On the existence of weak solutions to a coupled system of two turbulent flows
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Joachim Naumann
Jörg Wolf

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Abstract

In this paper, we study a model problem for the stationary turbulent motion of two fluids in disjoint bounded domains \( \Omega_1 \) and \( \Omega_2 \) such that \( \Gamma := \bar{\Omega}_1 \cap \bar{\Omega}_2 \neq \emptyset \). The specific difficulty of this problem arises from the boundary condition which characterizes the interaction of the fluid motions along \( \Gamma \).

We prove the existence of a weak solution to the problem under consideration which is more regular than the solution obtained in [3]. Moreover, we establish some regularity results for any weak solution. Our discussion is heavily based on the results in appendices 1 and 2 which seem to be of independent interest.
1. Introduction

Let $\Omega_1$ and $\Omega_2$ be bounded domains in $\mathbb{R}^d$ ($d = 2$ or $d = 3$) such that

\[
\Omega_1 \cap \Omega_2 = \emptyset, \quad \Gamma := \overline{\Omega}_1 \cap \overline{\Omega}_2 \neq \emptyset, \\
\partial \Omega_i \text{ Lipschitz,} \quad \Gamma \subset \partial \Omega_i \text{ relatively open } (i = 1, 2).
\]

We consider the following system of PDEs in $\Omega_i$ ($i = 1, 2$)

\[
\begin{align*}
\text{(1.1)} & \quad - \text{div}(\nu_i(k_i)D(u_i)) + \nabla p_i = f_i \text{ in } \Omega_i, \\
\text{(1.2)} & \quad \text{div} u_i = 0 \text{ in } \Omega_i, \\
\text{(1.3)} & \quad -\Delta k_i = \mu_i(k_i)|D(u_i)|^2 \text{ in } \Omega_i
\end{align*}
\]

where
\( \mathbf{u}_i = (u_{i1}, \ldots, u_{id}) = \) mean velocity, \( p_i = \) mean pressure, 
\( k_i = \) mean turbulent kinetic energy

are the unknown functions. For a vector field \( \mathbf{u} = (u_1, \ldots, u_d) \) we use the notations

\[
D(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top), \quad |D(\mathbf{u})|^2 = D(\mathbf{u}) : D(\mathbf{u}).
\]

The coefficients \( \nu_i \) and \( \mu_i \) are assumed to be uniformly bounded. We notice that the special case \( \nu_i(k_i) = \nu_{i0} + \nu_{iT}(k_i) \) where

\[
\nu_{i0} = \text{const} > 0 \quad \text{dynamic viscosity of the fluid},
0 \leq \nu_{iT}(k_i) \leq \text{const} \quad \text{eddy viscosity},
\]
as well as the two cases

\[ \nu_i(k_i) = \nu_i(k_i) \quad \text{or} \quad \mu_i(k_i) = \nu_{iT}(k_i) \]

are included in our discussion.

Finally, \( \mathbf{f}_i \) represents an external force in \( \Omega_i \).

The system (1.1) - (1.3) belongs to the class of one-equation RANS (Reynolds Averaged Navier-Stokes) models. The triple \( (\mathbf{u}_i, k_i, p_i) \) \( (i = 1, 2) \) characterizes the stationary turbulent motion of a viscous fluid in \( \Omega_i \), where the convection term in the fluid equations as well as in the turbulent kinetic energy equations is neglected.

A discussion of RANS models can be found in [2; pp. 304-316], [12; pp. 182-196, 216-252], [18; 319-337] (with \( \mu(k) = \nu_T(k) \)), and in [14] within the context of oceanography. Related problems (but without turbulence effects) are studied in [17]. The stationary turbulent motion of a fluid with unbounded eddy viscosities of the type \( \nu_T(k) = \alpha \sqrt{k} \) (Kolmogorov 1942, Prandtl 1945) has been studied in [7] and [13].

We complete (1.1) - (1.3) by the following boundary conditions which link both systems of PDEs in \( \Omega_1 \) and \( \Omega_2 \) through the interface \( \Gamma \):

\[
\left\{ \begin{array}{l}
\mathbf{u}_i = 0 \quad \text{on} \quad \partial \Omega_i \setminus \Gamma, \\
\mathbf{u}_i \cdot \mathbf{n}_i = 0 \quad \text{on} \quad \Gamma, \\
\nu_i(k_i)(\mathbf{D}(\mathbf{u}_i)\mathbf{n}_i)_\tau + |\mathbf{u}_i - \mathbf{u}_j|(\mathbf{u}_i - \mathbf{u}_j)_\tau = 0 \quad \text{on} \quad \Gamma \quad (i \neq j),
\end{array} \right.
\]

\[ ^1 \text{If} \mu_i = \nu_i, \text{system (1.1), (1.3) has some common features with the thermistor equations (see, e. g., Howison, S. D.; Rodrigues, J. F.; Shillor, M., Stationary solutions to the thermistor problem. J. Math. Analysis Appl. 174 (1993), 573-588; Cimatti, G., The stationary thermistor problem with a current limiting device. Proc. Royal Soc. Edinb. 116A (1990), 79-84). We notice that the assumption} \mu_i = \nu_i \text{significantly simplifies the arguments of the passage to the limit in (1.3) with approximate solutions (cf. [7] and Gallouet, T.; Lederer, J.; Lewandowski, R.; Murat, F.; Tartar, L., On a turbulent system with unbounded eddy viscosities. Nonlin. Analysis 52 (2003), 1051-1068).} \]
\[ k_i = 0 \text{ on } \partial \Omega_i \setminus \Gamma, \quad k_i = G_i(|u_1 - u_2|^2) \text{ on } \Gamma \]

where

\[ n_i = (n_{i1}, \ldots, n_{id}) = \text{unit outward normal on } \partial \Omega_i, \]

\[ \xi = \xi - (\xi \cdot n_i)n_i \quad (\xi \in \mathbb{R}^d), \]

\[ 0 \leq G_i(t) \leq c_0 t, \quad |G_i(t) - G_i(t^\prime)| \leq c_0 |t - t^\prime| \quad \forall t, t^\prime \in [0, +\infty) \quad (c_0 = \text{const} > 0) \]

\( (i = 1, 2) \). In (1.4), the boundary conditions on the (fixed) interface \( \Gamma \) model the situation when the interface is nonpermeable for both fluids which, however, do not completely adhere to the interface. Along this interface the fluids exhibit a partial slip which produces kinetic energy (cf. [3; pp. 69-73] for more details).

The boundary value problem (1.1) - (1.5) (with \( \nabla u_i \) in place of \( D(u_i) \) in (1.1), (1.3) and (1.4)) has been investigated in [3]. In this paper, the authors prove the existence of a solution \( \{u_1, k_1, p_1; u_2, k_2, p_2\} \) to (1.1)-(1.5) where (1.1) is satisfied in the usual weak sense (cf. our definition in Section 2), while (1.4) is satisfied in the sense of transposition of the Laplacean \(-\Delta\) under zero boundary conditions. The aim of the present paper is to give an existence proof for a weak solution to (1.1)-(1.5) (in the sense of the definition of Section 2). Our proof is shorter and more transparent than the one in [3]. Moreover, we establish some regularity results on \( (u_i, k_i) \).

Our paper is organized as follows. In Section 2, we introduce the notion of weak solution \( \{u_1, k_1; u_2, k_2\} \) to (1.1)-(1.5). By appealing to standard references, we show the existence of a pressure \( p_i \) associated with the pair \( (u_i, k_i) \) \( (i = 1, 2) \). Section 3 contains our main existence result. It's proof is based on a straightforward application of the Schauder \(^2\) fixed point theorem. A higher integrability result on \( \nabla u_i \) is established in Section 4. From this result we deduce the local existence of the second order derivatives of \( k_i \). In Appendix 1 we study in great detail the problem of whether a function which belongs to a Sobolev-Slobodeckij space over \( \Gamma \) and equals zero on \( \partial \Omega \setminus \Gamma \), is a trace of a Sobolev function defined in \( \Omega \). The solution of this problem is fundamental to the homogenization of the boundary condition (1.5). Finally, Appendix 2 is concerned with the inhomogeneous Dirichlet problem for the Poisson equation with right hand side in \( L^1 \).

### 2. Weak formulation of (1.1)-(1.5)

Let \( W^{1,q}(\Omega) \) \( (1 \leq q < +\infty) \) denote the usual Sobolev space. We define

\[ W^{1,q}_0(\Omega) := \{\varphi \in W^{1,q}(\Omega) : \varphi = 0 \text{ a. e. on } \partial \Omega\}. \]

\(^2\)We notice that the Schauder fixed point theorem has been also used in: Bernardi, C.; Chacon, T.; Lewandowski, R.; Murat, F., *Existence d'une solution pour un modèle de deux fluides turbulentes couplés*. C. R. Acad. Sci. Paris, Ser. I, 328 (1999), 993-998. In comparison with this paper, our existence theorem for a weak solution \( \{u_1, k_1, p_1; u_2, k_2, p_2\} \) to (1.1)-(1.5) (see Section 3) involves more regularity of \( k_1, k_2 \) (see Remark 2.2 for details).
Spaces of vector-valued function will be denoted by bold letters, e. g., \( L^q(\Omega) := [L^q(\Omega)]^d \), \( W^{1,q}(\Omega) := [W^{1,q}(\Omega)]^d \) etc. Next, define

\[
V_i := \{ v \in W^{1,2}(\Omega_i) : \text{div } v = 0 \text{ a. e. in } \Omega_i, \quad v = 0 \text{ a. e. on } \partial \Omega_i \setminus \Gamma, \quad v \cdot n_i = 0 \text{ a. e. on } \Gamma \}
\]

\((i = 1, 2)\).

