Parametric Linear Complementarity Problems *

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Abstract

We study linear complementarity problems depending on parameters in the right-hand side and (or) in the matrix. For the case that all elements of the right-hand side are independent parameters we give a new proof for the equivalence of three different important local properties of the corresponding solution set map in a neighbourhood of an element of its graph. For one- and multiparametric problems this equivalence does not hold and the corresponding graph may have a rather complicate structure. But we are able to show that for a generic class of linear complementarity problems depending linearly on only one real parameter the situation is much more easier.

1 Introduction

Linear complementarity problems with parameters in the right-hand side and in the matrix have been extensively studied by many authors (e.g. [1], [3], [5], [9], [17], [21]). Further interesting papers concerning more general problems contain essential consequences also for the special case of parametric linear complementarity problems (cf. [4], [15], [16], [18], [23], [24], [25]).
In our paper we consider parametric linear complementarity problems of the form

$$\mathcal{P}_0(\lambda) \qquad \text{Compute all } x \in R^n \text{ satisfying } \; q(\lambda) + K(\lambda)x \; \geq 0, \; x \geq 0, \; x'(q(\lambda) + K(\lambda)x) = 0,$$

for which generally both the vector $q \in R^n$ and the $(n \times n)$-matrix K depend on a parameter vector $\lambda \in R^d$. Concerning the kind of the parameter dependence we consider two cases, denoted by A_0 and A_1:
A_0 : $\quad q(\cdot) : R^d \to R^n$ and $K(\cdot) : R^d \to R^{n \times n}$ are locally Lipschitz.
A_1 : $\quad q(\mathbf{t}) = q^0 + t_1 q^1 + ... + t_d q^d$ and K is constant.

In the case that we assume A_1 we denote the parameter vector by $\mathbf{t}$ and the corresponding parametric linear complementarity problem by $\mathcal{P}_1(\mathbf{t})$.
Only few results will be devoted to the general problem $\mathcal{P}_0(\lambda)$ under assumption A_0. Our particular interest concerns its special cases

$$\mathcal{P}_{01}(q, \lambda) \qquad \text{Compute all } x \in R^n \text{ satisfying } \; q + K(\lambda)x \; \geq 0, \; x \geq 0, \; x'(q + K(\lambda)x) = 0,$$

where all components of q together with the components of λ are independent parameters, $\mathcal{P}_{02}(q)$ with q as parameter, $\mathcal{P}_{03}(q, K)$ with q and K as parameters and the one-dimensional special case $\mathcal{P}_{11}(t)$ of $\mathcal{P}_1(\mathbf{t})$

$$\mathcal{P}_{11}(t) \qquad \text{Compute all } x \in R^n \text{ satisfying } \; (q^0 + tq^1) + Kx \; \geq 0, \; x \geq 0, \; x'(q^0 + tq^1 + Kx) = 0.$$

*This work was supported by the Deutsche Forschungsgemeinschaft under grant Gu 304/1-4.

Let us denote the set of all solutions x of $\mathcal{P}_0(\lambda)$ ($\mathcal{P}_1(\mathbf{t}), \mathcal{P}_{01}(q,\lambda), ...$) for the corresponding parameter value by $\Psi(\lambda)$ ($\Psi(\mathbf{t}), \Psi(q,\lambda), ...$).
In section 2 of our paper we summarize some essentially known global results on the solution set maps of the considered parametric problems and give certain supplements concerning their polyhedral structure. Motivated by a recent equivalence statement of Dontchev and Rockafellar [4] concerning three different local properties of the solution set map of a more general class of parameter-depending problems (lower semicontinuity, pseudo-Lipschitz continuity and strong regularity) we present in section 3 for problem $\mathcal{P}_{01}(q,\lambda)$ another proof which gives a better insight into the situation. Especially, it will be clear, for which reason lower semicontinuity around a given point of the graph of the solution set map does not hold, if this map is not strongly regular at this point.
Counterexamples in the case of problem $\mathcal{P}_{11}(t)$ show, that in general lower semicontinuity around a given point of the graph and strong regularity at this point are not equivalent. In section 4 we show that generically the graph of the solution set map of $\mathcal{P}_{11}(t)$ has a much more easier structure as in the general case and that only six types of solutions may appear.
Throughout the whole paper we use the symbols "gph" for the graph of a set-valued map, "sgn" for the sign of a real number, "rg" for the rank of a matrix, "dim" for the dimension, "lin" for the linear hull, "int" for the interior, "bd" for the boundary and "ri" for the relative interior of a convex set. The symbol $\mathcal{B}$ stands for the closed unit ball in R^n.

2 Some global results on the solution set map

In the following theorem we summarize some known properties (cf. [3], [24]) of the solution set map Ψ of $\mathcal{P}_0(\lambda)$ and $\mathcal{P}_1(\mathbf{t})$, respectively.

Theorem 2.1
1. *For the problem $\mathcal{P}_0(\lambda)$ the map Ψ is closed (i.e., for each sequence $\{\lambda^\nu\}$, converging to any λ^0, and each sequence $\{x^\nu\}$ with $x^\nu \in \Psi(\lambda^\nu)$, converging to any x^0, we have $x^0 \in \Psi(\lambda^0)$).*
2. *For the problem $\mathcal{P}_1(\mathbf{t})$ the map Ψ is polyhedral (i.e., its graph is a union of a finite number of convex polyhedra).*
3. *For the problem $\mathcal{P}_1(\mathbf{t})$ the map Ψ is locally upper Lipschitz with a uniform modulus (i.e., there is a constant $c > 0$ and for each $\mathbf{t}^0 \in R^d$ there exists a neighbourhood V of $\mathbf{t}^0$ such that for all $\mathbf{t} \in V$ we have $\Psi(\mathbf{t}) \subseteq \Psi(\mathbf{t}^0) + c \parallel \mathbf{t} - \mathbf{t}^0 \parallel \mathcal{B}$.*

Obviously, Statement 3 of Theorem 2.1 has the following consequences.

Corollary 2.1 *For the problem $\mathcal{P}_1(\mathbf{t})$ the sets $Q_b = \{\mathbf{t} \ / \ \Psi(\mathbf{t})$ is bounded$\}$ as well as $Q_e = \{\mathbf{t} \ / \ \Psi(\mathbf{t}) = \emptyset\}$ are open.*

The following statements on the cardinality $\mid \Psi(q) \mid$ of the set of solutions of problem $\mathcal{P}_{02}(q)$ have been proven in [19] and [25].

Theorem 2.2 *For the problem $\mathcal{P}_{02}(q)$ it holds:*

1. *$\mid \Psi(q) \mid \geq 1 \ \forall q \in R^n \iff K$ is a Q-matrix (for the definition cf. [19]).*
2. *$\mid \Psi(q) \mid < \infty \ \forall q \in R^n \iff K$ is a N-matrix (i.e., all its principal subminors are nonzero).*
3. *$\mid \Psi(q) \mid = 1 \ \forall q \in R^n \iff K$ is a P-matrix (i.e., all its principal subminors are positive).*

Remark 2.1 *If the matrix K belongs to the class of Q-matrices (N-matrices, P-matrices, resp.) then for the problem $\mathcal{P}_1(\mathbf{t})$ we have $\mid \Psi(\mathbf{t}) \mid \geq 1 \ \forall \mathbf{t} \in R^d$ ($\mid \Psi(\mathbf{t}) \mid < \infty \ \forall \mathbf{t} \in R^d, \mid \Psi(\mathbf{t}) \mid = 1 \ \forall \mathbf{t} \in R^d$, resp.). But the reverse statements are not true in general.*

For any $\lambda \in R^d$ the set $P(\lambda) = \{x \in R^n \ / \ K(\lambda)x + q(\lambda) \geq 0, \ x \geq 0\}$ is a convex polyhedron associated with the problem $\mathcal{P}_0(\lambda)$. For any pair (I,J) of index sets $I, J \subseteq \{1, ..., n\}$ let be

$$P^{I,J}(\lambda) \ = \ \{x \in P(\lambda) \ / \ (K(\lambda)x + q(\lambda))_i = 0, \ i \in I, \ x_j = 0, \ j \in J\},$$

$$\tilde{P}^{I,J}(\lambda) \ = \ \{x \in P^{I,J}(\lambda) \ / \ (K(\lambda)x + q(\lambda))_i > 0, \ i \in \bar{I}, \ x_j > 0, \ j \in \bar{J}\},$$

$$\mathcal{A}^{I,J} = \{\lambda \in R^d \ / \ P^{I,J}(\lambda) \neq \emptyset\} \qquad \text{and} \qquad \tilde{\mathcal{A}}^{I,J} = \{\lambda \in R^d \ / \ \tilde{P}^{I,J}(\lambda) \neq \emptyset\},$$

where $\bar{I} = \{1, ..., n\} \setminus I$. Further let be $\mathcal{S} := \{(I, J)/I \cup J = \{1, ..., n\}\}$. For the special case that $J = \bar{I}$ we write shortly $P^I(\lambda), \tilde{P}^I(\lambda), P^I, \mathcal{A}^I, \tilde{\mathcal{A}}^I$ instead of $P^{I,\bar{I}}(\lambda), \tilde{P}^{I,\bar{I}}(\lambda), P^{I,\bar{I}}, \mathcal{A}^{I,\bar{I}}, \tilde{\mathcal{A}}^{I,\bar{I}}$.
Moreover, we introduce the set $\mathcal{A} = \{\lambda \in R^d / \Psi(\lambda) \neq \emptyset\}$, the matrix $V(I, J; \lambda)$ formed by the rows of $-K(\lambda)$ with the indices $i \in I$ and the rows of the $(n \times n)$-unit matrix with the indices $j \in J$ and the vector $p(I, J; \lambda)$ formed correspondingly by the components $q_i(\lambda), \ i \in I$ and $n- \mid I \mid$ zero components otherwise.
The following two theorems summarize and supplement corresponding results contained in [1], [3] and [25]. The proof of the first one is obvious.

