A modified standard embedding for linear complementarity problems[1]

S. ALLENDE ALLONSO[2], J. GUDDAT[3], D. NOWACK[4]

dedicated to the 85. birthday of
Prof. Dr. Dr. hc. František Nožička

Abstract

We propose a modified standard embedding for solving the linear complementarity problem (LCP). This embedding is a special one-parametric optimization problem $P(t), t \in [0,1]$. Under the conditions (A3) (the Mangasarian-Fromovitz Constraint Qualification is satisfied for the feasible set $M(t)$ depending on the parameter t), (A4) ($P(t)$ is Jongen-Jonker-Twilt regular) and two technical assumptions (A1) and (A2) there exists a path in the set of stationary points connecting the chosen starting point for $P(0)$ with a certain point for $P(1)$, and this point is a solution of the (LCP). This path may include types of singularities, namely points of Type 2 and Type 3 in the class of Jongen-Jonker-Twilt for $t \in [0,1)$. We can follow this path by using pathfollowing procedures (contained in the program package PAFO). In case that the condition (A3) is not satisfied, also points of Type 4 and 5 may appear. The assumption (A4) will be justified by a theorem. Illustrative examples are presented.

Keywords: Linear complementarity problem, standard embedding, Jongen-Jonker-Twilt regularity, Mangasarian-Fromovitz Constraint Qualification, pathfollowing methods

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[2] Professor, Faculdad de Matematica y Computacion, Universidad de la Habana,
[3] Professor, Institut für Mathematik, Humboldt-Universität zu Berlin
[4] Assistent Professor, Institut für Mathematik, Humboldt-Universität zu Berlin

1 Introduction

Let B be an $n \times n$-matrix, $q \in I\!R^n$, and

$$M^L := \{x \in I\!R^n | Bx + q \geq 0, x \geq 0, x^T Bx + q^T x \leq 0\}.$$

We consider the well-known linear complementarity problem (for its practical importance we refer e.g. to [11] and the papers cited there)

$$\text{(LCP)} \qquad \text{Find a point} \quad \hat{x} \in M^L. \tag{1.1}$$

If we introduce

$$B = \begin{pmatrix} b^{1^T} \\ \vdots \\ b^{n^T} \end{pmatrix} \text{ with } b^j \in I\!R^n \text{ and } b^j \neq 0, j = 1, \ldots, n,$$

then we can write M^L in the following form

$$M^L = \{x \in I\!R^n | b^{j^T} x + q_j \geq 0, x_j \geq 0, j \in J, x^T Bx + q^T x \leq 0\},$$

where $J := \{1, \ldots, n\}$.
We assume that

$$(A1) \qquad M^L \neq \emptyset.$$

Let $E(p) := \{x \in I\!R^n \mid \|x\|^2 \leq p\}$ be with $p \in I\!R$ and $p > 0$.

Then there exists a $p_0 > 0$ such that $M^L \cap E(p) \neq \emptyset$ for all $p > p_0$. (1.2)

If M^L is compact, then we even have: There exists a $p_0 > 0$ such that

$$M^L \subseteq E(p) \text{ for all } p > p_0.$$

Instead of the (LCP) (cf. (1.1)) we now consider the following optimization problem

$$(P^L) \quad \min \left\{ \frac{1}{2}(x - x^0)^T A(x - x^0) \mid x \in M^L \right\}, \tag{1.3}$$

where A is a symmetric $n \times n$ matrix ($A \in I\!R^{n(n+1)/2}$, here the space of symmetric $n \times n$ matrices coincides with $I\!R^{n(n+1)/2}$).

Now we introduce the well-known concept of embedding for the general nonlinear optimization problem

$$(P) \quad \min\{f(x) | x \in M\}, \tag{1.4}$$

where

$$M := \{x \in I\!R^n | g_j(x) \geq 0, j \in J\}, \tag{1.5}$$

$J := \{1, \ldots, s\}$ and $f, g_j \in C^3(I\!R^n, I\!R), j \in J$.
We choose a one-parametric optimization problem

$$P(t) \quad \min\{f(x,t) | x \in M(t)\}, t \in [0,1],$$

where

$$M(t) := \{x \in I\!R^n \mid g_j(x,t) \geq 0, j \in J\},$$

with the following properties:

(V1) A local minimizer for $P(0)$ is known and the corresponding Lagrange multipliers are known or easy to compute.

(V2) $P(t)$ has a global minimizer for all $t \in [0,1]$.

(V3) $P(1)$ is equivalent to (P).

(V1) and (V2) are the minimum of properties for finding a discretization of $[0,1]$:

$$0 = t_0 < \ldots < t_k < t_{k+1} < \ldots < t_N = 1 \tag{1.6}$$

and corresponding local minimizers, stationary or generalized critical points (g.c. point) $x(t_k)$ of $P(t_k)$, $k = 1, \ldots, N$. For the definition of a g.c. point we refer to [16] - [18].

Remark 1.1 *The concept of finding a discretization like (1.6) and of corresponding optimal points goes back to F. Nožička (see [22], [23] in linear one-parametric optimization problems).*

One of the classical standard embeddings of the problem (1.4), (1.5), is the following one

$$\tilde{P}^s(t) \quad \min\{tf(x) + (1-t)\|x - x^0\|^2 | \; x \in \tilde{M}^s(t)\}, \;\; t \in [0,1],$$

where

$$\tilde{M}^s(t) := \{x \in I\!R^n | t g_j(x) + (1-t) w_j^0 \geq 0, \; j \in J\}$$

with $w_j^0 > 0, \; j \in J$.
Then the problem (P^L) is embedded by

$$P^s(t) \quad \min\{(x - x^0)^T A (x - x^0) | x \in M^s(t)\}, \;\; t \in [0,1], \tag{1.7}$$

$$M^s(t) := \{x \in R^n | g_j(x,t) \geq 0, j = 0, 1 \ldots, n, h_i(x) \geq 0, i = 1 \ldots, n+1\}, \tag{1.8}$$

where

$$\begin{aligned} g_0(x,t) &:= t(-x^T B x - q^T x) + (1-t) w_0^0, \\ g_j(x,t) &:= t(b^{jT} x + q_j) + (1-t) w_j^0, \; j = 1, \ldots, n, \\ h_i(x) &:= x_i, \; i = 1, \ldots, n, \\ h_{n+1}(x) &:= p - \|x\|^2, \;\; p \text{ sufficiently large.} \end{aligned}$$