Without any further reference, throughout the paper we suppose

\[
\begin{cases}
\text{there exist constants } \nu_*, \nu^* \text{ and } \mu^* \text{ such that } \\
0 < \nu_i(t) \leq \nu^* < +\infty, \quad 0 \leq \mu_i(t) \leq \mu^* < +\infty \quad \forall t \in \mathbb{R} \quad (i = 1, 2).
\end{cases}
\]

**Definition** Let \( f_i \in L^{2^*}(\Omega_i) \) \(^3\) \((i = 1, 2)\). The functions \( \{u_1, k_1; u_2, k_2\} \) are called weak solution to (1.1)-(1.5) if

\[
(2.1) \quad (u_i, k_i) \in V_i \times \bigcap_{1 \leq q < \frac{d}{d-1}} W^{1,q}(\Omega_i) \quad (i = 1, 2),
\]

\[
(2.2) \begin{cases}
\int_{\Omega_1} \nu_1(k_1)D(u_1) : D(v_1) + \int_{\Omega_2} \nu_2(k_2)D(u_2) : D(v_2) + \\
+ \int_{\Gamma} |u_1 - u_2|(u_1 - u_2) \cdot (v_1 - v_2)dS = \\
= \int_{\Omega_1} f_1 \cdot v_1 + \int_{\Omega_2} f_2 \cdot v_2 \quad \forall (v_1, v_2) \in V_1 \times V_2,
\end{cases}
\]

\[
(2.3) \begin{cases}
\text{for some } r > d, \\
\int_{\Omega_i} \nabla k_i \cdot \nabla \varphi = \int_{\Omega_i} \mu_i(k_i)|D(u_i)|^2 \varphi \quad \forall \varphi \in W^{1,r}_0(\Omega_i) \quad \text{4),}
\end{cases}
\]

\[
(2.4) \quad k_i = 0 \text{ a. e. on } \partial \Omega_i \setminus \Gamma, \quad k_i = G_i(|u_1 - u_2|^2) \text{ a. e. on } \Gamma.
\]

**Remark 2.1 (existence of a pressure)** Define

\(^3\)By \( q^* \) we denote Sobolev embedding exponent for \( W^{1,q}(\Omega) \) \((\Omega \subset \mathbb{R}^N \text{ bounded, Lipschitzian; } N \geq 2)\), i. e. \( q^* = \frac{Nq}{N-q} \) if \( 1 \leq q < N \), and \( 1 \leq q^* < +\infty \) if \( q = N \). If \( q > N \), then \( W^{1,q}(\Omega) \subset C(\Omega) \) continuously.

\(^4\)Notice that \( r > N \) iff \( 1 < r < \frac{N}{N-1} \).
\[ W^{1,2}_{0,\Gamma}(\Omega_i) := \{ w \in W^{1,2}(\Omega_i) : w = 0 \quad a. \ e. \ on \ \partial \Omega_i \setminus \Gamma, \quad w \cdot n_i = 0 \quad a. \ e. \ on \ \Gamma \} \]

\( i = 1, 2 \). Clearly, \( V_i \) is a closed subspace of \( W^{1,2}_{0,\Gamma}(\Omega_i) \). We have:

\[ \text{Let } \{ u_1, k_1; u_2, k_2 \} \text{ be a weak solution to (1.1)-(1.5). Then there exists } p_i \in L^2(\Omega_i) \text{ with } \int_{\Omega_i} p_i = 0 \text{ such that} \]

\[
\begin{aligned}
\left\{ \begin{array}{l}
\int_{\Omega_i} \nu_i(k_i) D(u_i) : D(w) + (-1)^{i+1} \int_{\Gamma} |u_1 - u_2|(u_1 - u_2) \cdot w dS = \\
\int_{\Omega_i} f_i \cdot w + \int_{\Omega_i} p_i \text{ div } w \quad \forall \ w \in W^{1,2}_{0,\Gamma}(\Omega_i).
\end{array} \right.
\end{aligned}
\]

\( (2.2') \)

In addition, there holds

\[
(2.2'') \quad \| p_i \|_{L^2} \leq c \left( \| \nabla u_i \|_{L^2} + \| f_i \|_{L^{2^*}} \right).
\]

To prove this, we first note the following

**Proposition** Let \( \Omega \subset \mathbb{R}^N (N \geq 2) \) be a bounded Lipschitz domain and let \( 1 < r < +\infty \).

Then, for every \( f \in L^r(\Omega) \) with \( \int_{\Omega} f = 0 \), there exists \( v \in W^{1,r}_{0,\Gamma}(\Omega) \) such that

\[ \text{div } v = f \quad a. \ e. \ in \ \Omega, \]

\[ \| \nabla v \|_{L^r} \leq c \| f \|_{L^r}. \]

For a proof, see, e. g. [9; Chap. III, Thm. 3.2], [22; Chap. II, Lemma 2.1.1, a)].

We now proceed as follows. For \( w \in W^{1,2}_{0,\Gamma}(\Omega_i) \), define

\[ F_i(w) := \int_{\Omega_i} \nu_i(k_i) D(u_i) : D(w) + (-1)^{i+1} \int_{\Gamma} |u_1 - u_2|(u_1 - u_2) \cdot w dS - \int_{\Omega_i} f_i \cdot w \]

\( (i = 1, 2) \). It is easy to check that \( F_i \) is a linear continuous functional on \( W^{1,2}_{0,\Gamma}(\Omega_i) \). By \( (2.2) \), \( F_i(v) = 0 \) for all \( v \in V_i \).
Next, the above Proposition implies that the mapping

$$A : \mathbf{v} \mapsto A\mathbf{v} = \text{div } \mathbf{v}$$

is surjective from $W^{1,2}_{0,\Gamma}(\Omega_i)$ onto the space

$$\left\{ f \in L^2(\Omega_i) : \int_{\Omega_i} f = 0 \right\}.$$ 

Now, following word by word the arguments of the proof in [9; Chap. III, Thm. 5.2] or [22; Chap. II, Lemma 2.11, b)] we obtain the existence of a $p_i \in L^2(\Omega_i)$ with $\int_{\Omega_i} p_i = 0$ such that

$$\mathcal{F}_i(\mathbf{w}) = \int_{\Omega_i} p_i \text{div } \mathbf{w} \quad \forall \mathbf{w} \in W^{1,2}_{0,\Gamma}(\Omega_i),$$

i. e., (2.2') holds.

Estimate (2.2'') is readily seen.

Remark 2.2 In [3; Thm. 5.2, pp. 88-89] the notion of (weak) solution to (1.1)-(1.5) means that $k_i$ belongs to the Sobolev-Slobodeckij space $W^{s,2}(\Omega_i)$ ($0 < s < \frac{1}{2}$), and that (1.3) is satisfied in the sense of transposition of $-\Delta$ (cf. [3; p. 78]). In contrast to that paper, our definition of weak solution to (1.1)-(1.5) involves more regularity of $k_i$.  

Indeed, for any $0 < s < \frac{1}{2}$ we have $\frac{2d}{2d-2s} < \frac{d}{d-1}$. Thus, if

$$\frac{2d}{2d-2s} < q < \frac{d}{d-1},$$

then

$$1 - \frac{d}{q} > s - \frac{d}{2},$$

and therefore

$$W^{1,q}(\Omega_i) \subset W^{s,2}(\Omega_i)$$

(see, e. g., [24; p. 328]). Hence, $k_i \in \bigcap_{1 \leq q < \frac{d}{d-1}} W^{1,q}(\Omega_i)$ implies $k_i \in W^{s,2}(\Omega_i)$ for all

$$0 < s < \frac{1}{2}.$$ 

5) See also Appendix 2.
Finally, let \( k_i \in W^{1,q}(\Omega_i) \) \((1 \leq q < \frac{d}{d-1})\) satisfy (2.3) and (2.4). Integration by parts on the left hand side of (2.3) gives, for any \( \varphi \in W^{2,2}(\Omega_i) \cap W^{1,2}_0(\Omega_i) \),

\[- \int_{\Omega_i} k_i \Delta \varphi + \int_{\Gamma} G_i(|u_1 - u_2|^2) n_i \cdot \nabla \varphi dS = \int_{\Omega_i} \mu_i(k_i)|D(u_i)|^2 \varphi,\]

i. e., \( k_i \) satisfies (1.3) in the sense of transposition of \(-\Delta\) under zero boundary conditions on \( \varphi \) (cf. [3; p. 78]).

\section{Existence of a weak solution}

The following theorem is the main result of our paper.

**Theorem** Let \( \Omega_i \subset \mathbb{R}^d \) \((i = 1, 2; d = 2 \, \text{or} \, d = 3)\) be bounded domains of class \( C^1 \) \(^6\). Suppose that assumption (A) \(^7\) is satisfied.

Then, for every \( f_i \in L^2(\Omega_i) \) \((i = 1, 2)\) there exists a weak solution \( \{u_1, k_1; u_2, k_2\} \) to (1.1)-(1.5). In addition,

\begin{align}
\tag{3.1} k_i \geq 0 \quad &\text{a. e. in } \Omega_i, \\
\tag{3.2} \sum_{i=1}^{2} \|u_i\|^2_{W^{1,2}(\Omega_i)} + \int_{\Gamma} |u_1 - u_2|^3 dS &\leq c \sum_{j=1}^{2} \|f_j\|^2_{L^2(\Omega_j)}, \\
\tag{3.3} \begin{cases}
\quad \|k_i\|^2_{W^{1,q}(\Omega_i)} \leq c \sum_{j=1}^{2} \|f_j\|^2_{L^2(\Omega_j)} \\
\quad \text{for every } 1 \leq q < \frac{d}{d-1} \quad \text{there exists } c = \text{const} \quad \text{such that}
\end{cases}
\end{align}

\begin{align*}
\tag{3.3} &\text{for every } \Omega_i' \Subset \Omega_i \quad \text{and every } \delta > 0, \\
\tag{3.4} \int_{\Omega_i'} \frac{|\nabla k_i|^2}{(1 + k_i)^{1+\delta}} &\leq c \sum_{j=1}^{2} \|f_j\|^2_{L^2(\Omega_j)} \\
&\quad \text{where } c \to +\infty \quad \text{as } \text{dist} (\Omega_i', \partial \Omega_i) \to 0.
\end{align*}

\(^6\)The condition \( \Omega_i \in C^1 \) we need in order to apply Theorem A2.1.

