EQUILIBRIUM IN ABSTRACT ECONOMIES WITHOUT THE LOWER SEMI-CONTINUITY OF THE CONSTRAINT MAPS.

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Abstract. We use graph convergence of set valued maps to show the existence of an equilibrium for an abstract economy without assuming the lower semi continuity of the constraint maps.

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Introduction

In 1975 Shafer and Sonnenschein [1975] proved the existence of equilibria for abstract economies without ordered preferences. Over the last twenty years, more general existence results appeared in the literature. To make a partial list of these results we mention Borglin and Keiding [1976], Yannelis and Prabhakar [1983], Toussaint [1983], Yannelis [1987], Tarafdar [1989], and Tan and Yuan [1994]. All of these results however, assume directly or indirectly the lower semi-continuity of the set valued map representing the constraints for each agent.

In this paper, we will use the concept of graphical convergence of set-valued maps to show that an equilibrium for an abstract economy exists without such an assumption. Eliminating the requirement of the lower semi-continuity of the constraint map is not just technical improvement. It will actually make the setting of the abstract economy more realistic as we will demonstrate at the end of the paper.

We start by reviewing the mathematical and the economical concepts that we need. Then, we prove several results regarding the existence of equilibria for abstract economies without assuming the lower semi-continuity of the of the constraint map. Finally, we discuss some of the economical implications of the new results.

Preliminaries

In this section, we review some basic definitions and theorems regarding set convergence and set valued maps.

Set convergence

Let X be a topological vector space. Let 2^X be the collection of all subsets of X. For a subset K in 2^X we use the following notation:

$\operatorname{cl} K$ denotes the closure of K in X.

$\operatorname{int} K$ denotes the interior of K in X.

$\operatorname{con} K$ denotes the convex hull of K.

Let $C(X)$ be the collection of closed nonempty subsets of X. We are interested in set convergence in $C(X)$. Recall that a set D is directed by a relation $\geq$, if $\geq$ is reflexive and transitive and for all ν^1 and ν^2 in D, $\exists\, \nu^3 \in D$ such that $\nu^3 \geq \nu^1$ and $\nu^3 \geq \nu^2$. We define the collection of *cofinal subsets*, $\mathcal{N}^\infty(D)$, and the collection of *residual subsets* , $\mathcal{N}^\#(D)$, of a directed set D:

$$\mathcal{N}^\infty(D) = \{N \subset D | N \text{ contains all the elements of } D \text{ at or beyond some index } \nu \in D\},$$

$$\mathcal{N}^\#(D) = \{N \subset D | N \text{ contains some elements of } D \text{ at or beyond each index } \nu \in D\}.$$

Let $\{C_\nu | \nu \in D\}$ be a net in $C(X)$. We define the *Inner limit*, Li, and the *Upper limit*, Ls:

$$\operatorname{Li} C_\nu = \{x \in X | \exists N' \in \mathcal{N}^\infty(D) \text{ such that } x_{\nu'} \in C_{\nu'}, \forall \nu' \in N, \text{ and } x_{\nu'} \longrightarrow x\},$$

$$\operatorname{Ls} C_\nu = \{x \in X | \exists N' \in \mathcal{N}^{\#}(D) \text{ such that } x_{\nu'} \in C_{\nu'}, \forall \nu' \in N', \text{ and } x_{\nu'} \longrightarrow x\},$$

where in both definitions $x_{\nu'} \longrightarrow x$ indicates the convergence of $\{x_{\nu'}\}$ to x in X. We say that C_ν converges to C in $C(X)$, and we write $C_\nu \longrightarrow C$, when $\operatorname{Li} C_\nu = \operatorname{Ls} C_\nu = C$. This notion of set convergence is referred to in the literature as the Painlevé-Kuratowski set convergence. In general the Painlevé-Kuratowski set convergence is topological if and only if X is locally compact (cf. Klein and Thompson [1984]).
The following is a well known lemma