We assume

$$(A2) \qquad w_i^0 > 0,\ i = 0, 1, \ldots, n \text{ and } \|x^0\|^2 < p.$$

Here we use the pathfollowing procedure (cf. the Program Package PAFO in Chapter 2). We will see that we obtain a very good starting situation for $t = 0$. If we attain $t = 1$, we will have a solution of the (LCP). The use of pathfollowing methods for complementarity problems is not new (e.g. [4]-[10],[12],[13],[20],[21],[28],[29]) and the papers cited there). Modified standard embeddings (cf. [26]) are not new either. What is new is the application of this embedding to the (LCP). It will turn out that we attain $t = 1$ by using a pathfollowing procedure only. The matrix B could be indefinite in distinction to what was done in the papers cited above. Furthermore, the path we are following may include singularities. This is the real advantage of the approaches in [1] and here. From this point of view it is not necessary to compare our pathfollowing procedure with others for (LCP). Chapter 2 includes a summary of the theoretical background and a short description of the program package PAFO (only the part used here).
In Chapter 3 important properties of $P^s(t)$ (i.e., the starting situation and the singularities that may appear) will be discussed. Under the assumptions (A1)-(A4) there exists a path in the set of stationary points connecting the chosen starting point for $P^s(0)$ with a certain point for $P^s(1)$ and this point is a solution for the (LCP). The path may include types of singularities, namely points of Type 2 and Type 3 in the class of Jongen-Jonker-Twilt for $t \in [0, 1)$.
In Chapter 4 a theorem justifying the chosen approach is presented. Illustrative examples are given in Chapter 5, where we see that we attain $t = 1$ under the assumptions (A1)-(A4). Further, we present an example showing that we are successful even if (A3) is not satisfied. In the penalty embedding (cf. [1]) we have many more variables than in the standard embedding. This is a great advantage. Up to now, we have been successful with all our examples. Let us mention that the authors follow the same concept as for the penalty embedding in [1].

2 Theoretical Background and the Program Package PAFO

First, we present a very short version of 2.5, 2.6 from [17]. We consider the general one-parametric problem:

$$P(t) \qquad min\{f(x,t) | x \in M(t)\}, \ \ t \in [0, 1], \tag{2.1}$$

where

$$M(t) = \{x \in I\!R^n \mid h_i(x, t) = 0,\ i \in I,\ g_j(x, t) \geq 0,\ j \in J\} \tag{2.2}$$

$$f, h_i, g_j \in C^3(I\!R^n \times I\!R, I\!R), i \in I, j \in J$$

Furthermore, we introduce the following notations:

$$\begin{aligned}
\Sigma_{gc} &:= \{(x,t) \in I\!R^n \times I\!R \mid x \text{ is a g. c. point of } P(t)\}, \\
\Sigma_{\text{stat}} &:= \{(x,t) \in I\!R^n \times I\!R \mid x \text{ is a stationary point of } P(t)\}, \\
\Sigma_{\text{loc}} &:= \{(x,t) \in I\!R^n \times I\!R \mid x \text{ is a local minimizer of } P(t)\}, \\
H &:= (h_1, \ldots, h_m)^T, \quad G := (g_1, \ldots, g_s)^T.
\end{aligned}$$

The Linear Independence Constraint Qualification (briefly LICQ) is satisfied at $\bar{x} \in M(\bar{t})$ if the vectors $D_x h_i(\bar{x}, \bar{t})$, $i \in I$, $D_x g_j(\bar{x}, \bar{t})$, $j \in J_0(\bar{x}, \bar{t})$, are linearly independent ($J_0(x,t) := \{j \in J \mid g_j(x,t) = 0\}$ is the set of active constraints at (x,t).
The Mangasarian-Fromovitz Constraint Qualification (briefly MFCQ) is satisfied at $\bar{x} \in M(\bar{t})$ if:
(MF1) $D_x h_i(\bar{x}, \bar{t})$, $i \in I$, are linearly independent,
(MF2) there exists a vector $\xi \in R^n$ with

$$\begin{aligned}
D_x h_i(\bar{x}, \bar{t})\xi = 0, &\quad i \in I, \quad {}^1 \\
D_x g_j(\bar{x}, \bar{t})\xi > 0, &\quad j \in J_0(\bar{x}, \bar{t}).
\end{aligned}$$

Next, we cite our short characterization from [16]-[18] of the class $\mathcal{F}$, introduced by Jongen, Jonker and Twilt.
If $(f, H, G) \in \mathcal{F}$, then Σ_{gc} can be divided into 5 types.

Type 1: A point $(\bar{x}, \bar{t}) \in \Sigma_{\text{gc}}$ is of Type 1 (non-degenerate critical point), i.e., $(\bar{x}, \bar{t}) \in \Sigma^1_{\text{gc}}$, if the following conditions are satisfied:
There exist $\bar{\lambda}_i, \bar{\mu}_j \in I\!R$, $i \in I$, $j \in J_0(\bar{x}, \bar{t})$ with

$$\Big(D_x f + \sum_{i \in I} \bar{\lambda}_i D_x h_i + \sum_{j \in J_0(\bar{x}, \bar{t})} \bar{\mu}_j D_x g_j\Big)|_{(x,t)=(\bar{x},\bar{t})} = 0,$$

$$\text{the LICQ is satisfied at } \bar{x} \in \mathbf{M}(\bar{t}), \tag{2.3a}$$

(therefore $\bar{\lambda}_i$, $\bar{\mu}_j$, $i \in I$, $j \in J_0(\bar{x}, \bar{t})$ are uniquely defined)

$$\bar{\mu}_j \neq 0, \quad j \in J_0(\bar{x}, \bar{t}), \tag{2.3b}$$

$$D_x^2 L(\bar{x}, \bar{t})|_{T(\bar{x}, \bar{t})} \text{ is nonsingular,} \tag{2.3c}$$

where $D_x^2 L$ is the Hessian of the Lagrangian

$$L(x,t) = f(x,t) + \sum_{i \in I} \bar{\lambda}_i h_i(x,t) + \sum_{j \in J_0(\bar{x}, \bar{t})} \bar{\mu}_j g_j(x,t),$$

and the uniquely determined numbers $\bar{\lambda}_i, \bar{\mu}_j$ are taken from (2.3a) and

$$T(x,t) = \{\xi \in I\!R^n \mid D_x h_i(x,t)\xi = 0, i \in I, D_x g_j(x,t)\xi = 0, j \in J_0(x,t)\}$$

[1] We consider all gradients as a row vector.

is the tangent space at (x,t). $D_x^2 L(x,t)|_{T(x,t)}$ represents $V^T D_x^2 L V$, where V is a matrix whose columns form a basis of $T(x,t)$.
The points of the Types 2–5 represent four basic degeneracies (for details of the definition we refer to [16] - [18]):

Type 2 – violation of (2.3b),
Type 3 – violation of (2.3c),
Type 4 – violation of (2.3a) and $|I| + |J_0(\bar{x},\bar{t})| - 1 < n$,
Type 5 – violation of (2.3a) and $|I| + |J_0(\bar{x},\bar{t})| = n + 1$.

