU\|^2+|F|^{4/3}+\|G\|^2)\mathrm{d}y\mathrm{d}t$$

$$+c\int\limits_{Q^+_{8r}}(1+|F|^{4/3}+\|G\|^2)\mathrm{d}y\mathrm{d}t$$

for all $\varepsilon > 0$. (7.5) is now readily seen.

2. Assume (II). Let $n \geq 3$. From (6.4) and (A5) [Appendix] $\left(\text{with } s = \frac{2n}{n-2},\right.$ $\left. q = 2 \text{ resp. } s = q = \frac{2n}{n+2}\right)$ it follows that

$$\int\limits_{Q^+_{2r}} |U|^2 \mathrm{d}y\mathrm{d}t \leq$$

$$\leq c\Bigg\{ \int\limits_{Q^+_{12r}} (1+\|DU\|^2+\|G\|^2)\mathrm{d}y\mathrm{d}t + \Bigg(\int\limits_{Q^+_{12r}} |F|^{2(n+2)/(n+4)}\mathrm{d}y\mathrm{d}t\Bigg)^{(n+4)/(n+2)}\Bigg\}^{1/2} \times$$

$$\times \int\limits_{t_0-4r^2}^{t_0} \Bigg(\int\limits_{C^+_{2r}} |U|^{2n/(n-2)}\mathrm{d}y\mathrm{d}t\Bigg)^{(n-2)/4n} \Bigg(\int\limits_{C^+_{2r}} |U|^{2n/(n+2)}\mathrm{d}y\mathrm{d}t\Bigg)^{(n+2)/4n}$$

$$\leq cr^{3/2-1/n}\Bigg\{ \int\limits_{Q^+_{12r}} (1+\|DU\|^2+|F|^{2(n+2)/(n+4)}+\|G\|^2)\mathrm{d}y\mathrm{d}t\Bigg\}^{3/4} \times$$

$$\times \Bigg(\int\limits_{Q^+_{12r}} \|DU\|^{2n/(n+2)}\mathrm{d}y\mathrm{d}t\Bigg)^{(n+2)/(n+4)}.$$

We insert this estimate into the right-hand side of (6.2) and find (7.5).

If $n = 2$ we obtain (7.5) by arguing as in the case (I) above. ■

7.3 *Averaged gradient estimate over Q_r.*

Define

$$\fint\limits_{E} v \,\mathrm{d}y\mathrm{d}t = \frac{1}{\text{meas } E} \int\limits_{E} v \,\mathrm{d}y\mathrm{d}t.$$

We extend F and G from $C^+_{\rho_0}(0) \times (-T, T)$ onto $C^-_{\rho_0}(0) \times (-T, T)$ (for a.a. $t \in (-T, T)$) by reflection with respect to $\{y \mid y_n = 0\}$ (cf. (7.2)).

Let

$$y_0 \in C_{\rho_0/2}(0), \quad t_0 \in \left(-\frac{T}{2}, T\right), \quad 0 < r < \min\left\{\frac{1}{12}\sqrt{\frac{T}{2}}, \frac{\rho_0}{24}\right\}.$$

Let $U \in V_2^{1,0}(Q \times (-T,T); \mathbb{R}^N)$ satisfy (5.2), (5.3). *Then:*

$$\fint\limits_{Q_r} \|D\overline{U}\|^2 \mathrm{d}y\mathrm{d}t \leq$$

$$\leq c\Bigg(\fint\limits_{Q_{12r}} \|D\overline{U}\|^{2n/(n+2)}\mathrm{d}y\mathrm{d}t\Bigg)^{(n+2)/n} + \frac{1}{2} \fint\limits_{Q_{12r}} \|D\overline{U}\|^2\mathrm{d}y\mathrm{d}t \tag{7.6}$$

$$+c \fint\limits_{Q_{12r}} (1+|\overline{F}|^{2(n+2)/(n+4)} + \|\overline{G}\|^2)\mathrm{d}y\mathrm{d}t$$

($c = \text{const} > 0$ *independent of* r; $Q_r = C_r(y_0) \times (t_0 - r^2, t_0)$).

Proof. We have

$$\text{either} \quad \int_{Q_r} \|D\overline{U}\|^2 \mathrm{d}y\mathrm{d}t \le 2 \int_{C_r^+(y_0)\times(t_0-r^2,t_0)} \|DU\|^2 \mathrm{d}y\mathrm{d}t \quad (y_{0n} \ge 0)$$

$$\text{or} \quad \int_{Q_r} \|D\overline{U}\|^2 \mathrm{d}y\mathrm{d}t \le 2 \int_{C_r^+(y_0^*)\times(t_0-r^2,t_0)} \|DU\|^2 \mathrm{d}y\mathrm{d}t \quad (y_{0n} < 0)$$

(notice that $y_0^* = \{y_0', -y_{0n}\} \in C^+_{\rho_0/2}(0)$; cf. (7.3) resp. (7.4)).

