# Interior Integral Estimates on Weak Solutions of Nonlinear Parabolic Systems

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**Abstract**

This paper concerns various types of CACCIOPPOLI and POINCARÉ inequalities on weak solutions $u$ of nonlinear parabolic systems. The main result of the paper is the local integrability of the spatial gradient $Du$ to an exponent $p > 2$.

**Contents**



## 1 Introduction

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain, and let $0 < T < +\infty$ be fixed. Set $Q = \Omega \times (0, T)$.

We consider the following system of PDE's:

$$\frac{\partial u^i}{\partial t} - D_\alpha a_i^\alpha(x, t, u, Du) = f_i - D_\alpha g_i^\alpha \quad \text{in} \quad Q \ \ (i = 1, \dots, N), \tag{1.1}$$

[1)]

where:

$$\begin{aligned} u &= \{u^1, \dots, u^N\} \quad (N \geq 1), \\ Du &= \{D_\alpha u^i\} \quad \text{(matrix of first spatial derivatives of } u) \end{aligned}$$

$\Big(D_\alpha = \dfrac{\partial}{\partial x_\alpha};\ \alpha = 1, \dots, n\Big)$; $f = \{f_1, \dots, f_N\}$ and $g = \{g_i^\alpha\}$ are given functions in $Q$.

Let $a = \{a_i^\alpha\}$ be a matrix. Define

$$\|a\| = \Big( \sum_{\alpha=1}^{n} \sum_{i=1}^{N} (a_i^\alpha)^2 \Big)^{1/2}.$$

[1)] Throughout a repeated Greek resp. Latin index implies summation over $1, \dots, n$ resp. $1, \dots, N$.

Throughout the whole paper, we impose on the functions $a_i^\alpha$ in (1.1) the following conditions:

$$\text{(1.2)}\qquad \begin{cases} a_i^\alpha \quad \text{is a Carathéodory function on } \ Q\times I\!R^N \times I\!R^{nN} \\ (\alpha=1,\dots,n;\ i=1,\dots,N); \end{cases}$$

$$\text{(1.3)}\qquad \begin{cases} \|a(x,t,u,\xi)\| \le a_0(1+|u|^{(n+2)/n}+\|\xi\|) \\ \forall\,(x,t,u,\xi)\in Q\times I\!R^N\times I\!R^{nN} \quad (a_0=\text{const}); \end{cases}$$

$$\text{(1.4)}\qquad \begin{cases} a_i^\alpha(x,t,u,\xi)\xi_\alpha^i \ge \lambda_0\|\xi\|^2 \\ \forall\,(x,t,u,\xi)\in Q\times I\!R^N\times I\!R^{nN} \quad (\lambda_0=\text{const}>0). \end{cases}$$

■

Let

$$W_p^1(\Omega)=\{v\in L^p(\Omega): D_\alpha v\in L^p(\Omega)\,(\alpha=1,\dots,n)\}$$

denote the usual Sobolev space over $\Omega$. Next, define

$$W_2^{1,1}(Q)=\Big\{v\in L^2(Q): D_\alpha v,\ \frac{\partial v}{\partial t}\in L^2(Q)\,(\alpha=1,\dots,n)\Big\},$$

and

$$\begin{aligned} W_2^{1,0}(Q) &= \{v\in L^2(Q): D_\alpha v\in L^2(Q)\,(\alpha=1,\dots,n)\}, \\ V_2^{1,0}(Q) &= \Big\{v\in W_2^{1,0}(Q): \operatorname*{ess\,sup}_{(0,T)}\int_\Omega v^2(x,t)\mathrm{d}x<+\infty\Big\}. \end{aligned}$$

The following imbedding theorem is well-known (cf. e.g. [5]):

$$\text{(1.5)}\qquad \begin{cases} \textit{Let} \quad \omega\subset I\!R^n \quad \textit{be a bounded domain with Lipschitz} \\ \textit{boundary } \partial\omega.\ \textit{Let} \quad \gamma\subseteq\partial\omega \quad \textit{be relatively open. Then:} \\ \|v\|_{L^{2(n+2)/n}(\omega\times(0,T))} \le c_0\Big(\operatorname*{ess\,sup}_{(0,T)}\int_\omega v^2(x,t)\mathrm{d}x + \int_0^T\int_\omega |Dv|^2\mathrm{d}x\mathrm{d}t\Big)^{1/2} \\ \textit{for all } \ v\in V_2^{1,0}(\omega\times(0,T)),\ v=0 \ \textit{ a.e. on } \ \gamma\times(0,T) \ \ (c_0=\text{const}>0). \end{cases}$$

By $L^p(Q;I\!R^N)$, $W_2^{1,1}(Q;I\!R^N)$ etc. we denote the space of vector valued functions $v=\{v^1,\dots,v^N\}$ the components of which belong to $L^p(Q)$ resp. $W_2^{1,1}(Q)$ etc. ■

We introduce the notion of weak solution of (1.1) regardless of whether or not this solution is subject to any boundary or initial condition.

Let (1.2) and (1.3) be satisfied. Without any further reference, throughout the whole paper, we assume

$$g\in L^2(Q;I\!R^{nN}).$$

**DEFINITION 1.1** *Assume*

1° $f\in L^{2(n+2)/(n+4)}(Q;I\!R^N)$ *resp.*

2° $f \in L^1(Q; I\!R^N)$.

*The function* $u \in V_2^{1,0}(Q; I\!R^N)$ *is called a weak solution of* (1.1) *if*

$$\int_\Omega u^i(x,t)v^i(x,t)\mathrm{d}x - \int_0^t\int_\Omega u^i \frac{\partial v^i}{\partial s}\mathrm{d}x\mathrm{d}s + \int_0^t\int_\Omega a_i^\alpha(x,s,u,Du)D_\alpha v^i \mathrm{d}x\mathrm{d}s =$$

$$= \int_0^t\int_\Omega (f_i v^i + g_i^\alpha D_\alpha v^i)\mathrm{d}x\mathrm{d}s \tag{1.6}$$

*for a.a.* $t \in (0,T)$ *and all test functions*

1° $v \in W_2^{1,1}(Q; I\!R^N)$, $\mathrm{supp}(v) \subset \Omega \times (0,T]$ *resp.*

2° $v \in W_2^{1,1}(Q; I\!R^N) \cap L^\infty(Q; I\!R^N)$, $\mathrm{supp}(v) \subset \Omega \times (0,T]$.

The aim of the present paper is to prove various interior integral estimates on any weak solution $u$ of (1.1). Following [5], we first regularize $u$ and localize then (1.6) with respect to $t$. Then CACCIOPPOLI and POINCARÉ inequalities are readily obtained (Sections 3 and 4). Next, in Section 5 we prove an extended version of the preceeding CACCIOPPOLI inequality which is appearently new in the theory of parabolic systems. Our main result is the interior higher integrability of $Du$ (i.e. $Du \in L^p_{\mathrm{loc}}$ for a $p > 2$) which is presented in Section 6. It is based on the parabolic analogue of the well-known higher integrability by reverse Hölder inequality due to GEHRING-GIAQUINTA-MODICA.

This method has been used in [3], [6] to prove the higher integrability of the spatial gradient of *bounded* weak solutions to parabolic systems with quadratic growth nonlinearities. Under more restrictive assumptions on the coefficients $a_i^\alpha$, similar results have been obtained in [1], [2] by an entirely different technique.

The higher integrability of the spatial gradient of weak solution to a nonlinear parabolic system is of interest in itself. On the other hand, it is also a basic tool in the proof of partial regularity of weak solutions of nonlinear parabolic systems. ■

## 2 Regularization and localization

Let $f \in L^1(Q)$. Given any $t_0 \in (0,T)$ and $k > \dfrac{1}{T - t_0}$ we introduce the

*Steklov mean of* $f$ :

$$f_k(x,t) = k\int_t^{t+\frac{1}{k}} f(x,s)\mathrm{d}s \quad \text{for a.a.} \quad (x,t) \in \Omega \times (0,t_0).$$

We note some properties of $f_k$ which will be used in what follows. Let $t_0 \in (0,T)$.

**PROPOSITION 2.1** *Let* $f \in L^p(Q)$ $(1 \le p < +\infty)$ *and* $g \in L^{p'}(Q)$. *Then*

$$\int_0^{t_0+\frac{1}{k}}\int_\Omega f(x,t)\Big(\int_{t-\frac{1}{k}}^t g(x,s)\mathrm{d}s\Big)\mathrm{d}x\mathrm{d}t = \int_0^{t_0}\int_\Omega \Big(\int_t^{t+\frac{1}{k}} f(x,s)\mathrm{d}s\Big)g(x,t)\mathrm{d}x\mathrm{d}t$$

*for all* $k > \dfrac{1}{T - t_0}$.

**PROPOSITION 2.2** 1. *Let* $f \in L^p(Q)$ $(1 \le p \le +\infty)$. *Then*

$$\int_0^{t_0}\int_\Omega |f_k|^p \mathrm{d}x\mathrm{d}t \le \int_0^{t_0}\int_\Omega |f|^p \mathrm{d}x\mathrm{d}t \quad (1 \le p < +\infty),$$

$$\operatorname*{ess\,sup}_{\Omega\times(0,t_0)} |f_k| \le \operatorname*{ess\,sup}_{\Omega\times(0,t_0)} |f|$$

*for all* $k > \dfrac{1}{T - t_0}$.

2. *Let* $f \in L^p(Q)$ $(1 \le p < +\infty)$. *Then*

(i) $f_k \to f$ *in* $L^p(\Omega \times (0, t_0))$ *as* $k \to +\infty$;

(ii) $f_k(\cdot, t) \to f(\cdot, t)$ *in* $L^p(\Omega)$ *as* $k \to +\infty$ *for a.a.* $t \in (0, t_0)$.