Theorem 2.3 *For the problem $\mathcal{P}_0(\lambda)$ it holds:*

1. *For $\lambda \in \mathcal{A}^{I,J}$ the set $P^{I,J}(\lambda)$ is a closed facet of the convex polyhedron $P(\lambda)$ and for $\lambda \in \tilde{\mathcal{A}}^{I,J}$ the set $\tilde{P}^{I,J}(\lambda)$ is the corresponding open facet. For $(I', J') \neq (I'', J'')$ and any $\lambda \in R^d$ it holds $\tilde{P}^{I',J'}(\lambda) \cap \tilde{P}^{I'',J''}(\lambda) = \emptyset$ and $gph\tilde{P}^{I',J'} \cap gph\tilde{P}^{I'',J''} = \emptyset$.*

2. *The sets $\mathcal{A}$, $gph\Psi$ and $\Psi(\lambda)$ for any $\lambda \in R^d$ may be decomposed in the form:*

$$\mathcal{A} = \bigcup_{I \subseteq \{1,\ldots,n\}} \mathcal{A}^I \ = \ \bigcup_{(I,J) \in \mathcal{S}} \tilde{\mathcal{A}}^{I,J}, \qquad gph \ \Psi = \bigcup_{I \subseteq \{1,\ldots,n\}} gph \ P^I \ = \ \bigcup_{(I,J) \in \mathcal{S}} gph \ \tilde{P}^{I,J}$$

and

$$\Psi(\lambda) \ = \ \bigcup_{I \subseteq \{1,\ldots,n\}} P^I(\lambda) \ = \ \bigcup_{(I,J) \in \mathcal{S}} \tilde{P}^{I,J}(\lambda).$$

3. *If for any $\lambda \in R^d$ a point $x \in R^n$ is an isolated solution of $\mathcal{P}_0(\lambda)$ then x corresponds to a vertex $x(I, J; \lambda) := V^{-1}(I, J; \lambda) \, p(I, J; \lambda)$ of the convex polyhedron $P(\lambda)$, i.e., there is a pair (I,J) of index sets $I, J \subseteq \{1, ..., n\}$ with $\mid I \mid + \mid J \mid= n$ such that the system of (in x) linear equations $V(I, J; \lambda) \ x = p(I, J; \lambda)$ has a unique solution $x(I, J; \lambda)$ and this solution is also a solution of $\mathcal{P}_0(\lambda)$. Each vertex $x(I, J; \lambda)$ of the convex polyhedron $P(\lambda)$ depends locally Lipschitz continuous on the parameter λ.*

We note that for the problem $\mathcal{P}_1(\mathbf{t})$ the set $gphP$ is a convex polyhedron and the sets $gphP^I$ respectively $gph\tilde{P}^{I,J}$ are closed respectively open facets of $gphP$.
Now we use the submatrix K_{IJ} of K formed by the elements k_{ij} of K with $i \in I$ and $j \in J$.

Theorem 2.4 *For the problem $\mathcal{P}_{02}(q)$ and each pair $(I, J) \in \mathcal{S}$ it holds:*

1. *The set $\mathcal{A}^{I,J}$ is a nonempty polyhedral cone and $\tilde{\mathcal{A}}^{I,J} = ri\mathcal{A}^{I,J}$.*

2. *$dimP^{I,J}(q) = dim\tilde{P}^{I,J}(q) = d(I, J) \ \forall q \in \tilde{\mathcal{A}}^{I,J}$, where $d(I, J) =\mid \bar{J} \mid -rg(K_{I\bar{J}})$ does not depend on q.*

3. *$dim\mathcal{A}^{I,J} + d(I, J)+ \mid I \mid + \mid J \mid = 2n$ and $dim\mathcal{A}^{I,J} + d(I, J)+ \mid I \cap J \mid= n$.*

4. *$dim\mathcal{A}^{I,J} = n \iff J = \bar{I} \ \wedge \ d(I, J) = 0 \iff J = \bar{I} \ \wedge \ K_{II}$ is regular.*

5. *$dim\mathcal{A}^{I,J} = n - 1 \iff$ a) $(J = \bar{I} \ \wedge \ d(I, J) = 1) \ \vee$ b) $(\mid I \cap J \mid= 1 \ \wedge \ d(I, J) = 0)$.*

Proof: In our proof we apply the ideas already used in the proof of Theorem 5.4.3 of [1]. To prove Statement 1 we write $\mathcal{A}^{I,J}$ and $\tilde{\mathcal{A}}^{I,J}$ in the form $\mathcal{A}^{I,J} = \{q \in R^n \ / \ q = -Kx + y, \ x \geq 0, \ y \geq 0, x_j = 0, j \in J, y_i = 0, i \in I\}$ and $\tilde{\mathcal{A}}^{I,J} = \{q \in \mathcal{A}^{\mathcal{I},\mathcal{J}}/\S_| > \prime, | \in \not{\mathcal{J}}, \dagger_\rangle > \prime, \rangle \in \not{\mathcal{I}}\}$. Thus, the set $\mathcal{A}^{I,J}$ is the image of a closed facet of the polyhedral cone R^{2n}_+ under the linear map which is defined by the matrix (-K,E) and $\tilde{\mathcal{A}}^{I,J}$ the image of the corresponding open facet of R^{2n}_+. But this implies Statement 1. Because of $\tilde{P}^{I,J}(q) \neq \emptyset$ for all $q \in \tilde{\mathcal{A}}^{I,J}$ it follows $\tilde{P}^{I,J}(q) = riP^{I,J}(q)$ and $dim\tilde{P}^{I,J}(q) = dimP^{I,J}(q) = dimL^{I,J} = 2n- \mid I \mid - \mid J \mid -rgB(I,J) = 2n- \mid I \mid - \mid J \mid - \mid \bar{I} \mid -rg(K_{I\bar{J}}) = n- \mid J \mid -rg(K_{I\bar{J}}) = \mid \bar{J} \mid -rg(K_{I\bar{J}})$, where $L^{I,J} = \{(x,y) \in R^{2n}/ -Kx + y = 0, \ y_i = 0, i \in I, \ x_j = 0, j \in J\}$ and $B(I,J)$ is the $(n \times (\mid \bar{I} \mid + \mid \bar{J} \mid))$-matrix formed by the columns of -K with the numbers $i \in \bar{I}$ and by the columns of the $(n \times n)$ unit matrix with the numbers $j \in \bar{J}$. This implies Statement 2 and Statement 3, if we use the fact that $dim\mathcal{A}^{I,J} = rg \ B(I,J)$ holds. Statements 4 and 5 follow directly from Statement 3. q.e.d.

As an immediate consequence of Theorem 2.4 we mention the following fact.

Corollary 2.2 *For the problem $\mathcal{P}_{02}(q)$ and any $q \in R^n$ we have $\mid \Psi(q) \mid = \infty \iff q \in \bigcup_{(I,J) \in \mathcal{S}^*} \tilde{\mathcal{A}}^{I,J}$ with $\mathcal{S}^* := \{(I,J) \in \mathcal{S}/ \ d(I,J) \geq 1\}$, where for all $(I,J) \in \mathcal{S}^*$ it holds $dim\tilde{\mathcal{A}}^{I,J} < n$.*

For the application in the following sections we give now some additional properties of those sets $\mathcal{A}^{I,J}$ corresponding to problem $\mathcal{P}_{02}(q)$ which have the maximal dimension n. Note that the number of these sets is not zero, since for $I = \emptyset$ and $J = \{1, ..., n\}$ we have $\mathcal{A}^{I,J} = R^n_+$. All results follow from the Theorems 2.3 and 2.4 and from generally known facts on basic solutions in linear optimization.

Remark 2.2 *For each set $\mathcal{A}^{I,J}$ with the dimension n corresponding to problem $\mathcal{P}_{02}(q)$ it holds $J = \bar{I}$ and the corresponding matrix K_{II} must be regular (where for the index set $I = \emptyset$ this regularity condition is satisfied per definition). The corresponding set $P^{I,\bar{I}}(q) = P^I(q)$ is formed by a single point, which is a vertex $x(I,q) = \begin{pmatrix} x_I(I,q) \\ x_{\bar{I}}(I,q) \end{pmatrix}$ of $P^I(q)$, where $x_I(I,q) = -K_{II}^{-1}q_I$ and $x_{\bar{I}}(I,q) = 0$. Moreover, it holds $\mathcal{A}^I = \{ \ q \ / \ K_{II}^{-1}q_I \leq 0, \ q_{\bar{I}} + K_{\bar{I}I}K_{II}^{-1}q_I \geq 0\}$ and $\tilde{\mathcal{A}}^I = \{ \ q \ / \ K_{II}^{-1}q_I < 0, \ q_{\bar{I}} + K_{\bar{I}I}K_{II}^{-1}q_I > 0\}$.*
After introducing slack variables y we can write the convex polyhedron P(q) equivalently in the form $P'(q) = \{(x,y) \in R^{2n}/ -Kx + y = q, \ x \geq 0, \ y \geq 0\}$ and the complementarity slackness condition can be expressed by x'y=0. The y-part of the vertex $\begin{pmatrix} x(I,q) \\ y(I,q) \end{pmatrix}$ of the convex polyhedron P'(q), which corresponds to the vertex x(I,q) of P(q) is given by $y_I = 0$ and $y_{\bar{I}} = q_{\bar{I}} + K_{\bar{I}I}K_{II}^{-1}q_I$. The corresponding simplex table to this vertex of P' (q) with the vectors of basic variables x_I and $y_{\bar{I}}$ and the vectors of non-basic variables y_I and $x_{\bar{I}}$ is given by

$$\begin{array}{c|c|c|c} & y_I & x_{\bar{I}} & \\ \hline x_I & -K_{II}^{-1} & K_{II}^{-1}K_{I\bar{I}} & -K_{II}^{-1}q_I \\ \hline y_{\bar{I}} & K_{\bar{I}I}K_{II}^{-1} & K_{\bar{I}I}K_{II}^{-1}K_{I\bar{I}} - K_{\bar{I}\bar{I}} & q_{\bar{I}} + K_{\bar{I}I}K_{II}^{-1}q_I \end{array} . \tag{1}$$

