Global Weak Solutions and Well-Posedness of Weak Solutions for a Moving Boundary Problem for a Coupled System of Diffusion-Reaction Equations arising in the Corrosion-Modeling of Concrete (Part 2)

by

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Abstract

The evolution of the corrosion interface separating the uncorroded concrete part and the corroded part of a partially wet concrete wall of a pipe under the influence of hydrogen sulfide can be modelled by a moving boundary problem for three coupled one-dimensional diffusion equations. We show that the problem formulated via weak solutions is well-posed. The function describing the position of the moving boundary belongs to $W^{1,\infty}(\mathbb{R}^+)$. The paper generalizes previous results by relaxing the assumptions and by providing *global* weak solutions instead of local ones.

AMS-classification: 35 Q 80, 35 K 45, 35 K 57, 35 R 35, 35 D 05, 35 K 60, 35 B 30, 35 B 50

Key words: Moving boundary problem, system of reaction-diffusion equations, well-posedness, maximum estimates, one-dimensional, porous media, corrosion

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1 Introduction

Chemical corrosion of the cementious part of concrete surfaces exposed to hydrogen sulfide and humidity follows a common, generalizable pattern: The substance causing corrosion has to move through a layer of material, which is already corroded, before it reaches the location Γ where the actual corrosion takes place, or: The substance causing corrosion is formed in that layer and moves, driven by diffusion/reaction and/or convection, on to Γ. Moreover, on its way to Γ, it might be exposed to interactions (reactions, e.g.) with other substances.

As an example, we consider the corrosion of the cement wall of sewer pipes: The cross section of an uncorroded sewer pipe can (roughly) be identified with a ring (diameter 0.5m - 4m, thickness of the uncorroded concrete wall: 4 cm - 0.4 m). The sewage surface is at roughly 1/2 of the diameter of the pipe. One observes the most rapid corrosion at the crown and at or slighly above the (average) sewage surface and it is directed horizontally to the outside of the pipe. The wall of larger pipes is stabilized by steel reinforcement, which is at approximately one third of the pipe wall thickness, seen from the interior of the pipe. From an engineering point of view the pipe is lost once the corrosion front reaches the reinforcement, because then the steel begins to corrode rapidly, due to the exposure to chlorides. We consider the corrosion scenario at sewage level and describe a system of coupled diffusion-reaction equations with moving-boundaries, which is supposed to model the situation. The corrosion-causing substance will be identified with SO_4^{2-} (hydrogen-ions would be a *mathematically* equivalent alternative), the corrosion product will be gypsum, which we identify with $CaSO_4$ (a more complex approach would employ $CaS0_4 \cdot 2H_2O$ or Ettringite (cf. [GeWi], [ThiSa] for some practical details)).
Consider an uncorroded pipe and position the x-axis horizontally and radially pointing to the right with the origin being at the inner boundary of the wall before corrosion has begun. The position of the center of the pipe is denoted by $-L_2^*$, the one of the outer boundary of the wall is L_1^*. Since the corrosion product has a lower density than calcium carbonate, there is some expansion of the gypsum directed toward the center of the pipe to be expected. We envision the corrosion region as a sharp interface Γ (i.e. a surface in three dimension and a point at the x-axis in our one-dimensional setting). Let $s_2(t)$ denote the position of Γ at time t. $s_1(t)$ denotes the location of the inner boundary of the (partially corroded) pipe wall on the x-axis. Let $s_{0j} := s_j(0)$, $j = 1, 2$. We assume that at $t = 0$ there is already some corrosion product, i.e.

$$s_{01} < 0 \quad \text{and} \quad s_{02} > 0.$$

The sewage releases H_2S which moves to the wall of the pipe and ionizes according to

$$(1) \quad H_2S + 2O_2 \rightarrow H_2SO_4 \Longleftrightarrow 2H^+ + SO_4^{2-}.$$

Some of the acid gets washed away from the inner surface, some is available for diffusion into the porous gypsum matrix. Also, the non-oxidized hydrogen sulfide diffuses into the gypsum, which we consider as an interconnected system of pores embedded in a solid matrix. The pores are partially filled with water and we distinguish between the water filled part of the pores, Ω_{pw}, and the air filled (part of the) pores, Ω_{pa}. We also consider Ω_{pa} as well as Ω_{pw} as connected. We distinguish between three different concentrations: $v_1 :=$ concentration of H_2S in Ω_{pa}, $v_2 :=$ concentration of H_2S in Ω_{pw}, $v_3 :=$ concentration of SO_4^{2-} in Ω_{pw}. Invoking a two-compartment model, all three concentrations will be considered at *all* of $\Omega_g := \Omega_{pa} \cup \Omega_{pw} \cup$ solid gypsum matrix. Formally, we set $\Omega(t) := (s_1(t), s_2(t))$ (= the corroded part of the pipe wall) and $\Omega_c := (s_2(t), L_2^*)$ (= the uncorroded part of the pipe wall). An underlying assumption is that in every reference-volume (cf. [BeCo], e.g.) there are sufficiently many water- as well as air-filled pores. Note, that the reaction (1) seems to take place at the interior of the pipe surface and at

least in some region close to the interior surface.
H_2S diffuses in Ω_{pa} as well as in Ω_{pw} and SO_4^{2-} diffuses through Ω_{pw}. The SO_4^{2-}-flux arriving at the corrosion interface provokes a reaction resembling

$$(2)\quad 2H_2O + H^+ + SO_4^{2-} + Ca^{2+} + CO_3^{2-} \to CaSO_4 \cdot 2H_2O + HCO_3^-,$$

and creating gypsum.
Moreover, H_2S crosses the water-air interface between Ω_{pa} and Ω_{pw} within the pores. Invoking Henry's law (cf [StMo]), this yields a sink term $f_1 := K_1(v_2 - B_1 v_1)$ for the v_1-equation and a source term $f_{21} := K_2(B_2 v_1 - v_2)$ for the v_2-equation. The oxidation (1) in Ω_{pw} means a loss $f_{22} := -K_3 v_2$ for v_2 and a gain $f_3 := +K_3 v_2$ for v_3. The result is three diffusion equations for $v_i = v_i(x,t)$, $i = 1,2,3$ (cf. [BeCo]):

$$(3)\quad \frac{\partial v_i}{\partial t} - (A_i v_{ix})_x = f_i \quad \text{in} \quad \Omega(t) \quad \text{for each} \quad t > 0, \quad i = 1,2,3,$$

where

$$(4)\quad f_1, f_3 \quad \text{are as above,} \quad f_2 := f_{21} + f_{22}$$

and

$$(5)\quad A_i = \text{const.} > 0.$$

We complement this by initial conditions

$$(6i)\quad v_i(x,0) = v_{oi}(x) \quad \text{for} \quad x \in \Omega(0), \quad i = 1,2,3$$

and by Dirichlet-boundary conditions at $x = s_1(t)$:

$$(7)\quad v_i(s(t),t) = \lambda_i(t) \quad \text{for} \quad t > 0, \quad i = 1,2,3.$$

Boundary conditions at $x = s_2(t)$ are choosen as

$$(8)\quad -A_i v_{ix}(s_2(t),t) = g_i(s_1(t), s_2(t), v^*(t), v(s_2(t),t))$$

where $v = (v_1, v_2, v_3)^t$, $v^* = (v_1^*, v_2^*, v_3^*)^t$ and

$$(9), (10)\quad g_i(s_1, s_2, v^*, v) := \begin{cases} A_{ir}\{\dfrac{v_i - v_i^*}{s_2(t) - L_2^*}\}, & i = 1,2, \\ E_{31} v_3^m + E_{32} v_3^{m+1}, & i = 3, \end{cases}$$

where

$$(11)\quad m = \text{const.} > 0,\ A_{ir},\ E_{3j} = \text{const.} \geq 0, v_i^*(t) \geq 0, \ \lambda_i(t) \geq 0.$$

Furthermore, $s_2'(t)$ is proportional to the reaction rate for SO_4^{2-} in (2). Keeping in mind that the concentration of calcium-carbonate in Ω_c is considered to be constant, one obtains (cf. [BöDJR], [Devin])

$$(12)\quad s_2'(t) = L_2 v_3(s_2(t),t)^m \qquad \forall t \in (0,T)$$

with a const. $L_2 > 0$. $s_1(t)$ is determined by $s_2(t)$ and one has (cf. [BöDJR])

$$(13)\quad s_1'(t) = -c^* s_2'(t) \qquad \forall t \in (0,T),$$

$c^* = \text{const.} \geq 0$ and initial positions of the inner surface and of the reaction interface are supposed to be given, i.e.

(14) $s_j(0) = s_{oj}, \qquad j = 1, 2.$

An additional natural requirement is imposed by the geometrical situation and by (9):

(15) $-L_1^* \leq s_1(t) \leq s_2(t) < L_2^* \qquad \forall t \in [0, T].$

Finally, one expects concentrations to satisfy

(16) $v_i(y, t) \geq 0 \quad \text{a.e.}$

We will refer to (3)-(16) as the "original moving boundary problem ($\mathrm{P}_{\mathrm{orig}}$)".

