How Floquet-theory applies to differential-algebraic equations

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Abstract

Local stability of periodic solutions is established by means of a corresponding Floquet-theory for index-1 differential-algebraic equations. For this, linear differential-algebraic equations with periodic coefficients are considered in detail, and a natural notion of the monodromy matrix is figured out, which generalizes the well-known case of regular ordinary differential equations.

1 Introduction

Evaluating the stability of periodic solutions is of great interest from a theoretical point of view as well as against the background of applicability .
Surprisingly, differential-algebraic equations (DAEs) have not yet been investigated in this respect in detail. The demand for useful criteria in terms of the original data has not been satisfied by now.

In this paper we consider the question whether it will be possible to obtain stability criteria of periodic solutions for index-1 DAEs in a similar way as it can be done by means of the Floquet-theory for regular ordinary differential equations (ODEs). The answer is positive. The theorem of Floquet on the representation of the fundamental matrix [Fl1883] as well as the reduction theorem of Lyapunov [Ly1892] and, finally, the theorem on the stability of periodic solutions hold true for DAEs, too. On the basis of appropriate notions of equivalence, fundamental matrix, monodromy matrix and stability that directly generalize the case of regular ODEs, the results for DAEs sound as clear and simple as already for regular ODEs.

In distinction to regular ODEs, DAEs have only implicitly given dynamic parts. To work out and utilize this implicit structur we apply a decoupling technique which goes back to [GM86]. Via a proper decoupling, the classical procedures of ODE theory (e.g. [Po65], [Fa94]) become applicable.

Till now, only a few attempts have been made with respect to stability analysis for DAEs, and those few refer to stationary solutions and contractivity ([GM86], [Ti94], [Ma92], [Mu95], [LMM94] etc.).

In our paper we will make a first attempt towards an Floquet theory directly for DAEs. §2 presents a collection of fundamental facts on linear DAEs with continuous coefficients only. For the first time, we work with variable transformations which, like the solutions themselves, belong to the larger class C^1_N. This permits to apply the mentioned theorems of Floquet and Lyapunov, respectively, in §3. Each linear DAE with periodic coefficients turns out to be periodically equivalent to a constant coefficient DAE in Kronecker normal form. §4 contains a theorem on the stability of the trivial solution in case of nonautonomous DAEs with constant linear part and small nonlinearity.
Finally, the main result (§5) can be proved following the lines of the classical model.

2 Fundamentals

We start considering linear homogeneous DAEs

$$A(t)x'(t) + B(t)x(t) = 0, \tag{2.1}$$

where $A, B \in C(\mathbb{R}, L(\mathbb{R}^m))$.
Suppose the nullspace $N(t) := \ker A(t)$ to be smooth, i.e. to be spanned by continuously differentiable basis functions. In particular, $A(t)$ has constant rank then.
Obviously, all solutions of (2.1) belong to the subspace

$$S(t) := \{z \in \mathbb{R}^m : B(t)z \in \operatorname{im} A(t)\} \subset \mathbb{R}^m.$$

Assume that (2.1) is index-1-tractable, i.e.,

$$S(t) \cap N(t) = \{0\}.$$

Then, exactly one solution passes through each point of $S(t)$ at time t (cf. [GM86]).
Using any C^1 projector function $Q(t)$ onto $N(t)$ and $P(t) := I - Q(t)$ initial value problems (IVPs) are properly stated with the initial condition

$$P(0)(x(0) - x^0) = 0. \tag{2.2}$$

The initial value problem (IVP) (2.1), (2.2) is uniquely solvable for all $x^0 \in \mathbb{R}^m$.
In particular, for semi-explicit systems

$$\left.\begin{aligned} x_1'(t) + B_{11}(t)x_1(t) + B_{12}x_2(t) &= 0 \\ B_{21}(t)x_1(t) + B_{22}x_2(t) &= 0 \end{aligned}\right\} \tag{2.3}$$

we have $P(t) = \operatorname{diag}(I, 0)$,

$$\begin{aligned} S(t) &= \left\{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^m : B_{21}(t)z_1 + B_{22}(t)z_2 = 0\right\}, \\ N(t) &= \left\{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{R}^m : z_1 = 0\right\}, \\ S(t) \cap N(t) &= \left\{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : z_1 = 0, z_2 \in \ker B_{22}(t)\right\} \end{aligned}$$

and the index-1 condition holds if and only if $B_{22}(t)$ remains nonsingular. The initial condition (2.2) fixes now $x_1(0) - x_1^0 = 0$.
Solving the second equation in (2.3) for $x_2(t) = -B_{22}(t)^{-1}B_{21}(t)x_1(t)$ we know the component $x_2(t)$ to be continuous, naturally, due to the continuity of $B(t)$. However, for more regularity of the solution the coefficients $A(t)$, $B(t)$ must be smoother. As we consider linear systems arising via linearizations we are not interested in smoother coefficients in general. While for the semi-explicit DAE (2.3) the natural solution regularity is $x_1(.) \in C^1$, $x_2(.) \in C$, the solutions of the general DAE (2.1) should belong to the function space

$$C_N^1 := \{x \in C : Px \in C^1\}.$$

This is easily understood considering the identities

$$\begin{aligned} &A(t) = A(t)P(t), \quad A(t)Q(t) = 0, \\ &A(t)x'(t) = A(t)P(t)x'(t) = A(t)\{(Px)'(t) - P'(t)x(t)\}. \end{aligned}$$

In the following, for $x \in C_N^1$, we understand the expression $A(t)x'(t)$ to be nothing else an abbreviation of

$$A(t)\{(Px)'(t) - P(t)x(t)\}. \tag{2.4}$$

It should be stressed that the function space C_N^1 but also the value of expression (2.4) are independent of the choice of the projector function. Namely, let two C^1-projector functions P, $\bar{P}$ be given. Both $P(t)$ and $\bar{P}(t)$ project along $N(t)$. If $x \in C$, $Px \in C^1$ then $\bar{P}x = \bar{P}Px$ belongs to C^1 since $\bar{P}$ and Px do so. Moreover, we compute $A(t)\{(Px)'(t) - P'(t)x(t)\} = A(t)\bar{P}(t)\{(Px)'(t) - P'(t)x(t)\} = A(t)\{(\bar{P}Px)'(t) - \bar{P}'(t)P(t)x(t) - \bar{P}(t)P'(t)x(t)\} = A(t)\{(\bar{P}Px)'(t) - (\bar{P}P)'(t)x(t)\} = A(t)\{(\bar{P}x)'(t) - \bar{P}'(t)x(t)\}$.

