STRUCTURAL STABILITY OF VECTOR OPTIMIZATION PROBLEMS

by

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Abstract. — We study global stability properties for vector optimization problems of the type:
\[ \mathcal{VOP}(f, H, G): \min \{ f(x) = (f_1(x), \ldots, f_l(x)) \mid x \in M[H, G] \}, \]
where
\[ M[H, G] := \{ x \in \mathbb{R}^n \mid h_i(x) = 0, \quad g_j(x) \leq 0, \quad i \in I, j \in J \} \]
with
\[ I := \{ 1, \ldots, m \}, \quad J := \{ 1, \ldots, s \}, \quad L := \{ 1, \ldots, l \}. \]
We extend Guddat/Juengen’s [5] concept of structural stability of scalar nonlinear optimization problems to vector optimization problems. Under the assumption that \( M[H, G] \) is compact we prove the necessary condition for the structural stability of a vector optimization problem, i.e. the scalar problem
\[ \mathcal{P}^{\max}(f, H, G): \min \left\{ \max_{i \in L} f_i(x) \mid x \in M[H, G] \right\} \]
has to be structurally stable.

1. Introduction

We consider the vector optimization problem
\[ \mathcal{VOP}(f, H, G): \min \{ f(x) = (f_1(x), \ldots, f_l(x)) \mid x \in M[H, G] \}, \]
where
\[ M[H, G] := \{ x \in \mathbb{R}^n \mid h_i(x) = 0, \quad g_j(x) \leq 0, \quad i \in I, j \in J \} \]

Key words and phrases. — vector optimization, structural stability, strong stability, constraint qualification (Mangasarian-Fromovitz).

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with

\[ I := \{1, \ldots, m\}, \quad J := \{1, \ldots, r\}, \quad L := \{1, \ldots, t\}. \]

Unless otherwise stated, we will assume that the functions \( f_k, g_j \), \( k \in L, j \in J \) belong to \( C^2(\mathbb{R}^n, \mathbb{R}) \). The lower level set corresponding to \( u \in \mathbb{R}^{[L]} \) will be denoted as follows:

\[ L^u(f, H, G) := \{ x \in M[H, G] \mid f(x) \leq u \}, \quad u \in \mathbb{R}^{[L]}. \]

The partial ordering \( \leq \) is induced by a convex cone \( D \) with

\[ x \leq y \iff y - x \in D. \]

**Definition 1.** — The vector optimization problems \( \text{VOP}(f, H, G) \) and \( \text{VOP}(\tilde{f}, \tilde{H}, \tilde{G}) \) are called equivalent if there exist continuous mappings \( \phi : \mathbb{R}^{[L]} \times \mathbb{R}^n \to \mathbb{R}^n \) and \( \psi : \mathbb{R}^{[L]} \to \mathbb{R}^{[L]} \) with the properties P1 – P3:

P1 For every \( u \in \mathbb{R}^{[L]} \) the mapping \( \phi_u : \mathbb{R}^n \to \mathbb{R}^n \) is a homeomorphism from \( \mathbb{R}^n \) onto itself, where \( \phi_u := \phi(u, x) \).

P2 The mapping \( \psi \) is a homeomorphism from \( \mathbb{R}^{[L]} \) onto itself and \( \psi \) is \( D \)-monotonically increasing; i.e.,

\[ q_1 \leq q_2 \text{ implies } \psi(q_1) \leq \psi(q_2) \text{ for } q_1, q_2 \in \mathbb{R}^{[L]}. \]

P3 \( \phi_u[L^u(f, H, G)] = L^{\psi(u)}(\tilde{f}, \tilde{H}, \tilde{G}) \) for all \( u \in \mathbb{R}^{[L]} \).

**Definition 2.** — The vector optimization problems \( \text{VOP}(f, H, G) \) is called structurally stable if there exists a \( C^2_2 \)-neighbourhood \( O \) of \( (f, H, G) \) with the property that \( \text{VOP}(f, H, G) \) and \( \text{VOP}(\tilde{f}, \tilde{H}, \tilde{G}) \) are equivalent for all \( (f, H, G) \in O \).