Lemma 1.1. *Let $\{C_\nu | \nu \in D\}$ be a net in $C(X)$ that is decreasing with respect to inclusion, $\nu^2 \geq \nu^1$ in D implies $C_{\nu^2} \subset C_{\nu^1}$. Then,*

$$\operatorname{Ls} C_\nu \subset \bigcap_{\nu \in D} C_\nu$$

Proof. Let $x \in \operatorname{Ls} C_\nu$, then $\exists N' \in \mathcal{N}^{\#}(D)$ such that $\forall \nu' \in N'$, $x_{\nu'} \in C_{\nu'}$ and $x_{\nu'} \longrightarrow x$. C_ν are decreasing with respect to inclusion, hence for every $\nu_0 \in D$ we have

$$x_{\nu'} \in C_{\nu_0}, \forall \nu' \in N' \text{ such that } \nu' \geq \nu_0.$$

However, C_{ν_0} is closed and hence $x \in C_{\nu_0}$. Since this is true for any $\nu_0 \in D$, we get

$$x \in \bigcap_{\nu \in D} C_\nu.$$

□

For more on set convergence, see Beer[1993] and Klein and Thompson[1984].

<u>Set valued maps</u>

Let X and Y be two topological vector spaces. We write $F : X \rightrightarrows Y$ to indicate a set valued map from X to subsets of Y. We say F is *convex-valued*, *closed-valued* or *open-valued*, if $\forall x \in X$, $F(x)$ is convex, closed, or open respectively. We define the *domain* of F, $\operatorname{dom} F$, to be

$$\operatorname{dom} F = \{x \in X \,| F(x) \neq \emptyset\}.$$

If $\operatorname{dom} F$ is all of X, then we say F is *nonempty-valued*. Let U be a subset in X. $F|_U$ denotes the restriction of F to U. The *graph* of F, $\operatorname{gph} F$, is a subset of $X \times Y$ defined as follows

$$\operatorname{gph} F = \{(x, y) \in X \times Y \,| y \in F(x)\}.$$

We say F is *open* (resp. *closed*), if the graph of F is open (resp. closed) in $X \times Y$. Recall that $F : X \rightrightarrows Y$ is *lower semi-continuous*(lsc.), if for any open set G in Y the set

$\{x \in X | F(x) \cap G \neq \emptyset\}$ is open in X. F is *upper semi-continuous*(usc.), if for any open set G in Y the set $\{x \in X | F(x) \subset G\}$ is open in X. Furthermore, we say F has *open lower sections*(ols.), if for any $y \in Y$, the set $\{x \in X | y \in F(x)\}$ is open in X. Similarly, we say F has upper open section(uos.), if for all $x \in X$, the set $F(x)$ is open in Y.
We write $\mathrm{cl}\, F$ to indicate the map defined by $\mathrm{cl}\, F(x) = \mathrm{cl}(F(x))$, $\forall x \in X$. Similarly, we write $\mathrm{con}\, F$ to indicate the map defined by $\mathrm{con}\, F(x) = \mathrm{con}(F(x))$, $\forall x \in X$. Let F and G be two set valued maps from X to subsets in Y, then the map $F \cap G$ is defined by $F \cap G(x) = F(x) \cap G(x)$, $\forall x \in X$.

We will need the following facts about maps from X into subsets of Y,

Fact 1. F is open implies that F is ols., which in turn implies that F is lsc.

Fact 2. If F is lsc., then $\mathrm{dom}\, F$ is open in X.

Fact 3. If Y is compact, then F is closed valued and usc. if and only if F is closed. In particular, if F is such that $\mathrm{cl}\, F$ is usc., then $\mathrm{cl}\,\mathrm{gph}\, F = \mathrm{gph}\,\mathrm{cl}\, F$.

Fact 4. If F is lsc., then so are $\mathrm{cl}\, F$ and $\mathrm{con}\, F$.

Fact 5. If F is open and G is ols., then $F \cap G$ is ols.

Fact 6. If F is open and G is lsc., then $F \cap G$ is lsc.