Thus, by (7.5) with $\frac{\varepsilon}{2}$ in place of ε,

$$\int_{Q_r} \|D\overline{U}\|^2 \mathrm{d}y\mathrm{d}t \le$$

$$\le c(\varepsilon) r^{-2(n+2)/n} \Big(\int_{Q_{12r}} \|D\overline{U}\|^{2n/(n+2)} \mathrm{d}y\mathrm{d}t \Big)^{(n+2)/n} + \varepsilon \int_{Q_{12r}} \|D\overline{U}\|^2 \mathrm{d}y\mathrm{d}t$$

$$+ c \int_{Q_{12r}} (1 + |\overline{F}|^{2(n+2)/(n+4)} + \|\overline{G}\|^2) \mathrm{d}y\mathrm{d}t.$$

We divide this inequality by $\operatorname{meas} Q_r$ $(= 2^n r^{n+2})$ and choose $\varepsilon = \dfrac{1}{2 \cdot 12^{n+2}}$. Whence (7.6). ■

7.4 *Higher integrability of DU*. We deduce the integrability of DU with an exponent $p > 2$ from the following result on higher integrability by reverse Hölder inequality [5] (cf. also [4]):

Let $Q \subset \mathbb{R}^{n+1}$ *be a domain. Let*

$$\phi \in L^q_{\mathrm{loc}}(Q), \quad \psi \in L^{q_1}_{\mathrm{loc}}(Q) \quad (1 < q < q_1),$$
$$\phi \ge 0, \quad \psi \ge 0 \quad \text{a.e. in } Q.$$

Assume

$$(*) \qquad \fint_{Q_r} \phi^q \mathrm{d}x\mathrm{d}t \le a \Big\{ \Big(\fint_{Q_{2r}} \phi \mathrm{d}x\mathrm{d}t \Big)^q + \fint_{Q_{2r}} \psi^q \mathrm{d}x\mathrm{d}t \Big\} + \theta \fint_{Q_{2r}} \phi^q \mathrm{d}x\mathrm{d}t$$

for all Q_r *such that* $\overline{Q}_{2r} \subset Q$, *where* $a \ge 1$ *and* $0 < \theta < 1$ *are (fixed) constants* $(Q_r = C_r(x_0) \times (t_0 - r^2, t_0))$.

Then there exists a $\delta > 0$ *such that*

$$\phi \in L^p_{\mathrm{loc}}(Q) \quad \forall q < p < \min\{q + \delta, q_1\},$$

$$(**) \qquad \fint_{Q_r} \phi^p \mathrm{d}x\mathrm{d}t \le c \Big\{ \Big(\fint_{Q_{2r}} \phi^q \mathrm{d}x\mathrm{d}t \Big)^{p/q} + \fint_{Q_{2r}} \psi^p \mathrm{d}x\mathrm{d}t \Big\}$$

for all Q_r with $\overline{Q}_{2r} \subset Q$; here the constant c depends only on n, q, q_1, p, a and θ.

An inspection of the proof in [5] shows that this result continues to hold if Q_{2r} in $(*)$ (and thus in $(**)$) is replaced by Q_{12r} ($\overline{Q}_{12r} \subset Q$). ■

We make use of this result as follows. Set

$$
\begin{aligned}
Q &= C_{\rho_0/2}(0) \times \Big(-\frac{T}{2}, T \Big), \\
q &= \frac{n+2}{n}, \quad q^* = \min\Big\{ \frac{q_1(n+4)}{2n}, \frac{q_2(n+2)}{2n} \Big\}, \\
\phi &= \|D\overline{U}\|^{2n/(n+2)}, \quad \psi = (1 + |\overline{F}|^{2(n+2)/(n+4)} + \|\overline{G}\|^2)^{n/(n+2)}.
\end{aligned}
$$

If $Q_{12r} = C_{12r}(y_0) \times (t_0 - 144r^2, t_0)$ with $\overline{Q}_{12r} \subset Q$ then

$$
y_0 \in C_{\rho_0/2}(0), \quad t_0 \in \Big(-\frac{T}{2}, T \Big), \quad 0 < r < \min\Big\{ \frac{1}{12}\sqrt{\frac{T}{2}}, \frac{\rho_0}{24} \Big\},
$$

and (7.6) takes the form $(*)$:

$$
\fint_{Q_r} \phi^q \mathrm{d}y\mathrm{d}t \le c\Big(\fint_{Q_{12r}} \phi \mathrm{d}y\mathrm{d}t \Big)^q + \frac{1}{2} \fint_{Q_{12r}} \phi^q \mathrm{d}y\mathrm{d}t + c \fint_{Q_{12r}} \psi^q \mathrm{d}y\mathrm{d}t.
$$

