Index determination for DAEs

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Abstract

The index definition of DAEs with properly stated leading term bases on a matrix sequence with suitably chosen projectors. A way of realization of this matrix sequence is presented, it includes the calculation of suitable projectors using generalized inverses of the sequence matrices.

Keywords: DAE, index determination, generalized inverse

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1 Introduction

Most of the DAEs coming from application have the structure

$$A(x,t)(d(x,t))' + b(x,t) = 0 \quad t \in I, \tag{1.1}$$

where I describes the interval of interest. A linear equation of this structure is given by

$$A(t)(D(t)x(t))' + B(t)x(t) = q(t), \tag{1.2}$$

where all coefficients are supposed to be continuous matrix functions $A(t) \in \mathbb{R}^{n\times m}$, $D(t) \in \mathbb{R}^{m\times n}$ and $B(t) \in \mathbb{R}^{n\times n}$.
The coefficients $A(t)$ and $D(t)$ fulfils

Definition 1.1 *[Mär01] The leading term of (1.2) is stated properly if the coefficients $A(t)$ and $D(t)$ are well matched in the sense that*

$$\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{R}^m, \quad t \in I,$$

and there is a continuously differentiable projector $R(t) \in \mathbb{R}^{m\times m}$ such that $\operatorname{im} R(t) = \operatorname{im} D(t), \quad \ker R(t) = \ker A(t), \quad t \in I.$

For our further considerations we will drop the argument t.
To describe the structure of a DAE and to determine the index we form a sequence of matrices. For given coefficients A, D and B (A and D well matched) we define

$$\begin{aligned} G_0 &:= AD, \quad B_0 := B, \\ G_{i+1} &:= G_i + B_iQ_i = (G_i + W_iB_iQ_i)(I + G_i^- B_iQ_i), \\ B_{i+1} &:= (B_i - G_{i+1}D^-(DP_0 \ldots P_{i+1}D^-)'DP_0 \ldots P_{i-1})P_i, \end{aligned} \tag{1.3}$$

where Q_i denotes a projector function such that $\operatorname{im} Q_i = \ker G_i$, $P_i := I - Q_i$ and W_i is a projector function such that $\ker W_i = \operatorname{im} G_i$. D^- denotes the reflexive generalized inverse of D such that $D^-DD^- = D^-$, $DD^-D = D$, $DD^- = R$ and $D^-D = P_0$, and G_i^- is the reflexive generalized inverse of G_i with $G_i^-G_i = P_i$ and $G_iG_i^- = I - W_i$.

Definition 1.2 *[Mär01] An equation (1.2) with properly stated leading term is said to be a regular index μ DAE on the interval I, $\mu \in \mathbb{N}$, if there is a continuous matrix function sequence (1.3) such that*

(a) G_i has constant rank r_i on I,

(b) the projector Q_i fulfils $Q_iQ_j = 0, \quad 0 \le j < i$,

(c) $Q_i \in C(I, \mathbb{R}^{n\times n})$, $DP_0 \ldots P_iD^- \in C^1(I, \mathbb{R}^{m\times m})$, $i \ge 0$,

(d) $0 \le r_0 \le \cdots \le r_{\mu-1} < n$ and $r_\mu = n$.

Denoting $N_i := \ker G_i$, we know for a regular DAE that we can choose the projectors Q_i in such a way that

$$N_i \cap N_{i+1} = 0, \quad \forall i \ge 0, \tag{1.4}$$

(see [Mär01]). To use Definition 1.2 to determine the index of a DAE (pointwise) numerically we have to choose the projectors Q_i such that

$$Q_iQ_j = 0, \quad 0 \le j < i. \tag{1.5}$$

In the consequence, certain products of projectors also become projectors, e.g. P_0P_1 etc.
The paper aims at designing an algorithm to realize the matrix sequence (1.3) numerically. As the main problem we have to create projectors Q_i with (1.5).

