ON HALANAY-TYPE ANALYSIS OF EXPONENTIAL STABILITY FOR THE θ–MARUYAMA METHOD FOR STOCHASTIC DELAY DIFFERENTIAL EQUATIONS.

Dedicated to Donald Kershaw, Reader Emeritus, Lancaster University

Christopher T.H. Baker*
Mathematics Department, University College Chester,
Chester, CH1 4BJ, UK
cthbaker@na-net.ornl.gov
Evelyn Buckwar†
Institut für Mathematik, Humboldt Universität zu Berlin, Unter den Linden 6,
10099 Berlin, Germany
buckwar@mathematik.hu-berlin.de

Abstract

Using an approach that has its origins in work of Halanay, we consider stability in mean square of numerical solutions obtained from the θ–*Maruyama discretization* of a test stochastic delay differential equation

$$dX(t) = \{f(t) - \alpha X(t) + \beta X(t-\tau)\}\mathrm{d}t + \{g(t) + \eta\, X(t) + \mu X(t-\tau)\}\, \mathrm{d}W(t),$$

interpreted in the Itô sense, where $W(t)$ denotes a Wiener process. We focus on demonstrating that we may use techniques advanced in a recent report by Baker and Buckwar to obtain criteria for asymptotic and exponential stability, in mean square, for the solutions of the recurrence

$$\widetilde{X}_{n+1} - \widetilde{X}_n = \theta h\{f_{n+1} - \alpha\widetilde{X}_{n+1} + \beta\widetilde{X}_{n+1-N}\} +$$
$$+(1-\theta)h\{f_n - \alpha\widetilde{X}_n + \beta\widetilde{X}_{n-N}\} + \sqrt{h}(g_n + \eta\widetilde{X}_n + \mu\widetilde{X}_{n-N})\xi_n \quad (\xi_n \in \mathcal{N}(0,1)).$$

θ-Maruyama scheme; asymptotic and exponential stability; stochastic delay differential & difference equations; Halanay-type inequalities.
AMS Subject Classification: 65C30 60H35 34K20 34K50

1 Introduction

This work is an extension of previous work of Baker & Buckwar [2, 3]. We indicate how results for stability of solutions obtained from a θ–Maruyama method applied to a linear stochastic delay differential equation (SDDE), that serves as a test equation, can be derived. The use of such a test equation is commonplace in numerical analysis; see *e.g.* [1, 5] for deterministic delay differential equations (DDDEs); for stochastic ordinary differential equations (SODEs) see *e.g.* [9]; for SDDEs see [2, 7]. Though it may seem that standard test equations are often chosen for their amenability to investigation, we here accept without discussion that such simple equations generate a 'test-bed' for obtaining insight into related non-trivial problems. At the same time, analysis of deterministic problems can yield understanding of the analysis of stochastic equations, and we exploit this.

We assume a little familiarity with the related literature, but seek to present a self-contained discussion. We employ a strategy presented for stability analysis in [3], where we illustrated the investigation of numerical stability by examining the *Euler-Maruyama method*. As we remarked in an aside in [3], the technique for analyzing stability that we illustrated by reference to the Euler-Maruyama method can also be applied to some other methods, including some that are semi-implicit (*i.e.* drift-implicit). We justify this remark here; a number of the other comments suggest further insight not found (or not easily found) in the existing literature.

*Contact author. Research professor (professor emeritus), Department of Mathematics, The University, Manchester M13 9PL, UK.

†Honorary Research Fellow, The University, Manchester, UK.

The authors' collaboration was aided by the London Mathematical Society.

1.1 The test equation

With t_0, α, β, η, $\mu \in \mathbb{R}$ and $\tau \geq 0$, the Itô SDDE considered here is written

$$\begin{aligned} \mathrm{d}X(t) &= \{f(t) - \alpha X(t) + \beta\, X(t-\tau)\}\,\mathrm{d}t + \\ &\qquad + \{g(t) + \eta\, X(t) + \mu X(t-\tau)\}\,\mathrm{d}W(t)\ (t \geq t_0), \end{aligned} \tag{1a}$$

$$X(t) = \Phi(t), \quad t \in J, \quad J := [t_0 - \tau, t_0]. \tag{1b}$$

(Note the sign attached to α in (1a).) If $\tau > 0$, we can, by a change of variable, normalize it to unity, and replace $\{\alpha, \beta, \eta, \mu, \tau\}$ by $\{\alpha\tau, \beta\tau, \eta\sqrt{\tau}, \mu\sqrt{\tau}, 1\}$.
For $t \in [t_0, \infty)$, $X(t) \equiv X(\Phi; t)$ denotes the solution of the SDDE (1a) for a given initial function Φ in (1b).
The discussion in [3] was presented in terms of a more general equation

$$\mathrm{d}X(t) = F(t, X(t), X(t-\tau))\,\mathrm{d}t + G(t, X(t), X(t-\tau))\,\mathrm{d}W(t). \tag{2}$$