3. *Let* $f \in L^1(Q)$, $g \in L^\infty(Q)$. *Then there exists a subsequence* $\{k_j\}$ *such that*

$$\int_0^{t_0}\int_\Omega f_{k_j} g_{k_j} \mathrm{d}x\mathrm{d}t \longrightarrow \int_0^{t_0}\int_\Omega fg \mathrm{d}x\mathrm{d}t \quad as \quad j \to \infty.$$

The following result shows the effect of regularization with respect to $t$ of the Steklov mean.

**PROPOSITION 2.3** *Let* $f \in W_2^{1,0}(Q)$. *Then* $f_k \in W_2^{1,1}(\Omega \times (0, t_0))$ *and there holds*

$$D_\alpha f_k(x,t) = (D_\alpha f)_k(x,t) \qquad (\alpha = 1, \ldots, n),$$

$$\frac{\partial f_k}{\partial t}(x,t) = k\Big[f\Big(x, t + \frac{1}{k}\Big) - f(x,t)\Big]$$

*for a.a.* $(x,t) \in \Omega \times (0, t_0)$ *and all* $k > \dfrac{1}{T - t_0}$. ∎

We are now going to localize (1.6) with respect to $t$. To this end, let $t_0 \in (0, T)$ be arbitrary. We consider the Steklov mean with integers $k > \dfrac{1}{T - t_0}$.

**THEOREM 2.1** *Assume*

1° $f \in L^{2(n+2)/(n+4)}(Q;\mathbb{R}^N)$ *resp.*

2° $f \in L^1(Q;\mathbb{R}^N)$.

*Let* $u \in V_2^{1,0}(Q;\mathbb{R}^N)$ *be a weak solution of* (1.1). *Then*

$$\int_\Omega \frac{\partial u_k^i}{\partial t}\varphi^i \mathrm{d}x + \int_\Omega (a_i^\alpha)_k D_\alpha \varphi^i \mathrm{d}x = \int_\Omega ((f_i)_k\varphi^i + (g_i^\alpha)_k D_\alpha\varphi^i)\mathrm{d}x \tag{2.1}$$

*for a.a.* $t \in (0,t_0)$, *all integers* $k > \dfrac{1}{T-t_0}$ *and all test functions*

1° $\varphi \in W_2^1(\Omega;\mathbb{R}^N)$, $\mathrm{supp}(\varphi) \subset \Omega$ *resp.*

2° $\varphi \in W_2^1(\Omega;\mathbb{R}^N) \cap L^\infty(\Omega;\mathbb{R}^N)$, $\mathrm{supp}(\varphi) \subset \Omega$.

*Proof*. Fix an integer $m > \dfrac{n}{2}$. Let $\Omega_j$ $(j = 1,2,\ldots)$ be open sets such that:

$$\Omega_j \subset \overline{\Omega}_j \subset \Omega_{j+1} \subset \ldots \subset \Omega, \quad \partial\Omega_j \quad \text{smooth}, \quad \bigcup_{j=1}^\infty \Omega_j = \Omega.$$

The Sobolev imbedding theorem implies $W_2^m(\Omega_j) \subset C(\overline{\Omega}_j)$ $(j = 1,2,\ldots)$. Define

$$\overset{\circ}{W}{}_2^m(\Omega_j) = \left\{\varphi \in W_2^m(\Omega_j) : \varphi = \frac{\partial\varphi}{\partial\nu} = \ldots = \frac{\partial^{m-1}\varphi}{\partial\nu^{m-1}} = 0 \ \text{ a.e. on } \ \partial\Omega_j\right\}.$$

Let $\varphi \in \overset{\circ}{W}{}_2^m(\Omega_j;\mathbb{R}^N)$ be arbitrary. We extend $\varphi$ by zero onto $\Omega \setminus \Omega_j$ and denote this function in $\Omega$ again by $\varphi$. Let $\tau \in C(\mathbb{R})$ have its support in $(0,t_0)$. Then the function

$$v(x,t) = k\varphi(x)\int_{t-\frac{1}{k}}^{t} \tau(s)\mathrm{d}s, \quad (x,t) \in Q, \quad k \ \text{ integer } \ > \frac{1}{T-t_0}$$

is admissible in (1.6).

Let $t_1 \in \left(t_0 + \dfrac{1}{k}, T\right)$. Clearly, $v(\cdot,t_1) = 0$ and

$$\begin{aligned}\int_0^{t_1}\int_\Omega u^i(x,t)\frac{\partial v^i}{\partial t}(x,t)\mathrm{d}x\mathrm{d}t &= -k\int_0^{t_0}\int_\Omega \left[u^i\left(x,t+\frac{1}{k}\right) - u^i(x,t)\right]\varphi^i(x)\tau(t)\mathrm{d}x\mathrm{d}t \\ &= -\int_0^{t_0}\int_\Omega \frac{\partial u_k^i}{\partial t}(x,t)\varphi^i(x)\tau(t)\mathrm{d}x\mathrm{d}t\end{aligned}$$

(cf. Prop. 2.3). Now (1.6) with $t_1 \in \left(t_0 + \dfrac{1}{k}, T\right)$ gives

$$\int_0^{t_0}\int_\Omega \frac{\partial u_k^i}{\partial t}\varphi^i\tau\mathrm{d}x\mathrm{d}t + \int_0^{t_0}\int_\Omega (a_i^\alpha)_k D_\alpha\varphi^i\tau\mathrm{d}x\mathrm{d}t = \int_0^{t_0}\int_\Omega [(f_i)_k\varphi^i + (g_i^\alpha)_k D_\alpha\varphi^i]\tau\mathrm{d}x\mathrm{d}t.$$

Thus, by a standard argument,

$$\int\limits_{\Omega_j} \frac{\partial u_k^i}{\partial t}(x,t)\varphi^i(x)\mathrm{d}x + \int\limits_{\Omega} (a_i^\alpha)_k(x,t)D_\alpha\varphi^i(x)\mathrm{d}x =$$

$$(2.2)\qquad = \int\limits_{\Omega_j} [(f_i)_k(x,t)\varphi^i(x) + (g_i^\alpha)_k(x,t)D_\alpha\varphi^i(x)]\mathrm{d}x$$

for all $t \in (0,t_0)\setminus E_{j,k}$ where $\operatorname{meas} E_{j,k} = 0$ (notice that by virtue of the separability of $\overset{\circ}{W}{}_2^m(\Omega_j)$ the set $E_{j,k}$ does not depend on $\varphi$). Define $E = \bigcup\limits_{j=1}^{\infty} \bigcup\limits_{k>(T-t_0)^{-1}} E_{j,k}$. Then $\operatorname{meas} E = 0$, and (2.2) is true for a.a. $t \in (0,t_0)$, for $j = 1,2,\ldots$ and all integers $k > \dfrac{1}{T-t_0}$.

Let $\varphi$ satisfy 1° resp. 2° above. Fix $j$ such that $\operatorname{supp}(\varphi) \subset \Omega_j$. Let $\varphi_\rho$ be the standard mollification of $\varphi$. Then $\varphi_\rho \in \overset{\circ}{W}{}_2^m(\Omega_j;\mathbb{R}^N)$ for all $0 < \rho < \operatorname{dist}(\operatorname{supp}(\varphi),\partial\Omega_j)$ and $\varphi_\rho \to \varphi$ in $W_2^1(\Omega_j;\mathbb{R}^N)$ as $\rho \to 0$. If, in addition, $\varphi \in L^\infty(\Omega;\mathbb{R}^N)$ then there exists a subsequence of $\{\varphi_\rho\}$ (not relabelled) such that $\varphi_\rho \to \varphi$ a.e. in $\Omega_j$. Hence, inserting $\varphi_\rho$ in (2.2) and letting tend $\rho \to 0$ gives the claim. ■

# 3 CACCIOPPOLI inequalities

Define

$$\begin{aligned} B_r &= B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}, \\ Q_r &= Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r^2, t_0). \end{aligned}$$

Let $(x_0,t_0) \in Q$. Fix any $0 < r < \frac{1}{2}\sqrt{t_0}$ such that $\overline{B}_{2r} \subset \Omega$. Let $\zeta \in C^\infty(\mathbb{R}^n)$[2)] and $\tau \in C^\infty(\mathbb{R})$ be cut-off functions as follows:

$$\begin{cases} \zeta \equiv 1 \quad \text{on} \quad B_r, \quad \zeta \equiv 0 \quad \text{in} \quad \mathbb{R}^n \setminus B_{2r}, \\ 0 \le \zeta \le 1, \quad |D\zeta| \le \dfrac{c_0}{r} \quad \text{in} \quad \mathbb{R}^n; \end{cases}$$

$$\begin{cases} \tau \equiv 1 \quad \text{on} \quad (t_0 - r^2, +\infty), \quad \tau \equiv 0 \quad \text{on} \quad (-\infty, t_0 - 4r^2), \\ 0 \le \tau \le 1, \quad 0 \le \tau' \le \dfrac{c_0}{r^2} \quad \text{on} \quad \mathbb{R}^n; \end{cases}$$

($c_0 = \text{const} > 0$ independent of $r$).

