Asymptotic behaviour of solutions of semilinear hyperbolic systems in arbitrary domains

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Abstract:
In this paper the long time asymptotic behavior of solutions of semilinear symmetric hyperbolic system including Maxwell’s equations and the scalar wave-equation in an arbitrary domain are investigated. The possibly nonlinear damping term may vanish on a certain subset of the domain. It is shown that the solution decays weakly to zero if and only if the initial-state is orthogonal to all stationary states. In the case that the nonlinear damping is in addition montone, also strong local $L^q$-convergence is shown.

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1 Introduction

The subject of this paper the long time asymptotic behavior of solutions of semilinear hyperbolic systems of the form

$$\partial_t \mathbf{E} = E^{(1)} \left[ \left( \sum_{k=1}^{3} H_k \partial_k \mathbf{F} \right) - \mathbf{S}(t, \mathbf{E}) \right],$$

$$\partial_t \mathbf{F} = E^{(2)} \sum_{k=1}^{3} H_k \partial_k \mathbf{E},$$

with the initial-condition

$$\mathbf{E}(0, x) = \mathbf{E}_0(x), \mathbf{F}(0, x) = \mathbf{F}_0(x).$$

Here $\mathbf{E} \in C([0, \infty), L^2(\Omega, \mathbb{R}^M))$ and $\mathbf{F} \in C([0, \infty), L^2(\Omega, \mathbb{R}^N))$ are the unknown functions depending on the time $t \geq 0$ and the space-variable $x \in \Omega$. $\Omega \subset \mathbb{R}^3$ is an arbitrary domain. $H_k \in \mathbb{R}^{N \times M}$ are constant matrices, $E^{(1)} \in L^\infty(\Omega, \mathbb{R}^{M \times M})$ and $E^{(2)} \in L^\infty(\Omega, \mathbb{R}^{N \times N})$ are...
positive symmetric variable matrices, which depend on the space-variable \( x \in \Omega \) and satisfy \( E^{(1)} = 1 \) and \( E^{(2)} = 1 \) on \( \Omega_0 \equiv \Omega \setminus G \) with some subset \( G \subset \Omega \).

The generally nonlinear function \( S: [0, \infty) \times \Omega \times \mathbb{R}^M \rightarrow \mathbb{R}^M \) satisfies

\[
S(t, x, y) = 0 \text{ for all } x \in \Omega_0 = \Omega \setminus G
\]

and \( S(t, x, 0) = 0 \) for all \( x \in \Omega, t \in (0, \infty) \).

That means that the damping-term \( S(t, x, E) \) is only present on a certain subset \( G \subset \Omega \). The following coerciveness-assumption is imposed.

\[
yS(x, y) \geq \gamma(x) \min \{ |y|^p, |y| \} \text{ for all } y \in \mathbb{R}^M, x \in G.
\]

Here \( p \in [2, \infty) \) and \( \gamma \in L^\infty(G) \) is a positive function on \( G \), which does not necessarily have a uniform positive lower bound on \( G \).

This means that \( S(t, x, y) \) is allowed to be bounded as \( |y| \rightarrow \infty \) and \( |S(t, x, y)| \) behaves like \( |y|^{p-1} \) for small \( |y| \). For example \( S(t, x, E) \equiv \gamma(x)|E|^q(1 + |E|^q)^{-1}E \) with \( q \in [0, \infty) \) is possible.

A domain \( D(B) \subset L^2(\Omega, \mathbb{R}^{M+N}) \) containing \( C_0^\infty(\Omega, \mathbb{R}^{M+N}) \) is chosen, such that the operator

\[
B(E, F) \equiv \left( E^{(1)} \left( \sum_{k=1}^3 H_k^* \partial_k F \right), E^{(2)} \left( \sum_{k=1}^3 H_k^* \partial_k E \right) \right)
\]

is skew-adjoint on \( D(B) \), i.e. \( B^* = -B \) with respect to a weighted scalar-product. The choice of \( D(B) \) involves boundary-conditions on \( \partial \Omega \) supplementing 1.1-1.2.

A physically important example for this system are Maxwell's equations describing the propagation of the electromagnetic field

\[
\varepsilon \partial_t E = \text{curl } H - S(t, x, E), \quad \mu \partial_t H = -\text{curl } E, \quad (1.4)
\]

supplemented by the initial-boundary conditions

\[
\bar{n} \wedge E = 0 \text{ on } (0, \infty) \times \Gamma_1, \bar{n} \wedge H = 0 \text{ on } (0, \infty) \times \Gamma_2, \quad (1.5)
\]

\[
E(0, x) = E_0(x), \quad H(0, x) = H_0(x). \quad (1.6)
\]

In 1.5 \( \Gamma_1 \subset \partial \Omega \) and \( \Gamma_2 \equiv \partial \Omega \setminus \Gamma_1 \). \( E, H \) denote the electric and magnetic field respectively which depend on the time \( t \geq 0 \) and the space-variable \( x \in \Omega \) and \( S(t, x, E) \) describes a possibly nonlinear resistor. The dielectric and magnetic susceptibilities \( \varepsilon, \mu \in L^\infty(\Omega) \) are assumed to be uniformly positive.

For 1.4, 1.5 the operator \( B \) is defined in the space \( X \equiv L^2(\Omega, \mathcal{F}^0) \) by

\[
B(E, F) \equiv (\varepsilon^{-1} \text{curl } F, -\mu^{-1} \text{curl } E) \text{ for } (E, F) \in D(B) \equiv W_E \times W_H.
\]
Here $W_H$ is the closure of $C^\infty_0([\mathbb{R}^3 \setminus \Gamma_2, \mathbb{R}^3])$ in $H_{\text{curl}}(\Omega)$, where $H_{\text{curl}}(\Omega)$ is the space of all $\mathbf{E} \in L^2(\Omega, \mathbb{R}^3)$ with $\text{curl} \mathbf{E} \in L^2(\Omega)$.

$W_E$ denotes the set of all $\mathbf{E} \in H_{\text{curl}}(\Omega)$, such that

$$\int_\Omega \text{curl} \mathbf{F} - \mathbf{F} \text{curl} \mathbf{E} \, dx = 0 \text{ for all } \mathbf{F} \in W_H,$$

which includes a weak formulation of the boundary-condition $\bar{n} \wedge \mathbf{E} = 0$ on $\Gamma_1$, see [5].

Another example for 1.1-1.2 is the first-order system corresponding to the initial-boundary-value-problem of the scalar wave-equation with nonlinear damping, see [3], [4], [7].

\begin{equation}
\partial^2_t \varphi = \text{div} (E \nabla \varphi) - S(x, \partial_t \varphi)
\end{equation}

supplemented by the initial-boundary-conditions

\begin{equation}
\varphi = 0 \text{ on } (0, \infty) \times \partial \Omega
\end{equation}

\begin{equation}
\varphi(0, x) = f_0(x) \text{ and } \partial_t \varphi(0, x) = f_1(x)
\end{equation}

for initial-data $f_0 \in H^1(\Omega)$ and $f_1 \in L^2(\Omega)$. Here $E \in L^\infty(\Omega, \mathbb{R}^{3 \times 3})$ is a symmetric matrix-valued function satisfying $E = 1$ on $\Omega_0 = \Omega \setminus G$.

Note that $u \overset{\text{def}}{=} (\partial_t \varphi, E \nabla \varphi) \in C([0, \infty), L^2(\Omega, \mathbb{R}^4))$ solves the system

\begin{equation}
\partial_t u = (\text{div} (u_2, ..., u_4), E \nabla u_1) - (S(t, x, u_1), 0, 0, 0)
\end{equation}

which is of the form 1.1-1.3.

The set $\mathcal{N}$ of stationary states for 1.1-1.3 is the set of all $u \in D(B)$ with $Bu + F(t, u) = 0$ for all $t \geq 0$, where the nonlinear operator $F : (0, \infty) \times X \to X$ is defined by

\begin{equation}
F(t, u) \overset{\text{def}}{=} \left( E^{(1)} S(t, \cdot, u_1(\cdot)), 0 \right).
\end{equation}

From the assumptions on $S$ it follows

$$\{(E, F) \in \ker B : E = 0 \text{ on } G\} \subset \mathcal{N}.$$

Conversely assume $u = (E, F) \in D(B)$ with $Bu + F(t, u) = 0$. Since $B$ is skew-adjoint this yields $0 = \langle u, Bu \rangle_X = -\langle u, F(t, u) \rangle_X = \int_\Omega uS(t, x, E) \, dx$ which implies $E = 0$ on $G$ by the coerciveness assumption. Hence the set of stationary states is given by

$$\mathcal{N} = \{(E, F) \in \ker B : E = 0 \text{ on } G\}.$$

The aim of this paper is to show

$$\langle E(t), F(t) \rangle \overset{t \to \infty}{\rightharpoonup} 0 \text{ in } L^2(\Omega) \text{ weakly}$$

(1.11)
if and only if the initial-data \((\mathbf{E}_0, \mathbf{F}_0) \in L^2(\Omega)\) obey
\[
\int_{\Omega} \left( E^{(1)*} \mathbf{E}_0 e + E^{(2)*} \mathbf{F}_0 f \right) dx = 0 \text{ for all } (e, f) \in \mathcal{N}. \tag{1.12}
\]
In the case of Maxwell’s equations 1.4-1.6 the condition 1.12 on \((\mathbf{E}_0, \mathbf{F}_0)\) implies
\[
\text{div } (\varepsilon \mathbf{E}_0) = 0 \text{ on } \Omega_0 \text{ and } \text{div } (\mu \mathbf{H}_0) = 0 \text{ on } \Omega \tag{1.13}
\]
since \(\mathcal{N}\) contains all elements of the form \((\nabla \varphi, \nabla \psi)\) with \(\varphi \in C_0^\infty(\Omega_0)\) and \(\psi \in C_0^\infty(\Omega)\).