\(^7\)See p. 22
Proof. We consider the space $L^1(\Omega_1) \times L^1(\Omega_2)$ equipped with the norm

\[ \| (k_1, k_2) \| := \sum_{i=1}^{2} \| k_i \|_{L^1(\Omega_i)}. \]

For appropriate $R > 0$ which will be fixed below, we set

\[ K_R := \{ (k_1, k_2) \in L^1(\Omega_1) \times L^1(\Omega_2) : \| (k_1, k_2) \| \leq R \}. \]

Then, for any $(k_1, k_2) \in K_R$ we show that there exists exactly one $(u_1, u_2) \in V_1 \times V_2$ which satisfies (2.2). With $(u_1, k_1; u_2, k_2)$ at hand, we deduce from Theorem A2.1 the existence and uniqueness of a pair $(\hat{k}_1, \hat{k}_2) \in W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)$ ($1 < q < \frac{d}{d-1}$ arbitrary) which solves (2.3) with the given $L^1$-function $\mu_i(k_i)|D(u_i)|^2$ on the right hand side, and with given $G_i(\|u_1-u_2\|^2)$ on $\Gamma (i = 1, 2)$. This gives rise to introduce a mapping $T : K_R \rightarrow K_R$ by

\[ T(k_1, k_2) := (\hat{k}_1, \hat{k}_2). \]

We then prove:

(i) $T$ is continuous;

(ii) $T(K_R)$ is precompact.

From Schauder’s fixed it follows that there exists $(k^*_1, k^*_2) \in K_R$ such that $T(k^*_1, k^*_2) = (k^*_1, k^*_2)$.

Now, with the fixed point $(k^*_1, k^*_2)$ at hand, we obtain the existence and uniqueness of a pair $(u^*_1, u^*_2) \in V_1 \times V_2$ which satisfies (2.2) (with $(k^*_1, k^*_2)$ in place of $(k_1, k_2)$ therein). By the definition of $T$, the functions $\{u^*_1, k^*_1; u^*_2, k^*_2\}$ are a weak solution to (1.1)-(1.5).

We turn to the details of the proof.

Definition of $T : K_R \rightarrow K_R$. The space $V_1 \times V_2$ is a Hilbert space with respect to the scalar product

\[ \langle (u_1, u_2), (v_1, v_2) \rangle := \sum_{i=1}^{2} \int_{\Omega} \nabla u_i \cdot \nabla v_i. \]

By $\| \cdot \| := \langle \cdot, \cdot \rangle^\frac{1}{2}$ we denote the associated norm.

1) The mapping $(k_1, k_2) \mapsto (u_1, u_2)$. Given any $(k_1, k_2) \in L^1(\Omega_1) \times L^1(\Omega_2)$, we prove the existence and uniqueness of a pair $(u_1, u_2) \in V_1 \times V_2$ which satisfies (2.2). To do this, we replace (2.2) by an operator equation in $V_1 \times V_2$ to which an abstract existence and uniqueness theorem applies.
Firstly, for any (fixed) \((k_1, k_2) \in L^1(\Omega_1) \times L^1(\Omega_2)\) we introduce a linear bounded mapping \(A(k_1, k_2) : V_1 \times V_2 \rightarrow V_1 \times V_2\) by

\[
\langle A(k_1, k_2)(u_1, u_2), (v_1, v_2) \rangle := 2 \sum_{i=1}^{2} \int_{\Omega_i} \nu_i(k_i) D(u_i) : D(v_i).
\]

By Korn’s equality,

\[
\langle A(k_1, k_2)(u_1, u_2), (u_1, u_2) \rangle \geq c_0 \| (u_1, u_2) \|^2 \quad (c_0 = \text{const} > 0)
\]

for all \((u_1, u_2) \in V_1 \times V_2\) (\(c_0\) independent of \((k_1, k_2)\)).

Secondly, observing the continuity of the trace mapping \(\gamma : W^{1, q}(\Omega) \rightarrow L^4(\partial \Omega)\) \((d = 2 \text{ and } d = 3; \text{ see, e.g., } [8], [11], [24; pp. 281-282, 329-330])\) we obtain, for every \((u_1, u_2), (v_1, v_2) \in V_1 \times V_2\),

\[
\left| \int_{\Gamma} |u_1 - u_2|(u_1 - u_2) \cdot (v_1 - v_2) dS \right| \leq 
\leq \left( \int_{\Gamma} |u_1 - u_2|^2 dS \right)^\frac{3}{2} \left( \int_{\Gamma} |v_1 - v_2|^4 dS \right)^\frac{1}{4}
\leq c \left( \sum_{i=1}^{2} \| u_i \|_{L^\infty(\partial \Omega_i)}^2 \right) \sum_{j=1}^{2} \| v_j \|_{L^4(\partial \Omega_j)}^2
\leq c \| (u_1, u_2) \|^2 \| (v_1, v_2) \|.
\]

We now introduce a (nonlinear) mapping \(B : V_1 \times V_2 \rightarrow V_1 \times V_2\) by

\[
\langle B(u_1, u_2), (v_1, v_2) \rangle := \int_{\Gamma} |u_1 - u_2|(u_1 - u_2) \cdot (v_1 - v_2) dS.
\]

By elementary calculus,

\[
\langle B(u_1, u_2) - B(\bar{u}_1, \bar{u}_2), (u_1, u_2) - (\bar{u}_1, \bar{u}_2) \rangle \geq 
\geq \int_{\Gamma} (|u_1 - u_2|^2 - |\bar{u}_1 - \bar{u}_2|^2)(|u_1 - u_2| - |\bar{u}_1 - \bar{u}_2|) dS \geq 0
\]

\[8^)\] For notational simplicity, in this section we use the same notation for a function in \(W^{1, q}(\Omega)\) and its trace.

\[9^)\] Throughout the paper, we denote by \(c\) positive constants which may change their numerical value but do not depend on the functions under consideration.
and
\[
\|\mathcal{B}(\mathbf{u}_1, \mathbf{u}_2) - \mathcal{B}(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2)\| \leq c(\|\mathbf{u}_1, \mathbf{u}_2\| + \|\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2\|) \sum_{i=1}^{2} \|u_i - \bar{u}_i\|_{W^{1,2}(\Omega_i)}
\]
for all \((\mathbf{u}_1, \mathbf{u}_2), (\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2) \in V_1 \times V_2.

Thus,
\[
\begin{cases}
\mathcal{A}(k_1, k_2) + \mathcal{B} \text{ is continuous on the whole of } V_1 \times V_2 \\
\text{and maps bounded sets into bounded sets,}
\end{cases}
\]
\[
\begin{cases}
\langle (\mathcal{A}(k_1, k_2) + \mathcal{B})(\mathbf{u}_1, \mathbf{u}_2) - (\mathcal{A}(k_1, k_2) + \mathcal{B})(\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2), (\mathbf{u}_1, \mathbf{u}_2) - (\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2) \rangle \\
\geq c_0\|\mathbf{u}_1, \mathbf{u}_2\| - (\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2)\|^2 \quad \forall (\mathbf{u}_1, \mathbf{u}_2), (\bar{\mathbf{u}}_1, \bar{\mathbf{u}}_2) \in V_1 \times V_2.
\end{cases}
\]

From [27; Thm. 26.A, p. 557] it follows that for every \(f_i \in L^{2^*}(\Omega_i)\) \((i=1,2)\) there exists exactly one \((\mathbf{u}_1, \mathbf{u}_2) \in V_1 \times V_2\) such that
\[
\langle (\mathcal{A}(k_1, k_2) + \mathcal{B})(\mathbf{u}_1, \mathbf{u}_2), (\mathbf{v}_1, \mathbf{v}_2) \rangle = \sum_{i=1}^{2} \int_{\Omega_i} f_i \cdot \mathbf{v}_i \quad \forall (\mathbf{v}_1, \mathbf{v}_2) \in V_1 \times V_2,
\]
i. e., (2.2) holds with the given \((k_1, k_2) \in L^1(\Omega_1) \times L^1(\Omega_2).\) In addition, we have
\[
\sum_{i=1}^{2}\|\mathbf{u}_i\|_{W^{1,2}(\Omega_i)}^2 + \int_\Gamma |\mathbf{u}_1 - \mathbf{u}_2|^2 dS \leq c \sum_{j=1}^{2}\|f_j\|_{L^{2^*}(\Omega_i)}^2,
\]
where the constant \(c\) does not depend on \((k_1, k_2).\)

2) The mapping \((\mathbf{u}_1, \mathbf{u}_2) \mapsto (k_1, k_2).\) Let \(1 < q < \frac{d}{d-1}\). Let \((\mathbf{u}_1, \mathbf{u}_2) \in V_1 \times V_2\) denote the solution to (3.5) (uniquely determined by \((k_1, k_2) \in L^1(\Omega_1) \times L^1(\Omega_2))\) which has been obtained by the preceding step 1).

