Solvability Properties of Linear Elliptic Boundary Value Problems with Non-smooth Data

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In this paper linear elliptic boundary value problems of second order with non-smooth data (L^∞-coefficients, Lipschitz domain, mixed boundary conditions) are considered. It is shown that the weak solutions are Hölder continuous and that they depend smoothly – in the sense of Hölder spaces – on the coefficients of the equation.

1 Introduction

In this paper we consider boundary value problems for linear elliptic equations of the type

$$\left.\begin{array}{rcl} \sum_{i,j=1}^N [-\partial_j(a_{ij}\partial_i u + b_j u) + c_j \partial_j u + du] &=& -\sum_{j=1}^N \partial_j f_j + g \quad \text{in } \Omega \\ \sum_{i,j=1}^N (a_{ij}\partial_i u + b_j u)\nu_j &=& \sum_{j=1}^N f_j \nu_j \quad \text{on } \Gamma \\ u &=& 0 \quad \text{on } \partial\Omega \setminus \Gamma. \end{array}\right\} \tag{1.1}$$

In (1.1) Ω is a bounded Lipschitz domain in $\mathbb{R}^N$, and Γ is a subset of the boundary $\partial\Omega$ of Ω. By ∂_j we denote the partial derivative with respect to the j-th component of the space variable $(x_1, \ldots, x_N) \in \Omega$, and $(\nu_1, \ldots, \nu_N) : \partial\Omega \to \mathbb{R}^N$ is the unit outward normal field on $\partial\Omega$. The coefficients $a_{ij} = a_{ji}$, b_j, c_j and d are bounded measurable functions on Ω, and it is supposed that there exists an $\varepsilon > 0$ such that

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \varepsilon \sum_{j=1}^N \xi_j^2 \quad \text{for all } (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N \quad \text{and for almost all } x \in \Omega.$$

It is well-known that each weak solution u of the boundary value problem (1.1) is Hölder continuous up to the boundary if, for example,

$$f_j \in L^p(\Omega), \quad g \in L^{\frac{p}{2}}(\Omega) \quad \text{with} \quad p > N \tag{1.2}$$

and if Γ satisfies some regularity assumption (see, e.g., GILBARG, TRUDINGER [3] for the case of $\Gamma = \emptyset$, TROIANIELLO [13] for the case when Γ is open and closed in $\partial\Omega$ and STAMPACCHIA [12], MURTHY, STAMPACCHIA [7] for more general cases).

In the present paper we will prove that the weak solution u of (1.1) – if it is unique – depends smoothly in the sense of a Hölder space $C^{0,\alpha}(\bar{\Omega})$ on the coefficients $a_{ij}, b_j, c_j, d \in L^\infty(\Omega)$ (if (1.2) is satisfied and if $\Omega \cup \Gamma$ is regular in the sense of Definition 2.1 below). This result seems to be new (in case of $N > 2$) even if $\Gamma = \emptyset$ (pure Dirichlet boundary conditions) or if $\Gamma = \partial\Omega$ (pure natural boundary conditions). Moreover, this result is of some interest because it allows to apply theorems of the differential calculus (Implicit Function Theorem, Sard-Smale Theorem, Liapunov-Schmidt Procedure in bifurcation problems) to quasilinear elliptic boundary value problems with non-smooth data (cf. RECKE [9] and [10]).

In the case of $N = 2$ the smooth (in the sense of $C^{0,\alpha}(\bar{\Omega})$) dependence of the weak solution of (1.1) on the coefficients follows from the work of Gröger [4]. Moreover, in the case of $N = 2$ this result holds true for boundary value problems for linear elliptic systems as well, whereas in case of $N > 2$ there are examples of linear elliptic systems with bounded measurable coefficients which have unbounded weak solutions (cf., e.g., GIAQUINTA [1]).

In this work, we will show that the weak solutions of mixed boundary value problems for "weakly coupled" linear elliptic systems depend smoothly (in the sense of $C^{0,\alpha}(\bar{\Omega})$) on the L^∞-coefficients of the equations (if the right-hand sides of the equations are of the type (1.2) and if $\Omega \cup \Gamma$ is regular in the sense of Definition 2.1 below).

We do not consider the solution regularity for mixed boundary value problems for linear elliptic equations with smooth coefficients, see PRYDE [8], SIMANCA [11] and LIEBERMANN [6] for that question.

The present paper is organized as follows.

In the remaining part of Section 1 we introduce some notation and some results about Campanato spaces.

In Section 2 we prove a regularity result for weak solutions of (1.1) in the case of $b_j = c_j = 0$, $d = 1$.

In Section 3 we introduce two scales of Banach spaces $U_\lambda = W_o^{1,2,\lambda}(\Omega \cup \Gamma)$ and V_λ $(N-2 < \lambda < N)$ such that there are continuous embeddings $U_\lambda \hookrightarrow W_o^{1,2}(\Omega \cup \Gamma) \cap C^{0,\alpha}(\bar{\Omega})$ (with $\alpha = \frac{\lambda - N + 2}{2}$) and $V_\lambda \hookrightarrow W_o^{-1,2}(\Omega \cup \Gamma)$ and such that the operator associated with the boundary value problem (1.1) (with $b_j = c_j = 0$ and $d = 1$) is an isomorphism from U_λ onto V_λ (if λ is sufficiently close to $N-2$). U_λ is the space of all elements u of the Sobolev space $W_o^{1,2}(\Omega \cup \Gamma)$ such that ∇u belongs to the Campanato space $\mathcal{L}^{2,\lambda}(\Omega; \mathbb{R}^N)$, and V_λ is the image of U_λ with respect to the duality map of the Hilbert space

$W_o^{1,2}(\Omega \cup \Gamma)$.

In Section 4 we consider the case of arbitrary coefficients b_j, c_j and d, and we show that the operator associated with (1.1) is a Fredholm operator (index zero) from U_λ into V_λ and that it depends smoothly (in the sense of the operator norm in $\mathcal{L}(U_\lambda; V_\lambda)$) on the coefficients a_{ij}, b_j, c_j and d (if λ is sufficiently close to $N-2$).

Finally, in Section 5 we show that our results about the boundary value problems for linear elliptic equations of type (1.1) hold for "weakly coupled" linear elliptic systems as well.

Let us introduce some notation.

In this paper $N \geq 2$ is a natural number. For subsets G of $I\!R^N$ we denote by $\overset{\circ}{G}$, ∂G and $\bar{G}$ the interior, the boundary, and the closure of G, respectively.

A bijective map ϕ from one subset of $I\!R^N$ onto another is called a Lipschitz transformation if ϕ and ϕ^{-1} are Lipschitzian.

For $\xi, \eta \in I\!R^N$ we write $\xi \cdot \eta$ for their Euclidean scalar product, and $|\xi| := \sqrt{\xi \cdot \xi}$ is the Euclidean norm of ξ.

By $\mathbb{S}_N$ we denote the space of all real symmetric $N \times N$-matrices. If $A = (a_{ij}) \in \mathbb{S}_N$ and $\xi = (\xi_1, \ldots, \xi_N) \in I\!R^N$, then we write $A\xi \in I\!R^N$ for the vector with components $\sum\limits_{j=1}^{N} a_{ij}\xi_j$ $(i = 1, \ldots, n)$, i.e. for the application of A on ξ, and

$$|A| := \sup\{|A\xi| : \xi \in I\!R^N,\ |\xi| \leq 1\}$$

is the Euclidean operator norm of A.

Let Ω be a bounded open subset of $I\!R^N$.

We write $L^\infty(\Omega)$, $L^\infty(\Omega, I\!R^N)$ and $L^\infty(\Omega; \mathbb{S}_N)$ for the spaces of bounded measurable maps from Ω into $I\!R$, $I\!R^N$, and $\mathbb{S}_N$, respectively. The norms of these spaces are denoted by $\|\cdot\|_\infty$, for example

$$\|A\|_\infty := \inf\{r > 0 : |A(x)| \leq r \text{ for almost all } x \in \Omega\} \text{ for } A \in L^\infty(\Omega; \mathbb{S}_N).$$

Analogously, for $1 \leq p < \infty$ we write $\|\cdot\|_p$ for the norms in $L^p(\Omega)$ and $L^p(\Omega; I\!R^N)$. The gradient of $u \in L^p(\Omega)$ and the divergence of $f \in L^p(\Omega; I\!R^N)$ (derivatives in the sense of distributions) will be denoted by ∇u and div f, respectively, and $W^{1,p}(\Omega)$ is the usual Sobolev space with the norm

$$\|u\|_{1,p} := (\|u\|_p^p + \|\nabla u\|_p^p)^{\frac{1}{p}}.$$

Finally, for $0 < \alpha < 1$ we denote by $C^{0,\alpha}(\bar{\Omega})$ the space of all functions from $\bar{\Omega}$ into $I\!R$ that are Hölder continuous with exponent α. The norm of $u \in C^{0,\alpha}(\bar{\Omega})$ is

$$\sup\{|u(x)| : x \in \bar{\Omega}\} + \sup\Big\{\frac{|u(x) - u(y)|}{|x-y|^\alpha} : x, y \in \bar{\Omega},\ x \neq y\Big\}.$$

Now, let us recall some notation and facts about Campanato spaces (cf. KUFNER, JOHN, FUČIK [5], TROIANIELLO [13], GIAQUINTA [2]).

For $1 \le p < \infty$ and $0 \le \lambda \le N+p$ we denote by $\mathcal{L}^{p,\lambda}(\Omega)$ the space of all $u \in L^p(\Omega)$ such that

$$[u]_{p,\lambda} := \Big(\sup\Big\{r^{-\lambda}\int\limits_{\Omega(x,r)} |u(y)-u_{x,r}|^p \mathrm{d}y : x\in\Omega,\, r>0\Big\}\Big)^{\frac{1}{p}} < \infty. \tag{1.3}$$

In (1.3) we have used the notations

$$\begin{array}{rcl} \Omega(x,r) & := & \{y \in \Omega : |x-y| < r\} \\ u_{x,r} & := & \dfrac{1}{|\Omega(x,r)|}\displaystyle\int\limits_{\Omega(x,r)} u(y)\mathrm{d}y, \end{array}$$

where $|\Omega(x,r)|$ is the N-dimensional Lebesgue measure of $\Omega(x,r)$. The space $\mathcal{L}^{p,\lambda}(\Omega)$ is a Banach space with the norm

$$|u|_{p,\lambda} := (\|u\|_p^p + [u]_{p,\lambda}^p)^{\frac{1}{p}}. \tag{1.4}$$

Analogously, $\mathcal{L}^{p,\lambda}(\Omega;\mathbb{R}^N)$ is the space of all $f \in L^p(\Omega;\mathbb{R}^N)$ with components from $\mathcal{L}^{p,\lambda}(\Omega)$, and the norm in $\mathcal{L}^{p,\lambda}(\Omega;\mathbb{R}^N)$ is defined similarly to (1.3), (1.4) and denoted by $|\cdot|_{p,\lambda}$, too. Finally, for the sake of simplicity, we will use the notations

$$\left.\begin{array}{rcl} \mathcal{L}^{2,\lambda}(\Omega) & := & L^2(\Omega) \\ |u|_{2,\lambda} & := & \|u\|_2 \text{ for } u \in \mathcal{L}^{2,\lambda}(\Omega) \end{array}\right\} \text{ for } \lambda < 0,$$

and, if necessary, we indicate the dependence of the norms on the domain Ω by an additional index.

