On a generic class of regular one-parametric variational inequalities.

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Abstract

In this paper the regularity of one-parametric optimization problems in the sense of Jongen, Jonker and Twilt is extended to one-parametric variational inequalities. The five singularities are defined in this context and suitable indices are described around them. Many of the local properties of the singularities are also proved for this case.

The generic property of the defined class of regular one-parametric variational inequalities is proved and a corresponding linear quadratic perturbation result is stated.

Keywords: one-parametric variational inequalities, singularities, genericity

1 Introduction.

Let $F : M \to \mathbb{R}^n$ be a mapping and $M \subset \mathbb{R}^n$ be a closed set defined as

$$M = \{x \in \mathbb{R}^n | H(x) = 0, G(x) \geq 0\},$$

where $(H, G) : \mathbb{R}^n \to \mathbb{R}^{m+p}$. The variational inequality problem is defined as follows: Find some point $x \in M$ such that

$$F(x)(y - x) \geq 0, \quad \forall y \in M. \tag{1}$$

This problem has found wide application as an adequate mathematical framework for a number of economic, game-theoretic and equilibrium problems. For extensive surveys we cite e.g. [5] and [13].

Special cases of variational inequalities are for example equality systems ($M = I\!R^n$), nonlinear complementarity problems ($M = I\!R^n_+$), and convex minimization problems ($F = \nabla f$).

If M is convex ($-g_j$ are convex and h_i are affine linear) the solutions of the variational inequalities can be characterized, under constraint qualification, by a suitable version of the well-know KKT-system. This convex case have been intensively studied and various reformulations of the variational inequality problem are given, for example as a system of equations (e.g. the normal map), as a generalized equation (via the normal cone) and as an optimization problem (constrained and unconstrained). Many of the solution approaches proposed are very similar and base on the use of generalized Newton methods for nonsmooth equations (see e.g. [13], [5], [6], [3], [2], [8]).

A further approach was given recently (see e.g. [15], [18], [19] [17]), which base on the use of continuation methods. In this paper we are also interested in the application of such continuation techniques for the solution of variational inequalities. Our main purpose is to extend the well-known regularity of one-parametric optimization problems in the sense of Jongen, Jonker and Twilt (see [11]) to the case of one-parametric variational inequalities. This regularity deals with the local structure of the solution sets of one-parametric optimization problems and is an important theoretical issue, for instance by the so called path-following methods with jumps (see e.g. [4]).

The paper is organized as follows. In the second section we redefine the singularities, which characterize the regularity in the sense of Jongen, Jonker and Twilt, in the case of one-parametric varational inequalities and state the most important local properties. In the third section other properties are stated, for instance the generic character of the regular problems.

2 One-parametric variational inequalities.

Let us consider variational inequalities $(F(x,t), H(x,t), G(x,t))$ depending of a one-dimensional real parameter t. Such problems are defined by its data $(F, H, G) : I\!R^{n+1} \to I\!R^{n+m+p}$ and will also be denoted by $VI(F, H, G)$. The corresponding feasible set $M \subset I\!R^{n+1}$ will also be denoted by $M(H, G)$.

Let us denote $z = (x, t) \in I\!R^{n+1}$, $I = \{1, \ldots, m\}$ and $J = \{1, \ldots, p\}$. For a given parameter value t the notation $VI(F,H,G)(t)$ and $M(H,G)(t) \subset I\!R^n$ (or simply $M(t)$) state for the corresponding variational inequality and its feasible set.

Many well-known concepts from optimization theory concerning feasible sets apply also to variational inequalities, for instance constraint qualifications as LICQ or MFCQ, active index set, etc.

A feasible point $z \in M(H,G)$ of the $VI(F,H,G)$ is a *critical point* if there exist $(\lambda,\mu) \in I\!R^{m+|J_0(z)|}$ satisfying the following relation

$$\mathsf{L}(z,\lambda,\mu) = 0, \tag{2}$$

where $\mathsf{L}(z,\lambda,\mu) = F(z) - \sum_{i=1}^m \lambda_i D_x h_i(z) - \sum_{j\in J_0(z)} \mu_j D_x g_j(z)$. If $\mu_j \geq 0$, $\forall j \in J_0(z)$ then z is called a *stationary point.*

If the vectors $\{F(z), D_x h_i(z), D_x g_j(z), i \in I, j \in J_0(z)\}$ are linearly dependent, z is defined as a *generalized critical point*, shortly g.c. point (see [11]).

We consider in this paper mainly the sets

$$\begin{aligned} \Sigma_{gc} &= \left\{ z \in I\!R^{n+1} | z \text{ is a g.c. point of } VI(F,H,G) \right\} \\ \Sigma_{stat} &= \left\{ z \in I\!R^{n+1} | z \text{ is a stationary point of } VI(F,H,G) \right\} \end{aligned}$$

If necessary, the notation $\Sigma(F,H,G)$ will be used for recalling the data.

Definition 1
A g.c. point $\bar{z}$ of $VI(F,H,G)$ is called non-degenerated if the LICQ holds and the following conditions are fulfilled:

VU-ND1 *: The uniquely determined solution $(\bar{\lambda},\bar{\mu}) \in I\!R^{m+|J_0(\bar{z})|}$ of (2) satisfies $\bar{\mu}_j \neq 0$, $\forall j \in J_0(\bar{z})$.*

VU-ND2 *: The matrix $D_x\mathsf{L}(\bar{z},\bar{\lambda},\bar{\mu})|_{T_{\bar{x}}M(\bar{t})}$ is nonsingular.*

Here $T_{\bar{x}}M(\bar{t})$ is the tangential space to the set $M(\bar{t})$ at the point $\bar{x}$ and $D_x\mathsf{L}(\bar{z},\bar{\lambda},\bar{\mu})|_{T_{\bar{x}}M(\bar{t})}$ is a matrix of the form $V^T D_x\mathsf{L}(\bar{z},\bar{\lambda},\bar{\mu})V$, where the columns from V build a linear basis for $T_{\bar{x}}M(\bar{t})$. In general, $D_x\mathsf{L}(\bar{z},\bar{\lambda},\bar{\mu})$ is not symmetric. The *sign* $\left[\det(V^T D_x\mathsf{L}(\bar{z},\bar{\lambda},\bar{\mu})V)\right]$ is independent of the possible selection of V. Let us define for $K \subset J$ the following mapping

$$\mathsf{H}_K(z,\lambda,\mu) = \begin{pmatrix} F(z) - \sum_{i=1}^m \lambda_i D_x h_i(z) - \sum_{j\in K} \mu_j D_x g_j(z) \\ H(z) \\ G_K(z) \end{pmatrix}.$$

Remark 1
Under the LICQ the Condition **VU-ND2** *is equivalent to the regularity from* $D_{(x,\lambda,\mu)}\mathsf{H}_{J_0}(z,\lambda,\mu)$*. The set of critical points is given locally under* **VU-ND1** *by the projection onto the* z*-components of the zeros of* H_{J_0} *and, therefore, due to the non-degeneracy it builts a smooth one-dimensional manifold that is parametrizable in* t*.*

At a non-degenerated g.c. point the LI and LCI are uniquely determined (number of negative and positive entries of $\bar{\mu}$). $sign\left[\det(D_x\mathsf{L}(\bar{z},\bar{\lambda},\bar{\mu})|_{T_{\bar{x}}M(\bar{t})})\right]$ is also uniquely determined. Around the different singularities we are then going to describe the changes of the triple

$$(LI, LCI, sign\left[\det(D_x\mathsf{L}(z,\lambda,\mu)|_{T_xM(t)})\right]) \tag{3}$$

Taking into account the relation

$$sign\left[\det(D_x\mathsf{L}(\bar{z},\bar{\lambda},\bar{\mu})|_{T_{\bar{x}}M(\bar{t})})\right] = sign\left[\det(D_{(x,\lambda,\mu)}\mathsf{H}_{J_0}(\bar{z},\bar{\lambda},\bar{\mu}))\right],$$

we define $sign(\det(D_x\mathsf{L}(\bar{z},\bar{\lambda},\bar{\mu})|_{T_{\bar{x}}M(\bar{t})})) = +1$ for the case of $T_{\bar{x}}M(\bar{t}) = \{0\}$.

2.1 Singularities and local properties.

A g.c. point $\bar{z}$ is of Type 1 if it is non-degenerated as in Definition 1.

Remark 2
In the neighbourhood of a g.c. point $\bar{z}$ *of Type 1 the points of* Σ_{gc} *are non-degenerated and the triple (3) remain constant.*

Definition 2 *(see also [11])*
A g.c. point $\bar{z}$ *of* $VI(F,H,G)$ *is of Type 2 if the following conditions hold:*

Type2: 1-VU *: The LICQ is fulfilled*

Type2: 2-VU *: Let* $(\bar{\lambda},\bar{\mu}) \in I\!R^{m+|J_0(z)|}$ *solve (2), then there exists exactly one index* $l \in J_0(\bar{z})$*, such that* $\bar{\mu}_l = 0$*, and* $\bar{\mu}_k \neq 0$*,* $\forall k \in J_0(\bar{z}) \setminus \{\, l\}$*.*

Type2: 3-VU *:* $D_x\mathsf{L}(\bar{z},\bar{\lambda},\bar{\mu})|_{T_{\bar{x}}M(\bar{t})}$ *is nonsingular.*

Type2: 4-VU *:* $D_x\mathsf{L}(\bar{z},\bar{\lambda},\bar{\mu})|_{T^+_{\bar{x}}M(\bar{t})}$ *is nonsingular, where*

$$T^+_{\bar{x}}M(\bar{t}) = \left\{\xi \in I\!R^n | D_xH(\bar{z})\cdot\xi = 0, D_xG_{J^+_0(\bar{z})}(\bar{z})\cdot\xi = 0\right\}$$

and $J^+_0(\bar{z}) = \{j \in J_0(\bar{z}) | \bar{\mu}_l \neq 0\} = J_0(\bar{z}) \setminus \{l\}$*.*

Type2: 5-VU *: Let us denote*

- $\Pi_+ = (H, G_{J_0^+(\bar{z})})^T$, *and*
- W *is a matrix columns of which form a basis of* $T_{\bar{x}}^+ M$.

It holds that $\gamma \neq 0$, *where*

$$\begin{array}{rcl} \gamma &=& D_x g_l(\bar{z})(\alpha + \beta) + D_t g_l(\bar{z}) \\ \alpha &=& -((D_x \Pi_+(\bar{z}))^\dagger)^T \cdot D_t \Pi_+(\bar{z}), \\ \beta &=& -W(W^T \cdot D_x \mathsf{L} \cdot W)^{-1} W^T \cdot [D_x \mathsf{L} \cdot \alpha + D_t \mathsf{L}] \end{array}$$

and $B^\dagger = (B^T B)^{-1} B^T$ *represent the Moore-Penrose Inverse of* B.