2 The pseudo inverse

For a matrix $Z \in \mathbb{R}^{m\times n}$ we call $Z^- \in \mathbb{R}^{n\times m}$ a reflexive (generalized) inverse iff it fulfils

$$ZZ^-Z = Z \quad \text{and} \tag{2.1}$$
$$Z^-ZZ^- = Z^-. \tag{2.2}$$

The products $ZZ^- \in \mathbb{R}^{m\times m}$ and $Z^-Z \in \mathbb{R}^{n\times n}$ are projectors with the same rank r_Z. Let $P \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ be given projectors with rank r_Z.

Lemma 2.1 *With (2.1), (2.2) and the conditions*

$$Z^-Z = P \quad \textit{and} \tag{2.3}$$
$$ZZ^- = R \tag{2.4}$$

the reflexive inverse Z^- is uniquely determined.

Proof: Let Y be a further matrix fulfilling (2.1), (2.2), (2.3) and (2.4).

$$\begin{aligned} Y &\overset{(2.2)}{=} YZY \overset{(2.1)}{=} YZZ^-ZY \overset{(2.4)}{=} YRZY \\ &\overset{(2.4)}{=} YR \overset{(2.4)}{=} YZZ^- \overset{(2.3)}{=} PZ^- \overset{(2.2)}{=} Z^-. \end{aligned}$$

q.e.d.

To represent the pseudo inverse Z^- we want to use a decomposition of

$$Z = U \begin{pmatrix} S & \\ & 0 \end{pmatrix} V^{-1}$$

with nonsingular matrices U, V and S. The pseudo inverse is given by

$$Z^- = V \begin{pmatrix} S^{-1} & m_2 \\ m_1 & m_1Sm_2 \end{pmatrix} U^{-1} \tag{2.5}$$

with m_1 and m_2 being matrices of free parameters that fulfil

$$P = Z^- Z = V \begin{pmatrix} I & 0 \\ m_1 S & 0 \end{pmatrix} V^{-1}$$

and

$$R = Z Z^- = U \begin{pmatrix} I & S m_2 \\ 0 & 0 \end{pmatrix} U^{-1}.$$

(For details and different constructions of Z^- see [Zie79]).

3 Check of the well matched condition

A and D have to be well matched (see Def. 1.2). This is important in view of the representation of the DAE as a hand-made subroutine, which easily contains a programming error. From Def. 1.2 we obtain the relations

$$AD = ARD, \quad A = AR, \quad D = RD,$$

and

$$rank(A) = rank(D) = rank(AD). \tag{3.1}$$

Performing an SVD of A and D yields

$$A = U_A \begin{pmatrix} \Sigma_A & \\ & 0 \end{pmatrix} V_A^T,$$

$$D = U_D \begin{pmatrix} \Sigma_D & \\ & 0 \end{pmatrix} V_D^T.$$

We can check now that $rank(\Sigma_A) = rank(\Sigma_D)$.
Compute the matrix sequence, we need $G_0 := AD$. Using the decompositions we have

$$AD = U_A \begin{pmatrix} \Sigma_A & \\ & 0 \end{pmatrix} V_A^T U_D \begin{pmatrix} \Sigma_D & \\ & 0 \end{pmatrix} V_D^T \tag{3.2}$$

and, by $V_A^T U_D =: H = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}$, the relation (3.1) is fulfilled iff H_1 remains nonsingular. To compute the generalized inverse of D we use the relations $DD^- = R = A^- A$. We have

$$DD^- = U_D \begin{pmatrix} I & \Sigma_D m_{D_2} \\ 0 & 0 \end{pmatrix} U_D^T,$$

$$A^- A = V_A \begin{pmatrix} I & 0 \\ m_{A_1}\Sigma_A & 0 \end{pmatrix} V_A^T$$

and with $U_D = V_A H$ we obtain the relation

$$H \begin{pmatrix} I & \Sigma_D m_{D_2} \\ 0 & 0 \end{pmatrix} H^T = \begin{pmatrix} I & 0 \\ m_{A_1}\Sigma_A & 0 \end{pmatrix}.$$