By means of the fundamental matrix $X(t)$, which is the solution of the matrix-valued IVP

$$\begin{aligned} A(t)X'(t) + B(t)X(t) &= 0 \\ P(0)(X(0) - I) &= 0, \end{aligned}$$

we can write down the solutions of (2.1), (2.2) as

$$x(t; x^0) = X(t)x^0.$$

We will use a representation of the fundamental matrix X of the DAE using the fundamental matrix U of the inherent ODE (cf. [GM86])

$$\left.\begin{aligned} U' + [-P'P_{\text{can}} + P(A + BQ)^{-1}B]U &= 0 \\ U(0) &= I \in L(\mathbb{R}^m) \end{aligned}\right\}. \tag{2.5}$$

Here, $P_{\text{can}}(t)$ denotes the canonical projector along $N(t)$ onto $S(t)$. Then it holds

$$X(t) = P_{\text{can}}(t)U(t)P(0). \tag{2.6}$$

We emphasize that $X(t)$ is independent of the special projector P used in (2.5) and (2.6). In any case, we have

$$X(0) = P_{\mathrm{can}}(0).$$

Further, while U is C^1 the canonical projector $P_{\mathrm{can}}(t)$ is continuous but not of class C^1 in general.
For example, in the semi-explicit case (2.3) it holds that

$$P_{\mathrm{can}}(t) = \begin{pmatrix} I & 0 \\ B_{22}(t)^{-1}B_{21}(t) & 0 \end{pmatrix}.$$

In later sections we transform linear DAEs with periodic coefficients to constant-coefficient DAEs. This task is closely connected to transformations to Kronecker normal form.

Applying a scaling of the equations $E \in C(\mathbb{R}, L(\mathbb{R}^m))$ and a transformation of variables $x = F(t)\bar{x}$, $F \in C^1(\mathbb{R}, L(\mathbb{R}^m))$, E, F both nonsingular, the DAE (2.1) changes to

$$\bar{A}(t)\bar{x}'(t) + \bar{B}(t)\bar{x}(t) = 0, \tag{2.7}$$

where $$\bar{A} = EAF, \qquad \bar{B} = E(BF + AF'). \tag{2.8}$$

(2.7) is in Kronecker normal form if we attain

$$\bar{A}(t) = \begin{pmatrix} I & \\ & 0 \end{pmatrix}, \qquad \bar{B}(t) = \begin{pmatrix} W(t) & \\ & I \end{pmatrix}. \tag{2.9}$$

The relation between the characteristic subspaces and the canonical projections may be described by $\bar{N}(t) = F^{-1}(t)N(t)$, $\bar{S}(t) = F^{-1}(t)S(t)$, and $\bar{P}_{\mathrm{can}}(t) = F^{-1}(t)P_{\mathrm{can}}(t)F(t)$.
For the Kronecker normal form (2.9)) the projector onto $\bar{S}(t) := \left\{\binom{z_1}{z_2} : z_2 = 0\right\}$ along $\bar{N}(t) = \left\{\binom{z_1}{z_2} : z_1 = 0\right\}$ is $\bar{P}_{\mathrm{can}}(t) = \mathrm{diag}(I, 0)$. Hence, it comes out that starting with index 1 systems in Kronecker normal form and using C^1 transformations F we arive at DAEs with continuously differentiable canonical projectors.
In the consequence, looking for a Kronecker normal form for continuous coefficient DAEs, we should apply a larger class of transformations. In the following we show the class C^1_N to be the proper one for the transformations F also.

Lemma 2.1 *The transformation of the unknown function $x(t) = F(t)\bar{x}(t)$ with $F \in C^1_N$, F nonsingular, transforms the DAE* (2.1) *into* (2.7)*, where*

$$\bar{A} = AF, \qquad \bar{B} = BF + AF' \tag{2.10}$$

are continuous and $\bar{A}$ has a smooth nullspace again.

Note that we understand AF' again as an abbreviation of $A\{(PF)' - P'F\}$ with any P.

Proof. First, remark that, for $\bar{A} := AF$, we have $\bar{N} := \ker \bar{A} = \ker AF = F^{-1}\ker A = F^{-1}N$. We will present a C^1 projector $\bar{P}$ along $\bar{N}$ to show that $\bar{N}$ is smooth, i.e. C^1 again.

Therefore, let P be the orthogonal projector along N and set $\bar{P} := F^{-1}PF$. Then, clearly, $\bar{P}$ is a projector along $\bar{N}$ and due to $P = P^+$ and

$$\bar{P} = F^{-1}PF = F^{-1}PPF = F^{-1}P^+PF = (PF)^+(PF),$$

it follows that $\bar{P}$ is smooth.
Remark that $\bar{P}$ is not orthogonal in general. Note that also $\bar{P}F^{-1}$ is smooth, since $\bar{P}F^{-1} = F^{-1}PFF^{-1} = F^{-1}P = F^{-1}P^+ = (PF)^+$ is so.
Now, starting with (2.1) and substituting $x = F\bar{x}$ we obtain

$$\begin{aligned}
0 &= A\{(PF\bar{x})' - P'F\bar{x}\} + BF\bar{x} \\
&= A(PF\bar{x})' + (B - AP')F\bar{x} \\
&= AFF^{-1}(PF\bar{x})' + (B - AP')F\bar{x} \\
&= \bar{A}(\bar{P}F^{-1}PF\bar{x})' - \bar{A}(\bar{P}F^{-1})'PF\bar{x} + (B - AP')F\bar{x} \\
&= \bar{A}(\bar{P}\bar{x})' - \bar{A}(\bar{P}F^{-1})'PF\bar{x} + (B - AP')F\bar{x} \\
&= \bar{A}(\bar{P}\bar{x})' - \bar{A}\bar{P}'\bar{x} + \bar{A}\bar{P}F^{-1}(PF)'\bar{x} + (B - AP')F\bar{x} \\
&= \bar{A}\{(\bar{P}\bar{x})' - \bar{P}'\bar{x}\} + AFF^{-1}PFF^{-1}(PF)'\bar{x} + (B - AP')F\bar{x} \\
&= \bar{A}\{(\bar{P}\bar{x})' - \bar{P}'\bar{x}\} + A(PF)'\bar{x} + (B - AP')F\bar{x} \\
&= \bar{A}\{(\bar{P}\bar{x})' - \bar{P}'\bar{x}\} + [BF + A\{(PF)' - P'F\}]\bar{x} \\
&= \bar{A}\bar{x}' + \bar{B}\bar{x}. \qquad \square
\end{aligned}$$

Remark. A transformation of variables $x = F(t)\bar{x}$ with nonsingular $F \in C^1_N$ results further in

- $\bar{X}(t) = F^{-1}(t)X(t)F(0)$
- $\bar{S}(t) := \{z \in \mathbb{R}^m : \bar{B}(t)z \in \text{im } \bar{A}(t)\} = F^{-1}(t)S(t)$
- $\bar{P}_{\text{can}}(t) = F^{-1}(t)P_{\text{can}}(t)F(t)$

Because of $\bar{N}(t) \cap \bar{S}(t) = F^{-1}(t)(N(t) \cap S(t))$ the transformed DAE (2.7) is index 1 tractable if and only if (2.1) is so.

A nonsingular scaling $E \in C$ has influence on the coefficients of the DAE only, but not on the solutions.
Clearly (2.10) suggests an equivalence relation of linear continuous coefficient DAEs. Since we are interested in asymptotics we apply the notion of kinematical equivalence of the standard ODE-theory to the DAEs considered here.