Following [3] we define for any $v \in L^1(Q_{2r})$

$$\tilde{v}_{2r}(t) = \Big(\int\limits_{B_{2r}} \zeta^2 \mathrm{d}x\Big)^{-1} \int\limits_{B_{2r}} v(y,t)\zeta^2(y)\mathrm{d}y \quad \text{for a.a.} \quad t \in (t_0 - 4r^2, t_0).$$

Let $v \in W_2^{1,1}(Q_{2r})$. Then the function $t \mapsto \tilde{v}_{2r}(t)$ possesses a weak derivative $\dfrac{\mathrm{d}\tilde{v}_{2r}}{\mathrm{d}t} \in L^2(t_0 - 4r^2, t_0)$ and there holds

$$\frac{\mathrm{d}\tilde{v}_{2r}}{\mathrm{d}t}(t) = \Big(\widetilde{\frac{\mathrm{d}v}{\mathrm{d}t}}\Big)_{2r}(t) \quad \text{for a.a.} \quad t \in (t_0 - 4r^2, t_0). \qquad ■$$

[2)] To emphasize the dependence of $\zeta$ on $B_{2r}$, below we shall write $\zeta_{2r}$ in place of $\zeta$, $\zeta_r$ etc.

Let (1.2) – (1.4) be satisfied. In what follows, we consider the two cases [3)]

$$
(3.1)\qquad \begin{cases} f \in L^{2(n+2)/(n+4)}(Q;\mathbb{R}^N), \\ u \in V_2^{1,0}(Q;\mathbb{R}^N) \quad \textit{is a weak solution of} \quad (1.1), \end{cases}
$$

or

$$
(3.2)\qquad \begin{cases} f \in L^{1}(Q;\mathbb{R}^N), \\ u \in V_2^{1,0}(Q;\mathbb{R}^N) \cap L^\infty(Q;\mathbb{R}^N) \quad \textit{is a weak solution of} \quad (1.1). \end{cases}
$$

We begin by proving the following

**THEOREM 3.1** *Assume* (3.1) *or* (3.2). *Then, for every* $\varepsilon > 0$,

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{B_{2r}} |u(x,t)-\Lambda|^2\zeta^2(x)\mathrm{d}x\tau^2(t) + (\lambda_0-\varepsilon)\int\limits_{t_0-4r^2}^{t}\int\limits_{B_{2r}} \|Du\|^2\zeta^2\tau^2\mathrm{d}x\mathrm{d}s \le \\
(3.3)\qquad &\le c_1\Big(1+\frac{1}{\varepsilon}\Big)\frac{1}{r^2}\int\limits_{Q_{2r}} |u-\Lambda|^2\mathrm{d}x\mathrm{d}s + c_1\int\limits_{Q_{2r}}(1+|u|^{2(n+2)/n}\zeta^2\tau^2)\mathrm{d}x\mathrm{d}s \\
&\quad + \int\limits_{Q_{2r}} |f|\,|u-\Lambda|\zeta^2\tau^2\mathrm{d}x\mathrm{d}s + c_1\Big(1+\frac{1}{\varepsilon}\Big)\int\limits_{Q_{2r}}\|g\|^2\mathrm{d}x\mathrm{d}s
\end{aligned}
$$

*for a.a.* $t \in (t_0-4r^2, t_0)$ *and all* $\Lambda \in \mathbb{R}^N$*;*

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{B_{2r}} |u(x,t)-\tilde{u}_{2r}(t)|^2\zeta^2(x)\mathrm{d}x\tau^2(t) + (\lambda_0-\varepsilon)\int\limits_{t_0-4r^2}^{t}\int\limits_{B_{2r}} \|Du\|^2\zeta^2\tau^2\mathrm{d}x\mathrm{d}s \le \\
(3.4)\qquad &\le c_2\Big(1+\frac{1}{\varepsilon}\Big)\frac{1}{r^2}\int\limits_{Q_{2r}} |u-\tilde{u}_{2r}|^2\mathrm{d}x\mathrm{d}s + c_2\int\limits_{Q_{2r}}(1+|u|^{2(n+2)/n}\zeta^2\tau^2)\mathrm{d}x\mathrm{d}s \\
&\quad + \int\limits_{Q_{2r}} |f|\,|u-\tilde{u}_{2r}|\zeta^2\tau^2\mathrm{d}x\mathrm{d}s + c_2\Big(1+\frac{1}{\varepsilon}\Big)\int\limits_{Q_{2r}}\|g\|^2\mathrm{d}x\mathrm{d}s
\end{aligned}
$$

*for a.a.* $t \in (t_0-4r^2, t_0)$ *where the constants* $c_1$, $c_2$ *depend neither on* $r$ *nor on* $\varepsilon$.

*Proof*. Let $k$ be any integer $> \dfrac{1}{T-t_0}$. The function

$$
\varphi = (u_k(\cdot,t)-\Lambda)\zeta^2\tau^2(t), \quad t \in (t_0-4r^2, t_0)
$$

is admissible in (2.1). We obtain

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{B_{2r}} |u_k(t)-\Lambda|^2\zeta^2\mathrm{d}x\tau^2(t) + \int\limits_{t_0-4r^2}^{t}\int\limits_{B_{2r}} (a_i^\alpha)_k(D_\alpha u_k^i)\zeta^2\tau^2\mathrm{d}x\mathrm{d}s = \\
&= -2\int\limits_{t_0-4r^2}^{t}\int\limits_{B_{2r}} (a_i^\alpha)_k(u_k^i-\Lambda^i)\zeta(D_\alpha\zeta)\tau^2\mathrm{d}x\mathrm{d}s
\end{aligned}
$$

---

[3)] Recall that $g \in L^2(Q;\mathbb{R}^{nN})$ throughout.

$$+\int_{t_0-4r^2}^{t}\int_{B_{2r}}|u_k-\Lambda|^2\zeta^2\tau\tau'\mathrm{d}x\mathrm{d}s+\int_{t_0-4r^2}^{t}\int_{B_{2r}}(f_i)_k(u_k^i-\Lambda^i)\zeta^2\tau^2\mathrm{d}x\mathrm{d}s$$

$$+\int_{t_0-4r^2}^{t}\int_{B_{2r}}(g_i^\alpha)_k[(D_\alpha u_k^i)\zeta^2+2(u_k^i-\Lambda^i)\zeta(D_\alpha\zeta)]\tau^2\mathrm{d}x\mathrm{d}s$$

for all $t\in(t_0-4r^2,t_0)$. Using Prop. 2.2 we find by letting tend $k\to+\infty$

$$\frac{1}{2}\int_{B_{2r}}|u(t)-\Lambda|^2\zeta^2\mathrm{d}x\tau^2(t)+\lambda_0\int_{t_0-4r^2}^{t}\int_{B_{2r}}\|Du\|^2\xi^2\tau^2\mathrm{d}x\mathrm{d}s\le$$

$$\le\frac{2c_0}{r}\int_{t_0-4r^2}^{t}\int_{B_{2r}}(1+|u|^{(n+2)/n}+\|Du\|)|u-\Lambda|\zeta\tau^2\mathrm{d}x\mathrm{d}s$$

$$+\frac{c_0}{r^2}\int_{t_0-4r^2}^{t}\int_{B_{2r}}|u-\Lambda|^2\mathrm{d}x\mathrm{d}s+\int_{t_0-4r^2}^{t}\int_{B_{2r}}|f|\,|u-\Lambda|\zeta^2\tau^2\mathrm{d}x\mathrm{d}s$$

$$+\int_{t_0-4r^2}^{t}\int_{B_{2r}}\|g\|\Big(\|Du\|\zeta^2\tau^2+\frac{2c_0}{r}|u-\Lambda|\Big)\mathrm{d}x\mathrm{d}s.$$

Then (3.3) is readily seen by employing Young's inequality.

To prove (3.4), we first note that

$$\int_{B_{2r}}[u_k^i(x,t)-\widetilde{(u_k^i)}_{2r}(t)]\zeta^2(x)\mathrm{d}x=0\quad(i=1,\ldots,N)$$

for all $t\in(t_0-4r^2,t_0)$. Hence

$$\int_{B_{2r}}\frac{\partial u_k^i}{\partial t}(t)(u_k^i(t)-\widetilde{(u_k^i)}_{2r}(t))\zeta^2\mathrm{d}x=\frac{1}{2}\int_{B_{2r}}\frac{\mathrm{d}}{\mathrm{d}t}|u_k(t)-\widetilde{(u_k)}_{2r}(t)|^2\zeta^2\mathrm{d}x,$$

and therefore

$$\int_{t_0-4r^2}^{t}\int_{B_{2r}}\frac{\partial u_k^i}{\partial s}(u_k^i-\widetilde{(u_k^i)}_{2r})\zeta^2\tau^2\mathrm{d}x\mathrm{d}s=$$

$$=\frac{1}{2}\int_{B_{2r}}|u_k(t)-\widetilde{(u_k)}_{2r}(t)|^2\zeta^2\mathrm{d}x\tau^2(t)-\int_{t_0-4r^2}\int_{B_{2r}}|u_k-\widetilde{(u_k)}_{2r}|^2\zeta^2\tau\tau'\mathrm{d}x\mathrm{d}s$$

for all $t\in(t_0-4r^2,t_0)$.