This simplex table contains all coefficients which will be obtained if we transform the system of equations

$$-Kx + y = q \qquad or \qquad \begin{pmatrix} -K_{II} & 0 \\ -K_{\bar{I}I} & E \end{pmatrix} \begin{pmatrix} x_I \\ y_{\bar{I}} \end{pmatrix} + \begin{pmatrix} -K_{I\bar{I}} & E \\ -K_{\bar{I}\bar{I}} & 0 \end{pmatrix} \begin{pmatrix} x_{\bar{I}} \\ y_I \end{pmatrix} = \begin{pmatrix} q_I \\ q_{\bar{I}} \end{pmatrix}$$

in the equivalent form

$$\begin{pmatrix} x_I \\ y_{\bar I} \end{pmatrix} = -\begin{pmatrix} -K_{II}^{-1} & K_{II}^{-1}K_{I\bar I} \\ K_{\bar I I}K_{II}^{-1} & K_{\bar I I}K_{II}^{-1}K_{I\bar I} - K_{\bar I\bar I} \end{pmatrix}\begin{pmatrix} y_I \\ x_{\bar I} \end{pmatrix} + \begin{pmatrix} -K_{II}^{-1}q_I \\ q_{\bar I} + K_{\bar I I}K_{II}^{-1}q_I \end{pmatrix}.$$

3 Local properties

Besides global properties of the solution set maps of the considered parametric problems studied in the previous section also local properties are of interest. This means properties of the intersection of the corresponding graph with a sufficiently small neighbourhood of one of its elements. Because of the fact that we do not restrict our considerations to the case that the matrix K has only nonnegative principal subminors, the set of solutions must not be connected or even convex, such that local properties are not entirely determined by the global ones. Of course, if the set of all solutions for a fixed value of the parameter is finite, then necessarily each solution must be isolated, and, on the other hand, if locally the set of solutions is not finite, then this also must hold globally. But these trivial statements are already almost all relations between local and global properties.
As already done in the paper [4] we are interested to apply the following definitions for general set-valued maps $\Gamma : R^m \to 2^{R^n}$ to the solution set maps of our parametric linear complementarity problems.

Definition 3.1 *Let Γ be a set-valued map and $(u^0, v^0) \in gph\Gamma$. Then Γ is called*

(*) *lower semicontinuous around (u^0, v^0), if there are neighbourhoods U of u^0 and V of v^0 such that Γ is lower semicontinuous at every point $(u, v) \in (U \times V) \cap gph\Gamma$ (i.e., for every sequence $\{u^\nu\}$ converging to u there is a sequence $\{v^\nu\}$ with $v^\nu \in \Gamma(u^\nu)$ for ν sufficiently hight, converging to v).*

(**) *pseudo-Lipschitz at (u^0, v^0) with the constant $L > 0$, if there are neighbourhoods U of u^0 and V of v^0 such that $\Gamma(u^1) \cap V \subseteq \Gamma(u^2) + L \parallel u^1 - u^2 \parallel \mathcal{B} \qquad \forall u^1, u^2 \in U$.*

(***) *strongly regular at (u^0, v^0), if there are neighbourhoods U of u^0 and V of v^0 such that the map $u \to \Gamma(u) \cap V$ is single-valued and Lipschitz-continuous relative to U.*

The following lemma is the main basis to study parametric linear complementarity problems $\mathcal{P}_0(\lambda)$ locally. Suppose that (λ^0, x^0) is any element of the graph of the solution set map Ψ of $\mathcal{P}_0(\lambda)$ and we denote

$$I_1 = \{i \;/\; (q(\lambda^0) + K(\lambda^0)x^0)_i = 0,\; x_i^0 > 0\}, \qquad I_2 = \{i \;/\; (q(\lambda^0) + K(\lambda^0)x^0)_i = 0,\; x_i^0 = 0\}$$

$$\text{and} \qquad I_3 = \{i \;/\; (q(\lambda^0) + K(\lambda^0)x^0)_i > 0,\; x_i^0 = 0\}.$$

Lemma 3.1 *For each sufficiently small neighbourhood W of $(\lambda^0, x^0) \in gph\Psi$ we have*

$$W \cap gph\;\Psi = W \cap \bigcup_{I \in \mathcal{T}(\lambda^0, x^0)} gph\; P^I = W \cap \bigcup_{(I,J) \in \mathcal{S}(\lambda^0, x^0)} gph\; \tilde P^{I,J},$$

where $\mathcal{T}(\lambda^0, x^0) := \{I \;/\; I_1 \subseteq I \subseteq I_1 \cup I_2\}$ and $\mathcal{S}(\lambda^0, x^0) := \{(I, J) \in \mathcal{S} \;/\; I_1 \subseteq I \subseteq I_1 \cup I_2,\; I_3 \subseteq J \subseteq I_2 \cup I_3\}$.

Proof: Statement 2 of Theorem 2.3 implies $W \cap gph\Psi \supseteq W \cap \bigcup_{I \in \mathcal{T}(\lambda^0, x^0)} gphP^I$ and $W \cap gph\Psi \supseteq W \cap \bigcup_{(I,J) \in \mathcal{S}(\lambda^0, x^0)} gph\; \tilde P^{I,J}$. For each index set $I \notin \mathcal{T}(\lambda^0, x^0)$ and for each pair $(I, J) \notin \mathcal{S}(\lambda^0, x^0)$ it holds $(\lambda^0, x^0) \notin gphP^I$ and $(\lambda^0, x^0) \notin gphP^{I,J}$. Hence, since the set $gphP^I$ respectively $gphP^{I,J}$ is closed, we get $W \cap gphP^I = \emptyset$ respectively $W \cap gph\tilde P^{I,J} = \emptyset$, if we choose W sufficiently small (cf. also [23]). q.e.d.

Remark 3.1 *For each* $(\lambda^0, x^0) \in gph\Psi$ *there is a minimal subsystem* $\mathcal{Z}(\lambda^0, x^0) = \{P_1, ..., P_k\}$ *with* $k = k(\lambda^0, x^0)$ *of the system of convex polyhedra* $P^{I,J}(\lambda^0)$ *for* $(I, J) \in \mathcal{S}(\lambda^0, x^0)$ *such that* $\bigcup_{i=1}^{k} P_i = \bigcup_{(I,J)\in\mathcal{S}(\lambda^0,x^0)} P^{I,J}(\lambda^0)$ *and* $riP_{i_1} \cap riP_{i_2} = \emptyset$ *for* $i_1 \neq i_2$. *For* $i = 1, ..., k$ *let be* $\mathcal{S}_i(\lambda^0, x^0) = \{(I, J) \in \mathcal{S}(\lambda^0, x^0)/P^{I,J}(\lambda^0, x^0) = P_i\}$. *The sets* $\mathcal{S}_i(\lambda^0, x^0)$, $i = 1, ..., k$ *are pairwise disjoint and for each* $(I, J) \in \mathcal{S}(\lambda^0, x^0) \setminus \bigcup_{i=1}^{k} \mathcal{S}_i(\lambda^0, x^0)$ *the convex polyhedron* $P^{I,J}(\lambda^0)$ *is a closed facet of at least one of the convex polyhedra* P_i, $i = 1, ..., k$.

In the following proposition we summarize some immediate observations with respect to the application of Definition 3.1 to parametric linear complementarity problems.

Proposition 3.1 *1. For each set-valued map* Γ *we have* $(***) \Longrightarrow (**) \Longrightarrow (*)$.

2. *For any element* (λ^0, x^0) *of the graph of the solution set map of problem* $\mathcal{P}_0(\lambda)$ *condition (*) is equivalent with the existence of neighbourhoods* U *of* λ^0 *and* V *of* x^0 *satisfying:*
   (+) For each $(\tilde{\lambda}, \tilde{x}) \in (U \times V) \cap gph\Psi$ *there is a neighbourhood* $\tilde{U}$ *of* $\tilde{\lambda}$ *such that for each* $\lambda \in \tilde{U}$ *and each convex polyhedron* $P_i \in \mathcal{Z}(\tilde{\lambda}, \tilde{x})$ *there is at least one pair* $(I, J) \in \mathcal{S}_i(\tilde{\lambda}, \tilde{x})$ *with* $\lambda \in \mathcal{A}^{I,J}$ *and* $dimP_i \leq dimP^{I,J}(\lambda)$.

3. *For the solution set maps of the problems* $\mathcal{P}_{01}(q, \lambda)$ *as well as* $\mathcal{P}_1(\mathbf{t})$ *the conditions (*) and (**) are equivalent.*

4. *For any element* (λ^0, x^0) *of the graph of the solution set map of problem* $\mathcal{P}_0(\lambda)$ *condition (***) is equivalent with the property that there are neighbourhoods* U *of* λ^0 *and* V *of* x^0 *such that the map* $\lambda \to \Psi(\lambda) \cap V$ *is single-valued and continuous on* U.