Definition. The DAEs (2.1) and (2.7) are said to be kinematically equivalent if there are nonsingular matrix functions $F \in C^1_N$, $E \in C$ with (2.8), and if $\sup_{t \in \mathbb{R}} |F(t)| < \infty$, $\sup_{t \in \mathbb{R}} |F(t)^{-1}| < \infty$.

Kinematic equivalence does not alter the stability relations.

In this paper, we put emphasis on the stability of solutions of nonlinear DAEs

$$f(x'(t), x(t), t) = 0. \tag{2.11}$$

The function $f : \mathcal{G} \times \mathbb{R} \to \mathbb{R}^m$, $\mathcal{G} \subseteq \mathbb{R}^m \times \mathbb{R}^m$, is assumed to be continuous and to have partial Jacobians $f'_y(y, x, t)$, $f'_x(y, x, t)$ depending continuously on their arguments. Furthermore, the nullspace of $f'_y(y, x, t)$ is supposed to be invariant of (y, x), i.e.

$$\ker f'_y(y, x, t) =: N(t),$$

and to vary smoothly with t. Again, $P(t)$ denotes any C^1 projector function along $N(t)$. Let (2.11) be index-1 tractable, i.e.

$$N(t) \cap S(y, x, t) = \{0\}, \quad (y, x) \in \mathcal{G},\ t \in \mathbb{R},$$

where $S(y, x, t) := \{z \in \mathbb{R}^m : f'_x(y, x, t)z \in \operatorname{im} f'_y(y, x, t)\}$.
Then, as in the linear index-1-case, IVPs are stated properly with the initial condition

$$P(0)(x(0) - x^0) = 0. \tag{2.12}$$

Again, the appropriate solution space is C^1_N.
Suppose that there is a solution $x_* \in C^1_N([0, \infty))$ of (2.11), (2.12), whose stability properties we are interested in.
Then it is well-known that for every finite interval $[0, \bar{t}]$ there is a neighbourhood U of $x_*(0)$ such that for all $x^0 \in U$ (2.11), (2.12) has a unique solution $x(t; x^0)$ defined on $[0, \bar{t}]$, and with $x(0, x^0)$ in some neighbourhood of $x_*(0)$. In this paper, we are interested in what happens in the semiinfinite interval $[0, \infty)$.
As in the linear case, only the part $P(0)x^0$ of the initial condition influences the solutions $x(t; x^0)$. This is reflected in the following definition of stability in the sense of Lyapunov for DAEs.

Definition. x_* is stable in the sense of Lyapunov if there is a $\tau > 0$ and, to each $\varepsilon > 0$, a $\delta = \delta(\varepsilon) > 0$ such that

(i) for all x^0 with $|P(0)(x_*(0) - x^0)| \le \tau$ (2.11), (2.12) has a solution $x(t, x^0)$ defined on $[0, \infty)$ and

(ii) for all x^0 with $|P(0)(x_*(0) - x^0)| \le \delta$ we have $|x(t, x^0) - x_*(t)| \le \varepsilon$ for all $t > 0$.

Furthermore, x_* is called asymptotically stable in the sense of Lyapunov if, additionally, there is a $\sigma \in (0, \tau)$ such that

(iii) $\lim_{t\to\infty} |x(t; x^0) - x_*(t)| = 0$ for all x^0 with $|P(0)(x_*(0) - x^0)| < \sigma$.

Note that Tischendorf [Ti94] gives results on asymptotic stability of stationary solutions of autonomous index 1 DAEs.

3 Linear DAEs with periodic coefficients

First, we recall some well-known facts from linear ODE theory.
Consider the periodic coefficient ODE

$$x'(t) - W(t)x(t) = 0 \tag{3.1}$$

with $W \in C(\mathbb{R}, L(\mathbb{R}^m))$, $W(t) = W(t+T)$ for all $t \in \mathbb{R}$, and its fundamental solution matrix $X(t)$ with

$$X'(t) - W(t)X(t) = 0, \quad X(0) = I$$

Theorem of Floquet [Fl1883] :

The fundamental matrix $X(t)$ can be written in the form

$$X(t) = F(t)e^{tW_0},$$

where $F \in C^1(\mathbb{R}, L(\mathbb{C}^m))$ nonsingular, $F(t) = F(t+T)$ for all $t \in \mathbb{R}$, $W_0 \in L(\mathbb{C}^m)$.

To prove this, one may choose W_0 such that $X(T) = e^{TW_0}$ and

$$F(t) := X(t)e^{-tW_0} \tag{3.2}$$

fulfils the requirements.
Then, the characteristic exponents of (3.1) are defined to be the eigenvalues of W_0, whereas the characteristic multipliers of (3.1) are the eigenvalues of the monodromy matrix $X(T)$. Clearly, the real part of a characteristic exponent is negative iff the modulus of the corresponding characteristic multiplier is less than one.

Theorem of Lyapunov [Ly1892] :

(i) *Let $F \in C^1(\mathbb{R}, L(\mathbb{C}^m))$ be nonsingular and T-periodic. Then $x = F(t)\bar{x}$ transforms* (3.1) *into a homogeneous linear ODE with T-periodic coefficient matrix, whose characteristic multipliers coincide with those of* (3.1).

(ii) *There exists a nonsingular T-periodic $F \in C^1(\mathbb{R}, L(\mathbb{C}^m))$ (a nonsingular $2T$-periodic $F \in C^1(\mathbb{R}, L(\mathbb{R}^m))$ with $F(0) = I$ such that $x = F(t)\bar{x}$ transforms* (3.1) *into a homogeneous linear system with constant (real constant) coefficients.*

Let us turn to linear homogeneous DAEs with periodic coefficients

$$A(t)x'(t) + B(t)x(t) = 0, \tag{3.3}$$

where $A, B \in C(\mathbb{R}, L(\mathbb{R}^m))$, $A(t) = A(t+T)$, $B(t) = B(t+T)$ for all $t \in \mathbb{R}$.
Do those facts like the Theorems of Floquet and Lyapunov, respectively, apply to DAEs also? But in what sense? We are going to answer these questions.
The main difference to the ODE-case is that the inherent dynamics occur only in a lower-dimensional subsystem. Therefore, it is essential for the following proofs to find out and use the structure of the DAE.

We make use of the natural splitting

$$\mathbb{R}^m = N(t) \oplus S(t)$$

for index-1-tractable DAEs.
Note that $N(t)$ and $S(t)$ are T-periodic since the coefficients $A(t)$ and $B(t)$ are so. $N(t)$ is supposed to be smooth. We span it by T-periodic C^1-functions

$$N(t) = \text{span}\{n_{r+1}(t), \ldots, n_m(t)\}, \quad r = \text{rank } A(t).$$

$S(t)$ may be only continuous. Let $S(t)$ be spanned by T-periodic C-functions

$$S(t) = \text{span}\{s_1(t), \ldots, s_r(t)\}.$$

In all what follows we choose a projector $P(t)$ along $N(t)$ in the special way that P is not only smooth but periodic.
Since P_{can} acts onto S along N, we have the representation $P_{\text{can}}(t) = V(t)\begin{pmatrix} I & \\ & 0 \end{pmatrix} V^{-1}(t)$, where $V(t) := [s_1(t), \ldots, s_r(t), n_{r+1}(t), \ldots, n_m(t)] \in L(\mathbb{R}^m)$.