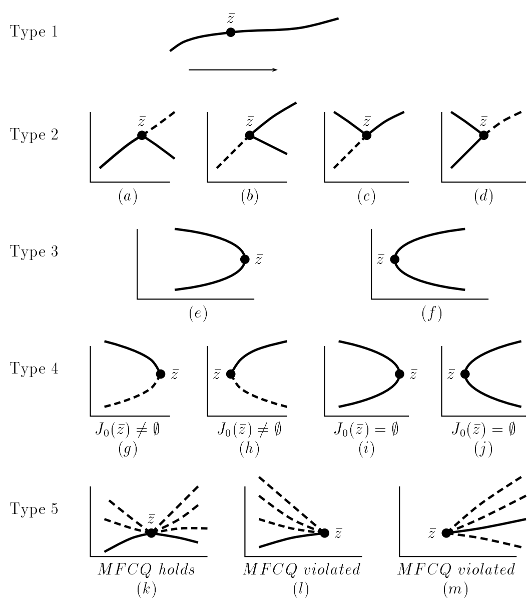

Figure 2.1: The full curve stands for the curve of stationary points $z = (x,t)$, and the dotted curve represents the curve of g.c. points that are not stationary points. For each of these five types Figure 2.1 illustrates the local structure.

For each of these five types Figure 2.2 illustrates the local structure of $\Sigma_{\rm gc}$ in the neighbourhood of stationary points.

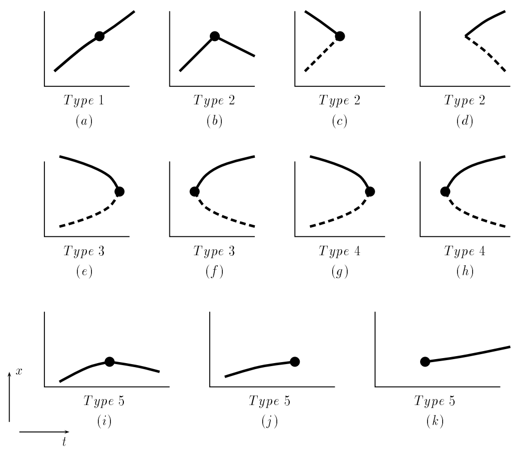

Figure 2.2: The full curve stands for a curve of local minimizers and the dotted curve in (c), (d), (e), (f) represents a curve of stationary points not being local minimizers. The dotted curve in (g), (h) stands for a curve of stationary points in case of $J_0(\bar{x},\bar{t}) = \emptyset$.

Remark 2.1 In Chapter 4 we need a complete description of a point of Type 4 and 5. Let $J_0(\bar{x},\bar{t}) = \{1,\ldots,p\}$ (w.l.o.g.).

$(\bar{x},\bar{t})$ is a point of Type 4 if the following conditions are satisfied:

a) $1 \le m+p \le n$ and it holds that

$$\text{rank} \begin{pmatrix} D_x h_1(\bar{x},\bar{t}) \\ \vdots \\ D_x h_m(\bar{x},\bar{t}) \\ D_x g_1(\bar{x},\bar{t}) \\ \vdots \\ D_x g_p(\bar{x},\bar{t}) \end{pmatrix} = m+p-1.$$

b) $\bar{q}_{\rm m+j} \neq 0$ for all $j \in \{1,\ldots,p\}$, where $\bar{q} \in I\!R^{m+p}$ is fixed and defined in

$$\sum_{i\in I} \bar{q}_i D_x h_i(\bar{x},\bar{t}) + \sum_{\rm j=1}^{p} \bar{q}_{\rm m+j} D_x g_j(\bar{x},\bar{t}) = 0, \qquad \bar{q} \neq 0.$$

c) $(\bar{x}, \bar{q}_1, ..., \bar{q}_{m+p-1}, \bar{t}, 0) \in I\!R^{n+m+p+1}$ is a non-degenerate critical point of the problem

$$(\hat{P}) \qquad min\{\hat{\mathcal{F}}(x,q,t,q_0) | \hat{\mathcal{G}}(x,q,t,q_0) = 0\},$$

where

$$\hat{\mathcal{F}}(x,q,t,q_0) = t, \qquad \hat{\mathcal{G}}(x,q,t,q_0) = \begin{pmatrix} D_x\mathcal{L}(x,q,t,q_0) \\ h_1(x,t) \\ \vdots \\ h_m(x,t) \\ g_1(x,t) \\ \vdots \\ g_p(x,t) \end{pmatrix},$$

and $\mathcal{L}(x,q,t,q_0) = q_0 f(x,t) - \sum_{i\in I} q_i h_i(x,t) - \sum_{j=1}^{p-1} q_{m+j} g_j(x,t) - \bar{q}_{m+p} g_p(x,t)$.

$(\bar{x}, \bar{t})$ is a point of Type 5 if the following three conditions hold:

(5a) $m + p = n + 1$ and
$$\text{rank }(Dh_1(\bar{x},\bar{t}), \ldots, Dh_m(\bar{x},\bar{t}), Dg_1(\bar{x},\bar{t}), \ldots, Dg_p(\bar{x},\bar{t})) = n + 1.$$

(5b) $\bar{q}_{m+j} \neq 0$ for each $j \in \{1, \ldots, p\}$, where $\bar{q}$ is fixed and defined as in (2.2.4).

(5c) For each $(\tilde{\lambda}, \tilde{\mu}) \in I\!R^m \times I\!R^p$ with
$$D_x f(\bar{x},\bar{t}) + \sum_{j\in I} \tilde{\lambda} D_x h_i(\bar{x},\bar{t}) + \sum_{j=1}^{p} \tilde{\mu} Dx g_j(\bar{x},\bar{t}) = 0$$

it holds that $|J_0(\bar{x},\bar{t}) \backslash J_+(\bar{\mu}| \leq 1$,
where
$J_+(\bar{\mu}) := \{j \subset \{1, \ldots, p\} | \bar{\mu}_j \neq 0\}$.

There are two theorems justifying that (f, H, G) belongs to the class $\mathcal{F}$ of Jongen, Jonker and Twilt.