The proof of 1.11 is based on a suitable modification of the approach in [3], [11] for the case that the operator \(B\) does not necessarily have purely discrete spectrum. The basic idea is to show that for each \(f \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})\) and \(g \in \omega_0(\mathbf{E}_0, \mathbf{F}_0)\) the function \(f(iB)g\) is real-analytic and vanishes on \(G\), where \(\omega_0(\mathbf{E}_0, \mathbf{F}_0)\) denotes the \(\omega\)-limit-set with respect to the weak topology of the orbit belonging to the initial-state \((\mathbf{E}_0, \mathbf{F}_0)\). This implies \(f(iB)g = 0\) for all \(f \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})\) and hence \(g \in \ker B\). (Here the operator \(f(iB)\) can be defined by the spectral-theorem, since \(iB\) is self-adjoint in \(L^2(\Omega, \mathcal{C}^{M+N})\).)

If \(\mathcal{S}\) is independent of \(t\) and monotone with respect to \(\mathbf{E}\) strong \(L^r\)-convergence is shown, i.e.
\[
\|\mathbf{E}(t)\|_{L^r(K)} + \|\mathbf{F}(t)\|_{L^2(K)} \xrightarrow{t \to \infty} 0 \text{ for all } 1 < r < 2, \text{ and compact sets } K \subset \Omega. \tag{1.14}
\]
if the initial-data \((\mathbf{E}_0, \mathbf{F}_0) \in L^2(\Omega)\) obey condition 1.12.

Finally 1.11 is used to prove that the solution the wave-equation 1.7-1.8 in an arbitrary domain \(\Omega \subset \mathbb{R}^3\) decays with respect to the energy-norm on each bounded subdomain of \(\Omega\). The case of a bounded domain has been considered in [3] and [11]. Here the domain \(\Omega\) is not necessarily bounded. For all \(R \in (0, \infty), f_0 \in H^1(\Omega)\) and \(f_1 \in L^2(\Omega)\) it is shown that
\[
\left( \|\nabla \varphi(t)\|_{L^2(\Omega \cap B_R)} + \|\partial_t \varphi(t)\|_{L^2(\Omega \cap B_R)} \right) \xrightarrow{t \to \infty} 0.
\]

\section{Notation, Assumptions}

For an arbitrary open set \(K \subset \mathbb{R}^3\) the space of all infinitely differentiable functions with compact support contained in \(K\) is denoted by \(C_0^\infty(K)\).

Let \(\Omega \subset \mathbb{R}^3\) be a (connected) domain and let \(\Omega_0 \subset \Omega\) be an open subset, such that \(G \equiv \Omega \setminus \Omega_0\) has nonempty interior. The variable matrices \(E^{(1)} \in L^\infty(\Omega, \mathbb{R}^{(M \times M)})\) and \(E^{(2)} \in L^\infty(\Omega, \mathbb{R}^{(N \times N)})\) assumed to be symmetric and uniformly positive in the sense that
\[
y^\perp \cdot E^{(1)}(x)y \geq c_0 |y|^2 \quad \text{and} \quad z^\perp \cdot E^{(2)}(x)z \geq c_0 |z|^2 \tag{2.15}
\]
for all \(x \in \Omega, y \in \mathbb{R}^M\) and \(z \in \mathbb{R}^N\) with some \(c_0 \in (0, \infty)\) independent of \(x, y, z\).

Next,
\[
E^{(1)}(x) = 1 \quad \text{and} \quad E^{(2)}(x) = 1 \text{ for all } x \in \Omega_0 = \Omega \setminus G. \tag{2.16}
\]
The assumptions on $S : [0, \infty) \times \Omega \times \mathbb{R}^M \rightarrow \mathbb{R}^M$ are the following.

$$S(t, x, y) = 0 \text{ if } x \in \Omega_0 = \Omega \setminus G,$$  \hspace{1cm} (2.17)

$$S(\cdot, \cdot, y) \text{ measurable for fixed } y \in \mathbb{R}^M,$$  \hspace{1cm} (2.18)

and Lipschitz-continuous, i.e. there exists $L \in (0, \infty)$, such that

$$|S(t, x, y) - S(t, x, \tilde{y})| \leq L|y - \tilde{y}| \text{ for all } y, \tilde{y} \in \mathbb{R}^M \text{ and } x \in \Omega.$$  \hspace{1cm} (2.19)

$$|S(t, x, y)|^2 \leq C_0 < y, S(t, x, y) > \text{ for all } t \geq 0, x \in G, y \in \mathbb{R}^M$$  \hspace{1cm} (2.20)

with some $C_0 \in (0, \infty)$. Moreover,

$$yS(x, y) \geq \gamma(x) \min \{|y|^p, |y|\} \text{ for all } y \in \mathbb{R}^M, x \in G.$$  \hspace{1cm} (2.21)

Here $\gamma \in L^\infty(G)$ with $\gamma > 0$ and $p \in [2, \infty)$. The function $\gamma$ does not necessarily have a uniform positive lower bound on $G$. It follows from the two latter assumptions that $S(t, x, y) = 0$ if and only if $y = 0$ for all $x \in G$.

In the sequel $L^q(K)$ denotes for a measurable subset $K \subset G$ the weighted $L^q$-space endowed with the norm

$$||u||_{L^q(K)} \overset{\text{def}}{=} \left( \int_K |u|^q x \gamma dx \right)^{1/q}$$

where $q \in [1, \infty)$ and $\gamma$ as in 2.21.

The matrices $H_j \in \mathbb{R}^{N \times M}$ obey the following algebraic condition, which is fulfilled in the examples 1.4-1.6 and 1.7-1.9.

$$\left( \sum_{k=1}^{3} \xi_k H_k \right) \left( \sum_{k=1}^{3} \xi_k H^*_k \right) = |\xi|^2 \left( \sum_{k=1}^{3} \xi_k H_k \right) \text{ for all } \xi \in \mathbb{R}^3.$$

(2.22)

Let $W_0 \subset L^2(\Omega, \mathcal{F}^M)$ be the space of all $e \in L^2(\Omega, \mathcal{F}^M)$ with $\sum_{k=1}^{3} \partial_k (H_k e) \in L^2(\Omega)$ in the sense of distributions endowed with the norm

$$||e||^2_{W_0} \overset{\text{def}}{=} ||e||^2_{L^2} + ||\sum_{k=1}^{3} \partial_k (H_k e)||^2_{L^2}.$$  

Furthermore, let $D(A)$ with $C^\infty(\Omega, \mathcal{F}^M) \subset D(A)$ be closed subspace of $W_0$ with respect to the above norm and

$$Ae \overset{\text{def}}{=} \sum_{k=1}^{3} \partial_k (H_k e) \text{ for } e \in D(A).$$  

(2.23)

Then the adjoint operator $A^*$ obeys $C^\infty(\Omega, \mathcal{F}^N) \subset D(A^*)$ and

$$A^*F = -\sum_{k=1}^{3} \partial_k (H^*_k F) \text{ for all } F \in D(A^*).$$  

(2.24)
For a vector $\mathbf{w} \in \mathcal{C}^{M+N}$ we denote by $\mathbf{w}_1$ the first $M$ and by $\mathbf{w}_2$ the last $N$ components of $\mathbf{w}$.

Now, the following operators are defined. Let $D(B_0) \overset{\text{def}}{=} D(A) \times D(A^*)$ and

$$B_0 \mathbf{w} \overset{\text{def}}{=} (-A^* \mathbf{w}_2, A \mathbf{w}_1) \quad \text{for} \quad \mathbf{w} \in D(B_0) = D(A) \times D(A^*).$$

Next, $B \overset{\text{def}}{=} E B_0$ with $E \overset{\text{def}}{=} \text{diag} \left( E^{(1)}, E^{(2)} \right)$, i.e. $D(B) \overset{\text{def}}{=} D(B_0)$ and

$$B \mathbf{w} \overset{\text{def}}{=} E B_0 \mathbf{w} = (-E^{(1)} A^* \mathbf{w}_2, E^{(2)} A \mathbf{w}_1) \quad (2.25)$$

for $\mathbf{w} \in D(B)$. It turns out that $B$ is a densely defined skew self-adjoint operator in the Hilbert-space $X \overset{\text{def}}{=} L^2(\Omega, \mathcal{C}^{M+N})$ endowed with the scalar-product

$$< \mathbf{F}, \mathbf{G} >_X \overset{\text{def}}{=} \int_{\Omega} E^{-1} \mathbf{F} \mathbf{G} dx$$

This follows from the closedness of $A$, which implies that $A^{**} = \overline{A} = A$. (It is advantageous for following considerations to consider a complex space $X$. But whenever the $\mathbf{S}(t, x, \mathbf{E})$ occurs in an equation, the function $\mathbf{E}$ is of course assumed to be real-valued.)

Now, let $\mathcal{N}$ be the set of all $\mathbf{a} \in \text{ker} \ B$ with $\mathbf{a}_i(x) = 0$ for all $x \in G$.