Denote by $(x^{J_0^+}(t), t, \lambda^{J_0^+}(t), \mu^{J_0^+}(t))$ (resp. $(x^{J_0}(t), t, \lambda^{J_0}(t), \mu^{J_0}(t))$) a t-parametrization for the zeros of $\mathsf{H}_{J_0^+}$ (resp. H_{J_0}).

Remark 3
It is easy to see that, around $\bar{z}$, *the set* Σ_{gc} *is given by the curve* $(x^{J_0}(t), t)$ *and the feasible part of* $(x^{J_0^+}(t), t)$. **Type2: 4-VU** *implies that the tangent lines of the two curves are not parallel.*

For the description of the change from the triple (3) we use the characteristic numbers $sign(\gamma)$ and $sign(\delta)$, where

$$\delta = \det(D_x \mathsf{L}(\bar{z}, \bar{\lambda}, \bar{\mu})|_{T_{\bar{x}} M(\bar{t})}) \cdot \det(D_x \mathsf{L}(\bar{z}, \bar{\lambda}, \bar{\mu})|_{T_{\bar{x}}^+ M(\bar{t})}).$$

Definition 3
Let A *be a matrix divided as follows* $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$, *where* B *and* E *are quadratic. Let* B *be regular. The matrix* $S(A|B) = E - DB^{-1}C$ *is called the Schur-complement of* B *in* A.

Lemma 1 *(see [14])*
Let A *be as in Definition 3. Then it holds:*

1. $\det(A) = \det(E) \cdot \det(S(A|B))$.
2. *If* A *is symmetric, it follows:* $In(A) = In(B) + In(S(A|B))$.

3. If A is regular, its inverse is given by:

$$A^{-1} = \begin{pmatrix} B^{-1} + B^{-1} \cdot C \cdot S(A|B)^{-1} \cdot D \cdot B^{-1} & -B^{-1} \cdot C \cdot S(A|B)^{-1} \\ -S(A|B)^{-1} \cdot D \cdot B^{-1} & S(A|B)^{-1} \end{pmatrix}.$$

Here $In(P) = (p(P), n(P), z(P))$ represents the so-called innertia triple of a symmetric quadratic matrix P, where $p(P)$ (resp. $n(P)$ and $z(P)$) means the number of positive (resp. negative or zero) eigenvalues of P.

Proposition 1
Let $\bar{z}$ be a g.c. point of Type 2 with vanishing multiplier $\bar{\mu}_l$. It holds then that $sign(\gamma) \cdot sign(\delta) = -sign(\dot{\mu}_l^{J_0}(\bar{t}))$.

Proof :
Let the columns of the matrix V_1 build a basis for $T_{\bar{x}}M(\bar{t})$ and the columns of $V_0 = \left[V_1 \vdots v_1\right]$ one for $T_{\bar{x}}^{+}M(\bar{t})$. It holds:

$$D_x \mathsf{L}|_{T_{\bar{x}}^{+}M(\bar{t})} = \begin{pmatrix} V_1^T D_x \mathsf{L} V_1 & V_1^T D_x \mathsf{L} v_1 \\ v_1^T D_x \mathsf{L} V_1 & v_1^T D_x \mathsf{L} v_1 \end{pmatrix}.$$

Type2: 3-VU implies the regularity of $V_1^T D_x \mathsf{L} V_1$. Lemma 1 (1.-) gives that $\det(D_x \mathsf{L}|_{T_{\bar{x}}^{+}M(\bar{t})}) = \det(D_x \mathsf{L}|_{T_{\bar{x}}M(\bar{t})}) \det(S(D_x \mathsf{L}|_{T_{\bar{x}}^{+}M(\bar{t})} | D_x \mathsf{L}|_{T_{\bar{x}}M(\bar{t})}))$, where

$$\begin{aligned} S(D_x \mathsf{L}|_{T_{\bar{x}}^{+}M(\bar{t})} | D_x \mathsf{L}|_{T_{\bar{x}}M(\bar{t})}) &= v_1^T D_x \mathsf{L} v_1 - \\ & \quad v_1^T D_x \mathsf{L} V_1 \left(V_1^T D_x \mathsf{L} V_1\right)^{-1} V_1^T D_x \mathsf{L} v_1. \end{aligned}$$

Consequently, it holds:

$$sign(\delta) = sign\left(S(D_x \mathsf{L}|_{T_{\bar{x}}^{+}M(\bar{t})} | D_x \mathsf{L}|_{T_{\bar{x}}M(\bar{t})})\right). \quad (4)$$

Since the derivatives of the components $(H, G_{J_0^+})$ in $\mathsf{H}_{J_0^+}$ and H_{J_0} coincide, it holds that $\dot{x}^{J_0^+}(\bar{t}) - \dot{x}^{J_0}(\bar{t}) \in T_{\bar{x}}^{+}M(\bar{t})$. Then there exists w_0 with $\dot{x}^{J_0^+}(\bar{t}) - \dot{x}^{J_0}(\bar{t}) = V_0 w_0$. Multiplying the Jacobi-matrix of L from the left-hand side by V_0 we obtain that $V_0^T D_x \mathsf{L} V_0 w_0 + \dot{\mu}_l^{J_0}(\bar{t}) V_0^T D_x g_l(\bar{z}) = 0$. Consequently,

$$\dot{x}^{J_0^+}(\bar{t}) - \dot{x}^{J_0}(\bar{t}) = -\dot{\mu}_l^{J_0}(\bar{t}) \cdot V_0 \left(V_0^T D_x \mathsf{L} V_0\right)^{-1} V_0^T D_x^T g_l(\bar{z}). \quad (5)$$

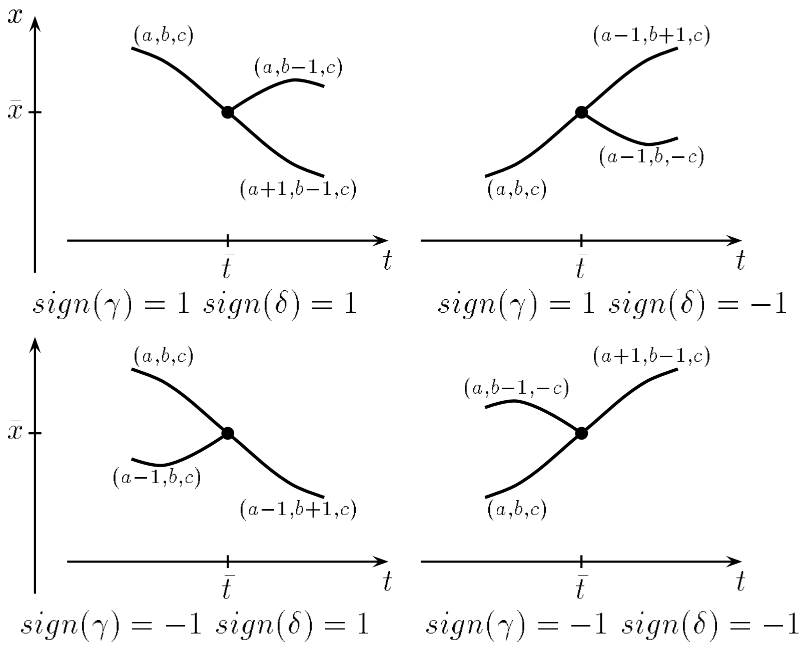

Figure 1: Type 2

Since $D_x g_l(\bar{z}) \cdot \dot{x}^{J_0}(\bar{t}) + D_t g_l(\bar{z}) = 0$, it follows that

$$\gamma = D_x g_l(\bar{z}) \cdot \dot{x}^{J_0^+}(\bar{t}) + D_t g_l(\bar{z}) = D_x g_l(\bar{z})(\dot{x}^{J_0^+}(\bar{t}) - \dot{x}^{J_0}(\bar{t})).$$

Now (5) provides $\gamma = -\dot{\mu}_l^{J_0}(\bar{t}) \cdot D_x g_l(\bar{z}) V_0 \left(V_0^T D_x \mathsf{L} V_0\right)^{-1} V_0^T D_x^T g_l(\bar{z})$. Since $V_0 = \left[V_1 \vdots v_1\right]$, it follows that

$$\gamma = -\dot{\mu}_l^{J_0}(\bar{t}) \cdot \left(0 \vdots D_x g_l(\bar{z}) v_1\right)^T \begin{pmatrix} V_1^T D_x \mathsf{L} V_1 & V_1^T D_x \mathsf{L} v_1 \\ v_1^T D_x \mathsf{L} V_1 & v_1^T D_x \mathsf{L} v_1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ v_1^T D_x^T g_l(\bar{z}) \end{pmatrix}.$$

Lemma 1 and (4) together give:

$$\gamma = -\dot{\mu}_l^{J_0}(\bar{t}) \cdot \left(D_x g_l(\bar{z}) v_1^T\right)^2 \left(D_x \mathsf{L}|_{T_{\bar{x}}^+ M(\bar{t})} | D_x \mathsf{L}|_{T_{\bar{x}} M(\bar{t})}\right)^{-1}.$$

Substituting (4) in the above relation concludes the proof. □

Using Proposition 1 we obtain the changes of the triple (3) as shown in Figure 1.

Definition 4 *(see also [11])*
A g.c. point $\bar{z}$ of $VI(F,H,G)$ is of Type 3 if the following conditions hold:

Type3: 1-VU *: The LICQ is fulfilled.*

Type3: 2-VU *:* **VU-ND1** *is fulfilled.*

Type3: 3-VU *:* $D_x \mathsf{L}(\bar{z}, \bar{\lambda}, \bar{\mu})|_{T_{\bar{x}} M(\bar{t})}$ *has rank* $n - m - |J_0(\bar{z})| - 1$.

Type3: 4-VU *: Let us denote*

- $\Pi = (H, G_{J_0(\bar{z})})$,
- W *is a matrix whose columns form a linear basis of* $T_{\bar{x}} M(\bar{t})$,
- $w_1 \neq 0$ *is a vector, with* $W^T D_x \mathsf{L}(\bar{z}, \bar{\lambda}, \bar{\mu}) W \cdot w_1 = 0$, $v_1 = W \cdot w_1$, *and*
- $w_2 \neq 0$ *is a vector, with* $W^T D_x^T \mathsf{L}(\bar{z}, \bar{\lambda}, \bar{\mu}) W \cdot w_2 = 0$, $v_2 = W \cdot w_2$.