Now we insert

$$\varphi=(u_k(\cdot,t)-\widetilde{(u_k)}_{2r}(t))\zeta^2\tau^2(t),\quad t\in(t_0-4r^2,t_0)$$

into (2.1) and find

$$\frac{1}{2}\int_{B_{2r}}|u_k(x,t)-\widetilde{(u_k)}_{2r}(t)|^2\zeta^2(x)\mathrm{d}x\tau^2(t)+\int_{t_0-4r^2}^{t}\int_{B_{2r}}(a_i^\alpha)_k(D_\alpha u_k^i)\zeta^2\tau^2\mathrm{d}x\mathrm{d}s=$$

$$= -2 \int\limits_{t_0-4r^2}^{t} \int\limits_{B^{2r}} (a_i^\alpha)_k (u_k^i - \widetilde{(u_k^i)}_{2r}) \zeta (D_\alpha \zeta) \tau^2 \mathrm{d}x\mathrm{d}s$$
$$+ \int\limits_{t_0-4r^2} \int\limits_{B_{2r}} |u_k - \widetilde{(u_k)}_{2r}|^2 \zeta^2 \tau\tau' \mathrm{d}x\mathrm{d}s + \int\limits_{t_0-4r^2} \int\limits_{B_{2r}} (f_i)_k (u_k^i - \widetilde{(u_k^i)}_{2r}) \zeta^2 \tau^2 \mathrm{d}x\mathrm{d}s$$
$$+ \int\limits_{t_0-4r^2}^{t} \int\limits_{B_{2r}} (g_i^\alpha)_k [(D_\alpha u_k^i)\zeta^2 + 2(u_k^i - \widetilde{(u_k^i)}_{2r})\zeta(D_\alpha\zeta)] \tau^2 \mathrm{d}x\mathrm{d}s$$

for all $t \in (t_0 - 4r^2, t_0)$. Letting $k \to +\infty$ and using an analogous reasoning as above we find (3.4). ■

From Theorem 3.1 we derive

**COROLLARY 3.1** (CACCIOPPOLI *inequalities*) *Assume* (3.1). *Then*

$$\operatorname*{ess\,sup}_{(t_0-r^2,t_0)} \int\limits_{B_r} |u(x,t) - \Lambda|^2 \mathrm{d}x + \int\limits_{Q_r} \|Du\|^2 \mathrm{d}x\mathrm{d}t \leq$$
$$\leq c_2 \left\{ \frac{1}{r^2} \int\limits_{Q_{2r}} |u - \Lambda|^2 \mathrm{d}x\mathrm{d}t + \int\limits_{Q_{2r}} (1 + |u|^{2(n+2)/n} + \|g\|^2) \mathrm{d}x\mathrm{d}t + \Big( \int\limits_{Q_{2r}} |f|^{2(n+2)/(n+4)} \mathrm{d}x\mathrm{d}t \Big)^{(n+4)/(n+2)} \right\} \tag{3.5}$$

*for all* $\Lambda \in I\!R^N$, *and*

$$\operatorname*{ess\,sup}_{(t_0-r^2,t_0)} \int\limits_{B_r} |u(x,t) - \tilde{u}_r(t)|^2 \mathrm{d}x + \int\limits_{Q_r} \|Du\|^2 \mathrm{d}x\mathrm{d}t \leq$$
$$\leq c_2 \left\{ \frac{1}{r^2} \int\limits_{Q_{2r}} |u - \tilde{u}_{2r}|^2 \mathrm{d}x\mathrm{d}t + \int\limits_{Q_{2r}} (1 + |u|^{2(n+2)/n} + \|g\|^2) \mathrm{d}x\mathrm{d}t + \Big( \int\limits_{Q_{2r}} |f|^{2(n+2)/(n+4)} \mathrm{d}x\mathrm{d}t \Big)^{(n+4)/(n+2)} \right\} \tag{3.6}$$

($c_2 = \mathrm{const} > 0$ *independent of* $r$).

*Proof*. Fix $\varepsilon = \dfrac{\lambda_0}{2}$. Then (3.3) implies

$$\frac{1}{2} \int\limits_{B_{2r}} |u(x,t) - \Lambda|^2 \zeta^2(x) \mathrm{d}x \tau^2(t) + \frac{\lambda_0}{2} \int\limits_{t_0-4r^2}^{t} \int\limits_{B_{2r}} \|Du\|^2 \zeta^2 \tau^2 \mathrm{d}x\mathrm{d}s \leq$$
$$\leq c \left\{ \frac{1}{r^2} \int\limits_{Q_{2r}} |u - \Lambda|^2 \mathrm{d}x\mathrm{d}s + \int\limits_{Q_{2r}} (1 + |u|^{2(n+2)/n} + \|g\|^2) \mathrm{d}x\mathrm{d}s + \int\limits_{Q_{2r}} |f|\,|u - \Lambda| \zeta^2 \tau^2 \mathrm{d}x\mathrm{d}s \right\} \tag{3.7}$$

for a.a. $t \in (t_0 - 4r^2, t_0)$. Hence, the essential supremum of the function

$$t \mapsto \int_{B_{2r}} |u(x,t) - \Lambda|^2 \zeta^2(x) \mathrm{d}x \tau^2(t)$$

over the interval $(t_0 - 4r^2, t_0)$, as well as the integral

$$\int_{Q_{2r}} \|Du\|^2 \zeta^2 \tau^2 \mathrm{d}x \mathrm{d}s$$

are bounded by the right-hand side of (3.7). Thus

$$\begin{aligned} &\operatorname*{ess\,sup}_{(t_0-4r^2,t_0)} \int_{B_{2r}} |u(x,t) - \Lambda|^2 \zeta^2(x) \mathrm{d}x \tau^2(t) + \int_{Q_{2r}} \|Du\|^2 \zeta^2 \tau^2 \mathrm{d}x \mathrm{d}s \le \\ &\le c \left\{ \frac{1}{r^2} \int_{Q_{2r}} |u - \Lambda|^2 \mathrm{d}x \mathrm{d}s + \int_{Q_{2r}} (1 + |u|^{2(n+2)/n} + \|g\|^2) \mathrm{d}x \mathrm{d}s \right. \\ &\qquad \left. + \int_{Q_{2r}} |f|\, |u - \Lambda| \zeta^2 \tau^2 \mathrm{d}x \mathrm{d}s \right\}. \end{aligned} \tag{3.8}$$

Next, we combine the imbedding (1.5) and Hölder's and Young's inequality to obtain

$$\begin{aligned} &\int_{Q_{2r}} |f|\, |u - \Lambda| \zeta^2 \tau^2 \mathrm{d}x \mathrm{d}s \le \\ &\le \delta \left\{ \operatorname*{ess\,sup}_{(t_0-4r^2,t_0)} \int_{B_{2r}} |u(x,t) - \Lambda|^2 \zeta^2(x) \mathrm{d}x \tau^2(t) + \int_{Q_{2r}} \|D((u - \Lambda)\zeta^2\tau^2)\|^2 \mathrm{d}x \mathrm{d}t \right\} \\ &\qquad + \frac{c}{\delta} \Big( \int_{Q_{2r}} |f|^{2(n+2)/(n+4)} \mathrm{d}x \mathrm{d}t \Big)^{(n+4)/(n+2)} \end{aligned}$$

for all $\delta > 0$. Thus, by choosing $\delta$ appropriately small, from (3.8) we obtain (3.5).

By an analogous argument, from (3.4) it follows that

$$\begin{aligned} &\operatorname*{ess\,sup}_{(t_0-4r^2,t_0)} \int_{B_{2r}} |u(x,t) - \tilde{u}_{2r}(t)|^2 \zeta^2(x) \mathrm{d}x \tau^2(t) + \int_{Q_{2r}} \|Du\|^2 \zeta^2 \tau^2 \mathrm{d}x \mathrm{d}t \le \\ &\le c \left\{ \frac{1}{r^2} \int_{Q_{2r}} |u - \tilde{u}_{2r}|^2 \mathrm{d}x \mathrm{d}s + \int_{Q_{2r}} (1 + |u|^{2(n+2)/n} \zeta^2 \tau^2 + \|g\|^2) \mathrm{d}x \mathrm{d}s \right. \\ &\qquad \left. + \Big( \int_{Q_{2r}} |f|^{2(n+2)/(n+4)} \mathrm{d}x \mathrm{d}s \Big)^{(n+4)/(n+2)} \right\}. \end{aligned} \tag{3.9}$$

We estimate the first term on the left of (3.9) from below. To this end, let $\zeta_r$ denote the cut-off function with respect to $B_r$ (analogously as $\zeta_{2r}$ with respect to $B_{2r}$; cf. footnote 2). We have

$$\int_{B_r} \zeta_r^2 \mathrm{d}x \ge |B_{r/2}| = \frac{1}{2^n} |B_r|.$$

Let $\varphi \in W_2^1(B_r)$. A simple calculation shows that the minimum of the function

$$\lambda \mapsto \int\limits_{B_r} (\varphi(x) - \lambda)^2 \zeta_r^2(x) \mathrm{d}x, \quad \lambda \in \mathbb{R}$$

is attained at the value

$$\lambda_r = \frac{1}{\int\limits_{B_r} \zeta_r^2 \mathrm{d}x} \int\limits_{B_r} \varphi \zeta_r^2 \mathrm{d}y.$$

We obtain

$$\int\limits_{B_r} |\tilde{u}_{2r}(t) - \tilde{u}_r(t)|^2 \zeta_r^2(x) \mathrm{d}x \le 4 \int\limits_{B_r} |u(x,t) - \tilde{u}_{2r}(t)|^2 \zeta_r^2(x) \mathrm{d}x$$

for a.a. $t \in (t_0 - r^2, t_0)$, and therefore

$$\begin{aligned}
&\int\limits_{B_r} |u(x,t) - \tilde{u}_r(t)|^2 \mathrm{d}x \le \\
&\le 2 \int\limits_{B_r} |u(x,t) - \tilde{u}_{2r}(t)|^2 \mathrm{d}x + 2^{n+1} \int\limits_{B_r} |\tilde{u}_{2r}(t) - \tilde{u}_r(t)|^2 \zeta_r^2 \mathrm{d}x \\
&\le 2(1 + 2^{n+2}) \int\limits_{B_r} |u(x,t) - \tilde{u}_{2r}(t)|^2 \mathrm{d}x \\
&\le 2(1 + 2^{n+2}) \int\limits_{B_{2r}} |u(x,t) - \tilde{u}_{2r}(t)|^2 \zeta_{2r}^2 \mathrm{d}x.
\end{aligned}$$