Moreover, let $X^0 \overset{\text{def}}{=} \mathcal{N}^\bot$ be the space of all $\mathbf{w} \in X$ with $< \mathbf{u}, \mathbf{w} >_X = 0$ for all $\mathbf{u} \in \mathcal{N}$.

For $\mathbf{w} \in L^2(\Omega, \mathcal{C}^{M+N})$ a function $\mathbf{u} \in C(\mathcal{I}, X)$ is called a weak solution to the problem 1.1-1.3, if

$$\frac{d}{dt} \langle \mathbf{u}(t), \mathbf{a} \rangle_X = -\langle \mathbf{u}(t), B \mathbf{a} \rangle_X + \langle F(t, \mathbf{u}(t)), \mathbf{a} \rangle_X \quad \text{for all} \quad \mathbf{a} \in D(B) \quad (2.26)$$

Here $F : (0, \infty) \times X \rightarrow X$ is defined by

$$F(t, \mathbf{u}) \overset{\text{def}}{=} -\left( E^{(1)} \mathbf{S}(t, \cdot, \mathbf{u}_i(\cdot)), 0 \right).$$

2.26 is equivalent to the variation of constant formula

$$\mathbf{u}(t) = \exp \left( tB \right) \mathbf{w} + \int_0^t \exp \left( (t - s)B \right) F(s, \mathbf{u}(s)) ds \quad (2.27)$$

where $\left( \exp \left( tB \right) \right)_{t \in \mathbb{R}}$ is the unitary group generated by $B$. Since $F(t, \cdot)$ is assumed to be Lipschitz-continuous in $X$ by assumption 2.19, it follows from a standard result that this integral-equation has a unique solution $\mathbf{u} \in C ([0, \infty), X)$, see [8], ch.7.

2.27 yields the energy-estimate

$$\frac{1}{2} \frac{d}{dt} \| \mathbf{u}(t) \|^2_X = \langle F(t, \mathbf{u}(t)), \mathbf{u}(t) \rangle_X = -\int_G \mathbf{S}(t, x, \mathbf{u}(t)_1) \cdot \mathbf{u}(t)_1 \, dx \leq 0. \quad (2.28)$$

In the sequel $T(\cdot) \mathbf{w} \in C ([0, \infty), X)$ denotes the unique solution to 1.1-1.3 in the sense of 2.26.
3 Weak convergence for $t \to \infty$

In the following lemma it is shown in particular that $T(t)w \in L^\infty((0, \infty), X)$, i.e. $\|T(t)w\|_X$ is bounded as $t \to \infty$.

**Lemma 1** Suppose $w \in X$ and $u(t) \overset{\text{def}}{=} T(t)w$. Then
\[
\int_0^\infty \langle u(t), F(t, u(t)) \rangle dt + \|u(t)\|^2_X \leq \|w\|^2_X, \tag{3.29}
\]
\[
\int_0^\infty \|F(t, u(t))\|^2_X dt \leq C_0 \|w\|^2_X
\]
with some $C_0 \in (0, \infty)$ independent of $w$. Moreover,
\[
u_t \in L^p((0, \infty), L^1_p(K)) \text{ for all bounded measurable subsets } K \subset G. \tag{3.30}
\]

**Proof:** Let $u(t) = (E(t), F(t)) \overset{\text{def}}{=} T(t)w$. By the assumptions 2.20 on $S$ one has
\[
\|F(t, f)\|^2_X \leq C_0 \langle F(t, f), f \rangle \text{ for all } f \in X
\]
with some $C_0 > 0$ independent of $f$. Therefore, the energy-estimate 2.28 yields
\[
\frac{1}{2} \frac{d}{dt}\|T(t)w\|^2_X \leq \langle F(t, f), f \rangle \leq -C_0^{-1} \|FT(t)w\|^2_X.
\]
This implies 3.29 by Gronwall’s lemma.

To prove 3.30 let $f \in X$ and define $a, b \in L^2((G, IR^M)$ by $a(x) \overset{\text{def}}{=} F_t(x)$ if $|F_t(x)| \leq 1$ and $a(x) \overset{\text{def}}{=} 0$ if $|F_t(x)| > 1$. Moreover, $b(x) \overset{\text{def}}{=} F_t(x)$ if $|F_t(x)| > 1$ and $b(x) \overset{\text{def}}{=} 0$ if $|F_t(x)| \leq 1$.

Then it follows from assumption 2.21 that
\[
a(x)S(t, x, a(x)) \geq \gamma(x)|a(x)|^p \text{ and } b(x)S(t, x, b(x)) \geq \gamma(x)|b(x)|
\]
for all $x \in G$. Hölder’s inequality yields
\[
\|F_t\|_{L^1_p(K)} \leq \|a\|_{L^p_p(K)} + \|b\|_{L^p_p(K)} \tag{3.31}
\]
\[
\leq C_{K, 1}\|a\|_{L^p_p(K)} + \|b\|_{L^p_p(K)} = C_{K, 1} \left( \int_G |a(x)|^p \gamma dx \right)^{1/p} + \int_G |b(x)| \gamma dx
\]
\[
\leq C_{K, 1} \left( \int_G a(x)S(t, x, a(x)) dx \right)^{1/p} + \int_G b(x)S(t, x, b(x)) dx
\]
\[
\leq C_{K, 1} \left( \int_G f(x)S(t, x, f(x)) dx \right)^{1/p} + \int_G f(x)S(t, x, f(x)) dx
\]
\[
= C_{K, 1} \left( (f, F(t, f)) \right)^{1/p} + (f, F(t, f)) \leq C_{K, 2} \left( 1 + \|f\|^2_X \right)^{1/p} \left( (f, F(t, f)) \right)^{1/p}
\]
Finally, the assertion follows from 3.29 and 3.31.

□
Lemma 2 $X^0 \cap D(B^n)$ is dense in $X^0 \cap D(B^m)$ for all $m,n \in \mathbb{N}$ with $m < n$.

Proof:
Let $w \in X^0 \cap D(B^m)$ and define $w_\tau \overset{\text{def}}{=} \tau^n (\tau - B)^{-n} w \in D(B^n)$ for $\tau > 0$. Then
\[
\|B^k (w_\tau - w)\|_X = \|B^k w - [\tau (\tau - B)^{-1}]^n B^k w\|_X \overset{\tau \to \infty}{\longrightarrow} 0 \quad \text{for all } k \in \{0, 1, \ldots, m\}.
\]
Suppose $a \in \mathcal{N}$. Then
\[
< w_\tau, a >_X = < w, \tau^n (\tau + B)^{-n} a >_X = < w, a >_X = 0.
\]
Hence $w_\tau \in X^0$. By 3.32 the proof is complete.

□

Lemma 3  
1) $\Delta w = B^2_0 w$ on $\Omega$ for all $w \in (\text{ran } B_0) \cap D(B^2_0)$, in particular
   $-\Delta e = A^* A e$ and $-\Delta f = A A^* f$ on $\Omega$ for all $e \in (\text{ran } A^*) \cap D(A)$ and $f \in (\text{ran } A) \cap D(A^*)$.
   with $A e \in D(A^*)$ and $A^* f \in D(A)$.
2) $\Delta w = B^2 w$ on $\Omega_0 = \Omega \setminus G$ for all $w \in X^0 \cap D(B^2)$.

Proof:
Let $u \in C_0^\infty(\Omega, \mathcal{C}^{M+N}) \subset D(B^n_0)$ for all $n \in \mathbb{N}$. Then it follows from 2.22 using Fourier-transform that
\[
\mathcal{F}(B^3_0 u)(\xi) = -i \left( \sum_{j=1}^3 \xi_j H_j \right) \left( \sum_{k=1}^3 \xi_k H_k \right) \mathcal{F}(u_2)(\xi)
\]
\[
= -i |\xi|^2 \left( \sum_{l=1}^3 \xi_l H_l \right) \mathcal{F}(u_3)(\xi)
\]
Analogously,
\[
\mathcal{F}(B^3_0 u)(\xi) = -i |\xi|^2 \left( \sum_{l=1}^3 \xi_l H_l \right) \mathcal{F}(u_1)(\xi)
\]
and hence
\[
B^3_0 u = B_0 \Delta u \quad \text{for all } u \in C_0^\infty(\Omega, \mathcal{C}^{M+N}).
\]

Now, assume $w \in (\text{ran } B_0) \cap D(B^2_0)$, i.e. $w = B_0 v$ with some $v \in D(B^2_0)$. Then
\[
\int_\Omega (B^3_0 w) u dx = \langle B^3_0 v, u \rangle_{L^2} = \langle w, B^3_0 u \rangle_{L^2}
\]
\[ \langle \mathbf{v}, B_0 \Delta \mathbf{u} \rangle_{L^2} = \langle \mathbf{w}, \Delta \mathbf{u} \rangle_{L^2} = \int_{\Omega} \mathbf{w} \Delta \mathbf{u} \, dx \]

for all \( u \in C_0^\infty(\Omega) \), which means \( B_0^2 \mathbf{w} = \Delta \mathbf{w} \) in the sense of distributions.

