Iterative operator-splitting with time overlapping algorithms: Theory and Application to constant and time-dependent wave equations.

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Abstract

In this article we consider wave equations with constant and linear time dependent diffusion-coefficients which are solved numerically by iterative operator splitting with interval overlapping algorithms. The benefits of overlappling for time dependent equations are discussed. A splitting analysis and the assemblation are presented. Numerical examples for 2D wave equations are discussed at the end of this paper.

Keyword numerical analysis, operator-splitting method, initial value problems, iterative solver method, stability analysis, overlapping schemes, convection-diffusion-reaction equation.

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1 Introduction

Our study is motivated by wave action models with time dependent diffusion coefficients where the decoupling algorithms are based on iterative splitting methods.

The paper is organised as follows. Mathematical models of constant and time dependent diffusion coefficients' wave equations are introduced in Section 2 and we provide analytical solutions as far as possible. In section 3 we give an overview to iterative operator-splitting methods in general, while in section 4 we discuss them with respect to wave equations. For the time dependent case we introduce overlapping schemes. We will do convergence and stability analysis

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of all methods in use. In section 5 we reformulate the methods for numerical applications, e.g. we give an appropriate discretisation and assembling. We present the numerical results in section **??** and finally, we discuss our future works in the area of splitting and decomposition methods.

2 Mathematical model

Motivated by simulating the propagation of a variety of waves, such as sound waves, light waves and water waves, we discuss a novel numerical scheme to solve the wave equation with time dependent diffusion coefficients, see [21]. We deal with a second-order linearly time dependent partial differential equation. It arises in fields such as acoustics, electromagnetics and fluid dynamics, see [5]. For example, when wave propagation models are physically more complex, due to combined propagations in three dimensions, time dependent equations of such dynamical models become the starting point of the analysis, see [3]. We concentrate on wave propagation models to obtain physically related results for time dependent diffusion parameters, see [5]. For the sake of completion we incorporate the constant case, too.

2.1 Wave Equations

In this section we present wave equations with constant and time dependent diffusion coefficients.

Wave equation with constant diffusion coefficients

First we deal with a wave equation that represents a simple model of a Maxwell equation which is needed for the simulation of electro-magnetic fields. We have a linear wave equation with constant coefficients given by:

$$\begin{aligned} \frac{\partial^2 c}{\partial t^2} &= D_1 \frac{\partial^2 c}{\partial x_1^2} + \ldots + D_d \frac{\partial^2 c}{\partial x_d^2} \quad \text{on } \Omega \times [0,T], \qquad (1)\\ c(x,0) &= c_0(x), \text{ and } \frac{\partial c}{\partial t}(x,0) = c_1(x) \quad \text{on } \Omega,\\ c(x,t) &= c_2(x,t) \text{ on } \partial\Omega_{Dirich} \times [0,T],\\ \frac{\partial c}{\partial n} &= 0 \text{ on } \partial\Omega_{Neum} \times [0,T], \end{aligned}$$

where c_0, c_1 are the initial conditions and c_3 the boundary condition for the Dirichlet boundary. We have $\partial\Omega_{Dirich} \cap \partial\Omega_{Neum} = \partial\Omega$.

For this PDE we can derive an analytical solution:

$$c(x_1, \ldots, x_d, t) = \sin(\frac{1}{\sqrt{D_1}}\pi x_1) \cdot \ldots \cdot \sin(\frac{1}{\sqrt{D_d}}\pi x_d) \cdot \cos(\sqrt{d}\pi t) \qquad (2)$$

where d is the spatial dimension.

Wave equation with time dependent diffusion coefficients

Mathematical models often need to have time dependent diffusion coefficients, e.g. hyperbolic differential equations. These are among others the Schrödinger equations or the wave equations with time dependent diffusion coefficients in fluid dynamics. In this paper we shall deal with the uncoupled wave equation with time dependent diffusion coefficients given by:

$$\begin{aligned}
\frac{\partial^2 c}{\partial t^2} &= D_1(t)\frac{\partial^2 c}{\partial x_1^2} + \ldots + D_d(t)\frac{\partial^2 c}{\partial x_d^2} \quad \text{on } \Omega \times [0,T], \\
c(x,0) = c_0(x),\ \frac{\partial c}{\partial t}(x,0) &= c_1(x) \quad \text{on } \Omega, \\
c(x,t) &= c_2(x,t) \quad \text{on } \partial\Omega_{Dirich} \times [0,T], \\
\frac{\partial c}{\partial n} &= 0 \quad \text{on } \partial\Omega_{Neum} \times [0,T],
\end{aligned} \tag{3}$$

where c_0, c_1 are the initial conditions and c_3 the boundary condition for the Dirichlet boundary. We have $\partial\Omega_{Dirich} \cap \partial\Omega_{Neum} = \partial\Omega$.

In general, we can not derive an analytical solution for arbitrary coefficients' functions. However, given linear diffusion functions, we can deliver an analytical solution with respect to a right hand side (inhomogeneous equation) where we may provide sufficient conditions for the right hand side to vanish in order to obtain an analytical solution for the homogeneous equation. Thus we have

$$\begin{aligned}
\frac{\partial^2 c}{\partial t^2} &= D_1(t)\frac{\partial^2 c}{\partial x_1^2} + \ldots + D_d(t)\frac{\partial^2 c}{\partial x_d^2} + f(x_1,\ldots,x_d,t) \quad \text{on } \Omega \times [0,T], \\
c(x,0) = c_{anal}(x,t) \ \text{ and } \ \frac{\partial c}{\partial t}(x,0) &= c_{anal}(x,t) \quad \text{on } \Omega, \\
c(x,t) &= c_{anal}(x,t) \quad \text{on } \partial\Omega \times [0,T],
\end{aligned} \tag{4}$$

where c_{anal} is the assumed analytical solution and $D_j(t) = a_j t + b_j$ with $a_j, b_j \in \mathbb{R}$.

Theorem 1. *We claim to have the following analytical solution for d dimensions:*

$$c(x_1,\ldots,x_d,t) = \sum_{j=1}^{d} \sin(\pi x_j)\big(\sin(\lambda_j(t)\pi)\big), \tag{5}$$

while the right hand side $f(x_1,\ldots,x_d,t)$ is given by

$$f(x_1,\ldots,x_d,t) = \sum_{j=1}^{d} \pi\frac{a_j}{2}(a_j t + b_j)^{-1/2}\sin(\pi x_j)\cos(\lambda_j(t)\pi), \tag{6}$$

and where

$$\lambda_j(t) = \frac{2}{3a_j}(a_j t + b_j)^{3/2}, \quad j = 1,\ldots,d. \tag{7}$$

Proof. We have the following derivatives

$$\frac{\partial^2 c}{\partial x_j^2} = -\pi^2 \sin(\pi x_j)\sin(\lambda_j(t)\pi), \quad j = 1, \ldots, d. \tag{8}$$

$$\frac{\partial^2 c}{\partial t^2} = \sum_{j=1}^{d} sin(\pi x_j)\left(\pi\lambda_j''(t)\cos(\lambda_j(t)\pi) - \pi^2(\lambda_j'(t))^2\sin(\lambda_j(t)\pi)\right), \tag{9}$$

where

$$\lambda_j'(t) = (a_j t + b_j)^{1/2},$$
$$\lambda_j''(t) = \frac{a_j}{2}(a_j t + b_j)^{-1/2}, \quad j = 1, \ldots, d.$$

Hence, by employing the derivatives (8)–(9) in (4) we obtian for $f(x_1, \ldots, x_d, t)$

$$f(t) = \sum_{j=1}^{d} \pi\frac{a_j}{2}(a_j t + b_j)^{-1/2}\sin(\pi x_j)\cos(\lambda_j(t)\pi). \tag{10}$$

□

Remark 1. *An analytical solution for the homogeneous equation (3) can be given for $x \in \Omega$ such that $f(x,t) = 0$, i.e.*

$$\Leftrightarrow \sin(\pi x_j) = 0 \quad j = 1, \ldots, d$$
$$\Leftrightarrow x_j \in \mathbb{Z} \quad j = 1, \ldots, d$$

Hence for $x \in \mathbb{Z}^d \cap \Omega$.

2.1.1 Existence of solutions for time dependent wave equations

We assume to have an analytical solution for the following equation.

$$\frac{\partial^2 c}{\partial t^2} = D_1(t)\frac{\partial^2 c}{\partial x_1^2} + \ldots + D_d(t)\frac{\partial^2 c}{\partial x_d^2} + f(x_1, \ldots, x_d, t) \quad \text{on } \Omega \times [0,T], \tag{11}$$
$$c(x,0) = c_{anal}(x,t) \text{ and } \frac{\partial c}{\partial t}(x,0) = c_{anal}(x,t) \quad \text{on } \Omega,$$
$$c(x,t) = c_{anal}(x,t) \quad \text{on } \partial\Omega \times [0,T],$$

where the analytical solution is given as $c_{anal}(x_1, \ldots, x_d, t) \in C^2(\Omega) \times C^2([0,T])$ and $f(x_1, \ldots, x_d, t) \in C^2(\Omega) \times C^2([0,T])$.