Then (3.9) implies (3.6). ■

For later use (i.e. parabolic systems with quadratic growth nonlinearities) we note the following CACCIOPPOLI inequality which immediately follows from (3.4). *Assume* (3.2). *Then, for any* $0 < \varepsilon < \lambda_0$,

$$\begin{aligned}
&\frac{1}{1 + 2^{n+2}} \int\limits_{B_r} |u(x,t) - \tilde{u}_r(t)|^2 \mathrm{d}x + (\lambda_0 - \varepsilon) \int\limits_{t_0 - 4r^2}^{t} \int\limits_{B_{2r}} \|Du\|^2 \zeta^2 \tau^2 \mathrm{d}x \mathrm{d}s \le \\
&\le c_1 \Big(1 + \frac{1}{\varepsilon}\Big) \frac{1}{r^2} \int\limits_{Q_{2r}} |u - \tilde{u}_{2r}|^2 \mathrm{d}x \mathrm{d}s + c_1 \int\limits_{Q_{2r}} \Big(1 + |u|^{2(n+2)/n} + \Big(1 + \frac{1}{\varepsilon}\Big) \|g\|^2\Big) \mathrm{d}x \mathrm{d}s \qquad (3.10) \\
&\quad + 2\|u\|_{L^\infty(Q;\mathbb{R}^N)} \int\limits_{Q_{2r}} |f| \zeta^2 \tau^2 \mathrm{d}x \mathrm{d}s
\end{aligned}$$

*for a.a.* $t \in (t_0 - 4r^2, t_0)$. ■

# 4 POINCARÉ inequalities

Let $B_r \subset I\!R^n$ be a ball. Fix $\eta \in C(\overline{B}_r)$ such that

$$\int\limits_{B_r} \eta \mathrm{d}x > 0.$$

Then the following inequality is well-known:

$$\text{(4.1)} \qquad \|v\|_{W^1_q(B_r)} \le c\Big\{ \int\limits_{B_r} |Dv|^q \mathrm{d}x + \Big( \int\limits_{B_r} \eta \mathrm{d}x \Big)^{-q} \Big| \int\limits_{B_r} v\eta \mathrm{d}x \Big|^q \Big\}^{1/q}$$

for all $v \in W^1_q(B_r)$ ($1 \le q < +\infty$; $c = \text{const} > 0$ independent of $r$).

We specialize (4.1) as follows. Firstly, let $\eta = \zeta_r$ be the cut-off function with respect to $B_r$ introduced in the preceding section. Then (4.1) (combined with a homothetical argument) implies the SOBOLEV-POINCARÉ inequality

$$\text{(4.2)} \qquad \Big( \int\limits_{B_r} |v - \tilde{v}_r|^s \mathrm{d}x \Big)^{1/s} \le c_0 r^{1+n/s-n/q} \Big( \int\limits_{B_r} |Dv|^q \mathrm{d}x \Big)^{1/q} \quad \forall\, v \in W^1_q(B_r),$$ [4)]

where

$$1 \le s \le \frac{nq}{n-q} \quad \text{if} \quad 1 \le q < n,$$
$$1 \le s < +\infty \quad \text{if} \quad q = n$$

($c_0 = \text{const} > 0$ independent of $r$).

Secondly, let $\eta \equiv 1$. Then from (4.1) we obtain the well-known POINCARÉ inequality

$$\text{(4.3)} \qquad \int\limits_{B_r} (v - v_{B_r})^2 \mathrm{d}x \le c_0^2 r^2 \int\limits_{B_r} |Dv|^2 \mathrm{d}x \quad \forall\, v \in W^1_2(B_r);$$

here $v_{B_r}$ denotes the usual integral mean of $v$ over $B_r$:

$$v_{B_r} = \fint\limits_{B_r} v \mathrm{d}y = \frac{1}{|B_r|} \int\limits_{B_r} v \mathrm{d}y. \qquad \blacksquare$$

Let $w \in W^{1,0}_2(Q_r)$. Set

$$\bar{w}_r = \frac{1}{r^2 \int\limits_{B_r} \zeta^2 \mathrm{d}y} \int\limits_{Q_r} w(x,s)\zeta^2(x) \mathrm{d}x\mathrm{d}s.$$

Using (4.3) we find

$$\int\limits_{Q_r} (w - w_{Q_r})^2 \mathrm{d}x\mathrm{d}t \le \int\limits_{Q_r} (w - \bar{w}_r)^2 \mathrm{d}x\mathrm{d}t$$

$$\le 2c_0^2 r^2 \int\limits_{Q_r} |Dw|^2 \mathrm{d}x\mathrm{d}t + 2|B_r| \int\limits_{t_0-r^2}^{t_0} (\tilde{w}_r(t) - \bar{w}_r)^2 \mathrm{d}t$$

$$\text{(4.4)} \qquad \le 2c_0^2 r^2 \int\limits_{Q_r} |Dw|^2 \mathrm{d}x\mathrm{d}t + 2|B_1| r^{n-2} \int\limits_{t_0-r^2}^{t_0} \int\limits_{t_0-r^2}^{t_0} (\tilde{w}_r(t) - \tilde{w}_r(s))^2 \mathrm{d}t\mathrm{d}s. \qquad \blacksquare$$

---

[4)] Recall $\tilde{v}_r = \Big( \int\limits_{B_r} \zeta_r^2 \mathrm{d}x \Big)^{-1} \int\limits_{B_r} v\zeta_r^2 \mathrm{d}x$.

Let $u \in V_2^{1,0}(Q;I\!R^N)$ be a weak solution of (1.1). Then the integral

$$\int_{t_0-r^2}^{t_0}\int_{t_0-r^2}^{t_0} |\tilde{u}_r(t)-\tilde{u}_r(s)|^2 \mathrm{d}t\mathrm{d}s$$

can be estimated by employing (2.1). Our result is the following

**THEOREM 4.1** *Assume* (3.1). *Then:*

$$\int_{Q_r} |u-u_{Q_r}|^2 \mathrm{d}x\mathrm{d}t \le$$

$$\le c_1 r^2 \left\{ \int_{Q_r} (1+|u|^{2(n+2)/n}+\|Du\|^2+\|g\|^2)\mathrm{d}x\mathrm{d}t + \Big(\int_{Q_r} |f|^{2(n+2)/(n+4)}\mathrm{d}x\mathrm{d}t\Big)^{(n+4)/(n+2)} \right\}, \tag{4.5}$$

*and*

$$\operatorname*{ess\,sup}_{(t_0-r^2,t_0)} \int_{B_r} |u(x,t)-\tilde{u}_r(t)|^2 \mathrm{d}x \le$$

$$\le c_2 \left\{ \int_{Q_{2r}} (1+|u|^{2(n+2)/n}+\|Du\|^2+\|g\|^2)\mathrm{d}x\mathrm{d}t + \Big(\int_{Q_{2r}} |f|^{2(n+2)/(n+4)}\mathrm{d}x\mathrm{d}t\Big)^{(n+4)/(n+2)} \right\}, \tag{4.6}$$

*where the constants $c_1$ and $c_2$ do not depend on $r$.*

*Proof* (cf. [7]). Let $\zeta = \zeta_r$ be the cut-off function with respect to $B_r$ (cf. p. 6). The function $\varphi = \{0,\dots,0,\zeta^2,0,\dots,0\}$ [$\zeta^2$ at the $i$-th place ($i=1,\dots,N$)] is admissible in (2.1). We find

$$\int_{B_r} \zeta^2 \mathrm{d}y\big(\widetilde{(u_k^i)}_r(t) - \widetilde{(u_k^i)}_r(s)\big) =$$

$$= \int_{B_r} [u_k^i(x,t)-u_k^i(x,s)]\zeta^2(x)\mathrm{d}x = \int_s^t \int_{B_r} \frac{\partial u_k^i}{\partial \tau}\zeta^2 \mathrm{d}x\mathrm{d}\tau =$$

$$= -2\int_s^t\int_{B_r} (a_i^\alpha)_k \zeta D_\alpha\zeta \mathrm{d}x\mathrm{d}\tau + \int_s^t\int_{B_r} ((f_i)_k\zeta^2 + 2(g_i^\alpha)_k\zeta D_\alpha \zeta)\mathrm{d}x\mathrm{d}\tau \tag{4.7}$$

for a.a. $s,t\in(t_0-r^2,t_0)$ $(s<t)$ and all integers $k > \dfrac{1}{T-t_0}$. Clearly,

$$\widetilde{(u_k^i)}_r(t) = k\int_t^{t+1/k} \left( \frac{1}{\int_{B_r}\zeta^2\mathrm{d}y} \int_{B_r} u^i(x,\tau)\zeta^2(x)\mathrm{d}x \right) \mathrm{d}\tau \longrightarrow \widetilde{(u^i)}_r(t)$$

for a.a. $t \in (t_0 - r^2, t)$ as $k \to \infty$ (cf. Proposition 2.2). Thus, letting tend $k \to \infty$ in (4.7) and multiplying the result by $\widetilde{(u^i)}_r(t) - \widetilde{(u^i)}_r(s)$ gives

$$
\begin{aligned}
|B_{r/2}|\, |\tilde{u}_r(t) - \tilde{u}_r(s)|^2 &\le \int\limits_{B_r} \zeta^2 \mathrm{d}y |\tilde{u}_r(t) - \tilde{u}_r(s)|^2 \\
&= -2 \int\limits_s^t \int\limits_{B_r} a_i^\alpha \zeta D_\alpha \zeta \mathrm{d}x \mathrm{d}\tau [\widetilde{(u^i)}_r(t) - \widetilde{(u^i)}_r(s)] \\
&\quad + \int\limits_s^t \int\limits_{B_r} (f_i \zeta^2 + 2 g_i^\alpha \zeta D_\alpha \zeta) \mathrm{d}x \mathrm{d}\tau [\widetilde{(u^i)}_r(t) - \widetilde{(u^i)}_r(s)] \\
&\le \frac{2c_0}{r} \int\limits_{Q_r} \|a\| \mathrm{d}x \mathrm{d}\tau |\tilde{u}_r(t) - \tilde{u}_r(s)| + \int\limits_{Q_r} \Big(|f| + \frac{2c_0}{r}\|g\|\Big) \mathrm{d}x \mathrm{d}\tau |\tilde{u}_r(t) - \tilde{u}_r(s)|.
\end{aligned}
$$