Define
\[
\tilde{h}_i := \begin{cases} 
G_i(|\mathbf{u}_1 - \mathbf{u}_2|^2) & \text{a. e. on } \Gamma, \\
0 & \text{a. e. on } \partial\Omega_i \setminus \Gamma
\end{cases}
\]
\((G_i\text{ as in (1.6); } i = 1, 2).\) By Corollary A1.1,
\[
\tilde{h}_i \in W^{1,\frac{2}{q}}(\partial\Omega_i), \quad \|\tilde{h}_i\|_{W^{1,\frac{2}{q}}(\partial\Omega_i)} \leq c \sum_{j=1}^{2}\|\mathbf{u}_j\|_{W^{1,2}(\Omega_j)}^2,
\]
Now, from Theorem A2.1 and Theorem A2.2, we obtain the existence and uniqueness of a pair \((\hat{k}_1, \hat{k}_2)\) \(\in W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)\) such that

\[
\hat{k}_i \geq 0 \quad \text{a.e. in } \Omega_i,
\]

\[
\int_{\Omega_i} \nabla \hat{k}_i \cdot \nabla \varphi_i = \int_{\Omega_i} \mu_i(k_i)|D(u_i)|^2 \varphi_i \quad \forall \varphi_i \in W^{1,q}(\Omega_i),
\]

\[
\hat{k}_i = \tilde{h}_i \quad \text{a.e. on } \partial \Omega_i,
\]

\[
\|\hat{k}_i\|_{W^{1,q}(\Omega_i)} \leq c\left(\|D(u_i)|^2\|_{L^1(\Omega_i)} + \|\tilde{h}_i\|_{W^{1-\frac{1}{q},q}(\partial \Omega_i)}\right),
\]

\[
\begin{cases}
\text{for every } \Omega'_i \subset \Omega_i \text{ and every } \delta > 0, \\
\frac{\nabla \hat{k}_i}{(1 + \hat{k}_i)^{1+\delta}} \in L^2(\Omega'_i), \\
\int_{\Omega_i} \frac{\nabla \hat{k}_i}{(1 + \hat{k}_i)^{1+\delta}} \leq \frac{c}{\delta} \left(\|D(u_i)|^2\|_{L^1(\Omega_i)} + \|\tilde{h}_i\|_{W^{1-\frac{1}{q},q}(\partial \Omega_i)}\right), \\
\text{where } c \to +\infty \text{ as } \text{dist}(\Omega'_i, \partial \Omega_i) \to 0.
\end{cases}
\]

We notice that the constants \(c\) in (3.7), (3.11) and (3.12) do not depend on \((k_1, k_2)\). By combining (3.7) and (3.11) we find

\[
\|(\hat{k}_1, \hat{k}_2)\| \leq c \sum_{i=1}^2 \|f_i\|^2_{L^2(\Omega_i)} = : R.
\]

3) Let us consider \(K_R\) with \(R\) as in (3.13). For \((k_1, k_2) \in K_R\), define

\[
\mathcal{T} : (k_1, k_2) \mapsto (u_1, u_2) \mapsto \mathcal{T}(k_1, k_2) := (\hat{k}_1, \hat{k}_2),
\]

where \((u_1, u_2)\) is as in step 1), \((\hat{k}_1, \hat{k}_2)\) as in step 2). Then \(\mathcal{T}\) is a well-defined (single valued) mapping of \(K_R\) into itself. \(^{11}\)

(i) \(\mathcal{T}\) is continuous. Let be \((k_{1m}, k_{2m}) \in K_R (m \in \mathbb{N})\) such that

\[
k_{im} \to k_i \quad \text{strongly in } L^1(\Omega_i) \text{ as } m \to \infty \ (i = 1, 2).
\]

\(^{10}\)Recall \(K_R := \{(k_1, k_2) \in L^1(\Omega_1) \times L^1(\Omega_2) : \|(k_1, k_2)\| \leq R\}.

\(^{11}\)In fact, \(\mathcal{T}\) maps the whole of \(L^1(\Omega_1) \times L^1(\Omega_2)\) into \(K_R\).
Clearly, \((k_1, k_2) \in \mathcal{K}_R\). Without loss of generality, we may assume that
\[
(3.14) \quad k_{im} \to k_i \text{ a. e. in } \Omega_i \text{ as } m \to \infty \quad (i = 1, 2).
\]

We prove that
\[
\mathcal{T}(k_{1m}, k_{2m}) \to \mathcal{T}(k_1, k_2) \text{ strongly in } L^1(\Omega_1) \times L^1(\Omega_2) \text{ as } m \to \infty.
\]

To begin with, we introduce the following notation. For \((k_{1m}, k_{2m})\), let \((u_{1m}, u_{2m}) \in V_1 \times V_2\) denote the uniquely determined solution of
\[
(3.5) \quad (A_{(k_{1m}, k_{2m})} + B)(u_{1m}, u_{2m}), (v_1, v_2) = \sum_{i=1}^{2} \int_{\Omega_i} f_i \cdot v_i \quad \forall (v_1, v_2) \in V_1 \times V_2.
\]

Clearly,
\[
(3.6) \quad \sum_{i=1}^{2} \|u_{im}\|_{W^{1,2}(\Omega_i)}^2 + \int_{\Gamma} |u_{1m} - u_{2m}|^2 dS \leq c \sum_{i=1}^{2} \|f_i\|_{L^2(\Omega_i)}^2.
\]

Analogously, for the limit element \((k_1, k_2)\), let \((u_1, u_2) \in V_1 \times V_2\) denote the uniquely determined solution to (3.5). This solution satisfies (3.6).

We claim
\[
(3.15) \quad (u_{1m}, u_{2m}) \to (u_1, u_2) \text{ strongly in } W^{1,2}(\Omega_1) \times W^{1,2}(\Omega_2) \text{ as } m \to \infty.
\]

To prove this, we first note that from (3.6) it follows that there exists a subsequence \(\{(u_{1ms}, u_{2ms})\} \) \((s \in \mathbb{N})\) such that
\[
(u_{1ms}, u_{2ms}) \to (\tilde{u}_1, \tilde{u}_2) \text{ weakly in } W^{1,2}(\Omega_1) \times W^{1,2}(\Omega_2) \text{ as } s \to \infty.
\]

Using the compactness of the embedding \(W^{1,2}(\Omega) \subset L^r(\partial \Omega) \) \((1 \leq r < 4; \ d = 2 \text{ resp. } d = 3)\), we obtain
\[
\langle B(u_{1ms}, u_{2ms}), (v_1, v_2) \rangle \to B(\tilde{u}_1, \tilde{u}_2), (v_1, v_2) \rangle \quad \forall (v_1, v_2) \in V_1 \times V_2
\]
as \(m \to \infty\). With the help of (3.14) the passage to the limit \(s \to \infty\) in (3.5) gives
\[
\langle (A_{(k_1, k_2)} + B)(\tilde{u}_1, \tilde{u}_2) = \sum_{i=1}^{2} \int_{\Omega} f_i \cdot v_i \quad \forall (v_1, v_2) \in V_1 \times V_2.
\]

Comparing this and (3.5) we find \(\tilde{u}_i = u_i \) \((i = 1, 2)\). Therefore the whole sequence \(\{(u_{1m}, u_{2m})\}\) converges weakly in \(W^{1,2}(\Omega_1) \times W^{1,2}(\Omega_2)\) to \((u_1, u_2)\).
We now form the difference between \((3.5_m)\) and \((3.5)\), and use the test function \(v_i = u_{im} - u_i \ (i = 1, 2)\). Observing the monotonicity of \(B\), we find

\[
\nu_i \sum_{i=1}^{2} \int_{\Omega_i} |D(u_{im} - u_i)|^2 \leq \sum_{i=1}^{2} \int_{\Omega_i} \nu_i(k_{im}) (D(u_{im}) - D(u_i)) : D(u_{im} - u_i)
\leq \sum_{i=1}^{2} \int_{\Omega_i} (-\nu_i(k_{im}) + \nu_i(k_i)) D(u_i) : D(u_{im} - u_i)
\rightarrow 0 \quad \text{as} \quad m \rightarrow \infty.
\]

Whence \((3.15)\).

Next, set \((\hat{k}_{1m}, \hat{k}_{2m}) := T(k_{1m}, k_{2m}) \ (m \in \mathbb{N})\) and \((\hat{k}_1, \hat{k}_2) := T(k_1, k_2)\). Let \(1 < q < \frac{d}{d-1}\). By the definition of \(T\), the pair \((\hat{k}_{1m}, \hat{k}_{2m}) \in W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)\) is uniquely determined by \((k_{1m}, k_{2m})\) and \((u_{1m}, u_{2m})\) through

\[
(3.9_m) \quad \int_{\Omega_i} \nabla \hat{k}_{im} \cdot \nabla \varphi_i = \int_{\Omega_i} \mu_i(k_{im}) |D(u_{im})|^2 \varphi_i \quad \forall \varphi_i \in W^{1,q}_0(\Omega_i),
\]

\[
(3.10_m) \quad \hat{k}_{im} = \tilde{h}_{im} \quad \text{a. e. on} \quad \partial \Omega_i,
\]

where \(\tilde{h}_{im} \in W^{1-\frac{1}{q},q}(\partial \Omega_i)\) is defined by

\[
\tilde{h}_{im} := \begin{cases} 
G_i(|u_{1m} - u_{2m}|) & \text{a. e. on} \quad \Gamma, \\
0 & \text{a. e. on} \quad \partial \Omega_i \setminus \Gamma
\end{cases}
\]

(see Theorem A2.1). From \((3.7)\) (with \(u_{im}\) in place of \(u_i\)) it follows that

\[
||\tilde{h}_{im}||_{W^{1-\frac{1}{q},q}(\partial \Omega_i)} \leq c \sum_{j=1}^{2} ||u_{jm}||_{W^{1,2}(\Omega_j)}^2 \leq \text{const.}
\]

We obtain

\[
(3.16) \quad \tilde{h}_{im} \rightarrow \tilde{h}_i \quad \text{weakly in} \quad W^{1-\frac{1}{q},q}(\partial \Omega_i) \quad \text{as} \quad m \rightarrow \infty,
\]

where \(\tilde{h}_i\) is defined as above, i.e.

\[
\tilde{h}_i := \begin{cases} 
G_i(|u_1 - u_2|^2) & \text{a. e. on} \quad \Gamma, \\
0 & \text{a. e. on} \quad \partial \Omega_i \setminus \Gamma
\end{cases}
\]

\((i = 1, 2)\). To see \((3.16)\), we first note that \((3.15)\) implies \(u_{im} \rightarrow u_i\) strongly in \(L^4(\partial \Omega_i)\) as \(m \rightarrow \infty \ (d = 2 \text{ resp.} \ d = 3)\). Therefore

\[
G_i(|u_{1m} - u_{2m}|^2) \rightarrow G_i(|u_1 - u_2|^2) \quad \text{strongly in} \quad L^2(\Gamma) \quad \text{as} \quad m \rightarrow \infty.
\]
Since $W^{1-\frac{1}{q},q}(\partial\Omega_i)$ is reflexive, (3.16) is now readily seen by routine arguments.