Remark 1. The H_2S-concentration in the pipe interior is very high and close to saturation. Therefore we choose Dirichlet boundary conditions (7) for $i = 1, 2$. In typical situations acid is reported to flow down at the interior surface of the pipe, indicating an oversupply of SO_4^{2-} at the inner surface and motivating the choice of Dirichlet-conditions for $i = 3$. Note, that under other, also typical, circumstances, zero-flux boundary conditions for v_3 make also sense [BöDJR]. In the latter situation the arguments in this paper would have to be only slightly changed to cover no-flux conditions, too.
In (9), (10) we specify the flux across the interface. The Ansatz for $i = 1, 2$ is a quasi-stationary one substituting the unknown gradient in Ω_c by the difference quotient, where v_i^* is a given concentration at the outside of the pipe. The coefficients A_{ir} are the effective diffusivities for the fluxes in Ω_c. The third boundary condition reads originally as

(17) $$-A_3 \frac{\partial v_3(s_2(t), t)}{\partial x} = cR_1 + s_2'(t) v_3(s_2(t), t),$$

where R_1 is the reaction rate for SO_4^{2-} in (2) and c is a proportionality constant. The second term at the right-hand side is the usual inertial term, the first one indicates that the SO_4^{2-} arriving at the corrosion interface is completely consumed by the reaction (2). In our situation we have $m = 1$. Subsequently we will assume $m \geq 1$ but there are indications [IwKH], that the case $0 < m < 1$ is also of practical interest. □

Remark 2. A more thorough discussion of the model can be found in [BöDJR] and in [BDMRW]. The whole area of non-metallic corrosion modelling is in flux and there are not many modelling attempts to include the corrosion layer itself in the corrosion scenario. Moreover, (2) is a simplification of the actually more complex corrosion reactions (cf. [ThiSa], [SaBo]) for extensive discussions of the chemical and biological processes being invloved in the corrosion theatre). Furthermore, in practice the corrosion rate $s_2'(t)$ depends on more than just the concentrations as in (12). In general, humidity, spatial distribution of the bacteria catalyzing (2), temperature and others can, to a certain extent, be formally included in the model (3)-(16) without too many changes. Furthermore, the pore geometry of the corrosion layer undergoes more or less drastic changes during the ongoing corrosion process. This will be addressed in a forthcoming paper.

□

Remark 3. (Standard-) references for single-equation moving boundary problems (mbp's) are [Ru] and [Meir] and the a-periodically appearing conference proceedings in [Pi]. For a treatment of single-equation mbp's involving nonlinear boundary conditions we refer to [FaPr], e.g.. Problems (diffusion through a layer of ash and subsequent reaction at one of the boundaries of the ash layer enlarging the layer) related to ours and leading to a *single*-equation mbp have been dealt with in [Ri]. A basic difference between single equations and systems is that maximum estimates for solutions (or alternatively: invariant-region results) are more difficult to obtain for systems. *Systems* of mbp's have been dealt with in [FrHu], [Zha]. It seems that each problem requires a new approach. In particular, we mention [FrRZ], where the authors deal with a mbp for a 1-D weakly coupled system of three reaction-diffusion equations. The main thrust in [FrRZ] is the discussion of the shut-down time (for the process) and, in some particular situations, the authors obtain global *classical* solutions, whereas we are interested in *weak solutions*, which give more flexibility for dealing with more elaborate settings. Furthermore we mention [AnRi], [CoRi] and [FaRi] for (single equation) mbp's in practically related contexts. [BöRo] deals with the same problem as this paper, but yields only local weak and strong solutions under slightly stronger assumptions. Local strong solutions in a variety of spaces can be obtained by using techniques which would even work for quasi-linear generalizations (cf. in particular [Am]). □

2 Technical Preliminaries and Results

2.1 Function Spaces, Notation

The following function spaces have been introduced and discussed in [GGZ], [KuJoF] and [Am], e.g. It is $H := L^2(0,1)$, normed by $|v| := (\int_0^1 v(y)^2 dy)^{1/2}$ and equipped with the standard scalar product $(\cdot,\cdot)$, $\mathbf{H} := H^3$, normed by $|v| := (\sum_{i=1}^3 |v_i|)^{1/2}$ for $v = (v_1, v_2, v_3)^t \in \mathbf{H}$ and equipped with the corresponding scalar product. (In our notation for the norms and scalar products we will not distinguish between H and $\mathbf{H}$. The context will make clear what is meant.). Set $V := \{v \in H^1(0,1) : \text{trace}_{j=1} v = 0\}$, normed by $\|v\| := |v_y|$, equipped with the scalar product $((\cdot,\cdot))$, $\mathbf{V} := V^3$ with $\|v\| := (\sum_{i=1}^3 |v_{iy}|^2)^{1/2}$ and $((\cdot,\cdot))$ denotes the scalar product, $W^{k,p}(0,1)$ and $H^s(0,1)$ are the usual Sobolev- and fractional-order Sobolev-spaces, $L^p(0,1)$ denotes the usual Lebesgue-space normed by $|\cdot|_p$. Set products of function spaces (as $\mathbf{H}$ and $\mathbf{V}$, e.g.) will always be equipped with the corresponding Euclidean product norm, if applicable. $C^{0,\lambda}([0,1])$ is the set of all Hölder-continuous functions defined on $[0,1]$ with Hölder exponent λ. If $\lambda \in (0,1]$, then $C^{0,\lambda^-}([0,1]) := \bigcap_{\mu \in (0,\lambda)} C^{0,\mu}([0,1])$.

Let $p \in [1,\infty], s \in I\!R^+, k \in I\!N, S' \subset I\!R$ - an interval, W - a Banach space, A and B - arbitrary sets. $[A \to B]$ denotes the set of all maps from A to B, $C(S';W) := \{w \in [S' \to W] : w$ is continuous $\}$, $L^p(S',W) := \{$ equivalence classes of maps $w \in [S' \to W] : w$ is Bochner-meazurable and $\|w\|_{L^p(S';W)} < \infty\}$, where $\|w\|_{L^p(S';W)} := (\int_{S'} \|w(\tau)\|_W^p d\tau)^{1/p}$ denotes the norm on $L^p(S';W)$. The usual modification applies for $p = \infty$. $H^1(S';W) := \{v \in L^2(S';W) : \exists$ the distributional derivative $v' \in L^2(S';W)\}$ is normed by $\|w\|_{H'(S;W)} := (\|w\|^2_{L^2(S;W)} + \|w'\|^2_{L^2(S;W)})^{1/2}$.

For functions $w = w(x,t),\ t \in I\!R,\ x \in I\!R$, we set $w(t) := w(\cdot,t)$, $w'(t) = \frac{\partial w}{\partial t}(\cdot,t)$, $w_y = \frac{\partial w}{\partial y}$.

There will be an abundance of constants. As long as it does not matter, they will generically be denoted by c.

References within the same section are made without prefix, references to sources in other sections include the number of that section. (2.3a) means reference (3a) in section 2, e.g.

2.2 $(\mathbf{P}_{\text{orig}})$ on a fixed domain

Let $u = (u_1, u_2, u_3)^t, \varphi = (\varphi_1, \varphi_2, \varphi_3)^t \in \mathbf{V}$, $s_1, s_2, s_1', s_2' \in I\!R$, $s := s_2 - s_1$, $s' := s_1' - s_2'$, $s \neq 0$, $s_2 \neq L_2^*$, $v^* = (v_1^*, v_2^*, v_3^*)^t \in I\!R^3$. Set

$$\begin{aligned}
a(s_1, s_2, u, \varphi) &:= s^{-2} \sum_{i=1}^{3} (A_i u_{iy}, \varphi_{iy}), \\
b_f(u, \varphi) &:= \sum_{i=1}^{3} (f_i(u), \varphi_i), \\
h_i(s_1, s_2, s_1', s_2', u, \varphi) &:= \frac{1}{s}((1-y)s_1' + y s_2') u_i, \varphi_i), \quad h := \sum_{i=1}^{3} h_i, \\
h_i(s_1, s_2, s_1', s_2', u) &:= h_i(s_1, s_2, s_1', s_2', u, \cdot), \\
e(s_1, s_2, v^*, u, \varphi) &:= \frac{1}{s} \sum_{i=1}^{2} g_i(s_1, s_2, v^*, u) \varphi_i(1) + \frac{1}{s} g_3(u) \varphi_3(1).
\end{aligned}$$

By means of the transformation

$$(1) \quad \begin{cases} (x,t) \in [s_1(t), s_2(t)] \times [0,T] \mapsto (y,t) \in [0,1] \times [0,T], \qquad y := \frac{x - s_1(t)}{s_2(t) - s_1(t)}, \\ u_i(y,t) := v_i(x,t) - \lambda_i(t), \quad i = 1,2,3, \quad u := (u_1, u_2, u_3)^t \end{cases}$$

the original problem (3)-(16) is transformed on a fixed domain. The resulting equations are

$$\begin{aligned}
&(2) & \frac{\partial u_i}{\partial t} - \frac{1}{s^2(t)} (A_i u_{iy})_y &= -\lambda_i'(t) + f_i(u(t) + \lambda(t)) + h_i(u(t)) \;\; y \in [0,1], t \in [0,T], \;\; i = 1,2,3 \\
&(3) & s_2'(t) &= L_2(u_3(1,t) + \lambda_3(t))^m, \;\; t \in (0,T), \\
&(4) & s_1'(t) &= -(c^*)^{-1} s_2'(t) \;\; \text{a.e.}, \;\; t \in (0,T), \\
&(5) & s_j(0) &= s_{0j}, \;\; j = 0,1, \\
&(6) & u_i(y,0) &= u_{i0}(y), \;\; y \in [0,1], \;\; i = 1,2,3, \\
&(7) & u_i(0,t) &= 0, \;\; t \in [0,T], \;\; i = 1,2,3, \\
&(8) & -\frac{1}{s(t)} A_i \frac{\partial u_i}{\partial y}(1,t) &= g_i(s_1(t), s_2(t), v^*(t), u(t) + \lambda(t)), \;\; t \in (0,T), \;\; i = 1,2, \\
&(9) & -\frac{1}{s(t)} A_3 \frac{\partial u_3(1,t)}{\partial y} &= g_3(u(1,t) + \lambda(t)), \;\; t \in (0,T),
\end{aligned}$$

where $u_0(y) := v_0(x) - \lambda_i(0)$ $(x = y(s_{02} - s_{01}) + s_{01}), \lambda = (\lambda_1, \lambda_2, \lambda_3)^t$, $s := s_2 - s_1$.