This fixes the free parameters

$$m_{D_2} = \Sigma_D^{-1} H_1^{-1} H_2 \tag{3.3}$$

and

$$m_{A_1} = H_3 H_1^{-1} \Sigma_A^{-1}.$$

4 The matrix sequence

To start the construction of the matrix sequence (1.3) we need the pseudo inverses D^-, G_0^- and the projectors Q_0, W_0 simultaneously. The following relations have to be taken into account $D^- D = I - Q_0$, $G_0^- G_0 = I - Q_0$ and $G_0 G_0^- = I - W_0$. For $G_0 = AD$ (3.2) provides the representation

$$G_0 = U_A \begin{pmatrix} Z & \\ & 0 \end{pmatrix} V_D^T \text{ with } Z = \Sigma_A H_1 \Sigma_D.$$

With an SVD of $Z = U_Z \Sigma_0 V_Z^T$ we have the SVD of G_0 as

$$G_0 = \underbrace{U_A \begin{pmatrix} U_Z & \\ & I \end{pmatrix}}_{U_0} \begin{pmatrix} \Sigma_0 & \\ & 0 \end{pmatrix} \underbrace{\begin{pmatrix} V_Z^T & \\ & I \end{pmatrix} V_D^T}_{V_0^T}.$$

Using the SVD of D and G_0 the pseudo inverses have the general representation

$$D^- = V_D \begin{pmatrix} \Sigma_D^{-1} & m_{D_2} \\ m_{D_1} & m_{D_1}\Sigma_D m_{D_2} \end{pmatrix} U_D^T,$$

and

$$G_0^- = V_0 \begin{pmatrix} \Sigma_0^{-1} & m_{0_2} \\ m_{0_1} & m_{0_1}\Sigma_0 m_{0_2} \end{pmatrix} U_0^T = V_D \begin{pmatrix} Z^{-1} & V_Z m_{0_2} \\ m_{0_1} U_Z^T & m_{0_1}\Sigma_0 m_{0_2} \end{pmatrix} U_A^T. \tag{4.1}$$

For

$$I - W_0 = U_0 \begin{pmatrix} I & \Sigma_0 m_{0_2} \\ 0 & 0 \end{pmatrix} U_0^T,$$

this yields

$$I - Q_0 = V_0 \begin{pmatrix} I & 0 \\ m_{0_1}\Sigma_0 & 0 \end{pmatrix} V_0^T = V_D \begin{pmatrix} I & 0 \\ m_{0_1}U_Z^T Z & 0 \end{pmatrix} V_D^T$$

and

$$D^- D = V_D \begin{pmatrix} I & 0 \\ m_{D_1}\Sigma_D & 0 \end{pmatrix} V_D^T,$$

which gives $m_{D_1} = m_{0_1}U_Z^T Z \Sigma_D^{-1}$ i.e., by m_{0_1} all parameters of D^- are fixed. Let us now assume that we have realized the matrix sequence up to G_i such that $Q_i Q_j = 0$ for $j < i$. We have to construct G_{i+1} and a reflexive generalized inverse G_{i+1}^- with

$$G_{i+1}G_{i+1}^- = I - W_{i+1}, \quad G_{i+1}^- G_{i+1} = I - Q_{i+1} \tag{4.2}$$

and

$$Q_{i+1}Q_j = 0 \text{ for } j < i+1. \tag{4.3}$$

First we give a representation of G_{i+1}^-. From the matrix sequence we have

$$G_{i+1} = G_i + B_i Q_i = (G_i + W_i B_i Q_i)F_i$$

with the nonsingular matrix $F_i = I + G_i^- B_i Q_i$. For the sequence matrix G_i we have a decomposition

$$G_i = \mathcal{U}_i \begin{pmatrix} S_i & \\ & 0 \end{pmatrix} \mathcal{V}_i^{-1}$$

with $\mathcal{U}_i$, S_i and $\mathcal{V}_i$ nonsingular matrices with $\mathcal{U}_0 = U_0$, $S_0 = \Sigma_0$ and $\mathcal{V}_0 = V_0$. The other components are given by