We use the linear inhomogeneous SDDE (1a), with $t_0 \leq t < \infty$, as a *test equation* for the discussion of stability and (on applying a numerical method) numerical stability. The functions $f(t)$ and $g(t)$ in (1a) satisfy conditions consistent with those normally assumed for F and G; their presence implies that the null function ($X(t) \equiv 0$ for $t \geq -\tau$) may not be a solution. We assume the standard infrastructure and notation [2, 3, 11]: (*i*) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathsf{P})$ is a complete probability space with the filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions, E denotes expectation with respect to P and $W(t)$ is a one-dimensional standard Wiener process on that probability space; (*ii*) the initial function $\Phi : J \times \Omega \to \mathbb{R}$ has continuous paths, is independent of the σ-algebra generated by $W(t)$ and satisfies $\mathsf{E}(\sup_{t\in J} |\Phi(t)|^2) < \infty$; (*iii*) there exists a unique strong solution of the SDDE (1) with $\mathsf{E}(\{X(t)\}^2) < \infty$ for bounded t. (A strong solution of (2) satisfies $X(t) = X(t_0) + \int_{t_0}^{t} F(s, X(s), X(s-\tau))\,\mathrm{d}s + \int_{t_0}^{t} G(s, X(s), X(s-\tau))\mathrm{d}W(s)$ almost surely, for $t \geq t_0$; for a full definition, and sufficient conditions for existence and uniqueness, see Mao [11] pp. 149–157. The convergence and stability results in [2, 3] require existence and uniqueness of solutions but do not require the global Lipschitz conditions in [11]. For a convergence proof for the θ-Maruyama method for general discrete delay SDDEs, including (2), see [6].)

1.2 The θ–Maruyama equations

Suppose that $\theta \in [0, 1]$ and choose a step $h = \tau/N$ where $N \in \mathbb{N}$. The Maruyama-type θ-method applied to (1) generates, where $\xi_n \in \mathcal{N}(0, 1)$ (ξ_n is normally distributed with zero mean and variance unity), the recurrence

$$\begin{aligned} \widetilde{X}_{n+1} - \widetilde{X}_n &= hf_n^\theta + h\Big\{\theta\{-\alpha\widetilde{X}_{n+1} + \beta\widetilde{X}_{n+1-N}\} + (1-\theta)\{-\alpha\widetilde{X}_n + \beta\widetilde{X}_{n-N}\}\Big\} + \\ &\qquad + \sqrt{h}(g_n + \eta\,\widetilde{X}_n + \mu\,\widetilde{X}_{n-N})\ \xi_n, \end{aligned} \tag{3a}$$

for approximations $\widetilde{X}_n \approx X(nh)$, where we use the shorthand notation

$$t_n = t_0 + nh,\ g_n = g(t_n),\ f_n = f(t_n) \text{ and } f_n^\theta := \theta f_{n+1} + (1-\theta) f_n\ (n \in \mathbb{Z}). \tag{3b}$$

Eq. (3a) is a drift-implicit formula that (if h is not an "exceptional value" – *i.e.*, provided $1 + \theta\alpha h \neq 0$) generates the sequence $\{\widetilde{X}_n\}_{n\geq 1}$, when given

$$\widetilde{X}_{-\ell} = \Phi(t_{-\ell}) \text{ for } \ell \in \mathcal{J} \quad \text{where } \mathcal{J} := \{0, 1, \cdots, N\}. \tag{3c}$$

To indicate the dependence on Φ we write $\widetilde{X}_n \equiv \widetilde{X}_n(\Phi)$, and our definition of stability relates to perturbations $\delta\widetilde{X}_n \equiv \delta\widetilde{X}_n(\Phi) := \widetilde{X}_n(\Phi+\delta\Phi) - \widetilde{X}_n(\Phi)$, that arise from perturbations $\delta\Phi(t_{-\ell})$ (for $\ell \in \mathcal{J}$) in the initial data. We will use the notation

$$\varrho_0 = \frac{1-(1-\theta)\alpha h}{1+\theta\alpha h},\ \varrho_1 = \frac{\theta\beta h}{1+\theta\alpha h},\ \varrho_2 = \frac{(1-\theta)\beta h}{1+\theta\alpha h},\ \varpi_0 = \frac{\eta\sqrt{h}}{1+\theta\alpha h},\ \varpi_1 = \frac{\mu\sqrt{h}}{1+\theta\alpha h}. \quad (4)$$