2.3 Weak Solutions and Results

The quadruple (s_2, u) is called a *weak solution of (2)-(9)*, if

(10) $s_2 \in W^{1,4}(S)$, $\quad (S := (0,T))$ $\quad$ and

(11) $u \in L^2(S; \mathbf{V}) \cap H^1(S; \mathbf{V}^*) \cap [\bar{S} \to L^\infty([0,1])^3) \cap L^\infty(S; C^{0,0.5-}([0,1])^3)$

satisfy

$$(12) \quad \begin{cases} (u'(t), \varphi) + a(s_1(t), s_2(t), u(t), \varphi) + e(s_1(t), s_2(t), v^*(t), u(t) + \lambda(t), \varphi) \\ \qquad = b_f(u(t) + \lambda(t), \varphi) + h(s_1(t), \ldots, s_2'(t), u(t), \varphi) - (\lambda'(t), \varphi) \\ \qquad \forall \varphi \in \mathbf{V}, \quad \text{for a.a.} \quad t \in S, \end{cases}$$

(13) $u(0) = u_0,$
(14) $s_2'(t) = L_2(u_3(1,t)+\lambda_3(t))^m$ for a.a. $t \in S,$
(15) $s_2(0) = s_{02},$

where

(16) $s_1'(t) = -(c^*)^{-1}s_2'(t), \ s_1(0) = s_{01}, \ s := s_2 - s_1.$

(10)-(16) will be referred to as *problem* (P).

Theorem Let

(17) $\lambda \in W^{1,2}(0,T)^3, \ \lambda_i(t) \geq 0 \ \forall t \in [0,T],$

(18) $u_0 \in L^\infty(0,T)^3, \ u_{0i}(y) + \lambda_i(0) \geq 0$ for a.a. $y \in [0,1],$

(19) $K_i = \text{const.} \geq 0, \ i = 1,2,3, \quad B_j = \text{const.} \geq 0, \ j = 1,2,$

(20) $A_i = \text{const.} > 0, \ i = 1,2,$

(21) $s_{01} \leq 0, \ s_{02} \in [0, L_2^*)$ and $s_{02} + s_{01} > 0,$

(22) $$\begin{cases} k_i \geq \max\{u_{0i}(y) + \lambda_i(t), \lambda_i(t), v_i^*(t), \ y \in [0,1], t \in [0,T]\}, \ i = 1,2, \\ k_3 \geq \max\{u_{03}(y) + \lambda_3(t), \lambda_3(t), \ y \in [0,1], t \in [0,T]\}, \end{cases}$$

(23) $v_i^* \in L^2(0,T), \ i = 1,2.$

Furthermore assume that

(24) $$\frac{K_2B_2}{K_2+K_3} \leq B_1,$$

(25) $$\frac{K_2B_2}{K_2+K_3} \leq \frac{k_2}{k_1} \leq B_1$$

and

(26) $TL_2k_3^m < \min\{c^*(L_1^* + s_{10}), L_2^* - s_{20}\}.$

(i) Then there is a unique solution (s_2, u) of problem (P). Moreover

(27) $0 \leq u_i(y,t) + \lambda_i(t) \leq k_i$ for a.a. $t \in [0,T]$ and for all $y \in [0,1].$

(ii) $s_j \in W^{1,\infty}(0,T), \ j = 1,2.$
(iii) If, in addition, $u_0 \in \mathbf{V}$, and if u_0 and λ satisfy compatibility conditions, then

$$s_2 \in C^1(\bar{S}), \ u \in L^2(S; W^{2,2}(0,1)^3) \cap C(\bar{S}; \mathbf{V}), \ u' \in L^2(S; \mathbf{H}).$$

Finally, if in addition $u_0 \in C^\infty(0,1)^3$, $v^* \in C^\infty(0,T)$, $\lambda \in C^\infty(0,T)^3$, then $s_j \in C^\infty(0,T)$.

Remark 1. (26) will be used to guarantee that $s_2(t) < L_2^*$ and $s_1(t) \geq -L_1^*$. In the case of homogeneous Neumann-conditions for u_1 and u_2 at $y = 1$, one can (formally) set $L_2^* = \infty$. In the case of unrestricted growth of the corrosion product, L_1^* can (formally) be set to be infinite. For the underlying corrosion process neglection of these fluxes would be justified if the interface is completely wetted. For the corrosion problem we have in mind, this can be assumed for most cases (cf. [BöDJR], [Devin], e.g.). This (partially) justifies the title "Global..." of this paper. □

Remark 2. $u_i(y,t)+\lambda_i(t)$, $\lambda_i(t)$, $u_{i0}(y)+\lambda_i(0)$ and $v_i^*(t)$ are concentrations and one might hope to get $[0,1]^3$ as an invariant region. Instead of that we obtain $[0,k_1]\times[0,k_2]\times[0,k_3]$ as an invariant region. Note, that for the underlying *practical* problem we have $B_1=B_2$, thus reducing (25) to $K_2\le K_2+K_3$, which is trivially fulfilled. Moreover, a common value for the Henry-law constant is 2.5 (cf. [StMo]), which implies that the physically reasonable assumption $k_i=1$, $i=1,2,3$, can be fulfilled. □

3 Proof of Theorem 1

The idea is the following: We begin with showing that there is a local solution (s_2,u) of (P) on a (possibly small) time interval $[0,d]$. This solution can be extended onto $[0,2d]$ by essentially repeating the existence argument on $[d,2d]$. In the same way one obtains a solution on $[0,3d]$ by verifying the existence on $[2d,3d]$. After finitely many steps one arrives at T with that procedure. The first part of the proof (in particular Lemma 1 - Lemma 3) will be devoted to the proof of the existence of local solutions on sufficiently small time intervals.
We are going to describe a generalization problem $P(S')$ of these local problems:
Let $t_0\in[0,T)$, $d>0$ (to be specified below in (59)), $S':=(t_0,t_0+d)$, k_i as in (2.22), (2.25),

$$(1)\quad \begin{cases} k_4:=K_3k_2,\ \ k^*:=\left(\sum_{i=1}^3 k_i^2\right)^{1/2},\ \ k_5:=(1+c^*)(b_1+s_{20})-(c^*s_{20}+s_{10}),\\ \hat{k}_i:=k_i+\sup_{t\in S}|\lambda_i(t)|,\ i=1,2,3,.\ \text{By (2.26), there is a constant } c_0^* \text{ with}\\ TL_2k_3^m\le L_2^*-s_{20}-\frac{1}{c_0^*}. \end{cases}$$

Let $\bar{s}_{0j},\bar{u}_{oj}$ satisfy

$$(2)\quad \bar{s}_{02}\in\left[s_{02},L_2^*-\frac{1}{c_0^*}\right),\ \ \bar{s}_{01}\le s_{01}\ (<0\,!),$$

$$(3)\quad \bar{u}_0=(\bar{u}_{01},\bar{u}_{02},\bar{u}_{03})^t\in L^\infty(0,1)^3\quad\text{such that}$$

$$(4)\quad |\bar{u}_{0i}+\lambda_i(t_0)|_\infty\le k_i\ \ (i=1,2,3).$$

Let

$$(5)\quad b_3:=L_2k_3^m\ (m\ge1),\quad b_1:=Tb_3,\quad b_2:=T^{1/2}b_3$$

and define the set

$$M(S'):=\{r_2\in W^{1,4}(S'):\quad r_2(t_0)=\bar{s}_{02},\ \ r_2(t)-s_{02}\in[0,b_1],\ \ r_2'(t)\ge0\text{ a.e.},\ \ |r_2'|_{L^2(S')}\le b_2\}.$$

With respect to $\varrho(r_{22},r_{12}):=|r_{22}'-r_{12}'|_{L^2(S')}$, $M(S')$ is a complete metric space. For any $s_2\in M(S')$ define the "*corresponding* s_1" by $s_1\in W^{1,4}(S')$ such that

$$(6)\quad s_2'(t)=-c^*s_1'(t)\ \text{ a.e.},\ \ s_1(t_0)=\bar{s}_{01}.$$

Problem $(P(S'))$ is (7)-(13):
Given $\bar{s}_{0j}, j=1,2$ and $\bar{u}_0$, find

$$(7)\quad s_2\in W^{1,\infty}(S')\cap M(S'),$$

(8) $\quad u \in L^2(S';\mathbf{V}) \cap H^1(S';\mathbf{V}^*) \cap L^\infty(S';C^{0,0.5-}([0,1])^3 \cap [\bar{S}' \to L^\infty(0,1)^3]$

such that, with s_1 being defined by (6) and with

(9) $\quad s := s_2 - s_1 :$

$$(10) \quad \begin{cases} (u'(t),\varphi) + a(s_1(t),s_2(t),\lambda(t),u(t),\varphi) + e(s_1(t),s_2(t),\lambda(t),u(t),\varphi) \\ \quad = b_f(\lambda(t),u(t),\varphi) + h(s_1(t),s_2(t),s_1'(t),s_2'(t),u(t),\varphi) - (\lambda'(t),\varphi) \\ \quad \text{for a.e.} \quad t \in S', \ \ \forall \varphi \in \mathbf{V}, \end{cases}$$

(11) $\quad u(t_0) = \bar{u}_0,$

(12) $\quad s_2'(t) = L_2(u_3(1,t) + \lambda_3(t))^{m_3} \quad$ a.e.,

(13) $\quad s_2(t_0) = \bar{s}_{02}.$

The main tool for obtaining a unique solution $\in P(S')$ is the following (fixed point) operator $\mathcal{T}$:

$$(14) \quad \begin{cases} \mathcal{T} : s_2 \in M(S') & \mapsto \quad s_1 \in \{(6)\} \mapsto s := s_2 - s_1 \\ & \mapsto \quad \text{solution } u \in \{(8)-(11)\} \\ & \mapsto \quad r_2 \in W^{1,4}(S') \text{ satisfying (15), i.e.} \end{cases}$$

(15) $\quad r_2'(t) = L_2(u_3(1,t) + \lambda_3(t))^m$ for a.a. $t \in S', r_2(t_0) = \bar{s}_{02}.$

Remark 1. By standard arguments (monotonicity, Galerkin procedure, cutting off the nonlinearities), one shows that (8)-(11) is uniquely solvable. The estimates of Lemma 2 will indicate how this works.
Note that in order to obtain $u' \in L^2(S';\mathbf{V}^*)$ we will use that $s_2' \in L^p(S') \quad (p \geq 4)$. □

The *further proceedings* are as follows: Lemma 1 collects several repeatedly used arguments, Lemma 3 will deal with self-mapping- and contraction properties of $\mathcal{T}$, Lemma 2 will provide estimates for $\mathcal{T}$. At the end the local solutions will be patched together to a global solution.