Proof: Statement 1 follows directly from Definition 3.1 (cf. [4]).
To prove the first direction of Statement 2 we assume that there are neighbourhoods U of λ^0 and V of x^0 having the property (+). Now let $(\tilde{\lambda}, \tilde{x})$ be an arbitrary element of $(U \times V) \cap gph\Psi$, $\{\lambda^\nu\}$ any sequence converging to $\tilde{\lambda}$ and $P_i \in \mathcal{Z}(\tilde{\lambda}, \tilde{x})$. Obviously, it holds $\tilde{x} \in P_i$. According to (+) for each ν sufficiently hight there exists a pair $(I, J) \in \mathcal{S}_i(\tilde{\lambda}, \tilde{x})$ (depending on ν) with $\lambda^\nu \in \mathcal{A}^{I,J}$ and $dimP_i \leq dimP^{I,J}(\lambda^\nu)$. As in the proof of Theorem 3.2.2 in [1] the sequence $\{x^\nu\}$, where x^ν minimizes the Euclidean distance between $\tilde{x}$ and $P^{I,J}(\lambda^\nu)$, converges to $\tilde{x}$. But this means that Ψ is lower semicontinuous at $(\tilde{\lambda}, \tilde{x})$ and, hence, condition (*) is fulfilled at (λ^0, x^0).
To prove the second direction of Statement 2 let us suppose that there do not exist any neighbourhoods U of λ^0 and V of x^0 with the property (+). This means that for each neighbourhoods U of λ^0 and V of x^0 there are an element $(\tilde{\lambda}, \tilde{x}) \in (U \times V) \cap gph\Psi$, a sequence $\{\lambda^\nu\}$ converging to $\tilde{\lambda}$ and a convex polyhedron $P_i \in \mathcal{Z}(\tilde{\lambda}, \tilde{x})$ (depending on ν) such for all $\nu = 1, 2, ...$ it holds either $\lambda^\nu \notin A^{I,J}$ $\forall$ $(I, J) \in \mathcal{S}_i(\tilde{\lambda}, \tilde{x})$ or for all pairs $(I, J) \in \mathcal{S}_i(\tilde{\lambda}, \tilde{x})$ with $\lambda^\nu \in A^{I,J}$ we have $dimP_i > dimP^{I,J}(\lambda^\nu)$. Hence, there must be an infinite subsequence of the sequence $\{\lambda^\nu\}$ (for simplicity we denote it again by $\{\lambda^\nu\}$) such that one of the following cases holds true. The first case is that there exists a convex polyhedron $P_i \in \mathcal{Z}(\tilde{\lambda}, \tilde{x})$ such that for all pairs $(I, J) \in \mathcal{S}_i(\tilde{\lambda}, \tilde{x})$ it holds $\lambda^\nu \notin A^{I,J}$. Lemma 3.1 and Remark 3.1 imply that for any element $(\tilde{\lambda}, x^*)$ with $x^* \in riP_i$ sufficiently near to $\tilde{x}$ there can not be any sequence $\{x^\nu\}$ with $x^\nu \in \Psi(\lambda^\nu)$ converging to x^* such that Ψ can not be lower semicontinuous at $(\tilde{\lambda}, x^*)$ and, consequently, (*) is not satisfied at (λ^0, x^0).
The second case is that there exists a convex polyhedron $P_i \in \mathcal{Z}(\tilde{\lambda}, \tilde{x})$ and a nonempty system $\tilde{\mathcal{S}} \subseteq \mathcal{S}_i(\tilde{\lambda}, \tilde{x})$ such that for $\nu = 1, 2, ...$ it holds $\tilde{\mathcal{S}} = \{(I, J) \in \mathcal{S}(\tilde{\lambda}, \tilde{x})/\lambda^\nu \in \mathcal{A}^{I,J}\}$ and $dimP^{I,J}(\lambda^\nu) < dimP^{I,J}(\tilde{\lambda})$ $\forall (I, J) \in \tilde{\mathcal{S}}$. For each pair $(I, J) \in \tilde{\mathcal{S}}$ let be $Q^{I,J} = \{x/\exists\{x^\nu\}, x^\nu \in P^{I,J}(\lambda^\nu), x^\nu \to x\}$. This set is obviously convex, contained in P_i and can only have a dimension less or equal to the minimal dimension of the sets $P^{I,J}(\lambda^\nu)$. To prove this last condition let us suppose the opposite. Then there must be an infinite subsequence of the sequence $\{\lambda^\nu\}$ (for simplicity

we denote it again by $\{\lambda^\nu\}$) such that $d = dimQ^{I,J} > dimP^{I,J}(\lambda^\nu)$ for $\nu = 1, 2, \ldots$. Thus, there must be d+1 linearly independent points z^l, $l = 0, 1, \ldots, d$ in $Q^{I,J}$, each of them limit of a sequence $\{x^{\nu l}\}$ with $x^{\nu l} \in P^{I,J}(\lambda^\nu)$. Because of our supposition $dimP^{I,J}(\lambda^\nu) < d$ for each ν there must be a normed vector $c^\nu \in R^d$ satisfying $\sum_{l=1}^{d} c_l^\nu (x^{\nu 0} - x^{\nu l}) = 0,\ \nu = 1, 2, \ldots$. The sequence $\{c^\nu\}$ must have an (again normed) accumulation point c and we obtain (using an infinite subsequence of the sequence $\{c^\nu\}$ converging to c) the relation $\sum_{l=1}^{d} c_l (z^0 - z^l) = 0$ which contradicts our supposition that the points z^l are linearly independent. Hence, it holds $dimQ^{I,J} \leq dimP^{I,J}(\lambda^\nu) < dimP_i$ and, consequently, $Q^{I,J} \subset P_i$ for each pair $(I, J) \in \tilde{\mathcal{S}}$. Using Lemma 3.1 and Remark 3.1 this relation implies that in each sufficiently small neighbourhood of $\tilde{x}$ there are elements of the convex polyhedron P_i which may not be a limit of any sequence $\{x^\nu\}$ with $x^\nu \in \Psi(\lambda^\nu)$. But this contradicts (*).
For the problem $\mathcal{P}_{01}(q, \lambda)$ Statement 3 follows from Theorem 1 of [4]. Now let us prove Statement 3 for the problem $\mathcal{P}_1(\mathbf{t})$. According to Statement 1 we have only to show $(*) \implies (**)$. We assume (*) at any element $(\mathbf{t}^0, x^0) \in gph\Psi$ and choose polyhedral neighbourhoods $U' \subset U$ of $\mathbf{t}^0$ and $V' \subset V$ of x^0 small enough such that for $W = U' \times V'$ Lemma 3.1 can be used. Consider the map Ψ_0 defined for $\mathbf{t} \in U'$ by $\Psi_0(\mathbf{t}) = V' \cap \bigcup_{(I,J) \in \mathcal{S}(\mathbf{t}^0, x^0)} P^{I,J}(\mathbf{t})$, which must be lower semicontinuous on int U'. The graph of Ψ_0 is a union of a finite number of convex polyhedra. Consider those edges of these convex polyhedra, which belong to the boundary of the graph but not to the set $bdU' \times V'$. Because of the lower semicontinuity of Ψ_0 all these edges can not be perpendicular to the parameter space R^d. For each such edge we consider its angel to the parameter space R^d. If we now choose L as the maximal absolute value of the tangent of all these angels we find that for arbitrary $\mathbf{t}^1, \mathbf{t}^2 \in U$ it holds $\Psi_0(\mathbf{t}^1) \subseteq \Psi_0(\mathbf{t}^2) + L \parallel \mathbf{t}^1 - \mathbf{t}^2 \parallel \mathcal{B}$ and, hence, condition (**).
The first direction of Statement 4 is trivial, since Lipschitz continuity implies continuity. On the other hand, Statement 3 of Theorem 2.3 implies that the vector function $x(\cdot)$ is a continuous selection of a finite number of vector functions $x(I, J; \cdot)$, which are locally Lipschitz, and is, thereby, locally Lipschitz itself. q.e.d.

Unlike the fact that properties (*) and (**) are equivalent for the problem $\mathcal{P}_1(\mathbf{t})$, the properties (*) and (***) differ generally. The following three examples of the type $\mathcal{P}_{11}(t)$ illustrate different possibilities which may appear although (*) is fulfilled.

Example 1: We define

$$\Psi_1(t) = \{x \in R_+^2 / t - x_1 + 2x_2 \geq 0, 3t + 2x_1 + x_2 \geq 0, x_1(t - x_1 + 2x_2) + x_2(3t + 2x_1 + x_2) = 0\}.$$

An easy computation shows $\Psi_1(t) = \begin{cases} \{(0, -3t)', (-t, -t)'\} & \text{for } t \leq 0 \\ \{(0, 0)', (t, 0)'\} & \text{for } t \geq 0 \end{cases}$ such that Ψ_1 satisfies condition (*) but not (***) at the solution (0,0)' for t=0. Locally (and in this case even globally) this solution for t=0 is unique but in each neighbourhood of (0,0)' and for each $t \neq 0$ sufficiently near to zero we have more than one element x (namely exactly two) with $(t, x) \in gph\Psi_1$.

Example 2: We define

$$\begin{aligned}\Psi_2(t) = \{x \in R_+^3 \quad / -2x_2 + 2x_3 \geq 0,\ 2t - 1 + x_1 + 2x_2 + 2x_3 \geq 0,\ -t + 1 - x_1 - x_2 \geq 0, \\ x_1(-2x_2 + 2x_3) + x_2(2t - 1 + x_1 + 2x_2 + 2x_3) + x_3(-t + 1 - x_1 - x_2) = 0\}.\end{aligned}$$

An easy computation shows $\Psi_2(t) = \begin{cases} \{(-0.5t + 1, -0.5t, -0.5t)'\} & \text{for } t \leq 0 \\ \{(x_1, 0, 0)' / 1 - 2t \leq x_1 \leq 1 - t\} & \text{for } 0 \leq t \leq 0.5 \end{cases}$ such that Ψ_2 satisfies condition (*) but not (***) at the solution (1,0,0)' for t=0. Locally (and again globally) this solution for t=0 is unique but in each neighbourhood of (1,0,0)' and for each sufficiently small $t > 0$ we have an infinite number of points x with $(t, x) \in gph\Psi_2$. But globally each connected component of $\Psi_2(t)$ having a nonempty intersection with a sufficiently small neighbourhood of (1,0,0)' is bounded.

Example 3: We define

$$\Psi_3(t) = \{x \in R^3_+ \;/\; \begin{array}{l} 2t - 2x_2 + 2x_3 \geq 0,\ -4t + x_1 + 2x_2 + 2x_3 \geq 0,\ -x_1 - x_2 \geq 0, \\ x_1(2t - 2x_2 + 2x_3) + x_2(-4t + x_1 + 2x_2 + 2x_3) + x_3(-x_1 - x_2) = 0\}. \end{array}$$

An easy computation shows $\Psi_3(t) = \left\{ \begin{array}{ll} \{(0,0,x_3)' \;/\; x_3 \geq -t\} & \text{for } t \leq 0 \\ \{(0,0,x_3)' \;/\; x_3 \geq 2t\} & \text{for } t \geq 0 \end{array} \right.$ such that also in this case Ψ_3 satisfies condition (*) but not (***) at each solution $(0,0,x_3)'$ with $x_3 \geq 0$ for t=0. Here we have the situation that the intersection of $gph\Psi_3$ with any neighbourhood of an arbitrary element $(0,0,0,x_3)$ of this graph consists of infinitely many points. But here one component of $\Psi_3(t)$ having a nonempty intersection with a sufficiently small neighbourhood of a solution $(0,0,x_3)$ for t=0 is unbounded.

Recently, Dontchev and Rockafellar [4] have shown a general equivalence statement for parametric variational inequalities over polyhedral convex sets, which we formulate here for problem $\mathcal{P}_{01}(q,\lambda)$.