The above higher integrability result implies the existence of a $q < \tilde{p} < q^*$ such that

$$
\phi \in L^{\tilde{p}}_{\mathrm{loc}}\Big(C_{\rho_0/2}(0) \times \Big(-\frac{T}{2}, T \Big) \Big),
$$

$$
\fint_{Q_r} \phi^{\tilde{p}} \mathrm{d}y\mathrm{d}t \le c\Big\{ \Big(\fint_{Q_{12r}} \phi^q \mathrm{d}y\mathrm{d}t \Big)^{\tilde{p}/q} + \fint_{Q_{12r}} \psi^{q^*} \mathrm{d}y\mathrm{d}t \Big\} \tag{7.7}
$$

for all Q_r with $\overline{Q}_{12r} \subset C_{\rho_0/2}(0) \times \Big(-\frac{T}{2}, T \Big)$.

Define

$$
p = \frac{2n\tilde{p}}{n+2}.
$$

Then $p > 2$, and (7.7) gives

$$\int\limits_{Q_r^+} \|DU\|^p \mathrm{d}y\mathrm{d}t \le \int\limits_{Q_r} \|D\overline{U}\|^p \mathrm{d}y\mathrm{d}t \le$$

$$\le c\Big\{ r^{(n+2)/(1-p/2)} \Big(\int\limits_{Q_{12r}^+} \|DU\|^2 \mathrm{d}y\mathrm{d}t \Big)^{p/2} + \int\limits_{Q_{12r}^+} (1+|F|^{q_1}+\|G\|^{q_2}) \mathrm{d}y\mathrm{d}t \Big\} \tag{7.8}$$

for all Q_r with $\overline{Q}_{12r} \subset C_{\rho_0/2}(0) \times \left(-\frac{T}{2}, T\right)$.

Fix $y_0 = 0$ and $\rho_1 = \min\left\{\frac{1}{13}\sqrt{\frac{T}{2}}, \frac{\rho_0}{25}\right\}$. Then from (7.8) we obtain, for any $t_0 \in (0,T)$,

$$\int\limits_{C_{\rho_1}^+(0)\times(t_0-\rho_1^2,t_0)} \|DU\|^p \mathrm{d}y\mathrm{d}t \le$$

$$\le K_0 \Big\{ \Big(\int\limits_{C_{12\rho_1}^+(0)\times(0,T)} \|DU\|^2 \mathrm{d}y\mathrm{d}t \Big)^{p/2} \tag{7.9}$$

$$+ \int\limits_{C_{12\rho_1}^+(0)\times(0,T)} (1+|F|^{q_1}+\|G\|^{q_2}) \mathrm{d}y\mathrm{d}t \Big\},$$

where the constant K_0[2)] depends on $\frac{1}{\rho_1}$ $\left(\text{and thus on } \frac{1}{\rho_0}\right)$. Next, we choose any $t_1 \in (T-\rho_1^2, T)$. Then the interval $(0, t_1]$ can be covered by a *finite number of intervals* $(t_0-\rho_1^2, t_0)$ $(t_0 \in (0,T))$ *independently of the position of* t_1. Summing the corresponding finitely many inequalities (7.9) and letting tend $t_1 \to T$ gives by the aid of the monotone convergence theorem

$$\int\limits_{C_{\rho_1}^+(0)\times(0,T)} \|DU\|^p \mathrm{d}y\mathrm{d}t \le$$

$$\le K_1 \Big\{ \Big(\int\limits_{C_{\rho_0}^+(0)\times(0,T)} \|DU\|^2 \mathrm{d}y\mathrm{d}t \Big)^{p/2} \tag{7.10}$$

$$+ \int\limits_{C_{\rho_0}^+(0)\times(0,T)} (1+|F|^{q_1}+\|G\|^{q_2}) \mathrm{d}y\mathrm{d}t \Big\};$$

Here $K_1 = \text{const}$ depends on $\frac{1}{\rho_0}$.

Remark. The preceding arguments remain true if $x_0 \in \Gamma_0$ or $x_0 \in \Gamma_1$, resp. Thus, (7.10) holds for any $x_0 \in \partial\Omega$ (with ρ_0 and K_1 depending on x_0). ■

[2)] Notice that $\mathrm{meas}(C_{12\rho_1}^+(0)\times(-\frac{T}{2},T)) < 2\mathrm{meas}(C_{12\rho_1}^+(0)\times(0,T))$.

