Consistency of Iterative Operator-Splitting Methods: Theory and Applications

Jürgen Geiser

geiser@mathematik.hu-berlin.de

Abstract. In this paper we describe a different operator-splitting method for decoupling complex equations with multi-dimensional and multi-physical processes for applications for porous media and phase-transitions. We introduce different operator-splitting methods with respect to their usability and applicability in computer codes. The error-analysis for the iterative operator-splitting methods is discussed. Numerical examples are presented.

Keyword numerical analysis, operator-splitting method, initial value problems, iterative solver method, consistency analysis, convection-diffusion-reaction equation.

AMS subject classifications. 35J60, 35J65, 65M99, 65N12, 65Z05, 74S10, 76R50.

1 Introduction

During the last period many works have been devoted to the research of the consistency and the order of different splitting methods, see [8, 17, 19]. In this paper, we present an iterative splitting method, which is a new toll, and, due to its high accuracy, can be widely used for several real-life problems. We study here complex models with coupled processes, e.g. transport and reaction-equations with nonlinear parameters. Such models are extensively discussed in [4, 14, 17, 19]. The ideas for these models came from the background to the simulation of chemical reaction and transport in bio-remediation and waste-disposal, see [4, 28]. In the past many software tools were developed for one-dimensional and simple physical problems, e.g. one-dimensional transport codes or reaction-codes based on ODEs. But future interests will lie in the coupling of such simple physical and one-dimensional problems to multi-physical and multi-dimensional problems. In this field the coupling of various software tools will be of major interest in simulating forced complex models. The software-recycling and re-engineering of such coupled models will reduce the programming time and lead complex models to adequate applications. The underlying mathematics are operator-splitting methods to couple the different equations together. We would like to introduce operator-splitting methods with respect to the application in software tools, e.g.

effectiveness and parallelisability. Here, we discuss the consistency of the iterative operator-splitting method, which can be used as a higher-order splitting method, see [15, 21].

We base our study on the computational simulation of bio-remediation [4] or radioactive contaminants [13].

The mathematical equations are given by:

$$\partial_t \, R \, c + \nabla \cdot (\mathbf{v}c - D\nabla c) = f(c) \; , \tag{1}$$

$$f(c) = c^p \; , \text{ chemical-reaction and } p > 0 \tag{2}$$

$$f(c) = \frac{c}{1-c} \; , \text{ bio-remediation} \tag{3}$$

The unknown $c = c(x,t)$ is considered in $\Omega \times (0,T) \subset \mathbb{R}^d \times \mathbb{R}$, and the space-dimension is given by d . The parameter is given as $R \in \mathbb{R}^+$ and is constant. and named as retardation factor. The other parameters $f(c)$ are nonlinear functions, for example, bio-remediation or chemical reaction. D is the Scheidegger diffusion-dispersion tensor and $\mathbf{v}$ is the velocity.

The aim of this paper is to present an iterative splitting method that can be applied to unbounded operators which occur in the presented partial differential equations.

We use the iterative operator-splitting method for decoupling into simpler equation parts and accelerate the solver process.

The outline of the paper is as follows. The operator-splitting-methods are introduced in the section 2 and the error-analysis of the operator-splitting methods are presented. In Section 3, we discuss the consistency analysis of the iterative methods. In Section 4 we introduce the application of our methods for existing software tools. Finally we discuss future works in the area of iterative methods.

2 Operator-Splitting Methods

Operator-splitting methods are used to solve complex models in geophysical and environmental physics: they are developed and applied in [26], [27] and [28]. The ideas in this article are based on solving simpler equations to obtain higher-order discretization methods for the remaining equations. To this end we use the operator-splitting method and decouple the equation as described below.

2.1 First-order Splitting methods for linear equations

First we describe the simplest operator-splitting, which is called *sequential splitting* for the following system of ordinary linear differential equations:

$$\partial_t c(t) = A \; c(t) \; + \; B \; c(t) \; , \tag{4}$$

whereby the initial conditions are $c^n = c(t^n)$. The operators A and B are spatially discretized operators, i.e. they correspond to the convection and diffusion operators (matrices) discretized in space. We assume, they can be considered in

a first analysis as bounded operators. A next section, we discuss the unbounded case.

The sequential operator-splitting method is introduced as a method which solves the two sub-problems sequentially, where the different sub-problems are connected via the initial conditions. This means that we replace the original problem (4) with the sub-problems

$$\begin{aligned} \frac{\partial c^*(t)}{\partial t} &= Ac^*(t) \ , \quad \text{with } c^*(t^n) = c^n \ , \\ \frac{\partial c^{**}(t)}{\partial t} &= Bc^{**}(t) \ , \quad \text{with } c^{**}(t^n) = c^*(t^{n+1}) \ , \end{aligned} \tag{5}$$

whereby the splitting time-step is defined as $\tau_n = t^{n+1} - t^n$. The approximated split solution is defined as $c^{n+1} = c^{**}(t^{n+1})$.

Clearly, the replacement of the original problems with the sub-problems usually results in some error, called *splitting error*. Obviously, the splitting error of the sequential splitting method can be derived as follows (cf. e.g.[12]):

$$\begin{aligned} \rho_n &= \frac{1}{\tau}(\exp(\tau_n(A+B)) - \exp(\tau_n B)\exp(\tau_n A))\ c(t^n) \\ &= \frac{1}{2}\tau_n[A,B]\ c(t^n) + \mathcal{O}(\tau^2) \ . \end{aligned} \tag{6}$$

whereby $[A,B] := AB - BA$ is the commutator of A and B. Consequently, the splitting error is $\mathcal{O}(\tau_n)$ when the operators A and B do not commute, otherwise the method is exact. Hence, by definition, the sequential splitting is called *first-order splitting method* .

2.2 Sequential splitting method for nonlinear problems

We could use the result for the general formulation of nonlinear ordinary differential equations:

$$c'(t) = F_1(t, c(t)) + F_2(t, c(t)) \ , \tag{7}$$

where the initial conditions are given as $c^n = c(t^n)$.

As before, we can decouple the above problem into two (typically simpler) sub-problems, namely

$$\frac{\partial c^*(t)}{\partial t} = F_1(t, c^*(t)) \text{ with } t^n \le t \le t^{n+1} \text{ and } c^*(t^n) = c^n \ , \tag{8}$$

$$\frac{\partial c^{**}(t)}{\partial t} = F_2(t, c^{**}(t)) \text{ with } t^n \le t \le t^{n+1} \text{ and } c^{**}(t^n) = c^*(t^{n+1}) \ , \tag{9}$$

where the initial values are given as $c^n = c(t^n)$ and the split approximation on the next time-level is defined as $c^{n+1} = c^{**}(t^{n+1})$.

In this case the splitting error can be defined by use of the Jacobians of the nonlinear mappings F_1 and F_2, namely

$$\rho_n = \frac{1}{2}\tau[\frac{\partial F_1}{\partial c}F_2, \frac{\partial F_2}{\partial c}F_1](t^n, c(t^n)) + \mathcal{O}(\tau_n^2) \ . \tag{10}$$

Hence, for the general case the splitting error has first order, i.e. $\mathcal{O}(\tau_n)$.

2.3 Higher-order splitting methods for linear operators

So far we have defined the sequential splitting which has first-order accuracy. In the practical computations, in many cases, however, we require splittings of higher-order accuracy.