Theorem 3.1 *For the problem $\mathcal{P}_{01}(q,\lambda)$ the properties (*), (**) and (***) are equivalent.*

Note that this assertion is valid also for the special cases $\mathcal{P}_{02}(q)$ and $\mathcal{P}_{03}(q,K)$ of $\mathcal{P}_{01}(q,\lambda)$. The essential assumption is only that at least all components of q are independent parameters.
The proof given in [4] is rather abstract and uses a reduction approach, known general properties of projections and normal as well as piecewise linear maps. However, it is not seen immediately, which requirements of (*) would be violated if (***) does not hold. Moreover, it will not intelligible why this proof can not be extended, for instance, to the problem $\mathcal{P}_1(\mathbf{t})$. For this reason we will give another proof at the end of this section after some preparations. A recent paper of Kummer [14] is devoted to a corresponding aim, however for the Karush-Kuhn-Tucker conditions for nonlinear and quadratic optimization problems.
According to the decomposition of the whole index set $\{1,...,n\}$ into the disjoint subsets I_1, I_2 and I_3 we also decompose K in the form $K = \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix}$.
The following necessary and sufficient condition for strong regularity is shown in [23] and [3] (if we use, additionally, Statement 4 of Proposition 3.1.)

Theorem 3.2 *For the problem $\mathcal{P}_{01}(q,\lambda)$ and each element $(q^0,\lambda^0,x^0) \in gph\ \Psi$ condition (***) is equivalent with*

$$K_{11} \textit{ is regular and the Schur-complement } N = K_{22} - K_{21}K_{11}^{-1}K_{12} \textit{ is a P-matrix.} \tag{2}$$

In the following for each index set $I \in \mathcal{T}(q^0,\lambda^0,x^0)$ we consider the Jacobian M_I of the linear system, which describes the set $P^I(q)$, namely $M_I = \begin{pmatrix} K_{II} & K_{I\bar{I}} \\ 0 & E \end{pmatrix}$. Obviously, it holds $detM_I = detK_{II}$. Now we are able to give another equivalent condition for strong stability in problems of the type $\mathcal{P}_{01}(q,\lambda)$, which is already known from [11] for the Karush-Kuhn-Tucker conditions of nonlinear parametric optimization problems. For this case the assertion of the following theorem is shown in [10].

Theorem 3.3 *For any element $(q^0, \lambda^0, x^0) \in gph\Psi$ condition (***) is equivalent to*

$$sgn\ detM_I = const \neq 0 \qquad \forall I \in \mathcal{T}(q^0, \lambda^0, x^0). \tag{3}$$

Proof: According to Theorem 3.2 we only have to show, that (2) and (3) are equivalent. Let (2) be satisfied. Then for $I = I_1$ we have $detM_{I_1} = detK_{I_1 I_1} = detK_{11} \neq 0$. For any index set I with $I_1 \subset I \subseteq I_1 \cup I_2$ we can write $K_{II} = \begin{pmatrix} K_{11} & K_{I_1 I'} \\ K_{I' I_1} & K_{I' I'} \end{pmatrix}$, where $I' = I \setminus I_1$. Moreover, using a known determinant rule for Schur complements (cf. [20]), we have $detK_{II} = detK_{11} \cdot detN'$, where $N' = K_{I'I'} - K_{I'I_1} K_{11}^{-1} K_{I_1 I'}$ is a principal submatrix of N having according to (2) a positive determinant. Hence, $sgn\ detM_I = sgn\ detK_{II} = sgn\ detK_{11} = const \neq 0$ as required in (3). The other direction of the proof is similar. We only mention the fact that any principal submatrix of N can be expressed in the form $K_{I'I'} - K_{I'I_1} K_{11}^{-1} K_{I_1 I'}$ with an index set I satisfying $I_1 \subset I \subseteq I_1 \cup I_2$. q.e.d.

Corollary 3.1 *If the solution set map Ψ of $\mathcal{P}_{01}(q, \lambda)$ does not satisfy condition (***) at any element $(q^0, \lambda^0, x^0) \in gph\Psi$ then one of the following two cases a) or b) holds true.*
Case a) There is an index set $I^* \in \mathcal{T}(q^0, \lambda^0, x^0)$ satisfying $rgK_{I^* I^*} < \mid I^* \mid$.
Case b) There are two index sets $I', I'' \in \mathcal{T}(q^0, \lambda^0, x^0)$ and one index $i' \notin I'$ such that $I'' = I' \cup \{i'\}$ and it holds $sgn\ detM_{I'} = -sgn\ detM_{I''} \neq 0$.

Proof: This assertion follows by negation of (3) taking into account that the condition in case a) is only a reformulation of the equation $detM_{I^*} = 0$ and that (if case a) does not come true) the existence of two different index sets $I', I'' \in \mathcal{T}(q^0, \lambda^0, x^0)$ with $sgn\ detM_{I'} = -sgn\ detM_{I''} \neq 0$ implies that there are also two index sets I' and I" and an index i' with the properties given in case b). q.e.d.

Remark 3.2

1. *As shown in [4] even parametric nonlinear complementarity problems satisfying certain differentiability properties can be characterized locally (especially concerning the property of strong regularity) in the same way as it was done here and in former papers for linear problems, namely by analyzing its corresponding linearization. Hence, many results of this section may be used, for instance, to study the Karush-Kuhn-Tucker conditions of nonlinear (and not only quadratic) optimization problems depending on parameters.*

2. *According to [3] problem $\mathcal{P}_{01}(q, \lambda)$ may be written equivalently as a Lipschitz continuous equation of the form*

$$F(z, \lambda) := \qquad K(\lambda) z^- + z^+ \ = \ q, \tag{4}$$

where $z^+ = max(0, z)$ and $z^- = min(0, z)$ componentwise.
The necessary and sufficient conditions (2), (3) (as well as all other equivalent conditions of other papers as [3] and [4]) are equivalent with the nondegeneracy of the projection of the generalized Jacobian $\pi_z \partial F(z, \lambda)$ (in the sense of Clarke [2]) onto the subspace of the z-variables. This follows from a result of [8] and from the fact that the vertices of this projection are closely related to the matrices K_{II} considered in our paper. As we know from [12] this nondegeneracy is in general only a sufficient (but not necessary) condition for the so-called Lipschitz invertibility of systems of the form (4). Only because of a special rank property of the vertices of the mentioned projection (which has been applied already in [8] for the Karush-Kuhn-Tucker condition for nonlinear parametric optimization problems described by C^2-functions) this nondegeneracy condition of Clarke is also necessary for Lipschitz invertibility and, hence, equivalent to a necessary and sufficient condition of Kummer [13] for Lipschitz invertibility of Lipschitz systems and (for the special case of problem $\mathcal{P}_1(\mathbf{t})$) to a corresponding necessary and sufficient condition of Scholtes [26] for piecewise linear systems.