From (3a), if $1+\theta\alpha h \neq 0$ (as we assume),

$$\delta\widetilde{X}_{n+1} = (\varrho_0 + \varpi_0\xi_n)\delta\widetilde{X}_n + \varrho_1\delta\widetilde{X}_{n+1-N} + (\varrho_2 + \varpi_1\xi_n)\delta\widetilde{X}_{n-N}. \quad (5)$$

2 Exponential Stability of Solutions by a Halanay-type Technique

There is a variety of approaches to the investigation of stability; we cannot overemphasize that each approach has its merits or demerits and each has its adherents. Halanay [8] provided a technique for examining the exponential stability of solutions of DDDE s. This was modified for difference equations by Tang [12] (see also the related publications, *e.g.* [4]). Baker and Buckwar [3] progressed the Halanay-type theory by applying it to establish conditions for p-th moment exponential stability of solutions of SDDE s and certain discretized versions.

2.1 Stability definitions

Our definitions of stability, asymptotic stability, and exponential stability in mean-square of solutions of (1) are consistent with usual definitions[a] to be found in the literature; *cf.* [3], or [10, 11]; they are analogues of the definitions of (asymptotic, exponential) stability of solutions of stochastic *recurrence relations* or *difference equations*. However, the general stability definitions associated with (2) and its discretization can be simplified when considering (1), (3).

Definition 1 *A solution of (3) is said to be (a)* stable in mean-square *(*SMS *) if, for each* $\varepsilon > 0$, *there exists a corresponding value* $\delta^+ > 0$ *such that* $\mathsf{E}(|\,\delta\widetilde{X}_n|^2) < \varepsilon$ *for* $n \in \mathbb{N}$, *whenever* $\mathsf{E}(\sup_{n\in\mathcal{J}} |\,\delta\Phi(t_n)|^2) < \delta^+$; *(b)* asymptotically stable in mean-square *(*ASMS *) if it is stable in mean-square and* $\mathsf{E}(|\,\delta\widetilde{X}_n|^2) \to 0$ *as* $n \to \infty$, *whenever* $\mathsf{E}(\sup_{n\in\mathcal{J}} |\,\delta\Phi(t_n)|^2)$ *is bounded; (c)* exponentially stable in mean-square *(*ESMS *) if it is stable in the mean-square and if, given* $\delta^+ > 0$, *there exist a finite* $C > 0$, *and a value* $\nu_h^+ > 0$ *such that, whenever* $\mathsf{E}(\sup_{n\in\mathcal{J}} |\,\delta\Phi(t_n)|^2) < \delta^+$, $\mathsf{E}(|\delta\widetilde{X}_n|^2) \le C\exp\{-\nu_h^+(t_n - t_0)\}$ *for all* n *sufficiently large. Given such a* $\nu_h^+ > 0$, *we then term the solution* ν_h^+-ESMS , *or* ESMS *with* exponent $-\nu_h^+$.

Emphasis in the numerical analysis literature is concentrated on (a) and (b) rather than (c); we consider results for ν-exponential stability (ν-ESMS). The definitions of stability for the *analytical solution* $X(\Phi;t)$ of (1) are natural analogues of those in Definition 1. Thus, exponential stability is defined as follows:

Definition 2 *The solution* $X(\Phi;t)$ *of the problem (1) is* exponentially mean-square stable, *with* exponent $-\nu^+$ *(*ν^+-ESMS *), if it is stable in the mean-square and if, given* $\delta^+ > 0$, *there exist a finite* $C > 0$ *and a value* $\nu^+ > 0$ *such that, whenever* $\mathsf{E}(\sup_{t\in J} |\,\delta\Phi(t)|^2) < \delta^+$, $\mathsf{E}(|\delta X(t)|^2) \le C\exp\{-\nu^+(t-t_0)\}$*(where* $\delta X(t) := X(\Phi+\delta\Phi;t) - X(\Phi;t)$*) for all* t *sufficiently large.*

[a] Different notions of stability will not be considered here. (Other notions relate to almost sure behaviour of $\{\delta\widetilde{X}_n\}$, or stability in probability; another class of definitions correspond to *persistent perturbations* – perturbations in the inhomogeneous terms – *e.g.* in $\{f_n\}$ – rather than in Φ.)

The terms ν^+-ESMS and *exponent* $-\nu^+$ appear to be nonstandard. The restriction of the definitions to solutions of DDDE s (omitting the words "mean square") is clear.

2.2 A discrete inequality of Halanay type

We appeal to some results used in [3], to which we refer for discussion and proofs.