To prove ii) let \( \mathbf{w} \in X^0 \cap D(B^2) \). Suppose \( u \in C_0^\infty(\Omega, \mathcal{A}^{M+N}) \) and define \( \tilde{u} \overset{\text{def}}{=} (B_0^2 - \Delta)u \in C_0^\infty(\Omega, \mathcal{A}^{M+N}) \subset D(B_0^2) \). Then \( 3.33 \) yields \( B_0 \tilde{u} = 0 \) and hence \( \tilde{u} \in \mathcal{N} \). In particular \( 0 = \langle \mathbf{w}, \tilde{u} \rangle \), because \( \mathbf{w} \in X^0 \). Since \( E = 1 \) on \( \Omega_0 \), it follows \( B_0 \tilde{u} = B_0^2 \tilde{u} \in D(B^2) \) and \( \tilde{u} = (B^2 - \Delta)u \). With \( \mathbf{w} \in X^0 \) and \( \tilde{u} \in \mathcal{N} \) one obtains

\[
0 = \langle \mathbf{w}, \tilde{u} \rangle_X = \langle \mathbf{w}, B^2 u \rangle_X - \langle \mathbf{w}, \Delta u \rangle_X = \langle B^2 \mathbf{w}, \mathbf{u} \rangle_X - \langle \mathbf{w}, \Delta \mathbf{u} \rangle_X.
\]

\[
= \int_{\Omega} \langle [B^2 \mathbf{w}] \nabla \mathbf{u} - \mathbf{w} \Delta \mathbf{u} \rangle \, dx
\]

Since for all \( u \in C_0^\infty(\Omega, \mathcal{A}^{M+N}) \) is arbitrary, the assertion follows.

\( \square \)

**Remark** 1 Due to the facts that generally \( E^{(j)} \neq 1 \) and \( a_1 = 0 \) on \( G \) for all \( a \in \mathcal{N} \) we have \( \Delta \mathbf{w}_1 \neq (B^2 \mathbf{w}_1) \), on \( G \) for all \( \mathbf{w} \in X^0 \cap D(B^2) \) in general.

For example is the case of Maxwell’s equations 1.4-1.6 all \( \mathbf{w} \in X^0 \cap D(B^2) \) obey

\[
\frac{\partial (B^2 \mathbf{w})}{\partial t} = -\varepsilon^{-1} \, \text{curl} (\mu^{-1} \, \text{curl} \, \mathbf{w}_1). 
\]

The condition \( \mathbf{w} \in X^0 \) implies

\[
\text{div} (\varepsilon \mathbf{w}_1) = 0 \quad \text{on} \ \Omega_0 \quad \text{and} \quad \text{div} (\mu \mathbf{w}_2) = 0 \quad \text{on} \ \Omega, 
\]

as mentioned in the introduction, but it does not provide any information on the divergence of \( \mathbf{w}_1 \) on the set \( G \), since \( a_1 = 0 \) on \( G \) for all \( a \in \mathcal{N} \).

In the next lemma it is shown that \( X^0 \) is an invariant space of \( T(t) \).

**Lemma 4** i) \( \langle T(t)\mathbf{w}, \mathbf{a} \rangle_X = \langle \mathbf{w}, \mathbf{a} \rangle_X \) for all \( \mathbf{w} \in X \), \( \mathbf{a} \in \mathcal{N} \) and \( t \geq 0 \).

ii) \( T(t)\mathbf{w} \in X^0 \) for all \( \mathbf{w} \in X^0 \) and \( t \geq 0 \).

**Proof:**
Suppose \( \mathbf{w} \in X^0 \) and \( \mathbf{a} \in \mathcal{N} \), that means \( \mathbf{a} \in \ker B \) and \( a_1 = 0 \) on \( G \). Then 3.51 and 2.27 yield

\[
\langle T(t)\mathbf{w}, \mathbf{a} \rangle_X = \langle \exp (tB)\mathbf{w} - \int_0^t \exp ((t-s)B) F(s, T(s)\mathbf{w}) \, ds, \mathbf{a} \rangle_X
\]

\[
= \langle \mathbf{w}, \exp (-tB)\mathbf{a} \rangle_X - \int_0^t \langle F(s, T(s)\mathbf{w}), (s-t)B\mathbf{a} \rangle_X \, ds
\]

\[
= \langle \mathbf{w}, \mathbf{a} \rangle_X - \int_0^t \langle F(s, T(s)\mathbf{w}), \mathbf{a} \rangle_X \, ds = \langle \mathbf{w}, \mathbf{a} \rangle_X.
\]

Hence, i) is proved. ii) follows from i) and the definition \( X^0 \overset{\text{def}}{=} \mathcal{N}^\perp \).

\( \square \)
In the sequel let $\omega_0(w)$ denote the $\omega$-limit-set of the solution $T(\cdot)w$ with respect to the weak topology of $X$, i.e., the set of all $g \in X$, such that there exists a sequence $t_n \to \infty$ with $T(t_n)w \to g$ in $X$ weakly, that means with $<T(t_n)w, f> = <g, f>$ for all $f \in X$.

Since the $T(\cdot)w \in L^\infty((0, \infty), X)$ by lemma 1 the weak $\omega$-limit-set $\omega_0(w)$ in nonempty for all $w \in X$.

**Theorem 1**  
\text{i)} Let $w \in X$. Then $\omega_0(w) \subset N$.

**Proof:**
Let $u(t) \overset{def}{=} T(t)w$ for $t \in \mathbb{R}$. Suppose $g \in X$ and $t_n \to \infty$ with $T(t_n)w \to g$ in $X^0$ weakly. Since

$$ u(t_n + t) = \exp(tB)u(t_n) + \int_{t_n}^{t_n + t} \exp((t_n + t - \tau)B) Fu(\tau)d\tau, $$

by 2.27, it follows from lemma 1, 3.29 that

$$ ||u(t_n + t) - \exp(tB)u(t_n)||_X \leq \int_{t_n}^{t_n + t} ||Fu(\tau)||_X d\tau $$

$$ \leq t^{1/2} \left( \int_{t_n}^{t_n + t} ||Fu(\tau)||_X^2 d\tau \right)^{1/2} \to 0 $$

for all $t \in \mathbb{R}$ and hence

$$ u(t_n + t) \to \exp(tB)g \text{ in } X \text{ weakly for all } t \in \mathbb{R}. \quad (3.34) $$

Suppose $\chi \in C_0^\infty(\mathbb{R})$ and define $f \overset{def}{=} \int_{\mathbb{R}} \chi(t) \exp(tB)gdtd$ and $f^{(n)} \overset{def}{=} \int_{\mathbb{R}} \chi(t)u(t,t)dt$. Then 3.34 yields by the dominated convergence-theorem

$$ \langle f^{(n)}, h \rangle_X = \int_{\mathbb{R}} \chi(t) \langle u(t,t) + h, h \rangle_X dt $$

$$ \overset{n \to \infty}{\longrightarrow} \int_{\mathbb{R}} \chi(t) \langle \exp(tB)g, h \rangle_X dt = \langle f, h \rangle_X $$

for all $h \in X$, i.e. $f^{(n)} \overset{n \to \infty}{\longrightarrow} f$ weakly. In particular

$$ f^{(n)}_{\mathbb{R}} \overset{n \to \infty}{\longrightarrow} f_{\mathbb{R}} \text{ in } L^2(G) \subset L^1_{\mathcal{G}}(K) \text{ weakly for all bounded } K \subset G. \quad (3.35) $$

On the other hand it follows from lemma 1 iii) that

$$ \|f^{(n)}_{\mathbb{R}}\|_{L^1_{\mathcal{G}}(K)} \leq \|\chi\|_{L^p(\mathbb{R})} \left( \int_{a+b}^{a+b+t_n} \|u(t)\|_{L^1_{\mathcal{G}}(K)} dt \right)^{1/p} \overset{n \to \infty}{\longrightarrow} 0 \quad (3.36) $$
for all \( t \in \mathbb{R} \). Here \( a, b \in \mathbb{R} \) with \( \text{supp } \chi \subset (a, b) \). 3.35 and 3.36 yield
\[
\mathbf{f}_1 = 0 \text{ on } K \text{ for all bounded } K \subset G \text{ and all } \chi \in C_0^\infty (\mathbb{R}),
\]
where \( \mathbf{f} \overset{\text{def}}{=} \int_{\mathbb{R}} \chi(t) \exp (tB)g dt \). This implies
\[
\left( \exp (tB)g \right)_1 = 0 \text{ on } G \text{ for all } t \in \mathbb{R}.
\] (3.37)

Since \( iB \) is self-adjoint in \( X \), \( f(iB) = \int_{\mathbb{R}} f(\lambda) dE_\lambda \) can be defined by the spectral-theorem for a Borel-measurable function \( f : \mathbb{R} \rightarrow \mathbb{C} \). Here \( (E_\lambda)_{\lambda \in \mathbb{R}} \) denotes the family of spectral-projectors of \( iB \). If \( f \in C_0^\infty (\mathbb{R}) \), then bounded operator \( f(iB) \) has the representation
\[
f(iB)u = \int_{\mathbb{R}} \hat{f}(t) \exp (-tB)udt \text{ for all } u \in X.
\] (3.38)

Here \( \hat{f} \) denotes the Fourier-transform of \( f \). To see this let \( u, v \in X \). Then
\[
\langle f(iB)u, v \rangle = \int_{\mathbb{R}} f(\lambda)d\langle E_\lambda u, v \rangle
\]
\[
= (2\pi)^{-1/2} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(t) \exp (it\lambda)dt \langle E_\lambda u, v \rangle
\]
\[
= (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{f}(t) \int_{\mathbb{R}} \exp (it\lambda)dt \langle E_\lambda u, v \rangle dt
\]
\[
= (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{f}(t) \langle \exp (-tB)u, v \rangle dt
\]
\[
= (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{f}(t) \exp (-tB)udt \langle u, v \rangle
\]