Theorem 2.2 (Genericity Theorem, cf. [18]). *Let $(f, H, G) \in C^3(I\!R^n \times I\!R, I\!R^{1+m+s})$. The class $\mathcal{F}$ is C^3_s-open and C^3_s-dense in $C^3(I\!R^n \times I\!R, I\!R)^{1+m+s}$ where C^3_s denotes the strong (or Whitney-) C^3_s-topology.*

The following theorem provides a special perturbation of (f, H, G) with additional parameters, which can be chosen such that the perturbed function vector belongs to the class $\mathcal{F}$. Let the space of symmetric $n \times n$-matrices be identified by $I\!R^{n(n+1)/2}$. Let Σ^ν_{gc}, $\nu \in \{1, \ldots, 5\}$ be the set of g.c. points of Type ν. The class $\mathcal{F}$ is defined by

$$\mathcal{F} = \Big\{(f, H, G) \in C^3(I\!R^n \times I\!R, I\!R)^{1+m+s} \mid \Sigma_{\text{gc}} \subset \bigcup_{\nu=1}^{5} \Sigma^\nu_{\text{gc}}\Big\}.$$

Theorem 2.3 (Perturbation Theorem, cf. [25]). *Let* $(f, H, G) \in C^3(I\!R^n \times R, R^{1+m+s})$. *Then, for almost all* $(b, A, c, D, e, F) \in I\!R^n \times I\!R^{n(n+1)/2} \times I\!R^m \times I\!R^{mn} \times I\!R^s \times I\!R^{sn}$, *we have*

$$(f(x,t) + b^T x + x^T Ax, H(x,t) + c + Dx, G(x,t) + e + Fx) \in \mathcal{F}.$$

Here "almost all" means: Each measurable subset of

$$\{(b, A, c, D, e, F) \mid (f(x,t) + b^T x + x^T Ax, H(x,t) + c + Dx, G(x,t) + e + Fx) \notin \mathcal{F}\}$$

has the Lebesgue-measure zero. □

Definition 2.4 *Let* $K \subseteq I\!R \cup \{\pm\infty\}$. *The problem* $P(t)$ *is called regular in the sense of Jongen-Jonker-Twilt (briefly JJT-regular) with respect to* K *if* $(f, H, G) \in \mathcal{F}\Big|_K \Big((I\!R^n \times K) \cap \Sigma_{\rm gc} \subseteq \bigcup_{\nu=1}^{5} \Sigma_{\rm gc}^{\nu}\Big)$.

Now, we present a theorem that is essential for our analysis.

Theorem 2.5 (follows from [14]). *We assume that*

(C1) $M(t)$ *is non-empty and there exists a compact set* C *with* $M(t) \subseteq C$ *for all* $t \in [0,1]$;

(C2) $P(t)$ *is JJT-regular with respect to* $[0,1]$;

(C3) *there exists a* $t_1 > 0$ *and a continuous function* $x : [0, t_1) \to I\!R^n$ *such that* $x(t)$ *is the unique stationary point for* $P(t)$ *for* $t \in [0, t_1)$;

(C4) *the MFCQ is satisfied for all* $x \in M(t)$ *for all* $t \in [0,1]$.

Then there exists a PC^2*-path in* $\Sigma_{\rm stat}$ *that connects* $(x^0, 0)$ *with some point* $(x^*, 1)$.

On the program package PAFO

We assume that (C1) and (C2) are satisfied.
PAFO is based on a pathfollowing method (called PATH III in 4.5 [17]) and jumps (called JUMP I in Chapter 5.2 [17] and JUMP II in Chapter 5.3 [17]).

Remark 2.6 *(i) Pathfollowing methods are also called homotopy- and continuation methods in the literature. The great amount of publications shows the international acceptance of this procedure not only for complementarity problems. There is also much numerical experience with such kind of methods. We refer to these statements: e.g. to [2], [4], [5], [10], [24], [27], [30], [31], [32].*

(ii) PAFO is the only program package that works in the class $\mathcal{F}$ *of Jongen, Jonker and Twilt, i.e., the types of singularities described above are admitted.*

We explain the main ideas of PATH III, but not those of JUMP I, II as we do not use them here.

PATH III

This algorithm computes a numerical description of a compact connected component in $\sum_{gc}$, i.e., in particular it finds a finite discretization of an interval $[t_A, t_B]$, $t_A < 0 < t_B$ (not necessarily $[t_A, t_B] \supset [0,1]$), and corresponding g.c. points starting at $(x^0, 0) \in \sum_{gc}$. The algorithm is based on the active index set strategy and is a so-called predictor-corrector scheme (we refer e.g. to [2],[24]) if the active index set is constant. A Newton-like corrector is used.
We note that we do not have any numerical difficulties walking around turning points of the Types 3 or 4. The main point of the approach consists in the computation of the new index sets for the possible continuations at points of Type 2 and 5. This is easily done without any numerical problems.

Remark 2.7 *If there exists a PC^2-path connecting $(x^0, 0)$ and a point $(x^*, 1)$, PAFO constructs a finite number of predictor steps in [0,1], i.e., a discretization*

$$0 = t_0 \leq ... \leq t_i \leq t_{i+1} \leq ... \leq t_N = 1$$

and, by corrector steps using Newton-like methods, corresponding approximations $\tilde{x}(t_i)$ of stationary points $x(t_i), i = 1, ..., N$, where the rate of convergence will be at least superlinear and the points $\tilde{x}(t_i)$ will be obtained by a finite number of Newton-like steps. This procedure is numerically stable.

3 Properties of the modified standard embedding

We consider the problem (P^L) (cf. (1.3)) and the corresponding modified standard embedding $P^s(t), t \in [0,1]$ (cf. (1.7),(1.8)).

Theorem 3.1 *Let (A1) and (A2) be satisfied. Then we have the following properties for $P^s(t)$:*

(i) If we choose the matrix A to be positive definite, then x^0 is a global minimizer, the unique stationary point for $P^s(0)$. Furthermore, x^0 is a nondegenerate critical point for $P^s(0)$.

(ii) $M^s(t)$ is non-empty for all $t \in [0,1]$.

(iii) $P^s(1) = (P^L)$.

We introduce the following assumptions:

(A3) The MFCQ is satisfied for all $x \in M^s(t)$ and all $t \in [0,1)$, (A4) $P^s(t)$ is JJT-regular with respect to $[0,1]$.

Remark 3.2 *We have to take into account that the MFCQ can be violated at points of $M^s(1) = M^L \cap E(p)$ because these points are points of Type 5.*

Using Theorem 2.4 we obtain

Theorem 3.3 *Let (A1), (A2), (A3) and (A4) be satisfied. Then there exists a PC^2-path in Σ_{stat} that connects $(x^0, 0)$ and some point $(\hat{x}, \hat{t})$ for all $\hat{t} \in (0,1)$, and only points of Type 1, 2 and 3 may appear.*

Remark 3.4 *Since the point-to-set mapping $t \to M^s(t)$ is closed at $t = 1$ (cf. e.g. [3]) and $M^s(t) \subseteq E(p)$ for $t \in [0,1]$, there exists a sequence $\{(x^k, t_k)\}$ with $x^k \in M^s(t_k)$ that converges to a point $(x^*, 1)$. From this point of view we are successful.*

4 A justification theorem for the JJT-regularity

We ask whether we can justify the very important assumption (A4).
For the general one-parametric optimization problem $P(t)$ (cf. (2.1)) we refer to the perturbation theorem (Theorem 2.2). We have to note that, from Theorem 2.2, we cannot directly derive a perturbation theorem for the special one-parametric optimization problem $P^s(t)$ (cf. (1.5)). Theoretically, for $P^s(t)$, other singularities than those we know in the class $\mathcal{F}$ could appear. From this point of view, we consider the vector $\mathcal{D} := (A, x^0, B, q, w^0)$, where $A \in I\!R^{\frac{1}{2}n(n+1)}, x^0 \in I\!R^n, B \in I\!R^{\frac{1}{2}n(n+1)}, q \in I\!R^n, w^0 \in I\!R^{n+1}$.
We consider the following perturbed embedding

$$\begin{aligned} P^s_{\mathcal{D}}(t): \quad \min\{(x-x^0)^T A(x-x^0) | t(-x^T Bx - q^T x) + (1-t)w^0_0 \geq 0, \\ t(b^j x - q_j) + (1-t)w^0_j \geq 0, \quad j \in J, \\ x_j \geq 0, \quad j \in J, \\ p - \|x\|^2 \geq 0\}, \quad t \in [0,1], \end{aligned}$$

where A is a symmetric regular matrix, $w^0_i > 0, i = 0, 1..., n \|x^0\|^2 < p$, and b^j denotes the j-th row of B.