The proofs of Facts 1 and 2 are fairly obvious. The proofs of Facts 3 and 4 can be found in chapter 11 of Border[1985] . The proofs of Facts 4, 5, and 6 can be found in Yannelis and Prabhakar [1983] and Yannelis [1987].

Finally we state a key lemma that we will use in proving our main results.

Lemma 1.2. *Let* $\{F_\nu | \nu \in D\}$ *be a net of set valued maps form a topological space* X *to subsets of a topological space* Y. *Suppose* $\mathrm{Ls}\,\mathrm{gph}\, F_\nu \subset \mathrm{gph}\, F$ *and suppose* F *is closed. Let* $\{x_{\nu'}\}$ *and* $\{y_{\nu'}\}$ *be such that* $y_{\nu'} \in F_{\nu'}(x_{\nu'})$, $\forall \nu' \in N' \in \mathcal{N}^{\#}(D)$. *Furthermore, suppose that* $x_{\nu'} \longrightarrow x$ *and* $y_{\nu'} \longrightarrow y$. *Then,* $y \in F(x)$.

Proof. Clearly, $(x_\nu, y_\nu) \in \mathrm{gph}\, F_\nu$ and $(x, y) \in \mathrm{Ls}\,\mathrm{gph}\, F_\nu$ by the definition of the Upper limit of F_ν. Hence, $(x, y) \in \mathrm{gph}\, F$. $\square$

<u>Selection theorems and abstract economies</u>

Most equilibria results in abstract economies are based on selection theorems that lead to fix point theorems. We recall that for a set valued map F, a selection is a continuous function f such that $\forall x \in X$, $f(x) \in F(x)$. We say F has a fixed point, if $\exists x^* \in X$ such that $x^* \in F(x^*)$. We use $\Gamma = (X_i, A_i, P_i, I)$ to denote an *abstract economy*. I is the set of agents. X_i is the strategy set for the ith agent. Let $X = \prod_{i\in I} X_i$, and for any $x \in X$, let x_i be the ith component of x. $P_i : X \rightrightarrows X_i$ is a map that represents the preferences of the ith agent. $A_i : X \rightrightarrows X_i$ is a map that represents the constraints for the ith agent. An *equilibrium* for this economy is a point $x^* \in X$ such that for all $i \in I$, $A_i \cap P_i(x^*) = \emptyset$ and $x_i^* \in \operatorname{cl} A_i(x^*)$.

The main result

The idea that we use for all the proofs in this section is the same and it is fairly simple. Instead of assuming the lower semi-continuity of the constraint maps, we "fatten" the graphs of theses maps. The "fattened" maps have open graphs, hence they are ols. and lsc. We apply the existing equilibrium results to the "fattened" maps and then we use Lemmas 1.1 and 1.2 to show that the equilibrium points of the "fattened" maps yield an equilibrium point for the original problem. Let X and Y be topological spaces. We index the local base of the zero element of the space $X \times Y$ with an ordered set D. Thus for any two elements V^{ν_1} and V^{ν_2} of the local base at the zero element of $X \times Y$, we have

$$V^{\nu_2} \subset V^{\nu_1} \iff \nu_1 \leq \nu_2,$$

where ν_1 and ν_2 are elements in D. Let $S : X \rightrightarrows Y$. We can construct the maps $S^\nu : X \rightrightarrows Y$, where

$$\operatorname{gph} S^\nu = \operatorname{gph} S + V^\nu.$$

We also can construct the maps $\bar{S}^\nu : X \rightrightarrows Y$, where

$$\operatorname{gph} \bar{S}^\nu = \operatorname{cl} \operatorname{gph} S^\nu.$$

S^ν is an open map for every $\nu \in D$ and clearly $\bar{A}^\nu$ is a closed map for every $\nu \in D$.