Symmetrically-weighted sequential splitting. In the following we introduce a weighted sequential splitting method, which is based on two sequential splitting methods with different ordering of the operators. I.e. we consider again the Cauchy problem (4) and we define the operator-splitting on the time-interval $[t^n, t^{n+1}]$ (where $t^{n+1} = t^n + \tau_n$) as follows:

$$\begin{aligned} \frac{\partial c^*(t)}{\partial t} &= Ac^*(t) \ , \quad \text{with } c^*(t^n) = c^n \ , \\ \frac{\partial c^{**}(t)}{\partial t} &= Bc^{**}(t) \ , \quad \text{with } c^{**}(t^n) = c^*(t^{n+1}) \ . \end{aligned} \tag{11}$$

and

$$\begin{aligned} \frac{\partial v^*(t)}{\partial t} &= Bv^*(t) \ , \quad \text{with } v^*(t^n) = c^n \ , \\ \frac{\partial v^{**}(t)}{\partial t} &= Av^{**}(t) \ , \quad \text{with } v^{**}(t^n) = v^*(t^{n+1}) \ . \end{aligned} \tag{12}$$

where c^n is known.

Then the approximation at the next time-level t^{n+1} is defined as:

$$c^{n+1} = \frac{c^{**}(t^{n+1}) + v^{**}(t^{n+1})}{2} \tag{13}$$

The splitting error of this operator-splitting method is derived as follows (cf. [6]):

$$\begin{aligned} \rho_n &= \frac{1}{\tau_n}\{\exp(\tau_n(A+B)) - \frac{1}{2}[\exp(\tau_n B)\exp(\tau_n A) + \exp(\tau_n A)\exp(\tau_n B)]\}\ c(t^n) \\ &= \mathcal{O}(\tau^2) \ . \end{aligned} \tag{14}$$

An easy computation shows that in general the splitting error of this method is $\mathcal{O}(\tau^2)$, i.e. the method is of second-order accuracy. (We note that in the case of commuting operators A and B the method is exact, i.e. the splitting error vanishes.)

Strang-Marchuk splitting method. One of the most popular and widely-used operator-splittings is the so-called *Strang splitting (or Strang-Marchuk splitting)*, defined as follows [25, 26]. The method is as follows:

$$\frac{\partial c^*(t)}{\partial t} = Ac^*(t) \ , \text{ with } t^n \le t \le t^{n+1/2} \text{ and } c^*(t^n) = c^n \ , \tag{15}$$
$$\frac{\partial c^{**}(t)}{\partial t} = Bc^{**}(t) \ , \text{ with } t^n \le t \le t^{n+1} \text{ and } c^{**}(t^n) = c^*(t^{n+1/2}) \ ,$$
$$\frac{\partial c^{***}(t)}{\partial t} = Ac^{***}(t) \ , \text{ with } t^{n+1/2} \le t \le t^{n+1} \text{ and } c^{***}(t^{n+1/2}) = c^{**}(t^{n+1}) \ ,$$

where $t^{n+1/2} = t^n + 0.5\tau_n$ and the approximation on the next time-level t^{n+1} is defined as $c^{n+1} = c^{***}(t^{n+1})$.

The splitting error of the Strang splitting is:

$$\rho_n = \frac{1}{24}(\tau_n)^2([B,[B,A]] - 2[A,[A,B]])\ c(t^n) + \mathcal{O}(\tau_n^4) \ . \tag{16}$$

(See, e.g. ([18].) This means that the global splitting error of this method is of second order, too. (We note that under some special conditions for the operators A and B, the Strang splitting has third-order accuracy and can even be exact [8].)

In our application the first-order splitting for the convection-reaction and the diffusion-dispersion term is applied, because of the dominance of the space-error. The time-error for this combination was only a constant in the total error.

In the next subsection we present the iterative-splitting method.

2.4 Iterative splitting method

The following algorithm is based on the iteration with fixed-splitting discretization step-size τ, namely, on the time-interval $[t^n, t^{n+1}]$ we solve the following sub-problems consecutively for $i = 0, 2, \dots 2m$. (cf. [17, 21].):

$$\frac{\partial c_i(t)}{\partial t} = Ac_i(t) \ + \ Bc_{i-1}(t), \text{ with } \ c_i(t^n) = c^n \tag{17}$$
$$\text{and } c_0(t^n) = c^n \ , \ c_{-1} = 0.0,$$
$$\frac{\partial c_{i+1}(t)}{\partial t} = Ac_i(t) \ + \ Bc_{i+1}(t), \tag{18}$$
$$\text{with } \ c_{i+1}(t^n) = c^n \ ,$$

where c^n is the known split approximation at the time-level $t = t^n$. The split approximation at the time-level $t = t^{n+1}$ is defined as $c^{n+1} = c_{2m+1}(t^{n+1})$. (Clearly, the function $c_{i+1}(t)$ depends on the interval $[t^n, t^{n+1}]$, too, but, for the sake of simplicity, in our notation we omit the dependence on n.)

In the following we will analyze the convergence and the rate of convergence of the method (17)–(18) for m tends to infinity for the linear operators $A, B : \mathbf{X} \to \mathbf{X}$, where we assume that these operators and their sum are generators of the C_0 semigroups. We emphasize that these operators aren't necessarily bounded, so, the convergence is examined in a general Banach space setting.

3 Consistency of the iterative operator-splitting method

In the following subsection, we discuss consistency with respect to the bounded and unbounded operators. We apply different proof techniques, which are adapted to each boundedness.

3.1 Bounded Operators

For the bounded operators we can apply the Taylor expansion of the operators.

Here we will analyze the consistency and the order of the local splitting error of the method (17)–(18) for the linear bounded operators $A, B : \mathbf{X} \to \mathbf{X}$, where $\mathbf{X}$ is a Banach space. In the following, we use the notation $\mathbf{X}^2$ for the product space $\mathbf{X} \times \mathbf{X}$ supplied with the norm $\|(u, v)^T\| = \max\{\|u\|, \|v\|\}$ $(u, v \in \mathbf{X})$.

We have the following consistency order of our iterative operator-splitting method.

Theorem 1. *Let $A, B \in \mathcal{L}(\mathbf{X})$ be given linear bounded operators. We consider the abstract Cauchy problem:*

$$\begin{aligned} \partial_t c(t) &= Ac(t) + Bc(t), \quad 0 < t \leq T, \\ c(0) &= c_0. \end{aligned} \tag{19}$$

Then the problem (19) *has a unique solution. The iteration* (17)–(18) *for $i = 1, 3, \ldots, 2m + 1$ is consistent with the order of the consistency $\mathcal{O}(\tau_n^{2m+1})$.*

Proof. Because $A + B \in \mathcal{L}(\mathbf{X})$, therefore it is a generator of a uniformly continuous semigroup, hence the problem (19) has a unique solution $c(t) = \exp((A + B)t)c_0$.

Let us consider the iteration (17)–(18) on the sub-interval $[t^n, t^{n+1}]$. For the local error function $e_i(t) = c(t) - c_i(t)$ we have the following relations:

$$\begin{aligned} \partial_t e_i(t) &= Ae_i(t) + Be_{i-1}(t), \quad t \in (t^n, t^{n+1}], \\ e_i(t^n) &= 0, \end{aligned} \tag{20}$$

and

$$\begin{aligned} \partial_t e_{i+1}(t) &= Ae_i(t) + Be_{i+1}(t), \quad t \in (t^n, t^{n+1}], \\ e_{i+1}(t^n) &= 0, \end{aligned} \tag{21}$$

for $i = 1, 3, 5, \ldots$, with $e_1(0) = 0$ and $e_0(t) = c(t)$. The elements $\mathcal{E}_i(t), \mathcal{F}_i(t) \in \mathbf{X}^2$ and the linear operator $\mathcal{A} : \mathbf{X}^2 \to \mathbf{X}^2$ are defined as follows:

$$\mathcal{E}_i(t) = \begin{bmatrix} e_i(t) \\ e_{i+1}(t) \end{bmatrix}, \quad \mathcal{F}_i(t) = \begin{bmatrix} Be_{i-1}(t) \\ 0 \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A & 0 \\ A & B \end{bmatrix}. \tag{22}$$