Theorem 4.1 *For almost all $\mathcal{D}$ the problem $P^s_{\mathcal{D}}(t)$ is JJT-regular with respect to $[0,1]$.*

Proof: We have to prove that, for almost all $\mathcal{D} = (A, x^0, \mathcal{B})$ with $\mathcal{B} := (B, q, w^0)$, each g.c. point of $P^s_{\mathcal{D}}(t)$ is one of the five types of the class $\mathcal{F}$. Now we introduce the following notations: $J_0 := J_0(x, t)$ active index set at (x, t),
$\tilde{J} = J_1 \cup J_2$ with $J_1 = \{j \in \{1, \ldots, n\} | g_j(x, t) = 0\}$ and $J_2 = \{j \in \{1, \ldots, n\} | x_j = 0\}$, where

$$g_0(x, t) := t(-x^T Bx - q^T x) + (1 - t)w^0_0,$$
$$g_j(x, t) := t(b^j x - q_j) + (1 - t)w^0_j, j \in J,$$
$$h_j(x) := x_j, j \in J,$$
$$h_{n+1}(x) := p - \|x\|^2.$$

We consider $\mathcal{D}$, $P_{\mathcal{D}}(t)$, a g.c. point (x, t) and the associated multipliers $\mu = (\lambda, \mu^1, \mu^2, \mu^c)$, where λ is the multiplier of the complementarity constraint, μ^1, μ^2 and μ^c the multiplier-vectors associated with J_1, J_2, and the compactification constraint, respectively. Then we distinguish two cases:

Case I: The LICQ is satisfied at the g.c. point (x, t).

Case II: The LICQ is not satisfied at the g.c. point (x, t).

<u>CASE I</u> In this case the corresponding Lagrange multipliers $\mu_j, j \in J_0$, are uniquely determined. We introduce the following set

$$J' := J_0 \cap \{j \mid \mu_j = 0\}.$$

Then the set of g.c. points is described as a union of sets satisfying the following systems

$$\begin{aligned} H(x, \mu, t) &= 0, && (4.1)\\ M(x, \mu, t) &= \Omega, && (4.2)\\ \Omega_1 &= \Omega_2 \Omega_4^{-1} \Omega_2^T, && (4.3)\\ \mu_j &= 0, j \in J' \subseteq J_0(x). && (4.4) \end{aligned}$$

$H(x, \mu, t) = D_{x,\mu} L(x, \mu, t) = 0$ corresponds to the definition of a critical point:

$$H(x, \mu, t) = \begin{pmatrix} 2A(x - x^0) + \lambda t[(B + B^T)x + q] + t\mu^1 B_1^T + \mu^2 I_2^T + 2\mu^c x \\ t(x^T Bx + q^T x) + (1 - t)w^0_0 \text{ if } 0 \in J_0(x, t) \\ t(b^j x + q_j) + (1 - t)w^0_j;\ j \in J_1 \subseteq J_0(x, t) \\ x_j, j \in\ J_2 \subseteq J_0(x, t) \\ \|x\|^2 - p \text{ if } \|x\|^2 = p \end{pmatrix},$$

where $B_1(I_2)$ are the rows of B (identity matrix I) corresponding to the index sets $I_1(I_2)$.

Let $M(x,t)$ be the Jacobian (see (4.2)) of $H(x,t)$:

$$M(x,\mu,t) = \begin{pmatrix} 2A+\lambda t(B+B^T)+2\mu^c I & t[(B+B^T)x+q] & tB_1^T & I_2^T & 2x \\ t[(B+B^T)x+q]^T & 0 & 0 & 0 & 0 \\ tB_1 & 0 & 0 & 0 & 0 \\ I_2 & 0 & 0 & 0 & 0 \\ 2x^T & 0 & 0 & 0 & 0 \end{pmatrix},$$

If the last (and/or first) constraint is not active, corresponding rows and columns of M are eliminated.
By $\hat{I}$ we denote the maximal set of independent columns of $M(x,t)$. Ω belongs to the $(n+|J_0|-|\hat{I}|)(n+|I_0|-|\hat{I}|+1)/2$ dimensional manifold described by (4.3), where

$$\Omega := \begin{pmatrix} \Omega_1 & \Omega_2 \\ \Omega_2^T & \Omega_4 \end{pmatrix},$$

and Ω_4 is symmetric, non-singular and has the rank of Ω.

We construct now the Jacobian of the system (4.1)-(4.4) with respect to $x, \mu, A, x^0, t, w^0 = (w_1^0, ..., w_n^0)^T, w_0^0$

$$\begin{array}{ccccccccccc} \partial_x & \partial_\lambda & \partial_{\mu^1} & \partial_{\mu^2} & \partial_{\mu^c} & \partial_A & \partial_\Omega & \partial_{x^0} & \partial_t & \partial_{w^0} & \partial_{w_0^0} \end{array}$$
$$\begin{pmatrix} \begin{smallmatrix}2A+\lambda \\ t(B+B^T)\end{smallmatrix} & \begin{smallmatrix}t[(B+B^T) \\ x+q]\end{smallmatrix} & tB_1^T & I_2^T & 2x & \otimes & 0 & -2A & \otimes & \otimes & 0 \\ \begin{smallmatrix}t[(B+B^T) \\ x+q]^T\end{smallmatrix} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \otimes & 0 & 1-t \\ tB_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & (t-1)I_1 & 0 \\ I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2x^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \otimes & 0 & 0 & 0 & 0 & \left[I_{\frac{n(n+1)}{2}}\right] & -I_* & 0 & \otimes & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & [I_{**}|\otimes] & 0 & 0 & 0 & 0 \\ 0 & & I_{J'} & & & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where
$I_*(I_{**})$ is the $N = \frac{(n+|I_0|+1)(n+|J_0|)}{2}$ identity matrix ($N' = \frac{n+|J_0|-|\hat{I}|)(n-|J_0|-|\hat{I}|+1)}{2}$ identity matrix) and I_1 denotes the rows of I corresponding to the index set I_1.
Expressions that do not play any role for the analysis we denote by $\otimes$.
We note that a linear combination of the rows of the matrix above, which gives the null vector, has coefficients corresponding to the first, second and third block equal to zero (because of the columns $\partial_{x^0}, \partial_{w_0^0}, \partial_{w^0}$, respectively). The relation between the structure of M and Ω implies that the coefficients corresponding to the sixth and seventh block