Therefore, by (1.3),

$$
\begin{aligned}
&|B_1| r^n |\tilde{u}_r(t) - \tilde{u}_r(s)|^2 \le \\
&\le c\Big\{ \int\limits_{Q_r} (1 + |u|^{2(n+2)/n} + \|Du\|^2 + \|g\|^2) \mathrm{d}x \mathrm{d}\tau \\
&\qquad + \Big( \int\limits_{Q_r} |f|^{2(n+2)/(n+4)} \mathrm{d}x \mathrm{d}\tau \Big)^{(n+4)/(n+2)} \Big\} \qquad (4.8)
\end{aligned}
$$

(obviously, (4.8) continues to hold for a.a. $s, t \in (t_0 - r^2, t_0)$ with $t < s$). Now, combining (4.4) and (4.8) gives (4.5).

In order to prove (4.6) we employ (4.2) ($s = q = 2$; $2r$ in place of $r$) to obtain

$$
\int\limits_{Q_{2r}} |u - \tilde{u}_{2r}|^2 \mathrm{d}x \mathrm{d}t \le \gamma_0^2 r^2 \int\limits_{Q_{2r}} \|Du\|^2 \mathrm{d}x \mathrm{d}t.
$$

Then (4.6) follows from (3.6). ■

# 5 Extended CACCIOPPOLI inequality

We maintain the notation introduced in Section 3. The aim of the present section is to establish an extended version of the CACCIOPPOLI inequality (3.6) which will be used for proving the local higher integrability of $Du$ below.

**THEOREM 5.1** *Assume* (3.1). *Then:*

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{(t_0 - r^2, t_0)} \int\limits_{B_r} |u - \tilde{u}_r(t)|^2 \mathrm{d}x + \int\limits_{Q_r} (|u|^{2(n+2)/n} + \|Du\|^2) \mathrm{d}x \mathrm{d}t \\
&\le c\Big\{ \frac{1}{r^2} \int\limits_{Q_{2r}} |u - \tilde{u}_{2r}|^2 \mathrm{d}x \mathrm{d}t + \int\limits_{Q_{2r}} (1 + |f|^{2(n+2)/(n+4)} + \|g\|^2) \mathrm{d}x \mathrm{d}t \Big\} \qquad (5.1) \\
&\quad + c r^{-2(n+2)/n} \Big( \int\limits_{Q_{2r}} |u|^2 \mathrm{d}x \mathrm{d}t \Big)^{(n+2)/n} + \sigma(r) \int\limits_{Q_{2r}} (|u|^{2(n+2)/(n+4)} + \|Du\|^2) \mathrm{d}x \mathrm{d}t,
\end{aligned}
$$

*where* $c = \text{const}$ *is independent of* $r$, *and*

$$\sigma(r) > 0, \quad \lim_{r\to 0} \sigma(r) = 0.$$

*Proof*. Adding the integral

$$\int\limits_{Q_{2r}} |u|^{2(n+2)/n} \zeta\tau \mathrm{d}x\mathrm{d}t$$

to both sides of (3.9) gives

$$\operatorname*{ess\,sup}_{(t_0-4r^2,t_0)} \int\limits_{B_{2r}} |u-\tilde{u}_{2r}(t)|^2\zeta^2 \mathrm{d}x\tau^2(t) + \int\limits_{Q_{2r}} \|Du\|^2\zeta^2\tau^2\mathrm{d}x\mathrm{d}t + \int\limits_{Q_{2r}} |u|^{2(n+2)/n}\zeta\tau\mathrm{d}x\mathrm{d}t \le$$

$$\le c\left\{\frac{1}{r^2}\int\limits_{Q_{2r}} |u-\tilde{u}_{2r}|^2\mathrm{d}x\mathrm{d}t + \int\limits_{Q_{2r}} (1+\|g\|^2)\mathrm{d}x\mathrm{d}t \right.$$
$$\left. + \Big(\int\limits_{Q_{2r}} |f|^{2(n+2)/(n+4)}\mathrm{d}x\mathrm{d}t\Big)^{(n+4)/(n+2)}\right\} + c\int\limits_{Q_{2r}} |u|^{2(n+2)/n}\zeta\tau\mathrm{d}x\mathrm{d}t. \tag{5.2}$$

We estimate the last integral on the right of (5.2). To begin with, we note that

$$\int\limits_{Q_{2r}} |u|^{2(n+2)/n}\zeta\tau\mathrm{d}x\mathrm{d}t \le$$
$$\le c\int\limits_{Q_{2r}} |u-\tilde{u}_{2r}(t)|^{2(n+2)/n}\zeta\tau\mathrm{d}x\mathrm{d}t$$
$$+c\int\limits_{Q_{2r}} \left|\tilde{u}_{2r}(t) - \frac{1}{4r^2\int\limits_{B_{2r}}\zeta^2\mathrm{d}y}\int\limits_{Q_{2r}} u\zeta^2\mathrm{d}x\mathrm{d}s\right|^{2(n+2)/n}\mathrm{d}x\mathrm{d}t$$
$$+c|Q_{2r}|\left|\frac{1}{4r^2\int\limits_{B_{2r}}\zeta^2\mathrm{d}y}\int\limits_{Q_{2r}} u\zeta^2\mathrm{d}x\mathrm{d}s\right|^{2(n+2)/n}$$

(for instance, $c = 3^{2(n+2)/n}$). By the imbedding (1.5),

$$\int\limits_{Q_{2r}} |u-\tilde{u}_{2r}(t)|^{2(n+2)/n}\zeta\tau\mathrm{d}x\mathrm{d}t \le$$
$$\le \Big(\int\limits_{Q_{2r}} |(u-\tilde{u}_{2r}(t))\zeta\tau|^{2(n+2)/n}\mathrm{d}x\mathrm{d}t\Big)^{n/2(n+2)}\Big(\int\limits_{Q_{2r}} |u-\tilde{u}_{2r}(t)|^{2(n+2)/n}\mathrm{d}x\mathrm{d}t\Big)^{(n+4)/2(n+2)}$$
$$\le \delta\Big(\operatorname*{ess\,sup}_{(t_0-4r^2,t_0)}\int\limits_{B_{2r}} |u-\tilde{u}_{2r}(t)|^2\zeta^2\mathrm{d}x\tau^2(t) + \int\limits_{Q_{2r}} \|Du\|^2\zeta^2\tau^2\mathrm{d}x\mathrm{d}t$$
$$+\frac{c}{r^2}\int\limits_{Q_{2r}} |u-\tilde{u}_{2r}|^2\mathrm{d}x\mathrm{d}t\Big) + \frac{c}{\delta}\Big(\int\limits_{Q_{2r}} |u-\tilde{u}_{2r}|^{2(n+2)/n}\mathrm{d}x\mathrm{d}t\Big)^{(n+4)/(n+2)}$$

for all $\delta > 0$. Hence, choosing $\delta$ sufficiently small, from (5.2) we obtain

$$\operatorname*{ess\,sup}_{(t_0-r^2,t_0)} \int_{B_r} |u - \tilde{u}_r(t)|^2 \mathrm{d}x + \int_{Q_r} (|u|^{2(n+2)/n} + \|Du\|^2) \mathrm{d}x\mathrm{d}t \leq$$

$$\leq c \left\{ \frac{1}{r^2} \int_{Q_{2r}} |u - \tilde{u}_{2r}|^2 \mathrm{d}x\mathrm{d}t + \int_{Q_{2r}} (1 + |f|^{2(n+2)/(n+4)} + \|g\|^2) \mathrm{d}x\mathrm{d}t \right\} \tag{5.3}$$

$$+c\Big( \int_{Q_{2r}} |u - \tilde{u}_{2r}|^{2(n+2)/n} \mathrm{d}x\mathrm{d}t \Big)^{(n+4)/(n+2)}$$

$$+c \int_{Q_{2r}} \left| \tilde{u}_{2r}(t) - \frac{1}{4r^2 \int_{B_{2r}} \zeta^2 \mathrm{d}y} \int_{Q_{2r}} u\zeta^2 \mathrm{d}x\mathrm{d}s \right|^{2(n+2)/n} \mathrm{d}x\mathrm{d}t$$

$$+cr^{n+2} \left| \frac{1}{4r^2 \int_{B_{2r}} \zeta^2 \mathrm{d}y} \int_{Q_{2r}} u\zeta^2 \mathrm{d}x\mathrm{d}s \right|^{2(n+2)/n};$$

here the constant $c$ at the first term on the right depends on $\|f\|_{L^{2(n+2)/(n+4)}(Q;\mathbb{R}^n)}$.

It remains to estimate the second, third and fourth term on the right of (5.3).