It holds that $\beta_1 \cdot \beta_2 \neq 0$, *where*

$$\begin{aligned} \beta_1 &= v_1^T (D_x^2 \mathsf{L} \cdot v_2) v_1 - 2 v_1^T D_x^T \mathsf{L} ((D_x^T \Pi)^\dagger)^T \cdot (v_1^T D_x^2 \Pi \cdot v_2) \\ &\quad - v_2^T D_x \mathsf{L} ((D_x^T \Pi)^\dagger)^T (v_1^T D_x^2 \Pi v_1), \\ \beta_2 &= D_t \mathsf{L} \cdot v_2 - D_t^T \Pi \cdot (D_x^T \Pi)^\dagger D_x^T \mathsf{L} \cdot v_2 \end{aligned}$$

and the following notation is used:

$$v_1^T (D_x^2 \mathsf{L} \cdot v_2) v_1 = v_1^T \left[D_x (D_x^T \mathsf{L} \cdot v_2) \right] v_1^T,$$
$$v_1^T D_x^2 \Pi v_2 = \left(v_1^T D_x^2 h_i v_2, i \in I, v_1^T D_x^2 g_j v_2, j \in J_0 \right)^T.$$

Remark 4 *(see also [16])*
Under the LICQ and **Type3: 3-VU** *the condition* **Type3: 4-VU** *is equivalent to the non-degeneracy of the vector* $(\bar{z}, \bar{\lambda}, \bar{\mu})$ *as a critical point of the optimization problem:*

$$\mathcal{P}_{VU} \quad \min_{(z, \lambda, \mu) \in \mathcal{X}_{VU}} \Lambda(z, \lambda, \mu) = t,$$
$$\mathcal{X}_{VU} = \left\{ (z, \lambda, \mu) \in I\!R^{n+m+|J_0(\bar{z})|+1} | \mathsf{H}_{J_0}(z, \lambda, \mu) = 0 \right\}.$$

Remark 5
Σ_{gc} *is locally described by the projection of* $\mathcal{X}_{VU}$ *onto the* z*-components. Consequently,* Σ_{gc} *can be parametrized around* $\bar{z}$ *by one of the* x*-components, and the local structure is as shown in Figure 2.*

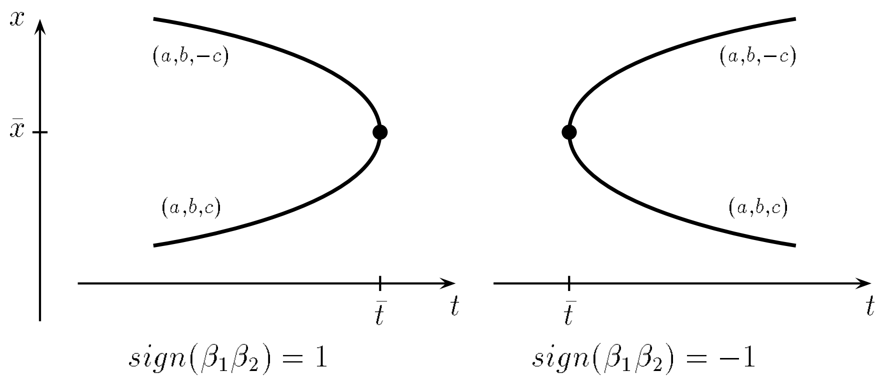

Figure 2: Type 3

In $\bar{z}$ the curve Σ_{gc} has a turning point in the variable t, since $(\bar{z}, \bar{\lambda}, \bar{\mu})$ is a locally maximizer or minimizer of $\mathcal{P}_{VU}$. The position of the turning point $\bar{z}$ ($t \geq \bar{t}$ as locally minimizer, the other $t \leq \bar{t}$ as locally maximizer) depends directly on $sign(\beta_1\beta_2)$ (see Figure 2). It is easy to note that $sign(\beta_1)$ and $sign(\beta_1\beta_2)$ are independent of the selected vectors v_1 and v_2 in **Type3: 4-VU**.

The linear indices remain locally constant along Σ_{gc}. The local changes of $sign\left[\det(D_x \mathsf{L}(z,\lambda,\mu)|_{T_xM(t)})\right]$ are explained in Theorem 1 and in Remark 17.

Definition 5 *(see also [11])*
A g.c. point $\bar{z}$ of $VI(F,H,G)$ is of Type 4 if the following conditions hold:

Type4: 1-VU *: $0 < m + |J_0(\bar{z})| < n+1$.*

Type4: 2-VU *: The matrix $D_x \begin{bmatrix} H(\bar{z}) \\ G_{J_0(\bar{z})}(\bar{z}) \end{bmatrix}$ has rank $m + |J_0(\bar{z})| - 1$.*

Type4: 3-VU *: Let $(\bar{\lambda}, \bar{\mu})$ satisfying $(\bar{\lambda}, \bar{\mu}) \cdot D_x \begin{bmatrix} H(\bar{z}) \\ G_{J_0(\bar{z})}(\bar{z}) \end{bmatrix} = 0$ be fixed. Then it holds that $\mu_j \neq 0$, $\forall j \in J_0(\bar{z})$.*

Type4: 4-VU *: $D_t\mathbf{L}(\bar{z}) \neq 0$ and the matrix $A = D_t\mathbf{L}(\bar{z}) \cdot W^T D_x^2 \mathbf{L}(\bar{z}) W$ is regular, where $\mathbf{L}(z) = \sum_{i\in I} \bar{\lambda}_i h_i(z) + \sum_{j\in J_0(\bar{z})} \bar{\mu}_j g_j(z)$ and the columns of W form a basis for the $(n-m-J_0(\bar{z})+1)$-dimensional linear space*

$$T = \{\eta \in I\!R^n | Dh_i(\bar{z})\eta = 0, Dg_j(\bar{z})\eta = 0, i \in I, j \in J_0(\bar{z})\}$$

Type4: 5-VU *:* $F^T(\bar{z})WA^{-1}W^TF(\bar{z}) \neq 0$.

Let us fix $l \in J_0(\bar{z})$ and denote $\theta = (\theta_0, \theta_i, \theta_j, i \in I, j \in J_0(\bar{z}) \setminus \{l\})$.

Remark 6 *(see also [16])*
If **Type4: 1-VU**, **Type4: 2-VU** *and* **Type4: 3-VU** *are fulfilled, the conditions* **Type4: 4-VU** *and* **Type4: 5-VU** *together are equivalent to the following non-degeneracy:*

The vector $(\bar{z}, \bar{\theta})$*, where* $\bar{\theta} = (0, \bar{\lambda}, \bar{\mu}_j, j \in J_0(\bar{z}) \setminus l)$*, is a non-degenerated critical point of the problem* $\widetilde{\mathcal{P}}_{VU}$*:*

$$\widetilde{\mathcal{P}}_{VU} \quad \min_{(z,\theta)\in\widetilde{\mathcal{X}}_{VU}} \quad \tilde{\Lambda}(z,\theta) = t,$$

$$\widetilde{\mathcal{X}}_{VU} = \left\{(z,\theta) \in I\!R^{n+m+|J_0(\bar{z})|+1} | \Gamma_{VU}(z,\theta) = 0\right\}$$

Here $\Gamma_{VU}(z,\theta)$ *is defined by:*

$$\Gamma_{VU}(z,\theta) = \begin{pmatrix} \theta_0 F(z) - \sum_{i\in I}\theta_i D_x h_i(z) - \sum_{j\in J_0(\bar{z})\setminus\{l\}} \theta_j D_x g_j(z) - \bar{\mu}_l D_x g_l(z) \\ H(z) \\ G_{J_0(\bar{z})}(z) \end{pmatrix}.$$

Remark 7
Since $W^TF(\bar{z}) \neq 0$*, the vectors*

$$\{F(\bar{z}), D_x h_i(\bar{z}), D_x g_j(\bar{z}), i \in I, j \in J_0(\bar{z}) \setminus \{k\}\} \tag{6}$$

are linearly independent for arbitrary $k \in J_0(\bar{z})$*. Therefore, the projection of* $\widetilde{\mathcal{X}}_{VU}$ *onto the z-components describes the set* Σ_{gc} *around* $\bar{z}$*.* $\widetilde{\mathcal{X}}_{VU}$ *can be parametrized with a smooth mapping* $(z(\tau), \theta(\tau))$ *defined around zero satisfying* $(z(0), \theta(0)) = (\bar{z}, \bar{\theta})$*. From the LICQ it follows that* $\dot{t}(0) = 0 \neq \dot{\theta}_0(0)$*. Consequently, LICQ is fulfilled around* $\bar{z}$ *at each point of* Σ_{gc} *different from* $\bar{z}$*. The linear indices change their values for* $\tau < 0$ *and* $\tau > 0$*, since the Lagrange-multipliers are given by* $\frac{1}{\theta_0(\tau)}(\theta_i(\tau), \theta_j(\tau), i \in I, j \in J_0(\bar{z}) \setminus \{l\}, \bar{\mu}_l)$*.*

Lemma 2 *(see [12])*
Let A *be a symmetric* $(n \times n)$*-matrix and* B *an* $(n \times m)$*- matrix with rank* $\bar{m}$*. If* $C = \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}$*, it follows* $In(C) = In(C|_{Kern B^T}) + In(\bar{m}, \bar{m}, m - \bar{m})$*.*

Here $KernB^T = \left\{\xi \in I\!R^n | B^T \cdot \xi = 0\right\}$ and $C|_{KernB^T}$ denotes a matrix of the form $\tilde{B}^T C \tilde{B}$, where the columns of $\tilde{B}$ form a linear basis for $KernB^T$.

Proposition 2
Let $(z(\tau), \theta(\tau))$ parametrize the curve $\widetilde{\mathcal{X}}_{VU}$ around a g.c. point $\bar{z}$ of Type 4 with $(z(0), \theta(0)) = (\bar{z}, \bar{\theta}))$. Then, for $\tau \neq 0$, $z(\tau)$ is a non-degenerated g.c. point. sign $\left[\det(D_x \mathsf{L}(z(\tau))|_{T_{x(\tau)}M(t(\tau))})\right]$ is the same (or different) for $\tau > 0$ and $\tau < 0$ if and only if $n - m - J_0(\bar{z})$ is even (resp. odd).

