On Artin’s L-functions. II: Dirichlet Coefficients

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Let $a_n$, $n \geq 1$, be complex numbers. If for a real number $\theta$ the summatory function $\sum_{n \leq x} a_n$ is $O(x^\theta)$ as $x \to \infty$, then the Dirichlet series $\sum_{n=1}^\infty \frac{a_n}{n^s}$ converges in the half-plane $\text{Re}(s) > \theta$. For a non-principal Dirichlet character modulo $m$ it holds for every $x \geq 1$ $|\sum_{n \leq x} \chi(n)| \leq \varphi(m),$

so $\sum_{n \leq x} \chi(n)$ is $O(1)$ and the series $\sum_{n=1}^\infty \frac{\chi(n)}{n^s}$ converges for $\text{Re}(s) > 0$. The main result of this paper is

**Theorem.** Let $K/\mathbb{Q}$ be a finite Galois extension, let $\chi$ be a character of the Galois group $G = \text{Gal}(K/\mathbb{Q})$ which does not contain the principal character, let $L_{ur}(s, \chi, K/\mathbb{Q})$ be the unramified part of the corresponding Artin L-function, and let

$$L_{ur}(s, \chi, K/\mathbb{Q}) \frac{1}{\chi(1)} = \sum_{n=1}^\infty \frac{a_n}{n^s}$$

for $\text{Re}(s) > 1$. Then:

(i) The coefficients $a_n$ are algebraic numbers of the field $\mathbb{Q}(e^{2\pi i/\varphi(m)})$ and $|a_n| \leq 1$ for every $n \geq 1$;

(ii) The summatory function $\sum_{n \leq x} a_n$ is $o(x)$ as $x \to \infty$.

The theorem says in the case $K = \mathbb{Q}(e^{2\pi i/\varphi(m)})$ that $\sum_{n \leq x} \chi(n)$ is $o(x)$ as $x \to \infty$, which is weaker than: $\sum_{n \leq x} \chi(n)$ is $O(1)$. It should be seen as an assertion...
about a general Artin $L$-function in the case of a non-abelian extension $K/Q$.

An evaluation $O(x^\theta)$ with $\theta < 1$ of the summatory function of the Dirichlet coefficients of an Artin $L$-function had as a consequence that its Dirichlet series converges in $\text{Re}(s) > \theta$, so the $L$-function is holomorphic in that half plane, which would be an important step towards Artin’s holomorphy conjecture. This is not reached here. It is the motivation for future research.

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Proof of the theorem: Let $d_K$ be the discriminant of the algebraic number field $K$, and let ([2], P. 297, (7))

\begin{equation}
L_{ur}(s, \chi, K/Q) := \prod_{(p, d_K) = 1} \frac{1}{\det(I - A_p/p^s)}
\end{equation}

be the product of the Euler factors of $L(s, \chi, K/Q)$ corresponding to the unramified primes $p$. This is absolutely convergent in $\text{Re}(s) > 1$. The dimension of the matrix $A_p$ is $r = \chi(1)$, and it holds

$$\det(I - A_p/p^s) = \prod_{j=1}^r (1 - \frac{\zeta_{j,p}}{p^s}),$$

where $\zeta_{1,p}, \ldots, \zeta_{r,p}$ are the eigenvalues of $A_p$, which are roots of unity of order dividing the order $|G|$ of the Galois group $G$. By Newton’s binomial formula it holds for $\text{Re}(s) > 0$

\begin{equation}
\frac{1}{\det(I - A_p/p^s)^k} = \prod_{j=1}^r \frac{1}{(1 - \frac{\zeta_{j,p}}{p^s})} = \prod_{j=1}^r (1 - \frac{\zeta_{j,p}}{p^s})^{-\frac{1}{k}} = \prod_{j=1}^r [1 + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{u=1}^{r} (1 + (u-1)r) \cdot \ldots \cdot (1 + (n-1)r) \frac{\zeta_{j,p}^n}{p^{n\text{th}_{l}}}] = \prod_{j=1}^r [1 + \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{u=1}^{r} (1 + (u-1)r) \cdot \ldots \cdot (1 + (n-1)r) \frac{\zeta_{j,p}^n}{p^{n\text{th}_{l}}}] = 1 + \sum_{k=1}^{\infty} a_{p^k} p^k,
\end{equation}