Lemma 1 (Some basic estimates). Let S' be as above, $\theta \in [\frac{1}{2},1), \varepsilon > 0$.

(i) There are constants $\hat{c} = \hat{c}(\theta), \ c_\varepsilon \geq 0$ such that

$$|v|_\infty \leq \hat{c}|v|^{1-\theta}\|v\|^\theta \leq \hat{c}(\varepsilon\|v\| + c_\varepsilon\|v\|) \quad \forall v \in V.$$

(ii) Let $s \in W^{1,1}(S'), \ v \in V, \ \varphi \in V$. Then

$$\begin{aligned} s'(t)(yv_y,\varphi) &= s'(t)\{v(1)\varphi(1) - (yv,\varphi_y) - (v,\varphi)\}, \\ s'(t)(yv_y,v) &= \frac{1}{2}s'(t)\{v(1)^2 - |v|^2\} \leq \frac{1}{2}|s'(t)|\{c_1|v|^{2(1-\theta)}\|v\|^{2\theta} - |v|^2\}. \end{aligned}$$

Let $\hat{c}$ as in (i), c_ε as above, $s, s' \in I\!R^+$, $v \in V$. Then

$$\text{(iii)} \quad \frac{1}{s}|v(1)|^2 \leq \frac{1}{s}|v|_\infty^2 \leq (\hat{c})^2 (s^{2\theta-1}|v|^{2(1-\theta)}(s^{-1}\|v\|)^{2\theta} \leq \frac{\varepsilon}{s^2}\|v\|^2 + c_\varepsilon(\hat{c})^{\frac{2}{1-\theta}} s^{\frac{2\theta-1}{1-\theta}}|v|^2.$$

$$\text{(iv)} \quad \frac{s'}{s}|v(1)|^2 \leq \frac{s'}{s}|v|_\infty^2 \leq (\hat{c})^2 s' \cdot s^{2\theta-1}|v|^{2(1-\theta)}(s^{-1}\|v\|)^{2\theta} \leq \frac{\varepsilon}{s^2}\|v\|^2 + c_\varepsilon(\hat{c})^{\frac{2}{1-\theta}} s^{\frac{2\theta-1}{1-\theta}}(s')^{\frac{1}{1-\theta}}|v|^2.$$

(v) Let $h \in H^1(S')$, $h(t_0) = 0$. Then $|h(\tau)|^2 \leq (\tau - t_0)\int\limits_{t_0}^{\tau}|h'(\xi)|^2 d\xi \leq d \int\limits_{t_0}^{t_0+d}|h'(\xi)|^2 d\xi$.

Let $s_i \in W^{1,p}(S'), p \geq 2, s_i(t_0) = \bar{s}_{i0}$, $c^* = \text{const.} > 0$, $s_2'(t) = -c^* s_1'(t)$ a.e., $s_2'(t) \geq 0$ a.e., $\bar{s}_{20} \geq 0$, $\bar{s}_{10} \leq 0$, $s_2(t) - s_2(t_0) \leq b_1$ $\forall t \in [t_0, t_0 + d]$. Then

(vi) $s_2(t) - s_1(t) = (1+c^*)s_2(t) - (c^*\bar{s}_{20} + \bar{s}_{10}) \leq (1+c^*)(b_1 + \bar{s}_{20}) - (c^*\bar{s}_{20} + \bar{s}_{10}) =: k_5$,

(vii) $s_2(t) - s_1(t) \geq \bar{s}_{20} - \bar{s}_{10} \quad \forall t \in \bar{S}'$,

(viii) $s_2'(t) - s_1'(t) = (1+c^*)s_2'(t)$ for a.a. $t \in S'$.

Proof of Lemma 1. (i) The case $\theta = \frac{1}{2}$ is dealt with in [Zie]. The case $\theta > \frac{1}{2}$ follows from $H^\theta(0,1) \hookrightarrow C([0,1])$ and interpolation, e.g. (ii) follows directly by integration by parts and subsequent application of (i). The second "$\leq$" in (iii) follows from (i) and by application of Young's inequality. (iv) follows similar as (iii). (v) follows from Hölder's inequality and (vi) - (viii) are obvious. □

The following LEMMA 2 summarizes most of the estimates for $\mathcal{T}$ needed in the sequel. In particular (i) and (ii) are "intermediate products" and will be used later. We need some notation:

$$c_0 := \min_{i=1,2,3}|A_i|, \qquad c_0^* \ \text{ as in (2)}, \ \ A_r^* := c_0^* \max_{i=1,2,3}|A_{ir}|,$$

$\varepsilon > 0$ and $\theta \in [\frac{1}{2}, 1)$ will be specified in the proof of LEMMA 2, c_ε is related to ε via Young's inequality, k_i and b_i were introduced in (1), (2.22), (2.23), (2.25) and in (5), resp., $\hat{c}$ arises in LEMMA 1(i), $K_4 := 4(1+K_1+K_2+K_3+K_1B_1+K_2B_2+\hat{c}+(\hat{c})^4+3A_r^*+(\hat{c}A_r^*)^{\frac{1}{1-\theta}}+c_\varepsilon+c_\varepsilon(A_r^*)^{\frac{1}{1-\theta}})$ (the "4" can be replaced by any other number ≥ 4),
$\hat{\psi}(t) := K_4\{|\lambda(t)|^2 + |\lambda'(t)|^2 + (s_{02} - s_{01})^{-1}|\lambda(t) - v^*(t)|^2)\}$ (s_{0i} is given in (2.21)),
$\hat{\Psi}(t) := 2\{|u(t_0)|^2 + \int\limits_{t_0}^{t}\hat{\psi}(\tau)d\tau\}$,

$\Psi(t) := 2\{2|u(t_0) + \lambda(t_0)|^2 + 2|\lambda(t_0)|^2 + \int\limits_{t_0}^{t}\hat{\psi}(\tau)d\tau\}$,

$\chi(t) := K_4\{1 + c_\varepsilon(1 + |s_2'(t)|^{\frac{1}{1-\theta}})s(t)^{\frac{2\theta-1}{1-\theta}}\}$,
$\hat{\chi}(t) := K_4\{1 + c_\varepsilon(1 + |s_2'(t)|^2)\}$ $(= \chi(t)$ with $\theta = \frac{1}{2})$.

Remark 2. Note that the following LEMMA also provides estimates for solutions on the *whole* interval $S = (0,T)$. □

Lemma 2. Let $s_2 \in M(S'), s_1, u, r_i$ as in (14), (15) and assume

(16) $u_0(t_0) \in L^\infty(0,1)^3$, $|u_0(t_0) + \lambda(t_0)|_\infty \leq k^*$ (k^* as in (1)).

Then

(i) $|u(t)|^2 \leq \frac{1}{2}\hat{\Psi}(t)\exp(\int\limits_{t_0}^{t}\chi(\tau)d\tau)$,

(ii) $\int\limits_{t_0}^{t}\|u(\tau)\|^2 d\tau \leq c_0^{-1}k_5^2\hat{\Psi}(t)\exp(\int\limits_{t_0}^{t}\chi(\tau)d\tau)$,

(iiia) $|u(t)|^2 \leq \frac{1}{2}\hat{\Psi}(t)\exp(\hat{\chi}(t))$,

(iiib) $|u(t)|^2 \leq \frac{1}{2}\Psi(t)\exp(\hat{\chi}(t))$,

(iva) $\int\limits_{t_0}^{t}\|u(\tau)\|^2 d\tau \leq c_0^{-1}k_5^2\hat{\Psi}(t)\exp(\hat{\chi}(t))$ ($k_5 -$ cf. LEMMA 1(vi)),

(ivb) $\int\limits_{t_0}^{t}\|u(\tau)\|^2 d\tau \leq c_0^{-1}k_5^2\Psi(t)\exp(\hat{\chi}(t))$,

(v) $u' \in L^2(S';\mathbf{V}^*)$, $u \in C(\bar{S}';\mathbf{H})$,

(vi) $u_i(y,t)+\lambda_i(t) \geq 0$ $\forall t$ and for a.a. $y \in [0,1]$,

(vii) $u_i(y,t)+\lambda_i(t) \leq k_i$ (a) for all $t \in \bar{S}'$ and for a.a. $y \in [0,1]$ and (b) for all $y \in [0,1]$ and for a.a. $t \in S'$. In particular: $u \in L^\infty(S';C^{0,0.5-}([0,1])^3) \cap [\bar{S}' \to L^\infty(0,1)^3]$,

(viii) $0 \leq r_2'(t) \leq L_2 k_3^m (= b_3\ !)$ and $|r_2'|_{L^2(S')} \leq T^{1/2}b_3 (= b_2\ !)$,

(ix) $0 \leq r_2(t) - s_{02} \leq Tb_3\ (= b_1\ !)$.