<u>Proof:</u>
Let $l \in J_0(\bar{z})$ be fixed. We use the following notations:
$B_0(z) = D_x \begin{pmatrix} H(z) \\ G_{J_0(\bar{z})\setminus\{l\}}(z) \\ F^T(z) \end{pmatrix}$, $B_1(z) = D_x \begin{pmatrix} H(z) \\ G_{J_0(\bar{z})\setminus\{l\}}(z) \end{pmatrix}$, and
$\mathcal{L}(z, \theta) = \theta_0 F(z) - \sum_{i \in I} \theta_i D_x h_i(z) - \sum_{j \in J_0(\bar{z})\setminus\{l\}} \theta_j D_x g_j(z) - \bar{\mu}_l D_x g_l(z)$.

Let $W_0(z)$ be defined in a neighbourhood $U_{\bar{z}}$ of $\bar{z}$ such that its columns form a linear basis for $KernB_0(z)$. $W_0(z)$ can be choosen depending from z as smooth as $B_0(z)$. Let the columns of a matrix $W_1 = \left[W_0(\bar{z}) \vdots b_0\right]$ form a basis for $KernB_1(z)$.

Consider τ near zero with $z(\tau) \in U_{\bar{z}}$. The columns of $W_0(z(\tau))$ form then a basis for $T_{x(\tau)}M(t(\tau))$ and, since LICQ and **VU-ND1** are fulfilled by Remark 7, the non-degeneracy of $z(\tau)$ holds if $W_0^T(z(\tau)) D_x \mathsf{L}(z(\tau)) W_0(z(\tau))$ is regular, where $D_x \mathsf{L}(z(\tau)) = \frac{1}{\theta_0(\tau)} D_x \mathcal{L}(z(\tau), \theta(\tau))$. Consequently, it is sufficient to show the regularity of

$$W_0^T(z(0)) D_x \mathcal{L}(z(0), \theta(0)) W_0(z(0)) = -W_0^T(\bar{z}) D_x^2 \mathbf{L}(\bar{z}) W_0(\bar{z}), \tag{7}$$

where $\mathbf{L}(z)$ is intended as in Definition 5. The equation

$$\det\left(W_0^T(z(\tau)) D_x \mathsf{L}(z(\tau)) W_0(z(\tau))\right) = \left(\frac{1}{\theta_0(\tau)}\right)^{(n-m-J_0(\bar{z}))} \det\left(W_0^T(z(\tau)) D_x \mathcal{L}(z(\tau), \theta(\tau)) W_0(z(\tau))\right) \tag{8}$$

explains the changes of $sign\left[\det(D_x \mathsf{L}(z(\tau))|_{T_{x(\tau)}M(t(\tau))})\right]$.

Using Lemma 2 we obtain that the matrix (7) is regular if and only if

$$\begin{pmatrix} D_x^2 \mathbf{L}(\bar{z}) & B_1(\bar{z}) & F(\bar{z}) \\ B_1^T(\bar{z}) & 0 & 0 \\ F^T(\bar{z}) & 0 & 0 \end{pmatrix} \tag{9}$$

is so, too, and that the submatrix $\begin{pmatrix} D_x^2\mathbf{L}(\bar{z}) & B_1(\bar{z}) \\ B_1^T(\bar{z}) & 0 \end{pmatrix}$ is regular, since A (as in Definition 5) is nonsingular. From Lemma 1 the regularity of (9) is obtained if

$$\begin{pmatrix} F^T(\bar{z}) & 0 \end{pmatrix} \begin{pmatrix} D_x^2\mathbf{L}(\bar{z}) & B_1(\bar{z}) \\ B_1^T(\bar{z}) & 0 \end{pmatrix}^{-1} \begin{pmatrix} F(\bar{z}) \\ 0 \end{pmatrix} \neq 0.$$

However the above number is equal to $F^T(\bar{z})W_1 \left[W_1^T D_x^2\mathbf{L}(\bar{z})W_1\right]^{-1} W_1^T F(\bar{z})$, and the proof is concluded by using **Type4: 5-VU**. $\square$

Remark 8
$\dot{x}(0) \neq 0$, since the vectors (6) are linearly independent and $\dot{t}(0) = 0$. Σ_{gc} is then parametrizable around $\bar{z}$ with respect to an x-component (see Figure 3).

Around $\bar{z}$ Σ_{gc} has a turning point in t, since $(\bar{z}, \bar{\theta})$ is a non-degenerated stationary point of $\widetilde{\mathcal{P}}_{VU}$. The geometric position of the turning point is determined by the number

$$D^2_{(z,\theta)} L(\bar{z}, \bar{\theta})|_{T_{(\bar{z},\bar{\theta})}\widetilde{\mathcal{X}}_{VU}} = \frac{-1}{D_t\mathbf{L}(\bar{z})} \cdot F^T(\bar{z})W \left[W^T D_x^2\mathbf{L}(\bar{z})W\right]^{-1} W^T F(\bar{z}),$$

where L is the Lagrange-function corresponding to $\widetilde{\mathcal{P}}_{VU}$ and W is intended as in Definition 5.

Since $sign(D^2_{(z,\theta)} L(\bar{z}, \bar{\theta})|_{T_{(\bar{z},\bar{\theta})}\widetilde{\mathcal{X}}_{VU}}) = -sign(F^T(\bar{z})WA^{-1}W^T F(\bar{z}))$, we can use $\alpha = sign(F^T(\bar{z})WA^{-1}W^T F(\bar{z}))$, for the characterization of the geometric position of the curve Σ_{gc} (see Figure 3).

Finally $sign(\beta) = (-1)^{n-m-J_0(\bar{z})}$ determines according to Proposition 2 the changes of $sign\left[\det(D_x \mathsf{L}(z(\tau))|_{T_{x(\tau)}M(t(\tau))})\right]$.

Remark 9
Let us consider a point $\bar{z}$ of Type 4 in the case that the sets $M(t)$ are convex (h_i linear affine and then $J_0(\bar{z}) \neq \emptyset$). Let $(z(\tau), \theta(\tau))$ be a parametrization as in the above Proposition 2, and such that $z(\tau)$ is a stationary point for $\tau < 0$. It follows that every $\bar{\mu}_j$, $j \in J_0(\bar{z})$, in Definition 5 has the same sign (i.e., the MFCQ is not fulfilled). Let us fix then $\bar{\mu}_j > 0, \forall j \in J_0(\bar{z})$. It follows that $\theta_0(\tau) > 0$ for $\tau < 0$.

The LICQ holds for $\tau < 0$ and the convexity of $M(t(\tau))$ implies that $D_x^2 g_j(z(\tau))$, $j \in J_0(\bar{z})$, are negative semidefinite on $T_{x(\tau)}M(t(\tau))$. Therefore,

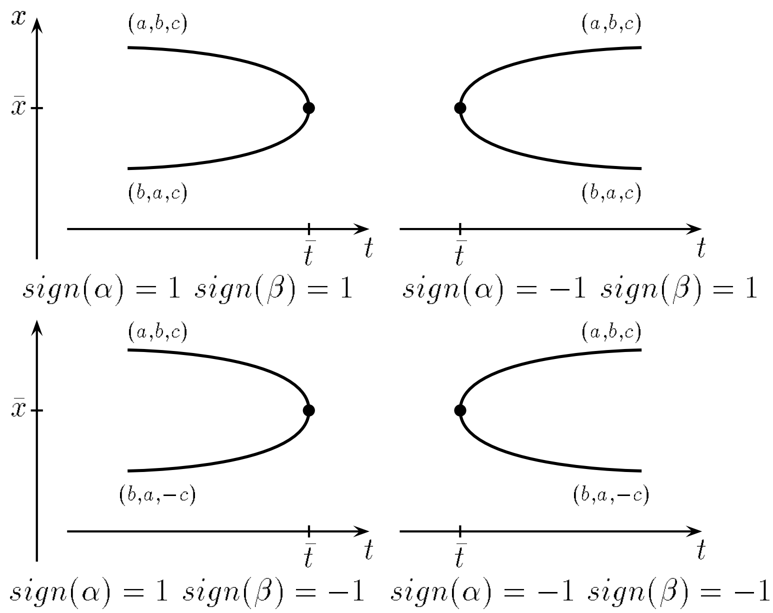

Figure 3: Type 4

$W_0^T D_x^2 \mathbf{L}(\bar{z}) W_0$ is negative semidefinite, where W_0 is intended as in the proof of the above Proposition 2 and $\mathbf{L}$ as in Definition 5. Since $W_0^T D_x^2 \mathbf{L}(\bar{z}) W_0$ is regular, it is then negative definite, and sign $\left[\det(-W_0^T D_x^2 \mathbf{L}(\bar{z}) W_0)\right] = +1$. The relations (7) and (8) imply then that sign $\left[\det(D_x \mathsf{L}(z(\tau))|_{T_{x(\tau)}M(t(\tau))})\right] = +1$ for $\tau < 0$.

On the other hand, $W^T D_x^2 \mathbf{L}(\bar{z}) W$, where W is intended as in Definition 5, has at most one positive eigenvalue. Applying the results from Theorem 5.3 (cases 2. and 3.) in [10] about the homotopic changes of parametric feasible sets it follows that $W^T D_x^2 \mathbf{L}(\bar{z}) W$ must be negative definite and that $M(t)$, by passing the parameter value $\bar{t}$, vanishes or is created.

Definition 6 *(see also [11])*
A g.c. point $\bar{z}$ of $VI(F,H,G)$ is of <u>*Type 5*</u> *if the following conditions hold:*

Type5: 1-VU *: $m + |J_0(\bar{z})| = n + 1$.*

Type5: 2-VU *: The matrix $D \begin{bmatrix} H(\bar{z}) \\ G_{J_0(\bar{z})}(\bar{z}) \end{bmatrix}$ has rank $n+1$.*

Type5: 3-VU *: It holds* **Type4: 3-VU** ($(\bar{\lambda}, \bar{\mu})$ *are fixed*) .

Type5: 4-VU *: Let $(\bar{\alpha}, \bar{\beta}) \in I\!R^m \times I\!R^{|J_0(\bar{z})|}$ solve the system*

$$\begin{pmatrix} F(\bar{z}) \\ 0 \end{pmatrix} - \sum_{i \in I} \bar{\alpha}_i D^T h_i(\bar{z}) - \sum_{j \in J_0(\bar{z})} \bar{\beta}_j D^T g_j(\bar{z}) = 0.$$

Then $\Delta_{jk} \neq 0$, $\forall j, k \in J_0(\bar{z})$, $j \neq k$, where $\Delta_{jk} = \bar{\beta}_j - \bar{\beta}_k \cdot \frac{\bar{\mu}_j}{\bar{\mu}_k}$.

Remark 10
From **Type5: 1-VU**, **Type5: 2-VU** *and* **Type5: 3-VU** *it follows that each g.c. point in a neighbourhood of $\bar{z}$ solves (with corresponding multipliers) one of the systems*

$$\mathsf{H}_{J_0(\bar{z}) \setminus \{k\}}(z, \lambda, \mu) = 0, \tag{10}$$

where $k \in J_0(\bar{z})$.