Lemma 1 *Denote by* $\mathcal{R}_N(\zeta;\ a,b)$ *the polynomial in* ζ*:*

$$\mathcal{R}_N(\zeta;\ a,b) := \zeta^{N+1} - (1-ah)\zeta^N - bh \quad (a,b \in \mathbb{R}; N \in \mathbb{N}), \tag{6a}$$

where $h = \tau/N > 0$*. If* $0 \le \beta_h < \alpha_h$ *and* $0 < \alpha_h h < 1$*, the polynomial* $\mathcal{R}_N(\zeta;\ \alpha_h, \beta_h)$ *has a single* positive *zero* ζ_h^+ *where*

$$\zeta_h^+ \in (1-(\alpha_h - \beta_h)h, 1), \quad \text{if } \beta_h > 0, \quad \text{and} \quad \zeta_h^+ = 1 - \alpha_h h, \ \text{if } \beta_h = 0; \tag{6b}$$

further, $\zeta_h^+ = \exp(-\nu_h^+ h)$ *where* $\nu_h^+ = -\ell n(\zeta_h^+)/h$ *lies in* $(0, \alpha_h]$.

Theorem 1 *Suppose, for some fixed integer* $N \ge 0$*, that* $t_n = t_0 + nh$ *for some* $h > 0$ *and* $\{v_n\}_{-N}^{\infty}$ *is a sequence of positive numbers that satisfies, where*

$$0 \le \beta_h < \alpha_h \text{ and } 0 < \alpha_h h < 1, \tag{7a}$$

the relation

$$\frac{v_{n+1} - v_n}{h} \le -\alpha_h v_n + \beta_h \max_{\ell \in \mathcal{J}} v_{n+\ell} \text{ for } n \in \mathbb{N} \tag{7b}$$

with $N = 0$ *if* $\beta_h = 0$*. Then* $v_n \le \{\max_{\ell \in \mathcal{J}} v_\ell\} \exp\{-\nu_h^+(t_n - t_0)\}$ *where* $\nu_h^+ > 0$ *is the value occurring in Lemma 1.*

Theorem 1 is similar in spirit to a result obtained by Halanay [8] in the context of DDDE s. The form of the result $0 < \nu_h^+ \le \alpha_h$ explains the presence of the scaling factor $1/h$ in (7b) – so that $\{v_{n+1} - v_n\}/h$ then simulates a derivative.

3 Deterministic Insight

Results for deterministic problems yield insight. Consider the DDDE

$$x'(t) = f(t) - \alpha x(t) + \beta x(t-\tau) \quad (\alpha, \beta \in \mathbb{R}). \tag{8}$$

Theorem 2 *Given* $\nu^+ > 0$*, solutions of (8) are* ν^+*–exponentially stable* if and only if *the zeros of the function* $\mathcal{Q}(\zeta;\ \alpha,\beta,\tau) := \zeta + \alpha - \beta\exp(-\zeta\tau)$ *lie in the left half–plane* $\Re(\zeta) \le -\nu^+$*; a* sufficient condition *for* ν_+-ESMS *for some* $\nu_+ > 0$ *is* $|\beta| < -\alpha$*.*

> **Remark:** The special form of (3.1) allows use of a type of "method of D-partitions" (a boundary locus technique, *cf.* [10]) to determine, given $\nu^+ > 0$, exact regions in $(\alpha\tau, \beta\tau)$ parameter space for which solutions are ν^+-exponentially stable.

In the following proof of Theorem 3, we give an analysis for the deterministic case that can be modified for the stochastic case. Suppose $Nh = \tau$ ($N \in \mathbb{N}$) and $1 + \alpha h\theta \ne 0$. The θ–method for (8) gives

$$x_{n+1} - x_n = hf_n^\theta - \alpha h\{\theta x_{n+1} + (1-\theta)x_n\} + \beta h\{\theta x_{n+1-N} + (1-\theta)x_{n-N}\}. \tag{9}$$