7.5 *Passage from U to u.* To begin with, we note that the construction of $\mathcal{T}$ implies the existence of an $0 < r_1 < r_0$ such that

$$\Omega \cap B_{r_1} \subset \mathcal{T}^{-1}(C^+_{\rho_1}(0)) \tag{7.11}$$

(cf. (4.2); $B_{r_1} = B_{r_1}(x_0)$, $x_0 \in \Gamma_0$, $x_0 \in \overline{\Gamma}_0 \cap \Gamma_1$ or $x_0 \in \Gamma_1$, resp.). Observing (4.1) and (7.11) we obtain

$$\int\limits_{\Omega\cap B_{r_1}} \|Du(x,t)\|^p \mathrm{d}x \leq$$

$$\leq \operatorname*{ess\,sup}_{\xi\in\Omega\cap B_{r_1}} \sum_{\alpha,\beta=1}^{n} \left|\frac{(\mathcal{T}\xi)_\alpha}{\partial \xi_\beta}\right|^p \int\limits_{\mathcal{T}^{-1}(C^+_{\rho_1}(0))} \|DU(\mathcal{T}x,t)\|^p \mathrm{d}x \text{ [3]}$$

$$= \operatorname*{ess\,sup}_{\xi\in\Omega\cap B_{r_1}} \sum_{\alpha,\beta=1}^{n} \left|\frac{(\mathcal{T}\xi)_\alpha}{\partial \xi_\beta}\right|^p \int\limits_{C^+_{\rho_1}(0)} \|DU(y,t)\|^p \mathrm{d}y \text{ [4]}$$

for a.a. $t \in (0,T)$. On the other hand, again observing (4.1) and (4.2) it follows that

$$\int\limits_{C^+_{\rho_0}(0)} \|DU(y,t)\|^2 \mathrm{d}y \leq$$

$$\leq \operatorname*{ess\,sup}_{z\in C^+_{\rho_0}(0)} \sum_{\alpha,\beta=1}^{n} \left|\frac{(\mathcal{T}^{-1}z)_\alpha}{\partial z_\beta}\right|^2 \int\limits_{\mathcal{T}(\Omega\cap B_{r_0})} \|Du(\mathcal{T}^{-1}y,t)\|^2 \mathrm{d}y$$

$$= \operatorname*{ess\,sup}_{z\in C^+_{\rho_0}(0)} \sum_{\alpha,\beta=1}^{n} \left|\frac{(\mathcal{T}^{-1}z)_\alpha}{\partial z_\beta}\right|^2 \int\limits_{\Omega\cap B_{r_0}} \|Du(x,t)\|^2 \mathrm{d}x$$

for a.a. $t \in (0,T)$.

We return from F and G to f resp. g by an analogous reasoning. Combining these estimates with (7.10) gives

$$\int\limits_{(\Omega\cap B_{r_1})\times(0,T)} \|Du\|^p \mathrm{d}x\mathrm{d}t \leq$$

$$\leq K_2\Bigg\{\bigg(\int\limits_{Q} \|Du\|^2 \mathrm{d}x\mathrm{d}t\bigg)^{p/2} + \int\limits_{Q}(1 + |f|^{q_1} + \|g\|^{q_2})\mathrm{d}x\mathrm{d}t\Bigg\}, \tag{7.12}$$

where both r_1 and K_2 depend on $x_0 \in \partial\Omega$ via $\mathcal{T}$. ■

[3] Recall $\hat{u}(x,t) = U(\mathcal{T}x,t)$ for a.a. $(x,t) \in (\Omega \cap B_{r_0}) \times (-T,T)$.

[4] Notice that $|\det(\mathcal{T}^{-1})'(y)| = 1$ for a.a. $y \in C^+_{\rho_1}(0)$ (cf. (4.1)).