To proceed, we note that $\hat{k}_{im}$ satisfies the estimate
\[
\|\hat{k}_{im}\|_{W^{1,q}(\Omega_i)} \leq c \left( \|D(u_{im})\|_{L^1(\Omega_i)}^2 + \|\tilde{h}_{im}\|_{W^{1-\frac{1}{q},q}(\partial\Omega_i)} \right) \quad [\text{cf. (3.11)}]
\]
\[
\leq c \sum_{j=1}^{2} \|u_{jm}\|_{W^{1,2}(\Omega_j)}^2
\]
\[
\leq c \sum_{j=1}^{2} \|f_j\|_{L^{2^*}(\Omega_j)}^2 \quad [\text{by (3.6m)}]
\]

\[(i = 1, 2; m \in \mathbb{N}).\] Hence there exists a subsequence $\{\hat{k}_{im}\}$ ($t \in \mathbb{N}$) such that
\[
\hat{k}_{im} \rightharpoonup \bar{k}_i \quad \text{weakly in } W^{1,q}(\Omega_i) \quad \text{as } t \to \infty.
\]

Using (3.14), (3.15) and (3.16) the passage to the limit $t \to \infty$ in (3.9m) and (3.10m) gives
\[
\int_{\Omega_i} \nabla \bar{k}_i \cdot \nabla \varphi_i = \int_{\Omega_i} \mu_i(k_i) |D(u_i)|^2 \varphi_i \quad \forall \varphi_i \in W^{1,q'}(\Omega_i),
\]
\[
\bar{k}_i = \tilde{h}_i \quad \text{a. e. on } \partial\Omega_i.
\]

Combining this and (3.9), (3.10) we get
\[
\int_{\Omega_i} \nabla (\bar{k}_i - \hat{k}_i) \cdot \nabla \varphi_i = 0 \quad \forall \varphi_i \in W^{1,q'}(\Omega_i),
\]
\[
\bar{k}_i - \hat{k}_i = 0 \quad \text{a. e. on } \partial\Omega_i.
\]

By theorem A2.1, $\bar{k}_i = \hat{k}_i$ a. e. in $\Omega_i$ ($i = 1, 2$). It follows that the whole sequence $\{\hat{k}_{im}\}$ converges weakly in $W^{1,q}(\Omega_i)$ to $\bar{k}_i$ as $m \to \infty$. Therefore, by the compactness of the embedding $W^{1,q}(\Omega) \subset L^1(\Omega)$,
\[
\hat{k}_{im} \rightharpoonup \bar{k}_i \quad \text{strongly in } L^1(\Omega_i) \quad \text{as } m \to \infty,
\]
i. e., $T$ is continuous.

(ii) $T(K_R)$ is precompact. Let $\hat{(k}_{1m}, \hat{k}_{2m}) \in T(K_R)$ ($m \in \mathbb{N}$). Then $\hat{(k}_{1m}, \hat{k}_{2m}) = T(k_{1m}, k_{2m})$, where $(k_{1m}, k_{2m}) \in K_R$. As above, let $(u_{1m}, u_{2m}) \in V_1 \times V_2$ denote the uniquely determined solutions to (3.5m). The existence and uniqueness argument used at the end of the proof of the continuity of $T$ (cf. Theorem A2.1), implies that $(\hat{k}_{1m}, \hat{k}_{2m})$ in $W^{1,q}(\Omega_1) \times W^{1,q}(\Omega_2)$ and $1 < q < \frac{d}{d-1}$ and (3.9m) and (3.10m) hold. It follows that
\[
\|\hat{k}_{im}\|_{W^{1,q}(\Omega_i)} \leq c \sum_{j=1}^{2} \|f_j\|_{L^{2^*}(\Omega_j)}^2 \quad (i = 1, 2; m \in \mathbb{N})
\]
(cf. above). By the compactness of the embedding $W^{1,q}(\Omega) \subset L^1(\Omega)$, there exists a subsequence $\{k_{im_s}\} (s \in \mathbb{N})$ and an element $(l_1, l_2) \in L^1(\Omega_1) \times L^1(\Omega_2)$ such that

$$\hat{k}_{im_s} \to l_i \quad \text{strongly in} \quad L^1(\Omega_i) \quad \text{as} \quad s \to \infty,$$

i.e. $T(\mathcal{K}_R)$ is precompact.

By Schauder’s fixed point theorem, there exists $(k_1^*, k_2^*) \in \mathcal{K}_R$ such that $T(k_1^*, k_2^*) = (k_1^*, k_2^*)$. The proof of the theorem is complete.

### 4. Regularity properties of weak solutions

In this section, we establish regularity properties for any weak solution $\{u_1, k_1; u_2, k_2\}$ to (1.1)–(1.5) (see Sect. 2 for the definition).

**Theorem 4.1 (Local regularity)** Let $f_i \in L^2(\Omega_i) (i = 1, 2)$. Then there exists $\sigma > 2$ such that for every weak solution $\{u_1, k_1; u_2, k_2\}$ to (1.1)–(1.5) there holds

$$\nabla u_i \in L^\sigma_{\text{loc}}(\Omega_i), \quad k_i \in W^{2,\frac{\sigma}{2}}_{\text{loc}}(\Omega_i).$$

Indeed, the local higher integrability of $\nabla u_i$ follows from [6; Prop. 4.1]. It follows $|D(u_i)|^2 \in L^\frac{\sigma}{2}_{\text{loc}}(\Omega_i)$. Then $k_i \in W^{2,\frac{\sigma}{2}}_{\text{loc}}(\Omega_i)$ is a consequence of Theorem A 2.1, (A2.7).

**Theorem 4.2 (global higher integrability of $\nabla u_i$)** Assume that

$$\Gamma \cap (\partial \Omega_i \setminus \Gamma) \quad \text{is Lipschitz} \quad (i = 1, 2)^{12}$$

Let $f_i \in L^2(\Omega_i)$. Then there exists $\rho > 2$ such that for every weak solution $\{u_1, k_1; u_2, k_2\}$ to (1.1)–(1.5) there holds

$$\nabla u_i \in L^\rho(\Omega_i).$$

This result is a special case of [26; Thm. 2.1].

We notice that the higher integrability of the gradient has been used in [3] for the uniqueness of the weak solution to (1.1)–(1.5) in the case $d = 2$. It has been also used in [4].

### Appendix 1. Extension of a function $g \in W^{s,q}(\Gamma)$ by zero onto $\partial \Omega \setminus \Gamma$

---

12) See [26; (1.24a), (1.24b)] for details.
Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded domain with Lipschitz boundary $\partial \Omega$. For $0 < s < 1$ and $1 < q < +\infty$ we consider the Sobolev-Slobodeckij space

$$W^{s,q}(\partial \Omega) := \left\{ w \in L^q(\partial \Omega) : \int_{\partial \Omega} \int_{\partial \Omega} \frac{|w(x) - w(y)|^q}{|x - y|^{N-1+sq}} dS_x dS_y < +\infty \right\}$$

with the norm

$$\|w\|_{W^{s,q}(\partial \Omega)} := \left( \|w\|_{L^q(\partial \Omega)}^q + \int_{\partial \Omega} \int_{\partial \Omega} \frac{|w(x) - w(y)|^q}{|x - y|^{N-1+sq}} dS_x dS_y \right)^{\frac{1}{q}}$$

(see, e. g., [8], [19] for details).

Let $\Gamma \subset \partial \Omega$ be relatively open. We have

1.1 Let $w \in W^{s,q}(\partial \Omega)$. If $w = 0$ a. e. on $\partial \Omega \setminus \Gamma$, then

$$\int_{\partial \Omega} \int_{\partial \Omega} \frac{|w(x) - w(y)|^q}{|x - y|^{N-1+sq}} dS_x dS_y =$$

$$= \int_{\Gamma} \int_{\Gamma} \frac{|w(x) - w(y)|^q}{|x - y|^{N-1+sq}} dS_x dS_y +$$

$$+ \int_{\Gamma} |w(y)|^q \left( \int_{\partial \Omega \setminus \Gamma} \frac{1}{|x - y|^{N-1+sq}} dS_x \right) dS_y$$

$$+ \int_{\partial \Omega \setminus \Gamma} \left( \int_{\Gamma} \frac{|w(x)|^q}{|x - y|^{N-1+sq}} dS_x \right) dS_y$$

This follows from the additivity of the integral.