The \( C^2_2 \)-topology above for the product \( \prod_{i=1}^r C^2(\mathbb{R}^n, \mathbb{R}) \) will be the product topology generated by the strong (or Whitney-) \( C^2 \)-topology \( C^2_2 \) on each factor \( C^2(\mathbb{R}^n, \mathbb{R}) \) (cf. [6], [9]). A typical base-neighbourhood of a function \( \rho \in C^2(\mathbb{R}^n, \mathbb{R}) \) is the set \( \rho + \mathcal{W}_\epsilon \), where \( \mathcal{W}_\epsilon \) is defined as follows with the aid of a continuous positive \( \epsilon : \mathbb{R}^n \to \mathbb{R} \).

\[ \mathcal{W}_\epsilon = \{ \eta \in C^2(\mathbb{R}^n, \mathbb{R}) \mid \| \eta(x) \| + \sum_i \left| \frac{\partial \eta}{\partial x_i}(x) \right| + \sum_{i,j} \left| \frac{\partial^2 \eta}{\partial x_i \partial x_j}(x) \right| < \epsilon(x), \forall x \in \mathbb{R}^n \}. \]

We have the chance of a more local control for data corresponding to the scalar problem \( \mathcal{P}(r, H, G), r \in C^2(\mathbb{R}^n, \mathbb{R}) \) by putting for any subset \( U \) of \( \mathbb{R}^n \):

\[ \text{norm}[(r, H, G), U] = \sup_{x \in U} \max_{\eta \in (r, h, l) \in I, g_j, j \in J} \left\{ \| \eta(x) \| + \sum_i \left| \frac{\partial \eta}{\partial x_i}(x) \right| + \sum_{i,j} \left| \frac{\partial^2 \eta}{\partial x_i \partial x_j}(x) \right| \right\}. \]
Remark 1. — If $|L| = 1$ and $D = \mathbb{R}^{[L]}$, then we have the structural stability for the scalar nonlinear problem introduced in [5].

We consider the following scalar optimization problem

$$\mathcal{P}^{\text{max}}(f, H, G): \min \left\{ \max_{t \in L} f_t(x) \mid x \in M[H, G] \right\}.$$  

For the corresponding lower level set we shall use the notation

$$L^t_{\text{max}}(f, H, G) := \left\{ x \in M[H, G] \mid \max_{t \in L} f_t(x) \leq t \right\}, \quad t \in \mathbb{R}.$$  

We use the notations given in Weber [14] and denote the canonical projection of the space $\mathbb{R}^{n+1}$ on the first $n$ components or on the last component by $pr_{(1,...,n)}$ and $pr_{n+1}$, respectively; i.e.

$$\bar{x} = (pr_{(1,...,n)}(\bar{x}), pr_{n+1}(\bar{x})) = (x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}.$$  

Let us now look at the following equivalent differentiable optimization problem which in particular reveals an objective function of class $C^2$.

$$\mathcal{P}^\prime(\bar{f}, \bar{H}, \bar{G}): \min \{ \bar{f}(\bar{x}) := pr_{n+1}(\bar{x}) \mid x \in M[\bar{H}, \bar{G}] \},$$  

where

$$M[\bar{H}, \bar{G}] := \{(x, x_{n+1}) \in \mathbb{R}^{n+1} \mid x \in M[H, G], f_l(x) - x_{n+1} \leq 0 \ (l \in L) \},$$  

namely with $h_i(\bar{x}) := h_i(x)(i \in I)$ and

$$\bar{g}_j(\bar{x}) := \begin{cases} g_j(x), & \text{for } j \in J, \\ f_l(x) - x_{n+1}, & \text{for } j = s + l, \ (l \in L). \end{cases}$$  

$\mathcal{P}^\prime(\bar{f}, \bar{H}, \bar{G})$ is strongly related to scalarization methods for vector optimization problems. As pointed out in Weidner [16], this scalarization method includes the known procedures of Brosowski and Conci [3], Helbig [8], Wierzbicki [17] and other papers cited in [7]. Roughly speaking, each minimal point of $\mathcal{P}^\prime(\bar{f}, \bar{H}, \bar{G})$ is weakly efficient and vice versa each weakly efficient and hence each efficient point solves the scalar problem $\mathcal{P}^\prime(\bar{f}, \bar{H}, \bar{G})$.

We note that the lower level sets $L^t_{\text{max}}(f, H, G)$ of $\mathcal{P}^{\text{max}}(f, H, G)$ are just the projection in the sense of $pr_{(1,...,n)}$ of the corresponding level sets $L^t(\bar{f}, \bar{H}, \bar{G})$ due to $\mathcal{P}^\prime(\bar{f}, \bar{H}, \bar{G})$. Both sets are homotopy-equivalent (cf. Jongen et al. [9], Lemma 4.2.1) and $M[\bar{H}, \bar{G}]$ is just the part of the epigraph $E(\max_{t \in L} f_t)$ over the feasible set $M[H, G]$. 