Given any $x_0 \in \partial\Omega$, let $r_1 = r_1(x_0) > 0$ be according to (7.11). Clearly, $\partial\Omega \subset \bigcup_{x_0\in\partial\Omega} B_{r_1}(x_0)$. Thus, by compactness, there exist $x_{0,k} \in \partial\Omega$ and $r_{1,k} = r_{1,k}(x_{0,k}) > 0$ $(k = 1, \dots, m)$ such that

$$\partial\Omega \subset \bigcup_{k=1}^{m} B_{r_{1,k}}(x_{0,k}).$$

Next, define

$$\Omega_\eta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \eta\}$$

($\eta > 0$ sufficiently small). Then $\overline{\Omega}_\eta \subset \Omega$, and it is readily seen that there exists an $\eta_0 > 0$ such that

$$\Omega = \overline{\Omega}_{\eta_0} \cup \left(\bigcup_{k=1}^{m} (\Omega \cap B_{r_{1,k}}(x_{0,k})) \right).$$

Taking $x_0 = x_{0,k}$ and $r_1 = r_{1,k}$ in (7.12) and summing on $k = 1, \dots, m$ gives

$$\int\limits_{(\Omega\setminus\overline{\Omega}_{\eta_0})\times(0,T)} \|Du\|^p \mathrm{d}x\mathrm{d}t \leq$$

$$\leq K_3 \left\{ \left(\int\limits_Q \|Du\|^2 \mathrm{d}x\mathrm{d}t \right)^{p/2} + \int\limits_Q (1 + |f|^{q_1} + \|g\|^{q_2}) \mathrm{d}x\mathrm{d}t \right\}, \tag{7.13}$$

where the constant K_3 depends on n, N, a_0, λ_0, q_1, q_2, $\|f\|_{L^{2(n+2)/(n+4)}(Q;\mathbb{R}^N)}$ and geometric properties of $\partial\Omega$ and $\overline{\Gamma}_0 \cap \Gamma_1$ (via the partial derivatives of $\mathcal{T}$ and $\mathcal{T}^{-1}$). ■

8 Proof of the Theorem completed

We fix an $0 < r_2 < \frac{1}{4}\sqrt{\frac{T}{2}}$ such that

$$\overline{B_{4r_2}(x_0)} \subset \Omega \quad \forall\, x_0 \in \overline{\Omega_{\eta_0}}. \tag{8.1}$$

It is easily seen that the arguments in [8] continue to hold with the above introduced functions $\hat{a}_i^\alpha$, $\hat{f}^i$, $\hat{g}^i$ and $\hat{u}^i$ (on $\Omega \times (-T, T)$) in place of a_i^α, f^i, g_i^α and u^i in [8]. Then from [8] we get the existence of a $p > 2$ such that

$$\|Du\| \in L^p(\Omega_{\eta_0} \times (0, T)),$$

$$\int\limits_{\Omega_{\eta_0}\times(0,T)} \|Du\|^p \mathrm{d}x\mathrm{d}t \leq$$

$$\leq K_4 \left\{ \left(\int\limits_Q \|Du\|^2 \mathrm{d}x\mathrm{d}t \right)^{p/2} + \int\limits_Q (1 + |f|^{q_1} + \|g\|^{q_2}) \mathrm{d}x\mathrm{d}t \right\}, \tag{8.2}$$

where the constant K_4 depends on n, N, a_0, λ_0, q_1, q_2, $\|f\|_{L^{2(n+2)/(n+4)}(Q;\mathbb{R}^N)}$ and $\frac{1}{r_2}$ (cf. (8.1)).

Finally, combining (7.12) and (8.2) gives (1.11). ■

Appendix

Let $x_0 \in \mathbb{R}^n$. Throughout we assume

$$x_{0n} > 0.$$

Define

$$\begin{aligned} C_r(x_0) &= \{x \in \mathbb{R}^n : |x_i - x_{0i}| < r \ (i = 1, \dots, n)\}, \\ C_r^+(x_0) &= \{x \in C_r(x_0) : x_n > 0\}. \end{aligned}$$

1. *Extension by reflection.* Assume $x_{0n} < r$. Let v be defined on $C_r^+(x_0)$. We extend v onto $C_r(x_0) \setminus C_r^+(x_0)$ by reflection with respect to $\{x \mid x_n = 0\}$:

$$\bar{v}(x) = \begin{cases} v(x', x_n) & \text{if} \quad x \in C_r(x_0), \ x_n \geq 0, {}^{5)} \\ v(x', -x_n) & \text{if} \quad x \in C_r(x_0), \ x_n < 0. \end{cases}$$

We have:

$$\begin{aligned} & v \in L^p(C_r^+(x_0)) \\ & \Longrightarrow \bar{v} \in L^p(C_r(x_0)), \quad \int\limits_{C_r(x_0)} |\bar{v}|^p \mathrm{d}x \leq 2 \int\limits_{C_r^+(x_0)} |v|^p \mathrm{d}x; \\ & v \in W_p^1(C_r^+(x_0)) \\ & \Longrightarrow \bar{v} \in W_p^1(C_r(x_0)), \quad \int\limits_{C_r(x_0)} |D\bar{v}|^p \mathrm{d}x \leq 2 \int\limits_{C_r^+(x_0)} |Dv|^p \mathrm{d}x \end{aligned}$$