The equation (11) can be reformulated into a system of first order PDEs. Then we can apply the variation of constants formula which is given by

$$C(x_1, \ldots, x_d, t) = K(x_1, \ldots, x_d, t) + \int_0^t K(x_1, \ldots, x_d, t-s)F(x_1, \ldots, x_d, s)\, ds, \tag{12}$$

where F and C are obtained by the reformulation of a system of first order PDEs. Then we assume that there exists a kernel $K(x_1, \ldots, x_d, t)$ with $C(x_1, \ldots, x_d, 0) = K(x_1, \ldots, x_d, 0)$.

Proof. The variation of constants formula is given by

$$C(x_1,\dots,x_d,t) = K(x_1,\dots,x_d,t) + \int_0^t K(x_1,\dots,x_d,t-s)F(x_1,\dots,x_d,s)\,ds,$$

Now we assume, given C and F such that we obtain an integral equation, where $K(x_1,\dots,x_d,t)$ is the unknown.

Based on the rewriting of the Voltera's integral equation there exists a solution when K is bounded, i.e.

$$|K(x_1,\dots,x_d,t) - K(x_1,\dots,x_d,t')| \leq L(x_1,\dots,x_d)|t-t'|,$$

for all $(x_1,\dots,x_d) \in \Omega$ and $t,t' \in [0,T]$. This is assumed in solving the solution and that the kernel is bounded, i.e. also for the case $F(x_1,\dots,x_d,t) \to 0$. □

Remark 2. *For $F(x_1,\dots,x_d,t) \equiv 0$ we obtain a solution for the homogeneous equation.Thus, there exists a solution for equation (3).*

3 Splitting methods

Splitting methods have been designed for accelerating solver processes and decomposing them into simpler solvable equation parts, see [24] and [16]. Other ways are to consider the physical behaviour and split it into simpler and solvable equation parts, e.g. symplectic schemes, [22] and [30]. The natural way to decouple a differential equation into simpler parts is done by:

$$\frac{dc(t)}{dt} = A_{\text{full}}\, c(t), \text{ for } t \in (t^n, T), \tag{13}$$

$$\frac{dc(t)}{dt} = (A+B)\, c(t), \text{ for } t \in (t^n, T), \tag{14}$$

$$c(t^n) = c^n, \text{ (initial condition)}, \tag{15}$$

where $t^n, T \in \mathbb{R}^+$ and $t^n \leq T$. The operator A_{full} can be decoupled into the operators A and B, cf. introduction in [28].

Based on these linear operators the equation (13) can be solved exactly. The solution is given by:

$$c(t^{n+1}) = \exp(\tau A_{\text{full}})\, c(t^n), \tag{16}$$

where the time step is $\tau = t^{n+1} - t^n$ and $t^{n+1} \leq T$.

The simplest operator splitting method is the sequential decoupling into two or more equations. The error for the linear case could be analysed by the Taylor expansion, see [1].

Remark 3. *The introduction for ordinary differential equations presented above can be extended for the abstract Cauchy problem of a parabolic equation by regarding the possibility of defining the operator A_{full} using a Friedrichs' extension. Thus, the mild solutions (or weak solutions) are possible and we can apply the notation of the* exp*-formulations, see [31].*

3.1 Iterative Operator Splitting Methods for Wave Equations

In the following we apply the iterative operator-splitting method as an extension to the traditional splitting methods for wave equations. The idea is to repeat the splitting steps with the improved computed solutions. We have to solve a fixed-point iteration and we gain higher-order results.

The iterative splitting method is given in the continous formulation as follows:

$$\begin{aligned} &\frac{\partial^2 c_i(t)}{\partial t^2} = Ac_i(t) \ + \ Bc_{i-1}(t) \ + \ f(t) \ , \\ &\text{with} \ \ c_i(t^n) = c_{\text{sp}}^n \ , \ c_i'(t^n) = c_{\text{sp}}'^n, \end{aligned} \tag{17}$$

$$\begin{aligned} &\frac{\partial^2 c_{i+1}(t)}{\partial t^2} = Ac_i(t) \ + \ Bc_{i+1}(t) \ + \ f(t) \ , \\ &\text{with} \ \ c_{i+1}(t^n) = c_{\text{sp}}^n \ , \ c_{i+1}'(t^n) = c_{\text{sp}}'^n, \end{aligned} \tag{18}$$

where $c_0(t), c_0'(t)$ are fixed functions for each iteration. Here $c_{\text{sp}}^n, c_{\text{sp}}'^n$ denote the known split approximations at time level $t = t^n$. The split approximation at time level $t = t^{n+1}$ is defined by $c_{\text{sp}}^{n+1} = c_{2m+1}(t^{n+1})$.

Remark 4. *The stop criteria is given by:*

$$|c_{k+1} - c_k| \leq \epsilon$$

for $k \in 1, 3, 5, \ldots$ *and* $\epsilon \in \mathbb{R}^+$. *Thus, the solution is given by* $c(t^{n+1}) = c_{k+2}$.

For the stability and consistency we can rewrite the equations (17)–(18) in continuous form as follows:

$$\partial_{tt} C_i = \mathcal{A} C_i + \mathcal{F}_i, \tag{19}$$

where $C_i = (c_i, c_{i+1})^t$ and the operators are given by

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ A & B \end{bmatrix} \ , \ \mathcal{F}_i = \begin{bmatrix} Bc_{i-1} \\ 0 \end{bmatrix} \ . \tag{20}$$

We discuss this equation with respect to stability and consistency.

4 Convergence Analysis

In the following we present the convergence analysis of the iterative splitting method for wave equations with constant and linear time dependent diffusion coefficients.

4.1 Stability and consistency for the constant case

The stability and consistency results can be done as for the parabolic case. The operator equation with second-order time derivatives can be reformulated into a system of first-order time derivatives.

4.1.1 Consistency

In the following we analyse the consistency and the order of the local splitting error for the linear bounded operators $A, B : \mathbf{X} \rightarrow \mathbf{X}$ where $\mathbf{X}$ is a Banach-space, see [31].

We assume our Cauchy-problem for two linear operators with second-order time derivative.

$$c_{tt} - Ac - Bc = 0 \ , \ t \in (0, T) \ , \tag{21}$$

$$\text{with} \quad c(0) = c_0 \ , \ c_t(0) = c_1 \ , \tag{22}$$

where c_0 and c_1 are the initial values, see equation (1).

We rewrite (21)–(22) to a system of first order time derivatives:

$$\partial_t c_1 - c_2 = 0 \text{ in } (0, T), \tag{23}$$

$$\partial_t c_2 - Ac_1 - Bc_1 = 0 \text{ in } (0, T), \tag{24}$$

$$\text{with} \quad c_1(0) = c_0, c_2(0) = c_1. \tag{25}$$

where $c_0 = c(0)$ and $c_1 = c_t(0)$ are the initial values. The iterative operator splitting method (17)–(18) is rewritten to a system of splitting methods. The method is given by:

$$\partial_t c_{1,i} = c_{2,i}, \tag{26}$$

$$\partial_t c_{2,i} = Ac_{1,i} + Bc_{1,i-1}, \tag{27}$$

$$\text{with} \quad c_{1,i}(t^n) = c_1(t^n), \ c_{2,i}(t^n) = c_2(t^n)$$

$$\partial_t c_{1,i+1} = c_{2,i+1}, \tag{28}$$

$$\partial_t c_{2,i+1} = Ac_{1,i} + Bc_{1,i+1}, \tag{29}$$

$$\text{with} \quad c_{1,i+1}(t^n) = c_1(t^n), \ c_{2,i+1}(t^n) = c_2(t^n).$$

We start with $i = 1, 3, 5, \ldots, 2m + 1$

We can obtain consistency with the underlying fundamental solution of the equation system.

Theorem 2. *Let $A, B \in \mathcal{L}(\mathcal{X})$ be linear bounded operators. Then the abstract Cauchy problem (21)–(22) has a unique solution and the iterative splitting method (26)–(29) by $i = 1, 3, \ldots, 2m + 1$ is consistent with the order of the consistency $\mathcal{O}(\tau_n^{2m})$. The error estimate is given by:*

$$\|e_i\| = K\|B\|\tau_n\|e_{i-1}\| + \mathcal{O}(\tau_n^2)., \tag{30}$$

where $K \in \mathbb{R}^+$, $e_i = \max\{|e_{1,i}|, |e_{i,2}|\}$ and $\|B\|$ is the norm of the bounded operator B. In general, we can do an estimation by recursive arguments:

$$\|e_i\| = \tilde{K}\tau_n^i\|e_0\| + \mathcal{O}(\tau_n^{i+1}), \tag{31}$$

where $\tilde{K} \in \mathbb{R}^+$ is the growth estimation.