Perturbing $\{x_\ell\}_{\ell \in \mathcal{J}}$, we find the consequent perturbations $\{\delta x_\ell\}_{\ell \ge 1}$ satisfy

$$\delta x_{n+1} = \frac{1 + (1-\theta)\alpha h}{1 + \theta\alpha h}\,\delta x_n + \frac{\beta h\theta}{1 + \alpha h\theta}\,\delta x_{n+1-N} + \frac{(1-\theta)\beta h}{1 + \alpha h\theta}\,\delta x_{n-N}, \tag{10}$$

for $n \geq 0$. With $\rho_{0,1,2}$ as in (4), $\delta x_{n+1}^2 - \delta x_n^2$ can be expressed as $\Big(\{\varrho_0 + 1\}\delta x_n + \varrho_1 \delta x_{n+1-N} + \varrho_2 x_{n-N}\Big) \times \Big(\{\varrho_0 - 1\}\delta x_n + \varrho_1 \delta x_{n+1-N} + \varrho_2 \delta x_{n-N}\Big)$; we deduce that

$$\delta x_{n+1}^2 - \delta x_n^2 = (\varrho_0^2 - 1)\ \delta x_n^2 + 2\varrho_0\varrho_1\ \delta x_n \delta x_{n-N+1} + 2\varrho_0\varrho_2\ \delta x_n \delta x_{n-N} + 2\varrho_1\varrho_2\ \delta x_{n+1-N}\delta x_{n-N} + \varrho_1^2\ \delta x_{n+1-N}^2 + \varrho_2^2\ \delta x_{n-N}^2. \tag{11}$$

If $uv \neq 0$, $|suv| \leq \frac{1}{2}\{v^2 + s^2u^2\}$, with equality if, and only if, $s = v/u$. Thus $|uv| = \inf_{s\in(0,\infty)} \dfrac{1}{2s}\{v^2 + s^2u^2\} \leq \frac{1}{2}\{\dfrac{v^2}{r} + ru^2\}$ for all $r \in (0,\infty)$. Then,

$$|\varrho_r\varrho_s \delta x_j \delta x_k| \leq \frac{|\varrho_r\varrho_s|}{2}\{r_{jk}\delta x_j^2 + \frac{1}{r_{jk}}\delta x_k^2\} \text{ for arbitrary } r_{jk} \in (0,\infty), \tag{12}$$

with equality for some r_{jk}. From (11) and (12) we obtain the inequality

$$\delta x_{n+1}^2 - \delta x_n^2 \leq (\varrho_0^2 - 1)\ \delta x_n^2 + |\varrho_0\varrho_1|\{\frac{1}{r}\delta x_n^2 + r\delta x_{n-N+1}^2\} + |\varrho_0\varrho_2|\{\frac{1}{r'}\delta x_n^2 + r'\delta x_{n-N}^2\} + |\varrho_1\varrho_2|\{\frac{1}{r''}\delta x_{n+1-N}^2 + r''\delta x_{n-N}^2\} + \varrho_1^2\ \delta x_{n+1-N}^2 + \varrho_2^2\ \delta x_{n-N}^2, \tag{13}$$

for arbitrary $r, r', r'' \in (0,\infty)$. Hence,

$$\frac{\delta x_{n+1}^2 - \delta x_n^2}{h} \leq -A^\natural\ \delta x_n^2 + B^\natural\ \max_{\ell\in\mathcal{J}} \delta x_{n-\ell}^2 \quad (\mathcal{J} = \{0,1,\cdots,N\}), \tag{14}$$

where, for arbitrary positive numbers $\{r, r'\}$ (and choosing $r'' = 1$) we may set

$$A_h^\natural \equiv A_h^\natural(r,r') = -\frac{1}{h}\{\varrho_0^2 - 1 + \frac{|\varrho_0\varrho_1|}{r} + \frac{|\varrho_0\varrho_2|}{r'}\}, \tag{15a}$$

$$B_h^\natural \equiv B_h^\natural(r,r') = \frac{1}{h}\{|\varrho_0\varrho_1|r + |\varrho_0\varrho_2|r' + (|\varrho_1| + |\varrho_2|)^2\}. \tag{15b}$$

($A_h^\natural$ and $B_h^\natural$ are functions of h and are $\mathcal{O}(1)$ as $h \to 0$.) We deduce:

Theorem 3 *For the deterministic case, the recurrence (9) is ν_+–exponentially stable for some $\nu_+ > 0$ if, for any choice of positive r,r', the values in (15) satisfy the conditions $hA^\natural \in (0,1)$ and $0 \leq B^\natural < A^\natural$.*

> **Remark:** Theorem 3 provides a sufficient condition for ν_+–ESMS, for some $\nu_+ > 0$. However, the recurrence (10) is special; it yields $x_{n+1} = \varrho_0 x_n + \varrho_1 x_{n+1-N} + \varrho_2 x_{n-N} + hf_n^\theta/\{1 + \alpha h\theta\}$ (where $\tau = Nh$) and, given $\nu^+ > 0$, its solutions are ν^+–exponentially stable if and only if the zeros of $S_N(\zeta;\alpha,\beta,\tau) := \zeta^{N+1} - \varrho_0\zeta^N - \varrho_1\zeta - \varrho_2$ lie within or on the circle in the complex plane that is centered on 0 and has radius $\exp(-\nu^+h)$, any on the circle being simple. Thus, the parameters that correspond to ν^+–exponential stability can here be computed by a boundary-locus technique.