$(1 \leq p < +\infty)$. ■

Let $\zeta_r \in C^\infty(\mathbb{R}^n)$ be a cut-off function for $C_r(x_0)$:

$$\begin{aligned} & \zeta_r \equiv 1 \ \text{ on } \ C_{r/2}(x_0), \ \ \zeta_r \equiv 0 \ \text{ in } \ \mathbb{R}^n \setminus C_r(x_0), \\ & 0 \leq \zeta_r \leq 1, \ \ |D\zeta_r| \leq \frac{c_0}{r} \quad \text{in } \ \mathbb{R}^n \\ & (c_0 = \text{const} > 0 \ \text{ independent of } \ r). \end{aligned}$$

In what follows, set $C_r \equiv C_r(x_0)$, $C_r^+ \equiv C_r^+(x_0)$. Define

$$\begin{aligned} \tilde{v}_{C_r} &= \Big(\int\limits_{C_r} \zeta_r^2 \mathrm{d}z \Big)^{-1} \int\limits_{C_r} v \zeta_r^2 \mathrm{d}x, \\ \tilde{v}_{C_r^+} &= \Big(\int\limits_{C_r^+} \zeta_r^2 \mathrm{d}z \Big)^{-1} \int\limits_{C_r^+} v \zeta_r^2 \mathrm{d}x. \end{aligned}$$

5) Recall $x = \{x', x_n\}$, $x' = \{x_1, \dots, x_{n-1}\}$.

PROPOSITION 1 *There holds:*

$$\text{(A1)}\qquad \int\limits_{C_r^+}(v-\tilde v_{C_r^+})^2\mathrm{d}x \le 2(1+2^{n+3})\int\limits_{C_{2r}}(\bar v-\tilde{\bar v}_{C_{2r}})^2\mathrm{d}x \quad \forall\, v\in L^2(C_{2r}^+);$$

$$\text{(A2)}\qquad \begin{cases} \displaystyle\int\limits_{C_r^+}(v-\tilde v_{C_r^+})^2\mathrm{d}x \le \gamma_0 r^2\int\limits_{C_{2r}^+}|Dv|^2\mathrm{d}x \quad \forall\, v\in W_2^1(C_{2r}^+)\\ (\gamma_0=\text{const}>0 \ \textit{independent of } r); \end{cases}$$

$$\text{(A3)}\qquad \begin{cases} \displaystyle\int\limits_{t_0-r^2}^{t_0}\int\limits_{C_r^+}|w|^{2(n+2)/n}\mathrm{d}x\mathrm{d}t \le \\ \displaystyle\le \gamma_1\Bigg(\operatorname*{ess\,sup}_{(t_0-r^2,t_0)}\int\limits_{C_r^+}w^2\mathrm{d}x+\int\limits_{t_0-r^2}^{t_0}\int\limits_{C_r^+}|Dw|^2\mathrm{d}x\mathrm{d}t\Bigg)^{(n+2)/n}\\ \forall\, w\in V_2^{1,0}(C_r^+\times(t_0-r^2,t_0)) \quad (\gamma_1=\text{const}>0\\ \ \textit{independent of } r). \end{cases}$$

Proof. – 1. Obviously, meas $C_r^+\le$ meas $C_r=2^n r^n$. Thus

$$\int\limits_{C_r^+}\zeta_r^2\mathrm{d}x \ge \text{meas}\, C_{r/2}^+ \ge \frac{1}{2}r^n \ge \frac{1}{2^{n+1}}\text{meas}\, C_r^+.$$

We obtain

$$\begin{aligned} &\int\limits_{C_r^+}(v-\tilde v_{C_r^+})^2\mathrm{d}x \le \\ &\le 2\int\limits_{C_r^+}(v-\tilde v_{C_{2r}^+})^2\mathrm{d}x + 2(\text{meas}\, C_r^+)(\tilde v_{C_{2r}^+}-\tilde v_{C_r^+})^2\\ &\le 2(1+2^{n+2})\int\limits_{C_r^+}(v-\tilde v_{C_{2r}^+})^2\mathrm{d}x + 2^{n+3}\int\limits_{C_r^+}(v-\tilde v_{C_r^+})^2\zeta_r^2\mathrm{d}x. \end{aligned}$$