Lemma 1.3. *Let $S : X \rightrightarrows Y$, where X and Y are compact topological vector spaces. Let $S^\nu : X \rightrightarrows Y$ be a collection of set valued maps defined by*

$$\operatorname{gph} S^\nu = \operatorname{gph} S + V^\nu.$$

Then,

$$\operatorname{Ls} \operatorname{gph} S^\nu \subset \operatorname{cl}(\operatorname{gph} S),$$

where the upper limit is taking over all $\nu \in D$.

Proof. In light of Lemma 1.1, we only need to show that $\cap_{\nu\in D} \operatorname{gph} S^\nu \subset \operatorname{cl} \operatorname{gph} S$. Let $x \notin \operatorname{cl} \operatorname{gph} S$. Since $\operatorname{cl}(\operatorname{gph} S)$ is compact, $\exists \nu_0 \in D$ such that $x \notin \operatorname{gph} S^{\nu_0}$ and therefore $x \notin \cap_{\nu\in D} \operatorname{gph} S^\nu$. ∎

We now prove a result that is based on a theorem of Tan and Yuan [1994] but with weaker conditions on the constraint maps. First, we recall the definition of locally $\mathcal{L}$-majorized maps. We say that $\tilde{S} : X \rightrightarrows Y$ is of class $\mathcal{L}$, if $\tilde{S}$ is ols. and $\forall x \in X$, $x \notin \operatorname{con} \tilde{S}(x)$. We say that $S : X \rightrightarrows Y$ is locally $\mathcal{L}$- majorized, if $\forall x \in X$, there exists open neighborhood V_x of x and a set valued map $\tilde{S}_x$ of class $\mathcal{L}$ such that $S(x') \in \tilde{S}_x(x')$, $\forall x' \in V_x$. We say that S is $\mathcal{L}-$majorized, if there exists an $\mathcal{L}$-class map $\tilde{S}$ such that $S(x) \subset \tilde{S}(x)$, $\forall\, x \in X$. We immediately have the following two lemmas

Lemma 1.4. *Let X and Y be topological spaces. Let $S : X \rightrightarrows Y$ be a locally $\mathcal{L}-$majorized set valued map. Let $T : X \rightrightarrows Y$ be an open map. Then, $S \cap T$ is $\mathcal{L}$-locally majorized.*

Proof. Let $x \in \operatorname{dom} S \cap T$. Since S is locally $\mathcal{L}$-majorized, there exists a neighborhood O_x of x and a class $\mathcal{L}$ map $\tilde{S}_x$ such that $S(x) \subset \tilde{S}_x(x)$, $\forall x \in O_x$. $\tilde{S}_x \cap T$ is ols. by Fact 5. Furthermore, $S \cap T(x) \subset \tilde{S}_x \cap T(x)$ and $x \notin \operatorname{con}(\tilde{S}_x \cap T)(x)$, $\forall x \in O_x$. Hence, $S \cap T$ is locally majorized.

□

Lemma 1.5. *[corollary 5.1, Yannelis and Prabhakar[1983]] Let X and Y be a linear topological spaces. Let $S : X \rightrightarrows Y$ be locally $\mathcal{L}$-majorized. Then, S is $\mathcal{L}$-majorized.*

Furthermore, it is clear that the converse of Lemma 1.7. is true. Hence, from now on we will not distinguish between locally $\mathcal{L}$-majorized maps and $\mathcal{L}$-majorized maps.
We now state a simplified version of a result of Tan and Yuan [1994].

Theorem 1.6. *[theorem 3.2, Tan and Yan [1994]]. Let $\Gamma = (X_i, B_i, C_i, P_i, I)$ be an abstract economy with two constraint maps, B_i and C_i for every agent $i \in I$. Suppose that for every $i \in I$, where I is possibly uncountable, the economy satisfy :*
(a) X_i is a nonempty, compact, convex subset of a locally convex Hausdorff space E_i.
(b) for each $x \in X$, B_i is lsc., nonempty, and convex valued map.
(c) $C_i : X \rightrightarrows X_i$ is usc. and $B_i(x) \subset C_i(x)$, $\forall x \in X$.
(d) $B_i \cap P_i$ is $\mathcal{L}$-majorized.
(e) $\operatorname{dom} B_i \cap P_i$ is open in X.
Then, Γ has an equilibrium; $\exists x^* \in X$ such that

$$B_i \cap P_i(x^*) = \emptyset \text{ and } x_i^* \in C_i(x^*), \forall i \in I.$$

