On the criterion of asymptotical stability for index-1 tractable DAEs

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Abstract

This paper considers the index-1 tractable differential-algebraic equation. The Lyapunov stability of the trivial solution is discussed. As a criterion of the asymptotical stability we propose a numerical parameter $æ(A, B)$ characterizing the property of the index-1 matrix pencil $\{A, B\}$ to have all finite eigenvalues within the negative complex half-plane. An algorithm for computing this parameter is described.

Introduction

The implicit systems of the differential-algebraic equations (DAEs)

$$f(x'(t), x(t), t) = 0$$

with the nontrivial nullspace of the Jacobian $f'_y(y, x, t)$ and its numerical solution have been discussed for many years [1-3] already. These equations often arise in various applications, e.g. in the simulation of electronic circuits, in control problems, in modelling constrained mechanical systems. Therefore the stability analysis is of great interest both from theoretical as well as from the practical point of view. The stability of DAEs was studied in [2, 4, 5].

In this paper we propose an approach to study the asymptotical stability of the trivial solution to the linear system of differential equations

$$Ax'(t) + Bx(t) = 0 \tag{1}$$

with constant matrix coefficients A and B. This problem is well investigated in the case of the nonsingular matrix A, when (1) turns into an explicit system of ordinary differential equations (ODEs)

$$x'(t) = Mx(t) \tag{2}$$

with the matrix $M = -A^{-1}B$. According to the classical Lyapunov stability theory the trivial solution of (2) is asymptotically stable if and only if the eigenvalues of M lie within the negative complex half-plane (see, e.g. [6]).

If the matrix A is singular, then the investigation of the spectrum of the matrix pencil $\{A, B\}$ is necessary. In [2] the trivial solution of the DAE (1) is shown to be asymptotically stable if all finite eigenvalues of the pencil $\{A, B\}$ have negative real parts. However, checking this condition in practice involves some difficulties due to the instability of the eigenvalues with respect to perturbations of the initial data.

As a matter of fact, as follows from the modern theory of Unsymmetric Eigenvalue Problem (see, e.g. [7]), it is more useful to deal not with individual eigenvalues but with some parameters that characterize their position with respect to a certain subset of the complex plane. For example, studying the asymptotical behaviour of solutions of equation (2) as a characteristic in [6] it was suggested a so-called dichotomy parameter $æ(M) = 2\|M\|\|H\|$, where the matrix H is the solution of the Lyapunov equation

$$HM + M^*H = -I.$$

In this paper we derive an analogous criterion of the asymptotical stability for the index-1 tractable DAEs. In Section 1 we recall the fundamental facts for the index-1 tractable DAE and some properties of the matrix pencils. Section 2 contains the investigation of the asymptotical stability of the trivial solution of (1) similar to the one of ODEs in [6]. In Section 3 we introduce a numerical parameter $æ(A, B)$ and show that this parameter can be used as a quantitative characteristic of the "quality" of the asymptotical stability. In Section 4 we derive a system of matrix equations generalizing the classical Lyapunov equation and permitting to work up an algorithm for computing $æ(A, B)$. This algorithm is discussed in Section 5. We also indicate methods of reducing the pencil $\{A, B\}$ with index 1 to the Kronecker canonical form and of computing the projectors onto the subspaces of $\{A, B\}$ associated with the finite and infinite eigenvalues, respectively.

The proposed criterion of numerical checking whether all finite eigenvalues of the matrix pencil $\{A, B\}$ belong to the negative complex half-plane can also be used by the investigation of the asymptotical stability of the stationary solution of the index-1 nonlinear autonomous equation [5] as well as of the nonautonomous equation with constant linear part and small nonlinearity [8].

1 The index-1 tractable DAE

Consider the differential-algebraic equation

$$Ax'(t) + Bx(t) = 0 \tag{3}$$

with constant matrix coefficients A and B of order m.

Definition 1 [2] *The equation (3) is called index-1 tractable if the matrix pencil* $\{A, B\}$ *is regular with index 1.*

Tractability with index 1 is characterized by the fact that, for any projector Q onto $\ker A$, the matrix $G = A + BQ$ is nonsingular and

$$\mathbb{R}^m = \ker A \oplus S,$$

where the subspace $S := \{z \in \mathbb{R}^m : \ Bz \in \operatorname{im} A\}$ [2, Theorem A13]. The projector Q onto the nullspace of A along S is called *the canonical projector.*

Let Q be the the canonical projector onto $\ker A$, then the relation $Q = QG^{-1}B$ is valid [2, Lemma A14]. Denote $P = I - Q$. We multiply equation (3) by G^{-1}. Taking into account the easily checked equalities $G^{-1}A = P$ and $G^{-1}B = PG^{-1}BP + Q$, we obtain the equation

$$(Px)'(t) + PG^{-1}BPx(t) + Qx(t) = 0.$$

Further, multiplying this equation by P and Q, respectively, we obtain the system

$$\begin{aligned} (Px)'(t) + PG^{-1}BPx(t) &= 0, \\ Qx(t) &= 0. \end{aligned}$$

Therefore, the solutions of (3) can be written as $x(t) = Px(t) + Qx(t) = Px(t)$.