Theorem 1.1 (i) *Let $p \le q$ and $\frac{N-\mu}{q} \le \frac{N-\lambda}{p}$. Then $\mathcal{L}^{q,\mu}(\Omega)$ is continuously embedded into $\mathcal{L}^{p,\lambda}(\Omega)$.*

(ii) *Let ϕ be a Lipschitz transformation from Ω onto $\tilde{\Omega}$. Then, for all $\lambda \le N$, there exists a $c > 0$ such that it holds for all $u \in \mathcal{L}^{2,\lambda}(\tilde{\Omega})$ that $u \circ \phi \in \mathcal{L}^{2,\lambda}(\Omega)$ and $\|u\circ\phi\|_{2,\lambda,\Omega} \le c\|u\|_{2,\lambda,\tilde{\Omega}}$.*

Theorem 1.2 *Let Ω have a Lipschitz boundary. Then the following is true:*

(i) *If $N < \lambda < N+2$, then $\mathcal{L}^{2,\lambda}(\Omega)$ is isomorphic to $C^{0,\alpha}(\bar{\Omega})$ with $\alpha = \frac{\lambda-N}{2}$.*

(ii) *For all $\lambda < N$ there exists a $c > 0$ such that for all $u \in \mathcal{L}^{2,\lambda}(\Omega)$ and $v \in L^\infty(\Omega)$ the product uv belongs to $\mathcal{L}^{2,\lambda}(\Omega)$ and $|uv|_{2,\lambda} \le c|u|_{2,\lambda}\|v\|_\infty$.*

(iii) *For all $\lambda < N$ there exists a $c > 0$ such that for all $u \in W^{1,2}(\Omega)$ with $\nabla u \in \mathcal{L}^{2,\lambda}(\Omega;\mathbb{R}^N)$ it holds that $u \in \mathcal{L}^{2,\lambda+2}(\Omega)$ and $|u|_{2,\lambda+2} \le c(\|u\|_2 + |\nabla u|_{2,\lambda})$.*

2 Regularity of the Solutions

In this section we consider mixed boundary value problems of the type

$$\left.\begin{array}{rcll} -\mathrm{div}\, A\nabla u + u & = & -\mathrm{div}\, f + g & \text{in } \Omega \\ A\nabla u\cdot\nu & = & f\cdot\nu & \text{on } \Gamma \\ u & = & 0 & \text{on } \partial\Omega\setminus\Gamma. \end{array}\right\} \tag{2.1}$$

In (2.1) $\Omega \subset I\!R^N$ is a bounded Lipschitz domain, Γ is a part of $\partial\Omega$, $\nu : \partial\Omega \to I\!R^N$ is the unit outward normal field on $\partial\Omega$, $f : \Omega \to I\!R^N$ and $g : \Omega \to I\!R$ are given right-hand sides, and $A \in L^\infty(\Omega; \mathcal{S}_N)$ is a given matrix valued function which is positive-definite almost everywhere.

In order to prove regularity properties for the weak solutions u of (2.1), one has to impose appropriate conditions on Γ. Following the work of Gröger [4] we will formulate such conditions in terms of the set

$$G := \Omega \cup \Gamma. \tag{2.2}$$

Definition 2.1 A bounded subset G of $I\!R^N$ will be called regular if, for every $x \in \partial G$, there exist subsets U and $\tilde{U}$ of $I\!R^N$ and a Lipschitz transformation ϕ from U onto $\tilde{U}$ such that U is an open neighbourhood of x in $I\!R^N$ and $\phi(G \cap U)$ is one of the following sets:

$$\begin{aligned} E_1 &:= \{x \in I\!R^N : |x| < 1,\ x_N < 0\} \\ E_2 &:= \{x \in I\!R^N : |x| < 1,\ x_N \le 0\}. \end{aligned} \tag{2.3}$$

Remark 2.2 The Definition 2.1 does not change if one supplies, e.g., the sets

$$\begin{aligned} E_3 &:= \{x \in E_2 : x_N < 0 \ \text{ or } \ x_1 > 0\} \\ E_4 &:= \{x \in E_2 : x_1 > 0\}, \end{aligned}$$

to the list (2.3), because there exist Lipschitz transformations from E_2 onto E_3 and from E_2 onto E_4, respectively. Thus, the regularity of G means, roughly speaking, that G is bounded, ∂G is a Lipschitzian hypersurface in $I\!R^N$, and $\Gamma := G \setminus \overset{\circ}{G}$ (cf. (2.2)) and $\partial \overset{\circ}{G} \setminus \Gamma = \bar{G} \setminus G$ are separated by a Lipschitzian hypersurface of ∂G.

Definition 2.3 (i) Let $G \subset I\!R^N$ be regular. For $1 \le p < \infty$ we denote by $W^{1,p}_o(G)$ the closure in $W^{1,p}(\overset{\circ}{G})$ of the set of the restrictions on $\overset{\circ}{G}$ of all smooth functions $u : I\!R^N \to I\!R$ with compact support supp u and such that supp $u \cap (\bar{G} \setminus G) = \emptyset$. The space $W^{1,2}_o(G)$ is equipped with the norm $\|\cdot\|_{1,2}$ of $W^{1,2}(\overset{\circ}{G})$.

(ii) Let $\Omega \subset I\!R^N$ and $\Gamma \subseteq \partial\Omega$ be such that $\Omega \cup \Gamma$ is regular. A function $u \in W^{1,2}_o(\Omega \cup \Gamma)$ is called a weak solution of the boundary value problem (2.1) if

$$\int_\Omega (A\nabla u \cdot \nabla v + uv)\mathrm{d}x = \int_\Omega (f \cdot \nabla v + gv)\mathrm{d}x \quad \text{for all } \ v \in W^{1,2}_o(G).$$

(iii) For $G \subset I\!R^N$ and $\varepsilon > 0$ we denote by $\mathcal{A}_\varepsilon(G)$ the set of all $A \in L^\infty(\overset{\circ}{G}; \mathcal{S}_N)$ such that

$$\frac{1}{\varepsilon}|\xi|^2 > A(x)\xi \cdot \xi > \varepsilon|\xi|^2 \ \text{ for all } \ \xi \in I\!R^N \setminus \{0\} \ \text{ and for almost all } \ x \in G.$$

(iv) A regular set $G \subset I\!R^N$ is called admissible if, for each $\varepsilon > 0$, there exists a $\lambda_\varepsilon > N-2$ such that for all $\lambda \leq \lambda_\varepsilon$, $A \in \mathcal{A}_\varepsilon(G)$, $f \in \mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$, $g \in \mathcal{L}^{2,\lambda-2}(\overset{\circ}{G})$ and $u \in W_o^{1,2}(G)$ with

$$\int_\Omega (A\nabla u \cdot \nabla v + uv)\mathrm{d}x = \int_\Omega (f \cdot \nabla v + gv)\mathrm{d}x \quad \text{for all } v \in W_o^{1,2}(G) \tag{2.4}$$

it holds that $\nabla u \in \mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$ and

$$|\nabla u|_{2,\lambda} \leq c(|f|_{2,\lambda} + |g|_{2,\lambda-2} + \|u\|_{1,2}), \tag{2.5}$$

where the constant c in (2.5) does not depend on A, f, g and u, but only on ε and λ.

Remark 2.4 Let $G \subset I\!R^N$ be regular.

The Lax-Milgram Lemma yields that for all $A \in \mathcal{A}_\varepsilon(G)$, $f \in L^2(\overset{\circ}{G}; I\!R^N)$ and $g \in L^2(\overset{\circ}{G})$ there exists exactly one weak solution $u \in W_o^{1,2}(G)$ of the boundary value problem (2.1) with $\Omega := \overset{\circ}{G}$ and $\Gamma := G \backslash \overset{\circ}{G}$, and the linear map

$$(f, g) \in L^2(\overset{\circ}{G}; I\!R^N) \times L^2(\overset{\circ}{G}) \mapsto u \in W_o^{1,2}(G)$$

is continuous.

Hence, G is admissible iff for each $\varepsilon > 0$ there exists a $\lambda_\varepsilon > N-2$ such that for all $\lambda \leq \lambda_\varepsilon$, $A \in \mathcal{A}_\varepsilon(G)$, $f \in \mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$ and $g \in \mathcal{L}^{2,\lambda-2}(\overset{\circ}{G})$ the gradient ∇u of the weak solution u of (2.1) (with $\Omega := \overset{\circ}{G}$ and $\Gamma := \bar{G} \setminus G$) belongs to $\mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$ and that the map

$$(f, g) \in \mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N) \times \mathcal{L}^{2,\lambda-2}(\overset{\circ}{G}) \mapsto (u, \nabla u) \in W_o^{1,2}(G) \times \mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$$

is continuous.

Therefore, Theorem 1.2(i) and (iii) imply the following: If G is admissible, then, for each $\varepsilon > 0$, there exists a $\lambda_\varepsilon > N-2$ such that for all $\lambda \in (N-2, \lambda_\varepsilon]$, $A \in \mathcal{A}_\varepsilon(G)$, $f \in \mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$ and $g \in \mathcal{L}^{2,\lambda-2}(\overset{\circ}{G})$ the weak solution u of (2.1) (with $\Omega := \overset{\circ}{G}$ and $\Gamma := G \backslash \overset{\circ}{G}$) belongs to $C^{0,\alpha}(\bar{G})$ with $\alpha = \frac{\lambda - N + 2}{2}$, and the map

$$(f, g) \in \mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N) \times \mathcal{L}^{2,\lambda-2}(\overset{\circ}{G}) \mapsto u \in C^{0,\alpha}(\bar{G})$$

is continuous.

In the remaining part of this section we will prove three lemmas which will lead to the

Theorem 2.5 *Each regular set $G \subset I\!R^N$ is admissible.*

A crucial point for the proof of Theorem 2.5 is the following regularity result for weak solutions of the boundary value problem (2.1) (Troianiello [13] Theorem 2.19).