4 Simulation of Stability of $X(t)$ by that of $\{\widetilde{X}_n\}$.

We now advance to the stochastic problem. It is natural to ask to what extent the stability of $\{X_n(\Phi)\}$ corresponds to the stability of the true solution $X(\Phi;t)$ that it is assumed to approximate. For (1) we have the following result (see, *e.g.* [3]).

Theorem 4 *Every solution of (1) is ν_+-ESMS for some $\nu_+ > 0$ when (i) $|\beta| < \alpha - \{|\eta|^2 + |\mu|^2\}$, or (ii) $\mu = 0$ and $|\beta| < \alpha - \frac{1}{2}|\eta|^2$.*

We seek an analogue of Theorem 4 for stability of the numerical solutions, given $1+\theta\alpha h \neq 0$. To analyze mean-square stability we first derive a relationship between the expectations $\{\mathsf{E}(\delta\widetilde{X}_n^2)\}$, starting from (5). We seek a suitable relationship

$$\mathsf{E}(\delta\widetilde{X}^2_{n+1}) - \mathsf{E}(\delta\widetilde{X}^2_{n}) \leq -\alpha_h h \mathsf{E}(\delta\widetilde{X}^2_{n}) + \beta_h h \max_{\ell\in\mathcal{J}} \mathsf{E}(\delta\widetilde{X}^2_{n-\ell}). \tag{16}$$

Lemma 2 $\mathsf{E}(\,\delta\widetilde{X}^2_{n+1}) - \mathsf{E}(\,\delta\widetilde{X}^2_{n})$ *can be written*

$$\Big\{(\varrho_0^2-1)\ \mathsf{E}(\,\delta\widetilde{X}^2_{n}) + 2\varrho_0\varrho_2\mathsf{E}(\,\delta\widetilde{X}_n\,\delta\widetilde{X}_{n-N}) + 2\varrho_0\varrho_1\mathsf{E}(\,\delta\widetilde{X}_n\,\delta\widetilde{X}_{n+1-N}) +$$
$$+2\varrho_1\varrho_2\mathsf{E}(\,\delta\widetilde{X}_{n-N}\,\delta\widetilde{X}_{n+1-N}) + \varrho_2^2\mathsf{E}(\,\delta\widetilde{X}^2_{n-N}) + \varrho_1^2\mathsf{E}(\,\delta\widetilde{X}^2_{n+1-N})\Big\}$$
$$+\Big(\varpi_0^2\mathsf{E}(\,\delta\widetilde{X}^2_{n}) + 2\varpi_0\varpi_1\mathsf{E}(\,\delta\widetilde{X}_{n-N}\,\delta\widetilde{X}_{n+1-N}) + \varpi_1^2\mathsf{E}(\,\delta\widetilde{X}^2_{n-N})\Big). \tag{17}$$

Proof: $\delta\widetilde{X}_{n+1} \pm \delta\widetilde{X}_n = (\varrho_0 \pm 1 + \varpi_0\xi_n)\delta\widetilde{X}_n + \varrho_1\delta\widetilde{X}_{n+1-N} + (\varrho_2 + \varpi_1\xi_n)\,\delta\widetilde{X}_{n-N}$. Hence, for appropriate coefficients $\mathfrak{a}_i$ etc. that are functions of $\{\varrho_i\}$ and $\{\varpi_j\}$ (and hence of α,β,η,μ, and h), $\delta\widetilde{X}^2_{n+1} - \delta\widetilde{X}^2_{n} = \{\mathfrak{a}_0 + \mathfrak{a}_1\xi_n + \mathfrak{a}_2\xi_n^2\}\delta\widetilde{X}^2_{n} + \{\mathfrak{b}_0' + \mathfrak{b}_1'\xi_n + \mathfrak{b}_2'\xi_n^2\}\delta\widetilde{X}_n\delta\widetilde{X}_{n+1-N} + \{\mathfrak{b}_0''+\mathfrak{b}_1''\xi_n+\mathfrak{b}_2''\xi_n^2\}\delta\widetilde{X}_n\delta\widetilde{X}_{n-N} + \{\mathfrak{b}_0'''+\mathfrak{b}_1'''\xi_n+\mathfrak{b}_2'''\xi_n^2\}\delta\widetilde{X}_{n-N}\delta\widetilde{X}_{n-N+1}$ $+\{\mathfrak{c}_0+\mathfrak{c}_1\xi_n+\mathfrak{c}_2\xi_n^2\}\delta\widetilde{X}^2_{n+1-N} + \{\mathfrak{d}_0+\mathfrak{d}_1\xi_n+\mathfrak{d}_2\xi_n^2\}\delta\widetilde{X}^2_{n-N}$. (We note that $\mathfrak{b}_2' = \mathfrak{b}_2''' = \mathfrak{c}_2 = 0$, and the coefficients with index 0 arise in the deterministic case.) If $-N \leq r,s \leq n$ $(r,s\in\mathbb{N})$, we have $\mathsf{E}(\xi_n\delta\widetilde{X}_r\delta\widetilde{X}_s) = 0$ and $\mathsf{E}(\xi_n^2\delta\widetilde{X}_r\delta\widetilde{X}_s) = \mathsf{E}(\delta\widetilde{X}_r\delta\widetilde{X}_s)$, and so the coefficients with index 1 vanish when we take expectations in the expression for $\delta\widetilde{X}^2_{n+1} - \delta\widetilde{X}^2_{n}$, and obtain