Suppose \( f \in C_0^\infty (\mathbb{R} \setminus \{0\}) \). Then 3.37 and 3.38 yield
\[
\left( f(iB)g \right)_1 = 0 \text{ on } G.
\] (3.39)

Moreover,
\[
\hat{f}(iB)g = Bf(iB)g = \left( -E^{(1)} A^* (f(iB)g)_2, E^{(2)} A (f(iB)g)_1 \right) \text{ on } \Omega,
\] (3.40)

where \( \hat{f}(\lambda) = \lambda f(\lambda) \). In particular 3.39 and 3.40 yield in the case that \( f \) is replaced by \( g(\lambda) \overset{\text{def}}{=} \lambda^{-1} f(\lambda) \in C_0^\infty (\mathbb{R} \setminus \{0\}) \) that
\[
\left( f(iB)g \right)_2 = E^{(2)} A (g(iB)g)_1 = 0 \text{ on } G
\]
and hence by 3.39
\[
f(iB)g = 0 \text{ on } G
\] (3.41)
Since \( E(x) = 1 \) on \( \Omega \setminus G \), 3.39 - 3.41 yield
\[
B_0 f(iB)g = B(f(iB)g) = \tilde{f}(iB)g \text{ for all } f \in C^\infty_0(\mathbb{R} \setminus \{0\})
\]  
(3.42)
with \( \tilde{f}(\lambda) = \lambda f(\lambda) \).
In particular it follows by induction
\[
f(iB)g \in (\text{ran } B_0) \cap D(B^0_n) \text{ with } B^0_n f(iB)g = B^n(f(iB)g)
\]  
(3.43)
for all \( f \in C^\infty_0(\mathbb{R} \setminus \{0\}) \) and \( n \in \mathbb{N} \).

The aim of the following considerations is to show that \( f(iB)g \) is real analytic on \( \Omega \). This will be achieved by means of a local integral representation.

Let \( f \in C^\infty_0(\mathbb{R} \setminus \{0\}) \) and choose \( \chi \in C^\infty_0(\mathbb{R} \setminus \{0\}) \) with \( \chi(\lambda) = 1 \) on \( \text{supp } f \). Define
\[
F(t) \overset{\text{def}}{=} \exp(-tB)\chi(iB)g = (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{\chi}(\xi) \exp((-t - \xi)B)gd\xi.
\]
Then 3.43 and lemma 3 i) yield
\[
\partial_t^2 F(t) = B^2 F(t) = B^2_0 F(t) = \Delta F,
\]  
(3.44)
in particular
\[
\partial_t^n F = B^n F(\cdot) \in L^\infty(\mathbb{R}, L^2(\Omega)) \text{ and }
\]
\[
\Delta^n F = B^n_0 F(\cdot) = B^{2n} F(\cdot) \in L^\infty(\mathbb{R}, L^2(\Omega)) \text{ for all } n \in \mathbb{N},
\]
which implies \( F \in C^\infty(\mathbb{R} \times \Omega) \) and
\[
\partial_t^j \partial^\alpha F \in L^\infty(\mathbb{R} \times \mathcal{K}) \text{ for all compact } \mathcal{K} \subset \Omega, j \in \mathbb{N}_0 \text{ and } \alpha \in \mathbb{N}^3.
\]  
(3.45)
Suppose \( x_0 \in \Omega \) and choose \( R > 0 \) with \( B_{2R} \subset \Omega \). Let
\[
K(x, \xi) \overset{\text{def}}{=} (4\pi|x|)^{-1}\tilde{f}(\xi - |x|) \text{ for } \xi \in \mathbb{R} \text{ and } x \in \mathbb{R}^3
\]
Then 3.45 yields for all \( x \in B_{R/2}(x_0) \)
\[
\lim_{r \to 0} \int_{\mathbb{R}} \int_{\partial B_r(x)} \bar{n}(y) [K(x - y, \xi) \nabla_y F_j(\xi, y) - F_j(\xi, y) \nabla_y K(x - y, \xi)] dS(y) d\xi
\]  
(3.46)
\[
= (4\pi)^{-1} \lim_{r \to 0} \left( r^{-3} \int_{\mathbb{R}} \tilde{f}(\xi - r) \int_{\partial B_r(x)} [\bar{n}(y)(x - y)] F_j(\xi, y) dS(y) d\xi \right)
\]
\[
= \int_{\mathbb{R}} \tilde{f}(\xi) F_j(\xi, x) d\xi = \int_{\mathbb{R}} \tilde{f}(\xi) (\exp(-\xi B)\chi(iB)g)_j(x) d\xi
\]
\[(2\pi)^{1/2}(f(iB)\chi(iB))_j(x) = (2\pi)^{1/2}(f(iB)g)_j(x).\]

For all \(x \in B_{R/2}(x_0)\) and all \(y \in B_{2R}(x_0)\) with \(y \neq x\) one has by 3.44
\[
\text{div}_y [K(x - y, \xi)\nabla_y F_j(\xi, y) - F_j(\xi, y)\nabla_y K(x - y, \xi)]
\]
\[
= K(x - y, \xi)\Delta_y F_j(\xi, y) - F_j(\xi, y)\Delta_y K(x - y, \xi)
\]
\[
= K(x - y, \xi)\partial^2_\xi F_j(\xi, y) - F_j(\xi, y)\partial^2_\xi K(x - y, \xi)
\]
\[
= \partial_\xi [K(x - y, \xi)\partial_\xi F_j(\xi, y) - F_j(\xi, y)\partial_\xi K(x - y, \xi)]
\]

and hence
\[
\int_{\mathbb{R}} \int_{\partial B_R(x_0)} \vec{n}(y) [K(x - y, \xi)\nabla_y F_j(\xi, y)
\]
\[
- F_j(\xi, y)\nabla_y K(x - y, \xi)] dS(y) d\xi \tag{3.47}
\]
\[
- \int_{\mathbb{R}} \int_{\partial B_r(x)} \vec{n}(y) [K(x - y, \xi)\nabla_y F_j(\xi, y) - F_j(\xi, y)\nabla_y K(x - y, \xi)] dS(y) d\xi
\]
\[
= \int_{\mathbb{R}} \int_{B_R(x_0) \setminus B_r(x)} \text{div}_y [K(x - y, \xi)\nabla_y F_j(\xi, y)
\]
\[
- F_j(\xi, y)\nabla_y K(x - y, \xi)] dy d\xi
\]
\[
= \int_{B_R(x_0) \setminus B_r(x)} \int_{\mathbb{R}} \partial_\xi [K(x - y, \xi)\partial_\xi F_j(\xi, y)
\]
\[
- F_j(\xi, y)\partial_\xi K(x - y, \xi)] d\xi dy = 0,
\]

since \(K(x - y, \xi) \xrightarrow{\|\xi\| \to \infty} 0\) and \(\partial_\xi K(x - y, \xi) \xrightarrow{\|\xi\| \to \infty} 0\), whereas \(F\) and \(\partial_\xi F\) remains bounded as \(|\xi| \to \infty\) by 3.45 for fixed \(y \neq x\).

Now, 3.46 and 3.47 yield for all \(x \in B_{R/2}(x_0)\)
\[
(2\pi)^{1/2}(f(iB)g)_j(x) = \int_{\mathbb{R}} \int_{\partial B_R(x_0)} \vec{n}(y) [K(x - y, \xi)\nabla_y F_j(\xi, y)
\]
\[
- F_j(\xi, y)\nabla_y K(x - y, \xi)] dS(y) d\xi \tag{3.48}
\]

Since \(f \in C_0^\infty(\mathbb{R})\), there exists a constant \(C_1 \in (0, \infty)\) with
\[\]
\[
(1 + |\xi|^2)||\hat{f}(k)(\xi)|| \leq C_1^k \text{ for all } \xi \in \mathbb{R} \text{ and } k \in \mathbb{N}.
\]

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Hence there exists a constant $C_2 \in (0, \infty)$ with
\[
\int_{\mathbb{R}} \int_{\partial B_R(x_0)} \left| \frac{d^k}{dt^k} K(x_0 + \tau \eta, \xi) \right| \\
+ \left| \frac{d^k}{dt^k} (\tilde{n}(y) \nabla_y K(x_0 + \tau \eta - y, \xi)) \right| dS(y) d\xi \leq C_2^k |\eta|^k
\]
for all $\eta \in \mathbb{R}^3$ with $|\eta| \leq R/2$, $\tau \in (-1, 1)$ and $k \in \mathbb{N}$. Now it follows from 3.45 and 3.48 and the previous estimate that there exists a constant $C_3 \in (0, \infty)$ with
\[
(k!)^{-1} \left| \frac{d^k}{dt^k} (f(iB)g)(x_0 + \tau \eta) \right| \leq (C_3 |\eta|^k)
\]
for all $\eta \in \mathbb{R}^3$ with $|\eta| \leq R/2$, $\tau \in (-1, 1)$ and $k \in \mathbb{N}$, which yields the analyticity of $f(iB)g$.