are also zero and, finally, the gradient vectors of the non-negativity and compactification constraints are linearly independent. Then the matrix has full rank.
Using Sard's Lemma, we see that the rows of the sub-matrix corresponding to $\partial_x, \partial_\mu, \partial_\Omega, \partial_t$ are linearly independent for almost all (A, x^0, w^0, w_0^0). Furthermore, the number of rows is less than or equal to $n + |J_0| + N + 1$. Therefore $N' + |J'| \leq 1$ and only three cases may occur:
$N' = 0 \quad |J'| = 0,$
$N' = 0 \quad |J'| = 1,$
$N' = 1 \quad |J|' = 0.$
They correspond to the points of Type 1, 2 and 3, respectively.
CASE II It is necessary to prove:
4a) For almost all $\mathcal{B}, M(\mathcal{B})$ is the union of a finite set of zero dimensional manifolds.
4b) Let (x, t) be a g.c. point. Then, for almost all $\mathcal{B}$, the set $\{D_{xt}g_j(x, t), j \in J_0(x, t)\}$ is linearly independent.
4c) For almost all $\mathcal{B}$ the Lagrange multipliers corresponding to the g.c. point (x, t) are non-zero.
In addition, let $J^* \subseteq J_0$ and S be the subspace generated by the gradient vectors D_x of the constraints corresponding to J^*.
4d) If S has a dimension less than or equal to $n-1$, then the gradient vector $2A(x-x^0)$ of the objective function does not belong to the subspace S for almost all (x^0, A).
Under these conditions we prove that the set $(A, x^0, \mathcal{B})$, where (x, t) is not a point of Type 4 or 5, has the Lebesque measure zero. Then the statements follows from Fubini's Theorem.
Now we prove 4a), 4b), 4c) and 4d).
We will consider all possible sets of indices of active constraints. We fix one of them and assume that the quadratic and the compactification constraints and some of the linear and nonnegativity constraints are active. If some of them are not active, the proof is analogous.
Let us consider a point (x, t), where the LICQ does not hold, and the associated multipliers $(\lambda, \mu^1, \mu^2, \mu^c)$, which describe the linear dependence. λ is the multiplier associated with the complementarity constraint, μ^1 the vector of multipliers of the inequalities in J_1, μ^2 that for the inequalities in J_2, and μ^c that for the compactification constraint. Then we obtain the following system:

$$
\begin{aligned}
t\lambda[(B + B^T)x + q] + t\mu^1 B_1 + \mu^2 I_2 + 2\mu^c x &= 0, \\
-t[x^T Bx + q^T x] + (1 - t)w_0^0 &= 0, \\
t(b^j x + q_j) + (1 - t)w_j^0 &= 0, \; j \in J_1, \\
x_j &= 0, \; j \in J_2, \\
\|x\|^2 &= p.
\end{aligned}
\tag{4.5}
$$

Since the gradient vectors of the non-negativity constraints and of the compactification constraint are linearly independent, either $\lambda \neq 0$ or $\mu^1 \neq 0$ holds.

If $\lambda = 1$, then the Jacobian with respect to (x, t), the multipliers, w_0^0, w^0, B, and q of the above system have the structure:

$$\begin{array}{ccccccccc} \partial_x & \partial_{\mu^1} & \partial_{\mu^2} & \partial_{\mu^c} & \partial_{w_0^0} & \partial_{w^0} & \partial_q & \partial_B & \partial_t \end{array}$$
$$\begin{pmatrix} \otimes & \otimes & I_2^T & 2x & 0 & 0 & tI & \otimes & \otimes \\ \otimes & 0 & 0 & 0 & 1-t & 0 & -tx & \otimes & \otimes \\ \otimes & 0 & 0 & 0 & 0 & (1-t)I_1 & tI_1 & \otimes & \otimes \\ I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2x^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

If $\lambda = 0$, let μ_p^1 be such that $\mu_p^1 = 1$, then the Jacobian of the system reads as follows:

$$\begin{array}{cccccccccc} \partial_x & \partial_\lambda & \partial_{\mu^1 \setminus \{p\}} & \partial_{\mu^2} & \partial_{\mu^c} & \partial_{w_0^0} & \partial_{w^0} & \partial_q & \partial_B & \partial_t \end{array}$$
$$\begin{pmatrix} \otimes & \otimes & \otimes & I_2^T & 2x & 0 & 0 & 0 & \otimes|\mu_p^1 I|\otimes & \otimes \\ \otimes & 0 & 0 & 0 & 0 & (1-t) & 0 & \otimes & \otimes & \otimes \\ \otimes & 0 & 0 & 0 & 0 & 0 & (1-t)I_1 & (1-t)I_1 & \otimes & \otimes \\ I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2x^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In both cases the matrices have full rank, because the gradients of the active non-negative constraints and the compactification constraint are linearly independent.
Sard's Lemma implies that, given a set of active constraints, the sub-matrix given by column blocks $\partial_x, \partial_\lambda, \partial_{\mu^1}, \partial_{\mu^2}, \partial_{\mu^c}, \partial_t$ has full rank for almost all w_0^0, w^0, B and $n + 1 + |\tilde{J}| - 1 = n + |J_0|$. Then the dimension of the set described by the system is 0.
4b) is a consequence of the previous analysis, considering the rows corresponding to the gradients of the constraints with respect to (x, t).
For proving 4c):
We consider the above system under additional conditions:
$\mu_j = 0, j \in J'$.
The Jacobian of the new system has now an additional block of rows:

$$\begin{array}{ccccccc} \partial_x & \partial_\mu & \partial_{w_0^0} & \partial_{w^0} & \partial_B & \partial_q & \partial_t \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & I_{J'} & 0 & 0 & 0 & 0 & 0 \end{array},$$

where $I_{J'}$ is a $|J'| \times |J_o|$-matrix whose (i, j) element is 1 if $i \in J', j = i$, and 0 otherwise. By the same arguments, the submatrix $\partial_x \partial_t \partial_\mu$ has full rank by the rows for almost all B, w_0^0, w^0, q.
Since the dimension of the space is $n + |J_0|$, it holds that

$n + |J_0| + |J'| \leq n + |J_0|$. Then we have $|J'| = 0$.