For each $k \in J_0(\bar{z})$ there exists a parametrization $(x^k(t), t, \lambda^k(t), \mu^k(t))$ around $(\bar{z}, \bar{\lambda}^k, \bar{\mu}^k)$ of the solution set from (10), with $(x^k(\bar{t}), \bar{t}, \lambda^k(\bar{t}), \mu^k(\bar{t})) = (\bar{z}, \bar{\lambda}^k, \bar{\mu}^k)$.

Remark 11
Σ_{gc} is described around $\bar{z}$ by the union of the feasible parts of the $n+1$ curves $(x^k(t), t)$. It also holds that $\frac{d}{dt} g_k(x^k(\bar{t}), \bar{t}) = D_x^T g_k(\bar{z}) \dot{x}^k(\bar{t}) + D_t g_k(\bar{z}) \neq 0$ for $k \in J_0(\bar{z})$. Since $D_x^T g_{k_1}(\bar{z}) \dot{x}^{k_2}(\bar{t}) + D_t g_{k_1}(\bar{z}) = 0$, it follows that $\dot{x}^{k_1}(\bar{t}) \neq \dot{x}^{k_2}(\bar{t})$ for $k_1 \neq k_2$ (see Figure 4).

Remark 12 *(see [11])*
If the MFCQ fails to hold in $\bar{z}$, then every $\bar{\mu}_k$ has the same sign. Then each curve belonging to Σ_{gc} runs in the same t-direction. Therefore, $\bar{z}$ is a turning point from Σ_{gc}. Analogously as for optimization problems it holds that Σ_{stat} around $\bar{z}$ consists of only one curve $(x^k(t), t)$ and $\bar{z}$ is a border point of Σ_{stat}.

If the MFCQ is fulfilled and $\bar{z}$ is a stationary point, then there exist exactly two curves $(x^{k_1}(t), t)$ and $(x^{k_2}(t), t)$ consisting of stationary points and with opposite t-directions. Hence, Σ_{stat} has a continuation.

The local structure of the sets Σ_{gc} and Σ_{stat} around a point of Type 5 is shown in figure 4.

Remark 13
If the $M(t)$ are convex (h_i affine and $J_0(\bar{z}) \neq \emptyset$, since $|I| <= n$) Theorems 4.1 (Type 3 and 4) and 5.3 (cases 2. and 3.) in [10] imply that the feasible set $M(t)$ in the parameter value $\bar{t}$ vanishes or appears when the MFCQ is not fulfilled.

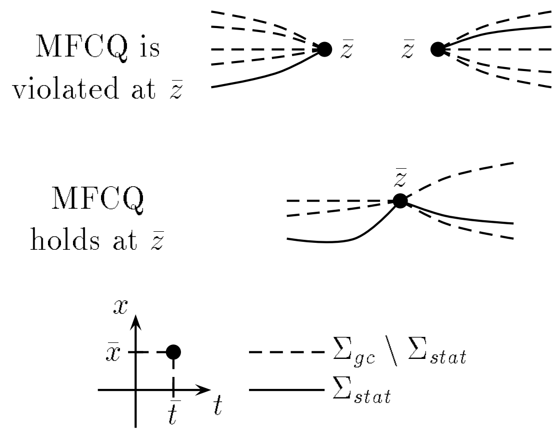

Figure 4: Typ 5

3 Some properties of regular one-parametric variational inequalities.

3.1 The points of Type 3.

In this section we prove a basic result similar to Theorem 10.2.2 in [9]. Let $q \leq n$ be fixed and consider mappings with the structure

$$T(x,u,t) = \begin{pmatrix} T^1(x,u,t) \\ T^2(x,t) \end{pmatrix}, \tag{11}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, $T : \mathbb{R}^{n+q+1} \rightarrow \mathbb{R}^{n+q}$, $T^1 : \mathbb{R}^{n+q+1} \rightarrow \mathbb{R}^n$ and therefore $T^2 : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^q$. Let us consider two matrices, $B_L = \begin{pmatrix} C_1 & C_2 \\ 0 & I_q \end{pmatrix}$ and $B_R = \begin{pmatrix} C_3 & 0 \\ C_4 & I_q \end{pmatrix}$, where C_1 and C_2 are quadratic regular matrices with size n, and I_q states for the identity matrix of size q. B_L and B_R are then quadratic and regular.

By $T_{(B_L,B_R)}(y,v,t) = B_L \cdot T(B_R \cdot (y,v),t) = \begin{pmatrix} T^1_{(B_L,B_R)}(y,v,t) \\ T^2_{(B_L,B_R)}(y,t) \end{pmatrix}$ let us denote another mapping. From the structure of B_L and B_R it follows that

$T_{(B_L,B_R)}$ has the same dependence on its variables as T (pointed out by the notations $T^1_{(B_L,B_R)}$ and $T^2_{(B_L,B_R)}$). It holds that $T(x,u,t) = 0$ if and only if $T_{(B_L,B_R)}(B_R^{-1} \cdot (x,u), t) = 0$. If we define $\Sigma(T) = \{(x,u,t) | T(x,u,t) = 0\}$, it follows that

$$\Sigma(T) = \begin{pmatrix} B_R & 0 \\ 0 & 1 \end{pmatrix} \Sigma(T_{(B_L,B_R)}). \tag{12}$$

Let us consider the following assumption for a mapping $T(x,u,t)$.

$$P_1(T): \qquad 0 \text{ is a regular value of } T$$

Remark 14
The relation

$$DT_{(B_L,B_R))}(y,v,t) = B_L \cdot DT(B_R \cdot (y,v), t) \cdot \begin{pmatrix} B_R & 0 \\ 0 & 1 \end{pmatrix} \tag{13}$$

implies that $P_1(T)$ holds if and only if $P_1(T_{(B_L,B_R)})$ holds, too.

Let us define $\Sigma^s(T) = \left\{(x,u,t) \in \Sigma(T) | \det(D_{(x,u)}T(x,u,t)) = 0\right\}$.

Proposition 3
Let $(x,u,t) \in \Sigma^s(T)$ and $(y,v,t) \in \Sigma^s(T_{(B_L,B_R)})$ be fixed with $(x,u) = B_R \cdot (y,v)$. Let $P_1(T)$ (and also $P_1(T_{(B_L,B_R)})$) be fulfilled around (x,u,t) (resp. (y,v,t)). Then, (x,u,t) is a non-degenerated critical point of $t|_{\Sigma(T)}$ if and only if (y,v,t) is a non-degenerated critical point of $t|_{\Sigma(T_{(B_L,B_R)})}$.

Here $t|_{\Sigma(T)}$ (analogously to $t|_{\Sigma(T_{(B_L,B_R)})}$) represents the optimization problem with the objective function $F(x,u,t) = t$ and the feasible set $\Sigma(T)$. Proposition 3 is proved easily writting down booth conditions.

Remark 15
Let $P_1(T)$ be fulfilled and $(y(\theta), v(\theta), t(\theta))$ be a local parametrization of the curve $\Sigma(T_{(B_L,B_R)})$ around $(y,v,t) \in \Sigma^s(T_{(B_L,B_R)})$, which is defined in a neighbourhood of zero and $(y(0), v(0), t(0)) = (y,v,t)$. By (12) the mapping

$$\begin{pmatrix} x(\theta) \\ u(\theta) \\ t(\theta) \end{pmatrix} = \begin{pmatrix} B_R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(\theta) \\ v(\theta) \\ t(\theta) \end{pmatrix}$$

is also a local parametrization of $\Sigma(T)$ around (x,u,t).

Since $(x,u,t) \in \Sigma^s(T)$ and $(y,v,t) \in \Sigma^s(T_{(B_L,B_R)})$, the mappings

$$\Psi(\theta) = \det\left[D_{(x,u)}T(x(\theta),u(\theta),t(\theta))\right] \tag{14}$$

and $\Psi_{(B_L,B_R)}(\theta) = \det\left[D_{(y,v)}T_{(B_L,B_R)}(y(\theta),v(\theta),t(\theta))\right]$ vanish at $\theta = 0$. Further, it holds that $\Psi_{(B_L,B_R)}(\theta) = \det[B_L]\cdot\det[B_R]\cdot\Psi(\theta)$. Consequently, $\Psi'(0) = 0$ if and only if $\Psi'_{(B_L,B_R)}(0) = 0$.

Definition 7
Let $\mathbf{M}_1(n+q)$ be the set of quadratic matrices of size $n+q$ having the submatrix formed by the last q rows and columns equal to zero. $\mathbf{M}_1(n+q)$ can be identified with $\mathbb{R}^{n(n+2q)}$. Let us define the open set

$$\mathbf{M}_2(n+q) = \left\{ \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathbf{M}_1(n+q) \mid \text{rank }(B) = \text{ rank }(C) = q \right\}$$

and the manifold $\mathbf{M}_3(n+q,k) = \{P \in \mathbf{M}_2(n+q) | \text{ rank }(P) = k\}$, where $2q \le k \le n+q$.

Due to the special dependence (11) of T and $T_{(B_L,B_R)}$ on their variables it holds that $D_{(x,u)}T, D_{(y,v)}T_{(B_L,B_R)} \in \mathbf{M}_1(n+q)$.

Remark 16
Let $\bar{P} \in \mathbf{M}_3(n+q,k)$ be fixed. Due to the structure of $\bar{P}$ there exist two index sets $\{n+1,\ldots,n+q\} \subset I_1, I_2 \subset \{1,\ldots,n+q\}$ such that $|I_1| = |I_2| = k$ and that the submatrix given by the intersection of the rows I_1 with the columns I_2 (and denoted by $\bar{P}_{(I_1,I_2)}$) is regular.

Let us denote $I_1^c = \{1,\ldots,n+q\} \setminus I_1$ (analogously I_2^c) and, by $\bar{P}_{(I_1^c,I_2)}$, $\bar{P}_{(I_1,I_2^c)}$,$\bar{P}_{(I_1^c,I_2^c)}$ the other submatrices of $\bar{P}$. Choose a neighbourhood $U(\bar{P}) \subset \mathbf{M}_1(n+q)$ of $\bar{P}$ such that $U(\bar{P}) \subset \mathbf{M}_2(n+q)$ holds and that the submatrix $P_{(I_1,I_2)}$ is regular $\forall P \in U(\bar{P})$.