Theorem 2.6 *If $\Omega \subset I\!R^N$ is a bounded domain with C^1-boundary and if $\Gamma \subseteq \partial\Omega$ is open and closed in $\partial\Omega$, then $\Omega \cup \Gamma$ is admissible.*

Remark 2.7 (i) In fact, we will use the assertion of Theorem 2.6 in the special case of $\Omega = \{x \in I\!R^N : |x| < 1\}$ and $\Gamma = \emptyset$, only.

(ii) It is obvious that $\Omega \cup \Gamma$ is regular if $\partial\Omega$ is C^1-smooth and if Γ is open and closed in $\partial\Omega$. Hence, in fact Theorem 2.6 asserts that (2.5) follows from (2.4) if $\partial\Omega$ is C^1-smooth and if Γ is open and closed in $\partial\Omega$. That is exactly the formulation of Theorem 2.19 of Troianiello [13].

Let us begin the sequence of the three lemmas with

Lemma 2.8 *E_1 and E_2 (cf. (2.3)) are admissible.*

Proof. Let $E_o := \{x \in I\!R^N : |x| < 1\}$. For $k = 1, 2$ and $u \in L^2(\overset{\circ}{E}_k)$ we define $S_k u \in L^2(E_o)$ by

$$(S_k u)(x) := \begin{cases} u(x) & \text{for } \ x \in E_k \\ (-1)^k u(x', -x_N) & \text{for } \ x = (x', x_N) \in E_o \setminus E_k. \end{cases}$$

Thus, $S_1 u$ and $S_2 u$ are the extensions of u to E_o "by antireflection" and "by reflection", respectively.

It is well-known that $u \in W_o^{1,2}(\overset{\circ}{E}_k)$ iff $S_k u \in W_o^{1,2}(E_o)$, and in this case

$$\|S_k u\|_{1,2,E_o} = \sqrt{2}\|u\|_{1,2,\overset{\circ}{E}_k}.$$

Moreover, for $0 < \lambda < N$ we have $u \in \mathcal{L}^{2,\lambda}(\overset{\circ}{E}_k)$ iff $S_k u \in \mathcal{L}^{2,\lambda}(E_o)$, and, in this case,

$$|S_k u|_{2,\lambda,E_o} \le \sqrt{2}|u|_{2,\lambda,\overset{\circ}{E}_k} \le \sqrt{2}|S_k u|_{2,\lambda,E_o}$$

(cf. Troianiello [13] p. 31 and p. 36).

Now, we extend vector valued maps $f = (f_1, \ldots, f_N) \in L^2(\overset{\circ}{E}_k; I\!R^N)$ to $S_k f = (f_1^{(k)}, \ldots, f_N^{(k)}) \in L^2(E_o; I\!R^N)$ and matrix valued maps $A = (a_{ij}) \in \mathcal{A}_\varepsilon(E_k)$ to $S_k A = (a_{ij}^{(k)}) \in \mathcal{A}_\varepsilon(E_o)$ by

$$\left.\begin{array}{lcl} f_j^{(k)}(x', x_N) & := & (-1)^k f_j(x', -x_N) \quad \text{for } \ j < N \\ f_N^{(k)}(x', x_N) & := & (-1)^{k+1} f_N(x', -x_N) \end{array}\right\}$$

$$\left.\begin{array}{lcl} a_{ij}^{(k)}(x', x_N) & := & (-1)^{k+1} a_{ij}(x', -x_N) \quad \text{for } \ i,j < N \ \text{ or } \ i = j = N \\ a_{ij}^{(k)}(x', x_N) & := & (-1)^k a_{ij}(x', -x_N) \quad \text{otherwise} \end{array}\right\}$$

for $(x', x_N) \in E_o \setminus E_k$. Then we get

$$\begin{array}{lcl} S_k(Af) & = & (S_k A)(S_k f) \\ S_k(\nabla u) & = & \nabla(S_k u) \end{array}$$

for all $A \in \mathcal{A}_\varepsilon(E_k)$, $f, \tilde{f} \in L^2(\overset{\circ}{E}_k; I\!R^N)$ and $u \in W^{1,2}(\overset{\circ}{E}_k)$.

Finally, for $k = 1, 2$ and $v \in W_o^{1,2}(E_o)$ we define $T_k \in W_o^{1,2}(E_k)$ by

$$(T_k v)(x', x_N) := \frac{1}{2}[v(x', x_N) + (-1)^k v(x', -x_N)] \quad \text{for } \ (x', x_N) \in E_k.$$

T_1v and T_2v are the restrictions to E_k of the antisymmetric and the symmetric part of v, respectively, and we have

$$\int_{E_o}[S_kf\cdot\nabla v+(S_kg)v]\mathrm{d}x = 2\int_{E_k}[f\cdot\nabla(T_kv)+gT_kv]\mathrm{d}x$$

for all $f\in L^2(\overset{\circ}{E}_k;I\!R^N)$, $g\in L^2(\overset{\circ}{E}_k)$ and $v\in W_o^{1,2}(E_o)$.

Now, take $A\in\mathcal{A}_\varepsilon(E_k)$, $f\in\mathcal{L}^{2,\lambda}(\overset{\circ}{E}_k;I\!R^N)$, $g\in\mathcal{L}^{2,\lambda-2}(\overset{\circ}{E}_k)$ and $u\in W_o^{1,2}(E_k)$ such that

$$\int_{E_k}(A\nabla u\cdot\nabla v+uv)\mathrm{d}x=\int_{E_k}(f\cdot\nabla v+gv)\mathrm{d}x\quad\text{for all }\ v\in W_o^{1,2}(E_k).$$

Then, for all $w\in W_o^{1,2}(E_o)$ it follows

$$\int_{E_o}[(S_kA)\nabla(S_ku)\cdot\nabla w+(S_ku)w]\mathrm{d}x = 2\int_{E_k}[A\nabla u\cdot\nabla(T_kw)+uT_kw]\mathrm{d}x=$$
$$=2\int_{E_k}[f\cdot\nabla(T_kw)+gT_kw]\mathrm{d}x=\int_{E_o}[S_kf\cdot\nabla w+(S_kg)w]\mathrm{d}x.$$

However E_o is admissible (Theorem 2.6). Therefore, there exists $\lambda_\varepsilon>N-2$ such that for all $\lambda\le\lambda_\varepsilon$ we have $\nabla(S_ku)=S_k(\nabla u)\in\mathcal{L}^{2,\lambda}(E_o;I\!R^N)$ (and, hence, $\nabla u\in\mathcal{L}^{2,\lambda}(\overset{\circ}{E}_k;I\!R^N)$) and

$$\begin{aligned}|\nabla u|_{2,\lambda,\overset{\circ}{E}_k} &\le |S_k\nabla u|_{2,\lambda,E_o}\le\\ &\le c\left(|S_kf|_{2,\lambda,E_o}+|S_kg|_{2,\lambda-2,E_o}+\|S_ku\|_{1,2,E_o}\right)\le\\ &\le \sqrt{2}c\left(|f|_{2,\lambda,\overset{\circ}{E}_k}+|g|_{2,\lambda-2,\overset{\circ}{E}_k}+\|u\|_{1,2,\overset{\circ}{E}_k}\right),\end{aligned}$$

where the constant $c>0$ does not depend on A, f, g and u, but only on ε and λ. Hence, E_k is admissible. ■

The next lemma is

Lemma 2.9 *Let $G\subset I\!R^N$ be admissible and ϕ be a Lipschitz transformation from G onto H. Then H is admissible.*

Proof. Obviously, H is regular.

Let us denote by $\phi'(x)$ the derivative of ϕ in x ($\phi'(x)$ exists for almost all $x\in G$, and ϕ' is a bounded measurable map from G into the space of the real $N\times N$-matrices). For the inverse and the inverse transpose of $\phi'(x)$ we write $\phi'(x)^{-1}$ and $\phi'(x)^{-*}$, respectively. If L_+ and L_- are Lipschitz constants of ϕ and ϕ^{-1}, respectively, then we have

$$\left.\begin{array}{l}|\phi'(x)^{-1}\xi|\ge L_+^{-1}|\xi|\quad\text{for all }\ \xi\in I\!R^N\\ L_-^N\le|\det\phi'(x)|\le L_+^N\end{array}\right\}\quad\text{for almost all }\ x\in G. \tag{2.6}$$

Now, take $A \in \mathcal{A}_\varepsilon(H)$, $f \in \mathcal{L}^{2,\lambda}(\overset{\circ}{H};\mathbb{R}^N)$, $g \in \mathcal{L}^{2,\lambda-2}(\overset{\circ}{H})$ and $u \in W_o^{1,2}(H)$ such that

$$\int_H (A\nabla u \cdot \nabla v + uv)\mathrm{d}x = \int_H (f \cdot \nabla v + gv)\mathrm{d}x \quad \text{for all } v \in W_o^{1,2}(H). \tag{2.7}$$

It is well-known that $u \circ \phi \in W_o^{1,2}(G)$ and

$$\|u \circ \phi\|_{1,2,\overset{\circ}{G}} \le \text{const}\, \|u\|_{1,2,\overset{\circ}{H}}, \tag{2.8}$$

where the constant in (2.8) does not depend on u. Moreover, Theorem 1.1(ii) yields $f\circ\phi \in \mathcal{L}^{2,\lambda}(\overset{\circ}{G};\mathbb{R}^N)$ and $g \circ \phi \in \mathcal{L}^{2,\lambda-2}(\overset{\circ}{G})$ and

$$\left.\begin{array}{rcl} |f \circ \phi|_{2,\lambda,\overset{\circ}{G}} & \le & \text{const}\, |f|_{2,\lambda,\overset{\circ}{H}} \\ |g \circ \phi|_{2,\lambda-2,\overset{\circ}{G}} & \le & \text{const}\, |g|_{2,\lambda-2,\overset{\circ}{H}}, \end{array}\right\} \tag{2.9}$$

where the constants in (2.9) do not depend on f and g. And finally, from (2.6) it follows that the map

$$x \in G \mapsto |\det \phi'(x)|\phi'(x)^{-1}A(\phi(x))\phi'(x)^{-*} \in \mathbb{S}_N$$

belongs to $\mathcal{A}_\delta(G)$ with $\delta = \varepsilon L_+^{-2}L_-^N$. Therefore, the chain rule for derivatives, the transformation rule for integrals and (2.7) imply for all $w \in W_o^{1,2}(G)$

$$\begin{aligned} &\int_G [\phi'(x)^{-1}A(\phi(x))\phi'(x)^{-*}\nabla(u\circ\phi)(x)\cdot\nabla w(x) + u(\phi(x))w(x)]\,|\det\phi'(x)|\mathrm{d}x = \\ &\quad = \int_H [A(y)\nabla u(y)\cdot\nabla(w\circ\phi^{-1})(y) + u(y)w(\phi^{-1}(y))]\mathrm{d}y = \\ &\quad = \int_H [f(y)\cdot\nabla(w\circ\phi^{-1})(y) + g(y)w(\phi^{-1}(y))]\mathrm{d}y = \\ &\quad = \int_G [\phi'(x)^{-1}f(\phi(x))\cdot\nabla w(x) + g(\phi(x))w(x)]\,|\det\phi'(x)|\mathrm{d}x. \end{aligned}$$

Hence, for all $w \in W_o^{1,2}(H)$ we obtain

$$\begin{aligned} &\int_G [|\det\phi'(x)|\phi'(x)^{-1}A(\phi(x))\phi'(x)^{-*}\nabla(u\circ\phi)(x)\cdot\nabla w(x) + u(\phi(x))w(x)]\mathrm{d}x = \\ &\quad = \int_G \{|\det\phi'(x)|\phi'(x)^{-1}f(\phi(x))\cdot\nabla w(x) + \\ &\quad\quad +[|\det\phi'(x)|(g(\phi(x)) - u(\phi(x))) + u(\phi(x))]w(x)\}\mathrm{d}x. \end{aligned} \tag{2.10}$$

Now, let us apply the assumption that G is admissible to the variational equation (2.10).