Proof of Theorem 3.1: Because of Statement 1 of Proposition 3.1 we only have to show $\neg(***) \Rightarrow \neg(*)$. Consider an arbitrary element $(q^0, \lambda^0, x^0) \in gph\Psi$ and suppose, that (***) is not satisfied there. Using Corollary 3.1 we have to study now more precisely the two cases a) and b) described there.
In the following we delete the dependence on the parameter λ by fixing $\lambda = \lambda^0$ and study the corresponding problem $\mathcal{P}_{02}(q)$. If we can show, that condition (*) is not satisfied at the point (q^0, x^0) of the graph of the solution set map of $\mathcal{P}_{02}(q)$, then, obviously, condition (*) is also not fulfilled at the point (q^0, λ^0, x^0) of the graph of the solution set map of $\mathcal{P}_{01}(q, \lambda)$.
Consider at first case a). According to our assumptions we have $(q^0, x^0) \in gphP^{I^*}$ and thus $q^0 \in \mathcal{A}^{I^*}$. Statements 2 and 3 of Theorem 2.4 applied to $I = I^*$ and $J = \bar{I}^*$ implies $d(I^*, \bar{I}^*) \geq 1$ and $dim\mathcal{A}^{I^*} < n$. Now we have to distinguish two subcases.
Subcase a_1): If $q^0 \in \tilde{\mathcal{A}}^{I^*}$ then $dimP^{I^*}(q^0) = d(I^*, \bar{I}^*) \geq 1$ and, hence, $\mid \Psi(q^0) \mid = \infty$. But according to Corollary 2.2 in each neighbourhood U of q^0 there must exist a parameter value q with $\mid \Psi(q) \mid < \infty$ and, hence, $dimP^{I^*}(q) < dimP^{I^*}(q^0)$, where because of $q^0 \in \tilde{\mathcal{A}}^{I^*}$ we have $\mathcal{T}(q^0, x^0) = \{I^*\}$, $\mathcal{Z}(q^0, x^0) = \{P_1\}$, $P_1 = P^{I^*}(q^0)$ and $\mathcal{S}_1 = \{(I^*, \bar{I}^*)\}$. But according to Statement 2 of Proposition 3.1 this contradicts condition (*).
Subcase a_2): If $q^0 \notin \tilde{\mathcal{A}}^{I^*}$ then q^0 belongs to the relative boundary of $\mathcal{A}^{I^*}$ and for any neighbourhoods U of q^0 and V of x^0 there are points $\tilde{q} \in U \cap \tilde{\mathcal{A}}^{I^*}$ and (because of the fact that the map P^{I^*} is continuous relative to $\mathcal{A}^{I^*}$) $\tilde{x} \in V \cap P^{I^*}(\tilde{q})$, for which our argumentation of subcase a_1) can be repeated.
Now we consider case b). According to our assumptions we have $(q^0, x^0) \in gphP^{I'', \bar{I}'}$ and thus $q^0 \in \mathcal{A}^{I'', \bar{I}'}$. The condition $sgn\ detM_{I'} = -sgn\ detM_{I''} \neq 0$ is equivalent with the condition $sgn\ detK_{I'I'} = -sgn\ detK_{I''I''} \neq 0$. Because of $I'' = I' \cup \{i'\}$ with $i' \notin I'$ the matrix $K_{I''I'}$ is formed by $K_{I'I'}$ refilled by one additional row. Hence, it holds $rg(K_{I''I'}) = \mid I' \mid$. Applying Statements 2 and 3 of Theorem 2.4 we get $d(I'', \bar{I}') = 0$ and $dim\mathcal{A}^{I'', \bar{I}'} = n - 1$. As in the case a) we want to distinguish two subcases.
Subcase b_1): If $q^0 \in \tilde{\mathcal{A}}^{I'', \bar{I}'}$ then $\mathcal{T}(q^0, x^0) = \{I', I''\}$, $\mathcal{Z}(q^0, x^0) = \{P_1\}$, $P_1 = \{x^0\}$ and $\mathcal{S}_1(q^0, x^0) = \{(I', \bar{I}'), (I'', \bar{I}''), (I'', \bar{I}')\}$. Let us consider the simplex tableaus (1') of the vertex x(I',t) as well as (1") of the vertex x(I",t) in the form described in (1). Because of $I'' = I' \cup \{i'\}$ tableau (1") can be generated from tableau (1') by exactly one pivot step with the element $d'_{i'i'}$ of the i'-th row and i'-th column in tableau (1') as pivot element. This pivot element is located in the main diagonal of the submatrix $K_{\bar{I}'I'}K^{-1}_{I'I'}K_{I'\bar{I}'} - K_{\bar{I}'\bar{I}'}$. Let us denote the linear functions of q in the last column in (1') (which is formed according to Remark 2.2 by the elements of the vectors $-K^{-1}_{I'I'}q_{I'}$ and $(q_{\bar{I}'} + K_{\bar{I}'I'}K^{-1}_{I'I'}q_{I'})$) by $d'_{i0}(q)$ and the corresponding functions in (1") by $d''_{i0}(q)$. According to the rules of the pivot technique it holds $d'_{i'0}(q) = d'_{i'i'}d''_{i'0}(q)$. Using the already mentioned determinant rule for Schur complements one can show that $detK_{I''I''} = -d'_{i'i'}detK_{I'I'}$ such that because of $sgn\ detK_{I''I''} = -sgn\ detK_{I'I'}$ necessarily $d'_{i'i'} > 0$ follows. According to Remark 2.2 it holds $\mathcal{A}^{I'} = \{\ q/\ d'_{i0}(q) \geq 0,\ i = 1, ..., n\}$ and $\mathcal{A}^{i''} = \{\ q/\ d''_{i0}(q) \geq 0,\ i = 1, ..., n\}$. Hence, both sets $\mathcal{A}^{I'}$ and $\mathcal{A}^{I''}$ are contained in the same halfspace $H_{i'} = \{q/d'_{i'0}(q) \geq 0\}$ and $\mathcal{A}^{I'', \bar{I}'}$ belongs to the corresponding hyperspace. But due to Statement 2 of Proposition 3.1 this contradicts (*).
Subcase b_2): If $q^0 \notin \tilde{\mathcal{A}}^{I'', \bar{I}'}$ then q^0 belongs to the relative boundary of $\mathcal{A}^{I'', \bar{I}'}$ and for any neighbourhoods U of q^0 and V of x^0 there are points $\tilde{q} \in U \cap \tilde{\mathcal{A}}^{I'', \bar{I}'}$ and (because of the fact that the map $P^{I'', \bar{I}'}$ is continuous relative to $\mathcal{A}^{I'', \bar{I}'}$) $\tilde{x} \in V \cap P^{I'', \bar{I}'}(\tilde{q})$, for which our argumentation of subcase b_1) can be repeated. q.e.d.

With other words the proof of Theorem 3.1 says: If at an element $(q^0, \lambda^0, x^0) \in gph\Psi$ condition (***) is violated, then in each neighbourhood of this element there is another element $(\tilde{q}, \lambda^0, \tilde{x})$ of this graph, at which the solution set map is not lower semicontinuous. In (q^0, λ^0, x^0) itself lower semicontinuity may hold or not. The violation of lower semicontinuity at $(\tilde{q}, \lambda^0, \tilde{x})$ may happen for two different reasons. One possibility is that for all sufficiently small neighbourhoods U of $\tilde{q}$ and V of $\tilde{x}$ there exists a value $q \in U$ such that there does not exist any solution x of $\mathcal{P}(q, \lambda^0)$ in V. The other possibility is that for all sufficiently small neighbourhoods U of $\tilde{q}$ and V of $\tilde{x}$ there exists a value $q \in U$ such that the number of solutions x of $\mathcal{P}(q, \lambda^0)$ in V is finite, whereas the number of solutions of $\mathcal{P}(\tilde{q}, \lambda^0)$ in V is infinite. Corollary 2.2 shows that this second possibility only leads to a contradiction to condition (*) if all components of q may be perturbed independently of each other. This would be not the case, for instance, in the problem $\mathcal{P}_1(\mathbf{t})$ if $d < n$.

4 Generic properties of one-parametric linear complementarity problems

Let us study in this section the one-parametric linear complementarity problem $\mathcal{P}_{11}(t)$. The examples given in the foregoing section show that even for problems of small size the solution set map of this problem may have a rather complicate structure. As the result of this section we will see that generically the graph of the solution set map has a very easy structure. We can show this with help of the results on the problem $\mathcal{P}_{02}(q)$ given in Theorem 2.4.

Lemma 4.1 *There is an open and dense subset $\mathbf{Q} \subseteq \mathbf{R^{2n}}$ such that for all $(q^0, q^1) \in \mathbf{Q}$ the set $g = \{q \in R^n / q = q^0 + tq^1,\ t \in R\}$ has the following two properties:*

1. *For all $(I, J) \in \mathcal{S}$ with $dim \mathcal{A}^{I,J} = n - 1$ it holds $g \not\subseteq lin\ \mathcal{A}^{I,J}$.*
2. *For all $(I, J) \in \mathcal{S}$ with $dim\ \mathcal{A}^{I,J} \leq n - 2$ it holds $g \cap \mathcal{A}^{I,J} = \emptyset$.*

Proof: We show that those values (q^0, q^1), for which 1 or 2 is violated, is contained in the union of a finite number of nondegenerated smooth manifold with dimension less or equal to 2n-1.
1. If $g \subseteq lin\ \mathcal{A}^{I,J}$ with $dim\ \mathcal{A}^{I,J} = n - 1$, then necessarily it follows that (q^0, q^1) belongs to the linear subspace $lin\ \mathcal{A}^{I,J} \times lin\ \mathcal{A}^{I,J}$ of R^{2n}, which has the dimension 2n-2.
2. If $g \cap \mathcal{A}^{I,J} \neq \emptyset$ with $dim\ \mathcal{A}^{I,J} \leq n - 2$, then also $g \cap lin\ \mathcal{A}^{I,J} \neq \emptyset$. According to our assumption on the dimension of $\mathcal{A}^{I,J}$ there must are two linear independent vectors $a, b \in R^n$ such that $lin\ \mathcal{A}^{I,J} \subseteq L^{n-2}$ with $L^{n-2} = \{q \in R^n / a'q = 0,\ b'q = 0\}$. This means that the two linear equations for one variable t, namely $a'(q^0 + tq^1) = 0$ and $b'(q^0 + tq^1) = 0$ must have a solution. But this implies that either (q^0, q^1) is an element of the linear subspace $L^{n-2} \times L^{n-2}$, which has the dimension 2n-4, or (q^0, q^1) belongs to the set described by $a'q^0b'q^1 - b'q^0a'q^1 = 0$, which is outside of $L^{n-2} \times L^{n-2}$ a nondegenerated quadratic manifold of dimension 2n-1. q.e.d.

The given proof shows that it suffices to disturb slightly only one of the both vectors q^0 or q^1 to reach the set $\mathbf{Q}$, if a given pair (q^0, q^1) originally would not belong to $\mathbf{Q}$. Only because of the possibility that $q^i \in L^{n-2}$ may come true we can not restrict our disturbations on $q^{i'}$ $(i = 0, 1;\ i' = 1, 0)$.
Using the set $\mathbf{Q}$ described in Lemma 4.1 we are now able to prove in the next theorem an essential generical property (in the sense that this property holds true for all vectors (q^0, q^1) from an open and dense subset $\mathbf{Q}$ of R^{2n}) for the graph of the solution set map Ψ of $\mathcal{P}_{11}(t)$. For a given element $(t, x) \in gph\Psi$ we use here the notation $I(t, x) = \{i/(Kx + q^0 + tq^1)_i = 0\}$ and $J(t, x) = \{j/x_j = 0\}$.

Theorem 4.1 *For all vectors $(q^0, q^1) \in \mathbf{Q}$ we have:*
1. Each connected component of $gph\Psi$ for $\mathcal{P}_{11}(t)$ is a crunode-free edge polygon, which may be either
a) homeomorphic to the real line or
b) homeomorphic to a circle or
c) an isolated point of $gph\Psi$.
2. Each element $(t, x) \in gph\Psi$ belongs to exactly one of the following six types:

Type 1: *$I(t,x) \cap J(t,x) = \emptyset$ (strict complementarity) $\wedge$ $rgK_{II} =\mid I \mid$, where $I = I(t,x)$.*

Type 2: *$I(t,x) \cap J(t,x) = \emptyset$ (strict complementarity) $\wedge$ $rgK_{II} =\mid I \mid -1$, where $I = I(t,x)$.*

Type 3: *$I(t,x) \cap J(t,x) = \{i'\}$ $\wedge$ $sgn\ detK_{I'I'} = sgn\ detK_{I''I''} \neq 0$, where $I' = I(t,x)$, $J' = J(t,x) \setminus \{i'\}$, $I'' = I(t,x) \setminus \{i'\}$, $J'' = J(t,x)$.*

Type 4: *$I(t,x) \cap J(t,x) = \{i'\}$ $\wedge$ $sgn\ detK_{I'I'} = -sgn\ detK_{I''I''} \neq 0$, where I', J', I"and J" are defined as above.*

Type 5: *$I(t,x) \cap J(t,x) = \{i'\}$ $\wedge$ $sgn\ detK_{I'I'} \neq 0$ $\wedge$ $sgn\ detK_{I''I''} = 0$ (or vice versa), where again I', J', I"and J" are defined as above.*

Type 6: *$I(t,x) \cap J(t,x) = \{i'\}$ $\wedge$ $sgn\ detK_{I'I'} = sgn\ detK_{I''I''} = 0$, where again I', J', I"and J" are defined as above.*

3. For almost all values of t only Type 1 occours. Almost all elements $(t,x) \in gph\Psi$ are of the types 1 and 2.

Proof: According to Lemma 4.1 for $(q^0, q^1) \in \mathbf{Q}$ the line g may intersect only those sets $\tilde{\mathcal{A}}^{I,J}$ with dimension n or n-1, where the second case may only accour for a finite number of values t. Hence, taking Theorem 2.4 and Remark 2.2 into account, the graph of Ψ will be formed by all points (t,x) satisfying

a) $(q^0 + tq^1) \in \tilde{\mathcal{A}}^I$ for any index set $I \in \{1, ..., n\}$ such that $dim\mathcal{A}^I = n$ and K_{II} is regular and $x = x(I,t) = \begin{pmatrix} x_I(I,t) \\ x_{\bar{I}}(I,t) \end{pmatrix}$ with $x_I(I,t) = -K_{II}^{-1}(q_I^0 + tq_I^1)$ and $x_{\bar{I}}(I,t) = 0$ or

b) $(q^0 + tq^1) \in \tilde{\mathcal{A}}^{I,J}$ for any pair $(I,J) \in \mathcal{S}$ such that $dim\mathcal{A}^{I,J} = n-1$ and $x \in P^{I,J}(q^0 + tq^1)$.