Definition 2 *A matrix-valued function $\mathcal{G}(t) \equiv \mathcal{G}(t, A, B) \in C^1$ is called the Green matrix of equation (3) if it satisfies the initial value problem*

$$\begin{aligned} \frac{d}{dt}\mathcal{G}(t) &= M\mathcal{G}(t), \qquad t > 0, \\ \mathcal{G}(0) &= P, \end{aligned}$$

with $M = -PG^{-1}B$.

Remark 1. The Green matrix $\mathcal{G}(t, A, B)$ is a generalization of the classical notion of the fundamental matrix of solution for linear ODEs (see, e.g. [6]).

It is easy to verify that the Green matrix can be represented as $\mathcal{G}(t) = Pe^{tM}$. Uniqueness of $\mathcal{G}(t)$ immediately follows from the theory of ordinary differential equations. Then the general solution of equation (3) is of the form

$$x(t) = \mathcal{G}(t)x_0 = Pe^{tM}x_0,$$

where x_0 is an arbitrary vector.

Thus, we have proved the following

Theorem 1 [2]. *Let $\{A, B\}$ be a regular pencil with index 1, Q be the canonical projector onto $\ker A$ along S and $P = I - Q$. Then the initial value problem*

$$\begin{aligned} Ax'(t) + Bx(t) &= 0, \\ P(x(0) - x_0) &= 0 \end{aligned}$$

for all $x_0 \in \mathbb{R}^m$ has a unique solution $x(t)$ given by

$$x(t) = Pe^{tM}x_0,$$

with the matrix $M = -P(A + BQ)^{-1}B$.

Now we recall some facts for the matrix pencil with index 1 which will be used in the sequel.

Let $\{A, B\}$ be a regular matrix pencil with index 1, $r = \operatorname{rank} A$. Then there are nonsingular matrices W and T such that

$$A = W \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} T^{-1} \qquad \text{and} \qquad B = W \begin{pmatrix} -B_1 & 0 \\ 0 & I_{m-r} \end{pmatrix} T^{-1}, \tag{4}$$

where I_r denotes the unit $(r \times r)$-matrix. The representation (4) is the Kronecker canonical form of the pencil $\{A, B\}$ [9].

It is easy to check that

$$Q = T \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} T^{-1} \tag{5}$$

is the canonical projector onto $\ker A$ along the subspace S. Indeed, the relations $AQ = 0$ and $Q = Q(A + BQ)^{-1}B$ are valid. Then

$$P = I - Q = T \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} T^{-1} \tag{6}$$

and the matrix M can be written in the form

$$M = -P(A + BQ)^{-1}B = T \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} T^{-1}. \tag{7}$$

Definition 3 *A complex value $\lambda \neq \infty$ is said to be a finite eigenvalue of the matrix pencil $\{A, B\}$ if $det(\lambda A + B) = 0$. If λ is an eigenvalue, then there is a vector $x \neq 0$ such that $\lambda A x = -Bx$. The vector x is called an eigenvector of the pencil.*

Definition 4 *The matrix pencil $\{A, B\}$ has the infinite eigenvalue $\lambda = \infty$ if there is a vector $x \neq 0$ such that $Ax = 0$. The vector x is called an eigenvector of the pencil $\{A, B\}$ corresponding to the eigenvalue $\lambda = \infty$.*

By the representation (4) we have

$$\lambda A + B = W \begin{pmatrix} \lambda I - B_1 & 0 \\ 0 & I \end{pmatrix} T^{-1}. \tag{8}$$

It follows from (7) and (8) that the matrix M has zero eigenvalue with multiplicity $m - r$ and the remaining eigenvalues of M are exactly the r finite eigenvalues of the matrix pencil $\{A, B\}$. Moreover, the matrices Q and P given by (5) and (6) are the projectors onto the invariant subspaces of the pencil $\{A, B\}$ associated with the infinite and finite eigenvalues, respectively [5].

2 Asymtotical stability

In this section we derive the necessary and sufficient condition for the asymptotical stability of the trivial solution of the index-1 tractable equation (3). The following definitions (see, e.g. [2]) describe the meaning of the Lyapunov stability for the linear differential-algebraic equation.

Definition 5 *The trivial solution $x(t) \equiv 0$ of (3) is stable in the sense of Lyapunov if, for a certain projector Π along the maximal invariant subspace of the matrix pencil $\{A, B\}$ associated with the infinite eigenvalue, the initial value problem*

$$\begin{aligned} Ax'(t) + Bx(t) &= 0, \\ \Pi(x(0) - x_0) &= 0 \end{aligned} \tag{9}$$

for all $x_0 \in \mathbb{R}^m$ has a solution $x(t, x_0)$ defined on $[0, \infty)$. Moreover, for each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\|x(t, x_0)\| < \varepsilon$ for all $t \geq 0$ and for all $x_0 \in \mathbb{R}^m$ with $\|\Pi x_0\| < \delta$.