The following theorem is our main result:

Theorem 1.7. *Let $\Gamma = (X_i, A_i, P_i, I)$ be an abstract economy satisfying for all $i \in I$:*
(a) X_i is a nonempty, compact and convex subset of a Hausdorff locally convex space E_i.
(b) A_i is nonempty, convex valued map such that $\operatorname{cl} A_i$ is usc.

(c) P_i *is* $\mathcal{L}$*-majorized.*

(d) $\operatorname{dom} A_i \cap P_i$ *is open in* X*.*

Then, Γ *has an equilibrium.*

Proof. We index the local base at the zero element of $X \times X_i$ with an ordered set D_i. Let $D = \prod_{i \in I} D_i$ and let ν_i denote the ith component of $\nu \in D$. We direct D in the following manner: for any ν^2 and ν^1 in D, we have

$$\nu^2 \geq \nu^1 \quad \Longleftrightarrow \quad \nu_i^2 \geq \nu_i^1, \forall i \in I.$$

Let $\pi_i : D \longrightarrow D_i$ be the projection map. We note that if $N \in \mathcal{N}^{\#}(D)$, then $\pi_i(N) \in \mathcal{N}^{\#}(D_i)$. We consider the relative topology on every X_i and the relative topology on X.
Now for every $\nu \in D$:
We construct $A_i^{\nu_i} : X \rightrightarrows X_i$:

$$\operatorname{gph} A_i^{\nu_i} = (\operatorname{gph} A_i + V^{\nu_i}) \cap (X \times X_i).$$

We also construct the maps $\bar{A}_i^{\nu_i} : X \rightrightarrows X_i$, where

$$\operatorname{gph} \bar{A}_i^{\nu_i} = \operatorname{cl} \operatorname{gph} A_i^{\nu_i}.$$

For all $i \in I$, $A_i^{\nu_i}$ is open since it has an open graph in $X \times X_i$. Because of Fact 3, $\bar{A_i^{\nu_i}}$ is closed and usc. We also know that $\forall i \in I$, P_i is $\mathcal{L}$-majorized by some map $\tilde{P}_i$. Let $\varphi_i^{\nu_i}(x) = A_i^{\nu_i} \cap \tilde{P}_i(x)$, $\forall x \in X$. Since $\tilde{P}_i$ is ols. and $A_i^{\nu_i}$ is open, $\varphi_i^{\nu_i}$ is $\mathcal{L}$-majorized by Lemma 1.4. Furthermore, the set $U_i^{\nu_i} = \{x \in X \,|\, \operatorname{dom} \varphi_i^{\nu_i} \neq \emptyset\}$ is open in X due to Fact 2. Now for every $\nu \in D$, we apply Theorem 1.6. by letting B_i and C_i be the maps $\varphi_i^{\nu_i}$ and $\bar{A}_i^{\nu_i}$ respectively. Thus, $\exists\, x^\nu \in X$ such that $x^\nu \notin U_i^{\nu_i}$ and $x_i^\nu \in \bar{A}_i^{\nu_i}(x^\nu)$, $\forall i \in I$. The reason $x^\nu \notin U_i^{\nu_i}$ is that otherwise $x_i^\nu \in \operatorname{con} \tilde{P}_i(x^\nu)$, which contradicts the fact that $\tilde{P}_i$ is a class $\mathcal{L}$ map .
Since X is compact, $\exists\, \hat{x} \in X$ and $N^* \in \mathcal{N}^{\#}(D)$ such that $x^{\nu^*} \longrightarrow \hat{x}$ and $\forall \nu^* \in N^*$,