As we have $X(t+T) = X(t)X(T)$ like in the ODE case we introduce $X(T)$ as the monodromy matrix for DAEs.

Aiming to construct a special transformation we choose for short the projector P such that $P(0) = P_{\text{can}}(0)$.
Applying (2.6) yields the fundamental matrix (cf. (2.5))

$$\begin{aligned} X(t) &= P_{\text{can}}(t)U(t)P(0) \\ &= P_{\text{can}}(t)U(t)P_{\text{can}}(0) \\ &= V(t)\begin{pmatrix} I & \\ & 0 \end{pmatrix} V^{-1}(t)U(t)V(0)\begin{pmatrix} I & \\ & 0 \end{pmatrix} V^{-1}(0) \\ &=: V(t)\begin{pmatrix} Z(t) & \\ & 0 \end{pmatrix} V^{-1}(0), \end{aligned} \tag{3.4}$$

where $Z \in C(\mathbb{R}, L(\mathbb{R}^r))$, $Z(0) = I$, and the monodromy matrix

$$X(T) = V(T)\begin{pmatrix} Z(T) & \\ & 0 \end{pmatrix} V^{-1}(0) = V(0)\begin{pmatrix} Z(T) & \\ & 0 \end{pmatrix} V^{-1}(0). \tag{3.5}$$

Since rank $X(t) = r$ is constant, $Z(t) \in L(\mathbb{R}^r)$ is nonsingular for all $t \in \mathbb{R}$. From linear algebra it is known that every nonsingular matrix $C \in L(\mathbb{R}^r)$ can be represented in the form

$$C = e^W \quad \text{with } W \in L(\mathbb{C}^r)$$

and

$$C^2 = e^{\bar{W}} \quad \text{with } \bar{W} \in L(\mathbb{R}^r),$$

where W and $\bar{W}$ are uniquely determined. Now, let

$$Z(T) = e^{TW_0}, \quad W_0 \in L(\mathbb{C}^r) \tag{3.6}$$

and

$$Z(2T) = Z(T)^2 = e^{2TW_0}, \quad W_0 \in L(\mathbb{R}^r), \tag{3.6'}$$

respectively. Here we concluded $Z(2T) = Z(T)^2$ from the corresponding property of X and the relation $V(2T) = V(T) = V(0)$.
Modifying the transformation of variables (3.2) in the Theorem of Floquet for ODEs we now set

$$\begin{aligned} F_K(t) &:= V(t)\begin{pmatrix} Z(t)e^{-tW_0} & \\ & I \end{pmatrix} & (3.7)\\ &= X(t)V(0)\begin{pmatrix} e^{-tW_0} & \\ & 0 \end{pmatrix} + V(t)\begin{pmatrix} 0 & \\ & I \end{pmatrix}. & (3.8)\end{aligned}$$

From (3.7) we see that this transformation is nonsingular.
If we treat the ODE (3.1) as a special case of the DAE (3.3), we may choose $V(t) = I$ and then (3.7) coincides with (3.2). We remark that the transformation (3.7) may not be smooth since the same is true for $S(t)$ and, hence, $V(t)$ and $X(t)$, too.
Now, it holds:

Theorem 3.1 *The fundamental matrix $X(t)$ of the DAE (3.3) can be written in the form*

$$X(t) = F(t)\begin{pmatrix} e^{tW_0} & \\ & 0 \end{pmatrix} F(0)^{-1}$$

where $F \in C^1_N(\mathbb{R}, L(\mathbb{C}^m))$ is nonsingular and T-periodic.

Proof. We will show that $F = F_K$ defined in (3.7) or, equivalently, in (3.8) fulfils the assertion.
1) F is T-periodic since

$$\begin{aligned} F(t+T) &= X(t+T)V(0)\begin{pmatrix} e^{-(t+T)W_0} & \\ & I \end{pmatrix} + V(t+T)\begin{pmatrix} 0 & \\ & I \end{pmatrix}\\ &= X(t)X(T)V(0)\begin{pmatrix} e^{-(t+T)W_0} & \\ & I \end{pmatrix} + V(t)\begin{pmatrix} 0 & \\ & I \end{pmatrix}\end{aligned}$$

and using the representation of the monodromy matrix (3.5) we obtain

$$\begin{aligned} F(t+T) &= X(t)V(0)\begin{pmatrix} Z(T) & \\ & 0 \end{pmatrix}\begin{pmatrix} e^{-(t+T)W_0} & \\ & I \end{pmatrix} + V(t)\begin{pmatrix} 0 & \\ & I \end{pmatrix}\\ &= X(t)V(0)\begin{pmatrix} e^{-tW_0} & \\ & I \end{pmatrix} + V(t)\begin{pmatrix} 0 & \\ & I \end{pmatrix}\\ &= F(t) \qquad \text{for all } \ t \in \mathbb{R}.\end{aligned}$$

2) $F \in C^1_N(\mathbb{R}, L(\mathbb{C}^m))$ since

$$(PF)(t) = P(t)X(t)V(0)\begin{pmatrix} e^{-tW_0} & \\ & I \end{pmatrix} + P(t)V(t)\begin{pmatrix} 0 & \\ & I \end{pmatrix}$$

and $P(t)X(t)$ is smooth and

$$P(t)V(t)\begin{pmatrix} 0 & \\ & I \end{pmatrix} = P(t)[0,\ldots,0,n_{r+1}(t),\ldots,n_m(t)] = 0. \qquad \square$$

Remark. From (3.4) we see once more that $\ker X(t) = N(0)$ for all $t \in \mathbb{R}$. The monodromy matrix $X(T)$ has $m-r$ zero eigenvalues with $N(0)$ as eigenspace.
The r nonzero eigenvalues of $X(T)$ have eigenvectors and possibly generalized eigenvectors belonging to $S(0)$.

The nonzero eigenvalues of the monodromy matrix $X(T)$ are said to be the characteristic multipliers of (3.3) and the eigenvalues of $W_0 \in L(\mathbb{C}^r)$ to be the characteristic exponents of (3.3).
As in the ODE-case we have the relation

$$\lambda = e^{T\mu}$$

between a characteristic multiplier λ and a corresponding characteristic exponent μ.

Next, we want to introduce the notion of (periodical) equivalence of two linear DAEs with T-periodic coefficients.