Definition 6 *The trivial solution $x(t) \equiv 0$ of (3) is asymptotically stable in the sense of Lyapunov if it is stable and if there is a $\delta_0 > 0$ such that for all $x_0 \in \mathbb{R}^m$ with $\|\Pi x_0\| < \delta_0$ the solution $x(t, x_0) \to 0$ for $t \to \infty$.*

Remark 2. The Lyapunov stability, as it was noted in [4], does not depend on the special choice of the projector Π, the only relevant characteristic feature is its nullspace, which is fully determined by the matrix pencil $\{A, B\}$.

Remark 3. For the ODE these definitions coincide with the classical notions of stability and asymptotical stability with $\Pi = I$.

The following theorem is already known and was shown in [2]. We prove it in another way using the representation of the solution by means of the matrix exponential.

Theorem 2 . *The trivial solution $x(t) \equiv 0$ of equation (3) is asymptotically stable if and only if all finite eigenvalues of the pencil $\{A, B\}$ have negative real parts.*

Proof. Assume that all finite eigenvalues of the pencil $\{A, B\}$ have negative real parts. Then, by (8), all eigenvalues of the matrix B_1 belong to the negative complex half-plane, i.e., $\operatorname{Re} \lambda_j(B_1) \leq -\sigma < 0$. In this case we have the following estimation for $t \geq 0$

$$\|e^{tB_1}\| \leq \gamma(r) \Big(\frac{\|B_1\|}{\sigma} \Big)^{r-1} e^{-t\sigma/2}, \tag{10}$$

where $\gamma(r)$ is a constant that depends on r only [6].

Further, taking into account that

$$Pe^{tM} = T \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} e^{tB_1} & 0 \\ 0 & I \end{pmatrix} T^{-1} = T \begin{pmatrix} e^{tB_1} & 0 \\ 0 & 0 \end{pmatrix} T^{-1} = Pe^{tM}P \tag{11}$$

we can estimate

$$\|Pe^{tM}\| \leq \|T\| \|T^{-1}\| \|e^{tB_1}\| \leq \gamma(r) \|T\| \|T^{-1}\| \Big(\frac{\|B_1\|}{\sigma} \Big)^{r-1} e^{-t\sigma/2}. \tag{12}$$

By Theorem 1 the initial value problem (9) with $\Pi = P$ has the unique solution $x(t) = Pe^{tM}x_0$. For each $\varepsilon > 0$ we assign

$$\delta = \frac{\varepsilon \sigma^{r-1}}{\gamma(r) \|T\| \|T^{-1}\| \|B_1\|^{r-1}}.$$

Then, for all $t \geq 0$ and $x_0 \in \mathbb{R}^m$ with $\|Px_0\| < \delta$, we obtain

$$\|x(t)\| = \|Pe^{tM}x_0\| < e^{-t\sigma/2}\varepsilon \leq \varepsilon,$$

i.e., the trivial solution of (3) is stable. Moreover, $x(t) \to 0$ for $t \to \infty$. This means that the solution $x(t) \equiv 0$ of equation (3) is asymptotically stable with $\Pi = P$.

Now, let the matrix pencil $\{A, B\}$ has a finite eigenvalue λ with nonegative real part, and $z \in \operatorname{im} \Pi$ be the corresponding eigenvector. Then $x(t) = e^{\lambda t} z$ is a solution of equation (3). In this case the trivial solution is not asymptotically stable since

$$\|x(t)\| = |e^{\lambda t}|\|z\| \geq \|z\| \neq 0.$$

The theorem is proved.

3 The criterion of asymptotical stability

Let all finite eigenvalues of the pencil $\{A, B\}$ with index 1 have negative real parts. Given the matrices M and P having the structures described above. We consider the Lyapunov equation

$$XM + M^*X = -P^*FP. \tag{13}$$

for the unknown matrix X. The matrix F is supposed to be hermitian and positive definite ($F^* = F > 0$).

Inequality (12) yields the convergence of the integral in the following form

$$H = \int_0^\infty e^{tM^*}P^*FPe^{tM}dt + Q^*FQ. \tag{14}$$

It is easy to verify that the matrix H is hermitian, positive definite and satisfies equation (13). Indeed, $H^* = H$ and

$$\begin{aligned} HM + M^*H &= \int_0^\infty e^{tM^*}P^*FPe^{tM}M dt + Q^*FQM + M^*Q^*FQ + \\ &+ \int_0^\infty M^*e^{tM^*}P^*FPe^{tM}dt = \int_0^\infty \frac{d}{dt}(e^{tM^*}P^*FPe^{tM})dt = -P^*FP. \end{aligned}$$

Here we used the property $M^*Q^* = QM = 0$. Further, by means of the Cholesky decomposition the matrix F can be represented as the product $F = L^*L$ with $\det L \neq 0$. Then we have for each vector z

$$(Hz, z) = \int_0^\infty (e^{tM^*}P^*FPe^{tM}z, z)dt + (Q^*FQz, z) = \int_0^\infty \|Le^{tM}Pz\|^2 dt + \|LQz\|^2.$$

Since the matrices L and e^{tM} are regular and the vectors Pz and Qz cannot vanish simultaneously for $z \neq 0$, we conclude that $(Hz, z) > 0$, i.e., the matrix H is positive definite.