Because of the continuous embedding

$$W_o^{1,2}(G) \hookrightarrow \begin{cases} L^{\frac{2N}{N-2}}(\overset{\circ}{G}) & \text{if } N > 2 \\ L^p(\overset{\circ}{G}) \text{ for all } p \ge 2 & \text{if } N = 2 \end{cases} \tag{2.11}$$

and because of Theorem 1.1(i) we have $u \circ \phi \in \mathcal{L}^{2,1}(\overset{\circ}{G})$. Therefore, Theorem 1.2(ii) yields that

$$x \mapsto |\det \phi'(x)| \, [g(\phi(x)) - u(\phi(x))] + u(\phi(x)) \text{ belongs to } \mathcal{L}^{2,\mu-2}(\overset{\circ}{G}) \tag{2.12}$$

with $\mu := \min\{\lambda, 3\}$. Analogously, the map $x \mapsto |\det \phi'(x)| \, \phi'(x)^{-1} f(\phi(x))$ belongs to $\mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$. Hence, (2.10) and the assumption that G is admissible imply that there exists a $\lambda_o > N-2$ such that for all $\lambda \le \lambda_o$ we have $\nabla(u \circ \phi) \in \mathcal{L}^{2,\mu}(\overset{\circ}{G}; I\!R^N)$ (and, hence $\nabla u \in \mathcal{L}^{2,\mu}(\overset{\circ}{H}; I\!R^N)$) and

$$|\nabla u|_{2,\mu,\overset{\circ}{H}} \le \text{const} \, (|f|_{2,\mu,\overset{\circ}{H}} + |g|_{2,\mu-2,\overset{\circ}{H}} + \|u\|_{1,2,\overset{\circ}{H}}) \tag{2.13}$$

(cf. (2.8) and (2.9)). Remark that λ_o depends on ε and ϕ, but not on f, g and u.

Now, we again apply (2.10) and the assumption that G is admissible. Because of $u \circ \phi \in \mathcal{L}^{2,\mu+2}(\overset{\circ}{H})$ (cf. Theorem 1.2(iii)) we have (2.12) with $\mu := \min\{\lambda, 7\}$. This implies $\nabla u \in \mathcal{L}^{2,\mu}(\overset{\circ}{H}; I\!R^N)$ and (2.13) with this new μ (and with a new constant in (2.13)).

Reiterating this procedure as often as necessary we obtain $\nabla u \in \mathcal{L}^{2,\lambda}(\overset{\circ}{H}; I\!R^N)$ and (2.13) with $\mu = \lambda$. Hence, H is admissible. ■

Finally, we prove the last lemma:

Lemma 2.10 *Let $\{U_o, U_1, \ldots, U_n\}$ be an open covering of a set $G \subset I\!R^N$ such that $U_j \cap G$ is admissible for all $j = 0, 1, \ldots, n$. Then G is admissible.*

Proof. Obviously, G is regular.

Let $\{\varphi_o, \varphi_1, \ldots, \varphi_n\}$ be a smooth partition of unity subordinate to the covering $\{U_o, U_1, \ldots, U_n\}$. Then, for each $j = 0, 1, \ldots, n$ and for $u \in W_o^{1,2}(G)$, we have $\varphi_j u \in W_o^{1,2}(U_j \cap G)$ (here and later on we use the symbol $\varphi_j u$ for the restriction to $U_j \cap G$ of the product $\varphi_j u$, too). Moreover, for $v \in W_o^{1,2}(U_j \cap G)$ we have $\varphi_j v \in W_o^{1,2}(G)$ (where $\varphi_j v$ is interpreted as a function defined on G and vanishing on $G \setminus U_j$).

Now, take $A \in \mathcal{A}_\varepsilon(G)$, $f \in \mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$, $g \in \mathcal{L}^{2,\lambda-2}(\overset{\circ}{G})$ and $u \in W_o^{1,2}(G)$ such that

$$\int_G (A\nabla u \cdot \nabla v + uv)\mathrm{d}x = \int_G (f \cdot \nabla v + gv)\mathrm{d}x \quad \text{for all } \; v \in W_o^{1,2}(G).$$

Then we have for all $w \in W_o^{1,2}(U_j \cap G)$

$$\begin{aligned} \int_{U_j \cap G} & [A(\nabla \varphi_j u) \cdot \nabla w + \varphi_j u w]\mathrm{d}x = \\ &= \int_G [A\nabla u \cdot \nabla(\varphi_j w) + A(u\nabla w - w\nabla u) \cdot \nabla \varphi_j + u\varphi_j w]\mathrm{d}x = \\ &= \int_G [f \cdot \nabla(\varphi_j w) + g\varphi_j w + A(u\nabla w - w\nabla u)\nabla\varphi_j]\mathrm{d}x = \\ &= \int_{U_j \cap G} [(\varphi_j f + uA\nabla\varphi_j) \cdot \nabla w + (\varphi_j g + f \cdot \nabla\varphi_j - A\nabla u \cdot \nabla\varphi_j)w]\mathrm{d}x. \end{aligned} \tag{2.14}$$

In order to apply the assumption that $U_j \cap G$ is admissible, on the variational equation (2.14), we use the continuous embedding (2.11), again. Thus, we have

$$\left.\begin{array}{l} \varphi_j f + uA\nabla\varphi_j \in \mathcal{L}^{2,\mu}(U_j \cap \overset{\circ}{G}; \mathbb{R}^N) \\ \varphi_j g + f\cdot\nabla\varphi_j - A\nabla u\cdot\nabla\varphi_j \in \mathcal{L}^{2,\mu-2}(U_j \cap \overset{\circ}{G}) \end{array}\right\} \tag{2.15}$$

with $\mu := \min\{\lambda, 2\}$. Therefore, there exists a $\lambda_o > N-2$ such that for all $\lambda \le \lambda_o$ from (2.14) it follows that $\nabla(\varphi_j u) \in \mathcal{L}^{2,\mu}(U_j \cap \overset{\circ}{G}; \mathbb{R}^N)$ and

$$|\nabla(\varphi_j u)|_{2,\mu,U_j\cap\overset{\circ}{G}} \le \text{const}\,(|f|_{2,\mu,\overset{\circ}{G}} + |g|_{2,\mu-2,\overset{\circ}{G}} + \|u\|_{1,2,\overset{\circ}{G}}) \tag{2.16}$$

(cf. Theorem 1.2(ii)). Note that λ_o and the constant in (2.16) depend on ε and j, but not on f, g and u.

Now, we again apply (2.14) and the assumption that $U_j \cap G$ is admissible. We have $\varphi_j u \in \mathcal{L}^{2,\mu+2}(U_j \cap \overset{\circ}{G})$ for all j and, hence,

$$u = \sum_{j=1}^{n} \varphi_j u \in \mathcal{L}^{2,\mu+2}(\overset{\circ}{G}).$$

Therefore, we get (2.15) with $\mu := \min\{\lambda, 4\}$. This implies $\nabla(\varphi_j u) \in \mathcal{L}^{2,\mu}(U_j \cap \overset{\circ}{G}; \mathbb{R}^N)$ and (2.16) with this new μ (and a new constant).

Reiterating this procedure as often as necessary we obtain $\nabla(\varphi_j u) \in \mathcal{L}^{2,\lambda}(U_j \cap \overset{\circ}{G}; \mathbb{R}^N)$ and (2.16) with $\mu = \lambda$. Hence,

$$\nabla u = \sum_{j=1}^{n} \nabla(\varphi_j u) \in \mathcal{L}^{2,\lambda}(\overset{\circ}{G}; \mathbb{R}^N)$$

and

$$\begin{aligned} |\nabla u|_{2,\lambda,\overset{\circ}{G}} &\le \sum_{j=1}^{n} |\nabla(\varphi_j u)|_{2,\lambda,U_j\cap\overset{\circ}{G}} \le \\ &\le \text{const}\,(|f|_{2,\lambda,\overset{\circ}{G}} + |g|_{2,\lambda-2,\overset{\circ}{G}} + \|u\|_{1,2,\overset{\circ}{G}}), \end{aligned} \tag{2.17}$$

where the constant in (2.17) depends on ε, but not on f, g and u. Therefore, G is admissible. ■

Now, we are able to prove Theorem 2.5.

Proof of Theorem 2.5. In view of Lemma 2.10 it suffices to find an open covering $\{U_o, U_1, \ldots, U_n\}$ of G such that all $U_j \cap G$ are admissible. Since G is regular and ∂G is compact, there exist open subsets $U_1, \ldots, U_n$ of $\mathbb{R}^N$ with

$$\partial G \subset \bigcup_{j=1}^{n} U_j$$

and Lipschitz transformations $\phi_j : U_j \to \tilde{U}_j$ such that $\phi_j(U_j \cap G) \in \{E_1, E_2\}$ (cf. Definition 2.1). Hence, the sets $U_j \cap G$ $(j = 1, \ldots, n)$ are admissible (cf. Lemma 2.8 and 2.9).