Definition. Two linear, homogeneous, T-periodic DAEs are said to be (periodically) equivalent iff the relation

$$\bar{A} = EAF \quad \text{and} \quad \bar{B} = E(BF + AF'), \tag{3.9}$$

where $F \in C^1_N$, $E \in C$ are T-periodic and nonsingular matrix functions, is true for their coefficients.
Periodic equivalence means kinematic equivalence by periodic transformations.

The following assertion generalizes the result of Lyapunov mentioned above.

Theorem 3.2

(i) *If two linear homogeneous T-periodic DAEs are (periodically) equivalent then their monodromy matrices are similar and, hence, their characteristic multipliers coincide.*

(ii) *If the monodromy matrices of two linear T-periodic DAEs are similar then the DAEs are (periodically) equivalent.*

(iii) *The DAE (3.3) is (periodically) equivalent to a T-periodic complex ($2T$-periodic real) linear system in Kronecker normal form with constant coefficients.*

Proof.
(i) follows immediately from $\bar{X}(T) = F^{-1}(T)X(T)F(0) = F^{-1}(0)X(T)F(0)$.

(iii) First, we apply the transformation of variables (3.5). In the proof of Theorem 3.1 we have already shown that F defined in (3.5) is nonsingular, T-periodic and belongs to

$C^1_N(\mathbb{R}, L(\mathbb{C}^m))$.
Then, for $\hat{A} := AF$, $\hat{B} := BF + AF'$ it holds

$$\hat{N}(t) =: \ker \hat{A}(t) = F^{-1}(t)N(t) = \begin{pmatrix} e^{tW_0}Z(t)^{-1} & \\ & I \end{pmatrix} V(t)^{-1}N(t)$$

and since $V^{-1}(t)n_k(t) = e_k$ for $k = r+1, \dots, m$, it follows that $\hat{N}(t) = \operatorname{span}\{e_{r+1}, \dots, e_m\}$.
Similarly, we have

$$\hat{S}(t) = F^{-1}(t)S(t) = \begin{pmatrix} e^{tW_0}Z(t)^{-1} & \\ & I \end{pmatrix} V(t)^{-1}S(t) = \mathbb{R}^r \times \{0\}^{m-r}$$

since $V^{-1}(t)s_k(t) = e_k$ for $k = 1, \dots, r$, Thus, the canonical projector $\hat{P}_{\mathrm{can}}$ onto $\hat{S}$ along $\hat{N}$ is

$$\hat{P}_{\mathrm{can}}(t) = \begin{pmatrix} I & \\ & 0 \end{pmatrix},$$

which is even orthogonal.
The next step is to find a suitable scaling E. We take

$$E := \hat{G}^{-1}, \tag{3.10}$$

where

$$\hat{G} := (\hat{A} + \hat{B}\hat{Q}_{\mathrm{can}}); \quad \hat{Q}_{\mathrm{can}} = I - \hat{P}_{\mathrm{can}} = \begin{pmatrix} 0 & \\ & I \end{pmatrix}.$$

Clearly, $\hat{G}^{-1}$ is T-periodic but nonsingular, due to the index-1-tractability. Applying (3.10) yields

$$\begin{aligned} \bar{A} &:= E\hat{A} = \hat{G}^{-1}\hat{A} = \hat{P}_{\mathrm{can}} = \begin{pmatrix} I & \\ & 0 \end{pmatrix}, \\ \bar{B} &:= E\hat{B} = \hat{G}^{-1}\hat{B} = \begin{pmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & \bar{B}_{22} \end{pmatrix}. \end{aligned}$$

Using the identities $\hat{P}_{\mathrm{can}}\hat{G}^{-1}\hat{B}\hat{Q}_{\mathrm{can}} = 0$ and $\hat{Q}_{\mathrm{can}}\hat{G}^{-1}\hat{B} = \hat{Q}_{\mathrm{can}}$ we obtain

$$\begin{pmatrix} I & \\ & 0 \end{pmatrix} \begin{pmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & \bar{B}_{22} \end{pmatrix} \begin{pmatrix} 0 & \\ & I \end{pmatrix} = 0, \quad \text{which implies } \bar{B}_{12} = 0$$

and

$$\begin{pmatrix} 0 & \\ & I \end{pmatrix} \begin{pmatrix} \bar{B}_{11} & 0 \\ \bar{B}_{21} & \bar{B}_{22} \end{pmatrix} = \begin{pmatrix} 0 & \\ & I \end{pmatrix}, \quad \text{which implies } \bar{B}_{21} = 0,\ \bar{B}_{22} = I.$$

Hence, $\bar{B} = \begin{pmatrix} \bar{B}_{11} & \\ & I \end{pmatrix}$.
Now, looking at the fundamental matrix $\bar{X}(t) = \hat{X}(t) = \begin{pmatrix} e^{tW_0} & \\ & 0 \end{pmatrix}$ we can conclude that $\bar{B}_{11} = -W_0$, which completes the proof of (iii).

(ii) Let us assume that, using E, F, (3.3) is transformed into Kronecker normal form

$$\begin{pmatrix} I & \\ & 0 \end{pmatrix} x' + \begin{pmatrix} -W_0 & \\ & I \end{pmatrix} x = 0$$

and, using $\bar{E}$, $\bar{F}$, the second DAE

$$\bar{A}(t)\bar{x}'(t) + \bar{B}(t)\bar{x}(t) = 0 \tag{3.11}$$

is transformed into Kronecker normal form

$$\begin{pmatrix} I & \\ & 0 \end{pmatrix} \bar{x}' + \begin{pmatrix} -\bar{W}_0 & \\ & I \end{pmatrix} \bar{x} = 0.$$

Since $X(T)$ and $\bar{X}(T)$ are similar, W_0 and $\bar{W}_0$ have the same size and are similar too. This reduces the assertion to the periodical equivalence of explicit linear ODEs with similar coefficient matrices, which is fulfilled, indeed.
Let D denote the similarity transformation $W_0 = D^{-1}\bar{W}_0 D$. With $\mathcal{D} = \text{diag}\,(D, I)$, by straightforward computations we check $\tilde{E} := \bar{E}^{-1}\mathcal{D}E$, $\tilde{F} := F\mathcal{D}^{-1}\bar{F}^{-1}$ to satisfy the relations needed, i.e.

$$\bar{A} = \tilde{E}A\tilde{F}, \quad \bar{B} = \tilde{E}B\tilde{F} + \tilde{E}A\tilde{F}'. \qquad \square$$