We have discussed properties related to the feasible set of the constraints. Before proving a property related to the objective function, we note that the following property of $M(t)$ is an immediate consequence of the above analysis:

Remark 4.2 *For any $t \in [0,1)$ and for almost all w_0^0 and w^0, at most $n+1$ constraints of the parametric problem $P_D^s(t)$ can be active at a feasible point.*

For proving 4d) we fix the g.c. point (there is a finite number of candidates):
Let $J^* \subseteq J_0$ be such that J^* generates S. We look for the solvability of the following system $S(\mu)$:

$$2A(x - x^0) + t\lambda^*[(B + B^T)x + q] + t\sum_{j\in J_1^*} \mu_j^{1*} b^j + \sum_{j\in J_2^*} \mu_j^{2*} e_j + 2\mu^{c*} x = 0,$$

where $J_i^* = J^* \cap J_i, i = 1, 2$.
The Jacobian with respect to x, the multipliers, A and x^0 is:

$$\begin{array}{cccccc} \partial_{\lambda^*} & \partial_{\mu^{1*}} & \partial_{\mu^{2*}} & \partial_{\mu^{c*}} & \partial_A & \partial_{x^0} \\ \otimes & \otimes & \otimes & \otimes & \otimes & -2A. \end{array}$$

Since A is regular, the last block of this matrix has rank n. So, using Sard's Lemma, the sub-matrix corresponding to $\lambda, \mu^{1^*}, \mu^{2^*}, \mu^{c^*}$ has full rank n for almost all x^0, which contradicts the assumption that S has a dimension less than n. Therefore, d) holds.
Due to Remark 4.2 we consider two possibilities:

i) $|J_0(x,t)| \leq n$,

ii) $|J_0(x,t)| = n + 1$.

From 4b) it follows that, in the first case, (x,t) satisfies the condition a) of a point of Type 4 (cf. Chapter 2). We check the conditions b), c) and d) of Type 4. The property c) implies condition b) for almost all B, w^0, w_0^0, q.
For proving c) we show that (x,t) is a critical point of $(\hat{P})$. The LICQ does not hold at (x,t), but the property 4b) implies that the gradients $D_{x,t}$ of the active constraints at (x,t) are linearly independent, then it holds that $\sum_{j\in J_0} \mu_j D_x \tilde{g}_j(x,t) = 0$, $\sum_{j\in J_0} \mu_j D_t \tilde{g}_j(x,t) \neq 0$,
where all coefficients are non-zero. Here, we denote by $\tilde{g}_j(x,t), j \in J_0$, all functions for the active constraints at (x,t) in $\tilde{P}^s(t)$ (cf. (1.7) and (1.8)). Without loss of generality we assume that $\sum_{j\in J_0} \mu_j D_t \tilde{g}_j(x,t) = 1$. The gradients of the active constraints form a submatrix of M with rank $n + |J_0|$ for almost all B. Hence, the LICQ is satisfied at (x,t). In addition the Hessian of the Lagrangian on the tangent space of the active constraints at (x,t) is nonsingular.
For proving it we note that the subspace $\hat{S}$ generated by the active constraints has the

dimension $n + |J_0|$ for almost all B, w^0, w_0^0 and q. Hence the orthogonal subspace of $\hat{S}$ $(\hat{S}^{\perp})$ has the dimension 1. Let u be a basis of $\hat{S}^{\perp}$ and using the same ideas of the proof of Theorem 6.18 in [16], we can prove that $u^T D_\mu^2 D u \neq 0$.
In the second case, $|J_0(x,t)| = n+1$, the conditions of the g.c. point of Type 5 follow immediately from the properties 4a), 4b), 4c) and 4d).
The theorem is proved. □

5 Illustrative Examples

EXAMPLE 1. We consider the (LCP) defined by

$$B = \begin{pmatrix} -4 & 1 & 1 \\ 2 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ -6 \\ -4 \end{pmatrix}.$$

B is an indefinite matrix. We have chosen $A = I_n$, the starting point $x^0 = \begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \end{pmatrix}$ and $p = 130$.

Passing 3 singularities of Type 2, we attain $t = 1$ at a point of Type 5, which is the solution $x^* = \begin{pmatrix} 0.68183 \\ 0.96969 \\ 0.75758 \end{pmatrix}$ of the (LCP):

	t	x_1	x_2	x_3
NEWS	0.00000	0.10000	0.10000	0.10000
TYPE 2	0.15875	0.10000	0.10000	0.10000
TYPE 2	0.29483	0.37697	0.65395	0.23849
TYPE 2	0.88693	0.51605	1.03258	0.70995
TYPE 5	1.00000	0.68183	0.96969	0.75758

In order to save space we show only Figure 5.1 with respect to x_1.

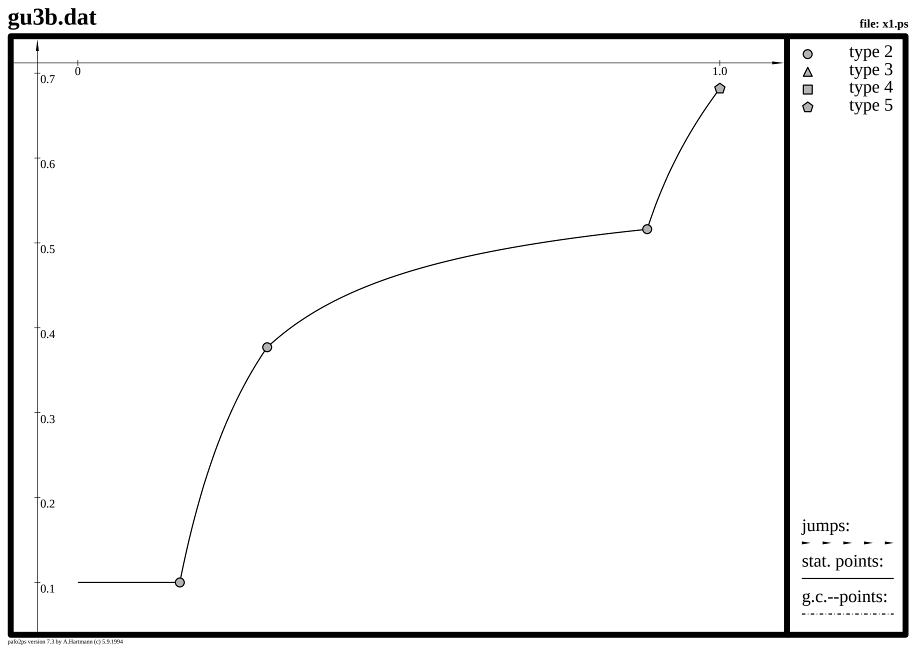

Figure 5.1

Example 2. We consider the (LCP) defined by

$$B = \begin{pmatrix} 0 & 2 & -3 & -2 \\ -2 & 0 & 1 & 2 \\ 3 & -1 & 0 & 4 \\ 2 & -2 & -4 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 9 \\ -5 \\ -9 \\ 14 \end{pmatrix}.$$