1) We have

$$\Big( \int_{Q_{2r}} |u - \tilde{u}_{2r}|^{2(n+2)/n} \mathrm{d}x\mathrm{d}t \Big)^{(n+4)/(n+2)} \leq$$

$$\leq c\Big( \int_{Q_{2r}} |u|^{2(n+2)/n} \mathrm{d}x\mathrm{d}t \Big)^{2/(n+2)} \int_{Q_{2r}} |u|^{2(n+2)/n} \mathrm{d}x\mathrm{d}t.$$

2) Obviously,

$$\left| \tilde{u}_{2r}(t) - \frac{1}{4r^2 \int_{B_{2r}} \zeta^2 \mathrm{d}y} \int_{Q_{2r}} u\zeta^2 \mathrm{d}x\mathrm{d}s \right| = \frac{1}{4r^2} \left| \int_{t_0-4r^2}^{t_0} (\tilde{u}_{2r}(t) - \tilde{u}_{2r}(s)) \mathrm{d}s \right|$$

$$\leq \frac{1}{2r} \Big( \int_{t_0-4r^2}^{t_0} |\tilde{u}_{2r}(t) - \tilde{u}_{2r}(s)|^2 \mathrm{d}s \Big)^{1/2}$$

for a.a. $t \in (t_0 - 4r^2, t_0)$. On the other hand, (4.8) implies

$$\Big( \int_{t_0-4r^2}^{t_0} |\tilde{u}_{2r}(t) - \tilde{u}_{2r}(s)|^2 \mathrm{d}s \Big)^{1/2} \leq$$

$$\leq cr^{1-n/2} \left\{ \int_{Q_{2r}} (1 + |u|^{2(n+2)/n} + \|Du\|^2 + |f|^{2(n+2)/(n+4)} + \|g\|^2) \mathrm{d}x\mathrm{d}s \right\}^{1/2}.$$

Thus

$$\int\limits_{Q_{2r}} \left| \tilde{u}_{2r}(t) - \frac{1}{4r^2 \int\limits_{B_{2r}} \zeta^2 \mathrm{d}y} \int\limits_{Q_{2r}} u\zeta^2 \mathrm{d}x\mathrm{d}s \right|^{2(n+2)/n} \mathrm{d}x\mathrm{d}t \le$$

$$\le c \left\{ \int\limits_{Q_{2r}} (1 + |u|^{2(n+2)/n} + \|Du\|^2 + |f|^{2(n+2)/(n+4)} + \|g\|^2) \mathrm{d}x\mathrm{d}s \right\}^{(n+2)/n}$$

$$\le c \left\{ \int\limits_{Q_{2r}} (|u|^{2(n+2)/n} + \|Du\|^2) \mathrm{d}x\mathrm{d}s \right\}^{2/n} \int\limits_{Q_{2r}} (|u|^{2(n+2)/n} + \|Du\|^2) \mathrm{d}x\mathrm{d}s$$

$$+ c \int\limits_{Q_{2r}} (1 + |f|^{2(n+2)/(n+4)} + \|g\|^2) \mathrm{d}x\mathrm{d}s.$$

3) Clearly,

$$r^{n+2} \left| \frac{1}{4r^2 \int\limits_{B_{2r}} \zeta^2 \mathrm{d}y} \int\limits_{Q_{2r}} u\zeta^2 \mathrm{d}x\mathrm{d}s \right|^{2(n+2)/n} \le$$

$$\le c r^{(n+2)[1-2(n+2)/n]} \Big( \int\limits_{Q_{2r}} |u| \mathrm{d}x\mathrm{d}s \Big)^{2(n+2)/n} \le c r^{-2(n+2)/n} \Big( \int\limits_{Q_{2r}} |u|^2 \mathrm{d}x\mathrm{d}s \Big)^{(n+2)/n}.$$

Inserting the estimates from 1), 2) and 3) into (5.3) gives (5.1) with

$$\sigma(r) = c \Big( \int\limits_{Q_{2r}} |u|^{2(n+2)/n} \mathrm{d}x\mathrm{d}t \Big)^{2/(n+2)} + c \Big\{ \int\limits_{Q_{2r}} (|u|^{2(n+2)/n} + \|Du\|^2) \mathrm{d}x\mathrm{d}t \Big\}^{2/n},$$

where the constant $c$ does not depend on $r$. ■

# 6 Local higher integrability of $Du$

**THEOREM 6.1** *Let* (1.2) – (1.4) *be satisfied. Assume*

$$f \in L^{q_1}(Q; \mathbb{R}^N) \quad \Big( q_1 > \frac{2(n+2)}{n+4} \Big), \qquad g \in L^{q_2}(Q; \mathbb{R}^{nN}) \quad (q_2 > 2).$$

*Let* $u \in V_2^{1,0}(Q; \mathbb{R}^N)$ *be a weak solution of* (1.1) (*cf. Definition* 1.1).

*Then there exists a* $p > 2$ *such that*

$$(|u|^{p(n+2)/n} + \|Du\|^p) \in L^1(\Omega' \times (t', T)) \quad \forall\, \Omega' \subset\subset \Omega,\ \forall\, t' \in (0, T),$$

$$\int\limits_{Q_r} (|u|^{p(n+2)/n} + \|Du\|^p) \mathrm{d}x\mathrm{d}t$$

$$\le c r^{(n+2)(1-p/2)} \Big\{ \int\limits_{Q_{4r}} (|u|^{2(n+2)/n} + \|Du\|^2) \mathrm{d}x\mathrm{d}t \Big\}^{p/2} \tag{6.1}$$

$$+ c \int\limits_{Q_{4r}} (1 + |f|^{q_1} + \|g\|^{q_2}) \mathrm{d}x\mathrm{d}t$$

*for all cylinders $Q_{4r} \subset \overline{Q}_{4r} \subset \Omega \times (0,T]$, where the constant $c$ does not depend on $r$.*

*Proof.* Let $n \geq 3$. We combine (4.6) (with $2r$ in place of $r$) and (4.2) to obtain

$$\int\limits_{Q_{2r}} |u - \tilde{u}_{2r}|^2 \mathrm{d}x\mathrm{d}t \leq$$

$$\leq \Big( \operatorname*{ess\,sup}_{(t_0-4r^2,t_0)} \int\limits_{B_{2r}} |u(x,s) - \tilde{u}_{2r}(s)|^2 \mathrm{d}x \Big)^{1/2} \int\limits_{t_0-4r^2}^{t_0} \Big( \int\limits_{B_{2r}} |u - \tilde{u}_{2r}|^2 \mathrm{d}x \Big)^{1/2} \mathrm{d}t$$

$$\leq c \left\{ \int\limits_{Q_{4r}} (1 + |u|^{2(n+2)/n} + \|Du\|^2 + \|g\|^2) \mathrm{d}x\mathrm{d}t + \right.$$

$$\left. + \Big( \int\limits_{Q_{4r}} |f|^{2(n+2)/(n+4)} \mathrm{d}x\mathrm{d}t \Big)^{(n+4)/(n+2)} \right\}^{1/2} \times$$

$$\times \int\limits_{t_0-4r^2}^{t_0} \Big( \int\limits_{B_{2r}} |u - \tilde{u}_{2r}|^{2n/(n-2)} \mathrm{d}x \Big)^{(n+2)/4n} \Big( \int\limits_{B_{2r}} |u - \tilde{u}_{2r}|^{2n/(n+2)} \mathrm{d}x \Big)^{(n+2)/4n} \mathrm{d}t$$

$$\leq c r^{1/2} \left\{ \int\limits_{Q_{4r}} (1 + |u|^{2(n+2)/n} + \|Du\|^2 + |f|^{2(n+2)/(n+4)} + \|g\|^2) \mathrm{d}x\mathrm{d}t \right\}^{1/2} \times$$

$$\times \int\limits_{t_0-4r^2}^{t_0} \Big( \int\limits_{B_{2r}} \|Du\|^2 \mathrm{d}x \Big)^{1/4} \Big( \int\limits_{B_{2r}} \|Du\|^{2n/(n+2)} \mathrm{d}x \Big)^{(n+2)/4n} \mathrm{d}t$$

$$\leq c r^{3/2-1/n} \left\{ \int\limits_{Q_{4r}} (1 + |u|^{2(n+2)/n} + \|Du\|^2 + |f|^{2(n+2)/(n+4)} + \|g\|^2) \mathrm{d}x\mathrm{d}t \right\}^{3/4} \times$$

$$\times \Big( \int\limits_{Q_{2r}} \|Du\|^{2n/(n+2)} \mathrm{d}x\mathrm{d}t \Big)^{(n+2)/4n},$$

the constant $c$ being dependent on $\|f\|_{L^{2(n+2)/(n+4)}(Q;\mathbb{R}^N)}$. Inserting this into the extended CACCIOPPOLI inequality (5.1) we get, for every $0 < \varepsilon \leq 1$,

$$\int\limits_{Q_r} (|u|^{2(n+2)/n} + \|Du\|^2) \mathrm{d}x\mathrm{d}t \leq$$

$$\leq c r^{-1/2-1/n} \left\{ \int\limits_{Q_{4r}} (1 + |u|^{2(n+2)/n} + \|Du\|^2 + |f|^{2(n+2)/(n+4)} + \|g\|^2) \mathrm{d}x\mathrm{d}t \right\}^{3/4} \times$$

$$\times \Big( \int\limits_{Q_{2r}} \|Du\|^{2n/(n+2)} \mathrm{d}x\mathrm{d}t \Big)^{(n+2)/4n} + c \int\limits_{Q_{4r}} (1 + |f|^{2(n+2)/(n+4)} + \|g\|) \mathrm{d}x\mathrm{d}t$$

$$+ c r^{-2(n+2)/n} \Big( \int\limits_{Q_{4r}} |u|^2 \mathrm{d}x\mathrm{d}t \Big)^{(n+2)/n} + \sigma(r) \int\limits_{Q_{4r}} (|u|^{2(n+2)/(n+4)} + \|Du\|^2) \mathrm{d}x\mathrm{d}t$$