Then, by means of the notations (22), the relations (20)–(21) can be written in the form

$$\begin{aligned} \partial_t \mathcal{E}_i(t) &= \mathcal{A}\mathcal{E}_i(t) + \mathcal{F}_i(t), \quad t \in (t^n, t^{n+1}], \\ \mathcal{E}_i(t^n) &= 0. \end{aligned} \tag{23}$$

Because of our assumptions, $\mathcal{A}$ is a generator of the one-parameter C_0 semigroup $(\exp \mathcal{A}t)_{t\geq 0}$. Hence, through the variations of constants formula, the solution of the abstract Cauchy problem (23) with homogeneous initial conditions can be written as:

$$\mathcal{E}_i(t) = \int_{t^n}^{t} \exp(\mathcal{A}(t-s))\mathcal{F}_i(s)ds, \quad t \in [t^n, t^{n+1}]. \tag{24}$$

Then, using the denotation

$$\|\mathcal{E}_i\|_\infty = \sup_{t\in[t^n,t^{n+1}]} \|\mathcal{E}_i(t)\|, \tag{25}$$

we have

$$\begin{aligned} \|\mathcal{E}_i(t)\| &\leq \|\mathcal{F}_i\|_\infty \int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\|ds \\ &= \|B\|\|e_{i-1}\| \int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\|ds, \quad t \in [t^n, t^{n+1}]. \end{aligned} \tag{26}$$

Because $(\mathcal{A}(t))_{t\geq 0}$ is a semigroup, therefore the so-called *growth estimation*,

$$\|\exp(\mathcal{A}t)\| \leq K \exp(\omega t); \quad t \geq 0, \tag{27}$$

holds with some numbers $K \geq 0$ and $\omega \in \mathbb{R}$.

– Assume that $(\mathcal{A}(t))_{t\geq 0}$ is a bounded or exponentially stable semigroup, i.e. (27) holds for some $\omega \leq 0$. Then obviously the estimate

$$\|\exp(\mathcal{A}t)\| \leq K, \quad t \geq 0, \tag{28}$$

holds, and hence, according to (26), we have the relation

$$\|\mathcal{E}_i\|(t) \leq K\|B\|\tau_n\|e_{i-1}\|, \quad t \in [t^n, t^{n+1}]. \tag{29}$$

– Assume that $(\exp \mathcal{A}t)_{t\geq 0}$ has an exponential growth with some $\omega > 0$. Using (27) we have

$$\int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\|ds \leq K_\omega(t), \quad t \in [t^n, t^{n+1}], \tag{30}$$

where

$$K_\omega(t) = \frac{K}{\omega}\left(\exp(\omega(t-t^n)) - 1\right), \quad t \in [t^n, t^{n+1}]. \tag{31}$$

Hence

$$K_\omega(t) \leq \frac{K}{\omega}\left(\exp(\omega\tau_n) - 1\right) = K\tau_n + \mathcal{O}(\tau_n^2). \tag{32}$$

The estimations (29) and (32) result in

$$\|\mathcal{E}_i\|_\infty = K\|B\|\tau_n\|e_{i-1}\| + \mathcal{O}(\tau_n^2). \tag{33}$$

Taking into account the definition of $\mathcal{E}_i$ and the norm $\|\cdot\|_\infty$ (supremum norm), we obtain

$$\|e_i\| = K\|B\|\tau_n\|e_{i-1}\| + \mathcal{O}(\tau_n^2), \tag{34}$$

and consequently,

$$\begin{aligned}\|e_{i+1}\| &= K\|B\|||e_i|| \int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\|ds, \\ &= K\|B\|\tau_n(K\|B\|\tau_n\|e_{i-1}\| + \mathcal{O}(\tau_n^2)), \\ &= K^2\tau_n^2\|e_{i-1}\| + \mathcal{O}(\tau_n^3).\end{aligned} \tag{35}$$

We apply the recursive argument that proves our statement.

Remark 1. When A and B are matrices (i.e. when (17)–(18) is a system of the ordinary differential equations), we can use the concept of the logarithmic norm for the growth estimation (27). Hence, for many important classes of matrices we can prove the validity of (27) with $\omega \leq 0$.

Remark 2. We note that a huge class of important differential operators generates a contractive semigroup. This means that for such problems, assuming the exact solvability of the split sub-problems, the iterative splitting method is convergent to the exact solution in second order.

Remark 3. We note that the assumption $A \in \mathcal{L}(X)$ can be formulated more weakly as it is enough to assume that the operator A is the generator of a C_0 semigroup.

Remark 4. When T is a sufficiently small number, then we do not need to partition the interval $[0, T]$ into the sub-intervals. In this case, the convergence of the iteration (17)–(18) to the solution of the problem (19) follows immediately from Theorem 1, and the rate of the convergence is equal to the order of the local splitting error.

Remark 5. Estimate (36) shows, that after the final iteration step ($i = 2m+1$), we have the estimation

$$\|e_{2m+1}\| = Km\|e_0\|\tau_n^{2m} + \mathcal{O}(\tau_n^{2m+1}). \tag{36}$$

This relation shows that the constant in the leading term strongly depends on the choice of the initial guess $c_0(t)$. When the choice is $c_0(t) = 0$ (see [21]), then $\|e_0\| = c(t)$ (where $c(t)$ is the exact solution of the original problem) and hence the error might be very significant.

Remark 6. In realistic applications, the final iteration steps $2m + 1$ and the time-step τ_n are chosen in an optimal relation to one another, such that the maximal time step τ_n can be chosen with at least three or five iteration steps. Additionally, a final stop criterion as an error bound, e.g. $|c_i - c_{i-1}| \leq \text{err}$ with, for example, $\text{err} = 10^{-4}$, helps to restrict the number of steps.

We can increase the order of accuracy by improving our choice of the initial iteration function, see [9].

From our previous assumption about the initial solutions, we start with exact solutions or an interpolated split solution and present our theory for the exactness of the method.

The Exact Solution of the Split Sub-problem

We derive the exact solution of the equations (17) and (18) by solving the first split problem,

$$c_i(t^{n+1}) = \exp(At)c^n + \sum_{s=0}^{\infty}\sum_{k=s+1}^{\infty}\frac{t^k}{k!}A^{k-s-1}Bc_{i-1}^{(s)}(t^n), \tag{37}$$

and the second split problem,

$$c_{i+1}(t^{n+1}) = \exp(Bt)c^n + \sum_{s=0}^{\infty}\sum_{k=s+1}^{\infty}\frac{t^k}{k!}B^{k-s-1}Ac_{i}^{(s)}(t^n), \tag{38}$$

where $\tau = t^{n+1} - t^n$ is the equidistant time-step and $c^n = c(t^n)$ is the exact solution at time t^n or at least approximately of local order $\mathcal{O}(\tau^{m+2})$. n is the number of time-steps ($n \in \{0,\ldots,N\}, N \in I\!N^+$) and $m > 0$ is the number of iteration steps.

Theorem 2. *Assume that for the functions $c_{i-1}(t^{n+1})$ and $c_i(t^{n+1})$ the conditions*

$$c_{i-1}^s(t^n) = (A+B)^s c^n,\ s = 0,1,\ldots,m+1, \tag{39}$$
$$c_{i}^s(t^n) = (A+B)^s c^n,\ s = 0,1,\ldots,m+2, \tag{40}$$

are satisfied. After $m+2$ iterations, the method has a local splitting error $\mathcal{O}(\tau^{m+2})$ and therefore the global error $\mathrm{err}_{\mathrm{global}}$ *is $\mathcal{O}(\tau^{m+1})$.*

Proof. We show that

$$\exp(\tau(A+B))c^n - c_{m+1}(t^{n+1}) = \mathcal{O}(\tau^{m+1}), \tag{41}$$
$$\exp(\tau(A+B))c^n - c_{m+2}(t^{n+1}) = \mathcal{O}(\tau^{m+2}). \tag{42}$$

Using the assumption and the exact solutions (37) and (38), we must prove the relations:

$$\sum_{p=0}^{m+1}\frac{1}{p!}\tau^p(A+B)^p = \sum_{p=0}^{m+1}\frac{1}{p!}\tau^p(A)^p + \sum_{s=0}^{m}\sum_{k=s+1}^{m+1}\frac{\tau^k}{k!}A^{k-s-1}B, \tag{43}$$

and

$$\sum_{p=0}^{m+2}\frac{1}{p!}\tau^p(A+B)^p = \sum_{p=0}^{m+2}\frac{1}{p!}\tau^p(B)^p + \sum_{s=0}^{m+1}\sum_{k=s+1}^{m+2}\frac{\tau^k}{k!}B^{k-s-1}A. \tag{44}$$

For the proof, we can use the mathematical induction, see [9].