We notice that the second and third integral on the right hand side of (A1.1) are equal. Indeed, we have

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\[
\int_{\Gamma} \left( \int_{\partial \Omega \setminus \Gamma} \frac{|w(y)|^q}{|x - y|^{N-1+sq}} dS_y \right) dS_x =
\]

\[
= \int_{\partial \Omega \setminus \Gamma} \left( \int_{\Gamma} \frac{|w(y)|^q}{|x - y|^{N-1+sq}} dS_y \right) dS_x \quad \text{[by Fubini-Tonelli]}
\]

(A1.2) \quad = \int_{\partial \Omega \setminus \Gamma} \left( \int_{\Gamma} \frac{|w(x)|^q}{|x - y|^{N-1+sq}} dS_x \right) dS_y

[change of notation of the variables \(x\) and \(y\)].

1.2 Let \(g \in L^q(\Gamma)\) \((1 < q < +\infty)\), let \(0 < s < 1\) and assume that

(A1.3) \quad \int_{\Gamma} \int_{\Gamma} \frac{|g(x) - g(y)|^q}{|x - y|^{N-1+sq}} dS_x dS_y < +\infty,

(A1.4) \quad \int_{\Gamma} |g(y)|^q \left( \int_{\partial \Omega \setminus \Gamma} \frac{1}{|x - y|^{N-1+sq}} dS_x \right) dS_y < +\infty.

Define

\[
\tilde{g} := \begin{cases} 
  g & \text{a. e. on } \Gamma, \\
  0 & \text{a. e. on } \partial \Omega \setminus \Gamma.
\end{cases}
\]

Then \(\tilde{g} \in W^{s,q}(\partial \Omega)\).

Indeed, firstly \(\tilde{g} \in L^q(\partial \Omega)\). Secondly, from (A1.3) and (A1.4) it follows
\[ +\infty > \int_\Gamma \left( \int_\Gamma \frac{|g(x) - g(y)|^q}{|x-y|^{N-1+sq}} dS_x + \int_{\partial\Omega \setminus \Gamma} \frac{|\tilde{g}(x) - g(y)|^q}{|x-y|^{N-1+sq}} dS_x \right) dS_y \]

\[ + \int_{\partial\Omega \setminus \Gamma} \left( \int_\Gamma \frac{|g(x) - \tilde{g}(y)|^q}{|x-y|^{N-1+sq}} dS_x + \int_{\partial\Omega \setminus \Gamma} \frac{|\tilde{g}(x) - \tilde{g}(y)|^q}{|x-y|^{N-1+sq}} dS_x \right) dS_y \]

[observe (A1.2) with \( g \) in place of \( w \)]

\[ = \int_{\partial\Omega} \left( \int_{\partial\Omega} \frac{|\tilde{g}(x) - \tilde{g}(y)|^q}{|x-y|^{N-1+sq}} dS_x \right) dS_y. \]

**Remark A1.1** Under the above assumptions, for \( y \in \Gamma \) define

\[ \omega(y) = \omega_{s,q}(y) := \int_{\partial\Omega \setminus \Gamma} \frac{1}{|x-y|^{N-1+sq}} dS_x. \]

We have

1) \( \omega \) is continuous on \( \Gamma \),

2) \( \omega(y) \leq \frac{\text{mes}(\partial\Omega \setminus \Gamma)}{(\text{dist}(y, \partial\Omega \setminus \Gamma))^{N-1+sq}} < +\infty \),

3) let \( x_0 \in \partial\Omega \setminus \Gamma \), \( \text{dist}(x_0, \Gamma) = 0 \); if there exists \( a_0 > 0, \rho_0 > 0 \) such that \( \text{mes}((\partial\Omega \setminus \Gamma) \cap B_\rho(x_0)) \geq a_0 \rho^{N-1} \) for all \( 0 < \rho \leq \rho_0 \)

\[ \lim_{y \in \Gamma; y \to x_0} \omega(y) = +\infty. \]

Condition (A1.4) reads

\[(A1.4') \quad \int_{\Gamma} \omega(y)|g(y)|^q dS_y < +\infty.\]

Thus, condition (A1.4) (resp. (A1.4')) expresses a decay property of \( g \) near the boundary \( \partial\Gamma \).

\[ ^{13)} B_\rho(x_0) = \{ \xi \in \mathbb{R}^N : |\xi - x_0| < \rho \} \]

We notice that the condition on \( \text{mes}((\partial\Omega \setminus \Gamma) \cap B_\rho(x_0)) \) occurs in the discussion of Campanato spaces; (see [8; pp. 209-245], [10; p. 32]) for more details.
The above discussion gives rise to introduce the following

**Definition** Let $0 < s < 1$, let $1 < q < +\infty$ and let $\omega$ as in Remark A1.1. Then

\[
W_{00}^{s,q}(\Gamma) := \left\{ g \in W^{s,q}(\Gamma) : \int_{\Gamma} \omega(y) |g(y)|^q dS_y < +\infty \right\}
\]

(cf. the definition of $H_{00}^{\frac{1}{2}}(\Omega)$ in [16; Chap. 1, Thm. 11.7 (with $\mu = 0$ therein)] and the notation $H_{00}^{\frac{1}{2}}(\Gamma)$ in [3; pp. 73, 80 etc.]).

Let $\gamma : W^{1,q}(\Omega) \to W^{1-\frac{1}{q},q}(\partial \Omega)$ ($1 < q < +\infty$) denote the trace mapping (see, e. g., [8], [11], [19], [24; pp. 281-282, 329-330]). To make things clearer, we also write $\gamma_{\Omega}$ in place of $\gamma$.

Summarizing our preceding discussion, we have:

1° Let $h \in W^{1,q}(\Omega)$ satisfy $\gamma(h) = 0$ a. e. on $\partial \Omega \setminus \Gamma$. Then

\[
\gamma(h)|_{\Gamma} \in W_{00}^{1-\frac{1}{q},q}(\Gamma).
\]

2° Let $g \in W_{00}^{1-\frac{1}{q},q}(\Gamma)$. Define

\[
\tilde{g} := \begin{cases} g & \text{a. e. on } \Gamma, \\ 0 & \text{a. e. on } \partial \Omega \setminus \Gamma. \end{cases}
\]

Then there exists $h \in W^{1,q}(\Omega)$ such that

\[
\gamma(h) = \tilde{g} \quad \text{a. e. on } \Gamma.
\]

Indeed, 1° follows immediately from [1.1]. To verify 2°, we notice that our above discussion gives $\tilde{g} \in W^{1-\frac{1}{q},q}(\partial \Omega)$. The claim then follows from the inverse trace theorem (see [8], [19], [24; p. 332]).

We now study the extension of any function $g \in W^{s,q}(\Gamma)$ by zero onto $\partial \Omega \setminus \Gamma$ (i.e. without the decay property (A1.4)).

Let $\{e_1, \ldots, e_n\}$ denote the standard basis in $\mathbb{R}^N$. We introduce

**Assumption (A)** For every $x \in \Gamma \cap (\partial \Omega \setminus \Gamma)$ there exists

(i) a Euclidean basis $\{f_1, \ldots, f_N\}$ in $\mathbb{R}^N$ \footnote{$f_1, \ldots, f_N$ originates from $\{e_1, \ldots, e_N\}$ by shift and rotation.},

(ii) an open cube $\Delta = \{ \tau \in \mathbb{R}^{N-1} : \max\{|\tau_1|, \ldots, |\tau_{N-1}| < \delta \}$,
(iii) a Lipschitz function \( a : \Delta \to \mathbb{R} \) such that in terms of local coordinates \( \xi \in \text{span}\{f_1, \ldots, f_N\} \)\(^{15}\) there holds

1) \( x = (0, \ldots, 0, a(0)) \),

2.1) \( \{\xi \in \mathbb{R}^d : \xi' \in \Delta, \ a(\xi') < \xi_N < a(\xi') + \delta\} \subset \Omega \),

2.2) \( \{\xi \in \mathbb{R}^d : \xi' \in \Delta, \ \xi_N = a(\xi')\} \subset \partial \Omega \),

2.3) \( \{\xi \in \mathbb{R}^d : \xi' \in \Delta, -\delta < \xi_{N-1} < 0, \ \xi_N = a(\xi')\} \subset \Gamma \)

(cf. figure 2).

For what follows we need some more notations.

\[
\Delta^- := \{\xi' \in \Delta : -\delta < \xi_{N-1} < 0\},
\]

\[
\Delta^+ := \{\xi' \in \Delta : 0 < \xi_{N-1} < \delta\}
\]

and

\[
\phi(\xi) := \begin{pmatrix}
\xi_1 \\
\vdots \\
\xi_{N-1} \\
a(\xi') + \xi_N
\end{pmatrix}, \quad \xi = (\xi', \xi_N) \in \Delta \times (-\delta, \delta),
\]

\[
U := \phi(\Delta \times (-\delta, \delta)).
\]

\(^{15}\)For \( \xi = \text{span}\{f_1, \ldots, f_N\} \) we write \( \xi = (\xi', \xi_N), \ \xi' = (\xi_1, \ldots, \xi_{N-1}) \).
Figure 2
We obtain
\[ \Delta = \Delta^- \cup \{ \xi' \in \Delta : \xi_N = 0 \} \cup \Delta^+, \]
\[ |\xi' - \xi'_0|_{\mathbb{R}^{N-1}} \leq |\phi(\xi) - \phi(\xi')|_{\mathbb{R}^N} \leq c_0 |\xi' - \xi'_0|_{\mathbb{R}^{N-1}} \quad \forall \xi, \hat{\xi} \in \Delta \times (-\delta, \delta), \]
\[ \phi^{-1}(\eta) := \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{N-1} \\ \eta_N - a(\eta') \end{pmatrix}, \quad \eta = (\eta', \eta_N) \in U. \]