Moreover, one can find an open subset $U_o \subseteq \overset{\circ}{G}$ with C^1-boundary such that

$$G \subset \bigcup_{j=0}^{n} U_j .$$

Then Theorem 2.6 implies that $U_o \cap G = U_o$ is admissible. ■

3 Smooth Dependence of the Solutions on the Coefficients

In the preceding section we proved the following fact:

If $\Omega \cup \Gamma$ is regular, then, for each $\varepsilon > 0$, there exists a $\lambda_\varepsilon > N-2$ such that for all $\lambda \in (N-2, \lambda_\varepsilon]$, $A \in \mathcal{A}_\varepsilon(\Omega)$, $f \in \mathcal{L}^{2,\lambda}(\Omega; I\!R^N)$ and $g \in \mathcal{L}^{2,\lambda-2}(\Omega)$ the weak solution u of (2.1) belongs to $C^{0,\alpha}(\bar{\Omega})$ with $\alpha = \frac{\lambda-N+2}{2}$ and that the linear map

$$(f,g) \in \mathcal{L}^{2,\lambda}(\Omega; I\!R^N) \times \mathcal{L}^{2,\lambda-2}(\Omega) \mapsto u \in C^{0,\alpha}(\bar{\Omega})$$

is continuous and, hence, smooth (cf. Remark 2.4).

In this section we will show that, moreover, the map

$$(A,f,g) \in \mathcal{A}_\varepsilon(\Omega) \times \mathcal{L}^{2,\lambda}(\Omega; I\!R^N) \times \mathcal{L}^{2,\lambda-2}(\Omega) \mapsto u \in C^{0,\alpha}(\bar{\Omega})$$

is smooth. Remark, that the set $\mathcal{A}_\varepsilon(\Omega)$ (cf. Definition 2.3(iii)) is open in $L^\infty(\Omega; \mathcal{S}_N)$. Here (and later on) we consider $\mathcal{A}_\varepsilon(\Omega)$ with the topology induced by $L^\infty(\Omega; \mathcal{S}_N)$.

Let us begin by introducing some notation.

In this section G is a fixed regular subset of $I\!R^N$. For each $\varepsilon > 0$ let $\lambda_\varepsilon > N-2$ be one of the numbers that are defined (for the fixed G) by Theorem 2.5.

As usual, we denote by $W^{-1,2}(G)$ the dual space of $W_o^{1,2}(G)$, and

$$\langle \cdot, \cdot \rangle : W^{-1,2}(G) \times W_o^{1,2}(G) \to I\!R$$

is the dual pairing between $W^{-1,2}(G)$ and $W_o^{1,2}(G)$.

For $A \in \mathcal{A}_\varepsilon(G)$ we denote by $L(A)$ the linear bounded operator from $W_o^{1,2}(G)$ into $W^{-1,2}(G)$ which is defined by

$$\langle L(A)u, v \rangle := \int_G (A\nabla u \cdot \nabla v + uv)\mathrm{d}x \quad \text{for all } u, v \in W_o^{1,2}(G). \tag{3.1}$$

Especially, if I_N is the unit $N \times N$-matrice, then

$$J := L(I_N) \tag{3.2}$$

is the duality map of the Hilbert space $W_o^{1,2}(G)$.

Because of the Cauchy-Schwarz inequality the definition (3.1) is correct, and the map

$$A \in \mathcal{A}_\varepsilon(G) \mapsto L(A) \in \mathcal{L}(W_o^{1,2}(G); W^{-1,2}(G)) \tag{3.3}$$

is continuous. Here (and later on) we use the symbol $\mathcal{L}(X;Y)$ for the space of all linear bounded operators from one normed vector space X into another normed vector space Y, equipped with the operator norm.

Obviously, the map (3.3) is affine. Therefore, it is smooth. Moreover, the Lax-Milgram Lemma implies that $L(A)$ is an isomorphism between $W_o^{1,2}(G)$ and $W^{-1,2}(G)$, and therefore the map

$$A \in \mathcal{A}_\varepsilon(G) \mapsto L(A)^{-1} \in \mathcal{L}(W^{-1,2}(G); W_o^{1,2}(G))$$

is smooth, too. This yields the well-known fact that the weak solutions of the boundary value problem (2.1) depend smoothly – in the sense of the space $W_o^{1,2}(\Omega \cup \Gamma)$ – on the coefficient matrix $A \in \mathcal{A}_\varepsilon(\Omega)$.

Definition 3.1 (i) By $U_\lambda(G) = W_o^{1,2,\lambda}(G)$ we denote the vector space of all $u \in W_o^{1,2}(G)$ such that $\nabla u \in \mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$, equipped with the norm

$$|u|_\lambda := \|u\|_{1,2} + |\nabla u|_{2,\lambda}.$$

(ii) By $V_\lambda(G)$ we denote the vector space of all $\varphi \in W^{-1,2}(G)$ such that there exists an $u \in U_\lambda(G)$ with $Ju = \varphi$ (cf. (3.2)). The norm of an element $\varphi = Ju$ in $V_\lambda(G)$ is defined to be equal to $|u|_\lambda$.

Lemma 3.2 *$U_\lambda(G)$ and $V_\lambda(G)$ are Banach spaces.*

Proof. Since $U_\lambda(G)$ and $V_\lambda(G)$ are isometrically isomorphic, it suffices to show that $U_\lambda(G)$ is complete.

Let (u_n) be a Cauchy sequence in $U_\lambda(G)$. Then (u_n) is a Cauchy sequence in $W_o^{1,2}(G)$, and (∇u_n) is a Cauchy sequence in $\mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$. As these two spaces are complete, we obtain $u_n \to u$ in $W_o^{1,2}(G)$ and $\nabla u_n \to f$ in $\mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$ for $n \to \infty$. This yields $\nabla u_n \to \nabla u$ a.e. and $\nabla u_n \to f$ a.e. for $n \to \infty$ and, hence, $\nabla u = f$ a.e. Thus we have $u \in U_\lambda(G)$ and $u_n \to u$ in $U_\lambda(G)$ for $n \to \infty$. ■

Remark 3.3 (i) From Theorem 1.2(i) and (iii) it follows immediately that for all $\lambda \in (N-2, N)$ the space $U_\lambda(G)$ is continuously embedded into the space $C^{0,\alpha}(\bar{\Omega})$ with $\alpha = \frac{\lambda - N + 2}{2}$.

(ii) Because of (3.2) and because the unit $N \times N$-matrix I_N belongs to $\mathcal{A}_{\frac{1}{2}}(G)$, for example (cf. Definition 2.3(iii)), Theorem 2.5 yields the following:

For all $\lambda \le \lambda_{\frac{1}{2}}$ the set $V_\lambda(G)$ equals the set of all $\varphi \in W^{-1,2}(G)$ such that there exist $f \in \mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$ and $g \in \mathcal{L}^{2,\lambda-2}(\overset{\circ}{G})$ with

$$\langle \varphi, v \rangle = \int_G (f \cdot \nabla v + gv) dx \quad \text{for all } v \in W_o^{1,2}(G). \tag{3.4}$$

Moreover, the map $(f,g) \in \mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N) \times \mathcal{L}^{2,\lambda-2}(\overset{\circ}{G}) \mapsto \varphi \in V_\lambda(G)$, defined by (3.4), is linear and continuous.

In the following lemma we show that (3.4) defines a continuous map not only from $\mathcal{L}^{2,\lambda}(\overset{\circ}{G};I\!\!R^N) \times \mathcal{L}^{2,\lambda-2}(\overset{\circ}{G})$ into $V_\lambda(G)$ (with $\lambda \le \lambda_{\frac{1}{2}}$) but also from $L^p(\overset{\circ}{G};I\!\!R^N) \times L^{\frac{p}{2}}(\overset{\circ}{G})$ into $V_\lambda(G)$ (with $p > N$ and $\lambda \le \min\{\lambda_{\frac{1}{2}}, N(1-\frac{2}{p})\}$:

Lemma 3.4 *Let $p > N$ and $\lambda \le \min\{\lambda_{\frac{1}{2}}, N(1-\frac{2}{p})\}$. Then, for each $f \in L^p(\overset{\circ}{G};I\!\!R^N)$ and $g \in L^{\frac{p}{2}}(\overset{\circ}{G})$, there exists a $\varphi \in V_\lambda(G)$ with (3.4), and the map $(f,g) \in L^p(\overset{\circ}{G};I\!\!R^N)\times L^{\frac{p}{2}}(\overset{\circ}{G}) \mapsto \varphi \in V_\lambda(G)$ is continuous.*

Proof. We have to show that there exists an $u \in U_\lambda(G)$ such that

$$\int_G (\nabla u \cdot \nabla v + uv)\mathrm{d}x = \int_G (f \cdot \nabla v + gv)\mathrm{d}x \quad \text{for all } v \in W_o^{1,2}(G) \tag{3.5}$$

and that u depends continuously on f and g.

In a first step consider the case of $g = 0$. Because of the assumptions $p > N$ and $\lambda \le N(1-\frac{2}{p})$ we have $p \ge 2$ and $\frac{N}{p} \le \frac{N-\lambda}{2}$. Therefore $L^p(\overset{\circ}{G};I\!\!R^N)$ is continuously embedded into $\mathcal{L}^{2,\lambda}(\overset{\circ}{G};I\!\!R^N)$ (Theorem 1.1(i)), and the assertion follows from Remark 3.3(ii).

In a second step consider the case of $f = 0$ and $p \ge 4$. In this case $L^{\frac{p}{2}}(\overset{\circ}{G})$ is continuously embedded into $\mathcal{L}^{2,\lambda-2}(\overset{\circ}{G})$, and the assertion follows from Remark 3.3(ii), again.

And finally, consider the case of $f = 0$ and $p < 4$.

If G is open (pure Dirichlet boundary conditions) and if ∂G is smooth, then the assertion is true: There exists a $u \in W^{2,\frac{p}{2}}(G) \cap W_o^{1,2}(G)$ such that (3.5) holds, and u depends continuously on g. Moreover, the assumption $p > N$ implies that $W^{1,\frac{p}{2}}(G, I\!\!R^N)$ is continuously embedded into $L^p(G;I\!\!R^N)$ and, hence, into $\mathcal{L}^{2,\lambda}(G;I\!\!R^N)$ (cf. the first step). Thus, u belongs to $U_\lambda(G)$ and depends continuously (in the sense of the norm of $U_\lambda(G)$) on $g \in L^{\frac{p}{2}}(G)$.

In the case of general regular G one has to apply the procedure of Section 2 in order to reduce this general case to the case of open G with smooth boundary. This procedure works because the operations "extension by reflection", "extension by antireflection" (cf. Lemma 2.8), "multiplication with a L^∞-function" and "superposition with a Lipschitz transformation" map $L^{\frac{p}{2}}$-functions into $L^{\frac{p}{2}}$-functions. ■

The main result of this section is

Theorem 3.5 (i) *The operator $L(A)$ maps $U_\lambda(G)$ continuously into $V_\lambda(G)$ if $\lambda \le \lambda_{\frac{1}{2}}$. Moreover, the map*

$$A \in \mathcal{A}_\varepsilon(G) \mapsto L(A) \in \mathcal{L}(U_\lambda(G); V_\lambda(G)) \tag{3.6}$$

is continuous.