In (14) we assign $F := G^{-*}G^{-1} = (A+BQ)^{-*}(A+BQ)^{-1}$ and define

$$æ(A,B) = 2\|A\|\|B\|\|H\|.$$

Here $^{-*}$ denotes the composition of inversion and conjugate transforms. If the matrix pencil $\{A, B\}$ has index 1 and all its finite eigenvalues belong to the negative complex half-plane, then $æ(A,B) < \infty$, obviously. We set $æ(A,B) = \infty$ if the index of $\{A, B\}$ is greater than 1 or the pencil $\{A, B\}$ has at least one eigenvalue with nonnegative real part.

It is interesting that the parameter $æ(A,B)$ can be used in pointwise estimates of solution of equation (3). More precisely, similar to [6], [7] the following estimation

$$\|x(t)\| \le \mu(G)\sqrt{æ(A,B)}\, e^{-t\|A\|\|B\|/(\|G\|^2 æ(A,B))}\|Px(0)\| \tag{15}$$

can be proved, where $\mu(G) = \|G\|\|G^{-1}\|$ is the condition number of the matrix G.

Indeed, if $æ(A,B) = \infty$, then there is nothing to prove. Let $æ(A,B) < \infty$ and H be the solution of the matrix equation (13) with $F = G^{-*}G^{-1}$. We note that the matrix H can be rewritten in terms of the Green matrix $\mathcal{G}(t) = Pe^{tM}$as follows

$$H = \int_0^\infty \mathcal{G}^*(t)F\mathcal{G}(t)dt + Q^*FQ.$$

Let us consider the matrix-valued function for $t > 0$

$$Y(t) = \int_t^\infty \mathcal{G}^*(s)F\mathcal{G}(s)ds.$$

Using the obvious properties of the Green matrix

$$\begin{aligned}
&\mathcal{G}(t) = \mathcal{G}(t)P = P\mathcal{G}(t),\\
&\mathcal{G}(t)Q = Q\mathcal{G}(t) = 0,\\
&\mathcal{G}(t+s) = \mathcal{G}(t)\mathcal{G}(s) = \mathcal{G}(s)\mathcal{G}(t),
\end{aligned}$$

we have

$$Y(t) = \int_t^\infty \mathcal{G}^*(s)F\mathcal{G}(s)ds = \mathcal{G}^*(t)\left\{\int_0^\infty \mathcal{G}^*(s)F\mathcal{G}(s)ds\right\}\mathcal{G}(t) = \mathcal{G}^*(t)H\mathcal{G}(t).$$

After differentiating the matrix $Y(t)$ taking into account the inequality

$$(Hz,z) \le \|H\|\|F^{-1}\|(Fz,z)$$

we obtain, for an arbitrary vector z, the estimation

$$\begin{aligned}
\frac{d}{dt}(Y(t)z,z) &= \Big(\mathcal{G}^*(t)(M^*H+HM)\mathcal{G}(t)z,z\Big) = -(\mathcal{G}^*(t)F\mathcal{G}(t)z,z) \le\\
&\le -\frac{(\mathcal{G}^*(t)H\mathcal{G}(t)z,z)}{\|H\|\|F^{-1}\|} = -\frac{(Y(t)z,z)}{\|H\|\|F^{-1}\|},
\end{aligned}$$

which implies
$$\frac{d}{dt}\Big(e^{t/(\|H\|\|F^{-1}\|)}(Y(t)z,z)\Big) \le 0.$$
Consequently,
$$\begin{aligned}(\mathcal{G}^*(t)H\mathcal{G}(t)z,z) &= (Y(t)z,z) \le e^{-t/(\|H\|\|F^{-1}\|)}(Y(0)z,z) = \\ &= e^{-t/(\|H\|\|F^{-1}\|)}(P^*HPz,z).\end{aligned} \tag{16}$$
Moreover, using the inequality $\|e^{tM}Pz\| \ge e^{-|t|\|M\|}\|Pz\|$ (see, e.g. [6]) we have
$$(HPz,Pz) = \int_0^\infty (Fe^{tM}Pz, e^{tM}Pz)dt \ge \frac{\|Pz\|^2}{\|F^{-1}\|}\int_0^\infty e^{-2t\|M\|}dt = \frac{\|Pz\|^2}{2\|M\|\|F^{-1}\|}. \tag{17}$$
Further, combining (16) and (17) for $z = x(t) = \mathcal{G}(t)x(0)$ we obtain
$$\begin{aligned}\|x(t)\|^2 &= \|\mathcal{G}(t)x(0)\|^2 \le 2\|M\|\|F^{-1}\|(H\mathcal{G}(t)x(0), \mathcal{G}(t)x(0)) \le \\ &\le 2\|M\|\|F^{-1}\|e^{-t/(\|H\|\|F^{-1}\|)}(HPx(0),Px(0)) \le \\ &\le 2\|M\|\|F^{-1}\|\|H\|e^{-t/(\|H\|\|F^{-1}\|)}\|Px(0)\|^2.\end{aligned}$$