Proof of (i)-(iv): (i) and (ii) are the usual energy-estimates: Choose $\varphi := (u_1(t),u_2(t),u_3(t))^t$ as test-function in (10) and shift some of the terms involving λ_i back to the left-hand side to obtain for $t \in S'$:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\varphi(t)|^2 + \sum_{i=1}^{3}|\sqrt{A_i}(u_i(t)+\lambda_i(t))_y|^2 + e(s_1(t),s_2(t),v^*(t),u(t)+\lambda(t),\ u(t)+\lambda(t))\\
&= \ e(s_1(t),s_2(t),\lambda(t),u(t),\lambda(t)-v^*(t)) + h(s_1(t),s_2(t),s_1'(t),s_2'(t),u(t),u(t))\\
&\quad + \sum_{i=1}^{3}(\lambda_i'(t),u_i(t)) + b_f(\lambda(t),u(t),u(t))\\
&\leq \ \frac{1}{s(t)}\sum_{i=1}^{2}A_{ir}\{\frac{u_i(1,t)+\lambda_i(t)-v_i^*(t)}{L^*-s_2(t)}\}(\lambda_i(t)-v_i^*(t)) + \frac{1}{2}(\frac{s_2'}{2s})\varphi^2(1,t) - \frac{1}{2}\frac{s'}{s}|\varphi(t)|^2\\
&\quad + \frac{1}{2}|\varphi(t)|^2 + \frac{1}{2}|\lambda'(t)|^2 - K_1B_1|u_1(t)|^2 - K_1B_1(\lambda_1(t),u_1(t))\\
&\quad + K_1(u_2(t),u_1(t)) + K_1(\lambda_2(t),u_1(t)) + K_2B_2(u_1(t),u_2(t)) + K_2B_2(\lambda_1(t),u_2(t))\\
&\quad - (K_2+K_3)|u_2(t)|^2 - (K_2+K_3)(\lambda_2(t),u_2(t)) + K_2(u_2,u_3) + K_2(\lambda_2,u_3)\\
&\leq \ \frac{A_r^*}{s(t)}\{|\varphi(1,t)| + |\lambda_i(t)-v_i^*(t)|\}|\lambda_i(t)-v_i^*(t)|\\
&\quad + \frac{1}{2}(\frac{s_2'}{2s})\varphi^2(1,t) - \frac{s'}{2s}|\varphi(t)|^2 + \frac{1}{2}|\varphi(t)|^2 + \frac{1}{2}|\lambda'(t)|^2 + K_1B_1|u_1(t)|^2\\
&\quad + K_1B_1(\frac{1}{2}|\lambda_1(t)|^2 + \frac{1}{2}|u_1(t)|^2) + \frac{K_1}{2}(|u_2(t)|^2 + (u,(t)|^2)\\
&\quad + \frac{K_2}{2}(|\lambda_2(t)|^2 + |u_1(t)|^2) + \frac{K_2}{2}(|u_2(t)|^2 + |u_3(t)|^2) + \frac{K_2}{2}(|\lambda_2(t)|^2 + |u_3(t)|^2)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{A_r^*}{s(t)}\{\frac{1}{2}\varphi^2(1,t)+\frac{3}{2}|\lambda_i(t)-v_i^*(t)|^2\}+\frac{1}{2}\frac{s_2'}{2s}\varphi^2(1,t)+(\frac{1}{2}-\frac{s'}{2s})|\varphi(t)|^2\\
&\quad +(K_1B_1+\frac{K_1B_1}{2}+\frac{K_1}{2}+\frac{K_2}{2})|u_1(t)|^2+(\frac{K_1}{2}+\frac{K_2}{2})|u_2(t)|^2\\
&\quad +(\frac{K_2}{2}+\frac{K_2}{2})|u_3(t)|^2+\frac{1}{2}|\lambda'(t)|^2+\frac{K_1B_1}{2}|\lambda_1(t)|^2+\frac{K_2}{2}|\lambda_2(t)|^2+\frac{K_2}{2}|\lambda_2(t)|^2\\
&\leq \frac{1}{s(t)}(\frac{\bar{A}_r^*}{2}+\frac{s_2'}{4})\varphi^2(1,t)\varphi(t)|^2+K_4|\varphi(t)|^2+\frac{1}{2}|\lambda'(t)|^2\\
&\quad +K_4|\lambda(t)|^2+\frac{3\bar{A}_r^*}{2s(t)}|\lambda(t)-v^*(t)|^2.
\end{aligned}
$$

By Lemma 1(iii) und (iv): Let $\theta\in[\frac{1}{2},1)$. Then there is a $c_\varepsilon=c_\varepsilon(\theta)$:

$$
\begin{aligned}
\frac{1}{s(t)}(\frac{A_r^*}{2}+\frac{s_2'(t)}{4})u^2(1,t) &\leq \frac{2\varepsilon}{s^2(t)}\|u(t)\|^2+c_\varepsilon s(t)^{\frac{2\theta-1}{1-\theta}}\Big[\Big(\frac{(\hat{c})^2A_r^*}{2}\Big)^{\frac{1}{1-\theta}}+|s_2'(t)|^{\frac{1}{1-\theta}}\Big]|u(t)|^2\\
&\leq \frac{2\varepsilon}{s^2(t)}\|u(t)\|^2+c_\varepsilon s^{\frac{2\theta-1}{1-\theta}}\Big[K_4+|s_2'(t)|^{\frac{1}{1-\theta}}\Big]|u(t)|^2.
\end{aligned}
$$

Choose $\varepsilon:=\frac{1}{4}c_0$, use Lemma 1 (vi) to obtain

$$s(t)\leq k_5,$$

integrate over $[t_0,t]$ and use Gronwall's inequality to arrive at

(17) $|u(t)|^2\leq \frac{1}{2}\hat{\Psi}(t)\exp(\hat{\chi}(t)),$

(18) $\int_{t_0}^{t}\|u(\tau)\|^2d\tau\leq c_0^{-1}\hat{k}_5^2\hat{\Psi}(t)\exp(\hat{\chi}(t)),$

which is (i) and (ii) of Lemma 2.
In order to get (iii) and (iv), we specify $\theta:=\frac{1}{2}$ and remember that $s_2\in M(S')$ implies $|s_2'|_{L^2(S')}\leq b_2$. Thus (17) and (18) yield (iii) and (iv) of the lemma. (v) is obtained from the variational formulation (8)-(11) by expressing $(u'(t),\varphi)$ in terms of all the other terms and by using the energy estimates (i), (ii). The only expression of concern is the term involving u_y and s_2':

$$s_2'(t)(yu_{iy},\varphi_i)=s_2'(t)\{u_i(1,t)\varphi_i(1)-(u_i,y\varphi_{iy})-(u_i,\varphi_i)\}.$$

We have $|s_2'(t)u_i(1,t)\varphi_i(1)|\leq|s_2'(t)|\hat{c}|u_i(t)|^{1-\theta}\|u_i(t)\|^{\theta}\|\varphi_i\|$, which implies $\|s_2'(t)u_i(1,\cdot)\|_{V^*}^2\leq \hat{c}^2|u_i|_{C(\bar{S}';H)}^{2(1-\theta)}|s_2'(t)|^2\|u_i(t)\|^{2\theta}\leq\frac{\hat{c}^2}{2}|u_i|_{C(\bar{S}';H)}^{2(1-\theta)}\{|s_2'(t)|^4+\|u_i(t)\|^{4\theta}\}$. Choosing $\theta=\frac{1}{2}$, one arrives at $s_2'u_i(1,\cdot)\in L^2(S';V^*)$. Similarly one concludes for the remaining expressions to arrive at $u'\in L^2(S;\mathbf{V}^*)$. This and $u\in L^2(S';\mathbf{V})$ imply $u\in C(\bar{S}';\mathbf{H})$.

Proof of (vii). Let $\varepsilon>0$ and shift $\lambda_i'(t)$ in (10) back to the left-hand side and choose $\varphi=(\varphi_1,\varphi_2,0)^t$ with $\varphi_i(y,t):=[u_i(1,t)+\lambda_i(t)-k_i]^+$ as test function in (10). The choice of the k_i's implies that $\varphi(t)\in\mathbf{V}$ for a.a. $t\in S'$ and that $\varphi(t_0)=0$ (cf. assumption (16) on $u(t_0)$). Moreover, one has

(19) $(u_i'(t)+\lambda_i'(t),\varphi_i(t))=\frac{1}{2}\frac{d}{dt}|\varphi_i(t)|^2\quad(i=1,2).$

Consider the two boundary-terms $e_i(...), i = 1, 2$ in (10): $s_2 \in M(S')$ and (2.26) imply $s_2(t) < L_2^*$, hence both boundary expressions are non-negative. As for the h-term in (10): The definition of φ and integration by parts yield

$$\begin{aligned}
&-h(s_1(t), s_2(t), s_1'(t), s_2'(t), u(t), \varphi)) = -h(s_1(t), \dots, s_2'(t), \varphi(t), \varphi(t)) \\
&= \frac{1}{2}\frac{s_2'(t)}{s(t)}\sum_{i=1}^{2}\varphi_i^2(1,t) - \frac{1}{2}\frac{s'(t)}{s(t)}|\varphi(t)|^2 \\
&\le \frac{\varepsilon}{s^2(t)}\|\varphi(t)\|^2 + c_\varepsilon K_4 s'(t))^2|\varphi(t)|^2 \quad (\text{Lemma 1 (iv) with } \theta = \frac{1}{2}).
\end{aligned}$$

The reaction terms involving the K_i's can be treated in the following way (note: We drop "t" for some time):

$$\begin{aligned}
&(K_2(v_2+\lambda_2 - B_1(v_1+\lambda_1), \varphi_1) + (K_2(B_2(v_1+\lambda_1) - (v_2+\lambda_2)\varphi_2) - (K_3(v_2+\lambda_2), \varphi_2) \\
&= -\sum_{i=1}^{2} K_i B_i|\varphi_i|^2 - K_3|\varphi_2|^2 + (K_1(k_2 - B_1 k_1), \varphi_i) + (K_2 B_2 k_1 - (K_2+K_3)k_2, \varphi_2) \\
&\quad +(K_1(v_2+\lambda_2-k_2),\varphi_1) + (K_2(B_2(v_1+\lambda_1-k_1),\varphi_2).
\end{aligned}$$

By assumption (2.25): The third and the fourth expression on the right-hand side are non-positive, i.e. the estimate continues as

$$\le K_1|\varphi_2|\cdot|\varphi_1| + K_2B_2|\varphi_1|\cdot|\varphi_2| \le K_4|\varphi|^2.$$

Finally, one has $(A_i u_{iy}, \varphi_i) = |\sqrt{A_i}\varphi_{iy}|^2$. Summarizing, we have

$$\frac{1}{2}\frac{d}{dt}|\varphi(t)|^2 + \frac{1}{s^2(t)}\sum_{i=1}^{2}|\sqrt{A_i}\varphi_{iy}(t)|^2 \le \frac{\varepsilon}{s^2(t)}\|\varphi(t)\|^2 + K_4(1 + c_\varepsilon|s'(t)|^2)|\varphi(t)|^2.$$

Choosing ε sufficiently small, integrating and observing $\varphi(t_0) = 0$, one obtains via Gronwall's inequality and because of $u \in C(\bar{S}'; \mathbf{H})$:

(20) $\varphi(t) = 0 \quad \text{in} \quad \mathbf{H} \quad \text{for all} \quad t \in \bar{S}'$,

hence

(21) $u_i(y,t) + \lambda_i(t) \le k_i \quad \text{for all} \quad t \in \bar{S}' \quad \text{and for a.a.} \quad y \in [0,1].$

On the other hand, $u \in L^2(S'; \mathbf{V}) \hookrightarrow L^2(S'; C^{0,0.5-}([0,1])^3), \lambda_i \in W^{1,2}([0,T])$, hence $u_i(t) + \lambda_i(t) \in C([0,1])$ for a.a. $t \in \bar{S}'$, say, for all $t \in S_1'$ with $|\bar{S}' \backslash S_1'| = 0$. Let $t_1 \in S_1'$. Continuity (in y) of $u_i(t_1) + \lambda_i(t_1)$ and (22) imply

$$(22) \quad \begin{cases} u_i(y,t) + \lambda_i(t) \le k_i \quad \text{for all} \quad t \in S_1' \quad (\text{i.e. for a.a.} \quad t \in S' \\ \text{and for all} \quad y \in [0,1]. \end{cases}$$

This can also be read as

(23) $u_i(t) \in L^\infty([0,1]) \quad \text{for a.a.} \quad t \in S' \quad \text{and} \quad |u_i(t)|_\infty \le k_i - \lambda_i(t).$

The proposed estimate for u_3 can be obtained in the following way: Introduce $\hat{u}_3 := u_3 - k_4(t - t_0)$ (for k_4 see (1)). $\hat{u}_3$ satisfies $\hat{u}_3(t_0) = u_3(t_0)$ and, by virtue of (10) and with $\varphi := (0, 0, \varphi_3)^t, \varphi_3 \in V$:

$$(24) \quad \begin{cases} (\hat{u}_3'(t), \varphi_3) + (A_3\hat{u}_{3y}, \varphi_3) + A_{31}(\hat{u}_3(1,t) + k_4(t-t_0) + \lambda_3(t))^m\varphi_3(1) \\ \quad +A_{32}(\hat{u}_3(1,t) + k_4(t-t_0) + \lambda_3(t))^{m+1}\varphi_3(1) \le A + B, \end{cases}$$

where

$$A := \frac{s_1'(t)}{s(t)}(\hat{u}_{3y}(t), \varphi_3), \quad B := \frac{s'(t)}{s(t)}(y\hat{u}_{3y}(t), \varphi_3).$$

Choose $\varphi_3(y,t) := [\hat{u}_3(y,t) + k_4(t-t_0) + \lambda_3(t) - k_3]^+$ as test-function in (25) and note that $\varphi_3(t_0) = 0$ (def. of k_3 !) and that $\varphi_3(0,t) = 0$. Copying the argumentation for u_1 and u_2, one arrives at $\varphi_3 = 0$, which implies (as before) (22), (23) and (vii) of Lemma 2. □□

Lemma 3. Let the assumptions of LEMMA 2 be fulfilled. Then:

(i) $\mathcal{T} : M(S') \to M(S')$ for any d and any t_0 s.th. $S' \subseteq S$.

(ii) There is a constant g^* such that

(25) $\varrho(\mathcal{T}s_{22}, \mathcal{T}s_{12}) \leq dg^* \varrho(s_{22}, s_{12}) \quad \forall s_{12}, s_{22} \in M(S')$.

g^* depends at most on $T, \sup_S |v^*(t)|, \|\lambda\|_{H^1(S)}, L_2, m, |u_0|_{L^\infty(S)^3}, K_1, \ldots, K_4, c^*, c_0^*$, $\max_{i=1,2,3} |A_{ir}|, \min_{i=1,2,3} |A_i|, \max_{j=1,2} E_{3j}, \; s_{i0} \; (i = 1,2)$. **Note:** All the data are considered on *all* of $\bar{S} = [0,T]$, whereas $\mathcal{T}$ acts only on functions defined on $\bar{S}' = [t_0, t_0+d]$. □□

Proof of Lemma 3. The **key observation** for the proof of (ii) is that estimates like (56), (57) below, combined with (27), yield

(26) $|(\mathcal{T}s_{22})' - (\mathcal{T}s_{12})'|_{L^4(S')} \leq g^* |s_{22}' - s_{12}'|_{L^2(S')}$,

i.e. $\mathcal{T}$ **improves integrability**. This will imply (58) ($\equiv$ (ii) of this lemma).

(i) By LEMMA 2 (vii), (viii) and by (15)

$$\begin{aligned} 0 \leq r_2(t) - r_2(t_0) &\leq (t-t_0)L_2 k_3^m \leq dL_2 k_3^m =: b_1, \\ |r_2'|_{L^2(t_0,t_0+d)} &\leq L_2 k_3^m d^{1/2} =: b_2. \end{aligned}$$

By (15) : $r_2 \in W^{1,\infty}(S') \hookrightarrow W^{1,p}(S')$ for all $p \in [1,\infty]$. All this yields $\mathcal{T} : M(S') \to M(S')$.

(ii) Let $s_{i2} \in M(S')$ be given and set $r_{i2} := \mathcal{T}(s_{i2}), r_{i1}'(t) := -c^* r_{i2}'(t)$ with $r_{i1}(t_0) = s_1(t_0)$ (fixed !), $i = 1,2$, $\delta_2 r := r_{22} - r_{21} = \mathcal{T}(s_{22}) - \mathcal{T}(s_{12})$. Furthermore, let $s_{1i}'(t) = -c^* s_{2i}'(t)$, $s_{1i}(t_0) = s_1(t_0)$ (the same initial value for $i = 1,2$!) and set $s_i := s_{2i} - s_{1i}$, $i = 1,2$. Finally, let $\hat{u}_i := (u_{i1}, u_{i2}, u_{i3})^t$ be the solutions of (8)-(11) corresponding to $s_{2i}, i = 1,2$.
For a short summary of the notations: We consider the map(s)

(27) $\mathcal{T} : s_{2i} \in M(S') \mapsto s_{2i} \mapsto s_i := s_{2i} \mapsto$ solution $w_i \mapsto r_{2i} \mapsto r_{1i}, \; i = 1,2$

and use the following abbreviations:

$$(28) \quad \begin{cases} \delta_j r := r_{j1} - r_{j2}, \;\; \delta_j r' := r_{j1}' - r_{j2}', \;\; j = 1,2, \\ \Delta r := r_{22} - r_{12}, \;\; \Delta r' := r_{22}' - r_{12}', \end{cases}$$

$\delta_j s, \cdots, \Delta s'$ in analogy, $\delta_0 s := \delta_1 s(t_0) = s_{12}(t_0) - s_{11}(t_0)$.

For the duration of this part of the calculation we set

$$\ell_1 := L_2 m k_3^{m-1} \hat{c} \;\; (\hat{c} \text{ as in Lemma 1(i)}).$$

Def. (15) of r_2' and the maximum estimates (22) imply via the mean-value theorem ($m \geq 1!$) for $t \in \bar{S}'$

$$(29)\quad \begin{cases} \int\limits_{t_0}^{t} |\Delta r'(\tau)|^4 d\tau & \leq \quad L_2^4 m^4 k_3^{4(m-1)} \int\limits_{t_0}^{t} |u_{23}(1,\tau) - u_{13}(1,\tau)|^4 d\tau \\ & \leq \quad \ell_1^4 \hat{c}^4 \sup\limits_{t_l e \tau \leq t} |w(\tau)|^2 \cdot \int\limits_{t_0}^{t} \|w(\tau)\|^2 d\tau. \end{cases}$$

Subtract the variational equations (10) for w_2 from the ones for w_1, choose $\varphi := w_2 - w_1$ as test-function and re-arrange to arrive at the following identity

$$(30)\quad \begin{cases} \frac{1}{2}\frac{d}{dt}|w(t)|^2 + \frac{1}{(\delta_1 s(t))^2} \sum\limits_{i=1}^{3} |\sqrt{A_i} w_{iy}(t)|^2 + \\ + \sum\limits_{j=1}^{2} \frac{E_{3j}}{(\delta_1 s(t))} \Big\{ (u_{23}(1,t) + \lambda_3(t))^{m+(r-1)} - (u_{13}(1,t) + \lambda_3(t))^{m+(j-1)} \Big\} \times \\ \times (u_{23}(1,t) - u_{13}(1,t)) + \frac{1}{(\delta_2 s(t))} \cdot \frac{1}{(L_2^* - s_2(t))} \times \\ \times \sum\limits_{i=1}^{2} A_{ir}(u_{2i}(1,t) - u_{1i}(1,t))^2 + (K_1 B_1 w_1, w_1) + (K_2 w_2, w_2) \\ = A(t) + B(t) + C(t) + D(t) + E(t) + F(t), \end{cases}$$

where

$$A(t) := \sum_{i=1}^{2} \frac{A_{ir}(u_{1i}(1,t) + \lambda_i(t) - v_i^*(t))}{(\delta_2 s(t))(L_2^* - s_{22})(L_2^* - s_{12})} \Delta s(t) w_3(1,t),$$

$$B(t) := \sum_{i=1}^{2} \frac{A_{ir}(u_{i1}(1,t) + \lambda_i(t) - v_i^*(t))}{(L_2^* - s_2(t))(\delta_1 s(t))(\delta_2 s(t))} (\delta_2 s(t) - \delta_1 s(t)) w_i(1,t),$$

$$C(t) := \Big(\frac{1}{(\delta_2 s(t))^2} - \frac{1}{(\delta_1 s(t))^2} \Big) \sum_{i=1}^{3} |(A_i u_{2iy}(t), w_{iy}(t))|,$$

$$D(t) := \Big(\frac{1}{(\delta_2 s(t))} - \frac{1}{(\delta_1 s(t))} \Big) \sum_{j=1}^{2} (u_{23}(1,t) + \lambda_3(t))^{m+j-1} w_3(1,t),$$

$$E(t) := (K_1 w_2, w_1) + (K_2 B_2 w_1, w_2) + (K_3 w_2, w_3),$$

$$F(t) := h(s_{11}, s_{12}, s_{11}', s_{12}', \hat{u}_1, w) - h(s_{21}, s_{22}, s_{21}', s_{22}', \hat{u}_2, w).$$

We will make use of the following estimates and identities:

(31) $\delta_k s(t) = s_{k2}(t) - s_{k1}(t) \geq s_2(t_0) - s_1(t_0) > 0, \quad k = 1, 2,$

(32) $s_{k2}(t) \leq b_1$ (because $s_{k2} \in M(S')$),

(33) $(L_2^* - s_{k2}(t))^{-1} \leq c_0^*$ (cf. (1) and $s_{k2} \in M(S')$),

$$(34)\quad \begin{aligned} (\delta_1 s'(t)) \; - \; (\delta_2 s'(t)) &= (s_{12}'(t) - s_{11}'(t)) - (s_{22}'(t) - s_{21}'(t)) \\ &= (s_{12}'(t) - s_{22}'(t)) - (s_{11}'(t) - s_{21}'(t)) \\ &= (1 + c^*)(\Delta s'(t)) \quad (\text{ cf. Lemma 1 (viii)}), \end{aligned}$$

(35) $(\delta_k s(t)) \leq k_5$ (cf. Lemma 1 (vi)).