Definition 3. — The Linear Independence Constraint Qualification (shortly LICQ) is said to hold at $x \in M[H, G]$ if the vectors $Dh_i(x), i \in I, Dg_j(x), j \in J_0(x)$, are linearly independent. The Mangasarian-Fromovitz Constraint Qualification (shortly MFCQ) is said to hold at $x \in M[H, G]$ if the following conditions $MF1$ and $MF2$ are satisfied:

$MF1$ The vectors $Dh_i(x), i \in I$ are linearly independent.
MF2 There exists a vector $\xi \in \mathbb{R}^n$ satisfying:

$$Dh_i(x)\xi = 0, \quad i \in I$$

$$Dg_j(x)\xi \leq 0, \quad j \in J_0(x).$$

A vector $\xi$ satisfying (6) and (7) will be called an MF-vector.

For $x \in \mathbb{R}^n$ and $\rho > 0$ let $B(x, \rho)$ denote the Euclidean ball in $\mathbb{R}^n$, centered at $x$ with radius $\rho$. Following M. Kojima ([11]) we define:

Definition 4. — Let $x^u \in M[H,G]$ be a Kuhn-Tucker point for $\mathcal{P}(r,H,G)$, $r \in C^2(\mathbb{R}^n, \mathbb{R})$. Then $x^u$ is called strongly stable if for some $\delta > 0$ and each $\delta \in (0, \delta]$ there exists an $\alpha > 0$ such that whenever $(\tilde{r}, \tilde{H}, \tilde{G})$ satisfies $\text{norm}[(r - \tilde{r}, H - \tilde{H}, G - \tilde{G})] \leq \alpha$, the ball $B(x^u, \delta)$ contains a Kuhn-Tucker point for $\mathcal{P}(\tilde{r}, \tilde{H}, \tilde{G})$ which is unique in $B(x^u, \delta)$.

Lemma 1 (Kojima [11]). — Let $x^u \in M[H,G]$ be a Kuhn-Tucker point for $\mathcal{P}(r,H,G)$, $r \in C^2(\mathbb{R}^n, \mathbb{R})$.

1. If LICQ is satisfied at $x^u$, then $x^u$ is a strongly stable Kuhn-Tucker point if and only if the matrix $D^2L[\lambda, \mu](x^u)$ has nonvanishing determinants with a common sign on the subspace $W(x^u, J)$ for all $J$ with $J_+(x^u) \subset J \subset J_0(x^u)$, where

$$J_+(x^u) = \{ j \in J_0(x^u) | \mu_j > 0 \}$$

$$W(x^u, J) = \{ \xi \in \mathbb{R}^n | Dh_i(x^u)\xi = 0, i \in I, Dg_j(x^u)\xi = 0, j \in J \}$$

$$L[\lambda, \mu](x) := r(x) + \sum_{i \in I} \lambda_i h_i(x) + \sum_{j \in J_+(x^u)} u_j g_j(x).$$

2. Let MFCQ be satisfied at $x^u$, but LICQ not. Then, $x^u$ is a strongly stable Kuhn-Tucker point if and only if for every $(\lambda, \mu)$ the matrix $D^2L[\lambda, \mu](x^u)$ is positive definite on the subspace $W(x^u, J_+(x^u))$.

Firstly we note that the structural stability of the problem $\mathcal{P}^\text{max}(f,H,G)$ was characterized in Weber [14].

Theorem 1 (Main result). — Let $D = \mathbb{R}^{|L|}_+$ and the vector optimization problem $\mathcal{VO}P(f,H,G)$ with compact feasible set $M[H,G]$ be structurally stable. Then the problem $\mathcal{P}^\text{max}(f,H,G)$ is structurally stable.

2. Lemmas and preliminary results

Lemma 2. — Let $t \in \mathbb{R}$, $D = \mathbb{R}^{|L|}_+$ and $u := (t, \ldots, t) = te, e = (1, \ldots, 1) \in \mathbb{R}^{|L|}$. Then the following relations between lower level sets are fulfilled:

$$\mathcal{L}^\text{max}(f,H,G) = \mathcal{P}r_{(1, \ldots, n)}\mathcal{L}^\text{vt}(f,H,G) = \mathcal{L}^u(f,H,G).$$

Proof. — The first equality is obvious. We show the second equality. If $f_t(x) \leq t, f \in [1, \ldots, L]$. This implies that $(x, t) \in M(H,G)$ and so $x \in \mathcal{P}r_{(1, \ldots, n)}\mathcal{L}^\text{vt}(f,H,G)$. 