Finally, taking into account that the relations $M = -PG^{-1}B$, $P = G^{-1}A$ and $F = G^{-*}G^{-1}$ imply the inequality
$$\|M\|\|F^{-1}\| \le \|A\|\|B\|\|G^{-1}\|^2\|G\|^2 = \|A\|\|B\|\mu^2(G)$$
to be valid, we obtain the required estimation
$$\|x(t)\| \le \mu(G)\sqrt{æ(A,B)}\, e^{-t\|A\|\|B\|/(\|G\|^2 æ(A,B))}\|Px(0)\|.$$

Remark 4. Note that the estimation (15) is a generalization of the classical one for explicit ODEs [6].

It follows from (15), in particular, that by $æ(A,B) < \infty$ the trivial solution of the index-1 tractable equation (3) is asymptotically stable. On the other hand, if the trivial solution of (3) is asymptotically stable, then all eigenvalues of the pencil $\{A,B\}$ with index 1 have negative real parts. In this case the matrix H defined in (14) has the finite norm, i.e., $æ(A,B) < \infty$.

Thus, the parameter $æ(A,B)$ can be used as a criterion of the asymptotical stability of the trivial solution of the index-1 tractable equation (3).

Concluding this section, we derive the integral representations for the matrix H and the projector P in terms of A and B
$$\begin{aligned}P &= \frac{1}{\pi}\int_{-\infty}^{\infty}(i\xi A + B)^{-1}Ad\xi, \\ H &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(A+BQ)^*(i\xi A+B)^{-*}P^*FP(i\xi A+B)^{-1}(A+BQ)d\xi + Q^*FQ.\end{aligned} \tag{18}$$

The first equation follows from the decompositions (4) and (8). In this formula we mean the principal value (in the sence of Caushy) of the integral. In order to prove the second equation we transfom the matrix H into

$$\begin{aligned} H &= \int_0^\infty e^{tM^*} P^* F P e^{tM} dt + Q^* F Q = \\ &= T^{-*} \Big\{ \int_0^\infty \begin{pmatrix} e^{tB_1^*} & 0 \\ 0 & 0 \end{pmatrix} T^* F T \begin{pmatrix} e^{tB_1} & 0 \\ 0 & 0 \end{pmatrix} dt + \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} T^* F T \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \Big\} T^{-1}. \end{aligned}$$

If the matrix T^*FT has the form

$$T^* F T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \tag{19}$$

where the matrices T_{11} and T_{22} are, obviously, hermitian and positive definite, then

$$H = T^{-*} \begin{pmatrix} \int_0^\infty e^{tB_1^*} T_{11} e^{tB_1} dt & 0 \\ 0 & T_{22} \end{pmatrix} T^{-1} = T^{-*} \begin{pmatrix} H_1 & 0 \\ 0 & T_{22} \end{pmatrix} T^{-1}. \tag{20}$$

Note that the matrix

$$H_1 = \int_0^\infty e^{tB_1^*} T_{11} e^{tB_1} dt$$

satisfies the Lyapunov equation

$$X B_1 + B_1^* X = -T_{11} \tag{21}$$

with unknown matrix X. By the Lyapunov theorem this equation is uniquely solvable for any matrix T_{11} if all eigenvalues of the matrix B_1 have negative real parts (see, e.g. [6]). Moreover, in [7] it was shown that the solution of (21) can be represented as

$$X = \frac{1}{2\pi} \int_{-\infty}^\infty (i\xi I - B_1)^{-*} T_{11} (i\xi I - B_1)^{-1} d\xi.$$

In view of the uniqueness of the solution we conclude that

$$H_1 = \frac{1}{2\pi} \int_{-\infty}^\infty (i\xi I - B_1)^{-*} T_{11} (i\xi I - B_1)^{-1} d\xi. \tag{22}$$

Consequently,

$$H = T^{-*} \begin{pmatrix} \frac{1}{2\pi} \int_{-\infty}^\infty (i\xi I - B_1)^{-*} T_{11} (i\xi I - B_1)^{-1} d\xi & 0 \\ 0 & T_{22} \end{pmatrix} T^{-1} =$$

$$= T^{-*} \left\{ \frac{1}{2\pi} \int_{-\infty}^\infty \begin{pmatrix} (i\xi I - B_1)^{-*} & 0 \\ 0 & 0 \end{pmatrix} T^* F T \begin{pmatrix} (i\xi I - B_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix} d\xi \right\} T^{-1} + Q^* F Q.$$

Finally, taking into account that

$$P(i\xi A + B)^{-1}(A + BQ) = T \begin{pmatrix} (i\xi I - B_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix} T^{-1},$$

we obtain the required integral representation for the matrix H.