With $\varepsilon, \bar{\varepsilon} > 0$, $c_\varepsilon \leq (4\varepsilon)^{-1}$ and constants summarized in (1) and after LEMMA 1 one has

$$
\begin{aligned}
A(t) &\leq c(A,S')|\Delta s(t)| \cdot |w_3(1,t)| \\
&\leq \varepsilon \cdot \frac{1}{(\delta_1 s(t))^2}\|w(t)\|^2 + c_\varepsilon(\delta_1(s(t)))^2 c(A,S')^2|\Delta s(t)|^2 \\
(c(A,S') &:= \max_{i=1,2} \{|A_{ir}|(k_i + |v_i^*|_{L^\infty(S')}\}(s_2(t_0) - s_1(t_0))^{-1}(c_0^*)^2)), \\
B(t) &\leq c(B,S') \cdot |\Delta s(t)| \cdot \|w(t)\| \\
&\leq \frac{\varepsilon}{(\delta_1 s(t))^2}\|w(t)\|^2 + c(B,S')^2(\delta_1 s(t))^2 c_\varepsilon |\Delta s(t)|^2, \\
(c(B,S') &:= \max_{i=1,2}\{|A_{ir}|(k_i + |v_i^*|_{L^\infty(S')})\}c_0^*(s_2(t_0) - s_1(t_0))^{-2}(1 + c^*)), \\
C(t) &\leq c(C,S')\sum_{k=1}^{2}(\delta_k s(t))^2\|\hat{u}_2(t)\| \cdot \|w(t)\| \cdot |\Delta s(t)| \\
&\leq \varepsilon \cdot \frac{1}{(\delta_1 s(t))^2}\|w(t)\|^2 + c_\varepsilon c(C,S')^2(\delta_1 s(t))^2(\sum_{k=1}^{2}(\delta_k s(t))^2)^2 \times \\
&\quad \times\|\hat{u}_2(t)\|^2 \cdot |\Delta s(t)|^2, \\
(c(C,S') &:= (1 + c^*) \cdot \max_{i=1,2,3}|A_i|), \\
D(t) &\leq c(D,S')\sum_{k=1}^{2}(\delta_k s(t))|\Delta s(t)| \cdot \|w(t)\| \\
&\leq \varepsilon \cdot \frac{1}{(\delta_1 s(t))^2}\|w(t)\|^2 + c_\varepsilon(c(D,S')^2(\sum_{k=1}^{2}\delta_k s(t))^2|\Delta s(t)|^2, \\
(c(D,S') &:= (1 + c^*)(k_3^m + k_3^{m+1})), \\
E(t) &\leq c(E,S')|w(t)|^2,\ c(E,S') := \frac{1}{2}(K_1 + K_2 B_2 + K_3), \\
F(t) &:= \sum_{j=1}^{3} F_j(t), \\
F_1(t) &:= \sum_{i=1}^{3}\Big(\Big[\frac{s'_{11} - y(\delta_1 s'(t))}{\delta_1 s(t))}\Big](-w_{iy}), w_i\Big) \\
&= \frac{1}{2(\delta_1 s(t))}\sum_{i=1}^{3}\Big[-s'_{11}(t)w_i(1,t)^2 + (\delta_1 s'(t))\Big\{w_i(1,t)^2 - |w_i(t)|^2\Big\}\Big] \\
&\leq \frac{1}{2(\delta_1 s(t))} \cdot \{|s'_{11}(t)| + |\delta_1 s'(t)|\}\{\hat{c}|w(t)|^{2(1-\theta)}\|w(t)\|^{2\theta} + |w(t)|^2\} \\
&= \frac{1}{(\delta_1 s(t))^2}\{|s'_{11}(t)| + |\delta_1 s'(t)|\}\{\delta_1 s(t)\}\{\frac{\hat{c}}{2}|w(t)|^{2(1-\theta)}\|w(t)\|^{2\theta} + \frac{1}{2}|w(t)|^2\} \\
&= \frac{1}{(\delta_1 s(t))} \cdot \frac{1}{2}\{|s'_{11}(t)| + |\delta_1 s'(t)|\} \cdot |w(t)|^2 \\
&\quad +\{|s'_{11}(t)| + |\delta_1 s'(t)|\} \cdot \{\delta_1(s(t))\} \cdot \{\frac{\hat{c}}{2}\}\Big[\frac{1}{\delta_1 s(t)}|w(t)|\Big]^{2(1-\theta)}\Big[\frac{1}{\delta_1 s(t)}\|w(t)\|\Big]^{2\theta} \\
&\leq \frac{1}{(\delta_1 s(t))} \cdot \frac{1}{2}\{|s'_{11}(t)| + |\delta_1 s'(t)|\}|w(t)|^2 \\
&\quad +\frac{\varepsilon}{(\delta_1 s(t))^2}\|w(t)\|^2 + c_\varepsilon c(F_1,S')\{|s'_{11}(t)| + |\delta_1 s'(t)|\}^{\frac{1}{1-\theta}} \cdot (\delta_1 s(t))^{2\theta-1}|w(t)|^2,
\end{aligned}
$$

$$
\begin{aligned}
F_2(t) &:= \frac{1}{(\delta_1 s(t))}\Big(\sum_{i=1}^{3}\Big(\Big([s_{11}'(t)-s_{21}'(t)]-y[(\delta_1 s'(t))-(\delta_2 s'(t))]\Big)u_{2iy},w_i\Big)\\
&= \frac{(\Delta s'(t))}{(d_1 s(t))}\sum_{i=1}^{3}([c^*-y(1+c^*)]u_{2iy},w_i)\\
&= \frac{(\Delta s'(t))}{(\delta_1 s(t))}\sum_{i=1}^{3}\{[c^*-(1+c^*)]u_{2i}(1,t)w_i(1,t)-(-(1+c^*)u_{2i},w_i)-\\
&\qquad -(c[c^*-y(1+c^*)u_{2i},w_{iy})\}\\
&= \sum_{j=1}^{3}F_{2j}(t),\\
F_{21}(t) &= -\frac{(\Delta s'(t))}{(\delta_1 s(t))}\sum_{i=1}^{3}u_{2i}(1,t)w_i(1,t)\le|\Delta s'(t)|\cdot\sum_{i=1}^{3}|u_{2i}(t)|_{L^\infty(S')}\hat{c}|w(t)|^{1-\theta}\times\\
&\qquad \times\Big(\frac{1}{\delta_1 s(t))}\|w(t)\|\Big)^{\theta}\cdot\frac{1}{(\delta_1 s(t))^{1-\theta}}\\
&\le \frac{\varepsilon}{(\delta_1 s(t))^2}\|w(t)\|^2+\bar{\varepsilon}|\Delta s'(t)|^2+c_{\bar{\varepsilon}}\{c_\varepsilon\Big[\sum_{i=1}^{3}\hat{k}_i\hat{c}\Big]^{4/3}\}^3\cdot\frac{1}{(\delta_1 s(t))^2}|w(t)|^2,\\
F_{22}(t) &:= \frac{(\Delta s'(t))}{\delta_1 s(t))}(1+c^*)\sum_{i=1}^{3}(u_{2i},w_i)\le(1+c^*)\Big(\sum_{i=1}^{3}\hat{k}_i\Big)|w_i(t)||\Delta s'(t)|\cdot\frac{1}{\delta_1 s(t)}\\
&\le \varepsilon|\Delta s'(t)|^2+c_\varepsilon\Big[(1+c^*)\sum_{i=1}^{3}\hat{k}_i\Big]^2\cdot\frac{1}{(\delta_1 s(t))^2}|w(t)|^2,\\
F_{23}(t) &:= -\frac{\Delta s'(t)}{(\delta_1 s(t))}c^*\Big(\sum_{i=1}^{3}\hat{k}_i\Big)\|w(t)\|\\
&\le \frac{\varepsilon}{(\delta_1 s(t))^2}\|w(t)\|^2+c_\varepsilon\Big[c^*\sum_{i=1}^{3}\hat{k}_i\Big]^2|\Delta s'(t)|^2.
\end{aligned}
$$

To further simplify the estimates, let

(36) $c_1^* := 2c_\varepsilon\max\{c(A,S')^2+c(B,S')^2, c(C,S')^2, c(D,S')^2\},\ c_0 := \min_{i=1,2,3}|A_i|,\ \varepsilon := \frac{1}{14}c_0, \bar{\varepsilon} > 0,$

(37) $g_4(t) := (\delta_1 s(t))^2 c_1^*\{1+(\|\hat{u}_2(t)\|^2)\Big(\sum_{k=1}^{2}(\delta_k s(t)\Big)^2\},$

(38) $c_2^* := 2\max\Big\{c(E,S'),\ c_\varepsilon c(F,S'), c_{\bar{\varepsilon}}c_3^3\hat{c}\sum_{i=1}^{3}\hat{k}_i+c_\varepsilon(1+c^*)\sum_{i=1}^{3}\hat{k}_i\Big\},$

(39) $g_5(S') := 2\Big\{\bar{\varepsilon}+\varepsilon+c_\varepsilon\Big[c^*\sum_{i=1}^{3}\hat{k}_i\Big]^2\Big\},$

(40) $g_6(t) := c_2^*\Big\{1+(|s_{11}'(t)|+|\delta_1 s'(t)|)^{\frac{1}{1-\theta}}(\delta_1 s(t))^{2\theta-1}+(\delta_1 s(t))^{-2}\Big\},$

(41) $g_7(t) := \Big\{(t-t_0)\int_{t_0}^{t}g_4(\tau)d\tau+g_5(S')\Big\}\int_{t_0}^{t}|\Delta s'(\tau)|^2 d\tau,$

$$
(42)\quad \begin{cases} g_8(S') := & 2c_0^{-1}k_5^2\{|u(t_0)|^2 + \int\limits_{t_0}^{t_0+d} K_6\Big[|\lambda(\tau)|^2 + |\lambda'(\tau)|^2 + \\ & +(s_{02}-s_{01})^{-1}|\lambda(\tau)-v^*(\tau)|^2\Big]d\tau \times \exp\{T(T(1+k_3^{\frac{2\theta-1}{1-\theta}})) + k_2^2\}. \\ g_9(S') := & \text{the preceding expression with } |u(t_0)|^2 \text{ replaced by} \\ & 2(k^*)^2 + 2|\lambda|^2_{L^\infty(S)^3}\ (k^* \text{– cf. (1)}). \end{cases}
$$

Note:

(43) $g_8(S') \leq g_9(S')$.