Proof. We derive the underlying consistency of the operator-splitting method. Let us consider the iteration (17)–(18) on the subinterval $[t^n, t^{n+1}]$. For the local error function $e_i(t) = c(t) - c_i(t)$ we have the relations

$$\begin{aligned}
&\partial_t e_{1,i}(t) = e_{2,i}(t), \quad t \in (t^n, t^{n+1}], \\
&\partial_t e_{2,i}(t) = Ae_{1,i}(t) + Be_{1,i-1}(t), \quad t \in (t^n, t^{n+1}], \\
&\partial_t e_{1,i+1}(t) = e_{2,i+1}(t), \quad t \in (t^n, t^{n+1}], \\
&\partial_t e_{2,i+1}(t) = Ae_{1,i}(t) + Be_{1,i+1}(t), \quad t \in (t^n, t^{n+1}],
\end{aligned} \tag{32}$$

for $m = 0, 2, 4, \ldots$, with $e_0(0) = 0$ and $e_{-1}(t) = c(t)$. We use the notations $\mathbf{X}^4$ for the product space $\mathbf{X} \times \mathbf{X} \times \mathbf{X} \times \mathbf{X}$ endowed with norm $||(u_1, u_2, u_3, u_4)^t|| = \max\{||u_1||, ||u_2||, ||u_3||, ||u_4||\}(u_1, u_2, u_3, u_4 \in \mathbf{X})$.

The elements $\mathcal{E}_i(t)$, $\mathcal{F}_i(t) \in \mathbf{X}^4$ and the linear operator $\mathcal{A} : \mathbf{X}^4 \to \mathbf{X}^4$ are defined as follows:

$$\mathcal{E}_i(t) = \begin{bmatrix} e_{1,i}(t) \\ e_{2,i}(t) \\ e_{1,i+1}(t) \\ e_{2,i+1}(t) \end{bmatrix}, \mathcal{F}_i(t) = \begin{bmatrix} 0 \\ Be_{1,i-1}(t) \\ 0 \\ 0 \end{bmatrix}, \mathcal{A} = \begin{bmatrix} 0 & I & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & I & 0 & I \\ A & 0 & B & 0 \end{bmatrix}. \tag{33}$$

Then, using the notations (33), the relations (32) can be written as:

$$\begin{aligned}
&\partial_{tt}\mathcal{E}_i(t) = \mathcal{A}\mathcal{E}_i(t) + \mathcal{F}_i(t), \quad t \in (t^n, t^{n+1}], \\
&\mathcal{E}_i(t^n) = 0.
\end{aligned} \tag{34}$$

Due to our assumptions, $\mathcal{A}$ is a generator of the one-parameter C_0 semi-group $(\exp \mathcal{A}t)_{t \geq 0}$. Hence, using the variations of constants formula, the solution to the abstract Cauchy problem (34) with homogeneous initial conditions can be written as:

$$\mathcal{E}_i(t) = c_0 \int_{t^n}^{t} \exp(\mathcal{A}(t-s))\mathcal{F}_i(s)ds, \tag{35}$$

with $t \in [t^n, t^{n+1}]$ (see, e.g. [6]). Hence, using the denotation

$$\|\mathcal{E}_i\|_\infty = \sup_{t \in [t^n, t^{n+1}]} \|\mathcal{E}_i(t)\| \ , \tag{36}$$

we have

$$\begin{aligned}
\|\mathcal{E}_i\|(t) &\leq \|\mathcal{F}_i\|_\infty \int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\|ds \\
&= \|B\|\|e_{1,i-1}\| \int_{t^n}^{t} \|\exp(\mathcal{A}(t-s))\|ds, \quad t \in [t^n, t^{n+1}].
\end{aligned} \tag{37}$$

Since $(\mathcal{A}(t))_{t \geq 0}$ is a semi-group, the so-called *growth estimation*

$$\|\exp(\mathcal{A}t)\| \leq K \exp(\omega t), \quad t \geq 0, \tag{38}$$

holds with numbers $K \geq 0$ and $\omega \in \mathbb{R}$, cf. [6].

The estimations (37) and (38) result in

$$\|\mathcal{E}_i\|_\infty = K\|B\|\tau_n\|e_{i-1}\| + \mathcal{O}(\tau_n^2), \tag{39}$$

where $||e_{i-1}|| = \max\{||e_{1,i-1}||, ||e_{2,i-1}||\}$.

Taking into account the definition of $\mathcal{E}_i$ and the norm $\|\cdot\|_\infty$, we obtain

$$\|e_i\| = K\|B\|\tau_n\|e_{i-1}\| + \mathcal{O}(\tau_n^2), \tag{40}$$

and hence

$$\|e_{i+1}\| = K_1\tau_n^2\|e_{i-1}\| + \mathcal{O}(\tau_n^3), \tag{41}$$

which proves our statement. □

Remark 5. *The proof is aligned to scalar temporal first-order derivatives, see [7]. The generalization for higher-order hyperbolic equations can also be done which are reformulated into first-order systems.*

4.1.2 Stability

The following stability theorem is given for the wave equation done with the iterative splitting method, see (26)–(29).

The convergence is examined in a general Banach space setting and we can prove the following stability theorem.

Theorem 3. *Let us consider the system of linear differential equation used for the spatial discretised wave equation*

$$\begin{aligned} &\partial_t c_1 = c_2, &(42)\\ &\partial_t c_2 = Ac_1 + Bc_1, &(43)\\ \textit{with}\quad &c_1(t^n) = c(t^n),\ c_2(t^n) = c_t(t^n)\ , \end{aligned}$$

where the operators $A, B : \mathbf{X} \to \mathbf{X}$ are linear and densely defined in the real Banach-space $\mathbf{X}$, see [32]. We can define a norm on the product space $\mathbf{X} \times \mathbf{X}$ with $||(u, v)^t|| = \max\{||u||, ||v||\}$. We rewrite the equation (42)–(43) and obtain

$$\begin{aligned} &\partial_t \tilde{c}(t) = \tilde{A}c(t) + \tilde{B}c(t)\ ,\\ &\tilde{c}(t^n) = \tilde{c}^n\ , \end{aligned} \tag{44}$$

where $\tilde{c}^n = (c(t^n), c_t(t^n))^T$ and $\tilde{A} = \begin{pmatrix} 0 & 1/2I \\ A & 0 \end{pmatrix}$ and $\tilde{B} = \begin{pmatrix} 0 & 1/2I \\ B & 0 \end{pmatrix}$. Let $\tilde{A}\ , \tilde{B} : \mathbf{X} \to \mathbf{X}$ be linear bounded operators which are generators of the C_0 semi-group and $c_0 \in \mathbf{X}$ a fixed element. To obtain a scalar estimation for the bounded operators A, B, we assume $\lambda_{\tilde{A}}$ as a maximal eigenvalue of $\tilde{A}$ and $\lambda_{\tilde{B}}$ as a maximal eigenvalue of $\tilde{B}$. Then the linear iterative operator-splitting method (26) - (29) is stable with the following result:

$$\|\tilde{c}_{i+1}(t^{n+1})\| \leq \tilde{K} \sum_{j=0}^{i+1} \|c_0\|\tau^j \lambda_{max}^j\ , \tag{45}$$

where $\tilde{K} > 0$ is a constant and $\tilde{c}_0 = (c(t^n), c_t(t^n))$ is the initital condition, $\tau = (t^{n+1} - t^n)$ the time step and λ_{max} the maximal eigenvalue of the linear and bounded operators $\tilde{A}$ and $\tilde{B}$.

Proof. Let us consider the iteration (26)–(29) on the subinterval $[t^n, t^{n+1}]$. Then we obtain the eigenvalues of the following linear and bounded operators. Due to the well-posed problem we have: $\lambda_{\tilde{A}}$ eigenvalue of $\tilde{A}$, $\lambda_{\tilde{B}}$ eigenvalue of $\tilde{B}$, see [32] and [13]. Then our iteration methods are given with the eigenvaules as follows:

$$\begin{aligned} &\partial_t \tilde{c}_i(t) = \lambda_{\tilde{A}} \tilde{c}_i(t)) + \lambda_{\tilde{B}} \tilde{c}_{i-1}(t), \quad t \in (t^n, t^{n+1}], \\ &\tilde{c}_i(t^n) = \tilde{c}_0 \ , \end{aligned} \tag{46}$$

and

$$\begin{aligned} &\partial_t \tilde{c}_{i+1}(t) = \lambda_{\tilde{A}} \tilde{c}_i(t) + \lambda_{\tilde{B}} \tilde{c}_{i+1}(t), \quad t \in (t^n, t^{n+1}], \\ &\tilde{c}_{i+1}(t^n) = \tilde{c}_0 \ , \end{aligned} \tag{47}$$

for $i = 1, 3, 5, \ldots$, with $\tilde{c}_0 = (c(t^n), c_t(t^n))^t$. The equations can be estimated:

$$\tilde{c}_i(t^{n+1}) = \exp(\lambda_{\tilde{A}} \tau) \tilde{c}_0 + \int_{t^n}^{t^{n+1}} \exp(\lambda_{\tilde{A}}(t-s)) \lambda_{\tilde{B}} \tilde{c}_{i-1}(s) ds,$$

where we can estimate

$$||\tilde{c}_i(t^{n+1})|| \leq K_1 ||\tilde{c}_0|| + \tau K_2 \lambda_{\tilde{B}} ||\tilde{c}_{i-1}(t^{n+1})||.$$