So, for each further iteration step, we conserve the order $\mathcal{O}(\tau^{m+1})$ for equation (43) or $\mathcal{O}(\tau^{m+2})$ for equation (44).

We assume for all local errors the order $\mathcal{O}(\tau^{m+2})$.

On this assumption, we obtain for the global error

$$\mathrm{err}_{\mathrm{global}}(t^{n+1}) = (n+1)\ \mathrm{err}_{\mathrm{local}}(\tau) = (n+1)\ \tau\ \frac{\mathrm{err}_{\mathrm{local}}(\tau)}{\tau} = \mathcal{O}(\tau^{m+1}), \quad (45)$$

where we assume equidistant time-steps, a time $t^{n+1} = (n+1)\ \tau$, and the same local error for all $n+1$ time-steps, see also [23].

Remark 7. The exact solution of the split sub-problem can also be extended to singular perturbed problems and unbounded operators. In these cases, a formal solution with respect to the asymptotic convergence of a power series, which is near the exact solution can be sought, see [2], [3].

Consistency Analysis of the Iterative Operator-Splitting Method with Interpolated Split Solutions

The algorithm (17)–(18) requires the knowledge of the functions $c_{i-1}(t)$ and $c_i(t)$ on the whole interval $[t^n, t^{n+1}]$. When, however, when we solve split sub-problems, we usually apply some numerical methods that allow us to know the values of the above functions only at some points of the interval. Hence, typically we can define only some interpolations with regard to the exact functions.

In the following we consider and analyze the modified iterative process

$$\frac{\partial c_i(t)}{\partial t} = A c_i(t)\ +\ B c_{i-1}^{\mathrm{int}}(t), \text{ with } \ c_i(t^n) = c_{\mathrm{sp}}^n, \quad (46)$$

$$\frac{\partial c_{i+1}(t)}{\partial t} = A c_i^{\mathrm{int}}(t)\ +\ B c_{i+1}(t), \text{ with } \ c_{i+1}(t^n) = c_{\mathrm{sp}}^n, \quad (47)$$

where $c_k^{\mathrm{int}}(t)$ (for $k = i-1, i$) denotes an approximation of the function $c_k(t)$ on the interval $[t^n, t^{n+1}]$ with the accuracy $\mathcal{O}(\tau_n^p)$. (For simplicity, we assume the same order of accuracy with the order p on each sub-interval.)

The iteration (46)–(47) for the error function $\mathcal{E}_i(t)$ recalls relation (22) with the modified right side, namely

$$\mathcal{F}_i(t) = \begin{bmatrix} B e_{i-1}(t) + B h_{i-1}(t) \\ A h_i(t) \end{bmatrix}, \quad (48)$$

where $h_k(t) = c_k(t) - c_k^{\mathrm{int}}(t) = \mathcal{O}(\tau_n^p)$ for $k = i-1, i$. Hence

$$\|\mathcal{F}_i\|_\infty \leq \max\{\|B\|\ \|e_{i-1}\| + \|h_{i-1}\|; \|A\|\ \|h_i\|\}, \quad (49)$$

which results in the estimation

$$\|\mathcal{F}_i\|_\infty \leq \|B\|\ \|e_{i-1}\| + C\ \tau_n^p. \quad (50)$$

Consequently, for these assumptions, the estimation (34) turns into the following:

$$\|e_i\| \leq K(\|B\|\tau_n\|e_{i-1}\| + C\,\tau_n^{p+1}) + \mathcal{O}(\tau_n^2). \tag{51}$$

So for these assumptions the estimation (36) takes the modified form

$$\|e_{i+1}\| \leq K_1\tau_n^2\|e_{i-1}\| + KC\tau_n^{p+2} + KC\tau_n^{p+1} + \mathcal{O}(\tau_n^3), \tag{52}$$

leading to the following theorem:

Theorem 3. *Let $A, B \in \mathcal{L}(X)$ be given linear bounded operators and consider the abstract Cauchy problem* (19)*. Then for any interpolation of order $p \geq 1$ the iteration* (46)–(47) *for $i = 1, 3, \ldots 2m+1$ is consistent with the order of consistency α where $\alpha = \min\{2m-1, p\}$.*

An outline of the proof can be found in [10].

Remark 8. Theorem 3 shows that the number of the iterations should be chosen according to the order of the interpolation formula. For more iterations, we expect a more accurate solution.

Remark 9. As a result, we can use the piecewise constant approximation of the function $c_k(t)$, namely $c_k^{\text{int}}(t) = c_k(t^n) = \text{const}$, which is known from the split solution. In this instance, it is enough to perform only two iterations in the case of a sufficiently small discretization step-size.

Remark 10. The above analysis was performed for the local error. The global error analysis is as usual and leads to the α-order convergence.

3.2 Unbounded Operators

Theorem 4. *Let us consider the abstract Cauchy problem in a Banach space* $\mathbf{X}$

$$\begin{gathered} \partial_t c(x,t) = Ac(x,t) + Bc(x,t), \quad 0 < t \leq T and\, x \in \Omega \\ c(x,0) = c_0(x)\; x \in \Omega \\ c(x,t) = c_1(x,t)\; x \in \partial\Omega \times [0,T], \end{gathered} \tag{53}$$

where $A, B : D(\mathbf{X}) \to \mathbf{X}$ are given linear operators which are generators of the C_0-semigroup and $c_0 \in \mathbf{X}$ is a given element. We assume A is bounded or more regular than B where B is unbounded. Further, we assume the estimations of an unbounded operator, see [20]:

$$||B^\alpha \exp(B\tau)|| \leq \kappa\tau^{-\alpha} \tag{54}$$

$$||B^\alpha \exp((A+B)\tau)|| \leq \kappa\tau^{-\alpha} \tag{55}$$

The error of the first time-step is of accuracy $\mathcal{O}(\tau^m)$, where $\tau = t^{n+1} - t^n$ and we have equidistant time-steps, with $n = 1, \ldots, N$. Then the iteration process (17)–(18) for $i = 1, 3, \ldots, 2m+1$ is consistent with the order of the consistency $\mathcal{O}(\tau_n^{m+\alpha m})$, where $0 \leq \alpha < 1$.

Proof. Let us consider the iteration (17)–(18) on the sub-interval $[t^n, t^{n+1}]$.