(44) $g_{10}(S') := k_5^2 c_1^*(1+k_5^2)d + k_5^2 c_1^* k_5^2 g_8(S')$.

Note:

(45) $g_{10}(S') \leq g_{11}(S') :=$ the expression $g_{10}(S')$ with $g_8(S')$ replaced by $g_9(S')$.

(46) $g_{12}(S') := Tg_{11}(S') + g_5(S')$,

(47) $g_{13} := c_2^*(1+(\delta_0 s)^{-2})T + 2(1+c^*)^2 b_2$,

(48) $g_{14} := g_{12}(S')\exp(g_{13})\max\{1, c_0^{-1}K_5^2\},\quad g_{15} := c_0^{-1}k_5^2 g_{14}$.

Returning to (30), we note that the third, fourth and the fifth terms on the left-hand side are non-negative. Collecting appropriate expressions on the right-hand side, substituting several of the constants and their combinations by c_1^* and by c_2^*, resp., and employing the special choice of $\varepsilon(=\frac{1}{14}c_0)$, we arrive at

$$
\frac{d}{dt}|w(t)|^2 + \frac{c_0}{(\delta_1 s(t))^2}\|w(t)\|^2 \leq g_4(t)|\Delta s(t)|^2 + g_5(S')|\Delta s'(t)|^2 + g_6(t)|w(t)|^2.
$$

Lemma 1 (v) implies $|\Delta s(t)|^2 \leq (t-t_0)\int\limits_{t_0}^{t}|\Delta s'(\tau)|^2 d\tau$, hence, Gronwall's inequality, applied to the preceding estimate, yields

$$
(49)\quad \begin{cases} |w(t)|^2 & \leq \int\limits_0^t g_4(\tau)|\Delta s(\tau)|^2 + g_5(S')|\Delta s'(\tau)|^2 d\tau \exp(\int\limits_{t_0}^t g_6(\tau)d\tau) \\ & \leq g_7(t)\exp(\int\limits_{t_0}^t g_6(\tau)d\tau), \end{cases}
$$

$$
(50)\quad \int\limits_{t_0}^{t} \frac{1}{(\delta_1 s(\tau))^2}\|w(\tau)\|^2 d\tau \leq c_0^{-1} g_7(t)\{1+\exp(\int\limits_{t_0}^{t} g_6(\tau)d\tau\}.
$$

By Lemma 1 (vii) and by the assumptions of Lemma 3 on $\delta_0 s := \delta_1 s(t_0) = s_{12}(t_0) - s_{11}(t_0)$:

(51) $\delta_1 s(\tau) \leq k_5, \quad \forall \tau \in S'$,

hence (50) implies

$$
(52)\quad \int\limits_{t_0}^{t} \|w(\tau)\|^2 d\tau \leq 2c_0^{-1}k_5^2 g_7(t)\exp\{\int\limits_{t_0}^{t} g_6(\tau)d\tau\}.
$$

We use LEMMA 2 (i) & (ii) to estimate $\hat{u}_2$ in $g_4(t)$, which appears in $g_7(t)$:

$$\int_{t_0}^{t} \|\hat{u}_2(\tau)\|^2 d\tau \le g_8(S') \le g_9(S') \quad \text{(cf. (42), (43))},$$

therefore

$$\text{(53)} \quad \int_{t_0}^{t} g_4(\tau) d\tau \le g_{10}(S') \le g_{11}(S') \quad \text{(cf. (44), (45))},$$

$$\text{(54)} \quad \begin{cases} g_7(t) & \le & \{T \int_{t_0}^{t} g_4(\tau)d\tau + g_5(S')\} \int_{t_0}^{t} |\Delta s'(\tau)|^2 d\tau \\ & \le & \{T g_{10}(S') + g_5(S')\} \int_{t_0}^{t} |\Delta s'(\tau)|^2 d\tau \\ & \le & g_{12}(S') \cdot \int_{t_0}^{t} |\Delta s'(\tau)|^2 d\tau \quad \text{(cf. (41) - (46))}. \end{cases}$$

Furthermore, we have $\delta_1 s(t) \ge \delta_0 s$ (cf. LEMMA 1 (viii) or (32), (33)) and

$$\int_{t_0}^{t} |s'_{11}(\tau)|^2 + |\delta_1 s'(\tau)|^2 d\tau \le [(c^*)^2 + (1 + c_1^*)^2] b_2 \le 2(1 + c^*)^2 b_2,$$

hence, with $\theta := \frac{1}{2}$,

$$\text{(55)} \quad \begin{cases} \int_{t_0}^{t} g_6(\tau) d\tau & \le & c_2^*(1 + (\delta_0 s)^{-2})(t - t_0) + 2(1 + c^*)^2 b_2 \\ & \le & g_{13} \quad \text{cf. (47))}. \end{cases}$$

Summarizing (49), (52) and (54), (55) we obtain

$$\text{(56)} \quad |w(t)|^2 \le g_{12}(S') \exp(g_{13}) \int_{t_0}^{t} |\Delta s'(\tau)|^2 d\tau \le g_{14} \int_{t_0}^{t} |\Delta s'(\tau)|^2 d\tau,$$

$$\text{(57)} \quad \int_{t_0}^{t} \|w(\tau\|^2 d\tau \le c_0^{-1} k_5^2 g_{14} \cdot \int_{t_0}^{t} |\Delta s'(\tau)|^2 d\tau = g_{15} \int_{t_0}^{t} |\Delta s'(\tau)|^2 d\tau.$$

To conclude the proof of LEMMA 3 (ii), we use (56), (57) and Hölder's inequality to continue from (29). We obtain

$$\text{(58)} \quad |\Delta r'|_{L^4(S')} \le \ell_1 g_{15}^{1/2} d |\Delta s'|_{L^4(S')},$$

which is identical with (25), if $g^* := \ell_1 g_{15}^{1/2}$. □□

Conclusion of the proof of THEOREM 1:

LEMMA 3 implies: By choosing

(59) $d < (g^*)^{-1}$,

$\mathcal{T}$ turns into a contraction. Therefore problem $P(S')$ ($\equiv$ (7)-(13)) has a solution for any $t_0 \in [0,T)$. Using arguments similar to the one leading to the proof of LEMMA 3 (ii), one shows that this solution is unique. In order to get a global solution of $[0,T]$, pick in a **first step** $t_0 := 0$, i.e. we get a solution of $(s_2, u) \in P((0,d))$. By LEMMA 2 (vii)(a): $u(d) \in L^\infty(0,1)^3$. Now choose in a **second step**: $t_0 := d, \bar{u}_0 := u(d)$, $\bar{s}_{02} := s_2(d)$. We show that the assumptions of LEMMA 1 through LEMMA 3 are fulfilled:
(a) (2) holds, because $s_2(t) - s_{02} \le b_1$ for all $t \in [0,d]$ and because of (1) and (2). $s_2 \in M(S')$ implies in (6): $s_1'(t) \le 0$ a.e., hence $\bar{s}_{01} := s_1(d) \le s_1(0) = s_{01}$.
(b) (3) & (4) hold because LEMMA 2 (vii) yields $\bar{u}_o := u(d) \in L^\infty(0,1)^3$ and $|u_i(d) + \lambda_i(d)| \le k_i$, $i = 1,2,3$.
(a) and (b) imply that LEMMA 3 holds with the SAME g^* as before in (60).
The rest is done by induction. In the last step one might have to choose a smaller time intervall in order to not shoot over T.

Proof of uniqueness:
Let $(s_{j2}, \hat{u}_j)$, $j = 1,2$ be two solutions satisfying (2.10) - (2.16), (2.27i) and (2.27ii), let d be fixed and satisfying (3.59). According to Lemma 3.3, the operator $\mathcal{T}$ in (3.14) is contractive. Moreover, it is easily seen that any solutions of (2.10) - (2.16), (2.27i) and (2.27ii) is also a solution of problem $P(S')$ (= (3.7) - (3.13)) with $\bar{S}' := [0,d]$. Contractivity of $\mathcal{T}$ implies $(s_{12}, \hat{u}_1) = (s_{22}, \hat{u}_2)$ on $[0,d]$. Repeating this argument on each of the intervals $[d,2d], ...[md,T]$, where m denotes the last step in the existence-proof procedure, one obtains uniqueness on each of these intervals. This proves uniqueness on the whole interval.

Proof of (ii) and (iii) of the Theorem: (2.4) and $u_3 \in L^\infty(0,T; C([0,1]))$ provide (ii). (ii) and (2.11) imply $h(s_1, s_2, s_1', s_2', u, \cdot) + b_f(u+\lambda, \cdot) \in L^2(0,T; \mathbf{H})$. The rest follows by parabolic regularity. □□□

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