\( \mathcal{L}^u(f, H, G) \supseteq \text{pr}_{(1, \ldots, n)} \mathcal{L}^{\nu t}(\bar{f}, \bar{H}, \bar{G}) \). Let \( x \in \text{pr}_{(1, \ldots, n)} \mathcal{L}^{\nu t}(\bar{f}, \bar{H}, \bar{G}) \). Then, there exists a \( z \in \mathbb{R} \) with \( (x, z) \in \mathcal{L}^{\nu t}(f, H, G) \). This implies that \( x \in M[H, G] \), \( f_l(x) \leq z \), \( l \in L \) and \( z \leq t \) and so \( f_l(x) \leq t \), \( l \in L \). That means \( x \in \mathcal{L}^u(f, H, G) \).

\[ \mathcal{L}^u(f, H, G) \supseteq \text{pr}_{(1, \ldots, n)} \mathcal{L}^{\nu t}(\bar{f}, \bar{H}, \bar{G}) \]

**Lemma 3.** Let \( D \) be a pointed, closed and convex cone. Then there exists a \( \bar{u} \in \mathbb{R}^{|L|} \) with:

\[ u_1 \leq \bar{u}, \quad u_2 \leq \bar{u} \]

for all \( u_1, u_2 \in \mathbb{R}^{|L|} \).

**Proof.** We follow the idea by Tanino in [12] (Proof of the Theorem 3.3.1). Pick \( d \in \text{int}D \)

and define

\[ y(\alpha) = \alpha d \quad \alpha \in \mathbb{R}. \]

We claim that, for any \( y \in \mathbb{R}^{|L|} \), there exists \( \alpha > 0 \) such that

\[ y \in y(\alpha) - D. \]

If it is not true, then applying the standard separation theorem to the ray \( \{y - \alpha d : \alpha > 0\} \) and the convex cone \( -D \), we deduce the existence of \( \nu \in \mathbb{R}^p \setminus \{0\} \) such that

\[ \langle \nu, y - \alpha d \rangle \geq 0 \quad \text{for all } \alpha > 0 \]

and

\[ \langle \nu, d' \rangle \geq 0 \quad \text{for all } d' \in D. \]

Since \( d \in \text{int}D \), \( \langle \nu, d \rangle > 0 \) from the latter inequality and so

\[ \langle \nu, y - \alpha d \rangle < 0 \]

for sufficiently large \( \alpha \), which is a contradiction.

We can therefore choose \( \alpha_1, \alpha_2 > 0 \) with

\[ u_1 \in \alpha_1 d - D \quad \text{and } u_2 \in \alpha_2 d - D. \]

Take \( \alpha_{\text{max}} := \max\{\alpha_1, \alpha_2\} \). Since \( \alpha_i d \leq \alpha_{\text{max}} d, i = 1, 2 \), we can observe that the inequalities (9) are satisfied by \( \bar{u} := \alpha_{\text{max}} d \).

3. **Proof of the Necessity Part**

Let us now assume that the problem \( \mathcal{OP}(f, H, G) \) is structurally stable according to the Definition 2. Then from the results given by Weber in [10] (see also [14] and [5]) we have to verify the following conditions:

\( (C1) \) The Mangasarian-Fromovitz constraint qualification is satisfied at every point of \( M[H, G] \).

\( (C2) \) Every Kuhn-Tucker point of \( \mathcal{P}^{\text{max}}(f, H, G) \) is strongly stable.