For the first iterations we have:

$$\partial_t c_1(t) = Ac_1(t), \quad t \in (t^n, t^{n+1}], \tag{56}$$

and for the second iteration we have:

$$\partial_t c_2(t) = Ac_1(t) + Bc_2(t), \quad t \in (t^n, t^{n+1}], \tag{57}$$

In general we have:

for the odd iterations: $i = 2m + 1$ for $m = 0, 1, 2, \ldots$

$$\partial_t c_i(t) = Ac_i(t) + Bc_{i-1}(t), \quad t \in (t^n, t^{n+1}], \tag{58}$$

where for $c_0(t) \equiv 0$.

for the even iterations: $i = 2m$ for $m = 1, 2, \ldots$

$$\partial_t c_i(t) = Ac_{i-1}(t) + Bc_i(t), \quad t \in (t^n, t^{n+1}], \tag{59}$$

We have the following solutions for the iterative scheme:

the solutions for the first two equations are given by the variation of constants:

$$c_1(t) = \exp(A(t^{n+1} - t))c(t^n), \quad t \in (t^n, t^{n+1}], \tag{60}$$

$$c_2(t) = \exp(Bt)c(t^n) + \int_{t^n}^{t^{n+1}} \exp(B(t^{n+1} - s))Ac_1(s)ds, \quad t \in (t^n, t^{n+1}], \tag{61}$$

For the recursive even and odd iterations we have the solutions: For the odd iterations: $i = 2m + 1$ for $m = 0, 1, 2, \ldots$

$$c_i(t) = \exp(A(t - t^n))c(t^n) + \int_{t^n}^{t} \exp(sA)Bc_{i-1}(t^{n+1} - s)\, ds, \quad t \in (t^n, t^{n+1}], \tag{62}$$

For the even iterations: $i = 2m$ for $m = 1, 2, \ldots$

$$c_i(t) = \exp(B(t - t^n))c(t^n) + \int_{t^n}^{t} \exp(sB)Ac_{i-1}(t^{n+1} - s)\, ds, \quad t \in (t^n, t^{n+1}], \tag{63}$$

The consistency is given as:

For e_1 we have:

$$c_1(\tau) = \exp(A)\tau)c(t^n), \tag{64}$$

$$\begin{aligned} c(\tau) &= \exp((A + B)\tau)c(t^n) = \exp(A\tau)c(t^n) \\ &+ \int_{t^n}^{t^{n+1}} \exp(As)B \exp((t^{n+1} - s)(A + B))c(t^n)\, ds. \end{aligned} \tag{65}$$

We obtain:

$$\begin{aligned} ||e_1|| = ||c - c_1|| &\leq || \exp((A + B)\tau)c(t^n) - \exp(A\tau)c(t^n)|| \\ &\leq ||B||\tau c(t^n). \end{aligned} \tag{66}$$

For e_2 we have:

$$c_2(\tau) = \exp(B)\tau)c(t^n) + \int_{t^n}^{t^{n+1}} \exp(Bs)A\exp((t^{n+1}-s)A)c(t^n)\,ds, \tag{67}$$

$$c(\tau) = \exp(B\tau)c(t^n) + \int_{t^n}^{t^{n+1}} \exp(Bs)A\exp((t^{n+1}-s)A)c(t^n)\,ds + \int_{t^n}^{t^{n+1}} \exp(Bs)A \int_{t^n}^{t^{n+1}-s} \exp(A\rho)B\exp((t^{n+1}-s-\rho)(A+B))c(t^n)\,d\rho\,ds. \tag{68}$$

We obtain:

$$\begin{aligned} ||e_2|| &\leq ||\exp((A+B)\tau)c(t^n) - c_2|| \\ &\leq ||B||\tau^{1+\alpha}c(t^n). \end{aligned} \tag{69}$$

For odd and even iterations, the recursive proof is given in the following: for the odd iterations: $i = 2m+1$ for $m = 0, 1, 2, \ldots$, for e_i we have :

$$\begin{aligned} c_i(\tau) &= \exp(A)\tau)c(t^n) \\ &+ \int_{t^n}^{t^{n+1}} \exp(As)B\exp((t^{n+1}-s)B)c(t^n)\,ds \\ &+ \int_{t^n}^{t^{n+1}} \exp(As_1)B\int_{t^n}^{t^{n+1}-s_1} \exp(s_2B)A\exp((\tau-s_1-s_2)A)c(t^n)\,ds_2\,ds_1 \\ &+ \ldots + \\ &+ \int_{t^n}^{t^{n+1}} \exp(As_1)B\int_{t^n}^{t^{n+1}-s_1} \exp(s_2B)A\exp((\tau-s_1-s_2)A)uc(t^n)\,ds_2\,ds_1 + \ldots + \\ &+ \int_{t^n}^{t^{n+1}} \exp(As_1)B\int_{t^n}^{t^{n+1}-\sum_{j=1}^{i-1}s_1} \exp(s_2B)A\exp((\tau-s_1-s_2)A)c(t^n)\,ds_2\,ds_1\ldots ds_i, \end{aligned} \tag{70}$$

$$\begin{aligned} c(\tau) &= \exp(B\tau) + \int_{t^n}^{t^{n+1}} \exp(Bs)A\exp((t^{n+1}-s)A)c(t^n)\,ds \\ &+ \ldots + \\ &+ \int_{t^n}^{t^{n+1}} \exp(As_1)B\int_{t^n}^{t^{n+1}-s_1} \exp(s_2B)A\exp((\tau-s_1-s_2)A)c(t^n)\,ds_2\,ds_1 + \ldots + \\ &+ \int_{t^n}^{t^{n+1}} \exp(As_1)B\int_{t^n}^{t^{n+1}-\sum_{j=1}^{i-1}s_1} \exp(s_2B)A\exp((\tau-s_1-s_2)A)c(t^n)\,ds_2\,ds_1\ldots \\ &\int_{t^n}^{t^{n+1}-\sum_{j=1}^{i}s_2} \exp(s_2B)A\exp((\tau-s_1-s_2)(A+B))c(t^n)ds_i, \end{aligned} \tag{71}$$

We obtain:

$$\begin{aligned} ||e_i|| &\leq ||\exp((A+B)\tau)c(t^n) - c_i|| \\ &\leq ||B||^m \tau^{m+1+m\alpha} c(t^n), \end{aligned} \tag{72}$$

where $\alpha = \min_{j=1}^{i}\{\alpha_1\}$ and $0 \leq \alpha_i < 1$.

The same proof idea can be applied to the even iterative scheme.

Remark 11. The same idea can be done with $A = \nabla D \nabla$ $B = -\mathbf{v} \cdot \nabla$, so that one operator is less unbounded
but we reduce the convergence order

$$||e_1|| = K||B||\tau^{\alpha_1}||e_0|| + \mathcal{O}(\tau^{1+\alpha_1}) \tag{73}$$

and hence

$$||e_2|| = K||B||||e_0||\tau^{1+\alpha_1+\alpha_2} + \mathcal{O}(\tau^{1+\alpha_1+\alpha}), \tag{74}$$

where $0 \leq \alpha_1, \alpha_2 < 1$.

Remark 12. If we assume the consistency of $\mathcal{O}(\tau^m)$ for the initial value $e_1(t^n)$ and $e_2(t^n)$, we can redo the proof and obtain at least a global error of the splitting methods of $\mathcal{O}(\tau^{m-1})$.

In the next section we describe the numerical results of our methods.

4 Numerical Results

We deal with some applications in this section to verify our theoretical results, described in the previous sections.

4.1 First example: time-dependent equation

In the first example we deal with a partial differential equation that is time-dependent (see [1]).

We examine a time-dependent 2-D equation

$$\partial_t u(x,y,t) = u_{xx} + u_{yy} - 4(1+y^2)e^{-t}e^{x+y^2}\ , \tag{75}$$

$$u(x,y,0) = e^{x+y^2} \text{ in } \Omega = [-1,1]\times[-1,1]\ , \tag{76}$$

$$u(x,y,t) = e^{-t}e^{x+y^2} \text{ on } \partial\Omega\ , \tag{77}$$

with the exact solution

$$u(x,y,t) = e^{-t}e^{x+y^2}\ . \tag{78}$$

We choose the time interval [0,1] and again use finite differences in space with $\Delta x = 2/19$.

The operators for our splitting methods are

$$Au = \begin{cases} u_{xx} + u_{yy} - 4(1+y^2)e^{-t}e^{x+y^2} & \text{for}(x,y) \in \Omega_1 \ , \\ 0 & \text{for}(x,y) \in \Omega_2 \ , \end{cases}$$

and

$$Bu = \begin{cases} 0 & \text{for}(x,y) \in \Omega_1 \ , \\ u_{xx} + u_{yy} - 4(1+y^2)e^{-t}e^{x+y^2} & \text{for}(x,y) \in \Omega_2 \ , \end{cases}$$

with $\Omega_1 = [-1,1] \times [-1,0]$ and $\Omega_2 = [-1,1] \times [0,1]$.