Remarks.
(i) Dealing with Theorem 3.2(ii) we may also write down a transformation of variables F that transforms $X(t)$ to $\bar{X}(t)$.
With the corresponding notations for (3.11) we have

$$\begin{aligned} X(T) &= V(0)\begin{pmatrix} Z(T) & \\ & 0 \end{pmatrix} V^{-1}(0), \\ \bar{X}(T) &= \bar{V}(0)\begin{pmatrix} \bar{Z}(T) & \\ & 0 \end{pmatrix} \bar{V}^{-1}(0), \end{aligned}$$

where $Z(T)$ and $\bar{Z}(T)$ are similar.
Let $\bar{Z}(T) = D^{-1}Z(T)D$ with $D \in L(\mathbb{C}^r)$ nonsingular.
Then $F(t) := V(t)\begin{pmatrix} Z(t)D\bar{Z}(t)^{-1} & \\ & I \end{pmatrix} \bar{V}(t)^{-1}$ fulfils the requirements.
First, we note that F is nonsingular.
Second, F is T-periodic, since V and $\bar{V}$ are T-periodic and

$$\begin{aligned} Z(t+T)D\bar{Z}(t+T)^{-1} &= Z(t)Z(T)D\bar{Z}(t+T)^{-1} \\ &= Z(t)D\bar{Z}(T)\bar{Z}(t+T)^{-1} = Z(t)D\bar{Z}(t)^{-1}. \end{aligned}$$

Third, we have

$$\begin{aligned} &F(t)^{-1}X(t)F(0) \\ &= \bar{V}(t)\begin{pmatrix} \bar{Z}(t)D^{-1}Z(t)^{-1} & \\ & I \end{pmatrix} V(t)^{-1}V(t)\begin{pmatrix} Z(t) & \\ & 0 \end{pmatrix} V(0)^{-1}V(0)\begin{pmatrix} D & \\ & I \end{pmatrix} \bar{V}(0)^{-1} \\ &= \bar{V}(t)\begin{pmatrix} \bar{Z}(t) & \\ & 0 \end{pmatrix} \bar{V}(0)^{-1} = \bar{X}(t). \end{aligned}$$

(ii) The scaling (3.10) which we have used in the proof is a natural generalization from the ODE-case, too.
Transforming (3.1) by a nonsingular C^1-transformation F we obtain

$$F\bar{x}' + (-WF + F')\bar{x} = 0.$$

Scaling with $E := F^{-1}$ leads to a new explicit ODE. Treating (3.1) as a special case of (3.3) we obtain $E = F^{-1}$ since $\hat{G} = \hat{A} = F$ then.
(iii) With the tools used above it is just as well possible to transform every linear index-1-tractable DAE into its Kronecker normal form.
To do so, simply use $F(t) := V(t)$ and $E(t) := \hat{G}(t)^{-1}$.

4 An auxiliary result

This section deals with nonlinear DAEs of the special form

$$Ax'(t) + Bx(t) + h(x'(t), x(t), t) = 0, \quad t \in J := [t_0, \infty), \tag{4.1}$$

where the linear part has constant coefficients $A, B \in L(\mathbb{R}^m)$ but the nonlinear one is small. More precisely, we assume $h : \mathcal{G} \times J \to \mathbb{R}^m$ to be continuous and to have the partial Jacobians $h'_y(y, x, t)$, $h'_x(y, x, t)$ continuously depending on their arguments. $\mathcal{G} \subseteq \mathbb{R}^m \times \mathbb{R}^m$ is open. Furthermore, let

$$N := \ker A \subseteq \ker h'_y(y, x, t), \quad (y, x, t) \in \mathcal{G} \times J, \tag{4.2}$$

such that with any projector $P \in L(\mathbb{R}^m)$ along N the identity

$$h(y, x, t) \equiv h(Py, x, t)$$

becomes true. In the consequence, the nonlinear part of (4.1) contains at most components of the derivative of $x(t)$ which do appear in the linear part.

Lemma 4.1 *Let $0 \in \mathcal{G}$, and for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $(y, x, t) \in \mathcal{G} \times J$, $|Py| + |x| \leq \delta(\varepsilon)$ imply*

$$|h(y, x, t)| \leq \varepsilon(|Py| + |x|) \tag{4.3}$$
$$|h'_x(y, x, t)| \leq \varepsilon, \quad |h'_y(y, x, t)| \leq \varepsilon. \tag{4.4}$$

Let the matrix pair $\{A, B\}$ be regular with index 1, and let all its finite eigenvalues belong to $\mathbb{C}^-$, i.e.

$$\det(\lambda A + B) = 0 \quad \textit{implies} \quad \lambda \in \mathbb{C}^-.$$

Then, the identically vanishing function $x_*(t) \equiv 0$ is an asymptotically stable solution of the DAE (4.1).

Proof. Obviously, $h(0,0,t) \equiv 0$ holds true so that $x_*(.)$ solves the DAE (4.1). Further, we have $h'_x(0,0,t) = 0$ and $h'_y(0,0,t) = 0$. Without loss of generality we choose P to represent the canonical projector onto $S := \{z \in I\!R^m : Bz \in \text{im } A\}$ along N. Denote $Q := I - P$, $G := A + BQ$ and recall the properties $G^{-1}A = P$, $G^{-1}BQ = Q$, $Q = QG^{-1}B$. Hence, with $u := Px$, $v := Qx$, the DAE (4.1) decouples into the system

$$u'(t) + PG^{-1}Bu(t) + PG^{-1}h(u'(t), u(t) + v(t), t) = 0, \tag{4.5}$$

$$v(t) + QG^{-1}h(u'(t), u(t) + v(t), t) = 0, \tag{4.6}$$

$$u(t_0) \in \text{im } P. \tag{4.7}$$

If the pair $u(.) \in C^1$, $v(.) \in C$ solves (4.5) – (4.7) on some interval $[t_0, T)$, then $v(t) \equiv Qv(t)$, $u(t) \equiv Pu(t)$, and $x(.) := u(.) + v(.)$ belongs to C^1_N and solves (4.1). Obviously, system (4.5) – (4.7) has the trivial solution $u_*(t) \equiv 0$, $v_*(t) \equiv 0$.
Now, equation (4.6) suggest to consider the equation for $v \in I\!R^m$

$$v = -QG^{-1}h(u', u + v, t),$$

where now $u', u \in I\!R^m$. Using standard arguments we find a function

$$\psi : \bar{B}(0, \varrho_{u'}) \times \bar{B}(0, \varrho_u) \times J \longrightarrow \bar{B}(0, \varrho_v)$$

with the following properties:

1) $\psi(u', u, t) = QG^{-1}h(u', u + \psi(u', u, t), t)$ for all $|u| \leq \varrho_u$, $|u'| \leq \varrho_{u'}$, $t \in J$,

2) $\psi(0, 0, t) = 0$,

3) $\psi(u', u, t) = Q\psi(u', u, t)$,

4) ψ is continuous together with its partial derivatives $\psi'_{u'}$, ψ'_u,

5) $\psi'_u(0, 0, t) = 0$, $\psi'_{u'}(0, 0, t) = 0$,

6) to each $\varepsilon > 0$ there is a $\sigma_\psi(\varepsilon) > 0$ such that $|u'| + |u| \leq \sigma_\psi(\varepsilon)$ implies $|\psi(u', u, t)| \leq \varepsilon(|u'| + |u|)$ uniformly for $t \in J$.