where for $k \geq 1$

$$a_{p^k} = \sum_{k_1 \geq 0, \ldots, k_r \geq 0, k_1 + \ldots + k_r = k} \prod_{j=1}^r c_{k_j} \zeta_{j,p}^{k_j} = \sum_{k_1 \geq 0, \ldots, k_r \geq 0, k_1 + \ldots + k_r = k} \prod_{j=1}^r c_{k_j} \prod_{j=1}^r \zeta_{j,p}^{k_j},$$

with

\begin{equation}
c_0 = 1, c_l = \frac{\prod_{u=1}^{l}(1 + (u-1)r)}{p^l \cdot l!}, l \geq 1,
\end{equation}
hence

\[ |a_{p^k}| \leq \sum_{k_1 \geq 0, \ldots, k_r \geq 0, k_1 + \ldots + k_r = k} | \prod_{j=1}^r c_{k_j} \cdot \prod_{j=1}^r \zeta_{k_j,p} | = \sum_{k_1 \geq 0, \ldots, k_r \geq 0, k_1 + \ldots + k_r = k} \prod_{j=1}^r c_{k_j}. \]

**Lemma.** For \( r \geq 1 \) and \( k \geq 1 \) it holds

\[ \sum_{k_1 \geq 0, \ldots, k_r \geq 0, k_1 + \ldots + k_r = k} \prod_{j=1}^r c_{k_j} = 1. \]

**Proof of the lemma:** For \( l \geq 0 \) let \( d_l = r^l \cdot l! \cdot c_l \). By (3) it holds

\[ d_0 = 1, \quad d_l = \prod_{u=1}^l (1 + (u - 1)r), \quad l \geq 1. \]

For \( k_1 \geq 0, \ldots, k_r \geq 0 \) with \( k_1 + \ldots + k_r = k \) it holds

\[ \prod_{j=1}^r c_{k_j} = \frac{\prod_{j=1}^r d_{k_j}}{r^k \cdot k_1! \cdot \ldots \cdot k_r!} = \frac{1}{r^k \cdot k!} \binom{k}{k_1 \ldots k_r} \prod_{j=1}^r d_{k_j}, \]

where

\[ \binom{k}{k_1 \ldots k_r} = \frac{k!}{k_1! \cdot \ldots \cdot k_r!} \]

is the multinomial coefficient, so (5) is equivalent to

\[ \sum_{k_1 \geq 0, \ldots, k_r \geq 0, k_1 + \ldots + k_r = k} \binom{k}{k_1 \ldots k_r} \prod_{j=1}^r d_{k_j} = r^k \cdot k!. \]

For \( k = 1 \) it holds by (6)

\[ d_1 + \ldots + d_1 = r \cdot d_1 = r \cdot 1, \]

so (7) is true. Suppose that (7) is true for a number \( k \geq 1 \). Then

\[ r^{k+1} \cdot (k + 1)! = r(k + 1)r^k \cdot k! = \]

\[ = r(k + 1) \sum_{k_1 \geq 0, \ldots, k_r \geq 0, k_1 + \ldots + k_r = k} \binom{k}{k_1 \ldots k_r} \prod_{j=1}^r d_{k_j} = \]

\[ = \sum_{k_1 \geq 0, \ldots, k_r \geq 0, k_1 + \ldots + k_r = k} (r^k + r) \binom{k}{k_1 \ldots k_r} \prod_{j=1}^r d_{k_j} = \]
\[
= \sum_{k_{1}, \ldots, k_{r} \geq 0, k_{1} + \ldots + k_{r} = k} \left( \frac{k}{k_{1} \ldots k_{r}} \right) [(rk_{1} + 1) + \ldots + (rk_{r} + 1)] \prod_{j=1}^{r} d_{k_{j}}.
\]

By (6) it holds for \( l \geq 0 \)
\[d_{l+1} = (1 + lr)d_{l},\]
so (8) is equivalent to
\[(9) \quad r^{k+1}.(k+1)! = \sum_{k_{1}, \ldots, k_{r} \geq 0, k_{1} + \ldots + k_{r} = k} \left( \frac{k}{k_{1} \ldots k_{r}} \right) \sum_{j=1}^{r} d_{k_{1}} \ldots d_{k_{j}+1} \ldots d_{k_{r}}.\]