Next this analyticity yields by 3.41 and the assumptions that $G$ has nonempty interior and $\Omega$ is connected that
\[
f(iB)g = 0 \text{ for all } f \in C_0^\infty(\mathbb{R} \setminus \{0\}). \quad (3.49)
\]
Choose a sequence $f_n \in C_0^\infty(\mathbb{R} \setminus \{0\}), n \in \mathbb{N}$ with $|f_n(\lambda)| \leq 1$ and $f_n(\lambda) \xrightarrow{\lambda \to \infty} 1$ for
\[
\lambda \in \mathbb{R} \setminus \{0\} \text{ and } f_n(0) \xrightarrow{n \to \infty} 0.
\]
By the spectral-theorem 3.49 implies
\[
0 = \langle f_n(iB)g, g \rangle_X \xrightarrow{n \to \infty} \langle (1 - P_{\ker B})g, g \rangle_X
\]
and hence $g = P_{\ker B}g \in \ker B$. Together with 3.37 this yields $g \in \mathcal{N}$, which completes the proof.

Theorem 2 Suppose $w \in X$. Then $T(t)w \xrightarrow{t \to \infty} 0$ in $X$ weakly, if and only if $w \in X_0 = \mathcal{N}^\perp$.

Proof:
First suppose that $T(t)w \xrightarrow{t \to \infty} 0$ in $X$ weakly. From lemma 4 i) it follows
\[
\langle w, a \rangle_X = \langle T(t)w, a \rangle_X \xrightarrow{t \to \infty} 0, \text{ for all } a \in \mathcal{N}, \text{ i.e. } w \in X_0.
\]
To prove the converse assume that $w \in X_0$. Then it follows from lemma 1 i) that $\omega_0(w) \neq \emptyset$. Hence it suffices to show that it can contain at most the zero-element. Since $w \in X_0$, lemma 4 yields $\omega_0(w) \subset X_0 = \mathcal{N}^\perp$. By the previous theorem it follows $\omega_0(w) \in \mathcal{N} \cap \mathcal{N}^\perp = \{0\}$.

Let $P$ be the orthogonal-projector on $\mathcal{N}$ in $X$. 

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Corollary 1  Let \( w \in X \) be arbitrary. Then \( T(t)w \xrightarrow{t \to \infty} Pw \) in \( X \) weakly.

Proof:
Define \( a \overset{\text{df}}{=} Pw \in \mathcal{N} \) and \( u^{(1)}(t) \overset{\text{df}}{=} T(t)(1 - P)w \). Since \( (1 - P)w \in \mathcal{N}^\perp = X^0 \), theorem 2 yields

\[
u^{(1)}(t) \xrightarrow{t \to \infty} 0 \text{ in } X \text{ weakly}.
\]

Since \( a_1 = 0 \) on \( G = \Omega \setminus \Omega_0 \), assumption 2.17 yields

\[
F(t, u^{(1)}(t) + a) = F(t, u^{(1)}(t)) \text{ for all } t \geq 0.
\]

Since \( u^{(1)} \) solves 1.1-1.3 and \( a \in \mathcal{N} \subset \ker B \), it follows with 3.51 that \( u^{(1)} + a \) also solves 1.1-1.3 in the sense of 2.26 with initial-condition \( u^{(1)}(0) + a = w \). Hence \( T(t)w = u^{(1)}(t) + a \) and the assertion follows from 3.50.

\[\square\]

4  Strong \( L^q \)-convergence of solutions

The aim of the following considerations is to find sufficient conditions for strong convergence. Assume that in addition \( S(t,x,y) \) is independent of \( t \), i.e. \( S(t,x,y) = S_0(x,y) \) and

\[
(S_0(x,y) - S_0(x,\tilde{y})) (y - \tilde{y}) \geq 0
\]

for all \( t \geq 0, y \in \mathbb{R}^M \) and \( x \in G \) with some function \( S_0 : \Omega \times \mathbb{R}^M \to \mathbb{R}^M \). The main purpose of this assumption is to ensure that \( T(t)w \in D(B) \), \( \partial_t(T(t)w) \in L^2(\Omega) \) and \( BT(\cdot)w \in L^\infty((0, \infty), X) \), i.e. \( ||BT(t)w||_X \) is bounded as \( t \to \infty \) if \( w \in D(B) \) as shown in the following lemma. (For example in the linear case \( S(t,x,y) = \sigma(t,x)y \) the condition that \( S \) is independent of \( t \) can be replaced by the weaker assumption

\[
\partial_t \sigma \in L^\infty((0, \infty) \times G) \text{ and } |\partial_t \sigma(t,x)| \leq C_1 \sigma(t,x)
\]

for all \( t \geq 0 \) and \( x \in G \) with some constant \( C_1 \) independent of \( t,x \).

Lemma 5  For all \( w \in D(B) \) one has

\[
T(\cdot)w \in W^{1,\infty}((0, \infty), X) \cap L^\infty((0, \infty), D(B))
\]

Proof:
It follows from the assumption that there is a nonlinear operator \( F_0 : X \to X \) with \( F(t,w) = F_0(w) \) and

\[
| < F_0(w) - F_0(\tilde{w}), w - \tilde{w} >_X | \leq C_2 \|w - \tilde{w}\|_X \text{ for all } w, \tilde{w} \in X
\]

for all \( w, \tilde{w} \in X \)
Suppose $w \in D(B)$ and set $u(t) \overset{\text{def}}{=} T(t)w$. Let $\tau > 0$. Then in analogy to 2.28 one has
\[
\frac{d}{dt} ||u(t + \tau) - u(t)||_X^2
\]
\[
= 2 < F(t + \tau, u(t + \tau)) - F(t, u(t)), u(t + \tau) - u(t) >_X
\]
\[
= 2 < F_0(u(t + \tau)) - F_0(u(t)), u(t + \tau) - u(t) >_X \leq 0
\]
Hence
\[
|\tau|^{-2} ||u(t + \tau) - u(t)||_X^p \leq |\tau|^{-2} ||u(\tau) - w||_X^p \tag{4.54}
\]
Now, it follows from 2.27 that
\[
u(\tau) - w = (\exp(\tau B) - 1)w - \int_0^\tau \exp((\tau - s)B)F(s, u(s))ds,
\]
Since $w \in D(B)$ one has $\tau^{-1}(\exp(\tau B) - 1)w \overset{\tau \to 0}{\to} Bw$ in $X$ strongly and hence
\[
\lim_{\tau \to 0} ||\tau^{-1}(u(\tau) - w)||_X \leq ||Bw||_X + \lim_{\tau \to 0} \left(\tau^{-1} \int_0^\tau ||F(s, u(s))||_X ds\right)
\]
\[
\leq ||Bw||_X + K_1 \lim_{\tau \to 0} \left(\tau^{-1} \int_0^\tau ||u(s)||_X ds\right) = ||Bw||_X + K_1 ||w||_X.
\]
Now, 4.54 and 4.55 yield $u \in W^{1,\infty}((0, \infty), X)$ and
\[
||\partial_t u||_{L^\infty((0, \infty), X)} \leq ||Bw||_X + K_1 ||w||_X.
\]
Next, 2.26 yields for all $c \in D(B)$
\[
|<u(t), Bc>_X| \leq (||\partial_t u(t)||_X + ||F(t, u(t))||) ||c||_X.
\]
\[
\leq (||Bw||_X + K_1 ||w||_X + ||F(t, u(t))||) ||c||_X.
\]
Finally, this implies $u(t) \in D(B^*) = D(B)$ and
\[
||Bu(t)||_X \leq ||Bw||_X + K_1 ||w||_X + ||S(t, u(t))||_X
\]
\[
\leq C_1 (||Bw||_X + ||w||_X)
\]
with some $C_1 \in (0, \infty)$ independent of $w$. 

\square
Lemma 6 i) Let \( K \subset \Omega_0 \) be a bounded open set with \( \overline{K} \subset \Omega_0 \). Then \( w \in H^1(K) \) and

\[
||w||_{H^1(K)} \leq C_K ||w||_{D(B)} \text{ for all } w \in X^0 \cap D(B).
\]

with some constant \( C_K \in (0, \infty) \) depending only on \( K \).

ii) Suppose in addition \( E(2) = 1 \) on all of \( \Omega \).

Let \( U \subset \Omega \) be a bounded open set with \( \overline{U} \subset \Omega \). Then \( F \in H^1(U) \) and

\[
||F||_{H^1(U)} \leq C_U ||w||_{D(B)} \text{ for all } w = (E,F) \in X^0 \cap D(B).
\]

with some constant \( C_U \in (0, \infty) \) depending only on \( U \).

Proof:

i) Let \( K \subset \Omega_0 \) be a bounded open set with \( \overline{K} \subset \Omega_0 \). Choose \( \chi \in C_0(\Omega_0) \) with \( \chi = 1 \) on \( K \). Suppose \( w \in X^0 \cap D(B^2) \). Then lemma 3 ii) yields \( w \in H^2_{\text{loc}}(\Omega_0) \) and

\[
\sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 |\nabla w_k|^2 dx = \sum_{k=1}^{M+N} \int_{\Omega_0} \text{div} (\chi^2 \nabla w_k) w_k dx
\]

\[
\leq C_{K,1} \sum_{k=1}^{M+N} \int_{\Omega_0} |\chi \nabla w_k| |w_k| dx + \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 \Delta w_k w_k dx
\]

\[
\leq C_{K,2} ||w||^2_X + 1/3 \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 |\nabla w_k|^2 dx + \langle \chi^2 (B^2 w), w \rangle_X
\]

\[
\leq C_{K,3} ||w||^2_X + 1/3 \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 |\nabla w_k|^2 dx + \langle \chi^2 (Bw), Bw \rangle_X
\]

\[
\leq C_{K,4} \left(||Bw||^2_X + ||w||^2_X\right) + 2/3 \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 |\nabla w_k|^2 dx
\]

by assumption 2.16. Hence

\[
||w||^2_{H^1(K)} \leq ||w||^2_X + \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 |\nabla w_k|^2 dx \leq 3C_{K,4} \left(||Bw||^2_X + ||w||^2_X\right)
\]

By lemma 2 the estimate holds for all \( w \in X^0 \cap D(B) \).