\( (C3) \) Different Kuhn-Tucker points have different \( (f-) \) values.
Proof of (C1): We show that the structural stability of $\mathcal{VOP}(f, H, G)$ implies (C1). Consider a $C^2_{a}$-neighbourhood $\mathcal{O}$ of $(f, H, G)$ with the property: for every $(\tilde{f}, \tilde{H}, \tilde{G}) \in \mathcal{O}$ the set $M[\tilde{H}, \tilde{G}]$ is compact and moreover $\mathcal{VOP}(f, \tilde{H}, \tilde{G})$ is equivalent with $\mathcal{VOP}(f, H, G)$. Choose an arbitrary element $(\tilde{f}, \tilde{H}, \tilde{G}) \in \mathcal{O}$. Let $\phi$ and $\psi$ be mappings which establish the equivalence according Definition 1. Take $\alpha, \tilde{\alpha} \in \mathbb{R}^{[L]}$

\begin{equation}
\alpha := (\alpha_1, \ldots, \alpha_{[L]}), \quad \alpha_i := \max_{x \in M[H, G]} f_i(x)
\end{equation}

\begin{equation}
\tilde{\alpha} := (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_{[L]}), \quad \tilde{\alpha}_i := \max_{x \in M[\tilde{H}, \tilde{G}]} \tilde{f}_i(x)
\end{equation}

and consider the vectors $\alpha$ and $\psi^{-1}(\tilde{\alpha})$. Lemma 9 implies the existence of a $\tilde{u} \in \mathbb{R}^{[L]}$ such that $\alpha \leq \tilde{u}$ and $\psi^{-1}(\tilde{\alpha}) \leq \tilde{u}$. We note that $\mathcal{L}^\alpha(f, H, G) = M[H, G]$. Since $\psi$ is monotonically increasing, we have $\tilde{\alpha} \leq \psi(\tilde{u})$. Thus $\mathcal{L}^{\psi(\alpha)}(\tilde{f}, \tilde{H}, \tilde{G}) = M[\tilde{H}, \tilde{G}]$.

From $\phi_0(\mathcal{L}^\alpha(f, H, G)) = \mathcal{L}^{\psi(\alpha)}(\tilde{f}, \tilde{H}, \tilde{G})$, we have $\phi_0(M[H, G]) = M[\tilde{H}, \tilde{G}]$. Hence, $\phi_0$ establishes a homeomorphism between $M[H, G]$ and $M[\tilde{H}, \tilde{G}]$. So, we have shown: for every $(H, G)$ in some $C^2_{a}$-neighbourhood of $(H, G)$ the sets $M[H, G]$ and $M[\tilde{H}, \tilde{G}]$ are homeomorphic. Then, with the help of the Stability Theorem from [5] we may equivalently say that MFCQ is satisfied at every point $x \in M[H, G]$. Hence, the condition (C1) is valid.

Proof of (C2): Let us assume that there is a stationary point which is not strongly stable, i.e. condition (C2) is not valid. In order to arrive at a contradiction, the perturbation result from Weber [14] plays a central part.

**Lemma 4 (Perturbation Lemma, Weber [14]).** — Let $M[H, G]$ be compact and suppose that (C1) holds. Suppose further, that $\mathcal{P}^{\max}(V, H, G)$ has a Kuhn-Tucker point $x^*$ which is not strongly stable. Then, there exist a number $k \in \mathbb{N}$ and functions $(V^1, \tilde{H}, \tilde{G})$, $(V^2, \tilde{H}, \tilde{G})$, both arbitrarily $C^2_{a}$-near $(V, H, G)$, with the following properties:

(i) Problem $\mathcal{P}^{\max}(V^1, \tilde{H}, \tilde{G})$ has $k$ Kuhn-Tucker points, exactly one of which is not strongly stable.

(ii) Problem $\mathcal{P}^{\max}(V^2, \tilde{H}, \tilde{G})$ has at least $k+1$ Kuhn-Tucker points, all of them being strongly stable.

(iii) Conditions C1 and C3 hold for problems $\mathcal{P}^{\max}(V^i, \tilde{H}, \tilde{G}), i = 1, 2$.

For $r \in C^2(\mathbb{R}^n, \mathbb{R})$ we denote the set $\{x \in M[H, G] \mid a \leq r(x) \leq b\}$ by $L^r_a(r, H, G)$ for $a \leq b, a, b \in \mathbb{R}$.

The next Lemma is proved by Guddat and Jongen in [5].

**Lemma 5.** — Let the feasible set $M[H, G]$ corresponding to the problem $\mathcal{P}(r, H, G)$ be compact and suppose that Condition C1 is fulfilled. Let $a, b \in \mathbb{R}$ be given with $a < b$.