(ii) *$L(A)$ is an isomorphism from $U_\lambda(G)$ onto $V_\lambda(G)$ if $A \in \mathcal{A}_\varepsilon(G)$ and $\lambda \le \min\{\lambda_\varepsilon, \lambda_{\frac{1}{2}}\}$.*

Proof. Let us fix $A \in \mathcal{A}_\varepsilon(G)$.

For $u \in U_\lambda(G)$ we have $L(A)u \in V_\lambda(G)$ iff there exists a $w \in U_\lambda(G)$ such that $L(A)u = Jw$, i.e. such that

$$\int_G (A\nabla u \cdot \nabla v + uv)\mathrm{d}x = \int_G (\nabla w \cdot \nabla v + wv)\mathrm{d}x \quad \text{for all } v \in W_o^{1,2}(G). \tag{3.7}$$

Because of Theorem 1.2(ii) we have $A\nabla u \in \mathcal{L}^{2,\lambda}(\overset{\circ}{G}; \mathbb{R}^N)$, and Theorem 1.2(iii) implies $u \in \mathcal{L}^{2,\lambda+2}(\overset{\circ}{G})$ and, hence, $u \in \mathcal{L}^{2,\lambda-2}(\overset{\circ}{G})$. Therefore, Theorem 2.5 yields that there exists exactly one $w \in U_\lambda(G)$ with (3.7) if $\lambda \le \lambda_{\frac{1}{2}}$, and the map $u \in U_\lambda(G) \mapsto w \in U_\lambda(G)$ is continuous (cf. Remark 2.4). Thus, $L(A)$ maps $U_\lambda(G)$ continuously into $V_\lambda(G)$ if $\lambda \le \lambda_{\frac{1}{2}}$.

Now, let us show that the map (3.6) is continuous.

Take $\tilde{A} \in \mathcal{A}_\varepsilon(G)$, $\lambda \le \lambda_{\frac{1}{2}}$ and $u, w \in U_\lambda(G)$ such that $[L(A) - L(\tilde{A})]u = Jw$, i.e. such that

$$\int_G (A - \tilde{A})\nabla u \cdot \nabla v \mathrm{d}x = \int_G (\nabla w \cdot \nabla v + wv)\mathrm{d}x \quad \text{for all } v \in W_o^{1,2}(G).$$

Then the Theorems 1.2(ii) and 2.5 imply (cf. Remark 2.4)

$$|w|_\lambda \le \text{const}\, |(A - \tilde{A})\nabla u|_{2,\lambda} \le \text{const}\, \|A - \tilde{A}\|_\infty |u|_\lambda, \tag{3.8}$$

where the constants in (3.8) do not depend on A, $\tilde{A}$, u and w. Hence, the map (3.6) is continuous.

Finally, we prove the assertion (ii).

As $L(A)$ is injective and because of the Open Mapping Theorem we have to show that $L(A)$ maps $U_\lambda(G)$ onto $V_\lambda(G)$ if $\lambda \le \min\{\lambda_\varepsilon, \lambda_{\frac{1}{2}}\}$. Thus, we have to show that for each $w \in U_\lambda(G)$ there exists an $u \in U_\lambda(G)$ with (3.7) if $\lambda \le \min\{\lambda_\varepsilon, \lambda_{\frac{1}{2}}\}$. However, this fact follows from Theorem 2.5. ∎

Let $p > N$, $\lambda \le \min\{\lambda_{\frac{1}{2}}, N(1 - \frac{2}{p})\}$ and denote by $E_{p,\lambda}$ the linear bounded operator from $L^p(\overset{\circ}{G}; \mathbb{R}^N) \times L^{\frac{p}{2}}(\overset{\circ}{G})$ into $V_\lambda(G)$, which is defined by Lemma 3.4. Then Theorem 3.5 implies that the map

$$A \in \mathcal{A}_\varepsilon(G) \mapsto L(A)^{-1} \circ E_{p,\lambda} \in \mathcal{L}(L^p(\overset{\circ}{G}; \mathbb{R}^N) \times L^{\frac{p}{2}}(\overset{\circ}{G}); U_\lambda(G))$$

is smooth. If, moreover, $\lambda > N - 2$, then $L(A)^{-1}$ maps $V_\lambda(G)$ into $C^{0,\alpha}(\bar{G})$ with $\alpha = \frac{\lambda - N + 2}{2}$ (cf. Remark 3.3(i)), and the map

$$A \in \mathcal{A}_\varepsilon(G) \mapsto L(A)^{-1} \circ E_{p,\lambda} \in \mathcal{L}(L^p(\overset{\circ}{G}; \mathbb{R}^N) \times L^{\frac{p}{2}}(\overset{\circ}{G}); C^{0,\alpha}(\bar{G}))$$

is smooth. Hence, the weak solution u of (2.1) depends smoothly in the sense of the space $C^{0,\alpha}(\bar{\Omega})$ on the coefficient matrix $A \in \mathcal{A}_\varepsilon(\Omega)$, and on the right-hand sides $f \in L^p(\Omega; \mathbb{R}^N)$ and $g \in L^{\frac{p}{2}}(\Omega)$ if $\Omega \cup \Gamma$ is regular.

Remark 3.6 Let us compare Theorem 3.5 with the results of GRÖGER [4]. There the following is shown to be true:

Let $G \subset I\!R^N$ be regular. Then, for each $\varepsilon > 0$, there exists a $p_\varepsilon > 2$ such that for all $p \in [2, p_\varepsilon]$ and $A \in \mathcal{A}_\varepsilon(G)$ the operator $L(A)$ is an isomorphism from $W_o^{1,p}(G)$ onto $W^{-1,p}(G)$, and the map

$$A \in \mathcal{A}_\varepsilon(G) \mapsto L(A) \in \mathcal{L}(W_o^{1,p}(G); W^{-1,p}(G))$$

is continuous. Here $W^{-1,p}(G) := W_o^{1,q}(G)^*$ is the dual space of $W_o^{1,q}(G)$, and $q := \frac{p}{p-1}$ is the exponent conjugate to p. Therefore, in case of $N = 2$, the continuous embedding $W_o^{1,p}(G) \hookrightarrow C^{0,\alpha}(\bar{G})$ with $\alpha = 1 - \frac{2}{p}$ (and $p > 2$) implies that the map

$$A \in \mathcal{A}_\varepsilon(G) \mapsto L(A)^{-1} \in \mathcal{L}(W^{-1,p}(G); C^{0,\alpha}(\bar{G}))$$

is smooth. This yields in the case of $N = 2$ that the weak solutions of (2.1) depend smoothly in the sense of $C^{0,\alpha}(\bar{\Omega})$ on A, again.

4 General Equations

In this section we consider boundary value problems for general linear elliptic equations of the type (1.1), i.e. of the type

$$\left.\begin{aligned} -\mathrm{div}(A\nabla u + bu) + c\cdot\nabla u + du &= -\mathrm{div}\, f + g && \text{in } \Omega \\ (A\nabla u + bu)\cdot\nu &= f\cdot\nu && \text{on } \Gamma \\ u &= 0 && \text{on } \partial\Omega\setminus\Gamma, \end{aligned}\right\} \tag{4.1}$$

when $\Omega \cup \Gamma \subset I\!R^N$ is regular, $A \in \mathcal{A}_\varepsilon(\Omega)$, $b, c \in L^\infty(\Omega; I\!R^N)$ and $d \in L^\infty(\Omega)$. We will show that each weak solution of (4.1), i.e. each $u \in W_o^{1,2}(\Omega\cup\Gamma)$ that satisfies

$$\left.\begin{aligned} \int_\Omega [(A\nabla u + bu)\cdot\nabla v + (c\cdot\nabla u + du)v]\mathrm{d}x = \int_\Omega (f\cdot\nabla v + gv)\mathrm{d}x \\ \text{for all } v \in W_o^{1,2}(\Omega\cup\Gamma), \end{aligned}\right\}$$

belongs to $U_\lambda(\Omega\cup\Gamma)$ and, hence, to $C^{0,\alpha}(\bar{\Omega})$ with $\alpha = \frac{\lambda - N + 2}{2}$ if $\lambda > N-2$ is close to $N-2$, $f \in \mathcal{L}^{2,\lambda}(\Omega; I\!R^N)$ and $g \in \mathcal{L}^{2,\lambda-2}(\Omega)$. Moreover, we prove that the weak solution of (4.1) – if it is unique – depends smoothly in the sense of $U_\lambda(\Omega\cup\Gamma)$ on A, b, c, d, f and g.

Firstly, let us introduce some notation.

In this section G is a fixed regular subset of $I\!R^N$, again, and $\lambda_\varepsilon > N-2$ is one of the numbers defined (for the fixed G and for $\varepsilon > 0$) by Theorem 2.5.

For $A \in \mathcal{A}_\varepsilon(G)$, $b, c \in L^\infty(\overset{\circ}{G}; I\!R^N)$ and $d \in L^\infty(\overset{\circ}{G})$ we denote by $L(A, b, c, d)$ the linear bounded operator from $W_o^{1,2}(G)$ into $W^{-1,2}(G)$ which is defined by

$$\left.\begin{aligned} \langle L(A,b,c,d)u, v\rangle &:= \int_G [(A\nabla u + bu)\cdot\nabla v + (c\cdot\nabla u + du)]\mathrm{d}x \\ &\quad \text{for all } u, v \in W_o^{1,2}(G). \end{aligned}\right\} \tag{4.2}$$

Because of the Cauchy-Schwarz inequality the definition (4.2) is correct, and the map

$$
\begin{aligned}
&(A, b, c, d) \in \mathcal{A}_\varepsilon(G) \times L^\infty(\overset{\circ}{G}; I\!R^N)^2 \times L^\infty(\overset{\circ}{G}) \mapsto \\
&\qquad \mapsto L(A, b, c, d) \in \mathcal{L}(W_o^{1,2}(G); W^{-1,2}(G))
\end{aligned}
$$

is continuous.

Theorem 4.1 (i) *The operator $L(A, b, c, d)$ maps $U_\lambda(G)$ continuously into $V_\lambda(G)$ if $\lambda \le \lambda_{\frac{1}{2}}$. Moreover, the map*

$$
\begin{aligned}
&(A, b, c, d) \in \mathcal{A}_\varepsilon(G) \times L^\infty(\overset{\circ}{G}; I\!R^N)^2 \times L^\infty(\overset{\circ}{G}) \mapsto \\
&\qquad \mapsto L(A, b, c, d) \in \mathcal{L}(U_\lambda(G); V_\lambda(G))
\end{aligned}
\tag{4.3}
$$

is continuous.