Further the second equation can be estimated by:

$$\tilde{c}_{i+1}(t^{n+1}) = \exp(\lambda_{\tilde{B}} \tau) \tilde{c}_0 + \int_{t^n}^{t^{n+1}} \exp(\lambda_{\tilde{B}}(t-s)) \lambda_{\tilde{A}} \tilde{c}_i(s) ds,$$

where can be estimated by:

$$||\tilde{c}_{i+1}(t^{n+1})|| \leq K_3 ||\tilde{c}_0|| + \tau K_4 \lambda_{\tilde{A}} ||| \tilde{c}_i(t^{n+1})||.$$

With a recursive argument and the maximum of the eigenvalues we can estimate the equations by:

$$\begin{aligned} ||\tilde{c}_{i+1}(t^{n+1})|| &\leq \sum_{j=0}^{i+1} K_j \tau^j ||\tilde{c}_0|| \lambda_{max}^j \ , \\ ||\tilde{c}_{i+1}(t^{n+1})|| &\leq \tilde{K} \sum_{j=0}^{i+1} \tau^j ||\tilde{c}_0|| \lambda_{max}^j \ , \end{aligned}$$

where $\tilde{K}$ is the maximum of all constants and $\lambda_{max} = \max\{\lambda_{\tilde{A}}, \lambda_{\tilde{B}}\}$. □

Remark 6. *We have stability for sufficient small time steps τ. Based on the estimation with the eigenvalues we can do the same technique also for unbounded operators that are boundable locally. More accurate estimates can be derived by using the techniques of the mild or weak solutions, see [32].*

4.2 Stability and Consistency analysis for time dependent case

We propose the following numerical methods to solve the time dependent wave equation: iterative splitting method with embedded semi-analytical initial conditions and overlapping intervals.

Assumption 1. *For the analysis with the linear time dependent coefficients we assume to have convergence of the semi-analytical solutions to the analytical.*

We give an error estimate for the standard splitting method in Theorem 4. For the iterative operator splitting method with embedded semi-analytical solutions we deal with two time partitions. An inner time partition in which we compute the numerical results and an outer time partition on which we compute the semi-analytical solution.

We have the following definition:

Definition 1. *We part the time interval* $[0, T]$ *as follows*

$$t_{i,j} = i \cdot \tau^{out} + j \cdot \tau^{in}, \quad i = 0, \dots, M-1 \text{ and } j = 0, \dots, N, \tag{48}$$

$$\tau^{out} = \frac{T}{M}, \quad \tau^{in} = \frac{\tau^{out}}{N} \tag{49}$$

where τ^{out} *denotes the outer time step size and* τ^{in} *the inner.*

For all outer time points the semi-analytical solutions are computed to initialise the inner time interval, where we apply the iterative splitting method.

Theorem 4. *The error estimation of the iterative splitting method with embedded semi-analytical solutions is given by*

$$||u_{num,i}(\tau_n) - u_{semi-anal}(\tau_n)|| \leq KN\mathcal{O}(\tau_n^i)||u(t^m)|| + \tilde{K}||e_{out}(\tau)||, \tag{50}$$

$K, N \in \mathbb{R}^+$. $u_{semi-anal}(t)$ *is the semi-analytical solution and* $u_{num,i}(t)$ *is the numerical solution with i-th iterative steps computed. The overlapping does not cause any significant error in terms of that it would be higher than without overlapping.*

We prove theorem 4 for the 2-dimensional equation in the following lemma:

Lemma 1. *The wave equation 3, with parameters* $D_1(t) = \tilde{D}_1 \frac{T-t}{T} + \tilde{D}_2 \frac{t}{T}$ *and* $D_2(t) = \tilde{D}_1 \frac{t}{T} + \tilde{D}_2 \frac{T-t}{T}$, $t \in (0, T)$ *and* $\tilde{D}_1, \tilde{D}_2 \in \mathbb{R}^+$ *is given with the semi-analytical solution by*

$$\begin{aligned}
&u_{semi-anal}(x, y, t) = \sin(\frac{1}{\sqrt{\hat{D}_1}}\pi x)\sin(\frac{1}{\sqrt{\hat{D}_2}}\pi x)\cos(\sqrt{2}\pi t),\\
&t \in (t_m, t_{m+1}), (x, y) \in \Omega,\\
&\hat{D}_1 = \tilde{D}_1(t_m), \hat{D}_2 = \tilde{D}_2(t_m), \quad m = 0, \dots, M
\end{aligned}$$

We apply the iterative splitting scheme (17)–(18) with the η-method in the sub-time-intervalls $t \in (t_m, t_{m+1})$, with time step $N\tau_n = \tau$. The local error is given by:

$$||u_{num,i}(t^{n+1}) - u_{anal}(t^n)|| \leq NKO(\tau^i)||u(t^n)|| + \tilde{K}||e_{out}(t)||, \qquad (51)$$

Proof. The error analysis is given in two parts:

- Error of the outer intervall $||u_{anal}(t) - u_{semi-anal}(t)|| \leq \tilde{K}||e_{out}(\tau)||$
- Error of the inner intervall $||u_{num,i}(\tau_n) - u_{semi-anal}(\tau_n)|| \leq KNO(\tau_n^i)||u(t^m)|| + \tilde{K}||e_{out}(\tau)||$

Part 1.)

The differential equations are given by:

$$\frac{\partial^2 u_{anal}}{\partial t^2} = D_1(t)\frac{\partial^2 u_{anal}}{\partial x^2} + D_2(t)\frac{\partial^2 u_{anal}}{\partial y^2},\ t \in (t^m, t^{m+1}) \qquad (52)$$

$$\frac{\partial^2 u_{semianal}}{\partial t^2} = \hat{D}_1\frac{\partial^2 u_{semianal}}{\partial x^2} + \hat{D}_2\frac{\partial^2 u_{semianal}}{\partial y^2},\ t \in (t^m, t^{m+1}) \qquad (53)$$

We define $e_{out}(t) = u_{anal} - u_{semi-anal}$ and obtain:

$$\frac{\partial^2 e_{out}}{\partial t^2} = (D_1(t^{m+1}) - D_1(t^m))\frac{\partial^2 e_{out}}{\partial x^2} + (D_2(t^{m+1}) - D_2(t^m))\frac{\partial^2 e_{out}}{\partial y^2},\ t \in (t^m, t^{m+1}) \qquad (54)$$

We assume $e_{out}(t) \to 0$ for $\tau \to 0$.

For all other solutions, we assume that the analytical solution $u_{semi-anal}(t^m) \leq u_{anal} \leq u_{semi-anal}(t^{m+1})$ and we estimate

$$\begin{aligned} ||e_{out}(t)|| \leq || &\sin\left(\frac{1}{\sqrt{D_1(t^{m+1}) - D_1(t^m)}}\pi x\right) \cdot \\ &\sin\left(\frac{1}{\sqrt{D_2(t^{m+1}) - D_2(t^m)}}\pi x\right) \cos(\sqrt{2}\pi t) e_{out}(t^m)||, \\ &t \in (t_m, t_{m+1}), (x, y) \in \Omega, \end{aligned}$$

where $e_{out}(t^m)$ is sufficient small as an initial condition.

Part 2.)

Here we have to use the local error of the iterative splitting method, see previous section 4.1:

$$\|e_i\| = K\|B\|\tau_n\|e_{i-1}\| + \mathcal{O}(\tau_n^2),$$

where $e_i = \max\{|e_{1,i}|, |e_{i,2}|\}$.

Thus, the combination of both parts reflects the numerical error. □

For the additional overlapping scheme we have only to prove the error of the overlapping regions.