The real function

$$\lambda\mapsto\int\limits_{C_r^+}(v-\lambda)^2\zeta_r^2\mathrm{d}x \quad (\lambda\in\mathbb{R})$$

attains its absolute minimum over $\mathbb{R}$ at the value $\lambda_0=\tilde v_{C_r^+}$. It follows that

$$\begin{aligned} \int\limits_{C_r^+}(v-\tilde v_{C_r^+})^2\mathrm{d}x \ &\le\ 2(1+2^{n+3})\int\limits_{C_r^+}(v-\tilde v_{C_{2r}^+})^2\mathrm{d}x\\ &\le\ 2(1+2^{n+3})\int\limits_{C_{2r}^+}(v-\tilde{\tilde v}_{C_{2r}^+})^2\zeta_{2r}^2\mathrm{d}x\\ &\le\ 2(1+2^{n+3})\int\limits_{C_{2r}}(\bar v-\tilde{\bar v}_{C_{2r}})^2\mathrm{d}x. \end{aligned}$$

2. The POINCARÉ inequality with weighted mean reads

$$\int_{C_r} (w - \tilde{w}_{C_r})^2 \mathrm{d}x \le \sigma_0 r^2 \int_{C_r} |Dw|^2 \mathrm{d}x \quad \forall\, w \in W_2^1(C_r)$$

($\sigma_0 = \text{const} > 0$ independent of r; cf (7.1)). Then (A2) follows from (A1) ($\gamma_0 = 16(1 + 2^{n+3})\sigma_0$).

3. Let $z \in W_2^1(C_r(x_0))$. The well-known GAGLIARDO-NIRENBERG inequality combined with a homothetical argument gives

$$\int_{C_r(x_0)} |z|^q \mathrm{d}x \le$$

$$\le c\Big(\int_{C_r(x_0)} z^2 \mathrm{d}x\Big)^{(1-\alpha)q/2} \Big(\int_{C_r(x_0)} |\nabla z|^2 \mathrm{d}x\Big)^{\alpha q/2} + \frac{c}{r^2}\Big(\int_{C_r(x_0)} z^2 \mathrm{d}x\Big)^{q/2}$$

$\left(c = \text{const} > 0 \text{ independent of } r;\ q = \dfrac{2(n+2)}{n},\ \alpha = \dfrac{2}{q} = \dfrac{n}{n+2}\right)$.

Given any $w \in V_2^{1,0}(C_r^+(x_0) \times (t_0 - r^2, t_0))$ let $\bar{w}(\cdot, t)$ denote the extension of $w(\cdot, t)$ from $C_r^+(x_0)$ onto $C_r(x_0) \setminus C_r^+(x_0)$ by reflection with respect to $\{x \mid x_n = 0\}$ (for a.a. $t \in (t_0 - r^2, t_0)$). We obtain

$$\int_{C_r^+(x_0)} |w(x,t)|^{2(n+2)/n} \mathrm{d}x \le \int_{C_r(x_0)} |\bar{w}(x,t)|^{2(n+2)/n} \mathrm{d}x \le$$

$$\le c\Big\{ \operatorname*{ess\,sup}_{(t_0 - r^2, t_0)} \Big(\int_{C_r(x_0)} |\bar{w}(x,s)|^2 \mathrm{d}x\Big)^{2/n}\Big\} \int_{C_r(x_0)} |\nabla \bar{w}(x,t)|^2 \mathrm{d}x$$

$$+ \frac{c}{r^2} \operatorname*{ess\,sup}_{(t_0 - r^2, t_0)} \Big(\int_{C_r(x_0)} |\bar{w}(x,s)|^2 \mathrm{d}x\Big)^{(n+2)/n}$$

$$\le 2^{(n+2)/n} c \Big\{ \operatorname*{ess\,sup}_{(t_0 - r^2, t_0)} \Big(\int_{C_r^+(x_0)} |w(x,s)|^2 \mathrm{d}x\Big)^{2/n}\Big\} \int_{C_r^+(x_0)} |\nabla w(x,t)|^2 \mathrm{d}x$$

$$+ 2^{(n+2)/n} c r^{-2} \operatorname*{ess\,sup}_{(t_0 - r^2, t_0)} \Big(\int_{C_r^+(x_0)} |w(x,s)|^2 \mathrm{d}x\Big)^{(n+2)/n}$$

(for a.a. $t \in (t_0 - r^2, t_0)$). Integrating this inequality over the interval $(t_0 - r^2, t_0)$ and employing Young's inequality gives (A3). ■

2. *A* SOBOLEV *inequality over* $C_r^+(x_0)$.