$$G_i^- = \mathcal{V}_i \begin{pmatrix} S_i^{-1} & m_{i,2} \\ m_{i,1} & m_{i,1}S_i m_{i,2} \end{pmatrix} \mathcal{U}_i^{-1},$$

$$W_i = \mathcal{U}_i \begin{pmatrix} 0 & -S_i m_{i,2} \\ & I \end{pmatrix} \mathcal{U}_i^{-1} = \mathcal{U}_i T_{u,i}^{-1} \begin{pmatrix} 0 & \\ & I \end{pmatrix} \mathcal{U}_i^{-1}, \tag{4.4}$$

$$Q_i = \mathcal{V}_i \begin{pmatrix} 0 & \\ -m_{i,1}S_i & I \end{pmatrix} \mathcal{V}_i^{-1} = \mathcal{V}_i \begin{pmatrix} 0 & \\ & I \end{pmatrix} T_{l,i}^{-1} \mathcal{V}_i^{-1} \tag{4.5}$$

with the upper and lower triangle matrices

$$T_{u,i} := \begin{pmatrix} I & S_i m_{i,2} \\ & I \end{pmatrix} \text{ and } T_{l,i} := \begin{pmatrix} I & \\ m_{i,1} S_i & I \end{pmatrix}.$$

Using the detailed structure of the different matrices we find for

$$G_{i+1} = \mathcal{U}_i T_{u,i}^{-1} \left(\begin{pmatrix} S_i & \\ & 0 \end{pmatrix} + \begin{pmatrix} 0 & \\ & I \end{pmatrix} \underbrace{\mathcal{U}_i^{-1} B_i \mathcal{V}_i}_{\bar{B}_i} \begin{pmatrix} 0 & \\ & I \end{pmatrix} \right) T_{l,i}^{-1} \mathcal{V}_i^{-1} F_i.$$

If we structure $\bar{B}_i = \begin{pmatrix} b_{11}^i & b_{12}^i \\ b_{21}^i & b_{22}^i \end{pmatrix}$, we obtain the SVD of $b_{22}^i = \tilde{U}_{i+1} \begin{pmatrix} \Sigma_{i+1} & \\ & 0 \end{pmatrix} \tilde{V}_{i+1}^T$. Using this decomposition we have

$$G_{i+1} = \underbrace{\mathcal{U}_i T_{u,i}^{-1} \begin{pmatrix} I & \\ & \tilde{U}_{i+1} \end{pmatrix}}_{=:\mathcal{U}_{i+1}} \begin{pmatrix} S_i & & \\ & \Sigma_{i+1} & \\ & & 0 \end{pmatrix} \underbrace{\begin{pmatrix} I & \\ & \tilde{V}_{i+1}^T \end{pmatrix} T_{l,i}^{-1} \mathcal{V}_i^{-1} F_i}_{=:\mathcal{V}_{i+1}^{-1}}$$

and we define $S_{i+1} := \begin{pmatrix} S_i & \\ & \Sigma_{i+1} \end{pmatrix}$. The pseudo inverse of G_{i+1} is then given by

$$G_{i+1}^- = \mathcal{V}_{i+1} \begin{pmatrix} S_{i+1}^{-1} & m_{i+1,2} \\ m_{i+1,1} & m_{i+1,1} S_{i+1} m_{i+1,2} \end{pmatrix} \mathcal{U}_{i+1}^{-1}.$$