We note that B is an antisymmetric indefinite matrix. We choose $A = I_n$, the starting point $x^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $p = 100$. Passing 4 singularities of Type 2 we attain $t = 1$ in a singularity of Type 5. This is a solution $x^* = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix}$ of the (LCP):

	t	x_1	x_2	x_3	x_4
NEWS	0.00000	1.00000	1.00000	1.00000	1.00000
TYPE 2	0.10000	1.00000	1.00000	1.00000	1.00000
TYPE 2	0.19529	0.88534	1.06370	1.11466	0.82164
TYPE 2	0.20213	0.91570	1.07146	1.19511	0.84424
TYPE 2	0.63637	0.92857	1.21429	2.85715	1.71429
TYPE 5	1.00000	1.00000	2.00000	3.00000	2.00000

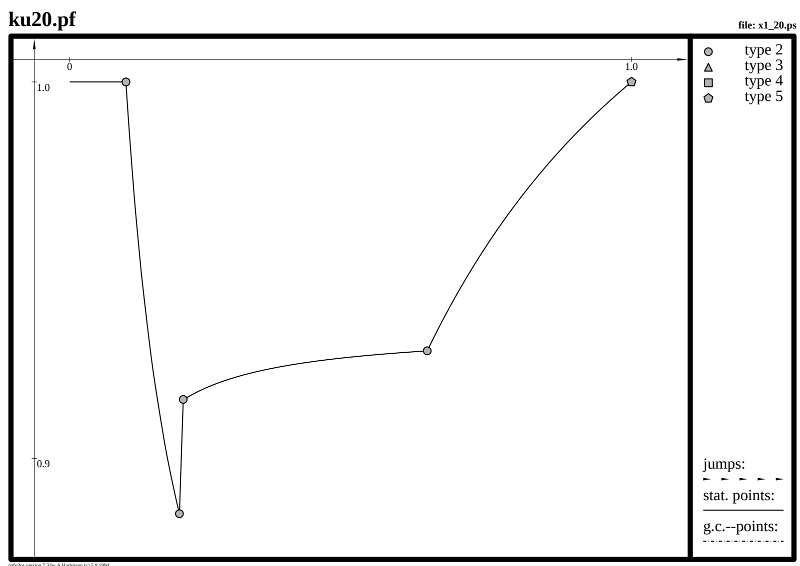

Figure 5.2

Figure 5.2 shows the curves of stationary points connecting x^0 at $t = 0$ with the solution x^* at $t = 1$ with respect to x_1.

EXAMPLE 3. We consider the (LCP) with

$$B = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 1 & -1 \\ 3 & -2 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix},$$

where B is indefinite. If we choose $A = I_n,\ p = 100$, and the starting point

$$x^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

then we attain the solution $x^* = \begin{pmatrix} 1.42855 \\ 0.85709 \\ 0.00000 \end{pmatrix}$ at $t = 1$, passing 3 singularities of Type 2:

	t	x_1	x_2	x_3
NEWS	0.00000	1.00000	1.00000	1.00000
TYPE 2	0.20001	1.00000	1.00000	1.00000
TYPE 2	0.55956	0.95655	0.60270	0.44809
TYPE 2	0.90401	1.40657	0.79854	0.09164
TYPE 5	1.00000	1.42855	0.85709	0.00000

Furthermore, beginning at the first singularity of Type 2, we follow g. c. points and, at $t = 1$, we obtain a further solution $x^{**} = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}$ of the (LCP). On this path we also have singularities of the Types 3, 4, 5:

	t	x_1	x_2	x_3
NEWP	0.20001	1.00000	1.00000	1.00000
TYPE 2	0.01060	3.91175	0.00000	4.44276
TYPE 2	0.00356	6.77931	0.00000	7.35126
TYPE 4	0.00356	6.74284	0.00000	7.38472
TYPE 5	0.00909	0.00000	0.00000	10.00000
TYPE 2	0.12603	0.00000	0.00000	2.18041
TYPE 3	0.32143	0.00000	0.99945	1.33333
TYPE 4	0.31250	0.00000	1.39988	1.60000
TYPE 2	0.44974	0.00000	2.40026	2.57681
TYPE 5	1.00000	0.00000	3.00000	5.00000

The above table illustrates that the assumption (A3) is not satisfied. Figure 5.3 shows these curves with respect to x_1, but we are also successful.

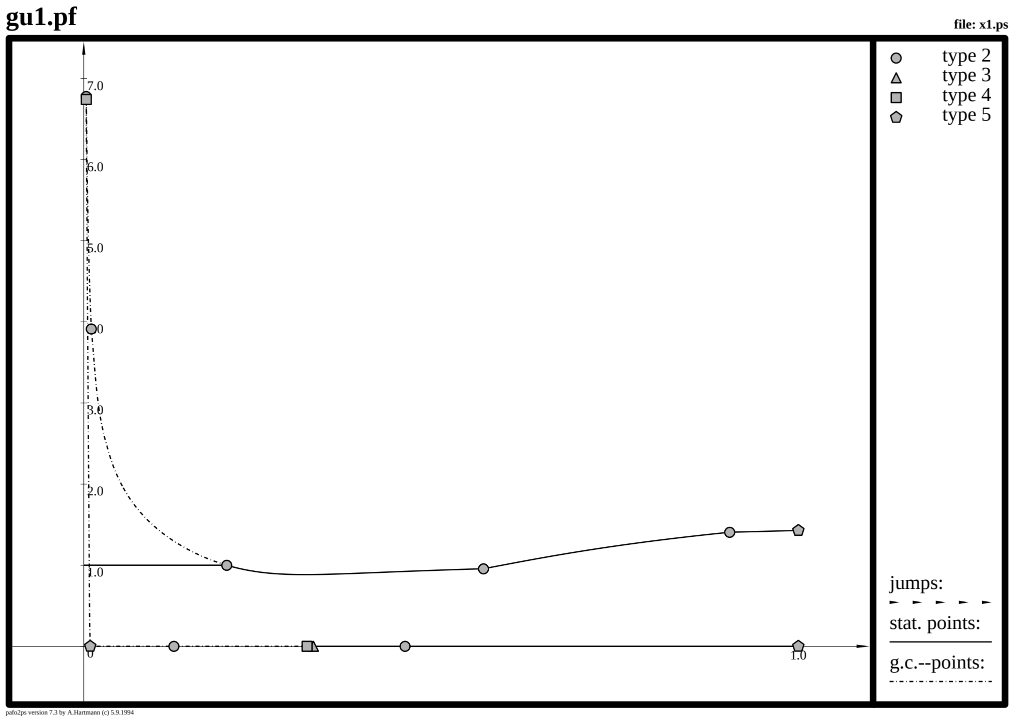
gu1.pf
file: x1.ps
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
1.0
type 2
type 3
type 4
type 5
jumps:
stat. points:
g.c.--points:
pafo2ps version 7.3 by A.Hartmann (c) 5.9.1994

Figure 5.3

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