4 Matrix equations

Consider the problem of numerical checking whether all finite eigenvalues of the pencil $\{A, B\}$ belong to the negative complex half-plane. In [6], [7] this problem is completely investigated for the case of $A = I$. For the general case in [7] it is proposed to reduce the problem of linear dichotomy of the matrix pencil $\{A, B\}$ to the problem dichotomy with respect to the unit circle of the pencil $\{A+B, A-B\}$. However, it is easy to see that for the index-1 pencil $\{A, B\}$ the number $\lambda = 1$ is an eigenvalue of the pencil $\{A+B, A-B\}$. In this section we investigate this independently, essentialy using ideas from [7].

Using the decomposition (20) for the matrix H we obtain the following relations

$$\begin{aligned} P^*H &= T^{-*}\begin{pmatrix} H_1 & 0 \\ 0 & 0 \end{pmatrix} T^{-1} = HP = P^*HP, \\ Q^*H &= T^{-*}\begin{pmatrix} 0 & 0 \\ 0 & T_{22} \end{pmatrix} T^{-1} = HQ = Q^*HQ = Q^*FQ. \end{aligned}$$

Then

$$H = P^*HP + Q^*HQ. \tag{23}$$

On the other hand, we have

$$\begin{aligned} B^*(A+BQ)^{-*}P^*HP(A+BQ)^{-1}A &= T^{-*}\begin{pmatrix} -B_1^*H_1 & 0 \\ 0 & 0 \end{pmatrix} T^{-1}, \\ B^*(A+BQ)^{-*}Q^*HQ(A+BQ)^{-1}A &= 0, \\ A^*(A+BQ)^{-*}P^*HP(A+BQ)^{-1}B &= T^{-*}\begin{pmatrix} -H_1B_1 & 0 \\ 0 & 0 \end{pmatrix} T^{-1}, \\ A^*(A+BQ)^{-*}Q^*HQ(A+BQ)^{-1}B &= 0. \end{aligned}$$

Therefore, from (23) we obtain

$$\begin{aligned} &A^*(A+BQ)^{-*}H(A+BQ)^{-1}B + B^*(A+BQ)^{-*}H(A+BQ)^{-1}A = \\ &\qquad = T^{-*}\begin{pmatrix} -H_1B_1 - B_1^*H_1 & 0 \\ 0 & 0 \end{pmatrix} T^{-1}. \end{aligned} \tag{24}$$

Since the matrix H_1 given by (22) satisfies the Lyapunov equation (21) and

$$T^{-*}\begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} T^{-1} = T^{-*}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} T^{-1} = P^*FP,$$

we can rewrite (24) as

$$A^*(A+BQ)^{-*}H(A+BQ)^{-1}B + B^*(A+BQ)^{-*}H(A+BQ)^{-1}A = P^*FP. \tag{25}$$

Denote

$$\begin{aligned} Z &:= (A+BQ)^{-*}H(A+BQ)^{-1} = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi A+B)^{-*}P^*FP(i\xi A+B)^{-1}d\xi + (A+BQ)^{-*}Q^*FQ(A+BQ)^{-1}. \end{aligned}$$

By (4) and (20) we have

$$Z = W^{-*} \begin{pmatrix} H_1 & 0 \\ 0 & T_{22} \end{pmatrix} W^{-1}.$$

Then (25) implies

$$A^* Z B + B^* Z A = P^* F P.$$

Moreover, the matrix Z satisfies the relations

$$(BQ)^* Z (A + BQ) = (A + BQ)^* Z BQ \quad \text{and} \quad (BQ)^* Z BQ = Q^* F Q,$$

which immediately follow from (23).

Theorem 3 . *Let $\{A, B\}$ be a regular matrix pencil with index 1 and F be a hermitian, positive definite matrix. Assume that there exist the matrix Q and the hermitian, positive definite matrix Z ($Z^* = Z > 0$) which satisfy the matrix equations*

$$A^* Z B + B^* Z A = (I - Q)^* F (I - Q), \tag{26}$$

$$AQ = 0, \quad Q = Q(A + BQ)^{-1} B, \tag{27}$$

$$(BQ)^* Z (A + BQ) = (A + BQ)^* Z BQ, \tag{28}$$

$$(BQ)^* Z BQ = Q^* F Q. \tag{29}$$

Then all finite eigenvalues of the pencil $\{A, B\}$ have negative real parts and $I - Q$ is the projector onto the subspace corresponding to the finite eigenvalues of $\{A, B\}$.