$$x^{\nu^*} \notin U_i^{\nu_i^*}, \forall i \in I.$$

Since

$$U_i \subset U_i^{\nu_i^*}, \forall i \in I,$$

we get

$$x^{\nu^*} \notin U_i, \forall i \in I,$$

and since for all $i \in I$, U_i is open, we get

$$\hat{x} \notin U_i, \forall i \in I. \tag{1}$$

Moreover, for all $\nu \in N^*$,

$$x_i^{\nu^*} \in \bar{A}_i^{\nu_i^*}(x^{\nu^*}), \ \forall i \in I.$$

Hence,

$$(\hat{x}, \hat{x}_i) \in \operatorname{Ls} \operatorname{gph} \bar{A}_i^{\nu_i^*} \subset \operatorname{cl}(\operatorname{gph} A_i) = \operatorname{gph} \operatorname{cl} A_i, \ \forall i \in I \tag{2}$$

where the upper limit is taking over all $\nu_i^* \in \pi_i(N^*)$ and the first set inclusion holds because of Lemma 1.3. and the last equality holds because of Fact 3.
Finally, from (1) and (2) we conclude that for all $i \in I$, $\hat{x} \notin U_i$ and $\hat{x}_i \in \operatorname{cl} A_i(\hat{x})$ and thus, $\hat{x}$ is an equilibrium point for Γ.

□

Note that in the proof of theorem 3.2 in Tan and Yan [1994], the idea was to "fatten" the values of the constraint maps, whereas the idea of our proof of theorem 1.7. is to "fatten" the graphs of the set valued maps.

The finite dimensional case

Suppose I is a countable set and for all $i \in I$, let E_i be $\mathbb{R}^n$. We then can have a different set of conditions that guarantee the existence of an equilibrium. Furthermore, since $\mathbb{R}^n$ is first countable, we will consider sequences instead of nets and for all $i \in I$, we will take D_i to be the set of natural numbers . Let X be $\mathbb{R}^n$ and for any $C \in C(X)$ and for any $x \in X$, let $d_C(x) = \inf_{y \in C} d(x, y)$, where d is the usual metric of $\mathbb{R}^n$. For a fixed $C \in C(X)$, the function $d_C(.)$ is a continuous function from X to $\mathbb{R}$. The continuity of this function is immediate from the following observation: Let $x_n \longrightarrow x$ in X, then the triangle inequalities

$$d_C(x_n) \leq d_C(x) + d(x_n, x) \ \text{ and } \ d_C(x) \leq d_C(x_n) + d(x_n, x),$$

and the fact that $d(x_n, x) \longrightarrow 0$ yield

$$\limsup_n d_C(x_n) \leq d_C(x) \leq \liminf_n d_C(x_n).$$

We define the $\varepsilon-$*open fattening* of $C \in C(X)$ by

$$S_\varepsilon[C] = \{x \in X \, | d_C(x) < \varepsilon\}.$$

We also define the $\varepsilon-$*closed fattening*, $\overline{S}_\varepsilon[C]$, by

$$\overline{S}_\varepsilon[C] = \operatorname{cl} S_\varepsilon[C].$$

Due to the continuity of $d_C(.)$, $S[C]_\varepsilon$ is an open set in X and clearly $\overline{S}_\varepsilon[C]$ is a closed set of X.

The following lemmas are Lemma 1.1. and Lemma 1.3. in a metric setting.