To use G_{i+1}^- for calculations we have to determine the free parameters $m_{i+1,1}$ and $m_{i+1,2}$ in such a way that (4.3) is fulfilled. From (4.5) we see that only $m_{i+1,1}$ influences Q_{i+1}, and from (4.4) that only $m_{i+1,2}$ influences W_{i+1}. Up to now we have no special conditions to the projectors W_j. What is a criterion for (4.3)?
From (4.2) we have $I - G_{i+1}^- G_{i+1} = Q_{i+1}$ and it follows that $Q_j - G_{i+1}^- G_{i+1} Q_j = 0$ has to be fulfilled for $j < i+1$, and using the structure of G_{i+1} we obtain

$$G_{i+1}^- B_j Q_j = Q_j, \quad j = 0, \dots, i. \tag{4.6}$$

Are these conditions helpful for a determination of $m_{i+1,1}$?
With $Q_j = \mathcal{V}_j \begin{pmatrix} 0 & \\ & I_j \end{pmatrix} T_{l,j}^{-1} \mathcal{V}_j^{-1}$ condition (4.6) reads in detail, after multiplying by $\mathcal{V}_j T_{l,j}$ from the right,

$$\mathcal{V}_j \begin{pmatrix} 0 & \\ & I_j \end{pmatrix} = \mathcal{V}_{i+1} \begin{pmatrix} S_{i+1}^{-1} & m_{i+1,2} \\ m_{i+1,1} & m_{i+1,1} S_{i+1} m_{i+1,2} \end{pmatrix} \mathcal{U}_{i+1}^{-1} B_j \mathcal{V}_j \begin{pmatrix} 0 & \\ & I_j \end{pmatrix}.$$

Introducing $\bar{U}_k := T_{u,k-1}\begin{pmatrix} I & \\ & \tilde{U}_k \end{pmatrix}$ it follows that

$$\underbrace{\mathcal{V}_{i+1}^{-1}\mathcal{V}_j}_{=:\tilde{w}_j}\begin{pmatrix} 0 & \\ & I_j \end{pmatrix} =$$
$$\begin{pmatrix} S_{i+1}^{-1} & m_{i+1,2} \\ m_{i+1,1} & m_{i+1,1}S_{i+1}m_{i+1,2} \end{pmatrix} \bar{U}_{i+1}^{-1}\dots\bar{U}_{j+2}^{-1}\begin{pmatrix} I & \\ & \tilde{U}_{j+1}^T \end{pmatrix} T_{u,j}^{-1}\bar{B}_j\begin{pmatrix} 0 & \\ & I_j \end{pmatrix}. \tag{4.7}$$

With $\tilde{w}_j =: \begin{pmatrix} w_j^{11} & w_j^{12} \\ w_j^{21} & w_j^{22} \end{pmatrix}$ we have for $j = 0, \dots, i$ the relation

$$\underbrace{\begin{pmatrix} w_j^{12} \\ w_j^{22} \end{pmatrix}}_{=:w_j} = \begin{pmatrix} S_{i+1}^{-1} & m_{i+1,2} \\ m_{i+1,1} & m_{i+1,1}S_{i+1}m_{i+1,2} \end{pmatrix} \underbrace{\bar{U}_{i+1}^{-1}\dots\bar{U}_{j+2}^{-1}\begin{pmatrix} I & \\ & \tilde{U}_{j+1}^T \end{pmatrix} T_{u,j}^{-1}\begin{pmatrix} b_{12}^j \\ b_{22}^j \end{pmatrix}}_{=:z_j}. \tag{4.8}$$

Let's have a look at the special structure of z_j.

$$\begin{aligned} z_j &= \bar{U}_{i+1}^{-1}\dots\bar{U}_{j+2}^{-1}\begin{pmatrix} I & \\ & \tilde{U}_{j+1}^T \end{pmatrix}\begin{pmatrix} b_{12}^j - S_j m_{j,2} b_{22}^j \\ b_{22}^j \end{pmatrix} \\ &= \bar{U}_{i+1}^{-1}\dots\bar{U}_{j+2}^{-1}\begin{pmatrix} b_{12}^j - S_j m_{j,2} b_{22}^j \\ \begin{pmatrix} \Sigma_{j+1} & 0 \end{pmatrix}\tilde{V}_{j+1}^T \\ 0 \end{pmatrix} \quad \}n - r_{j+1}. \end{aligned}$$