$$\mathsf{E}(\,\delta\widetilde{X}^2_{n+1}) - \mathsf{E}(\,\delta\widetilde{X}^2_{n}) =$$
$$\{\mathfrak{a}_0 + \mathfrak{a}_2\}\ \mathsf{E}(\,\delta\widetilde{X}^2_{n}) + \mathfrak{b}_0'\mathsf{E}(\,\delta\widetilde{X}_n\,\delta\widetilde{X}_{n-N}) + (\mathfrak{b}_0'' + \mathfrak{b}_2'')\mathsf{E}(\,\delta\widetilde{X}_n\,\delta\widetilde{X}_{n+1-N}) +$$
$$\mathfrak{b}_0'''\mathsf{E}(\,\delta\widetilde{X}_{n-N}\,\delta\widetilde{X}_{n+1-N}) + \mathfrak{c}_0\mathsf{E}(\,\delta\widetilde{X}^2_{n+1-N}) + \{\mathfrak{d}_0 + \mathfrak{d}_2\}\mathsf{E}(\,\delta\widetilde{X}^2_{n-N}). \tag{18}$$

Expressing the coefficients in (18) in terms of (4) we establish the lemma.

4.1 Application of the general Halanay-type theory

Eq. (17) reduces to (11) in the deterministic case, and the first term – the term in braces $\Big\{\ \Big\}$ – in (17) can be treated in the manner used to prove (13) from (11), when obtaining (14). We now bound the terms in (18) that involve $\varpi_{0,1}$, using $|2\mathsf{E}(\,\delta\widetilde{X}_{n-N}\,\delta\widetilde{X}_{n+1-N})| \leq\ \mathsf{E}(\,\delta\widetilde{X}^2_{n-N}) + \mathsf{E}(\,\delta\widetilde{X}^2_{n+1-N})$, to obtain

$$\Big|\varpi_0^2\mathsf{E}(\,\delta\widetilde{X}^2_{n}) + 2\varpi_0\varpi_1\mathsf{E}(\,\delta\widetilde{X}_{n-N}\,\delta\widetilde{X}_{n+1-N}) + \varpi_1^2\mathsf{E}(\,\delta\widetilde{X}^2_{n-N})\Big| \leq$$
$$(\varpi_0^2 + 2|\varpi_0\varpi_1| + \varpi_1^2)\sup\big\{\mathsf{E}(\,\delta\widetilde{X}^2_{n}), \mathsf{E}(\,\delta\widetilde{X}^2_{n+1-N}), \mathsf{E}(\,\delta\widetilde{X}^2_{n-N})\big\}. \tag{19}$$

We thus obtain a delay-difference inequality of Halanay type, and hence, using Theorem 1, a condition for ν_+-ESMS:

Theorem 5 *Given arbitrary positive numbers* $\{r,r'\}$, *set* $A_h(r,r') = A_h^\natural(r,r')$, $B_h(r,r') = B_h^\natural(r,r') + (|\varpi_0| + |\varpi_1|)^2$, *where* $A_h^\natural(r,r')$ *and* $B_h^\natural(r,r')$ *are the values in (15), the deterministic case. Then*

$$\frac{\mathsf{E}(\delta\widetilde{X}^2_{n+1}) - \mathsf{E}(\delta\widetilde{X}^2_{n})}{h} \leq -A_h(r,r')\ \mathsf{E}(\delta\widetilde{X}^2_{n}) + B_h(r,r')\ \max_{\ell\in\mathcal{J}} \mathsf{E}(\delta\widetilde{X}^2_{n-\ell}), \tag{20}$$

where $\mathcal{J} = \{0,1,\cdots,N\}$. *If, for any* $r,r' \in (0,\infty)$, $0 < hA_h(r,r') < 1$ *and* $0 \leq B_h(r,r') < A_h(r,r')$, $\{\widetilde{X}_n(\Phi)\}$ *is* $\nu^+(r,r')$-ESMS *for a value* $\nu^+(r,r') > 0$.