Lemma 2. *In general, we assume that in each time step the error can be bounded by:*

$$\epsilon \geq \epsilon(t) = \sum_{i=1}^{m} |u_{num,i}(t) - u_{ana,i}| \Delta x \Delta y$$

where $m = (x_{steps} + 1)(y_{steps} + 1)$ *is the number of the spatial steps in the overlapping area. So that the error for the overlapped time steps* t_1 *and* t_2 *can be bounded by:*

$$\epsilon_{overlap} \leq \epsilon$$

Proof. The error in t_1 and t_2 is given by

$$\epsilon_{overlap}(t_1) = \sum_{i=1}^{m} |u_{num,i}(t_1) - u_{ana,i}(t_1)| \Delta x \Delta y \leq \epsilon$$
$$\epsilon_{overlap}(t_2) = \sum_{i=1}^{m} |u_{num,i}(t_2) - u_{ana,i}(t_2)| \Delta x \Delta y \leq \epsilon$$

We obtain by overlapping, i.e. averaging the values in t_1 and t_2

$$\frac{\epsilon_{overlap}(t_1) + \epsilon_{overlap}(t_2)}{2} \leq \epsilon$$

Hence, the average error is also in the assumed error bound for the overlapped area. Thus, for a given error bound ϵ, the overlap error is below the error bound for sufficient large m. □

5 Discretization and Assembling

We exercise our theory for the two dimensional wave equation. More dimensional equations can be treated analogously, particularly, we did an implementation for the three dimensional case, too. First we present methods (finite difference and iterative methods) to solve the wave equation with constant diffusion coefficients. Besides their benefit for their own, the methods provide basic tools to solve the linearly time dependent equation. By adding overlapping schemes we have all tools at hand to deal with the latter case which we present in a second step.

5.1 Wave equation with constant coefficients

5.1.1 Finite Difference Discretization

For getting an iterative method for the two dimensional wave equation we first have to apply the η-method together with the mixed forward-backward difference method in time and get:

$$\begin{aligned} c^{n+1} &- 2c^n + c^{n-1} \\ &= \Delta t^2 \left(D_1(t^{n+1})\eta \frac{\partial^2 c^{n+1}}{\partial x^2} + D_1(t^n)\,(1-2\eta)\,\frac{\partial^2 c^n}{\partial x^2} + D_1(t^{n-1})\eta \frac{\partial^2 c^{n-1}}{\partial x^2} \right) \\ &+ \Delta t^2 \left(D_2(t^{n+1})\eta \frac{\partial^2 c^{n+1}}{\partial y^2} + D_2(t^{n+1})\,(1-2\eta)\,\frac{\partial^2 c^n}{\partial y^2} + D_2(t^{n-1})\eta \frac{\partial^2 c^{n-1}}{\partial y^2} \right). \end{aligned} \tag{55}$$

Now we can apply the mixed forward-backward difference method in space defined as:

$$\begin{aligned} \frac{\partial^2 c(x_i, y_j)}{\partial x^2} &\approx \frac{1}{\Delta x^2}\left(c\left(x_i + \Delta x, y_j\right) - 2c\left(x_i, y_j\right) + c\left(x_i - \Delta x, y_j\right)\right), \\ \frac{\partial^2 c(x_i, y_j)}{\partial y^2} &\approx \frac{1}{\Delta y^2}\left(c\left(x_i, y_j + \Delta y\right) - 2c\left(x_i, y_j\right) + c\left(x_i, y_j - \Delta y\right)\right), \end{aligned}$$

end get for the uncoupled wave equation (**??**):

$$\begin{aligned} c^{n+1} &- 2c^n + c^{n-1} \\ &= \frac{\Delta t^2}{\Delta x^2} D_1(\eta(c^{n+1}_{i+1,j} - 2c^{n+1}_{i,j} + c^{n+1}_{i-1,j}) + (1-2\eta)(c^n_{i+1,j} - 2c^n_{i,j} + c^n_{i-1,j}) \\ &\quad +\eta(c^{n-1}_{i+1,j} - 2c^{n-1}_{i,j} + c^{n-1}_{i-1,j})) \\ &+ \frac{\Delta t^2}{\Delta y^2} D_2(\eta(c^{n+1}_{i,j+1} - 2c^{n+1}_{i,j} + c^{n+1}_{i,j-1}) + (1-2\eta)(c^n_{i,j+1} - 2c^n_{i,j} + c^n_{i,j-1}) \\ &\quad +\eta(c^{n-1}_{i,j+1} - 2c^{n-1}_{i,j} + c^{n-1}_{i,j-1})). \end{aligned} \tag{56}$$

5.2 Application of the Sequential Operator-Splitting Method

In the classical operator-splitting method we part every time step into two smaller substeps to get a better accuracy.
In the first step the partial derivative in x will be discretised implicit. The partial derivative in y will be discretised explicit. In the second step it is the other way around.
We exemplify the idea with the two dimensional wave equation. Thus we get form (55):

$$\begin{aligned} 1)\,\tilde{c} &- 2c^n + c^{n-1} \\ &= \Delta t^2 D_1 \left(\eta \frac{\partial^2 \tilde{c}}{\partial x^2} + (1-2\eta)\,\frac{\partial^2 c^n}{\partial x^2} + \eta \frac{\partial^2 c^{n-1}}{\partial x^2} \right) \\ &+ \Delta t^2 D_2 \frac{\partial^2 c^n}{\partial y^2}, \end{aligned} \tag{57}$$

$$
\begin{aligned}
2)\, c^{n+1} &- 2c^n + c^{n-1} \\
&= \Delta t^2 D_1 \left(\eta \frac{\partial^2 \tilde{c}}{\partial x^2} + (1-2\eta) \frac{\partial^2 c^n}{\partial x^2} + \eta \frac{\partial^2 c^{n-1}}{\partial x^2} \right) \\
&+ \Delta t^2 D_2 \left(\eta \frac{\partial^2 c^{n+1}}{\partial y^2} + (1-2\eta) \frac{\partial^2 c^n}{\partial y^2} + \eta \frac{\partial^2 c^{n-1}}{\partial y^2} \right).
\end{aligned} \tag{58}
$$

5.3 Application of the Iterative Operator-Splitting Method

In the two dimensional iterative operator-splitting method we will change the algorithm (57)-(58) in such a way that the result of the second step will be again used in the first step. Thus we get an iterative method with an initial value c^{i-1}:

$$
\begin{aligned}
1)\, c^{i} &- 2c^n + c^{n-1} \\
&= \Delta t^2 D_1 \left(\eta \frac{\partial^2 c^{i}}{\partial x^2} + (1-2\eta) \frac{\partial^2 c^n}{\partial x^2} + \eta \frac{\partial^2 c^{n-1}}{\partial x^2} \right) \\
&+ \Delta t^2 D_2 \left(\eta \frac{\partial^2 c^{i-1}}{\partial y^2} + (1-2\eta) \frac{\partial^2 c^n}{\partial y^2} + \eta \frac{\partial^2 c^{n-1}}{\partial y^2} \right),
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
2)\, c^{i+1} &- 2c^n + c^{n-1} \\
&= \Delta t^2 D_1 \left(\eta \frac{\partial^2 c^{i}}{\partial x^2} + (1-2\eta) \frac{\partial^2 c^n}{\partial x^2} + \eta \frac{\partial^2 c^{n-1}}{\partial x^2} \right) \\
&+ \Delta t^2 D_2 \left(\eta \frac{\partial^2 c^{i+1}}{\partial y^2} + (1-2\eta) \frac{\partial^2 c^n}{\partial y^2} + \eta \frac{\partial^2 c^{n-1}}{\partial y^2} \right).
\end{aligned} \tag{60}
$$

Now we have an iterative operator-splitting method that stops by achieving a given iteration depth or a given error tolerance

$$||c^i - c^{i-2}|| \leq TOL.$$

Hereafter the numerical result for the function c at time point $n+1$ is given by:

$$c^{n+1} := c^{i+1,n+1} = c^{i+1}.$$

For the stability of the function it is important to start the iterative algorithm with a good initial value $c^{i-1,n+1} = c^{i-1}$. Some options for their choice are given in the following subsection.

5.3.1 Initial Conditions for the Iteration

I.1)
The easiest initial condition for our $c^{i-1,n+1}$ is given by the zero vector, $c^{i-1,n+1} \equiv 0$, but it might be a bad choice, if the stability depends on the initial value.

I.2)
A better variant would be to set the initial value to be the result of the last step, $c^{i-1,n+1} = c^n$. Thus the initial value might be next to c^{n+1}, which would be a better start for the iteration.
I.3)
With using the average growth of the function depending on the time, the function at the time point $n+1$ might be even better guessed: $c^{i-1,n+1} = c^n + \frac{1}{\Delta t} \cdot (c^n - c^{n-1})$.
I.4)
A better initial value can be achieved by calculating it with using a method for the first step. The easier one is the explicit method,

$$\begin{aligned} c^{i-1,n+1} &- 2c^n + c^{n-1} \\ &= \Delta t^2 (D_1 \frac{\partial^2 c^n}{\partial x^2} + D_2 \frac{\partial^2 c^n}{\partial y^2}). \end{aligned}$$

I.5)
The prestepping method might be the best of the ones described in this section because the iteration starts next to the value of c^{n+1}.