To prove ii) consider first \( f \in D(A^*) \cap \ker A^* \) with \( A^* f \in D(A) \).

Since \( \ker A^* = \overline{\text{ran}A} \) lemma 3 i) yields \( \Delta f = -AA^* f \). From a similar cut-off argument as in the proof of the first part it follows that

\[
||f||^2_{H^1(U)} \leq C_{U,A} \left(||A^* f||^2_{L^2} + ||f||^2_{L^2}\right)
\] (4.56)
Since the set of all $f \in D(A^\ast) \cap (\ker A^\ast)^\perp$ with $A^\ast f \in D(A)$ is dense in $D(A^\ast) \cap (\ker A^\ast)^\perp$, 4.56 holds for all $f \in D(A^\ast) \cap (\ker A^\ast)^\perp$.

Now let $(\mathbf{E}, \mathbf{F}) \in X^0 \cap D(B)$.

Since $(0, g) \in \mathcal{N}$ it follows from the assumption $E^{(2)} = 1$ on $\Omega$ that

$$\langle \mathbf{F}, g \rangle_{L^2} = \langle (\mathbf{E}, \mathbf{F}), (0, g) \rangle_X = 0$$

for all $g \in (\ker A)^*$, in particular $F \in D(A^\ast) \cap (\ker A^\ast)^\perp$. Finally, the assertion follows from 4.56.

$\square$

**Remark 2** As described in remark 1 the $H^1_{\text{loc}}$-regularity of $\mathbf{u}_1$ for $\mathbf{w} \in X^0 \cap D(B)$ does generally not hold on the set $G = \Omega \setminus \Omega_0$ even if $E^{(j)} = 1$ on $\Omega$.

**Lemma 7** Suppose $E^{(2)} = 1$ on $\Omega$.

Then $(E(t), F(t)) \overset{\text{def}}{=} T(t)w$ obeys

$$\left( \|E(t)\|_{L^r(K)} + \|F(t)\|_{L^r(U)} \right) \overset{t \to \infty}{\longrightarrow} 0,$$

for all compact sets $K \subset \Omega_0$ and $U \subset \Omega$, $\mathbf{w} = (\mathbf{E}_0, \mathbf{F}_0) \in X^0 \cap D(B)$ and $r \in [2, 6)$.

**Proof:** Lemma 4 and lemma 5 yield

$$u = (\mathbf{E}, \mathbf{F}) \overset{\text{def}}{=} T(\cdot)w \in L^\infty((0, \infty), D(B) \cap X^0) \tag{4.57}$$

Hence, it follows from lemma 6 and Sobolev’s imbedding theorem that

$$\{E(t) : t \geq 0\} \text{ is precompact in } L^r(K) \text{ and } \{F(t) : t \geq 0\} \text{ is precompact in } L^r(U).$$

Therefore, theorem 2 yields

$$\|E(t)\|_{L^r(K)} + \|F(t)\|_{L^r(U)} \overset{t \to \infty}{\longrightarrow} 0.$$

$\square$

In the next lemma the strong $L^r_{\text{loc}}$-convergence of $u_1$ on the set $G$ is proved, which in general does not follow from lemma 6, see remark 2.

**Lemma 8** Suppose $\mathbf{w} = (\mathbf{E}_0, \mathbf{F}_0) \in X$, $R > 0$ and $r \in [1, 2)$.

Then $(E(t), F(t)) \overset{\text{def}}{=} T(t)w$ satisfies

$$\|E(t)\|_{L^r(G \cap B_R)} \overset{t \to \infty}{\longrightarrow} 0.$$
Proof: Let $u(t) = (E(t), F(t)) \overset{\text{def}}{=} T(t)w$ with $w \in D(B)$, and define $G^{(R)} \overset{\text{def}}{=} G \cap B_R$ and $M \overset{\text{def}}{=} \|u\|_{L^\infty([0, \infty), L^2(\Omega))}$. Suppose $\delta > 0$. With $\gamma > 0$ as in 2.21 one has $G = \bigcup_{n \in \mathbb{N}} \{x \in G : \gamma(x) > 1/n\}$. Therefore there exists a subset $G_{\delta}^{(R)} \subset G^{(R)}$, such that
\[
M|G^{(R)} \setminus G_{\delta}^{(R)}|^{1/r-1/2} \leq \delta/2, \tag{4.58}
\]
and
\[
\gamma(x) \geq c_\delta \text{ for all } x \in G_{\delta}^{(R)} \tag{4.59}
\]
with some positive constant $c_\delta > 0$. In 4.58 $|G^{(R)} \setminus G_{\delta}^{(R)}|$ denotes the Lebesgue-measure of this set.

From 4.59 and lemma 1 one obtains
\[
E \in L^p((0, \infty), L^1(G_{\delta}^{(R)})) \subset L^p((0, \infty), L^1(G_{\delta}^{(R)})). \tag{4.60}
\]
Lemma 5 yields
\[
E \in W^{1, \infty}((0, \infty), L^2(\Omega)) \subset W^{1, \infty}((0, \infty), L^1(G_{\delta}^{(R)})). \tag{4.61}
\]
By 4.60 and 4.61 the function $t \to ||E(t)||^p_{L^1(G_{\delta}^{(R)})}$ is uniformly continuous and integrable over $(0, \infty)$ and hence
\[
||E(t)||_{L^1(G_{\delta}^{(R)})} \xrightarrow{t \to \infty} 0.
\]
Since $r \in (1, 2)$, this yields
\[
||E(t)||_{L^r(G_{\delta}^{(R)})} \leq ||E(t)||^{\theta}_{L^2(G_{\delta}^{(R)})}||E(t)||^{1-\theta}_{L^1(G_{\delta}^{(R)})} \leq M^\theta||E(t)||^{1-\theta}_{L^1(G_{\delta}^{(R)})} \xrightarrow{t \to \infty} 0. \tag{4.62}
\]
where $1/r = \theta/2 + 1 - \theta$. Next it follows from 4.58 that
\[
||E(t)||_{L^r(G^{(R)} \setminus G_{\delta}^{(R)})} \leq ||E(t)||_{L^2(\Omega)}|G^{(R)} \setminus G_{\delta}^{(R)}|^{1/r-1/2} \tag{4.63}
\]
\[
\leq M|G^{(R)} \setminus G_{\delta}^{(R)}|^{1/r-1/2} \leq \delta/2.
\]
Finally, the assertion follows from 4.62 and 4.63, since $\delta > 0$ is arbitrary.

\[
\square
\]

Theorem 3 Suppose $E^{(2)} = 1$ on $\Omega$. Then it follows for all $q \in [1, 2)$, $w = (E_0, F_0) \in X^0$ and all compact $U \subset \Omega$ that
\[
\left(||E(t)||_{L^q(U)} + ||F(t)||_{L^2(U)}\right) \xrightarrow{t \to \infty} 0.
\]
where $(E(t), F(t)) \overset{\text{def}}{=} T(t)w$. 

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Proof:
First consider \( \mathbf{w} = (\mathbf{E}_0, \mathbf{F}_0) \in X^0 \cap D(B) \), \((\mathbf{E}(t), \mathbf{F}(t)) \text{ def } \mathbf{u}(t) \text{ def } T(t)\mathbf{w} \) and define \( M \text{ def } \|\mathbf{u}\|_{L^\infty((0,\infty);L^2(\Omega))} \).
Suppose \( \delta > 0 \). Choose a compact set \( K \subset U \cap \Omega_0 \) with \( M||(U \cap \Omega_0) \setminus K||^{1/q-1/2} \leq \delta \). Then Hölder’s inequality yields
\[
\|\mathbf{E}(t)\|_{L^q(U)} \leq \|\mathbf{E}(t)\|_{L^q(U \cap G)} + \|\mathbf{E}(t)\|_{L^q(K)} + \|\mathbf{E}(t)\|_{L^q(U \cap \Omega_0) \setminus K}^{1/q-1/2} \]
\[
\leq \|\mathbf{E}(t)\|_{L^q(U \cap G)} + \|\mathbf{E}(t)\|_{L^q(K)} + \delta.
\]
Now, lemma 7 and lemma 8 yield \( \lim \sup_{t \to \infty} \|\mathbf{w}(t)\|_{L^q(U)} \leq \delta \), which proves the assertion for \( \mathbf{w} = (\mathbf{E}_0, \mathbf{F}_0) \in X^0 \cap D(B) \).
In order to prove the theorem for all \( \mathbf{w} \in X^0 \) note that from 4.52 and a similar estimate as in 2.28 one obtains
\[
\frac{d}{dt} \|T(t)\mathbf{w} - T(t)\bar{\mathbf{w}}\|_{X}^2 \]
\[
= 2 < F(t, T(t)\mathbf{w}) - F(t, T(t)\bar{\mathbf{w}}), T(t)\mathbf{w} - T(t)\bar{\mathbf{w}} >_{X} \leq 0
\]
and therefore
\[
\|T(t)\mathbf{w} - T(t)\bar{\mathbf{w}}\|_{X} \leq \|\mathbf{w} - \bar{\mathbf{w}}\|_{X}
\]
With \( ||\mathbf{u}_1||_{L^q(U)} + ||\mathbf{u}_2||_{L^q(U)} \leq C_{q,U} ||\mathbf{u}||_{X} \) for all \( \mathbf{u} \in X \), the assertion follows for arbitrary \( \mathbf{w} = (\mathbf{E}_0, \mathbf{F}_0) \in X^0 \) from the density of \( X^0 \cap D(B) \) in \( X^0 \) as shown in lemma 2.

\[ \square \]

In the case of Maxwell’s equations 1.4-1.6 the assumption \( E^{(2)} = 1 \) on \( \Omega \) can be omitted using the compactness-result in [5] [9] and [12].