Proof. Assume that the matrix pencil $\{A, B\}$ is reduced to the Kronecker canonical form (4). Let the matrices

$$Q = T \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} T^{-1} \qquad \text{and} \qquad Z = W^{-*} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} W^{-1}$$

satisfy the equations (26)-(29). The equality $AQ = 0$ implies $Q_{11} = Q_{12} = 0$. Since the matrix $A + BQ$ is nonsingular, Q_{22} is so, too. Further, from the second relation of (27) it follows that

$$\begin{pmatrix} 0 & 0 \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ -Q_{22}^{-1} Q_{21} & Q_{22}^{-1} \end{pmatrix} \begin{pmatrix} B_1 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

i.e. $Q_{21} = 0$ and $Q_{22} = I$. Thus, we see that the matrix

$$I - Q = T \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} T^{-1}$$

is the projector onto the subspace corresponding to the finite eigenvalues of $\{A, B\}$.

Denote

$$R := BQ(A + BQ)^{-1} = W \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} W^{-1}.$$

Equality (28) implies $R^*Z = ZR$. Then

$$R^*Z(I-R) = R^*Z - R^*ZR = 0, \qquad (I-R)^*ZR = (R^*Z(I-R))^* = 0.$$

Further, we obtain

$$Z = (R+(I-R))^*Z(R+(I-R)) = R^*ZR+(I-R)^*Z(I-R) = W^{-*}\begin{pmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{pmatrix} W^{-1},$$

where the matrices Z_{11} and Z_{22} are hermitian and positive definite.

From the relation (26) it follows that

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} T^*FT \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{pmatrix}\begin{pmatrix} -B_1 & 0 \\ 0 & I \end{pmatrix} +$$

$$+\begin{pmatrix} -B_1^* & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -Z_{11}B_1 - B_1^*Z_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

in addition there exists the hermitian matrix Z_{11} that satisfies the Lyapunov equation

$$Z_{11}B_1 + B_1^*Z_{11} = -T_{11}. \tag{30}$$

In this case by the Lyapunov theorem all eigenvalues of the matrix B_1 have negative real parts [6], i.e., all finite eigenvalues of the pencil $\{A, B\}$ belong to the negative complex half-plane. The theorem is proved.

The converse is also true.

Theorem 4 . *If all finite eigenvalues of the matrix pencil $\{A, B\}$ with index 1 have negative real parts, the system of matrix equations (26)-(29) with hermitian, positive definite matrix F for the unknown matrices Q and Z has an unique solution such that the matrix Z is hermitian and positive definite. Moreover, the solution can be represented as follows*

$$\begin{aligned} Q &= I - \frac{1}{\pi}\int_{-\infty}^{\infty}(i\xi A + B)^{-1}A d\xi, \qquad (31)\\ Z &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi A + B)^{-*}(I-Q)^*F(I-Q)(i\xi A + B)^{-1}d\xi + \\ &+ (A+BQ)^{-*}Q^*FQ(A+BQ)^{-1}. \end{aligned}$$

Proof. Equations (27) imply that Q is the canonical projector onto N along S. Then the representation (31) for Q follows immediately from (18).

The proof of Theorem 3 has shown that any solution Z of (26)-(29) is of the following form

$$Z = W^{-*}\begin{pmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{pmatrix} W^{-1},$$

where the matrix Z_{11} satisfies the Lyapunov equation (30). As mentioned above, the equation (30) has exactly one solution, which is given by

$$Z_{11} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(i\xi I - B_1)^{-*}T_{11}(i\xi I - B_1)^{-1}d\xi.$$

The matrix Z_{22} can be computed from equation (29). It is uniquely defined and given by $Z_{22} = T_{22}$. The theorem is proved.

5 Computing the parameter $\ae(A, B)$

In this section we propose a method for reducing the regular index-1 pencil $\{A, B\}$ to the Kronecker canonical form (4) and discuss the computational aspects of the criterion of the asymptotical stability $\ae(A, B)$.

Let

$$A = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^* \tag{32}$$

be the singular value decomposition of A, where U and V are unitary matrices, Σ is a regular $(r \times r)$-matrix, $r = \operatorname{rank} A$. Then

$$Q_1 = V \begin{pmatrix} 0 & 0 \\ 0 & I_{m-r} \end{pmatrix} V^*$$

is the orthogonal projector onto $N = \ker A$. We can use it to determine the canonical projector $Q = Q_1(A + BQ_1)^{-1}B$. If the matrix B has the form

$$B = U \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} V^*,$$

then the matrix $G = A + BQ_1$ is given by

$$G = U \begin{pmatrix} \Sigma & B_{12} \\ 0 & B_{22} \end{pmatrix} V^*.$$

The block B_{22} is nonsingular, since G is supposed to be nonsingular. Then the projectors Q and $P = I - Q$ can be represented as

$$Q = V \begin{pmatrix} 0 & 0 \\ B_{22}^{-1}B_{21} & I \end{pmatrix} V^* \qquad \text{and} \qquad P = V \begin{pmatrix} I & 0 \\ -B_{22}^{-1}B_{21} & 0 \end{pmatrix} V^*.$$