5.4 Discretization and Assembling

Discretising the algorithm of the iterative operator-splitting method (59)–(60) analogously to (56), we get the following scheme for the two dimensional wave equation:

$$\begin{aligned} 1)\Delta x^2 \Delta y^2 c^i_{k,l} &- \Delta t^2 \Delta y^2 D_1(t^{n+1}) \eta (c^i_{k+1,l} - 2c^i_{k,l} + c^i_{k-1,l}) \\ &= 2\Delta x^2 \Delta y^2 c^n_{k,l} + \Delta t^2 \Delta y^2 D_1(t^n)(1-2\eta)(c^n_{k+1,l} - 2c^n_{k,l} + c^n_{k-1,l}) \\ &\quad + \Delta t^2 \Delta x^2 D_2(t^n)(1-2\eta)(c^n_{k,l+1} - 2c^n_{k,l} + c^n_{k,l+1}) \\ &- \Delta x^2 \Delta y^2 c^{n-1}_{k,l} + \Delta t^2 \Delta y^2 D_1(t^{n-1}) \eta (c^{n-1}_{k+1,l} - 2c^{n-1}_{k,l} + c^{n-1}_{k-1,l}) \quad (61) \\ &\quad + \Delta t^2 \Delta x^2 D_2(t^{n-1}) \eta (c^{n-1}_{k,l+1} - 2c^{n-1}_{k,l} + c^{n-1}_{k,l-1}) \\ &+ \Delta t^2 \Delta x^2 D_2(t^{n+1}) \eta (c^{i-1}_{k,l+1} - 2c^{i-1}_{k,l} + c^{i-1}_{k,l-1}), \end{aligned}$$

$$\begin{aligned} 2)\Delta x^2 \Delta y^2 c^{i+1}_{k,l} &- \Delta t^2 \Delta x^2 D_2(t^{n+1}) \eta (c^{i+1}_{k,l+1} - 2c^{i+1}_{k,l} + c^{i+1}_{k,l-1}) \\ &= 2\Delta x^2 \Delta y^2 c^n_{k,l} + \Delta t^2 \Delta y^2 D_1(1-2\eta)(c^n_{k+1,l} - 2c^n_{k,l} + c^n_{k-1,l}) \\ &\quad + \Delta t^2 \Delta x^2 D_2(t^n)(1-2\eta)(c^n_{k,l+1} - 2c^n_{k,l} + c^n_{k,l+1}) \\ &- \Delta x^2 \Delta y^2 c^{n-1}_{k,l} + \Delta t^2 \Delta y^2 D_1(t^{n-1}) \eta (c^{n-1}_{k+1,l} - 2c^{n-1}_{k,l} + c^{n-1}_{k-1,l}) \quad (62) \\ &\quad + \Delta t^2 \Delta x^2 D_2(t^{n-1}) \eta (c^{n-1}_{k,l+1} - 2c^{n-1}_{k,l} + c^{n-1}_{k,l-1}) \\ &+ \Delta t^2 \Delta y^2 D_1(t^{n+1}) \eta (c^i_{k+1,l} - 2c^i_{k,l} + c^i_{k-1,l}). \end{aligned}$$

This can be written in a matrix scheme as follows:

1) $c^i = (Sys_i^1)^{-1}(t^{n+1}) \cdot (Sys_{Alt}(t^{n+1}) \cdot c^{i-1} + InterB^1(t^n) \cdot c^n + InterC^1(t^{n-1}) \cdot c^{n-1})$,
2) $c^{i+1} = (Sys_{Neu})^{-1}(t^{n+1}) \cdot (Sys_i^2(t^{n+1}) \cdot c^i + InterB^2(t^n) \cdot c^n + InterC^2(t^{n-1}) \cdot c^{n-1})$.

With this scheme the sequence c_i can be calculated only with the results of the last steps. It ends when the given error tolerance is achieved. The matrices only have to be calculated once in the program. They do not change during the iteration.

The matrices $Sys_i^d, Sys_{Oldj}^d, Sys_{Newj}^d, InterB^d$ and $InterC^d$ depend on the solutions at different time levels, i.e. $c^{i-1}, c^i, c^{i+1}, c^{n-1}$ and c^n.

5.5 Wave equation with linear time dependent diffusion coefficients

The main idea to solve the time dependent wave equation with linear diffusion functions is to part the time domain $[0, T]$ into sub-intervals at which we assume equations with constant diffusion coefficients on each of the sub-intervals. Hence, we reduce the problem of the time depedent wave equation to the one with constant diffusion coefficients.

Mathematically, given:

$$\frac{\partial^2 c}{\partial t^2} = D_1(t)\frac{\partial^2 c}{\partial x^2} + D_2(t)\frac{\partial^2 c}{\partial y^2}, \quad (x, y, t) \in \Omega \times [0, T] \tag{63}$$

$$D_1(t) = \frac{d_2 - d_1}{T} + d_1, \tag{64}$$

$$D_2(t) = \frac{d_1 - d_2}{T} + d_2, \quad d_1, d_2 \in [0, 1]. \tag{65}$$

The partition of $[0, T]$ is given by:

$$t_{i,j} = i \cdot \tau^{out} + j \cdot \tau^{in}, \quad i = 0, \ldots, M-1 \text{ and } j = 0, \ldots, N, \tag{66}$$

$$\tau^{out} = \frac{T}{M}, \quad \tau^{in} = \frac{\tau^{out}}{N} \tag{67}$$

where τ^{out} denotes the outer time step size and τ^{in} the inner.

We have the following system of wave equations with constant diffusion coefficients on the sub-intervals $[t_{i,0}, t_{i,N}]$ $(i = 0, \ldots, M-1)$:

$$\frac{\partial^2 c^i}{\partial t^2} = D_1(t_{i,0})\frac{\partial^2 c^i}{\partial x^2} + D_2(t_{i,0})\frac{\partial^2 c^i}{\partial y^2}, \quad (x, y, t) \in \Omega \times [t_{i,0}, t_{i,N}]. \tag{68}$$

$$\tag{69}$$

For each sub-interval $[t_{i,0}, t_{i,N}]$ $(i = 0, \ldots, M-1)$ we can make use of the results in 4.1. In particular, we can give an analytical solution by:

$$c^i_{anal}(x,y,t) = \sin(\frac{1}{\sqrt{D_1(t_{i,0})}}\pi x) \cdot \sin(\frac{1}{\sqrt{D_2(t_{i,0})}}\pi y) \cdot \cos(\sqrt{2}\pi t), \quad (70)$$

$$(x,y,t) \in \Omega \times [t_{i,0}, t_{i,N}], \quad i = 0, \ldots, M-1. \quad (71)$$

$$(72)$$

Thus we assume for each $i = 0, \ldots, M-1$ following initial and boundary conditions for 68:

$$c^0(x,y,0) = c^0_{anal}(x,y,0), \quad (x,y) \in \Omega, \quad (73)$$

$$c^i(x,y,t) = c^i_{anal}(x,y,t), \quad \text{on } \partial\Omega \times [0,T]. \quad (74)$$

Furthermore, we can make use of the numeric methods, developed for the wave equation with constant diffusion-coefficients, to give a discretisation and assembling for each sub-interval, see 5.1. We obtain a numerical, resp. semi-analytical, solution for the time depedent equation (63) in $\Omega \times [0,T]$ by joining the results c^i of all sub-intervals $[t_{i,0}, t_{i,N}]$ $(i = 0, \ldots, M-1)$. In 4.2 we show that the semi-analytical solution converges to the presumed analytical solution for $\tau^{out} \to 0$. We need the semi-analytical solution as reference solution in order to be able to evaluate the numerical.

In order to reach a more accurate result we propose an interval-overlapping method. Let $p \in \{0, \ldots, [\frac{N}{2}]\}$. We solve the following system:

$$\frac{\partial^2 c^0}{\partial t^2} = D_1(t_{0,0})\frac{\partial^2 c^0}{\partial x^2} + D_2(t_{0,0})\frac{\partial^2 c^0}{\partial y^2}, \quad (75)$$

$$(x,y,t) \in \Omega \times [0, t_{0,N} + p\tau^{in}],$$

$$\frac{\partial^2 c^i}{\partial t^2} = D_1(t_{i,0})\frac{\partial^2 c^i}{\partial x^2} + D_2(t_{i,0})\frac{\partial^2 c^i}{\partial y^2}, \quad (76)$$

$$(x,y,t) \in \Omega \times [t_{i,0} - p\tau^{in}, t_{i,N} + p\tau^{in}], \quad i = 1, \ldots, M-2,$$

$$\frac{\partial^2 c^{M-1}}{\partial t^2} = D_1(t_{M-1,0})\frac{\partial^2 c^{M-1}}{\partial x^2} + D_2(t_{M-1,0})\frac{\partial^2 c^{M-1}}{\partial y^2}, \quad (77)$$

$$(x,y,t) \in \Omega \times [t_{M-1,0} - p\tau^{in}, T],$$

while the initial and boundary conditions are as previously set.