In $U(\bar{P})$ the manifold $\mathbf{M}_3(n+q,k)$ can be described by the equalities

$$P_{(I_1^c,I_2^c)} - P_{(I_1^c,I_2)}\left[P_{(I_1,I_2)}\right]^{-1} P_{(I_1,I_2^c)} = 0.$$

The gradients of these mappings are linearly independent (the partial derivatives with respect to the components of the matrix $P_{(I_1^c,I_2^c)}$ form an identity matrix of size $(n+q-k)^2$). Therefore, the set $\mathbf{M}_3(n+q,k)$ is a differentiable manifold with codimension $(n+q-k)^2$.

Let us introduce the following notation

$$\begin{aligned} M_3^{-1} &= M_3(n+q, n+q-1) \\ \hat{M}_3^{-1} &= \{0_{(n+q)}\} \times M_3^{-1}. \end{aligned}$$

Consider for T as in (11) the following condition

$$P_2(T): \qquad (T, D_{(x,u)}T) \,\bar{\pitchfork}\, \hat{M}_3^{-1}.$$

The structure of B_L and B_R implies that $(T(\bar{x},\bar{u},\bar{t}), D_{(x,u)}T(\bar{x},\bar{u},\bar{t})) \in \hat{M}_3^{-1}$ if and only if $(T_{(B_L,B_R)}(\bar{y},\bar{v},\bar{t}), D_{(y,v)}T_{(B_L,B_R)}(\bar{y},\bar{v},\bar{t})) \in \hat{M}_3^{-1}$, where $(\bar{y},\bar{v},\bar{t}) = (B_R^{-1}\cdot(\bar{x},\bar{u}),\bar{t})$.

Proposition 4
Let $(T(\bar{x},\bar{u},\bar{t}), D_{(x,u)}T(\bar{x},\bar{u},\bar{t}))$ and $(T_{(B_L,B_R)}(\bar{y},\bar{v},\bar{t}), D_{(y,v)}T_{(B_L,B_R)}(\bar{y},\bar{v},\bar{t}))$ belong to $\hat{M}_3^{-1}$ with $(\bar{y},\bar{v},\bar{t}) = (B_R^{-1}\cdot(\bar{x},\bar{u}),\bar{t})$. $P_2(T)$ holds in a neighbourhood of $((\bar{x},\bar{u},\bar{t}), 0_{(n+q)}, D_{(x,u)}T(\bar{x},\bar{u},\bar{t}))$ if and only if $P_2(T_{(B_L,B_R)})$ holds around $((\bar{y},\bar{v},\bar{t}), 0_{(n+q)}, D_{(y,v)}T_{(B_L,B_R)}(\bar{y},\bar{v},\bar{t}))$.

Proof:
Let $z_1 \in I\!R^{n+q}$ and $z_2 \in \mathbf{M}_1(n+q)$ be two variables. Denote by $\Phi(z_2)$ a differentiable mapping whose zero-set describes $\mathbf{M}_3^{-1}$ around $D_{(x,u)}T(\bar{x},\bar{u},\bar{t})$. $P_2(T)$ around $((\bar{x},\bar{u},\bar{t}), 0_{(n+q)}, D_{(x,u)}T(\bar{x},\bar{u},\bar{t}))$ is equivalent to the following fact: at the zero $(\bar{x},\bar{u},\bar{t}, 0_{(n+q)}, D_{(x,u)}T(\bar{x},\bar{u},\bar{t}))$ the mapping

$$\Omega_1(x,u,t,z_1,z_2) = \begin{pmatrix} z_1 - T(x,u,t) \\ z_2 - D_{(x,u)}T(x,u,t) \\ z_1 \\ \Phi(z_2) \end{pmatrix},$$

has a Jacobian-matrix with linearly independent rows.
Consider the linear transformation of coordinates

$$\begin{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\ t \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} B_R \begin{pmatrix} y \\ v \end{pmatrix} \\ t \\ B_L^{-1}\gamma_1 \\ B_L^{-1}\gamma_2 B_R^{-1} \end{pmatrix}.$$

Here $\gamma_1 \in I\!R^{n+q}$ and $\gamma_2 \in \mathbf{M}_1(n+q)$ hold. The mapping $\Lambda(\gamma_2) = B_L^{-1}\gamma_2 B_R^{-1}$ is linear and regular. Since B_L^{-1} and B_R^{-1} preserve the same structure as B_L and B_R, it holds that $\Lambda(\gamma_2) \in \mathbf{M}_3^{-1}$ if and only if $\gamma_2 \in \mathbf{M}_3^{-1}$.

If $\bar{\gamma}_2 = B_L D_{(x,u)} T(\bar{x}, \bar{u}, \bar{t}) B_R \in \mathbf{M}_3^{-1}$, there exists a neighbourhood of $\bar{\gamma}_2$ in $\mathbf{M}_1(n+q)$ such that the zero-set of the mapping $\Psi(\gamma_2) = \Phi(B_L^{-1}\gamma_2 B_R^{-1})$ describes $\mathbf{M}_3^{-1}$ around $\bar{\gamma}_2$. $\Psi(\gamma_2)$ is differentiable and regular at $\bar{\gamma}_2$. In the new coordinates Ω_1 is given by:

$$\Omega_2(y, v, t, \gamma_1, \gamma_2) = \begin{pmatrix} B_L^{-1}\gamma_1 - T(B_R(y, v), t) \\ B_L^{-1}\gamma_2 B_R^{-1} - D_{(x,u)} T(B_R(y, v), t) \\ B_L^{-1}\gamma_1 \\ \Phi(B_L^{-1}\gamma_2 B_R^{-1}) \end{pmatrix}.$$

At $(\bar{y}, \bar{v}, \bar{t}, 0_{(n+q)}, \bar{\gamma}_2) = (B_R^{-1}(\bar{x}, \bar{u}), \bar{t}, 0_{(n+q)}, B_L D_{(x,u)} T(\bar{x}, \bar{u}, \bar{t}) B_R)$, the rows of the Jacobian-matrix of Ω_2 are also linearly independent. The set of zeros of Ω_2 around $(\bar{y}, \bar{v}, \bar{t}, 0_{(n+q)}, \bar{\gamma}_2)$ is the same as the one corresponding to the following mapping Ω_3 (obtained from Ω_2 by a linear transformation in target space).

$$\begin{aligned} \Omega_3(y, v, t, \gamma_1, \gamma_2) &= \begin{pmatrix} \gamma_1 - B_L T(B_R(y, v), t) \\ \gamma_2 - B_L D_{(x,u)} T(B_R(y, v), t) B_R \\ \gamma_1 \\ \Psi(\gamma_2) \end{pmatrix} \\ &= \begin{pmatrix} \gamma_1 - T_{(B_L, B_R)}(y, v, t) \\ \gamma_2 - D_{(y,v)} T_{(B_L, B_R)}(y, v, t) \\ \gamma_1 \\ \Psi(\gamma_2) \end{pmatrix}. \end{aligned}$$

At $(\bar{y}, \bar{v}, \bar{t}, 0_{(n+q)}, \bar{\gamma}_2) = (\bar{y}, \bar{v}, \bar{t}, T_{(B_L, B_R)}(\bar{y}, \bar{v}, \bar{t}), D_{(y,v)} T_{(B_L, B_R)}(\bar{y}, \bar{v}, \bar{t}))$ the mapping Ω_3 vanishes and its Jacobian-matrix has linearly independent rows at this point. Hence, $P_2(T_{(B_L, B_R)})$ holds around $(\bar{y}, \bar{v}, \bar{t}, 0_{(n+q)}, \bar{\gamma}_2)$. $\square$

Theorem 1 *(for gradient-mappings see Theorem 10.2.2 in [9])*
Let T be as in (11) and $T(\bar{x}, \bar{u}, \bar{t}) = 0$ such that:

- $D_{(x,u,t)} T(\bar{x}, \bar{u}, \bar{t})$ *has rank* $n + q$.
- $(T(\bar{x}, \bar{u}, \bar{t}), D_{(x,u)} T(\bar{x}, \bar{u}, \bar{t})) \in \hat{M}_3^{-1}$

Then the following three conditions are equivalent:

1. *$(\bar{x},\bar{u},\bar{t})$ is a non-degenerated critical point of $t|_{\Sigma(T)}$.*

2. *Let $(x(\theta),u(\theta),t(\theta))$, where $(x(0),u(0),t(0))=(\bar{x},\bar{u},\bar{t})$, parametrize the curve $\Sigma(T)$ locally and $\Psi(\theta)$ be as in (14). Then $\Psi'(0)\neq 0$.*

3. *$P_2(T)$ holds in a neighbourhood of $((\bar{x},\bar{u},\bar{t}),0_{(n+q)},D_{(x,u)}T(\bar{x},\bar{u},\bar{t}))$.*

Proof:
From the assumptions on $(\bar{x},\bar{u},\bar{t})$ it follows that:

- $(\bar{x},\bar{u},\bar{t})\in\Sigma^s(T)$ and then $\Psi(0)=0$.
- $(\bar{x},\bar{u},\bar{t})$ is a critical point of $t|_{\Sigma(T)}$, where the LICQ is fulfilled.

Since $D_{(x,u)}T(\bar{x},\bar{u},\bar{t})\in\mathbf{M}_3^{-1}$, two matrices B_L and B_R can be selected such that:

$$B_L\cdot D_{(x,u)}T(\bar{x},\bar{u},\bar{t})\cdot B_R=\begin{pmatrix}0 & 0_{n-1}^T & 0_q^T\\ 0_{n-1} & A_1 & B_1\\ 0_q & B_2^T & 0\end{pmatrix}. \tag{15}$$

Here 0_{n-1} and 0_q are the zero vectors in $I\!R^n$ and $I\!R^q$ respectively, A_1 is a quadratic matrix of size $(n-1)$, and B_1,B_2 are matrices of size $(n-1)\times q$. The submatrix $\begin{pmatrix}A_1 & B_1\\ B_2^T & 0\end{pmatrix}$ is regular.

Taking into account the Remarks 14 and 15 and the Propositions 3 and 4, it can be supposed w.l.o.g. that $D_{(x,u)}T(\bar{x},\bar{u},\bar{t})$ possesses the structure given in (15).

Let us denote by $T_1^1(x,u,t)$ the first component of the mapping T^1, which is part of T. From the full rank of $D_{(x,u,t)}T(\bar{x},\bar{u},\bar{t})$ it follows that $D_tT_1^1(\bar{x},\bar{u},\bar{t})\neq 0$.