Under the general assumptions considered so far it cannot be expected that the assertion of the previous theorem holds for \( q = 2 \) or sets \( U \) which may overlap the boundary \( \partial \Omega \). However, for the system corresponding to the scalar wave-equation the result can be improved in this direction. Consider
\[
\partial_t^2 \varphi = \text{div} (E \nabla \varphi) - S(x, \partial_\alpha \varphi)
\]
(4.64)
supplemented by the initial-boundary-conditions
\[
\varphi = 0 \text{ on } (0, \infty) \times \partial \Omega
\]
(4.65)
\[
\varphi(0, x) = f_0(x) \text{ and } \partial_t \varphi(0, x) = f_1(x)
\]
(4.66)
Here the nonlinear function \( S : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3 \) obey the assumptions 2.15 - 2.21. According to 4.52 it is assumed that \( S \) is independent of \( t \) and monotone with respect to \( y \in \mathbb{R}^3 \).

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For a domain $\Omega_1 \subset \Omega$ let $H^1(\Omega_1)$ be the usual first order Sobolev space and $H^1_0(\Omega_1)$ denotes the closure of $C_0^\infty(\Omega_1)$ in $H^1(\Omega_1)$.

Next, $D(A) \subset H^1(\Omega)$ is defined as the set of all $f \in H^1(\Omega)$, such that

$$Af \overset{\text{def}}{=} \operatorname{div}(E\nabla f) \in L^2(\Omega).$$

It is well known that for $f_0 \in H^1_0(\Omega)$ and $f_1 \in L^2(\Omega)$ problem 4.64 - 4.66 admits a unique solution $\varphi \in C([0, \infty), H^1(\Omega))$ with $\partial_t \varphi \in C([0, \infty), L^2(\Omega))$. The usual energy-estimate yields

$$\partial_t \varphi \in L^\infty((0, \infty)L^2(\Omega)), \nabla \varphi \in L^\infty((0, \infty), L^2(\Omega)). \tag{4.67}$$

If in addition $f_1 \in H^1_0(\Omega)$ and $f_0 \in D(A)$ then $\varphi \in C([0, \infty), D(A))$ and $\partial_t \varphi \in C([0, \infty), H^1(\Omega))$ with

$$\partial_t \varphi, \partial_t^2 \varphi \in L^\infty((0, \infty)L^2(\Omega)), \operatorname{div}(E\nabla \varphi) = A\varphi(\cdot) \in L^\infty((0, \infty), L^2(\Omega)). \tag{4.68}$$

In order to consider problem 4.64 - 4.66 is the setting of section 2 the following operators are introduced. Let $D(A) \overset{\text{def}}{=} H^1_0(\Omega, \mathcal{C})$, $A\varphi \overset{\text{def}}{=} \nabla \varphi$, $D(A^*)$ is the space of all vector-fields $a \in L^2(\Omega, \mathcal{C}^g)$ with $A^*a = - \operatorname{div} a \in L^2(\Omega)$. Next, $D(B) \overset{\text{def}}{=} D(A) \times D(A^*)$ and

$$B(w_1, ..., w_4) \overset{\text{def}}{=} -A^*(w_2, ..., w_4), EAw_1) = (\operatorname{div}(w_2, ..., w_4), E\nabla w_1)$$

for $w \in D(B)$.

Suppose $\varphi \in C([0, \infty), H^1_0(\Omega))$ is for $f_0 \in H^1_0(\Omega)$ and $f_1 \in L^2(\Omega)$ a solution of problem 4.64 - 4.66. Then $u \overset{\text{def}}{=} (\partial_t \varphi, E\nabla \varphi) \in C([0, \infty), L^2(\Omega, \mathbb{R}^4))$ is a weak solution of 2.26, i.e.

$$\frac{d}{dt}(u(t), a)_X = -\langle u(t), Ba \rangle_X + \langle F_0(u(t)), a \rangle_X \text{ for all } a \in D(B)$$

where $F_0 : L^2(\Omega, \mathbb{R}^4) \rightarrow L^2(\Omega, \mathbb{R}^4)$ is defined by

$$F_0(u) \overset{\text{def}}{=} -(S(\cdot, u_1(\cdot)), 0).$$

If $f_0 \in D(A)$ and $f_1 \in H^1_0(\Omega)$ then $u(0) \in D(B)$ and hence by lemma 5 $u \in L^\infty((0, \infty), D(B))$, whence again 4.68.

Next it is shown that

$$\nabla \varphi(t) \overset{t \rightarrow \infty}{\longrightarrow} 0 \text{ and } \partial_t \varphi(t) \overset{t \rightarrow \infty}{\longrightarrow} 0 \text{ in } L^2(\Omega) \text{ weakly.} \tag{4.69}$$
for all \( f_0 \in H^1(\Omega) \) and \( f_1 \in L^2(\Omega) \). For this purpose let \( w \overset{\text{def}}{=} (f_1, E\nabla f_0) \in L^2(\Omega, \mathbb{R}^4) \). Then \((\partial_t \varphi(t), E\nabla \varphi(t)) = u(t) = T(t)w\) solves 2.26. In order to apply theorem 2 it suffices to show
\[
\mathbf{w} \in X^0
\]
(4.70)

Suppose \( a \in \mathcal{N} \). Then \( a_1 \in H^1(\Omega) \), with \( \nabla a_1 = 0 \), which implies \( a_1 = 0 \). Moreover, \( \text{div}(a_2, \ldots, a_4) = 0 \) by the definition of \( A, B \). Hence
\[
\langle w, a \rangle = \int_{\Omega} [E^{-1}(w_2, \ldots, w_4)](a_2, \ldots, a_4)dx = \int_{\Omega} (a_2, \ldots, a_4)\nabla f_0dx = 0
\]
since \( f_0 \in H^1(\Omega) \). Thus, 4.70 and 4.69 are proved. In the following theorem local strong convergence in the energy-norm is shown.

**Theorem 4** For all \( R \in (0, \infty) \), \( f_0 \in H^1(\Omega) \) and \( f_1 \in L^2(\Omega) \) one has
\[
\left( \|\nabla \varphi(t)\|_{L^2(\Omega \cap B_R)} + \|\partial_t \varphi(t)\|_{L^2(\Omega \cap B_R)} \right) \xrightarrow{t \to \infty} 0.
\]

**Proof:**

By a density-argument it suffices to consider \( f_0 \in D(A) \) and \( f_1 \in H^1(\Omega) \).

Choose \( \chi \in C_0^\infty(B_{2R}) \) with \( \chi(x) = 1 \) on \( B_R \) and define \( \Omega_R \overset{\text{def}}{=} \Omega \cap B_{2R} \) and \( \varphi_R(t, x) \overset{\text{def}}{=} \chi(x) \varphi(t, x) \). It follows easily from 4.68 using Poincaré’s inequality that
\[
\varphi_R \in L^\infty((0, \infty), H^1(\Omega \cap B_{2R})) \text{ and } \partial_t \varphi_R \in L^\infty((0, \infty), H^1(\Omega \cap B_{2R})).
\]

Since \( \Omega \cap B_{2R} \), is bounded, the imbedding \( H^1(\Omega \cap B_{2R}) \hookrightarrow L^2(\Omega \cap B_{2R}) \) is compact. Hence
\[
\{\varphi(t) : t \geq 0\} \text{ is precompact in } L^2(\Omega \cap B_R)
\]
(4.71)
and
\[
\{\partial_t \varphi(t) : t \geq 0\} \text{ is precompact in } L^2(\Omega \cap B_R).
\]
(4.72)

for all \( R \in (0, \infty) \). Next, one obtains by 2.15 and the definition of \( A \) that
\[
\omega_0 \|\nabla(\varphi(t_1) - \varphi(t_2))\|_{L^2(B_R)}^2 \leq \int_{\Omega} \chi E\nabla(\varphi(t_1) - \varphi(t_2))\nabla(\varphi(t_1) - \varphi(t_2))dx
\]
\[
= -\int_{\Omega} (\varphi(t_1) - \varphi(t_2)) \text{ div } (\chi E\nabla(\varphi(t_1) - \varphi(t_2)))dx
\]
\[
\leq \|\varphi(t_1) - \varphi(t_2)\|_{L^2(B_{2R})} (\|A(\varphi(t_1) - \varphi(t_2))\|_{L^2(\Omega)}
\]
\[
+ K_R \|\nabla(\varphi(t_1) - \varphi(t_2))\|_{L^2(\Omega)}
\]
for all \( t_1, t_2 \geq 0 \).

which implies by 4.67, 4.68 and 4.71 also
\[
\{\nabla \varphi(t) : t \geq 0\} \text{ is precompact in } L^2(\Omega \cap B_R)
\]
(4.73)

Finally, the result follows from 4.69, 4.72 and 4.73.

\[\square\]
References