We present the interval-overlapping for the analytical solutions of (75)–(77). Hence, $c_{semi-anal}(x,y,t)$ is

$$\begin{cases} c^0_{anal}(x,y,t), & \text{for } (x,y,t) \in \Omega \times [0, t^0_N - p\tau^{in}]; \\ c^i_{anal}(x,y,t), & \text{for } (x,y,t) \in \Omega \times [t^i_0 + p\tau^{in}, t^i_N - p\tau^{in}], \\ & i = 1, \ldots, M-1; \\ \frac{1}{2}(c^{i-1}_{anal}(x,y,t) + c^i_{anal}(x,y,t)) & \text{for } (x,y,t) \in \Omega \times [t^{i-1}_N - p\tau^{in}, t^i_0 + p\tau^{in}], \\ & i = 1, \ldots, M-1; \\ c^{M-1}_{anal}(x,y,t) & \text{for } (x,y,t) \in \Omega \times [t^{M-1}_0 + p\tau^{in}, T]. \end{cases}$$

The same can be done analogously for the numerical solution.

6 Numerical experiments

We test our methods for the two dimensional wave equation. First we analyse test series for the constant coefficient wave equation. Here, we give some general remarks on how to carry out the experiments, e.g. choise of parameters, and how to interpret the test series correctly, e.g. CFL condition. Moreover, we present a method how to obtain acceptable accuracy with a minimum of cost. In a second step we do an error analysis for the wave equation with linearly time dependent diffusion coefficients. The tables are given at the end of the paper.

6.1 Wave equation with constant diffusion coefficients

The PDE to solve with our numerical methods is given by:

$$\frac{\partial^2 c}{\partial t^2} = D_1 \frac{\partial^2 c}{\partial x^2} + D_2 \frac{\partial^2 c}{\partial y^2}.$$

We assume Dirichlet boundary conditions:

$$u = u_D \text{ on } \partial\Omega_{Dirich} \quad \text{with}$$
$$u_D(x,y) = sin(\frac{1}{\sqrt{D_1}}\pi x) \cdot sin(\frac{1}{\sqrt{D_2}}\pi y)$$

We can derive an analytical solution which we will use as reference solution for the error estimates:

$$c(x,y,t) = \sin(\frac{1}{\sqrt{D_1}}\pi x_1) \cdot \sin(\frac{1}{\sqrt{D_2}}\pi y) \cdot \cos(\sqrt{2}\pi t)$$

The analytical solution is periodic. Thus it suffices to do the error analysis for the following domain:

$$x \in [0, 2 \cdot D_1]$$
$$y \in [0, 2 \cdot D_2]$$
$$t \in [0, \sqrt{2}]$$

Remark 7. *The analytical solutions for the constant coefficients are given exact solutions for* $t = \frac{n}{\sqrt{2}}$*, for this we obtain the boundary conditons of the solutions. The extrem values are given with respect to* $\cos(\sqrt{2}\pi t) = \pm 0.5$.

We consider stiff and non stiff equations with $D_1, D_2 \in [0, 1]$. In section 5 we gave some options for the initial condition to start the iterative method. In [12] we discussed the optimization with respect to the initialisation process. Here the best initialisation is obtained by a prestep first order method, **I.5**. However, this option needs one more iteration step. Thus we take the explicit method **I.4** for our experiment which delivers almost optimal results.

As already mentioned above we take the analytical solution as reference function and consider an average of L^1-errors over time calculated by:

$$err_{L_1}(t^n) := \sum_{i,j} |u(x^i, y^j, t^n) - u_{anal}(x^i, y^j, t^n)| \cdot \Delta x \cdot \Delta y \tag{78}$$

$$err_{L_1} := \sum_{n} err_{L_1}(t^n) \cdot \Delta t \tag{79}$$

We exercised experiments for non stiff (table (1) and (2)) and stiff (table (3) and (4)) equations while we changed the parameters η and Δt for constant spatial discretisation. Generally, we see that the test series for the stiff equation deliver better results than the one for the non stiff equation. This can be deduced to the smaller spatial grid, see domain restrictions.

In table (1)–(4) we observe that we obtain the best result for $\eta = 0$ and $tsteps = 16$, e.g. for the explicit method. However, for smaller time steps we can always find an η, e.g. implicit method, so that the L^1-error is within an acceptable range. The benefit of the implicit methods is the reduction in computational time, see table (6), with a small loss in accuracy.

During our experiments we observed a correlation between η and Δt. It appears that for each given number of time steps there is an η that minimises the L^1-error indepedently of the equation's stiffness. In tables (1)–(4) we have just listed these numerically computed η's with some additional values to see the movement. We experimented with up to three decimal places for η. We assume, however, that you can minimise the error more if you increase the number of decimal places. This leads us to the idea that for each given time step size there may exist a weight function ω of Δt with which we can obtain a optimal η to reduce the error. We assume that this phenomenon is closely related to the CFL condition and shall give a brief survey on it in the follwing section.

6.2 CFL condition

We look at the CFL condition for the methods in use, see [12], which is given by:

$$\Delta t \leq \frac{1}{\sqrt{D_{max}}} \frac{x_{min}}{\sqrt{2}} \frac{1}{\sqrt{1-2\eta}},$$

where Δt, $D_{max} = \max\{D_1, D_2\}$, $x_{min} = \min\{\Delta x, \Delta y\}$ for $\Delta x = \frac{2 \cdot D_1}{xsteps}$ and $\Delta y = \frac{2 \cdot D_2}{ysteps}$. Based on the observations in tables (1)–(4) we assume that we need to take an additional value into account to achieve optimal results:

$$\omega(\Delta t) = \frac{x_{min}}{\sqrt{2(\Delta t)^2 D_{max}(1-2\eta)}},$$

where ω may be thought of as a weight function of the CFL condition. In table (5) we calculated ω for the numerically obtained optimal pairs of η and $tsteps$ from the tables (1) and (2). Then, we applied a linear regression to the values in table (5) with respect to Δt and found the linear function

$$\omega(\Delta t) = 9.298 \Delta t + 0.2245. \tag{80}$$

With this function at hand, we can determine an ω for every Δt. We can use this ω to calculate an optimal η with respect to Δt in order to minimise the numerical error. Hence, we have a tool to minimise costs without loosing much accuracy. We think that it is even possible to have more accurate ω-functions based on the accuracy of the optimal η with respect to $tsteps$ which we had calculated before to gain ω via linear regression. We will follow this interesting issue in our future work.

Finally, we present test series where we changed the number of iterations in table (7). For different number of time steps we choose the correlated η with the smallest error and exercise on them different types of iteration. We do not observe any significant difference.

Remark 8. *In the numerical experiments we can see the benefit of applying less iterative steps, because of the sufficient accuracy of the method. Thus $i = 2, 3$ is sufficient. The optimal iterative steps are realted to the order of the time- and spatial discretisation, see [12]. This means that with time and spatial discretisation orders of $O(\Delta t^q)$ and Δx^p the number of iterative steps are $i = \min p, q$, while we assume to have optimal CFL condition. The optimisation in the spatial and time discretisation can be derived from the CFL condition. Here we obtain at least second order methods. The explicit methods are more accurate but need higher computational time, so that we have to balance between sufficient accuracy of the solutions and low computational time achieved by implicit methods, where we can minimise the error using the wight function ω.*

6.3 Wave equation with linearly time dependent diffusion coefficients

We carried out the experiments for the following time dependent PDE:

$$\frac{\partial^2 c}{\partial t^2} = D_1(t)\frac{\partial^2 c}{\partial x^2} + D_2(t)\frac{\partial^2 c}{\partial y^2}, \quad (x,y,t) \in [0,2]\times[0,2]\times[0,\sqrt{2}]$$
$$D_1(t) = \frac{1/1000 - 1}{T} + 1,$$
$$D_2(t) = \frac{1 - 1/1000}{T} + 1/1000.$$

For the experiments we fixe the spatial step sizes Δx and Δy, the iteration depths, η and the inner time step size τ^{in} and change the length of the overlapped region p and the number of outer time steps. We proved that the smaller τ^{out} the closer the numerical (resp. semi-analytical) solution to the assumed analytical. For all subintervals we choose one η and τ^{in} optimally in accordance with our analysis in section 6.2.

We consider L^1-errors over the complete time domain, see (78)–(79), while we take as compare functions the semi-analytical solutions.

In table (8) we compare the L^1-error for different values of p and $tsteps^{out}$. We do not see any significant difference when altering p. This may be a reassurement of what we proved in lemma 2. However, we can observe a considerable decrease of the L^1-error increasing the outer time steps.

Thus, in our next experiment, reflected in table (9), we fixe $p = 4$, too, and only alter $tsteps^{out}$. We can observe that the error diminshes significantly while raising the number of outer time steps.

7 Conclusions and Discussions

We have presented a new iterative splitting methods to solve time dependent wave equations. Based on a overlapping scheme we could obtain more accurate results of the splitting scheme.

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8 Appendix: Regression (Least Square Approximation) for extrapolation of functions

Here we have points with values and we assume to have a best approximation with respect to following minimisation:

$$S = \sum_{k=1}^{m} (y_k - L_n(x_k))^2,$$

where $m \geq n$, y_k are the values for the regression and L_n is a function, e.g. polynom, exponential function, etc. that is constructed with the least square algorithm, see [**?**].