(i) If $L^r_a(r, H, G)$ contains no Kuhn-Tucker point, then $L^r_a(r, H, G)$ and $L^r_b(r, H, G)$ are homeomorphic.
(ii) Suppose that \( \mathcal{L}_k^a(r, H, G) \) contains exactly one Kuhn Tucker point \( \bar{x}^a \) with \( a < r(\bar{x}^a) < b \), and suppose that \( \bar{x}^a \) is strongly stable. Then the sets \( \mathcal{L}_k^a(r, H, G) \) and \( \mathcal{L}_k^b(r, H, G) \) are not homeomorphic.

**Lemma 6.** Let \( M[H, G] \) be compact and suppose that Condition C1 is fulfilled. Let \( V^1, V^2 \in C^2(\mathbb{R}^n, \mathbb{R})^I \) be functions satisfying Condition C3 w.r.t. \( \mathcal{P}^\text{max}(V^i, H, G), i = 1, 2 \), with the following additional properties:

(i) Problem \( \mathcal{P}^\text{max}(V^1, H, G) \) has \( k \) Kuhn-Tucker points, exactly one of which is not strongly stable.

(ii) Problem \( \mathcal{P}^\text{max}(V^2, H, G) \) has at least \( k + 1 \) Kuhn-Tucker points, all of them being strongly stable.

Then, the vector optimization problems \( \mathcal{VOP}(V^i, H, G), i = 1, 2 \) are not equivalent.

**Proof.** We refer to the \( n \)-dimensional problems \( \mathcal{P}^\text{max}(V^i, H, G), i = 1, 2 \) due to the corresponding \((n+1)\)-dimensional problems \( \mathcal{P}^\text{max}(f^i, H, G), i = 1, 2 \). For the latter problems, we study the topological behaviour of the lower level sets, when the parameter \( t \) increasingly traverses the line of real numbers. Taking account of Condition C3 formulated for both problems and the Lemma 5, it follows that the homeomorphism type of the lower level set \( \mathcal{L}^V(f^i, H, G) \) changes at least \( k + 1 \) and at least \( k - 1 \), but (only) at most \( k \) such changes for \( \mathcal{L}^V(f^1, H, G) \). The pair-wise homotopy equivalence between the lower level sets in \( \mathbb{R}^n \) and \( \mathbb{R}^{n+1} \) gives rise to the analogous description in dimension \( n \) of the topological behavior of \( \mathcal{L}^V(V^i, H, G), i = 1, 2 \). From Lemma 2 (8) it follows that \( \mathcal{L}^V(V^i, H, G), i = 1, 2 \) have two different numbers for the changes of the topological type in \( \mathbb{R}^n \), as we move along the line \( u(t) = te, e \in \mathbb{R}^{1|I}, t \in \mathbb{R} \).

If the problems \( \mathcal{VOP}(V^i, H, G), i = 1, 2 \) would be equivalent, then these two numbers have to coincide.

We now apply the perturbational result for our indirect proof of Condition C2. If this condition does not hold, we can apply Lemma 4 and Lemma 6 in order to obtain two non-equivalent problems \( \mathcal{VOP}(f^i, H, G), i = 1, 2 \) with each \( f^i, H, G \) arbitrarily \( C^2 \)-near to \( (f, H, G) \). This is in contradiction with structural stability.

**Proof of (C3):** We know the validity of C1 and C2. If \( \mathcal{P}^\text{max}(f, H, G) \) does not satisfy Condition C3, whenever the equation \( \max_{i \in I} f(x^i_k) = \max_{i \in I} f(\bar{x}^i_0) \) holds for two different Kuhn-Tucker points of \( \mathcal{P}^\text{max}(f, H, G) \) then an addition of one small positive and one small negative constant to all functions \( f_k \) \((k \in L)\) locally at \( x^i_k \) and \( \bar{x}^i_0 \) respectively, leads to a problem \( \mathcal{P}^\text{max}(f, H, G) \) satisfying Condition C3. Let \( k \) the number of Kuhn-Tucker points for \( \mathcal{P}^\text{max}(f, H, G) \). Then, reasoning as in the proof of Lemma 6, the homeomorphism type of \( \mathcal{L}^\text{max}(f, H, G) \) for increasing \( t \), changes exactly \( k \)-times. The corresponding number of changes w.r.t. \( \mathcal{P}^\text{max}(f, H, G) \) is less than \( k \). So, as in Lemma 6, \( \mathcal{VOP}(f, H, G) \) and \( \mathcal{VOP}(f, H, G) \) are not equivalent. This leads again to a contradiction with the condition on structural stability of \( \mathcal{VOP}(f, H, G) \).
References


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