Lemma 1.8. *Let C_n be a sequence in $C(X)$ that is decreasing with respect to inclusion, then*

$$\operatorname{Ls} C_n \subset \bigcap_{n\geq 1} C_n.$$

lemma 1.9. *Let C be a compact set in $C(X)$. Let ε_n be a sequence of reals that monotonically decrease to 0. Then,*

$$\operatorname{Ls} S_{\varepsilon_n}[C] \subset C.$$

Theorem 1.10. *Let $\Gamma = (X_i, A_i, P_i, I)$ be an abstract economy. Assume that for every $i \in I$, where I is a countable set of agents, Γ satisfies :*
(a) X_i be compact and convex subset of $I\!R^n$.
(b) $A_i : X \rightrightarrows X_i$ is such that $\operatorname{cl} A_i$ is nonempty, convex valued, and usc.
(c) $P_i : X \rightrightarrows X_i$ is lsc.
(d) $U_i = \operatorname{dom} A_i \cap P_i \neq \emptyset$ is open in X.
(e) $\forall x \in X$, $x \notin \operatorname{con} P_i(x)$.
Then, the abstract economy Γ has an equilibrium.

Proof. The proof follows the outlines of the proofs of Theorem 1.7. and theorem 6.1. of Yannelis and Prabhakar [1983]. For all $i \in I$ and $\forall n \in I\!N$, We construct the maps $A_i^n : X \rightrightarrows X_i$ and $\bar{A}_i^n : X \rightrightarrows X_i$, where

$$\operatorname{gph} A_i^n = S_{\frac{1}{n}}[\operatorname{gph} A_i],$$

$$\operatorname{gph} \bar{A}_i^n = \bar{S}_{\frac{1}{n}}[\operatorname{gph} A_i].$$

We know now that $\forall i \in I$ and $\forall n \in I\!N$, A_i^n is open and $\bar{A}_i^n$ is usc. We also know from Lemma 1.9. that $\forall i \in I$,

$$\operatorname{Ls} \operatorname{gph} \bar{A}_i^n \subset \operatorname{gph} A_i. \tag{3}$$

Now let $\varphi_i^n(x) = A_i^n \cap P_i(x)$, $\forall x \in X$. Since A_i^n is open, φ_i^n is lsc. by Fact 6. Furthermore, the set $U_i^n = \{x \in X | \operatorname{dom} \varphi_i^n \neq \emptyset\}$ is open in X and $\forall n \in I\!N$,

$$U_i \subset U_i^n, \forall i \in I. \tag{4}$$

X is a mertizable space and U_i^n is paracompct. By theorem 3.2" in Michael [1955], the restriction of φ_i^n to U_i^n has a selection f_i^n.
Let

$$F_i^n = \begin{cases} f_i^n(x), & \text{if } x \in U_i^n \text{ ;} \\ \bar{A}_i^n(x), & \text{otherwise .} \end{cases}$$

Then, F_i^n is usc.(lemma 6.1 in Yannelis and Prabhakar [1983]) with closed, convex, and nonempty values. Finally, $\forall n \in I\!N$, the map $F_n : X \rightrightarrows X$, where $F_n = \prod_{i\in I} F_i^n$, is also usc. by lemma 3 in [Fan (1952, p. 124)]. Hence, $\exists x^n$ such that $x^n \in F_n(x^n)$. Therefore, $\forall i \in I$, we have $x_n \notin U_i^n$ (otherwise $x_i^n \in \text{con}\, P_i$) and $x_i^n \in \bar{A}_i^n(x_i^n)$. Since X is compact, $\exists x^* \in X$ such that $x_n \longrightarrow x^*$. ¿From (4) and from the fact that U_i is open, we have $x^* \notin U_i$ and

$$A_i \cap P_i(x^*) = \emptyset, \forall i \in I. \tag{5}$$

¿From (3), we have

$$(x^*, x_i^*) \in \text{Ls}\, \text{gph}\, \bar{A}_i^n \subset \text{gph}\, \bar{A}_i,$$

where the upper limit is taking over all $n \in \mathbb{N}$. Hence,

$$x_i^* \in \bar{A}_i(x^*), \forall i \in I \qquad (6).$$

¿From (5) and (6) we conclude that the economy has an equilibrium. □

Note that above theorem is not a sepcial case of Theorem 1.7 since condition (c) of Theorem 1.7 does not imply condition (c) of Theorem 1.10.