The approximation error is computed with the maximum error and given by Max-error $= \max_{i,j} ||u_{exact}(x_i, y_j, T) - u_{approx}(i\Delta x, j\Delta x, T)||$

Iterative steps	Number of splitting-partitions	Max-error
1	1	2.7183e+000
4	1	2.5147e+000
5	1	1.8295e+000
10	1	6.8750e-001
20	1	8.7259e-002
25	1	2.5816e-002
30	1	5.3147e-003
35	1	2.8774e-003

Table 1. Numerical results for the first example with the iterative-operator-splitting method and Backward Differential Formula 3 (BDF3) with $h = 10^{-1}$.

The relaxation error is smoothed, as given in Figure 1 following.

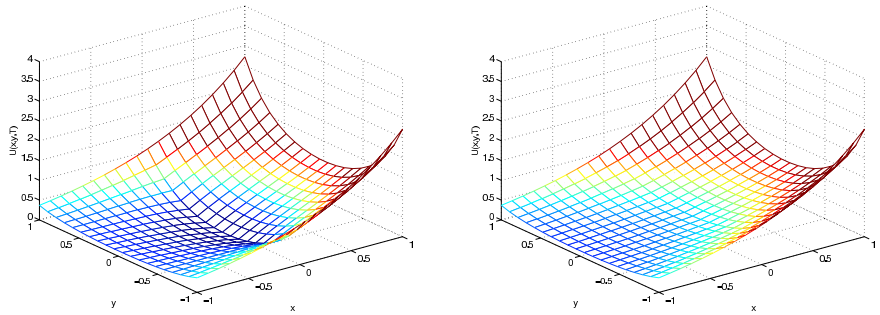

Fig. 1. The numerical results of the first example after ten iterations (left) and twenty iterations (right).

4.2 Second example: convection-reaction equation with sparsity pattern

We consider the one-dimensional convection-diffusion-reaction equation, where the reaction terms strongly couple the equations.

This is given by:

$$\begin{aligned} R\partial_t u + v\partial_x u - D\partial_{xx} u = -\lambda u \ , \text{ on } \Omega \times [t_0, t_{\text{end}}) \ , \\ u(x, t_0) = u_{\text{exact}}(x, t_0) \ , \\ u(0, t) = u_{\text{exact}}(0, t) \ , \ u(L, t) = u_{\text{exact}}(L, t). \end{aligned}$$

We choose $x \in [0, 30]$, and $t \in [10^4, 2 \cdot 10^4]$.
Further, we have $\lambda = 10^{-5}$, $v = 0.001$, $D = 0.0001$ and $R = 1.0$. The analytic solution is given by:

$$u_{\text{exact}}(x, t) = \frac{1}{2\sqrt{D\pi t}} \exp(-\frac{(x - vt)^2}{4Dt}) \exp(-\lambda t) \ .$$

To avoid the singular point of the exact solution, we start from the time-point $t_0 = 10^4$.

Our split operators are:

$$A = \frac{D}{R}\partial_{xx} u \ , \ B = -\frac{1}{R}(\lambda u + v\partial_x u) \ . \tag{79}$$

For the spatial discretization we use finite differences with $\Delta x = \frac{1}{10}$.

Iterative steps	Number of splitting-partitions	error $x = 18$	error $x = 20$	error $x = 22$
1	10	9.8993e-002	1.6331e-001	9.9054e-002
2	10	9.5011e-003	1.6800e-002	8.0857e-003
3	10	9.6209e-004	1.5782e-002	2.2922e-004
4	10	8.7208e-004	1.4100e-002	1.5168e-004

Table 2. Numerical results for the second example with the iterative operator-splitting method and BDF3 with $h = 10^{-2}$.

The relaxation error can be reduced with more iterations, as given in the Figure 2 following.

Additionally, we can reduce the relaxation error with an improved discretization method, for example, BDF3 method (see Figure 3).

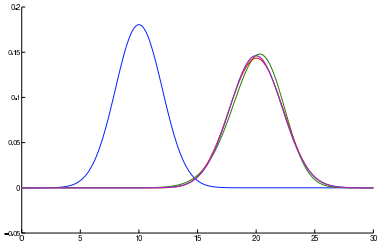

Fig. 2. Numerical results for the third example with the number of iterations. The plot on the left side is at $t = 0$, while that on the right side is at $t = T$ with iterations $i = 1, 2, 3$.

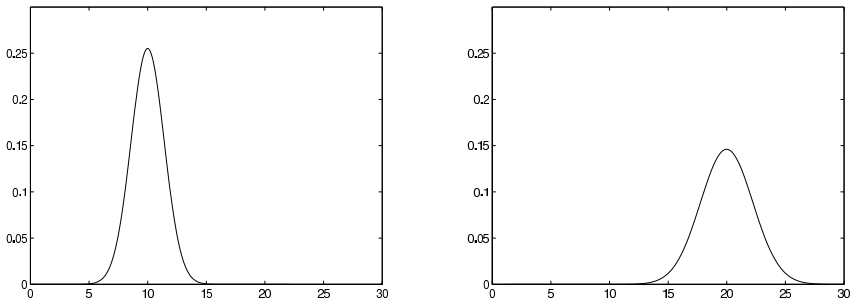

Fig. 3. Numerical results for the second example with the iterative splitting method and BDF3. The plot on the left is at $t = 0$, while that on the right is at $t = T$.

4.3 Test example 3: momentum equation (molecular flow)

We deal with an example of a momentum equation, that is used to model the viscous flow of a fluid.

$$\partial_t \mathbf{u} = -\mathbf{u} \cdot \nabla \mathbf{u} + 2\mu \nabla (D(\mathbf{u}) + 1/3 \nabla \mathbf{u}) + \mathbf{f}(x, y, t), \ (x, y, t) \in \Omega \times [0, T] \quad (80)$$

$$\mathbf{u}(x, y, 0) = \mathbf{g_1}(x, y), \ (x, y) \in \Omega, \quad (81)$$

$$\text{with } \mathbf{u}(x, y, t) = \mathbf{g_2}(x, y, t) \text{ on} \partial\Omega \times [0, T] \ (enclosedflow), \quad (82)$$

where $\mathbf{u} = (u_1, u_2)^t$ is the solution and $\Omega = [0, 1] \times [0, 1]$, $T = 1.25$, $\mu = 5$, and $\mathbf{v} = (0.001, 0.001)^t$ are the parameters and I is the unit matrix.

The nonlinear function $D(\mathbf{u}) = \mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u}$ is the viscosity flow, and $\mathbf{v}$ is a constant velocity.

We can derive the analytical solution with respect to the first two test examples with the functions:

$$u_1(x,y,t) = (1+\exp(\frac{x+y-t}{2\mu}))^{-1} + \exp(\frac{x+y-t}{2\mu}), \tag{83}$$

$$u_2(x,y,t) = (1+\exp(\frac{x+y-t}{2\mu}))^{-1} + \exp(\frac{x+y-t}{2\mu}). \tag{84}$$

For the splitting method our operators are given as:

$A(\mathbf{u})\mathbf{u} = -\mathbf{u}\nabla\mathbf{u} + 2\mu\nabla D(\mathbf{u})$ (the nonlinear operator), and
$B\mathbf{u} = 2/3\mu\Delta\mathbf{u}$ (the linear operator).
We first deal with the one-dimensional case,

$$\partial_t u = -u\cdot\partial_x u + 2\mu\partial_x(D(u) + 1/3\partial_x u) + f(x,t),\ (x,t)\in\Omega\times[0,T] \tag{85}$$

$$u(x,0) = g_1(x), (x)\in\Omega \tag{86}$$

$$\text{with } u(x,t) = g_2(x,t)\ \text{on}\partial\Omega\times[0,T](enclosedflow), \tag{87}$$

where u is the solution and $\Omega = [0,1]$, $T = 1.25$, $\mu = 5$, and $v = 0.001$ are the parameters.