Then conditions 1) and 2.1)- 2.3) can be equivalently stated as follows:

1’) \( \phi(0) = (0, \ldots, 0, a(0))^\top, \)
2.1’) \( \phi(\Delta \times (0, \delta)) = \Omega \cap U, \)
2.2’) \( \phi(\Delta \times \{0\}) = \partial \Omega \cap U, \)
2.3’) \( \phi(\Delta^- \times \{0\}) = \Gamma \cap U. \)

**Theorem A1.1** Let assumption (A) be satisfied and let \( 1 < q < +\infty. \) For \( g \in W^{s,q}(\Gamma), \) define
\[ \tilde{g} := \begin{cases} g & \text{a. e. on } \Gamma, \\ 0 & \text{a. e. on } \partial \Omega \setminus \Gamma. \end{cases} \]

If \( s < \frac{1}{q}, \) then \( \tilde{g} \in W^{s,q}(\partial \Omega) \) and
\[ ||\tilde{g}||_{W^{s,q}(\partial \Omega)} \leq c ||g||_{W^{s,q}(\Gamma)}. \]

**Proof** The definition of the Lipschitz continuity of \( \partial \Omega \) implies the existence of Euclidean coordinate systems \( \{f_{\alpha1}, \ldots, f_{\alpha N}\} \) in \( \mathbb{R}^N, \) open cubes \( \Delta_\alpha \subset \mathbb{R}^{N-1} \) and Lipschitz functions \( a_\alpha : \Delta_\alpha \to \mathbb{R} \) \( (\alpha = 1, \ldots, m) \) such that 2.1) and 2.2) hold with \( \Delta_\alpha \) and \( a_\alpha \) in place of \( \Delta \) and \( a, \) respectively (see, e. g. [8; pp. 304-306], [10; pp. 21-25], [11; pp. 5-7]). It follows
\[ \partial \Omega \subset \bigcup_{\alpha=1}^m U_\alpha, \] where
\[ U_\alpha := \phi_\alpha(\Delta_\alpha \times (-\delta_\alpha, \delta_\alpha)) \]
(recall $\phi_\alpha(\xi) = (\xi', a_\alpha(\xi') + \xi_N)^T$, $\xi = (\xi', \xi_N) \in \Delta \times (-\delta, \delta)$). By 2.2), $x_\alpha = (0, \ldots, 0, a_\alpha(0)) \in \partial \Omega$.

If $\Gamma \cap U_\alpha \subset \Gamma$ or $(\partial \Omega \setminus \Gamma) \cap U_\alpha \subset \partial \Omega \setminus \Gamma$ there is nothing to prove. Therefore, it suffices to consider a local representation $\{\{f_{\alpha 1}, \ldots, f_{\alpha N}\}, \Delta_\alpha, a_\alpha\}$ of $\partial \Omega$ such that $x_\alpha \in \bar{\Gamma} \cap (\partial \Omega \setminus \Gamma)$. Then 2.3) of assumption (A) implies

$$\{\xi \in \mathbb{R}^d : \xi' \in \Delta, \ -\delta_\alpha < \xi_{N-1} < 0, \ \xi_N = a_\alpha(\xi')\} = \Gamma \cap U_\alpha.$$ 

For notational simplicity, in what follows we omit the index $\alpha$.

Let $g \in W^{s,q}(\Gamma)$. By 2.3),

$$\int_{\Gamma \cap U} \int_{\Gamma \cap U} \frac{|g(x) - g(y)|^q}{|x - y|^{N-1+sq}} dS_x dS_y = \int_{\Delta^-} \int_{\Delta^-} \frac{|\phi(\xi', a(\xi')) - \phi(\eta', a(\eta'))|^q}{|\phi(\xi', a(\xi')) - \phi(\eta', a(\eta'))|^{N-1+sq}} \sqrt{1 + |\nabla a(\xi')|^2} \sqrt{1 + |\nabla a(\eta')|^2} d\xi' d\eta' \geq c \int_{\Delta^-} \int_{\Delta^-} \frac{|g \circ \phi(\xi', a(\xi')) - g \circ \phi(\eta', a(\eta'))|^q}{|\xi' - \eta'|^{N-1+sq}} d\xi' d\eta'.$$

Next, define $z(\xi') := g \circ \phi(\xi', a(\xi'))$ for a.e. $\xi' \in \Delta^-$, and

$$\tilde{z} := \begin{cases} z & \text{a.e. in } \Delta^-, \\ 0 & \text{a.e. in } \Delta^+. \end{cases}$$

Then $z \in W^{s,q}(\Delta^-)$, and

$$\tilde{z} = \tilde{g} \circ \phi \text{ a.e. in } \Delta, \quad \tilde{g} = \tilde{z} \circ \phi^{-1} \text{ a.e. in } \partial \Omega \cap U.$$

Now from [25; Thm. 3.5] (see also [16; Chap. 1, Thm. 11.4] for $q = 2$) it follows that

$$z \in W^{s,q}(\Delta), \quad \|z\|_{W^{s,q}(\Delta)} \leq c \|z\|_{W^{s,q}(\Delta^-)}.$$ 

We obtain
The proof of the theorem is now easily completed by standard arguments.

**Remark A1.2** If \( s = \frac{1}{q} \), then the statement of Theorem A1.1 fails.

2 Let \( \Omega_1, \Omega_2 \subset \mathbb{R}^d \) (\( d = 2 \) or \( d = 3 \)) be bounded domains such that

\[ \Omega_1 \cap \Omega_2 = \varnothing, \quad \Gamma := \overline{\Omega}_1 \cap \overline{\Omega}_2 \neq \varnothing, \]

\( \partial \Omega_i \) Lipschitz, \( \Gamma \) relatively open in \( \partial \Omega_i \) \((i = 1, 2)\) (cf. Section 1). Let \( \gamma_{\Omega_i} : W^{1,q}(\Omega_i) \rightarrow W^{1,\frac{q}{2}}(\partial \Omega_i) \) \((1 < q < +\infty)\) denote the trace mapping (cf. above). In what follows, we write \( \gamma_i = \gamma_{\Omega_i} \). For \( u_i \in W^{1,2}(\Omega_i) \) the trace \( \gamma_i(u_i) \) is understood componentwise. By Sobolev’s embedding theorem,

\begin{align*}
(A1.7) \quad \left\{ \begin{array}{l}
|u_i|^2 \in W^{1,q}(\Omega_i) \quad \text{where} \\
1 \leq q < 2 \quad \text{arbitrary if} \quad d = 2, \quad q = \frac{3}{2} \quad \text{if} \quad d = 3.
\end{array} \right.
\end{align*}

Then \( \gamma_i(|u_i|^2) \in W^{1,\frac{q}{2}}(\partial \Omega_i) \).

Let us consider

\[ u_i \in W^{1,2}(\Omega_i), \quad \gamma_i(u_i) = 0 \quad \text{a.e. on} \quad \partial \Omega_i \setminus \Gamma. \]
For notational simplicity, set \( v_i := \gamma_i(u_i) \) a. e. on \( \Gamma \). Then \( v_i \in W^{1,2}(\Gamma), \ |v_i|^2 \in W^{1-\frac{2}{q},q}(\Gamma) \)

and

\[
(A1.8) \quad \int_{\Gamma} |v_i(y)|^{2q} \left( \int_{\partial \Omega_i \setminus \Gamma} \frac{1}{|x-y|^{d-2+q}} dS_x \right) dS_y < +\infty
\]

(cf. (A1.1). To homogenize boundary condition (2.3), we have to consider the following

**Problem (P)** Define \( g := |v_1 - v_2|^2 \) a. e. on \( \Gamma \), and

\[
\bar{g}_i := \begin{cases} 
g & \text{a. e. on } \Gamma, \\
0 & \text{a. e. on } \partial \Omega_i \setminus \Gamma. \end{cases}
\]

Does there exist \( \tilde{h}_i \in W^{1,q}(\Omega_i) \) such that \( \gamma_i(\tilde{h}_i) = \bar{g}_i \) a. e. on \( \partial \Omega_i \)?

An answer to this problem can be given by imposing the following condition on the geometry of \( \Omega_1 \) and \( \Omega_2 \) "near to the interface \( \Gamma = \Omega_1 \cap \Omega_2 \):

**Assumption (B)** For every \( y \in \Gamma \), there holds

\[
\int_{\partial \Omega_1 \setminus \Gamma} \frac{1}{|x-y|^{d-2+q}} dS_x = \int_{\partial \Omega_2 \setminus \Gamma} \frac{1}{|x-y|^{d-2+q}} dS_x \quad \text{for all } y \in \Gamma
\]

\((q \text{ as in (A1.7)})\)

We obtain the following result.

Let assumption (B) be satisfied. Let be \( u_i \in W^{1,2}(\Omega_i), \gamma_i(u_i) = 0 \) a. e. on \( \Omega_i \setminus \Gamma \) \((i = 1, 2)\). Set \( v_i := \gamma_i(u_i) \) a. e. on \( \Gamma \). If

\[
(A1.9) \quad |v_1 - v_2|^2 \in W^{1-\frac{2}{q},q}(\Gamma) \quad (q \text{ as in (A1.7)}),
\]

then there exists \( \tilde{h}_i \in W^{1,q}(\Omega_i) \) such that

\[
\gamma_i(\tilde{h}_i) = \bar{g}_i \quad \text{a. e. on } \partial \Omega_i.
\]

\(^{16}\)The definition of the trace mapping implies

\[
(\gamma(\varphi))^2 = (\varphi|_\Gamma)^2 = \varphi^2|_\Gamma = \gamma(\varphi^2)
\]

for every \( \varphi \in C^{1}(\Omega) \). Thus, by approximation

\[
|u_i|^2 = \sum_{i=1}^{d} (\gamma_i(u_{il}))^2 = \sum_{i=1}^{d} \gamma_i(u_{il}^2) = \gamma_i \left( \sum_{i=1}^{d} u_{il}^2 \right) = \gamma_i(|u_i|^2).
\]
Indeed, combining (A1.8) and assumption (B) we find
\[ \int_{\Gamma} |(v_1 - v_2)(y)|^{2q} \left( \int_{\partial \Omega_1 \setminus \Gamma} \frac{1}{|x-y|^{d-2+q}} dS_x \right) dS_y < +\infty \quad (i = 1, 2). \]

Observing (A1.9) we see that (A1.3) and (A1.4) are satisfied with \( g = |v_1 - v_2|^2 \), \( N = d \), \( s = 1 - \frac{1}{q} \), and \( \Omega = \Omega_i \). The claim follows from [1.2] above.