The condition $hA_h(r, r') \in (0, 1)$ is a condition in Theorem 3.
Given (r, r') for which $hA_h(r, r') \in (0, 1)$ and $B_h(r, r') \in (0, A_h(r, r'))$, an estimate of $\nu^+(r, r')$ can obtained (by Lemma 1) from the positive zero $\zeta^+(r, r')$ of $\mathcal{R}_N(\zeta; A_h(r, r'), B_h(r, r'))$; in principle, one can then seek the maximum value $\nu^+(r, r')$ over all such pairs (r, r').
It is clear that to emulate the result $|\beta| < -\alpha + |\eta|^2 + |\mu|^2$, with $\alpha \in (0, \infty)$, that holds in the case of the test equation (1) (*cf.* Theorem 4 (*i*)) it is advantageous if $\varrho_0 \to 0$ as $\alpha \to \infty$. Thus, when considering ν-ESMS properties, it appears that $\theta \in (\frac{1}{2}, 1]$ (corresponding to an underlying L-stable deterministic θ-formula) may be preferable to $\theta = \frac{1}{2}$ (where the deterministic formula is only A-stable and $|\varrho_0| \to 1$ as $\alpha \to \infty$) or to $\theta \in [0, \frac{1}{2})$. However, an L-stable formula can be stable when the DDDE is unstable.

5 Summary

Affine constant-coefficient test equations with constant lags (such as that in (1)) are special, and they allow a more complete stability analysis than is in general possible. A justification of the use of (1a) for insight for more general equations can be formulated if the theory of approximating linear equations is analyzed; theories involving approximation by deterministic problems can also be found in the literature [10]. Theorems 3 and 5 are new, and they accomplish our main objective of demonstrating the applicability of Halanay-type inequalities. However, restrictions on space have limited our discussion; it has not been possible to demonstrate the advantages of an approach based upon Halanay-type inequalities. These lie in the opportunity to consider solutions of test equations with time-dependent coefficients and lags and certain types of non-linearity. On the other hand, the perceived advantages come at a price, *e.g.* some loss of precision in special cases such as those where necessary and sufficient conditions can be found from other approaches.

References

[1] C. T. H. Baker, *Retarded differential equations*, *J. Comput. Appl. Math.* **125** (2000) 309–335.

[2] C. T. H. Baker and E. Buckwar, *Numerical analysis of explicit one-step methods for stochastic delay differential equations*, *LMS J. Comput. Math.* **3** (2000) 315–335. URL: `http://www.lms.ac.uk/jcm/`

[3] C. T. H. Baker and E. Buckwar, *Exponential Stability in p-th Mean of Solutions, and of Convergent Euler-type Solutions, of Stochastic Delay Differential Equations* (submitted for publication); see also MCCM Technical Report 390, 2001/2002. ISSN: 1360 1725. URL: `http://www.maths.man.ac.uk/~nareports/`

[4] C. T. H. Baker and A. Tang, *Stability analysis of continuous implicit Runge-Kutta methods for Volterra integro-differential systems with unbounded delays*, *Appl. Numer. Math.* **24** (1997) 153–173.

[5] A. Bellen and M. Zennaro, *Numerical methods for delay differential equations* (Oxford University Press, 2003).

[6] E. Buckwar and R. Winkler *Multi-step Maruyama methods for stochastic delay differential equations*, submitted for publication.

[7] Wanrong Cao, Mingzhu Liu and Zhencheng Fan, *MS-stability of the Euler-Maruyama method for stochastic differential delay equations*, *Applied Mathematics and Computation* (in press, available online 27 November 2003).

[8] A. Halanay, *Differential Equations — Stability, Oscillations, Time Lags.* (Academic Press, New York, 1966).

[9] D. J. Higham, *Mean-square and asymptotic stability of the stochastic theta method*, *SIAM J. Numer. Anal.* **38** (2000) 753–769.

[10] V. B. Kolmanovskii and V. R. Nosov, *Stability of Functional Differential Equations* (Academic Press, New York, 1986).

[11] X. Mao, *Stochastic Differential Equations and their Applications* (Horwood Publishing Ltd., Chichester, 1997).

[12] A. Tang, *Analysis and numerics of delay Volterra integro-differential equations.* PhD thesis, Manchester University, 1995.