All factors in $\bar{U}_k^{-1}$ have the structure $\begin{pmatrix} I & \star \\ & \star \end{pmatrix}$, where the number of columns in $\begin{pmatrix} \star \\ \star \end{pmatrix}$ is less than or equal to $n - r_{j+1}$, which means that

$$z_j = \begin{pmatrix} b_{12}^j - S_j m_{j,2} b_{22}^j \\ \begin{pmatrix} \Sigma_{j+1} & 0 \end{pmatrix}\tilde{V}_{j+1}^T \\ 0 \end{pmatrix}. \tag{4.9}$$

This forms the following linear system

$$\underbrace{\begin{pmatrix} w_0 & \dots & w_i \end{pmatrix}}_{=:W} = \begin{pmatrix} S_{i+1}^{-1} & m_{i+1,2} \\ m_{i+1,1} & m_{i+1,1}S_{i+1}m_{i+1,2} \end{pmatrix} \underbrace{\begin{pmatrix} z_0 & \dots & z_i \end{pmatrix}}_{=:Z}. \tag{4.10}$$

Let us investigate the properties of Z in detail.

Lemma 4.1 *Let the DAE (1.2) be regular with index* μ*, then the matrix* $Z := \begin{pmatrix} z_0 & \dots & z_i \end{pmatrix}$ *has full (column) rank for* $i \geq 0$.

Proof: Due to of the regularity of (1.2) it holds that

$$\begin{aligned} 0 = N_k \cap N_{k+1} &= \ker G_k \cap (\ker G_k \cap \ker B_k Q_k) \\ &= \ker G_k \cap \ker B_k Q_k \quad \text{(see [Mär01]).} \end{aligned} \tag{4.11}$$

Using the decomposition of

$$G_k = \mathcal{U}_k \begin{pmatrix} S_k & \\ & 0 \end{pmatrix} \mathcal{V}_k^{-1} = \mathcal{U}_k \begin{pmatrix} S_k & \\ & 0 \end{pmatrix} T_{l,k}^{-1} \mathcal{V}_k^{-1}$$

and the structure of

$$B_k Q_k = B_k \mathcal{V}_k \begin{pmatrix} 0 & \\ & I \end{pmatrix} T_{l,k}^{-1} \mathcal{V}_k^{-1} = \mathcal{U}_k \begin{pmatrix} 0 & b_{12}^k \\ & b_{22}^k \end{pmatrix} T_{l,k}^{-1} \mathcal{V}_k^{-1}.$$

$\ker G_k$ has the representation

$$\ker G_k = \left\{ v : v = \mathcal{V}_k T_{l,k} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, v_1 = 0 \right\}$$

and

$$\ker B_k Q_k = \left\{ v : v = \mathcal{V}_k T_{l,k} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} b_{12}^k \\ b_{22}^k \end{pmatrix} v_2 = 0 \right\}.$$

The condition (4.11) means now that $\begin{pmatrix} b_{12}^k \\ b_{22}^k \end{pmatrix}$ has to have full column rank. It follows immediately that $z_k = \bar{U}_{i+1}^{-1} \dots \bar{U}_{k+2}^{-1} \begin{pmatrix} I & \\ & \tilde{U}_{k+1}^T \end{pmatrix} T_{u,k}^{-1} \begin{pmatrix} b_{12}^k \\ b_{22}^k \end{pmatrix}$ has full rank, too. Every component of Z has full rank. Now let us have a look to the rank of Z itself.

If we have reached the level i of the matrix sequence, we can compute G_{i+1} and we want to compute the nullspace projector Q_{i+1} with property (b) of Definition 1.2. We know that there exists a (reflexive) generalized inverse G_{i+1}^- with $I - G_{i+1}^- G_{i+1} = Q_{i+1}$ and (4.6). The projectors $Q_0, \dots, Q_i$ were chosen in such a way that

$$N_0 \cap \dots \cap N_i = \{0\}. \tag{4.12}$$

Consider a nontrivial linear combination of the columns of Z and let us assume that it is identically zero. With $\lambda := \big(\lambda_0, \dots, \lambda_i\big)^T$ and $\lambda_j := \big(\lambda_{j1}, \dots, \lambda_{jk_j}\big)^T$ and k_j being equal to the number of columns of z_j, which is identical with rank Q_j, we can reformulate