In a subsequent paper, we hope to show that the method of "fattening" the constraint maps can be used in abstract economies where the set of agents is a measure space and thus we hope to be able to weaken some of the conditions of the existence results of Yannelis [1987].

Budget constraints and Walrasian equilibrium

In this section, we show that eliminating the requirement of the lower semi-continuity of the constraint map is significant economically. One of the important ways to show the existence of a Walrasian equilibrium is to use the abstract economy approach where the agents are divided into consumers, producers, and an auctioneer.(see Debreu[1959] and Gale and Mas-Colell[1975]). Let us consider the case where the commodity space is $\mathbb{R}^m$ and the set of agents is finite, $I = 1, ..., n$. Let X_i denote the ith consumer's consumption set, w_i denote his initial endowment, and U_i denote his preference relation on X_i. For $j = i, ..., k$, Y_j denotes the jth producer's production set. $\alpha_j^i(p)$ is a function that denotes the share of the ith consumer in the profits of jth supplier at the given price level p. The auctioneer is a player with a strategy set that consist of a simplex $\Delta \subset \mathbb{R}^m$ that represents the set of normalized prices. Our main concern in this model is the constraint map (the budget map) A_i for the ith consumer. We let $X = \prod_{i=1}^n X_i$, $Y = \prod_{j=1}^k Y_j$ and for every $p \in \Delta$ and every j, we let $\delta_j(p) = \sup_{y_j \in Y_j} p.y_j$. Now the budget map for the ith consumer is $A_i : X \times Y \times \Delta \rightrightarrows X_i$, where

$$A_i(x, y, p) = \{x_i \in X_i | p.x_i \leq p.w_i + \sum_{j=1}^k \alpha_j^i(p)\delta_j(p)\}.$$

In order to use the equilibrium results that currently exist in the literature, the budget map has to be lsc. However, to guarantee the lower semi continuity of such a map, one has to assume the continuity of each $\alpha_j^i(.)$. More importantly, one has to assume that $\forall i$ and for all $p \in \Delta$, $\exists x_i \in X_i$ such that

$$p.x_i < p.w_i + \sum_{j=1}^{k} \alpha_j^i(p)\delta_j(p). \tag{$\dagger$}$$

The most common way to satisfy the above condition is to assume that $\alpha_j^i(p)$ are non negative and $\forall i$ and $\forall p$,

$$p.w_i > \inf_{x_i \in X_i} p.x_i. \tag{$\ddagger$}$$

The strict inequalities in ($\dagger$) and ($\ddagger$) mean that consumers can not function when their consumption is compatible with their minimal income. The same problem occurs even when we use two sets of constraint maps and we assume that one of them is lsc. For example, Toussaint [1984] assumed that every agent $i \in I$ has two set valued constraint maps. The first, A_i, is osc. The second, B_i, is nonempty and lsc. Furthermore, the two were related by the following

$$\operatorname{cl} B_i(x) = A_i(x), \ \forall x \in X.$$

In the context of a Walrasian equilibrium, A_i and B_i are

$$A_i(x, y, p) = \{x_i \in X_i | p.x_i \leq p.w_i + \sum_{j=1}^{k} \alpha_j^i(p)\delta_j(p)\},$$

$$B_i(x, y, p) = \{x_i \in X_i | p.x_i < p.w_i + \sum_{j=1}^{k} \alpha_j^i(p)\delta_j(p)\}.$$

Clearly, assuming that B_i is nonempty and lsc. causes the same problem that we discussed earlier. For the results of this paper, no lower semi-continuity conditions are required for the constraint maps. Furthermore, we can allow for discontinuities in the functions $\alpha_j^i(.)$ which might arise from taxes or welfare checks. In fact we can assume that

$$A_i(x, y, p) = \{x_i \in X_i | p.x_i \leq \beta_i(p)\},$$

where $\beta_i(p)$ is the income function of the ith consumer. In order to satisfy condition (b) of Theorem 1.7.(or Theorem 1.8.), we only need $\beta_i(p)$ to be an upper semi-continuous function of p.

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