(ii) *$L(A, b, c, d)$ is a Fredholm operator (index zero) from $U_\lambda(G)$ into $V_\lambda(G)$ if $N - 2 < \lambda \le \min\{\lambda_\varepsilon, \lambda_{\frac{1}{2}}\}$ and $A \in \mathcal{A}_\varepsilon(G)$.*

Proof. The proof of part (i) is similar to the proof of Theorem 3.5(i).

Let us fix $A \in \mathcal{A}_\varepsilon(G)$, $b, c \in L^\infty(\overset{\circ}{G}; I\!R^N)$ and $d \in L^\infty(\overset{\circ}{G})$.

For $u \in U_\lambda(G)$ we have $L(A, b, c, d)u \in V_\lambda(G)$ iff there exists a $w \in U_\lambda(G)$ such that $L(A, b, c, d)u = Jw$, i.e. such that

$$
\left.
\begin{aligned}
\int_G [(A\nabla u + bu) \cdot \nabla v + (c \cdot \nabla u + dn)v]\mathrm{d}x = \int_G (\nabla w \cdot \nabla v + wv)\mathrm{d}x \\
\text{for all } \ v \in W_o^{1,2}(G).
\end{aligned}
\right\}
\tag{4.4}
$$

The functions $A\nabla u + bu$ and $c \cdot \nabla u + du$ belong to $\mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$ and $\mathcal{L}^{2,\lambda-2}(\overset{\circ}{G})$, respectively, and depend continuously on u. Therefore, Theorem 2.5 yields that there exists exactly one $w \in U_\lambda(G)$ with (4.4) if $\lambda \le \lambda_{\frac{1}{2}}$, and w depends continuously on u. Hence, $L(A, b, c, d)$ maps $U_\lambda(G)$ continuously into $V_\lambda(G)$ if $\lambda \le \lambda_{\frac{1}{2}}$.

Now, let us show that the map (4.3) is continuous.

Take $\tilde{A} \in \mathcal{A}_\varepsilon(G)$, $\tilde{b}, \tilde{c} \in L^\infty(\overset{\circ}{G}; I\!R^N)$, $\tilde{d} \in L^\infty(\overset{\circ}{G})$, $\lambda \le \lambda_{\frac{1}{2}}$ and $u, v \in U_\lambda(G)$ such that

$$
[L(A, b, c, d) - L(\tilde{A}, \tilde{b}, \tilde{c}, \tilde{d})]u = Jw,
$$

i.e. such that

$$
\begin{aligned}
&\int_G \{[(A - \tilde{A})\nabla u + (b - \tilde{b})u] \cdot \nabla v + [(c - \tilde{c}) \cdot \nabla u + (d - \tilde{d})u]v\}\mathrm{d}x = \\
&\qquad = \int_G (\nabla w \cdot \nabla v + wv)\mathrm{d}x \quad \text{for all } \ v \in W_o^{1,2}(G).
\end{aligned}
$$

Then the Theorems 1.2(ii) and 2.5 imply (cf. Remark 2.4)

$$|w|_\lambda \leq \text{const}\,(\|A-\tilde{A}\|_\infty + \|b-\tilde{b}\|_\infty + \|c-\tilde{c}\|_\infty + \|d-\tilde{d}\|_\infty)|u|_\lambda,$$

where the constant does not depend on A, $\tilde{A}$, b, $\tilde{b}$, c, $\tilde{c}$, d, $\tilde{d}$, u and w. Hence, the map (4.3) is continuous.

Finally, we prove the assertion (ii).

For $\varrho \in I\!R$ we have

$$L(A,b,c,d) = L(A,0,c,\varrho) + L(0,b,0,d-\varrho). \tag{4.5}$$

Hence, it suffices to show that $L(0,b,0,d-\varrho)$ is completely continuous from $U_\lambda(G)$ into $V_\lambda(G)$ if $N-2<\lambda\leq\lambda_{\frac{1}{2}}$, and that $L(A,0,c,\varrho)$ is an isomorphism from $U_\lambda(G)$ onto $V_\lambda(G)$ if $\lambda\leq\min\{\lambda_\varepsilon,\lambda_{\frac{1}{2}}\}$ and if $\varrho>0$ is sufficiently large.

Let us prove the first assertion.

Take $\lambda\in(N-2,\lambda_{\frac{1}{2}}]$ and a sequence (u_n) which is bounded in $U_\lambda(G)$. Then (u_n) is bounded in $\mathcal{L}^{2,\lambda+2}(\overset{\circ}{G}) \approx C^{0,\alpha}(\bar{G})$ $(\alpha = \frac{\lambda-N+2}{2})$ because of Theorem 1.2(i) and (iii). But $C^{0,\alpha}(\bar{G})$ is completely continuously embedded into $C^{0,\beta}(\bar{G}) \approx \mathcal{L}^{2,\mu+2}(\overset{\circ}{G})$ $(\mu = 2\beta+N-2)$ for each $\beta\in(0,\alpha)$. Therefore, there exists a subsequence (u_{n_k}) that converges in $\mathcal{L}^{2,\lambda}(\overset{\circ}{G})$.

For $n\in I\!N$ let w_n denote the element of $U_\lambda(G)$ which is defined by $L(0,b,0,d-\varrho)u_n = Jw_n$, i.e. by

$$\int_G [bu_n\cdot\nabla v + (d-\varrho)uv]\mathrm{d}x = \int_G (\nabla w_n\cdot\nabla v + w_n v)\mathrm{d}x \quad \text{for all } v\in W_o^{1,2}(G).$$

Theorem 2.5 yields that such functions w_n exist and that the sequence (∇w_{n_k}) converges in $\mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$. Analogously, the Lax-Milgram Lemma implies that (w_{n_k}) converges in $W_o^{1,2}(G)$. Hence, (Jw_{n_k}) converges in $V_\lambda(G)$ and, therefore, $L(0,b,0,d-\varrho)$ is completely continuous from $U_\lambda(G)$ into $V_\lambda(G)$.

Now, let us prove the second assertion.

Obviously, the operator $L(A,0,c,\varrho)$ is injective if $\varrho>0$ is sufficiently large. Hence, it remains to show that it is surjective from $U_\lambda(G)$ onto $V_\lambda(G)$ for such ϱ and for $\lambda\leq\min\{\lambda_\varepsilon,\lambda_{\frac{1}{2}}\}$.

Thus, take an arbitrary $\varphi = Jw\in V_\lambda(G)$, $w\in U_\lambda(G)$. Because of the Lax-Milgram Lemma there exists exactly one $u\in W_o^{1,2}(G)$ such that $L(A,0,c,\varrho)u = Jw$, i.e. such that

$$\left.\begin{aligned} \int_G (A\nabla u\cdot\nabla v + uv)\mathrm{d}x = \int_G \{\nabla w\cdot\nabla v + [w - c\cdot\nabla u + (1-\varrho)u]v\}\mathrm{d}x \\ \text{for all } v\in W_o^{1,2}(G). \end{aligned}\right\} \tag{4.6}$$

Now, Theorem 2.5 works if $\lambda\leq\min\{\lambda_\varepsilon,\lambda_{\frac{1}{2}}\}$: Because of $\nabla w\in\mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N)$ and $w-c\cdot\nabla u+(1-\varrho)u\in L^2(\overset{\circ}{G})$ we get $\nabla u\in\mathcal{L}^{2,\mu}(\overset{\circ}{G}; I\!R^N)$ and, hence, $u\in\mathcal{L}^{2,\mu+2}(\overset{\circ}{G})$ with $\mu := \min\{\lambda, 2\}$. Applying (4.6) once more we obtain $u \in \mathcal{L}^{2,\mu}(\overset{\circ}{G}; I\!R^N)$ with $\mu := \min\{\lambda, 4\}$ and so on. This gives, finally, $u \in U_\lambda(G)$. Hence, $L(A, 0, c, \varrho)$ is surjective from $U_\lambda(G)$ onto $V_\lambda(G)$ if $\lambda \le \min\{\lambda_\varepsilon, \lambda_{\frac{1}{2}}\}$ and if $\varrho > 0$ is sufficiently large. ■

A Fredholm operator is an isomorphism iff it is injective. Therefore, the assertions about the solvability properties of the boundary value problem (4.1) from the beginning of this section follow immediately from Theorem 4.1:

Corollary 4.2 *Let $\Omega \cup \Gamma$ be regular, $A \in \mathcal{A}_\varepsilon(\Omega)$, $b, c \in L^\infty(\Omega; I\!R^N)$, $d \in L^\infty(\Omega)$, $\lambda \le \min\{\lambda_\varepsilon, \lambda_{\frac{1}{2}}\}$, $f \in \mathcal{L}^{2,\lambda}(\Omega; I\!R^N)$ and $g \in \mathcal{L}^{2,\lambda-2}(\Omega)$. Then each weak solution of* (4.1) *belongs to $U_\lambda(G)$. Moreover, if the weak solution of* (4.1) *is unique, then it depends smoothly in the sense of $U_\lambda(G)$ on A, b, c, d, f and g.*

5 Weakly Coupled Systems

In this last section we consider boundary value problems for systems of linear elliptic equations of the type

$$\left.\begin{aligned} -\mathrm{div}\Big(A_i\nabla u_i+\sum_{j=1}^n b_{ij}u_j\Big)+\sum_{j=1}^n(c_{ij}\nabla u_j+d_{ij}u_j) &= -\mathrm{div}\, f_i+g_i \quad \text{in } \Omega \\ (A_i\nabla u_i+\sum_{j=1}^n b_{ij}u_j)\cdot\nu &= f_i\cdot\nu \quad \text{on } \Gamma \\ u_i &= 0 \quad \text{on } \partial\Omega\setminus\Gamma, \end{aligned}\right\} i=1,\ldots,n \tag{5.1}$$

when $\Omega \cup \Gamma \subset I\!R^N$ is regular, $A_i \in \mathcal{A}_\varepsilon(\Omega)$, $b_{ij}, c_{ij} \in L^\infty(\Omega; I\!R^N)$ and $d_{ij} \in L^\infty(\Omega)$ for $i, j = 1, \ldots, n$. The system (5.1) is a "weakly coupled" system of n linear elliptic equations because it is coupled via terms that contain lower order derivatives, only.