$$\begin{aligned}
0 &= Z\lambda = \sum_{j=0}^{i} z_j \lambda_j = \sum_{j=0}^{i} \begin{pmatrix} 0 & z_j \end{pmatrix} \begin{pmatrix} 0 \\ \lambda_j \end{pmatrix} \\
&= \sum_{j=0}^{i} \left(0 \quad \bar{U}_{i+1}^{-1} \dots \bar{U}_{j+2}^{-1} \begin{pmatrix} I & \\ & \tilde{U}_{j+1}^T \end{pmatrix} T_{u,j}^{-1} \begin{pmatrix} b_{12}^j \\ b_{22}^j \end{pmatrix} \right) \begin{pmatrix} 0 \\ \lambda_j \end{pmatrix} \\
&= \sum_{j=0}^{i} \bar{U}_{i+1}^{-1} \dots \bar{U}_{j+2}^{-1} \begin{pmatrix} I & \\ & \tilde{U}_{j+1}^T \end{pmatrix} T_{u,j}^{-1} \mathcal{U}_j^{-1} B_j \underbrace{\mathcal{V}_j \begin{pmatrix} 0 & \\ & I_j \end{pmatrix} T_{u,j}^{-1} \mathcal{V}_j^{-1}}_{=Q_j} \underbrace{\mathcal{V}_j T_{u,j} \begin{pmatrix} 0 \\ \lambda_j \end{pmatrix}}_{=:v_j} \\
&= \sum_{j=0}^{i} \mathcal{U}_{i+1}^{-1} B_j Q_j v_j .
\end{aligned}$$

Multiplying the last expression by $G_{i+1}^{-}\mathcal{U}_{i+1}$ leads to

$$\begin{aligned}
0 &= G_{i+1}^{-} \mathcal{U}_{i+1} \sum_{j=0}^{i} \mathcal{U}_{i+1}^{-1} B_j Q_j v_j \\
&= \sum_{j=0}^{i} G_{i+1}^{-} B_j Q_j v_j \quad \text{and using (4.6)} \\
&= \sum_{j=0}^{i} Q_j v_j . \qquad (4.13)
\end{aligned}$$

Because of (4.12) the elements of N_j are independent. This means that every addend $Q_j v_j$ of (4.13) is zero, hence, due to the structure of v_j, $\lambda_j = 0$. This contradicts our assumption and Z has full (column) rank. q.e.d.

Let us consider the solution of the linear system (4.10)

$$W = \begin{pmatrix} S_{i+1}^{-1} & m_{i+1,2} \\ m_{i+1,1} & m_{i+1,1} S_{i+1} m_{i+1,2} \end{pmatrix} Z.$$

Using the structure of $Z = \begin{pmatrix} \tilde{Z} \\ 0 \end{pmatrix} \}m - r_{i+1}$ (see (4.9)) we can reformulate (4.10) as

$$W = \begin{pmatrix} S_{i+1}^{-1} & 0 \\ m_{i+1,1} & 0 \end{pmatrix} Z.$$

First we discover that the parameter $m_{i+1,2}$ does not influence the computation of $m_{i+1,1}$ and, second, we can represent a solution X of $W = XZ$ by $X = WZ^-$ with an arbitrary (generalized) reflexive inverse Z^-, since $Z^-Z = I$ is valid for full column rank matrices. The appropriate part of X in the left lower corner gives us a value for $m_{i+1,1}$. Which parameter set we select depends on the used inverse Z^-. If we look at a Householder decomposition of a full column rank matrix $Z = U \begin{pmatrix} R \\ 0 \end{pmatrix}$ with nonsingular R, the (generalized) reflexive inverse is given by

$$Z^- = \begin{pmatrix} R^{-1} & \tilde{m} \end{pmatrix} U^T \tag{4.14}$$

with the free parameter $\tilde{m}$.