Then the operators are given as:

$A(u)u = -u\partial_x u + 2\mu\partial_x D(u)$ (the nonlinear operator), and
$Bu = 2/3\mu\partial_{xx}u$ (the linear operator).

For the iterative operator-splitting as fixed point scheme, we have the following results, see Tables 3 and 5. The result for the iterative operator-splitting method plus Newton's method as linearization technique, see [16], is given in Table 4.

Figure 4 presents the profile of the 1D momentum equation.
We have the following results for the 2D case, see Tables 6, 7, and 8.

Figure 5 presents the profile of the 2D momentum equation.
For the Newton operator-splitting method we obtain the following functional matrices for the one-dimensional case,

$$DF(u) = (4\mu - 1)\partial_x u, \tag{88}$$

and

$$\begin{aligned} D(F(u)) &= -\begin{pmatrix} \partial_{u_1}F_1(u) & \partial_{u_2}F_1(u) \\ \partial_{u_1}F_2(u) & \partial_{u_2}F_2(u) \end{pmatrix} \\ &= -\begin{pmatrix} -\partial_x u_1 + 4\mu\partial_x u_1 & -\partial_x u_2 + 4\mu\partial_x u_2 \\ -\partial_y u_1 + 4\mu\partial_y u_1 & -\partial_y u_2 + 4\mu\partial_y u_2 \end{pmatrix} \\ &= (4\mu-1)\nabla u \end{aligned} \tag{89}$$

Δx	Δt	err_{L_1}	$\mathrm{err}_{\max}$	ρ_{L_1}	$\rho_{\max}$
1/10	1/20	0.0213	0.0495		
1/20	1/20	0.0203	0.0470	0.0689	0.0746
1/40	1/20	0.0198	0.0457	0.0401	0.0402
1/80	1/20	0.0195	0.0450	0.0216	0.0209
1/10	1/40	0.0134	0.0312		
1/20	1/40	0.0117	0.0271	0.1957	0.2009
1/40	1/40	0.0108	0.0249	0.1213	0.1211
1/80	1/40	0.0103	0.0238	0.0682	0.0674
1/10	1/80	0.0094	0.0217		
1/20	1/80	0.0073	0.0169	0.3591	0.3641
1/40	1/80	0.0062	0.0143	0.2451	0.2448
1/80	1/80	0.0056	0.0129	0.1478	0.1469

Table 3. Numerical results for the 1D momentum equation with $\mu = 5$, $v = 0.001$, initial condition $u_0(t) = c_n$, and two iterations per time step.

Δx	Δt	err_{L_1}	$\mathrm{err}_{\max}$	ρ_{L_1}	$\rho_{\max}$
1/10	1/20	0.0180	0.0435		
1/20	1/20	0.0120	0.0276	0.5867	0.6550
1/40	1/20	0.0095	0.0227	0.3311	0.2870
1/80	1/20	0.0085	0.0208	0.1706	0.1231
1/10	1/40	0.0172	0.0459		
1/20	1/40	0.0125	0.0305	0.4652	0.5884
1/40	1/40	0.0108	0.0253	0.2366	0.2698
1/80	1/40	0.0097	0.0235	0.1191	0.1111
1/10	1/80	0.0166	0.0475		
1/20	1/80	0.0132	0.0338	0.3327	0.4917
1/40	1/80	0.0119	0.0280	0.1640	0.2734
1/80	1/80	0.0112	0.0265	0.0802	0.0779

Table 4. Numerical results for the 1D momentum equation with $\mu = 5$, $v = 0.001$, initial condition $u_0(t) = c_n$, two iterations per time step and $K = 1$ using Newton iterative method.

for the two-dimensional case using

$$A(\mathbf{u})\mathbf{u} = -\mathbf{u}\nabla\mathbf{u} + 2\mu\nabla D(\mathbf{u}), \tag{90}$$

$$= -\begin{pmatrix} u_1\partial_x u_1 + u_2\partial_x u_2 \\ u_1\partial_y u_1 + u_2\partial_y u_2 \end{pmatrix} + 2\mu\begin{pmatrix} 2u_1\partial_x u_1 + 2u_2\partial_x u_2 + v_1\partial_x u_1 + v_2\partial_x u_2 \\ 2u_1\partial_y u_1 + 2u_2\partial_y u_2 + v_1\partial_y u_1 + v_2\partial_y u_2 \end{pmatrix} \tag{91}$$

Here, we do not need the linearization and apply the standard iterative splitting method.

Δx	Δt	err_{L_1}	$\mathrm{err}_{\max}$	ρ_{L_1}	$\rho_{\max}$
1/10	1/20	$2.7352 \cdot 10^{-6}$	$6.4129 \cdot 10^{-6}$		
1/20	1/20	$2.3320 \cdot 10^{-6}$	$5.4284 \cdot 10^{-6}$	0.2301	0.2404
1/40	1/20	$2.1144 \cdot 10^{-6}$	$4.9247 \cdot 10^{-6}$	0.1413	0.1405
1/80	1/20	$2.0021 \cdot 10^{-6}$	$4.6614 \cdot 10^{-6}$	0.0787	0.0793
1/10	1/40	$2.1711 \cdot 10^{-6}$	$5.2875 \cdot 10^{-6}$		
1/20	1/40	$1.7001 \cdot 10^{-6}$	$4.1292 \cdot 10^{-6}$	0.3528	0.3567
1/40	1/40	$1.4388 \cdot 10^{-6}$	$3.4979 \cdot 10^{-6}$	0.2408	0.2394
1/80	1/40	$1.3023 \cdot 10^{-6}$	$3.1694 \cdot 10^{-6}$	0.1438	0.1423
1/10	1/80	$1.6788 \cdot 10^{-6}$	$4.1163 \cdot 10^{-6}$		
1/20	1/80	$1.1870 \cdot 10^{-6}$	$2.9138 \cdot 10^{-6}$	0.5001	0.4984
1/40	1/80	$9.1123 \cdot 10^{-7}$	$2.2535 \cdot 10^{-6}$	0.3814	0.3707
1/80	1/80	$7.6585 \cdot 10^{-7}$	$1.9025 \cdot 10^{-6}$	0.2507	0.2443

Table 5. Numerical results for the 1D momentum equation with $\mu = 50$, $v = 0.1$, initial condition $u_0(t) = c_n$, and two iterations per time step.

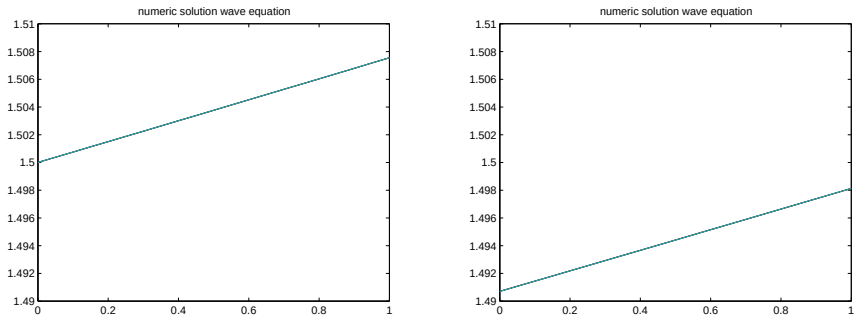

Fig. 4. 1D momentum equation at initial time $t = 0.0$ (left figure) and end time $t = 1.25$ (right figure) for $\mu = 5$ and $v = 0.001$.

We only linearize the first split step and therefore we can relax this step with the second linear split step. Therefore we obtain stable methods, see [22].