9 Tables

η	ErrL1	ErrL1	ErrL1	ErrL1	ErrL1
tsteps	2	3	4	6	8
0.0	3.4461e+01	4.9988e+00	7.1888e-01	1.9701e-01	8.9679e-02
0.02	2.1240e+01	2.9023e+00	4.9343e-01	1.4000e-01	6.1068e-02
0.037	1.4416e+01	1.7632e+00	3.3421e-01	9.4631e-02	3.7458e-02
0.052	1.0303e+01	1.0795e+00	2.1622e-01	5.6866e-02	1.7159e-02
0.065	7.6778e+00	6.5839e-01	1.2939e-01	2.5795e-02	3.5487e-05
0.076	5.9393e+00	3.9440e-01	6.6075e-02	6.7007e-04	1.4458e-02
0.09	4.1978e+00	1.5079e-01	2.3614e-03	3.0836e-02	3.2885e-02
0.102	3.1107e+00	4.7987e-03	5.7883e-02	5.8181e-02	4.8735e-02
0.154	5.0867e-02	5.4006e-01	3.2383e-01	1.7908e-01	1.1781e-01
0.2	1.3615e+00	9.7161e-01	5.6802e-01	2.8732e-01	1.7912e-01
0.3	2.6516e+00	1.6138e+00	1.0523e+00	5.1841e-01	3.1147e-01
0.4	3.2841e+00	1.9514e+00	1.4344e+00	7.3493e-01	4.4047e-01
0.5	3.7528e+00	2.1258e+00	1.7175e+00	9.3132e-01	5.6435e-01

Table 1: $D_1 = 1$, $D_2 = 1$, $\Delta x = \Delta y = \frac{1}{8}$, iter_depth= 2

η	ErrL1	ErrL1	ErrL1	ErrL1	ErrL1
tsteps	10	12	14	16	32
0.0	4.5568e-02	2.2517e-02	8.8274e-03	1.0083e-15	2.1641e-02
0.02	2.7865e-02	1.0372e-02	5.2088e-05	6.8018e-03	2.3353e-02
0.037	1.3056e-02	1.4938e-04	7.5747e-03	1.2583e-02	2.4807e-02
0.052	1.7048e-04	8.8323e-03	1.4213e-02	1.7683e-02	2.6091e-02
0.065	1.0931e-02	1.6621e-02	1.9967e-02	2.2103e-02	2.7203e-02
0.076	2.0338e-02	2.3214e-02	2.4836e-02	2.5842e-02	2.8144e-02
0.09	3.2324e-02	3.1607e-02	3.1032e-02	3.0600e-02	2.9341e-02
0.102	4.2609e-02	3.8803e-02	3.6342e-02	3.4678e-02	3.0367e-02
0.154	8.7245e-02	6.9988e-02	5.9344e-02	5.2335e-02	3.4811e-02
0.2	1.2674e-01	9.7551e-02	7.9667e-02	6.7935e-02	3.8739e-02
0.3	2.1221e-01	1.5725e-01	1.2371e-01	1.0176e-01	4.7271e-02
0.4	2.9654e-01	2.1645e-01	1.6748e-01	1.3542e-01	5.5789e-02
0.5	3.7915e-01	2.7491e-01	2.1089e-01	1.6887e-01	6.4294e-02

Table 2: $D_1 = 1$, $D_2 = 1$, $\Delta x = \Delta y = \frac{1}{8}$, iter_depth= 2

η	ErrL1	ErrL1	ErrL1	ErrL1	ErrL1
tsteps	2	3	4	6	8
0.0	1.0897e+00	1.5808e-01	2.2733e-02	6.2300e-03	2.8359e-03
0.02	6.7167e-01	9.1778e-02	1.5604e-02	4.4272e-03	1.9311e-03
0.037	4.5587e-01	5.5758e-02	1.0569e-02	2.9925e-03	1.1845e-03
0.052	3.2582e-01	3.4137e-02	6.8376e-03	1.7983e-03	5.4261e-04
0.065	2.4279e-01	2.0820e-02	4.0918e-03	8.1571e-04	1.1222e-06
0.076	1.8782e-01	1.2472e-02	2.0895e-03	2.1189e-05	4.5719e-04
0.09	1.3274e-01	4.7685e-03	7.4675e-05	9.7511e-04	1.0399e-03
0.102	9.5563e-02	1.5175e-04	1.8304e-03	1.8399e-03	1.5412e-03
0.154	1.6086e-03	1.7078e-02	1.0240e-02	5.6631e-03	3.7256e-03
0.2	4.3053e-02	3.0725e-02	1.7962e-02	9.0860e-03	5.6644e-03
0.3	8.3850e-02	5.1033e-02	3.3278e-02	1.6394e-02	9.8494e-03
0.4	1.0385e-01	6.1709e-02	4.5360e-02	2.3241e-02	1.3929e-02
0.5	1.1867e-01	6.7225e-02	5.4311e-02	2.9451e-02	1.7846e-02

Table 3: $D_1 = 1$, $D_2 = 1/1000$, $\Delta x = \frac{1}{8}\Delta y = \frac{1}{8 \cdot \sqrt{1000}}$, iter_depth= 2

η	ErrL1	ErrL1	ErrL1	ErrL1	ErrL1
tsteps	10	12	14	16	32
0.0	1.4410e-03	7.1204e-04	2.7915e-04	1.0391e-16	6.8434e-04
0.02	8.8117e-04	3.2800e-04	1.6472e-06	2.1509e-04	7.3847e-04
0.037	4.1288e-04	4.7238e-06	2.3953e-04	3.9791e-04	7.8448e-04
0.052	5.3910e-06	2.7930e-04	4.4946e-04	5.5919e-04	8.2506e-04
0.065	3.4567e-04	5.2560e-04	6.3142e-04	6.9896e-04	8.6023e-04
0.076	6.4313e-04	7.3408e-04	7.8538e-04	8.1720e-04	8.8998e-04
0.09	1.0222e-03	9.9950e-04	9.8132e-04	9.6767e-04	9.2783e-04
0.102	1.3474e-03	1.2271e-03	1.1492e-03	1.0966e-03	9.6028e-04
0.154	2.7589e-03	2.2132e-03	1.8766e-03	1.6550e-03	1.1008e-03
0.2	4.0079e-03	3.0848e-03	2.5193e-03	2.1483e-03	1.2250e-03
0.3	6.7107e-03	4.9728e-03	3.9121e-03	3.2179e-03	1.4948e-03
0.4	9.3775e-03	6.8447e-03	5.2963e-03	4.2823e-03	1.7642e-03
0.5	1.1990e-02	8.6935e-03	6.6690e-03	5.3403e-03	2.0332e-03

Table 4: $D_1 = 1$, $D_2 = 1/1000$, $\Delta x = \frac{1}{8}\Delta y = \frac{1}{8 \cdot \sqrt{1000}}$, iter_depth= 2

tsteps	η	ω
16	0	1.0000
14	0.02	1.1198
12	0.037	1.2831
10	0.052	1.5145
8	0.065	1.8655
6	0.076	2.4557
4	0.09	3.6222
3	0.102	4.7583
2	0.154	6.6549

Table 5: Calculating ω for different values of dt and η. $D1 = D2 = 1$, $dx = dy = 1/8$, $ttop = sqrt(2)$.

η	tsteps	Comp.Time [msec]
0.154 (implicit)	2	133
0 (explicit)	16	308

Table 6: Computational time of the explicit and implicit schemes.

# iter	ErrL1	ErrL1	ErrL1	ErrL1
tsteps	4	8	12	16
1	1.1990e-03	1.3095e-04	1.3503e-04	1.0083e-15
2	2.3614e-03	3.5487e-05	1.4938e-04	1.0083e-15
4	2.3873e-03	3.5486e-05	1.4938e-04	1.0083e-15
6	2.3873e-03	3.5486e-05	1.4938e-04	1.0083e-15

Table 7: $\Delta x = \Delta y = \frac{1}{8}$. For each tsteps we take the η with the best result from table 1 and 2

$tsteps^{out}$	ErrL1	ErrL1	ErrL1	ErrL1
p	4	8	16	32
5	0.5551	0.5773	0.5754	0.4877
10	0.3178	0.3453	0.3853	0.4133
20	0.1695	0.1864	0.2169	0.2633
40	0.0865	0.0957	0.1133	0.1446

Table 8: $\Delta x = \Delta y = \frac{1}{64}$, iter_depth= 2, $\eta = 0$ and tsteps= 64.

$tsteps^{out}$	ErrL1
5	0.5551
10	0.3178
20	0.1695
40	0.0865
80	0.0435

Table 9: $\Delta x = \Delta y = \frac{1}{64}$, iter_depth= 2, $\eta = 0$, tsteps= 64 and $p = 4$.