Let $(\bar{x},\bar{u},\bar{t})$ be a non-degenerated critical point of $t|_{\Sigma(T)}$. The corresponding multiplier-vector is $\frac{1}{D_tT_1^1(\bar{x},\bar{u},\bar{t})}(1,0_{n+q-1})$. On the other hand, the tangent space $T_{(\bar{x},\bar{u},\bar{t})}\Sigma(T)$ is generated by the vector $(1,0_{n+q})$. The second order condition for the non-degeneracy of $(\bar{x},\bar{u},\bar{t})$ with respect to $t|_{\Sigma(T)}$ is then equivalent to the condition

$$D_{x_1}^2T_1^1(\bar{x},\bar{u},\bar{t})\neq 0. \tag{16}$$

The rest consists in proving that the other two conditions are equivalent to (16).

Let $(x(\theta), u(\theta), t(\theta))$ and $\Psi(\theta)$ be as in the second condition. Given two indices $k, l \in \{1, \ldots, n+q\}$, denote by $\Psi_{k,l}(\theta)$ the determinant of the (k,l)-minor of the matrix $D_{(x,u)}T(x(\theta), u(\theta), t(\theta))$. Then it holds:

$$\Psi(\theta) = \sum_{j=1}^{n} (-1)^{j+1} D_{x_j} T_1^1(\theta) \cdot \Psi_{1,j}(\theta) + \sum_{k=1}^{q} (-1)^{n+k+1} D_{u_k} T_1^1(\theta) \cdot \Psi_{1,n+k}(\theta).$$

From (15) it follows:

$$\begin{aligned} \Psi_{1,1}(0) &\neq 0 \\ \Psi_{1,j}(0) &= 0, \ \forall j = 2, \ldots, n+q \\ D_{x_j} T_1^1(\bar{x}, \bar{u}, \bar{t}) &= 0, \ \forall j = 1, \ldots, n \\ D_{u_k} T_1^1(\bar{x}, \bar{u}, \bar{t}) &= 0, \ \forall k = 1, \ldots, q \end{aligned}$$

Moreover,

$$\begin{aligned} \Psi'(0) &= \frac{d}{d\theta}\left[D_{x_1} T_1^1(x(0), u(0), t(0))\right] \cdot \Psi_{1,1}(0) \\ &= D_{(x,u,t)} D_{x_1} T_1^1(\bar{x}, \bar{u}, \bar{t}) \cdot \begin{pmatrix} \dot{x}(0) \\ \dot{u}(0) \\ \dot{t}(0) \end{pmatrix} \cdot \Psi_{1,1}(0). \end{aligned}$$

Since $(\dot{x}(0), \dot{u}(0), \dot{t}(0))$ belongs to $T_{(\bar{x},\bar{u},\bar{t})}\Sigma(T)$, it follows that $\Psi'(0) = D^2_{x_1} T_1^1(\bar{x}, \bar{u}, \bar{t}) \cdot \dot{x}_1(0) \cdot \Psi_{1,1}(0)$, where $\dot{x}_1(0) \neq 0$ and $\Psi_{1,1}(0) \neq 0$. Now it is obvious that the second condition is equivalent to (16).

Let $z \in I\!R^{n+q}$ and $\gamma \in M_2(n+q)$. (15) implies that M_3^{-1} can be described around $D_{(x,u)}T(\bar{x}, \bar{u}, \bar{t})$ by the zeros of $\Lambda_4(\gamma) = \gamma_{1,1} - \tilde{\Lambda}_4(\gamma)$. Then $D_\gamma \tilde{\Lambda}_4(D_{(x,u)}T(\bar{x}, \bar{u}, \bar{t})) = 0$ holds, too (see remark 16).

The third condition is then equivalent to the fact that the Jacobian-matrix of

$$\Lambda = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \end{pmatrix} = \begin{pmatrix} z - T(x,u,t) \\ \gamma - D_{(x,u)}T(x,u,t) \\ z \\ \gamma_{1,1} - \tilde{\Lambda}_4(\gamma) \end{pmatrix}$$

has full rank at $(\bar{x}, \bar{u}, \bar{t}, \bar{z}, \bar{\gamma}) = (\bar{x}, \bar{u}, \bar{t}, 0_{n+q}, D_{(x,u)}T(\bar{x}, \bar{u}, \bar{t}))$.

This Jacobian-matrix has the following structure

$$\begin{array}{c} \\ \Lambda_1 \\ \Lambda_2 \\ \Lambda_2 \\ \Lambda_3 \\ \Lambda_4 \end{array} \begin{array}{c} \begin{array}{ccccc} D_z & D_{x_1} & D_{(x,u,t)\backslash\{x_1\}} & D_{\gamma_{1,1}} & D_{\gamma\backslash\{\gamma_{1,1}\}} \end{array} \\ \left[\begin{array}{ccccc} I_{n+q} & 0_{n+q} & D_{(x,u,t)\backslash\{x_1\}}T(\bar{x},\bar{u},\bar{t}) & 0 & 0 \\ 0 & -D^2_{x_1}T^1_1(\bar{x},\bar{u},\bar{t}) & \otimes & 1 & 0^T_{n(n+2q)-1} \\ 0 & \otimes & \otimes & 0_{n(n+2q)-1} & I_{n(n+2q)-1} \\ I_{n+q} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0^T_{n(n+2q)-1} \end{array}\right] \end{array}$$

where 0_k represents the $k-$dimensional zero vector (as column), and $\otimes$ a submatrix not written in detail. $D_{(x,u,t)\backslash\{x_1\}}T(\bar{x},\bar{u},\bar{t})$ is regular, and a simple consideration of the above matrix provides the equivalence of its regularity (by rows) with (16).□

Remark 17

From Remark 4 it is known that the first condition of the above Theorem 1 is fulfilled at the points of Type 3. Here $T(x,u,t) = \mathsf{H}_{J_0}(z,u)$ *with* $u = (\lambda,\mu)$, $z = (x,t)$ *and*

$$T_2(z) = \left[\begin{array}{c} H(z) \\ G_{J_0}(z) \end{array}\right].$$

Let $(z(\theta),u(\theta))$ *be a local parametrization of the set* $\{\mathsf{H}_{J_0} = 0\}$ *around* $(\bar{z},\bar{\lambda},\bar{\mu}) = (z(0),u(0))$. *We use the notations* $D_x\mathsf{L}(\theta) = D_x\mathsf{L}(z(\theta),u(\theta))$ *and* $D_xT_2(\theta) = D_xT_2(z(\theta))$. *Then the sign of*

$$\Psi(\theta) = \det\left[\begin{array}{ll} D_x\mathsf{L}(\theta) & -D^T_xT_2(\theta) \\ D_xT_2(\theta) & 0 \end{array}\right]$$

is different for $\theta < 0$ *and* $\theta > 0$. *Let* $W(\theta)$ *be matrices deppending smoothly from* θ, *whose columns form for* θ *around zero a basis of* $T_{\bar{x}(\theta)}M(t(\theta))$. *The equation*

$$\left[\begin{array}{ll} W^T(\theta) & 0 \\ D_xT_2(\theta) & 0 \\ 0 & I_{m+|J_0(\bar{z})|} \end{array}\right]\left[\begin{array}{ll} D_x\mathsf{L}(\theta) & -D^T_xT_2(\theta) \\ D_xT_2(\theta) & 0 \end{array}\right]\left[\begin{array}{lll} W(\theta) & D^T_xT_2(\theta) & 0 \\ 0 & 0 & I_{m+|J_0(\bar{z})|} \end{array}\right]$$

$$= \begin{bmatrix} W^T(\theta) D_x \mathsf{L}(\theta) W(\theta) & W^T(\theta) D_x \mathsf{L}(\theta) D_x^T T_2(\theta) & 0 \\ D_x T_2(\theta) D_x \mathsf{L}(\theta) W(\theta) & D_x T_2(\theta) D_x \mathsf{L}(\theta) D_x^T T_2(\theta) & -D_x T_2(\theta) D_x^T T_2(\theta) \\ 0 & D_x T_2(\theta) D_x^T T_2(\theta) & 0 \end{bmatrix}$$

implies the relation

$$\Psi(\theta) \left(\det \left[\begin{array}{cc} W(\theta) & D_x^T T_2(\theta) \end{array} \right]\right)^2 = \det \left[W^T(\theta) D_x \mathsf{L}(\theta) W(\theta)\right] \left(\det \left[D_x T_2(\theta) D_x^T T_2(\theta)\right]\right)^2$$

This fact explains the changes of $\det(D_x \mathsf{L}(z(\theta), u(\theta))|_{T_{\bar{x}(\theta)} M(t(\theta))})$ *over the curve* Σ_{gc} *around the points of Type 3.*

3.2 Genericity and other results.

Definition 8
Given $VI(F, H, G)$ *define for* $i = 1, \ldots, 5$ *the sets:*

$$\Sigma_{gc}^i(F, H, G) = \{z \in \Sigma_{gc}(F, H, G) | z \text{ is of Type } i\}.$$

We also use the notation Σ_{gc}^i.

Let us denote

$$\mathcal{D} = C^2(I\!R^{n+1}, I\!R^n) \times C^3(I\!R^{n+1}, I\!R^{m+p})$$

and then define the set

$$\mathcal{V} = \left\{(F, H, G) \in \mathcal{D} | \Sigma_{gc} = \cup_{i=1}^5 \Sigma_{gc}^i\right\}.$$

Definition 9
A one-parametric variational inequality $(F, H, G) \in \mathcal{D}$ *belonging to* $\mathcal{V}$ *will be called regular (in the sense of Jongen, Jonker and Twilt).*

An important property of the regularity is its openness and its density with respect to the strong Whitney topology (see e.g. [1], [7], [9]). Let us state the local openness in the following.

Theorem 2
Let $(\bar{F}, \bar{H}, \bar{G}) \in \mathcal{D}$ *and* $\bar{z} \in \Sigma_{gc}^i(\bar{F}, \bar{H}, \bar{G})$ *(or* $\bar{z} \notin \Sigma_{gc}(\bar{F}, \bar{H}, \bar{G})$*),* $i \in \{1, \ldots, 5\}$.

Then there exist an open neighbourhood $U_{\bar{z}}$ *from* $\bar{z}$ *and a positive number* $r_{\bar{z}}$ *such that* $\Sigma_{gc}(\tilde{F},\tilde{H},\tilde{G}) \cap U_{\bar{z}} \subset \Sigma^{i}_{gc}(\tilde{F},\tilde{H},\tilde{G}) \cup \Sigma^{1}_{gc}(\tilde{F},\tilde{H},\tilde{G})$ *(resp.* $\Sigma_{gc}(\tilde{F},\tilde{H},\tilde{G}) \cap U_{\bar{z}} = \emptyset$*) holds* $\forall(\tilde{F},\tilde{H},\tilde{G})$ *such that*

$$\sup_{z\in U_{\bar{z}}}\left\{\max_{\alpha\in\{0,1,2\}}\left|D^{\alpha}(\tilde{F}-\bar{F})(z)\right|\right\} < r_{\bar{z}}$$
$$\sup_{z\in U_{\bar{z}}}\left\{\max_{\alpha\in\{0,1,2,3\}}\left|D^{\alpha}\left(\begin{bmatrix}\tilde{H}\\ \tilde{G}\end{bmatrix}-\begin{bmatrix}\bar{H}\\ \bar{G}\end{bmatrix}\right)(z)\right|\right\} < r_{\bar{z}}.$$

This Theorem can be proved by continuity arguments taken into account the definition of the 5 singularities.