According to Theorem 2.4 and Remark 2.2 the points $(t,x) \in gph\Psi$ satisfying a) are just those elements of $gph\Psi$ of the type 1 and form, together with their boundary points, a first finite system of edges of the set $gph\Psi$. These boundary points also must belong to $gph\Psi$, since this set is closed. At these boundary points necessarily b) must be satisfied.

For the finite number of values $\bar{t}$, for which there is a point $\bar{x}$ such that $(\bar{t}, \bar{x}) \in gph\Psi$ satisfies b) we can use Statements 2 and 5 of Theorem 2.4 and Lemma 3.1. According to Statement 5 of Theorem 2.4 we must study two subcases of case b).

In the subcase b_1) we have $I \cap J = \emptyset$ and $d(I,J) = 1$. Hence, the points $(t,x) \in gph\Psi$ satisfying b_1) are just the elements of $gph\Psi$ of the type 2 and form together with their boundary points a second finite system of edges of the set $gph\Psi$. Also these boundary points must belong to $gph\Psi$ and are in this case just the elements of $gph\Psi$ of the type 5.

In the subcase b_2) we have the situation $I \cap J = \{i'\}$ and $d(I,J) = 0$ such that the set $\{(\bar{t}, x)/x \in P^{I,J}(q^0 + \bar{t}q^1)\}$ is a singleton. Obviously, with the types 3-6 all possibilities for $sgn\ detK_{I'I'}$ and $sgn\ detK_{I''I''}$ under the assumption $I \cap J = \{i'\}$ are exhausted such that the points $(t,x) \in gph\Psi$ satisfying b_2) are just the elements of $gph\Psi$ of the types 3-6. The points of the types 3 and 4 are common boundary points of exactly two adjacent edges of the first system, one of them given by the vertex $x(I', t)$ of the convex polyhedron P(t), the other one by the vertex $x(I'', t)$. This follows by Lemma 3.1. Again by Lemma 3.1 we see that the points of the type 5 are the common boundary points of exactly one edge of the first system and exactly one of the second system. Finally, the points of the type 6 are isolated points of $gph\Psi$. Both systems of edges together with the isolated points of type 6 form the whole graph of Ψ. This completes the proof. q.e.d.

Remark 4.1 *1. For all elements $(t,x) \in gph\Psi$ of the types 1 and 3-6 the x-part is a vertex of the convex polyhedron P(t). For all elements $(t,x) \in gph\Psi$ of the type 2 the x-part is an inner point of an edge of the convex polyhedron P(t).*

2. *For all elements* $(\bar t,\bar x) \in gph\Psi$ *of the type 6 the corresponding vertices* $(\bar x,\bar y)$ *of the convex polyhedron P'(t) with* $\bar y = q^0 + \bar t q^1 + K\bar x$ *are exactly those vertices of P'(t) which actually satisfy the complementarity condition* $\bar x'\bar y = 0$, *but for which there does not exist any simplex tableau of the form (1), i.e., there does not exist any basis solution with the property that for each i=1,...,n exactly one of the varibles* x_i *and* y_i *is a basic variable and the other one a non-basic variable. Under different assumptions on the matrix K such vertices and, hence, elements* $(t,x) \in gph\Psi$ *of the type 6 can not exist (cf. [1]).*

3. *For an arbitrary element* $(\bar t,\bar x) \in gph\Psi$ *of one of the types 1 or 3-5 consider a corresponding simplex tableau (1). As in the proof of Theorem 3.1 let us denote the elements of (1) by* d_{ij} *and the linear functions of t in the last column of (1) by* $d_{i0}(t)$. *According to Remark 2.2 we assume* $d_{i0}(\bar t) \geq 0,\ i = 1,...,n$. *With help of the data of (1) we can characterize uniquely the type of this point as follows:*
a) $(\bar t,\bar x)$ *is of the type 1* $\iff d_{i0}(\bar t) > 0,\ i = 1,...,n.$
b) $(\bar t,\bar x)$ *is of the type 3* $\iff$ *there is exactly one index* $i' \in \{1,...,n\}$ *such that* $d_{i'0}(\bar t) = 0$ *and it holds* $d_{i'i'} < 0$.
c) $(\bar t,\bar x)$ *is of the type 4* $\iff$ *there is exactly one index* $i' \in \{1,...,n\}$ *such that* $d_{i'0}(\bar t) = 0$ *and it holds* $d_{i'i'} > 0$.
d) $(\bar t,\bar x)$ *is of the type 5* $\iff$ *there is exactly one index* $i' \in \{1,...,n\}$ *such that* $d_{i'0}(\bar t) = 0$ *and it holds* $d_{i'i'} = 0$.

4. *The unique open edge of the second class formed by solutions* $(\bar t,x) \in gph\Psi$ *of the type 2 with an element* $(\bar t,\bar x) \in gph\Psi$ *of the type 5 (which satisfies d) of Statement 3) as one boundary point can be constructed with help of the corresponding simplex table (1) to* $(\bar t,\bar x)$ *as follows: We put* $z_{Bi'} = 0,\ z_{Bi} = d_{i0}(\bar t) - d_{ii'}s,\ i \neq i',\ z_{Ni'} = s,\ 0 < s < \bar s,\ z_{Ni} = 0, i \neq i'$, *where* z_B *stands for the vector of basic variables in (1),* z_N *for the vector of non-basic variables and* $\bar s = sup\{s / d_{i0}(\bar t) - d_{ii'}s \geq 0,\ i \neq i'\}$.

5. *Concerning the edges of* $gph\Psi$ *of the second class there are three different cases to distinguish:*
Case a) If all elements $d_{ii'},\ i \neq i'$, *are nonpositive then this edge is unbounded* ($\bar s = \infty$) *and is, hence, the first or last edge of the corresponding edge polygon, to which it belongs.*
Otherwise the edge is bounded and must have a second boundary point $(\bar t',\bar x')$. *Let be* $i'' \in \{i \neq i' / \frac{d_{i0}(\bar t)}{d_{ii'}} = min_{j:d_{ji'}>0} \frac{d_{j0}(\bar t)}{d_{ji'}}\}$ *and* $\bar s = \frac{d_{i''0}(\bar t)}{d_{i''i'}}$. *For* $(q^0,q^1) \in Q$ *the index i" is uniquely determined and it holds* $d_{i'i''} \neq 0$. *Hence, we can obtain a simplex tableau (1') to* $(\bar t',\bar x')$ *from (1) by one* (2×2)*-pivot step with the* (2×2)*-matrix* $\begin{pmatrix} d_{i'i'} & d_{i'i''} \\ d_{i''i'} & d_{i''i''} \end{pmatrix}$ *as pivot matrix. Because of* $d_{i'i'} = 0,\ d_{i''i'} > 0$ *and* $d_{i'i''} \neq 0$ *this matrix is regular, if* $(q^0,q^1) \in Q$. *The index set I', which corresponds to (1'), is formed by I, i' and i" in the following way. First we put* $I^* = I \setminus \{i'\}$ *if* $i' \in I$ *respectively* $I^* = I \cup \{i'\}$ *otherwise. Analoguesly, we set* $I' = I^* \setminus \{i''\}$ *if* $i'' \in I$ *respectively* $I' = I^* \cup \{i''\}$ *otherwise. The corresponding matrix* $K_{I'I'}$ *will always be regular. With respect to sufficiently small neighbourhoods W of* $(\bar t,\bar x)$ *and W' of* $(\bar t',\bar x')$ *the following two possibilities b) and c) may appear.*
Case b) If $d_{i'i''} < 0$ *then* $sgn\ det K_{II} = sgn\ det K_{I'I'}$ *and for* $t < \bar t$ *the intersection of W with* $gph\Psi$ *consists of all points* $(t,x(I,t))$ *and is empty for* $t > \bar t$ *and the intersection of W' with* $gph\Psi$ *is empty for* $t < \bar t$ *and consists for* $t > \bar t$ *of all points* $(t,x(I',t))$ *(or vice versa).*
Case c) If $d_{i'i''} > 0$ *then* $sgn\ det K_{II} = -sgn\ det K_{I'I'}$ *and for* $t < \bar t$ *the intersection of W with* $gph\Psi$ *consists of all points* $(t,x(I,t))$ *and is empty for* $t > \bar t$ *and the intersection of W' with* $gph\Psi$ *is empty for* $t > \bar t$ *and consists for* $t < \bar t$ *of all points* $(t,x(I',t))$ *(or vice versa).*

6. *At all elements* $(t, x) \in gph\Psi$ *of the types 1 and 3 condition (***) is satisfied, whereas at all other types 2 and 4-6 even condition (*) is not fulfilled.*

7. *With respect to an open and dense subset of the* $(n \times n)$*-dimensional Euclidean space of all elements* k_{ij} *the matrix* K *is an N-matrix. Hence, for the problem* $\mathcal{P}_{11}(t)$ *only the types 1, 3 and 4 remains generic, if we permit to disturb beside the vectors* q^0 *and* q^1 *also the elements of the matrix* K*.*

8. *If all principal minors of the matrix* K *are nonnegative, then the types 4 and 6 as well as the case c) described in Statement 5 can not appear, the graph of the solution set map of* $\mathcal{P}_{11}(t)$ *consists of exactly one edge polygon and is always homeomorphic to the real line (cf. [1]). If the matrix* K *is even a P-matrix, then also the types 2 and 5 can not appear, all elements* $(t, x) \in gph\Psi$ *satisfy (***) and* $gph\Psi$ *is formed only by edges of the first class.*

In the following theorem we characterize the local structure of the graph of the solution set map Ψ of $\mathcal{P}_{11}(t)$ for the six different types given above. We use the notations I', I" and x(I,t) as in Theorem 4.1.