5 Numerical realization with MATLAB

To realize the matrix sequence (1.3) we need the matrices A, D and B. If the DAE is given by

$$f((d(x(t),t))', x(t), t) = 0, \tag{5.1}$$

the related matrices are

$$A := f'_y, \quad B := f'_x \text{ and } D := d'_x.$$

f'_y means the derivative of f with respect to the first argument. Very often theoretical investigations consider a quasilinear structure (1.1), but the algorithm is realized for more general equations (5.1).

The algorithm is realized in MATLAB. In the first step the well matched condition (see Def.1.1) of A and D is checked (see Section 3). The SVD of A and D is used to perform an SVD of the first matrix sequence element $G_0 = AD$. The free parameter $m_{0,1}$ in G_0^- (see (4.1)) is set to zero (but a parameter in the routines) and the second parameter set $m_{i,2}$, which influences the projector W_i, is set to zero. The algorithm follows the description in Section 4. That means that we have to perform an SVD of the dimension

$d_i = n - \text{rank } G_i$ in every step up to the condition that $d_i = 0$.
The computation of G_{i+1}^- needs the solution of (4.10). We check the full rank condition of Z, which checks the regularity of the DAE (see Def.1.2). The pseudo-inverse of Z is computed by (4.14) with the free parameter $\tilde{m} = 0$. All differentiations (numerical approximation of A, B or D, time differentiations in the matrix sequence to compute B_i) are performed by the MATLAB routine *numjac*.

6 Examples

We will present a few examples, that illustrate the algorithm. At first we have to describe a problem by a specially structured MATLAB routine. The structure is similar to the description of ODEs in MATLAB. The algorithm and the example files are available under http://www.mathematik.hu-berlin/~lamour/software.

The following examples are tested:

		Index	Dimension
1.	Example 2.1 from [Mär01]	3	3
2.	Classical mathematical pendulum	3	5
3.	Andrew's squeezing mechanism from [CWI]	3	27
4.	Aircraft from [CWI]	5	8
5.	Discharge pressure control from [EH89]	2	7
6.	Robotic arm from [CWI]	5	8
7.	Electronic circuit [Tis]	2	2

The index of the examples was verified by the algorithm. One of the reasons for developing this algorithm was to compute projectors Q_i with (1.5). The following table summarizes the dimensions of the matrices Σ_j of the different levels, the compliance with the property (1.5) and the projector property $Q_j^2 - Q_j = 0$.

Ex.	Σ_0	Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	$\max\limits_{\substack{0\le k\le index-1\\ j<k}} \|Q_kQ_j\|$	$\max\limits_{j} \|Q_j^2 - Q_j\|$
1.	2	0	0	1			0	0
2.	4	0	0	1			4.8e-16	3.9e-16
3.	14	7	0	6			8.7e-12	1.1e-11
4.	6	1	0	0	0	1	6.7e-15	4.3e-15
5.	3	3	1				5.0e-15	7.9e-15
6.	6	0	0	1	0	1	6.8e-14	4.4e-14
7.	1	0	1				0	0

References

[CWI] CWI, http://www.cwi.nl/ftp/IVPtestset/. *Test Set for IVP Solvers.*

[EH89] Michel Roche Ernst Hairer, Christian Lubich. *The Numerical Solution of Differential-Algebraic Systems by Runge-Kutta Methods.* Number 1409 in Lecture Notes in Mathematics. Springer-Verlag, 1989.

[Mär01] R. März. The index of differential algebraic systems with properly stated leading term. Preprint 2001-7, Humboldt-Universität, Institut für Mathematik, Berlin, http://www.mathematik.hu-berlin.de/publ/pre/2001/M-01-7.html, 2001.

[Tis] Caren Tischendorf. private communication.

[Zie79] Gerhard Zielke. Motivation und Darstellung von verallgemeinerten Matrixinversen. *Beiträge zur Numerischen Mathematik*, 7:177–218, 1979.