PROPOSITION 2 *Assume*

(A4) $\qquad C_r(x_0) \cap \{x \in I\!R^n : x_1 \le 0,\, x_n = 0\} \ne \emptyset.$

Then:

$$\text{(A5)} \qquad \Bigg(\int\limits_{C_r^+(x_0)} |v|^s \mathrm{d}x \Bigg)^{1/s} \le \gamma_2 r^{1+n/s-n/p} \Bigg(\int\limits_{C_{3r}^+(x_0)} |Dv|^p \mathrm{d}x \Bigg)^{1/p}$$

for all $v \in W_p^1(C_{3r}^+(x_0))$ *such that*

$$v = 0 \quad \text{a.e. on} \;\; \partial C_{3r}^+(x_0) \cap \{x \in I\!R^n : x_1 \le 0,\, x_n = 0\},$$

where:

$$1 \le s \le \frac{np}{n-p} \quad \text{if} \;\; 1 \le p < n,$$
$$1 \le s < +\infty \qquad \text{if} \;\; p = n$$

($\gamma_2 = \text{const} > 0$ *independent of* r).

Proof. – Firstly, by SOBOLEV's imbedding theorem,

$$\text{(A6)} \qquad \Bigg(\int\limits_{C_1^+(0)} |\varphi|^s \mathrm{d}z \Bigg)^{1/s} \le \sigma_1 \Bigg(\int\limits_{C_1^+(0)} |D\varphi|^p \mathrm{d}z \Bigg)^{1/p}$$

for all $\varphi \in W_p^1(C_1^+(0))$, $\varphi = 0$ a.e. on $\partial C_1^+(0) \cap \{z \in I\!R^n : z_1 \le 0,\, z_n = 0\}$ ($\sigma_1 = \text{const} > 0$).

From (A4) we infer: $x_{0n} < r$.

1) Let $x_{01} \le 0$. Obviously,

$$C_r^+(x_0) \subset C_{2r}^+(\{x_0', 0\}) \subset C_{3r}^+(x_0)\ ^{6)}.$$

We introduce new independent variables by

$$z_i = \frac{x_i - x_{0i}}{2r} \quad (i = 1, \ldots, n-1), \quad z_n = \frac{x_n}{2r}.$$

Then

$$x \in C_{2r}^+(\{x_0', 0\}) \Longleftrightarrow z \in C_1^+(0),$$
$$x_1 \le x_{01} \Longleftrightarrow z_1 \le 0, \quad x_n = 0 \Longleftrightarrow z_n = 0.$$

Next, we define a function φ by

$$\varphi(z) = v(x) \quad \text{for a.a.} \;\; z \in C_1^+(0).$$

Clearly, $\varphi \in W_p^1(C_1^+(0))$. From $v = 0$ a.e. on

[6] $x_0' = \{x_{01}, \ldots, x_{0,n-1}\}$.

$$\partial C^+_{2r}(\{x'_0, 0\}) \cap \{x \in I\!R^n : x_1 \le x_{01},\, x_n = 0\}$$

(for $x_{01} \le 0$) it follows that $\varphi = 0$ a.e. on

$$\partial C^+_1(0) \cap \{z \in I\!R^n : z_1 \le 0,\, z_n = 0\}.$$

Thus, by (A6),

$$\int_{C^+_r(x_0)} |v|^s \mathrm{d}x \;\le\; (2r)^n \int_{C^+_1(0)} |\varphi|^s \mathrm{d}z \le$$

$$\le\; \sigma_1^s (2r)^n \Bigg(\int_{C^+_1(0)} |D\varphi|^p \mathrm{d}z \Bigg)^{s/p}$$

$$\le\; \sigma_1^s (2r)^{n+s-ns/p} \Bigg(\int_{C_{3r}(x_0)} |Dv|^p \mathrm{d}x \Bigg)^{s/p}.$$

Whence (A5) ($\gamma_2 = 2^{1+n/s-n/p}\sigma_1$).

2) Let $x_{01} > 0$. (A4) implies $x_{01} < r$. Therefore,

$$C^+_r(x_0) \subset C^+_{2r}(\{0, x''_0, 0\}) \subset C^+_{3r}(x_0).^{7)}$$

In the present case we introduce new independent variables by

$$z_1 = \frac{x_1}{2r}, \quad z_i = \frac{x_i - x_{0i}}{2r} \ (i = 2, \ldots, n-1), \quad z_n = \frac{x_n}{2r}.$$

Again defining

$$\varphi(z) = v(x) \quad \text{for a.a.} \;\; z \in C^+_1(0),$$

we get $\varphi = 0$ a.e. on

$$\partial C^+_1(0) \cap \{z \in I\!R^n : z_1 \le 0,\, z_n = 0\}.$$

By an analogous argument as above we obtain (A5). ■

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7) $x''_0 = \{x_{02}, \ldots, x_{0,n-1}\}$.

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