Next, rewrite (4.5), (4.6) to

$$u'(t) + PG^{-1}Bu(t) + PG^{-1}h(u'(t), u(t) + \psi(u'(t), u(t), t), t) = 0, \tag{4.8}$$

$$v(t) = \psi(u'(t), u(t), t).$$

Again we use standard arguments to transform (4.8) into an explicit ODE

$$u'(t) = g(u(t), t). \tag{4.9}$$

The function $g : \bar{B}(0, \varrho_u) \times J \to \bar{B}(0, \varrho_{u'})$ has the following properties:

1) $g(u, t) + PG^{-1}Bu + PG^{-1}h(g(u, t), u + \psi(g(u, t), u, t), t) = 0$ for $u \in B(0, \varrho_u)$, $t \in J$,

2) $g(0, t) = 0$,

3) $g(u,t) = Pg(u,t)$,

4) g is continuous together with its partial derivative g'_u,

5) $g'_u(0,t) = -PG^{-1}B$,

6) to each $\varepsilon > 0$ there is a $\sigma_g(\varepsilon) > 0$ such that $|u| \leq \sigma_g(\varepsilon)$ implies $|g(u,t) + PG^{-1}Bu| \leq \varepsilon|u|$ uniformly for all $t \in J$.

Introduce $\tilde{g}(u,t) := g(u,t) + PG^{-1}Bu$ and reformulate (4.9) to

$$u'(t) = -PG^{-1}Bu(t) + \tilde{g}(u(t),t). \tag{4.10}$$

The matrix $M := -PG^{-1}B = -PG^{-1}BP$ has zero as eigenvalue with multiplicity $m-r$, $r = \operatorname{rank} A$ and N is the respective $m-r$ dimensional eigenspace. The other eigenvalues are exactly the r finite eigenvalues of the pair $\{A, B\}$. Due to our assumption they belong to $\mathbb{C}^-$. Applying [GHM92, p. 57], we find a scalar product resp. $C \in L(\mathbb{C}^m)$, $\langle z_1, z_2\rangle_C := \langle C^{-1}z_1, C^{-1}z_2\rangle_2$ such that

$$\mathrm{Re}\langle Mz, z\rangle_C \leq -\beta|z|_C^2, \quad z \in \operatorname{im} P = S$$

holds true with a constant $\beta > 0$. Naturally, $W(u) := |u|_C^2$ represents the Lyapunov function for the linear system $u' = Mu$ on the invariant subspace $\operatorname{im} P = S$.
With sufficiently small $u_0 \in \operatorname{im} P$, the initial value problem (4.10), $u(t_0) = u_0$ has an solution; say on $[t_0, T)$. For $t \in [t_0, T)$ we have now

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}W(u(t)) &= 2\mathrm{Re}\langle C^{-1}u'(t), C^{-1}u(t)\rangle_2 \\ &\leq -2\beta|u(t)|_C^2 + \gamma|\tilde{g}(u(t),t)|_2|u(t)|_C \\ &\leq (-2\beta + \gamma\varepsilon)|u(t)|_C^2 \end{aligned}$$

where γ is a constant fixed by C. Choosing $\varepsilon_* > 0$ small enough we obtain

$$-2\beta + \gamma\varepsilon_* \leq -2\beta_0 \quad \text{with} \quad \beta_0 > 0.$$

Consequently

$$W(u(t)) \leq e^{-2\beta_0(t-t_0)}W(u(t_0)).$$

Therefore, the solution $u(.)$ may be continued onto $[t_0, \infty)$.
It comes out, that there is a $\tau \in \sigma_g(\varepsilon_*)$ so that the initial value problem

$$(4.1), \quad Px(t_0) = Px^0 =: u_0, \quad |u_0| \leq \tau$$

has a solution defined on $[t_0, \infty)$, namely

$$x(t) = u(t) + \psi(g(u(t),t), u(t), t), \quad t \in [t_0, \infty).$$

Moreover, there are constants K_1, K_2 such that

$$\begin{aligned} |x(t)| &\leq K_1|u(t)|_C \leq e^{-\beta_0(t-t_0)}K_1|Px^0|_C \\ &\leq e^{-\beta_0(t-t_0)}K_2|Px^0|. \end{aligned} \qquad \square$$

Remark. Naturally our proof follows the lines of Tischendorf [Ti94], where a respective criterion on asymptotical stability of a stationary solution of an autonomeous DAE with a C^2 function is considered.

5 Nonlinear periodic DAEs

Now, we consider the nonlinear case

$$f(x'(t), x(t), t) = 0, \tag{5.1}$$

where $f : \mathcal{G} \times I\!R \to I\!R^m$, $\mathcal{G} \subseteq I\!R^m \times I\!R^m$ open and connected, and $f(y, x, t) = f(y, x, t+T)$ for all $(x, y) \in \mathcal{G}$, $t \in I\!R$. We suppose that f and the partial derivatives f'_y, f'_x, f''_{yy}, f''_{xx}, f''_{yx} exist and are continuous on $\mathcal{G} \times I\!R$. Further, let $\ker f'_y(y, x, t) =: N(t)$ be smooth, $P(t)$ be a smooth and periodic projector along $N(t)$, and suppose that (5.1) is tractable with index 1.

Now, let $x_* \in C^1_N$ be the T-periodic solution of (5.1), whose stability properties are to be considered. We want to present a theorem like the well-known theorem of Lyapunov for ODEs to guarantee that this periodic solution is stable under certain conditions.
For this reason we consider the homogeneous linearized equation

$$\left.\begin{array}{rcl} A(t)X'(t) + B(t)X(t) & = & 0 \\ P(0)(X(0) - I) & = & 0 \end{array}\right\} \tag{5.2}$$

with

$$A(t) := f'_y(x'_*(t), x_*(t), t), \qquad B(t) := f'_x(x'_*(t), x_*(t), t) \tag{5.3}$$

and the monodromy matrix $X(T)$.
Again, we emphasize that, naturally, the coefficients A and B are expected to be continuous only. To guarantee smoother A and B we would need not only a smoother function f, but also a smoother solution x_*.

Theorem 5.1 *Let the monodromy matrix $X(T)$ have all its eigenvalues in $\{z \in \mathbb{C} : |z| < 1\}$, then the periodic solution x_* is asymptotically stable in the sense of Lyapunov.*

Proof. First, we use Theorem 3.2(iii) to find appropriate $F \in C^1_N(I\!R, L(\mathbb{C}^m))$ and $E \in C(I\!R, L(\mathbb{C}^m))$, that are both T-periodic and nonsingular and transform the linear DAE

$$A(t)x'(t) + B(t)x(t) = 0 \tag{5.4}$$

into Kronecker normal form with constant coefficients

$$\begin{pmatrix} I & \\ & 0 \end{pmatrix} \bar{x}'(t) + \begin{pmatrix} -W_0 & \\ & I \end{pmatrix} \bar{x}(t) = 0.$$