$$\leq c\Big(1+\frac{1}{\varepsilon^3}\Big)r^{-2(n+2)/n}\left\{\int\limits_{Q_{4r}}(|u|^2+\|Du\|^{2n/(n+2)})\mathrm{d}x\mathrm{d}t\right\}^{(n+2)/n}$$

$$+(\varepsilon+\sigma(r))\int\limits_{Q_{4r}}(|u|^{2(n+2)/n}+\|Du\|^2)\mathrm{d}x\mathrm{d}t$$

$$+c\int\limits_{Q_{4r}}(1+|f|^{2(n+2)/(n+4)}+\|g\|^2)\mathrm{d}x\mathrm{d}t.$$

Dividing by $|Q_r|$ gives

$$\fint\limits_{Q_r}(|u|^{2(n+2)/n}+\|Du\|^2)\mathrm{d}x\mathrm{d}t\leq$$

$$\leq c\Big(1+\frac{1}{\varepsilon^3}\Big)\left\{\fint\limits_{Q_{4r}}(|u|^2+\|Du\|^{2n/(n+2)})\mathrm{d}x\mathrm{d}t\right\}^{(n+2)/n} \tag{6.2}$$

$$+4^{n+2}(\varepsilon+\sigma(r))\fint\limits_{Q_{4r}}(|u|^{2(n+2)/n}+\|Du\|^2)\mathrm{d}x\mathrm{d}t$$

$$+c\fint\limits_{Q_{4r}}(1+|f|^{2(n+2)/(n+4)}+\|g\|^2)\mathrm{d}x\mathrm{d}t$$

for all $0<\varepsilon\leq 1$.

Let $n=2$. Again combining (4.6) and (4.2) yields

$$\int\limits_{Q_{2r}}|u-\tilde{u}_{2r}|^2\mathrm{d}x\mathrm{d}t\leq$$

$$\leq c\left\{\int\limits_{Q_{4r}}(1+|u|^4+\|Du\|^2+|f|^{3/4}+\|g\|^2)\mathrm{d}x\mathrm{d}t\right\}^{1/2}\int\limits_{Q_{4r}}\|Du\|\mathrm{d}x\mathrm{d}t.$$

Thus, by (5.1),

$$\int\limits_{Q_r}(|u|^4+\|Du\|^2)\mathrm{d}x\mathrm{d}t\leq$$

$$\leq cr^{-2}\left\{\int\limits_{Q_{4r}}(1+|u|^4+\|Du\|^2+|f|^{3/4}+\|g\|^2)\mathrm{d}x\mathrm{d}t\right\}^{1/2}\int\limits_{Q_{4r}}\|Du\|\mathrm{d}x\mathrm{d}t$$

$$+c\int\limits_{Q_{4r}}(1+|f|^{3/4}+\|g\|^2)\mathrm{d}x\mathrm{d}t+cr^{-4}\Big(\int\limits_{Q_{4r}}|u|^2\mathrm{d}x\mathrm{d}t\Big)^2$$

$$+\sigma(r)\int\limits_{Q_{4r}}(|u|^4+\|Du\|^2)\mathrm{d}x\mathrm{d}t$$

$$\leq c\Big(1+\frac{1}{\varepsilon}\Big)r^{-4}\left\{\int\limits_{Q_{4r}}(|u|^2+\|Du\|)\mathrm{d}x\mathrm{d}t\right\}^2$$

$$+(\varepsilon+\sigma(r))\int\limits_{Q_{4r}}(|u|^4+\|Du\|^2)\mathrm{d}x\mathrm{d}t+c\int\limits_{Q_{4r}}(1+|f|^{4/3}+\|g\|^2)\mathrm{d}x\mathrm{d}t$$

for all $0<\varepsilon\le 1$. Dividing this inequality by $|Q_r|$ gives

$$\fint\limits_{Q_r}(|u|^4+\|Du\|^2)\mathrm{d}x\mathrm{d}t\le$$

$$\le c\Big(1+\frac{1}{\varepsilon}\Big)\left\{\fint\limits_{Q_{4r}}(|u|^2+\|Du\|)\mathrm{d}x\mathrm{d}t\right\}^2 \tag{6.3}$$

$$+4^4(\varepsilon+\sigma(r))\fint\limits_{Q_{4r}}(|u|^4+\|Du\|^2)\mathrm{d}x\mathrm{d}t+c\fint\limits_{Q_{4r}}(1+|f|^{4/3}+\|g\|^2)\mathrm{d}x\mathrm{d}t.$$

We *fix* $0<\varepsilon\le 1$ and $r_0>0$ (depending on $n\ge 2$) such that

$$4^{n+2}(\varepsilon+\sigma(r))\le\frac{1}{2}\qquad\forall\, 0<r\le r_0.$$

Thus

$$\fint\limits_{Q_r}(|u|^{2(n+2)/n}+\|Du\|^2)\mathrm{d}x\mathrm{d}t\le$$

$$\le c\left\{\fint\limits_{Q_{4r}}(|u|^2+\|Du\|^{2n/(n+2)})\mathrm{d}x\mathrm{d}t\right\}^{(n+2)/n} \tag{6.4}$$

$$+\frac{1}{2}\fint\limits_{Q_{4r}}(|u|^{2(n+2)/n}+\|Du\|^2)\mathrm{d}x\mathrm{d}t+c\fint\limits_{Q_{4r}}(1+|f|^{2(n+2)/(n+4)}+\|g\|^2)\mathrm{d}x\mathrm{d}t$$

for all $0<r\le r_0$ such that $\overline{Q}_{4r}\subset\Omega\times(0,T]$ $(n\ge 2)$. ■

Inequality (6.4) implies the integrability of $|u|^{(n+2)/n}+\|Du\|$ to an exponent $p>2$. To see this, we recall the following

**THEOREM** (cf. [4]). *Let $Q\subset\mathbb{R}^{n+1}$ be open. Let*

$$F\in L^q_{\mathrm{loc}}(Q),\quad G\in L^{q_1}_{\mathrm{loc}}(Q)\quad(1<q<q_1),$$
$$F,G\ge 0\quad a.e.\ in\quad Q\qquad\qquad .$$

*Suppose that*

$$\fint\limits_{Q_r}F^q\mathrm{d}x\mathrm{d}t\le a\left\{\Big(\fint\limits_{Q_{2r}}F\mathrm{d}x\mathrm{d}t\Big)^q+\fint\limits_{Q_{2r}}G^q\mathrm{d}x\mathrm{d}t\right\}+\theta\fint\limits_{Q_{2r}}F^q\mathrm{d}x\mathrm{d}t \tag{$*$}$$

*for all $Q_r$ such that $\overline{Q}_{2r}\subset Q$, where $a\ge 1$ and $0<\theta<1$ are (fixed) constants.*

*Then there exists an $\varepsilon > 0$ such that*

$$F \in L^p_{\mathrm{loc}}(Q) \quad \forall\, q < p < \min\{q+\varepsilon; q_1\},$$

$$\fint_{Q_r} F^p \mathrm{d}x\mathrm{d}t \le c\left\{\Big(\fint_{Q_{2r}} F^q \mathrm{d}x\mathrm{d}t\Big)^{p/q} + \fint_{Q_{2r}} G^p \mathrm{d}x\mathrm{d}t\right\}$$

*for all $Q_r$ with $\overline{Q}_{2r} \subset Q$; here the constant $c$ depends only on $n$, $q$, $q_1$, $p$, $a$ and $\theta$.*

An inspection of the proof in [4] shows that this result continues to hold with $Q_{4r}$ ($\overline{Q}_{4r} \subset Q$) in place of $Q_{2r}$ on the right of $(*)$. ■

Define

$$q = \frac{n+2}{n}, \quad q^* = \min\Big\{\frac{q_1(n+4)}{2n}, \frac{q_2(n+2)}{2n}\Big\},$$

$$F = |u|^2 + \|Du\|^{2n/(n+2)}, \quad G = (1 + |f|^{2(n+2)/(n+4)} + \|g\|^2)^{n/(n+2)} \quad \text{a.e. in } Q.$$

Then

$$q^* > q$$

$$F^q \le 2^q(|u|^{2(n+2)/n} + \|Du\|^2), \quad G \in L^{q^*}(Q),$$

and (6.4) takes the form

$$\fint_{Q_r} F^q \mathrm{d}x\mathrm{d}t \le c\Big(\fint_{Q_{4r}} F \mathrm{d}x\mathrm{d}t\Big)^q + \frac{1}{2}\fint_{Q_{4r}} F^q \mathrm{d}x\mathrm{d}t + c\fint_{Q_{4r}} G^q \mathrm{d}x\mathrm{d}t$$

($0 < r \le r_0$, $\overline{Q}_{4r} \subset Q$).

From the above theorem we get the existence of a $\tilde{q} \in (q, q^*]$ such that

$$F \in L^{\tilde{q}}(\Omega' \times (t', T)) \quad \forall\, \Omega' \subset\subset \Omega, \quad \forall\, t' \in (0, T),$$

$$\fint_{Q_r} F^{\tilde{q}} \mathrm{d}x\mathrm{d}t \le c\left\{\Big(\fint_{Q_{4r}} F^q \mathrm{d}x\mathrm{d}t\Big)^{\tilde{q}/q} + \fint_{Q_{4r}} G^{q^*} \mathrm{d}x\mathrm{d}t\right\} \tag{6.5}$$

Set $p = \dfrac{2n\tilde{q}}{n+2}$. Then

$$p > 2, \quad |u|^{p(n+2)/n} + \|Du\|^p \le 2F^{\tilde{q}} \quad \text{a.e. } Q$$

and (6.1) is readily deduced from (6.5). ■

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