For the next perturbation theorem we fix the following notations. Let $VI(\bar{F},\bar{H},\bar{G})$ be fixed. Let us identify the space of every $k\times l$ matrix with $I\!R^{kl}$ and consider the following parameters: $A\in I\!R^{n^2}$ a quadratic matrix, $b\in I\!R^{n}$ a column vector, $C\in I\!R^{(m+p)n}$ a matrix with $m+q$ rows and n columns, and $d\in I\!R^{m+q}$ a column vector. Since only A and b are used for the perturbation of F, and C and d for the perturbation of (H,G), we consider two parameter vectors $\mathcal{A}=(A,b)$ and $\mathcal{C}=(C,d)$. As a vector containing all perturbation parameters we use the notation $\mathcal{P}=(\mathcal{A},\mathcal{C})\in I\!R^{n^2+n+n(m+p)+m+p}$.

For a parameter value $\mathcal{P}$ we denote by

$$(\bar{F},\bar{H},\bar{G},\mathcal{P}) = (\bar{F}(x,t)+Ax+b,(\bar{H}(x,t),\bar{G}(x,t))+Cx+d)$$

the resulting perturbed one-parametric variational inequality. $(\bar{H},\bar{G},\mathcal{C})$, $(\bar{H},\mathcal{C})_i$, $i\in I$ and $(\bar{G},\mathcal{C})_j$, $j\in J$, represent the perturbed restrictions and their components. Analogously $(\bar{F},\mathcal{A})$ denotes the perturbation of $\bar{F}$.

Theorem 3 *(see also [16])*
Let $VI(\bar{F},\bar{H},\bar{G})$ *be fixed with smooth data. Each measurable subset of*

$$\left\{\mathcal{P}\in I\!R^{n^2+n+n(m+p)+m+p}|(\bar{F},\bar{H},\bar{G},\mathcal{P})\notin\mathcal{V}\right\}$$

has the Lebesgue measure zero.

The proof follows the same lines as the corresponding result for the class $\mathcal{F}$ in [16]. We give here only the main ideas of the extensive proof in [16] and indicate where new arguments are needed for our case.

Proof:

Lemma 1 in [16] can be used in our case without any changes. The considerations around the points of Type 4 and 5 in the second step also apply without major differences. Let us only note that for almost all $\mathcal{A} = (A, b) \in I\!R^{n^2+n}$ (with non symmetric A) the following holds

$$\begin{array}{rcl} F(\bar{x}) + A\bar{x} + b & \notin & T \\ (F(\bar{x}) + A\bar{x} + b)^T Q(F(\bar{x}) + A\bar{x} + b) & \neq & 0, \end{array}$$

where Q is a fixed nonvanishing $n \times n$-matrix, $T \subset I\!R^n$ is a fixed proper linear subspace and $\bar{x}, F(\bar{x}) \in I\!R^n$ are two fixed vectors.

For the first step consider two fixed index sets $J_1 \subset J_0 \subset \{1, \ldots, p\}$ with $0 \leq q = m + |J_0| \leq n$ and two numbers η and s with $\eta \in \{0, 1\}$, respectively, $0 \leq 2(q + \eta - 1) \leq s \leq n + q + \eta - 1$. In the variables $(x, \lambda, \mu, t, \gamma_1, \gamma_2)$, where $\mu \in I\!R^{J_0}$, $\gamma_1 \in I\!R^n$ and $\gamma_2 \in M_2(n + q + \eta - 1)$, consider the manifold

$$M_4^\eta = I\!R^n \times I\!R^m \times \{\mu \in I\!R^{J_0} | \mu_j = 0, j \in J_1\} \times \{0_{n+q}\} \times M_3(n + q + \eta - 1, s),$$

which has codimension $|J_1| + n + q + (n + q + \eta - 1 - s)^2$.

Let us fix an index in J_0 (w.l.o.g. 1) and define the mappings

$$\begin{array}{rcl} \mathsf{L}^\eta_{J_0}(\mathcal{P},x,\lambda,\mu,t) & = & (\bar{F}, \mathcal{A}) - \sum_{i=1}^{m} \lambda_i D_x^T(\bar{H}, \mathcal{C})_i - \eta\mu_1 D_x^T(\bar{G}, \mathcal{C})_1 \\ & & - \sum_{j \in J_0 \setminus \{1\}} \mu_j D_x^T(\bar{G}, \mathcal{C})_j \\ \mathsf{H}^\eta_{J_0}(\mathcal{P},x,\lambda,\mu,t) & = & \left[\begin{array}{c} \mathsf{L}^\eta_{J_0}(\mathcal{P},x,\lambda,\mu,t) \\ (\bar{H}, \mathcal{C}) \\ (1-\eta)\mu_1 + (\bar{G}, \mathcal{C})_1 \\ (\bar{G}, \mathcal{C})_{J_0 \setminus \{1\}} \end{array} \right] \\ \mathsf{M}^1(\mathcal{P},x,\lambda,\mu,t) & = & D_{(x,\lambda,\mu)} \mathsf{H}^1_{J_0}(\mathcal{P},x,\lambda,\mu,t) \in M_1(n+q) \\ \mathsf{M}^0(\mathcal{P},x,\lambda,\mu,t) & = & \left[\begin{array}{lll} D_x \mathsf{L}^1_{J_0} & D_\lambda \mathsf{L}^0_{J_0} & D_{\mu \setminus \{\mu_1\}} \mathsf{L}^0_{J_0} \\ D_x(\bar{H}, \mathcal{C}) & 0 & 0 \\ D_x(\bar{G}, \mathcal{C})_{J_0 \setminus \{1\}} & 0 & 0 \end{array} \right] \in M_1(n+q-1) \end{array}$$

It can be shown that $Graf(\mathsf{H}^\eta_{J_0}, \mathsf{M}^\eta) \,\bar{\pitchfork}\, I\!R^{n^2+n+n(m+p)+m+p} \times M_4^\eta$ for each possible selection of J_1, J_0, η, s and of the fixed index. By use of the well-known parametrized theorem of Sard it follows for almost all fixed $\mathcal{P}$ that, for each of the possible selections above, it holds that

$$Graf(\mathsf{H}^\eta_{J_0}, \mathsf{M}^\eta) \,\bar{\pitchfork}\, M_4^\eta. \quad (17)$$

The codimension of $Graf(\mathsf{H}^{\eta}_{J_0}, \mathsf{M}^{\eta})$ is $n+q+n(n+2q)$. If (17) is not the empty set, then $|J_1| + (n+q+\eta-1-s)^2 \leq 1$.

For each g.c. point $\bar{z}$ of the problem $(\bar{F}, \bar{H}, \bar{G}, \mathcal{P})$ ($\mathcal{P}$ is already fixed and (17) holds), where LICQ holds, there exists an active index set $J_0 = J_0(\bar{z})$, multipliers $(\bar{\lambda}, \bar{\mu})$ and $J_1 = \{j \in J_0 | \bar{\mu}_j = 0\}$ such that (17) holds for $\eta = 1$.

If $s = n+q-1$ it follows $J_1 = \emptyset$ (**VU-ND1**) and due to the definition of $M_3(n+q, n+q-1)$ it holds also **Type3: 3-VU**. Now (17) implies that $D_{(z,\lambda,\mu)}\mathsf{H}^1_{J_0}(\mathcal{P}, \bar{z}, \bar{\lambda}, \bar{\mu})$ has full rank. From Theorem 1 and Remark 4 we know that **Type3: 4-VU** holds and then $\bar{z}$ is of Type 3. In case $s = n+q$ the same arguments as in [16] lead to points of Type 1 or 2. □

Theorem 4
The set $\mathcal{V}$ is open and dense in $\mathcal{D}$.

Here $\mathcal{D}$ is endowed with the product topology obtained by considering the C^2-strong (or Whitney) topology over $C^2(\mathbb{R}^{n+1}, \mathbb{R}^n)$ and the C^3-strong topology over $C^3(\mathbb{R}^{n+1}, \mathbb{R}^{m+p})$.

Proof:

The proof follows from the Theorems 2 and 3 by use of the partition of the unity. The needed local→global construction is standard in the differential topology (see e.g. [1], [9]). □

The following remarks about the structure of the critical curves hold with basically the same proof as in the case of one-parametric optimization problems (see e.g. [11]).

Remark 18
Let $VI(F,H,G) \in \mathcal{V}$. Remarks 1 and 2 imply the openness of the set Σ^1_{gc} in Σ_{gc}. On the other hand, from Remark 3 (for Type 2), Remark 5 (for Type 3), the Remarks 7 and 8, Proposition 2 (for Type 4) and the Remarks 10 and 11 (for Type 5), it follows that every point in $\Sigma_{gc} \setminus \Sigma^1_{gc}$ is isolated and a border points of Σ^1_{gc}. Therefore:

- *Σ^1_{gc} is open and dense in Σ_{gc},*
- *$\Sigma_{gc} \setminus \Sigma^1_{gc} = \cup_{i=2}^{5} \Sigma^i_{gc}$ is a discrete set.*

Remark 19
Let $VI(F,H,G) \in \mathcal{V}$. Remark 2 implies the openness of Σ^1_{stat} in Σ_{stat}. The changes of the linear indices around points of Type 2 and 3, shown in the figures 1 and 2, imply that such points are not border points of the set Σ_{stat}.

By definition the points of Type 4 do not belong to the set Σ_{stat}. *If* $z \in \Sigma_{gc}^4$ *lies in* $\overline{\Sigma_{stat}}$, *then the MFCQ is violated and* $\overline{\Sigma_{stat}}$ *form a one-dimensional manifold (possibly with border) around* z. *In this case* z *is a border point of* $\overline{\Sigma_{stat}}$ *if and only if* $J_0(z) \neq \emptyset$. *On the other hand, it follows from Remark 12 that points of Type* 5 *in* Σ_{stat} *are border points of* Σ_{stat} *if and only if the MFCQ fails to hold.*

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