Remark 13. In the more realistic examples of a 1D and 2D momentum equations, we can also observe the stiffness problem, which we obtain with a more hyperbolic behaviour. In the 1D experiments we deal with a more hyperbolic behaviour and could obtain at least first order convergence with 2 iterative steps. In the 2D experiments we obtain nearly second order convergence results with 2 iterative steps, if we increase the parabolic behaviour, e.g. larger μ and $\mathbf{v}$ values. For such methods, we have to balance the usage of the iterative steps, refinement in time and space with respect to the hyperbolicity of the equations. At least we can obtain a second order method with more than 2 iterative steps. So the stiffness influence the number of iterative steps.

Δx $= \Delta y$	Δt	err_{L_1} 1st c.	$\mathrm{err}_{\max}$ 1st c.	ρ_{L_1} 1st c.	$\rho_{\max}$ 1st c.	err_{L_1} 2nd c.	$\mathrm{err}_{\max}$ 2nd c.	ρ_{L_1} 2nd c.	$\rho_{\max}$ 2nd c.
1/5	1/20	0.0027	0.0112			0.0145	0.0321		
1/10	1/20	0.0016	0.0039	0.7425	1.5230	0.0033	0.0072	2.1526	2.1519
1/20	1/20	0.0007	0.0022	1.2712	0.8597	0.0021	0.0042	0.6391	0.7967
1/5	1/40	0.0045	0.0148			0.0288	0.0601		
1/10	1/40	0.0032	0.0088	0.5124	0.7497	0.0125	0.0239	1.2012	1.3341
1/20	1/40	0.0014	0.0034	1.1693	1.3764	0.0029	0.0054	2.1263	2.1325
1/5	1/80	0.0136	0.0425			0.0493	0.1111		
1/10	1/80	0.0080	0.0241	0.7679	0.8197	0.0278	0.0572	0.8285	0.9579
1/20	1/80	0.0039	0.0113	1.0166	1.0872	0.0115	0.0231	1.2746	1.3058

Table 6. Numerical results for the 2D momentum equation with $\mu = 2$, $v = (1, 1)^t$, initial condition $u_0(t) = c_n$, and two iterations per time step.

Δx $= \Delta y$	Δt	err_{L_1} 1st c.	$\mathrm{err}_{\max}$ 1st c.	ρ_{L_1} 1st c.	$\rho_{\max}$ 1st c.
1/5	1/20	$1.5438 \cdot 10^{-5}$	$3.4309 \cdot 10^{-5}$		
1/10	1/20	$4.9141 \cdot 10^{-6}$	$1.0522 \cdot 10^{-5}$	1.6515	1.7052
1/20	1/20	$1.5506 \cdot 10^{-6}$	$2.9160 \cdot 10^{-6}$	1.6641	1.8513
1/5	1/40	$2.8839 \cdot 10^{-5}$	$5.5444 \cdot 10^{-5}$		
1/10	1/40	$1.3790 \cdot 10^{-5}$	$2.3806 \cdot 10^{-5}$	1.0645	1.2197
1/20	1/40	$3.8495 \cdot 10^{-6}$	$6.8075 \cdot 10^{-6}$	1.8408	1.8061
1/5	1/80	$3.1295 \cdot 10^{-5}$	$5.5073 \cdot 10^{-5}$		
1/10	1/80	$1.7722 \cdot 10^{-5}$	$2.6822 \cdot 10^{-5}$	0.8204	1.0379
1/20	1/80	$7.6640 \cdot 10^{-6}$	$1.1356 \cdot 10^{-5}$	1.2094	1.2400

Table 7. Numerical results for the 2D momentum equation for the first component with $\mu = 50$, $v = (100, 0.01)^t$, initial condition $u_0(t) = c_n$, and two iterations per time step.

5 Conclusions and Discussions

We have presented an iterative operator-splitting method and analyze the consistency error for the bounded and unbounded operators. We derived the local splitting error and showed the decreasing consistency with respect to the unbounded operators. The application of the splitting methods has also been discussed. We can confirm that a complex model could be simulated with the help of different splitting and discretization methods. In the future we will focus us on the development of improved operator-splitting methods with respect to their application in nonlinear convection-diffusion-reaction equations.

Δx $= \Delta y$	Δt	err_{L_1} 2nd c.	$\mathrm{err}_{\max}$ 2nd c.	ρ_{L_1} 2nd c.	$\rho_{\max}$ 2nd c.
1/5	1/20	$4.3543 \cdot 10^{-5}$	$1.4944 \cdot 10^{-4}$		
1/10	1/20	$3.3673 \cdot 10^{-5}$	$7.9483 \cdot 10^{-5}$	0.3708	0.9109
1/20	1/20	$2.6026 \cdot 10^{-5}$	$5.8697 \cdot 10^{-5}$	0.3717	0.4374
1/5	1/40	$3.4961 \cdot 10^{-5}$	$2.2384 \cdot 10^{-4}$		
1/10	1/40	$1.7944 \cdot 10^{-5}$	$8.9509 \cdot 10^{-5}$	0.9622	1.3224
1/20	1/40	$1.5956 \cdot 10^{-5}$	$3.6902 \cdot 10^{-5}$	0.1695	1.2783
1/5	1/80	$9.9887 \cdot 10^{-5}$	$3.3905 \cdot 10^{-4}$		
1/10	1/80	$3.5572 \cdot 10^{-5}$	$1.3625 \cdot 10^{-4}$	1.4896	1.3153
1/20	1/80	$1.0557 \cdot 10^{-5}$	$4.4096 \cdot 10^{-5}$	1.7525	1.6275

Table 8. Numerical results for the 2D momentum equation for the second component with $\mu = 50$, $v = (100, 0.01)^t$, initial condition $u_0(t) = c_n$, and two iterations per time step.

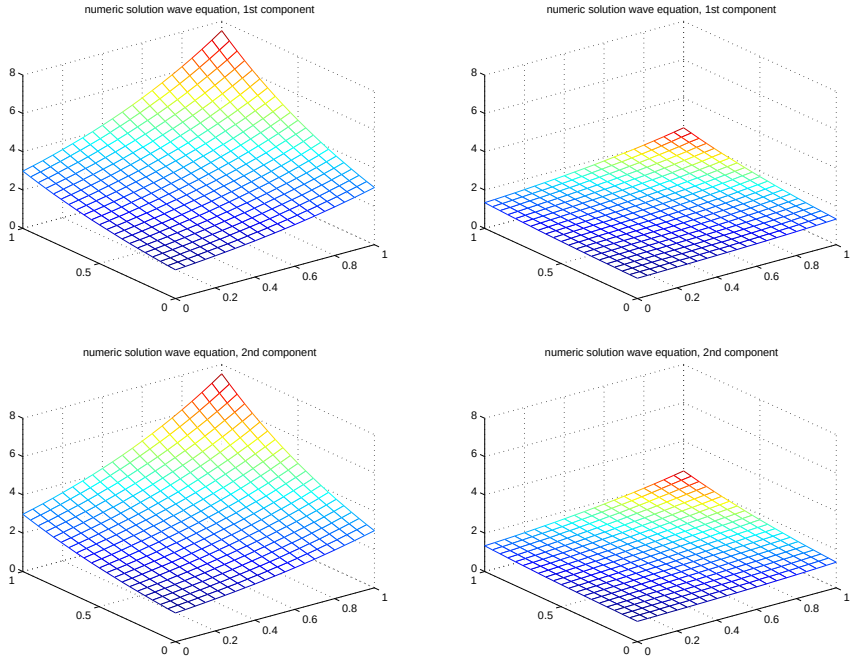

Fig. 5. 2D momentum equation at initial time $t = 0.0$ (left figure) and end time $t = 1.25$ (right figure) for $\mu = 0.5$ and $v = (1, 1)^t$ for the first and second component of the numerical solution.

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