It holds
\[(10) \quad \sum_{k_{1}, \ldots, r \geq 0, k_{1} + \ldots + k_{r} = k} \left( \frac{k}{k_{1} \ldots k_{r}} \right) \sum_{j=1}^{r} d_{k_{1}} \cdot \ldots \cdot d_{k_{j}+1} \cdot d_{k_{r}} =\]
\[= \sum_{l_{1}, \ldots, l_{r} \geq 0, l_{1} + \ldots + l_{r} = k+1, l_{j} \geq 1, j = 1, \ldots, r} \left( \frac{k}{l_{1} \ldots l_{r} - 1 \ldots l_{r}} \right) d_{l_{1}} \cdot \ldots \cdot d_{l_{r}} =\]
\[= \sum_{l_{1}, \ldots, l_{r} \geq 0, l_{1} + \ldots + l_{r} = k+1} \left( \frac{k + 1}{l_{1} \ldots l_{r}} \right) d_{l_{1}} \cdot \ldots \cdot d_{l_{r}}.\]

From (8), (9) and (10) it follows that (7) is true for \( k+1 \). □

By (4) and the preceding lemma it holds for every \( k \geq 1 \)
\[|a_{p^{k}}| \leq 1.\]

Let \( a_{1} := 1, a_{n} := 0 \) for \( (n, d_{K}) \neq 1, a_{n} := a_{p_{1}}^{k_{1}} \ldots a_{p_{m}}^{k_{m}} \) for \( n = p_{1}^{k_{1}} \ldots p_{m}^{k_{m}} > 1, (n, d_{K}) = 1. \) Then it holds by (1) and (2)
\[L_{ur}(s, \chi, K/\mathbb{Q}) \frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{a_{n}}{n^{s}},\]
with
\[(12) \quad a_{n} \in \mathbb{Q}(e^{i\pi r}), \quad |a_{n}| \leq 1,\]
for every \( n \geq 1. \) The first part of the theorem is proved. For the second part apply

**The Tauberian Theorem of Wiener-Ikehara.** Let \( G(s) = \sum_{n=1}^{\infty} \frac{a_{n}}{n^{s}} \) be a Dirichlet series. Suppose there exists a Dirichlet series \( H(s) = \sum_{n=1}^{\infty} \frac{b_{n}}{n^{s}} \) with positive real coefficients such that
\( a) \quad |a_{n}| \leq b_{n} \) for all \( n \geq 1; \)
b) the series $H(s)$ converges for $\text{Re}(s) > 1$; 
c) the function $H(s)$ (respectively $G(s)$) can be extended to a meromorphic function in the region $\text{Re}(s) \geq 1$ having no poles except (respectively except possibly) for a simple pole at $s = 1$ with residue $C \geq 0$ (respectively $c$).

Then

$$\sum_{n \leq x} a_n = cx + o(x)$$

as $x \to \infty$. In particular, if $G(s)$ is holomorphic at $s = 1$, then $c = 0$ and

$$\sum_{n \leq x} a_n = o(x) \text{ as } x \to \infty. \text{ ([3], Theorem 1.1, P. 7)}$$

Take

$$G(s) = L_{ur}(s, \chi, K/\mathbb{Q})^\chi = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and

$$H(s) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

The assumptions of the tauberian theorem of Wiener-Ikehara are satisfied:

a) It was proved above that $|a_n| \leq 1$ for every $n \geq 1$;

b) the series $H(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for $\text{Re}(s) > 1$;

c) the Riemann zeta function $H(s) = \zeta(s)$ is meromorphic in the whole complex plane with exactly one pole at $s = 1$, which is simple with residue $C = 1 \geq 0$. The Artin $L$-function $L_{ur}(s, \chi, K/\mathbb{Q})$ is holomorphic and has no zeroes for $\text{Re}(s) \geq 1$ ([1], Satz 3, P. 105), so the function $L_{ur}(s, \chi, K/\mathbb{Q})^{\chi}$ is defined and holomorphic for $\text{Re}(s) \geq 1$. It has no pole at $s = 1$: $c = 0$.

By the theorem of Wiener-Ikehara it holds

$$\sum_{n \leq x} a_n = o(x).$$
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References


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