Theorem 4.2 *Let be* $(\bar{t}, \bar{x}) \in gph\Psi$ *and* W *a sufficiently small neighbourhood of* $(\bar{t}, \bar{x})$*.*

1. *If* $(\bar{t}, \bar{x})$ *is of the type 1 then we have* $W \cap gph\Psi = \{(t, x) \in W \ / \ x = x(I, t)\}$*, where* $I = I(\bar{t}, \bar{x})$*.*

2. *If* $(\bar{t}, \bar{x})$ *is of the type 2 then we have* $W \cap gph\Psi = \{(t, x) \in W \ / \ t = \bar{t},\ x \in P^I(\bar{t})\}$*, where* $I = I(\bar{t}, \bar{x})$*.*

3. *If* $(\bar{t}, \bar{x})$ *is of the type 3 then we have* $W \cap gph\Psi = \{(t, x) \in W \ / \ t \le \bar{t},\ x = x(I', t)\} \cup \{(t, x) \in W \ / \ t \ge \bar{t},\ x = x(I'', t)\}$ *or* $W \cap gph\Psi = \{(t, x) \in W \ / \ t \ge \bar{t},\ x = x(I', t)\} \cup \{(t, x) \in W \ / \ t \le \bar{t},\ x = x(I'', t)\}$*. For* $t = \bar{t}$ *it holds* $x(I', t) = x(I'', t)$*.*

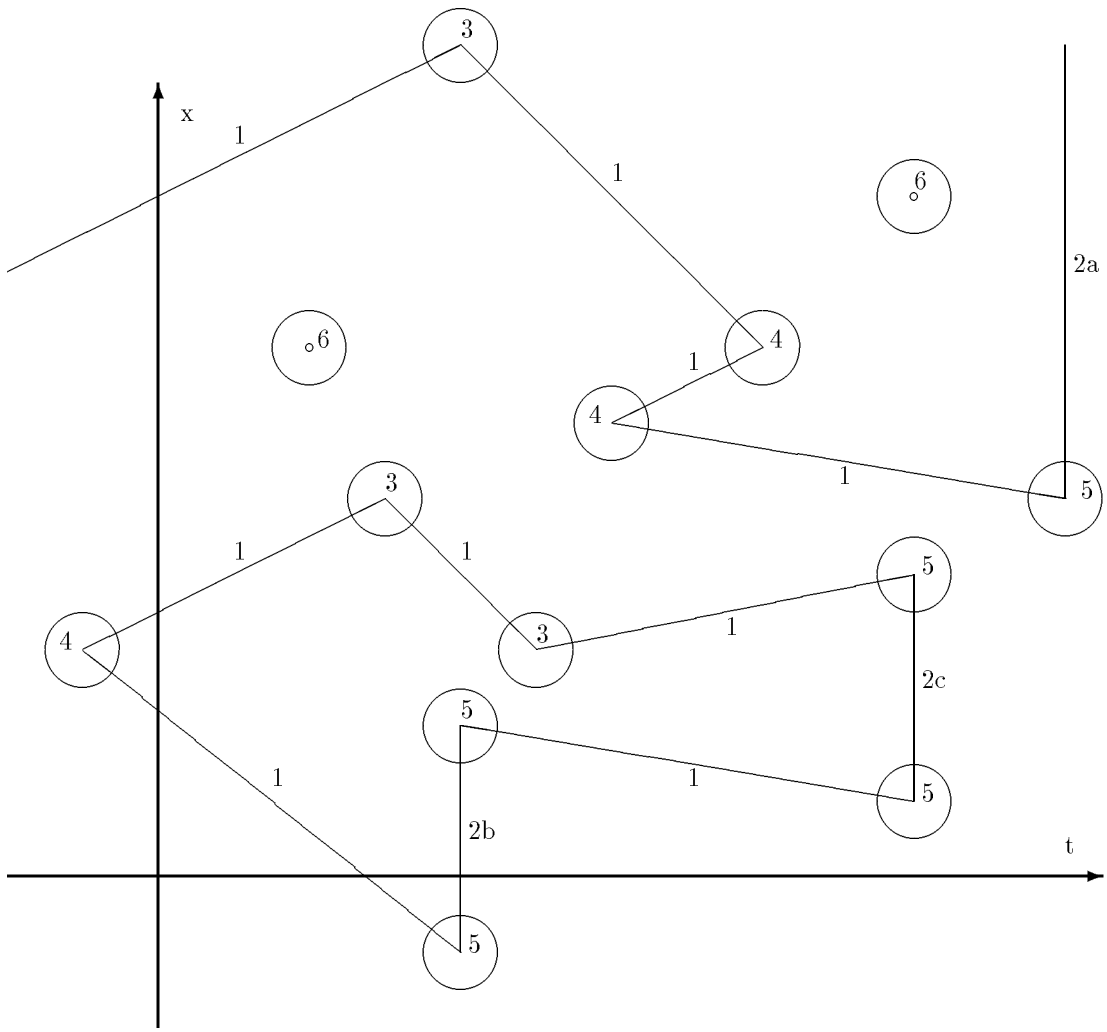
4. *If* $(\bar{t}, \bar{x})$ *is of the type 4 then we have* $W \cap gph\Psi = \{(t, x) \in W \ / \ t \le \bar{t},\ x = x(I', t)\} \cup \{(t, x) \in W \ / \ t \le \bar{t},\ x = x(I'', t)\}$ *or* $W \cap gph\Psi = \{(t, x) \in W \ / \ t \ge \bar{t},\ x = x(I', t)\} \cup \{(t, x) \in W \ / \ t \ge \bar{t},\ x = x(I'', t)\}$*. For* $t = \bar{t}$ *it holds* $x(I', t) = x(I'', t)$*.*

5. *If* $(\bar{t}, \bar{x})$ *is of the type 5 then we have* $W \cap gph\Psi = \{(t, x) \in W \ / \ t \le \bar{t},\ x = x(I', t)\} \cup \{(t, x) \in W \ / \ t = \bar{t},\ x \in P^{I''}(\bar{t})\}$ *or* $W \cap gph\Psi = \{(t, x) \in W \ / \ t \ge \bar{t},\ x = x(I', t)\} \cup \{(t, x) \in W \ / \ t = \bar{t},\ x \in P^{I''}(\bar{t})\}$ *or* $W \cap gph\Psi = \{(t, x) \in W \ / \ t \le \bar{t},\ x = x(I'', t)\} \cup \{(t, x) \in W \ / \ t = \bar{t},\ x \in P^{I'}(\bar{t})\}$ *or* $W \cap gph\Psi = \{(t, x) \in W \ / \ t \ge \bar{t},\ x = x(I'', t)\} \cup \{(t, x) \in W \ / \ t = \bar{t},\ x \in P^{I'}(\bar{t})\}$*.*

6. *If* $(\bar{t}, \bar{x})$ *is of the type 6 then we have* $W \cap gph\Psi = \{(\bar{t}, \bar{x})\}$*.*

Proof: The proof follows by Theorem 4.1 and Remark 4.1. q.e.d.

The following picture illustrates the given six types of solutions and the possible situations concerning the structure of the graph of the solution set map. In this example this graph has fore connected components, namely two isolated points, one edge polygon which is homeomorphic to the real line and one edge polygon which is homeomorphic to a circle. Moreover, there are three edges of the second class, one of them corresponds to case a) described in Statement 5 of Remark 4.1, one to case b) and one to case c).



Generic properties of the Karush-Kuhn-Tucker conditions for one-parametric quadratic optimization problems are the common subject of Section 4 of our paper with the papers [7] of Jongen et al and [6] of Henn et al. For the special case of one-parametric linear optimization problems we refer also to the relevant paper [22] of Patewa. The results are partially similar but not identical because of the following essential differences in the assumptions. Firstly, in the papers [7] and [6] the dependence on the parameter t is assumed to be more general (of the type C^3 respectively C^1), whereas we restrict our considerations to the case that only the vector q depends on t and this dependence is linear. This point is connected with the second difference, namely with the fact, that our notion "generic" is based only on small pertubations of the problem in the finite dimensional

space R^{2n} of the data q^0 and q^1 with the corresponding topology, whereas in [7] and [6] small perturbations of all underlying functions in the strong C^3-topology respectively strong C^1-topology are allowed. Finally, an exact comparison of the three papers would require, on the one hand, to include the Lagrange multipliers and their dependence on the parameter into the considerations of [7] and [6] and, on the other hand, to include all aspects of [7] which are only relevant for the Karush-Kuhn-Tucker conditions of an optimization problem into our considerations.
Remember that in [7] five types of (generalized) critical points of one-parametric nonlinear optimization problems described by C^3-functions have been identified to be generic, whereas the paper [6] shows that in the corresponding special case of quadratic respectively linear optimization only the types 1, 2 and 5 respectively 1 and 5 from [7] remain generic.
Taking into consideration all differences between the approaches in [7] and in our Section 4 mentioned above, we can see, that the types 1 of both papers are identical and that the two subcases of type 2 of [7] correspond to our types 3 and 4. Type 3 of [7] has some common properties as our type 2 but both are not identical . Finally, type 5 of [7] is related to our types 2 and 5 but is not identical. All other types of both papers differ essentially of each other.

Acknowledgement

The author is indepted to Bernd Kummer and Diethard Klatte for helpful discussions on the subject of this paper.

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