In the next step we apply the same matrix functions F and E to the nonlinear equation. Before doing so we shift the variables and split the linear terms.
Let

$$f(x'_*(t) + y, x_*(t) + x, t) = A(t)y + B(t)x + h(y, x, t), \tag{5.5}$$

which defines a function h for (x,y) in some neighbourhood of 0 and $t \in I\!R$. h is as smooth as f and it holds

$$h(0,0,t)=0, \quad h'_y(0,0,t)=0, \quad h'_x(0,0,t)=0 \quad \text{for all } t$$

and

$$|h(y,x,t)| \le C(|x|+|y|)^2. \tag{5.6}$$

Further, we have $h(y,x,t)=h(P(t)y,x,t)$ and again, for $x \in C^1_N$, we use $h(x'(t),x(t),t)$ as an abbreviation for $h((Px)'(t)-P'(t)x(t),x(t),t)$.
Now, we look for solutions of

$$A(t)x'(t)+B(t)x(t)+h(x'(t),x(t),t)=0. \tag{5.7}$$

Scaling by $E(t)$ and transforming $x(t)=F(t)\bar{x}(t)$ we obtain

$$\begin{pmatrix} I & \\ & 0 \end{pmatrix} \bar{x}'(t) + \begin{pmatrix} -W_0 & \\ & I \end{pmatrix} \bar{x}(t) + \bar{h}(\bar{x}'(t),\bar{x}(t),t)=0, \tag{5.8}$$

where

$$\bar{h}(\bar{y},\bar{x},t)=E(t)h(P(t)F(t)\bar{y}-P(t)F'(t)\bar{x},F(t)\bar{x},t).$$

The relation

$$\bar{h}'_{\bar{y}}(\bar{y},\bar{x},t)=E(t)h'_y(P(t)F(t)\bar{y}-P(t)F'(t)\bar{x},F(t)\bar{x},t)P(t)F(t)$$

shows that $\bar{N} := \ker \bar{A} \subset \bar{h}'_{\bar{y}}(\bar{y},\bar{x},t)$ is satisfied since $\ker P(t)F(t) = \ker A(t)F(t) = \ker \bar{A} = \bar{N}$. Hence, relation (4.2) is fulfilled.
Furthermore, (5.6) implies

$$|\bar{h}(\bar{y},\bar{x},t)| \le \bar{C}(|\bar{P}\bar{y}|+|\bar{x}|)^2.$$

Hence, Lemma 4.1 applies to the DAE (5.8) and $x(t)=F(t)\bar{x}(t)$ solves (5.7). □

Finally, let us illustrate by means of a small example.

Example 1 *Consider the DAE*

$$\left.\begin{array}{l} x_1' + x_1 - x_2 - x_1x_3 + (x_3-1)\sin t = 0 \\ x_2' + x_1 + x_2 - x_2x_3 + (x_3-1)\cos t = 0 \\ x_1^2 + x_2^2 + x_3 - 1 = 0 \end{array}\right\}, \tag{5.9}$$

which has the 2π*-periodic solution*

$$x_*(t) = (\sin t, \cos t, 0)^T. \tag{5.10}$$

Due to $N \cap S(x) = \{0\}$ *for all* $x \in I\!R^3$

$N := \{z \in I\!R^3 : z_1 = 0,\ z_2 = 0\}$,

$S(x) := \{z \in I\!R^3 : 2x_1z_1 + 2x_2z_2 + z_3 = 0\}$,

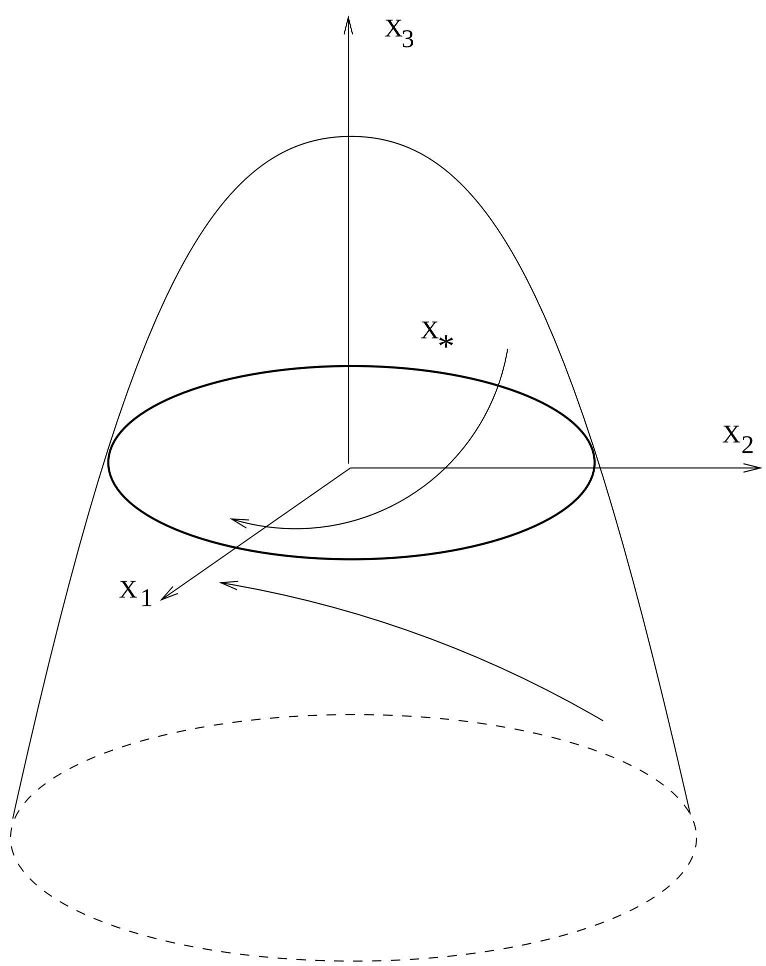
the DAE has globally index 1.
The linearized along $x_*(.)$ DAE has the coefficients

$$A_*(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_*(t) = \begin{pmatrix} 1 & -1 & -\sin t \\ 1 & 1 & -\cos t \\ -2\sin t & -2\cos t & -1 \end{pmatrix}$$

and the monodromy matrix reads

$$X(2\pi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{pmatrix} \exp\Big\{ -2\pi \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Big\},$$

it has the eigenvalues $\lambda_{1,2} = e^{-2\pi(2\pm i)} = e^{-4\pi}$.
Hence, the solution (5.10) is asymptotically stable in the sense of Lyapunov.



6 Conclusions

No daubt, our illustrative example (5.9) is extremely simple. In particular, in case we have application problems, the situation is by far not so clear. The state manifold - in our

case the paraboloid - is just not known and can move in time. Furthermore, the solution $x_*(.)$ to be analyzed can only be otained numerically.
Well, our calculus allows to proceed for DAEs with index 1 exactly as for regular ODEs (e.g. [Se94]), but the resulting numerical methods will inherit all difficulties typical of the regular ODEs.
Now the question arises whether our calculus can also be applied to DAEs with higher index. As the problem of linearization, which is the basis for the Floquet-theory, has been sufficiently solved for DAEs with index 2 in the meantime ([Ma95], [Ti96]), we are optimistical concerning this case, which, however, is considerably more complicated in its details.

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