Further, for the matrix $M = -P(A + BQ)^{-1}B$ we obtain

$$M = V \begin{pmatrix} -\Sigma^{-1}(B_{11} - B_{12}B_{22}^{-1}B_{21}) & 0 \\ B_{22}^{-1}B_{21}\Sigma^{-1}(B_{11} - B_{12}B_{22}^{-1}B_{21}) & 0 \end{pmatrix} V^* = V \begin{pmatrix} M_1 & 0 \\ -B_{22}^{-1}B_{21}M_1 & 0 \end{pmatrix} V^*,$$

where $M_1 = -\Sigma^{-1}(B_{11} - B_{12}B_{22}^{-1}B_{21})$.

Let us consider the nonsingular matrix

$$T = V \begin{pmatrix} I & 0 \\ -B_{22}^{-1}B_{21} & I \end{pmatrix}. \tag{33}$$

Then the matrix M is of the following form

$$M = T \begin{pmatrix} I & 0 \\ B_{22}^{-1}B_{21} & I \end{pmatrix} \begin{pmatrix} M_1 & 0 \\ -B_{22}^{-1}B_{21}M_1 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -B_{22}^{-1}B_{21} & I \end{pmatrix} T^{-1} = T \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix} T^{-1}$$

and for the canonical projectors we obtain the representations

$$Q = T \begin{pmatrix} I & 0 \\ B_{22}^{-1}B_{21} & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ B_{22}^{-1}B_{21} & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -B_{22}^{-1}B_{21} & I \end{pmatrix} T^{-1} = T \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} T^{-1}$$

and

$$P = T \begin{pmatrix} I & 0 \\ B_{22}^{-1}B_{21} & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -B_{22}^{-1}B_{21} & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -B_{22}^{-1}B_{21} & I \end{pmatrix} T^{-1} = T \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} T^{-1}.$$

Let us now consider the nonsingular matrix

$$W = U \begin{pmatrix} \Sigma & B_{12} \\ 0 & B_{22} \end{pmatrix}. \tag{34}$$

Then the matrices A and B can be rewritten as

$$A = W \begin{pmatrix} \Sigma^{-1} & -\Sigma^{-1}B_{12}B_{22}^{-1} \\ 0 & B_{22}^{-1} \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -B_{22}^{-1}B_{21} & I \end{pmatrix} T^{-1} = W \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} T^{-1},$$

and

$$\begin{aligned} B &= W \begin{pmatrix} \Sigma^{-1} & -\Sigma^{-1}B_{12}B_{22}^{-1} \\ 0 & B_{22}^{-1} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ -B_{22}^{-1}B_{21} & I \end{pmatrix} T^{-1} = \\ &= W \begin{pmatrix} \Sigma^{-1}(B_{11} - B_{12}B_{22}^{-1}B_{21}) & 0 \\ 0 & I \end{pmatrix} T^{-1}. \end{aligned}$$

Thus, we obtain the representation of the matrix pencil $\{A, B\}$ in Kronecker canonical form (4) with nonsingular matrices T and W in the form (33) and (34), respectively, and $B_1 = -\Sigma^{-1}(B_{11} - B_{12}B_{22}^{-1}B_{21}) = M_1$.

By the decomposition (20) the computation of the matrix H is reduced to calculating the matrices T_{11}, T_{22} and solving the Lyapunov equation

$$H_1B_1 + B_1^*H_1 = -T_{11}. \tag{35}$$

It follows from (19) that

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = T^*FT = (TW^{-1}T)^*TW^{-1}T.$$

For the solving of equation (35) we can use the quickly convergent procedure described in detail in [7]. First, one has to compute the axiliary matrices

$$\begin{aligned} L &= e^{\tau B_1} = I + \tau B_1 + \frac{\tau^2}{2!}B_1^2 + \ldots, \\ C &= \int_0^\tau e^{\tau B_1^*}T_{11}e^{\tau B_1} = \tau D_0 + \frac{\tau^2}{2!}D_1 + \frac{\tau^3}{3!}D_2 + \ldots, \end{aligned}$$

where $D_0 = T_{11}$, $D_j = D_{j-1}B_1 + B_1^* D_{j-1}$ and τ is chosen to be not too large, for example $\tau = 1/2\|B_1\|$. Then it is easy to verify that the

$$H_1 = C + \sum_{j=1}^{\infty} (L^*)^j C L^j$$

is the solution of equation (35).

Acknowledgement: The author would like to thank R.März and V.I.Kostin for valuable advice and useful discussions.

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