We will show that each weak solution of (5.1), i.e. each tuple $(u_1, \ldots, u_n) \in W_o^{1,2}(\Omega \cup \Gamma)^n$ that satisfies

$$\sum_{i,j=1}^n \int_\Omega [(A_i\nabla u_i + b_{ij}u_j)\cdot\nabla v_i + (c_{ij}\cdot\nabla u_j + d_{ij}u_j)v_i]\mathrm{d}x =$$
$$= \sum_{i=1}^n \int_\Omega (f_i\cdot\nabla v_i + g_i v_i)\mathrm{d}x \quad \text{for all } (v_1, \ldots, v_n) \in W_o^{1,2}(\Omega\cup\Gamma)^n,$$

belongs to $U_\lambda(\Omega\cup\Gamma)^n$ and, hence, to $C^{0,\alpha}(\bar\Omega)^n$ with $\alpha = \frac{\lambda-N+2}{2}$ if $\lambda > N-2$ is close to $N-2$, $f_i \in \mathcal{L}^{2,\lambda}(\Omega; I\!R^N)$ and $g_i \in \mathcal{L}^{2,\lambda-2}(\Omega)$ for $i = 1, \ldots, n$. Moreover, we prove that the weak solution of (5.1) – if it is unique – depends smoothly in the sense of $U_\lambda(\Omega\cup\Gamma)^n$ on the coefficients A_i, b_{ij}, c_{ij}, d_{ij} and on the right-hand sides f_i and g_i.

Let $G \subset I\!R^N$ be a regular set and $\lambda_\varepsilon > N-2$ be one of the numbers defined (for G and $\varepsilon > 0$) by Theorem 2.5.

For

$$
\begin{aligned}
A &= (A_1, \ldots, A_n) \in \mathcal{A}_\varepsilon(G)^n \\
b &= (b_{ij})_{i,j=1}^n,\ c = (c_{ij})_{i,j=1}^n \in L^\infty(\overset{\circ}{G}; I\!R^N)^{n^2} \\
d &= (d_{ij})_{i,j=1}^n \in L^\infty(\overset{\circ}{G})^{n^2}
\end{aligned}
$$

we denote by $L(A, b, c, d)$ the linear bounded operator from $W_o^{1,2}(G)^n$ into $W^{-1,2}(G)^n$ which is defined by

$$
\left.
\begin{aligned}
&\langle L(A, b, c, d)u, v\rangle := \\
&:= \sum_{i,j=1}^n \int_\Omega [(A_i \nabla u_i + b_{ij} u_j) \cdot \nabla v_i + (c_{ij} \nabla u_j + d_{ij} u_j) v_i] \mathrm{d}x \\
&\qquad \text{for all } \ u = (u_1, \ldots, u_n),\ v = (v_1, \ldots, v_n) \in W_o^{1,2}(G)^n.
\end{aligned}
\right\}
\tag{5.2}
$$

In (5.2) $\langle \cdot, \cdot \rangle : W^{-1,2}(G)^n \times W_o^{1,2}(G)^n \to I\!R$ is the dual pairing between $W^{-1,2}(G)^n \approx [W_o^{1,2}(G)^n]^*$ and $W_o^{1,2}(G)^n$. Again, the Cauchy-Schwarz inequality implies that the definition (5.2) is correct and that $L(A, b, c, d)$ depends continuously (in the sense of the operator norm in $\mathcal{L}(W_o^{1,2}(G)^n; W^{-1,2}(G)^n)$) on (A, b, c, d).

Theorem 5.1 (i) *The operator $L(A, b, c, d)$ maps $U_\lambda(G)^n$ continuously into $V_\lambda(G)^n$ if $\lambda \le \lambda_{\frac{1}{2}}$. Moreover, the map*

$$
\begin{aligned}
(A, b, c, d) \in \mathcal{A}_\varepsilon(G)^n \times L^\infty(\overset{\circ}{G}; I\!R^N)^{n^2} \times L^\infty(\overset{\circ}{G}; I\!R^N)^{n^2} \times L^\infty(\overset{\circ}{G})^{n^2} \mapsto \\
\mapsto L(A, b, c, d) \in \mathcal{L}(U_\lambda(G)^n; V_\lambda(G)^n)
\end{aligned}
\tag{5.3}
$$

is continuous.

(ii) *$L(A, b, c, d)$ is a Fredholm operator (index zero) from $U_\lambda(G)^n$ into $V_\lambda(G)^n$ if $N - 2 < \lambda \le \min\{\lambda_\varepsilon, \lambda_{\frac{1}{2}}\}$ and $A \in \mathcal{A}_\varepsilon(G)^n$.*

Proof. The proof is similar to the proof of Theorem 4.1.

Firstly, let us show that $L(A, b, c, d)$ maps $U_\lambda(G)^n$ into $V_\lambda(G)^n$ if $\lambda \le \lambda_{\frac{1}{2}}$.

We have $L(A, b, c, d)u \in V_\lambda(G)^n$ iff there exists a tuple $(w_1, \ldots, w_n) \in U_\lambda(G)^n$ such that

$$
\left.
\begin{aligned}
&\sum_{i,j=1}^n \int_\Omega [(A_i \nabla u_i + b_{ij} u_j) \cdot \nabla v_i + (c_{ij} \cdot \nabla u_j + d_{ij} u_j) v_i] \mathrm{d}x = \\
&= \sum_{i,j=1}^n \int_\Omega (\nabla w_i \cdot \nabla v_i + w_i v_i) \mathrm{d}x \quad \text{for all } \ (v_1, \ldots, v_n) \in W_o^{1,2}(G)^n.
\end{aligned}
\right\}
\tag{5.4}
$$

Let us denote by E_λ the linear continuous map from $\mathcal{L}^{2,\lambda}(\overset{\circ}{G}; I\!R^N) \times \mathcal{L}^{2,\lambda-2}(\overset{\circ}{G})$ into $V_\lambda(G)$ which is defined by (3.4), and let us use the notation (3.2). Then (5.4) is equivalent to

$$
E_\lambda\Big(A_i \nabla u_i + \sum_{j=1}^n b_{ij} u_j, \sum_{j=1}^n (c_{ij} \cdot \nabla u_j + d_{ij} u_j)\Big) = J w_i, \quad i = 1, \ldots, n. \tag{5.5}
$$

Hence, similar to the proof of Theorem 4.1 we obtain (if $\lambda \le \lambda_{\frac{1}{2}}$) that (5.4) has exactly one solution $(w_1, \dots, w_n) \in U_\lambda(G)^n$ if $(u_1, \dots, u_n) \in U_\lambda(G)^n$, that the linear operator

$$(u_1, \dots, u_n) \in U_\lambda(G)^n \mapsto (w_1, \dots, w_n) \in U_\lambda(G)^n$$

is continuous and that this linear operator depends continuously (in the sense of the operator norm in $\mathcal{L}(U_\lambda(G)^n)$) on A_i, b_{ij}, c_{ij} and d_{ij}.

Now, let us prove the assertion (ii).

Let us denote by I_n the unit $n \times n$-matrix. From definition (5.2) we get (cf. (4.5))

$$L(A, b, c, d) = L(A, 0, c, \varrho I_n) + L(0, b, 0, d - \varrho I_n)$$

for all $\varrho \in \mathbb{R}$.

Similar to the proof of Theorem 4.1 one shows that $L(0, b, 0, d - \varrho I_n)$ maps $U_\lambda(G)^n$ completely continuously into $V_\lambda(G)^n$ (if $N - 2 < \lambda < \lambda_{\frac{1}{2}}$) and that $L(A, 0, c, \varrho I_n)$ is injective for sufficiently large $\varrho > 0$.

Now, take $A = (A_1, \dots, A_n) \in \mathcal{A}_\varepsilon(G)^n$, $\lambda \le \min\{\lambda_\varepsilon, \lambda_{\frac{1}{2}}\}$ and $\varrho > 0$ sufficiently large. It remains to show that $L(A, 0, c, \varrho I_n)$ is surjective from $U_\lambda(G)^n$ onto $V_\lambda(G)^n$.

Thus, take an arbitrary $\varphi = (\varphi_1, \dots, \varphi_n) \in V_\lambda(G)^n$. The Lax-Milgram Lemma yields that there exists an $u = (u_1, \dots, u_n) \in W_o^{1,2}(G)^n$ such that

$$L(A, 0, c, \varrho I_n) u = \varphi. \tag{5.6}$$

Moreover, there exist $w_i \in U_\lambda(G)^n$ such that $J w_i = \varphi_i$ for $i = 1, \dots, n$. Hence, (5.6) is equivalent to

$$\left.\begin{aligned} &\sum_{i,j=1}^{n} \int_\Omega [A_i \nabla u_i \cdot \nabla v_i + (c_{ij} \cdot \nabla u_j + \varrho u_i) v_i] \mathrm{d}x = \\ &= \sum_{i=1}^{n} \int_\Omega (\nabla w_i \cdot \nabla v_i + w_i v_i) \mathrm{d}x \quad \text{for all } (v_1, \dots, v_n) \in W_o^{1,2}(G)^n. \end{aligned}\right\} \tag{5.7}$$

However, (5.7) is equivalent to (cf. (3.1) and (5.5))

$$L(A_i) u_i = E_\lambda\Big(\nabla w_i, w_i + (1 - \varrho) u_i - \sum_{j=1}^{n} c_{ij} \cdot \nabla u_j\Big) \quad \text{for } i = 1, \dots, n.$$

Therefore, similar to the proof of Theorem 4.1 one obtains that $u_i \in U_\lambda(G)$. ∎

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List of Symbols

a_{ij}, A_j, A, $\tilde{A}$

b_{ij}, b_j, b, $\tilde{b}$

c_{ij}, c_j, c, $\tilde{c}$

d_{ij}, d, $\tilde{d}$

E_j, E_λ, $E_{p,\lambda}$

f_j, f

g_j, g, $G, \mathring{G}$, $\bar{G}$, ∂G

H

I_n, I_N

J

L_+, L_-, $L(A)$, $L(A,b,c,d)$

N

u_j, u, U_j, $\tilde{U}_j$

v_j, v

w_j, w

x_j, x

α, β, δ, ε, λ, λ_o, λ_ε, μ, ν_j, ν, ϱ, ξ_j, ξ, η, φ_j, φ

Γ, ϕ, Ω, $\partial\Omega$, $\tilde{\Omega}$

$\mathbb{R}$, $\mathbb{R}^N$, $\mathbb{S}_N$, $\mathcal{A}_\varepsilon$, $\mathcal{L}$

L^p, L^∞, $W^{1,p}$, $W_o^{1,p}$, $W^{-1,p}$, $\mathcal{L}^{p,\lambda}$, $C^{0,\alpha}$, U_λ, V_λ

$|\cdot|$, $|\cdot|_\lambda$, $|\cdot|_{p,\lambda}$, $\|\cdot\|_p$, $\|\cdot\|_\infty$, $\|\cdot\|_{1,p}$, $[\cdot]_{p,\lambda}$