It is easily verified that this result continues to hold for \( G_i(|v_1 - v_2|^2) \) in place of \( |v_1 - v_2|^2 \).

We notice that assumption (B) is satisfied if \( \Omega_1 \) and \( \Omega_2 \) obey an appropriate symmetry property with respect to \( \Gamma \).

**Remark A1.2** Assumption (A1.9) is equivalent to
\[(A1.9') \quad v_1 \cdot v_2 \in W^{1,\frac{1}{q}}(\Gamma).\]

This is readily seen when observing the elementary identity
\[ |a - b|^2 - |\hat{a} - \hat{b}|^2 = |a|^2 - |\hat{a}|^2 + (|b|^2 - |\hat{b}|^2) - 2(a \cdot b - \hat{a} \cdot \hat{b}) \]
\((a, \hat{a}, b, \hat{b} \in \mathbb{R}^d).\)

**Remark A1.3** We notice that (A1.9') is true in case \( d = 2 \). To see this, first observe that \( W^{\frac{1}{2},2} (\partial \Omega_i) \subset L^r(\partial \Omega_i) \) \((1 \leq r < +\infty \) arbitrary). We obtain, for every \( 1 \leq q < 2 \),
\[ \int_{\Gamma} \int_{\Gamma} \frac{|v_i(x) - v_i(y)|^q}{|x-y|^q} |v_j(x)|^q dS_x dS_y \leq \]
\[ \leq \int_{\Gamma} \left( \int_{\Gamma} \frac{|v_i(x) - v_i(y)|^2}{|x-y|^2} dS_x \right)^{\frac{q}{2}} \left( \int_{\Gamma} |v_j(x)|^{\frac{2q}{2-q}} dS_x \right)^{\frac{2-q}{2}} dS_y \]
\[ \leq (\text{mes } \Gamma)^{\frac{2-q}{2}} \frac{2}{3} \frac{q}{2-q} \|v_i\|_{W^{\frac{1}{2},2}(\Gamma)}^q \|v_j\|_{L^{\frac{2q}{2-q}}(\Gamma)}^q \]
\((i, j = 1, 2; i \neq j)\). Whence (A1.9').

We obtain: if \( d = 2 \) and assumption (B) holds, then problem \((P)\) has a solution.

**Theorem A1.2** Suppose that \( \Gamma \cap (\partial \Omega_i \setminus \Gamma) \) \((i = 1, 2)\) satisfies assumption (A). Let \( v_1, v_2 \in W^{\frac{1}{2},2}(\Gamma) \). Define \( g := |v_1 - v_2|^2 \) a. e. on \( \Gamma \), and
\[ \tilde{g}_i := \begin{cases} g & \text{a.e. on } \Gamma, \\ 0 & \text{a.e. on } \partial\Omega_i \setminus \Gamma. \end{cases} \]

Then

\[ \tilde{g}_i \in \bigcap_{1 \leq q < \frac{d}{d-1}} W^{1, \frac{1}{2}q}(\partial\Omega_i), \]

\[ \|\tilde{g}_i\|_{W^{1, \frac{1}{2}q}(\partial\Omega_i)} \leq c \|v_1 - v_2\|_{W^{\frac{1}{2}, 2}(\Gamma)} \|v_1 - v_2\|_{L^r(\Gamma)}, \]

where

\[ c = c(q) \to +\infty \text{ as } q \to \frac{d}{d-1}, \quad \left( 1 \leq q < \frac{d}{d-1} \right), \]

\[ r = \frac{2q}{2 - q} \text{ if } d = 2, \quad r = 4 \text{ if } d = 3. \]

**Proof** \[ d = 2 \] First, notice \( W^{\frac{1}{2}, 2}(\partial\Omega) \subset L^r(\partial\Omega) \) \((1 \leq r < +\infty)\) continuously. Observing that

\[ \|a - b\|^2 - |\hat{a} - \hat{b}|^2 \leq |a - b - (\hat{a} - \hat{b})| |a - b + (\hat{a} - \hat{b})|, \quad a, \hat{a}, b, \hat{b} \in \mathbb{R}^N, \]

we obtain by the aid of Hölder’s inequality, for every \(1 \leq q < 2\),

\[ \int_{\Gamma} \int_{\Gamma} \frac{|g(x) - g(y)|^q}{|x - y|^q} dS_x dS_y \leq \]

\[ \leq \left( \int_{\Gamma} \int_{\Gamma} \frac{|v_1(x) - v_2(x) - (v_1(y) - v_2(y))|^2}{|x - y|^2} dS_x dS_y \right)^{\frac{q}{2}} \times \]

\[ \times \left( \int_{\Gamma} \int_{\Gamma} \frac{|v_1(x) - v_2(x) + (v_1(y) - v_2(y))|^{2q}}{|x - y|^{2q}} dS_x dS_y \right)^{\frac{2-q}{q}} \]

\[ \leq c \|v_1 - v_2\|^q_{W^{\frac{1}{2}, 2}(\Gamma)} \|v_1 - v_2\|^q_{L^{\frac{2q}{d}}(\Gamma)}. \]

Thus, \( g \in W^{1, \frac{1}{2}q}(\Gamma) \) and
\[
\|g\|_{W^{1-\frac{1}{q}}(\Gamma)}^q \leq \left( \int_{\Gamma} |v_1 - v_2|^{2q} \right)^{2-q} + \int_{\Gamma} \int_{\Gamma} \frac{|g(x) - g(y)|^q}{|x - y|^q} \, dS_x dS_y \\
\leq c\|v_1 - v_2\|_{W^{q}_{2}}(\Gamma)\|v_1 - v_2\|_{L^{2q}}(\Gamma).
\]

On the other hand, Theorem A1.1 (with \(\Omega = \Omega_i\), \(s = 1 - \frac{1}{q}, s < \frac{1}{q}\)) gives

\[
\tilde{g}_i \in W^{1-\frac{1}{q}}(\partial\Omega_i), \quad \|\tilde{g}_i\|_{W^{1-\frac{1}{q}}(\partial\Omega_i)} \leq c\|g\|_{W^{1-\frac{1}{q}}(\Gamma)}.
\]

Whence the claim.

\[d = 3\] Then \(W^{1,2}(\partial\Omega) \subset L^4(\partial\Omega)\) continuously. Hence \(g \in L^2(\Gamma)\). We divide the proof into two steps.

**Step 1** For every \(0 < \delta < 1\), there holds

\[
\left\{ \begin{array}{l}
g \in W^{1-\frac{\delta}{q}}(\Gamma), \\
\int_{\Gamma} \int_{\Gamma} \frac{|g(x) - g(y)|^{\frac{q}{3}}}{|x - y|^{2 + \frac{2(1-\delta)}{3}}} dS_x dS_y \leq c\|v_1 - v_2\|_{W^{1-\frac{1}{q}}(\Gamma)}\|v_1 - v_2\|_{L^{4}}(\Gamma).
\end{array} \right.
\]

Indeed, with the help of the above inequality for \(a, \hat{a}, b, \hat{b} \in \mathbb{R}^N\) and Hölder’s inequality we find

\[
\int_{\Gamma} \int_{\Gamma} \frac{|g(x) - g(y)|^{\frac{q}{3}}}{|x - y|^{2 + \frac{2(1-\delta)}{3}}} dS_x dS_y \leq \\
\leq c \int_{\Gamma} \int_{\Gamma} \frac{|v_1(x) - v_2(x) - |v_1(y) - v_2(y)|^2|^{\frac{q}{3}}}{|x - y|^{2 + \frac{2(1-\delta)}{3}}} dS_x dS_y \\
\leq c \int_{\Gamma} \int_{\Gamma} \frac{|v_1(x) - v_2(x) - (v_1(y) - v_2(y))|^{\frac{q}{3}}}{|x - y|^2} \times \\
\times \frac{|v_1(x) - v_2(x) + (v_1(y) - v_2(y))|^{\frac{q}{3}}}{|x - y|^{\frac{2(1-\delta)}{3}}} dS_x dS_y \\
\leq c \int_{\Gamma} \int_{\Gamma} \frac{|v_1(x) - v_2(x) + (v_1(y) - v_2(y))|^{\frac{q}{3}}}{|x - y|^{\frac{2(1-\delta)}{3}}} dS_x dS_y
\]

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\[ (A1.11) \leq c \| \mathbf{v}_1 - \mathbf{v}_2 \|_{W^{3,2}(\Gamma)} ^{\frac{4}{3}} \times \]
\[ \left( \int_{\Gamma} \int_{\Gamma} \frac{(|\mathbf{v}_1(x) - \mathbf{v}_2(x)| + |\mathbf{v}_1(y) - \mathbf{v}_2(y)|)^4}{|x-y|^{2(1-\delta)}} dS_x dS_y \right)^{\frac{3}{4}} \]

(notice that \( W^{\frac{1}{2},2}(\Gamma) \subset L^4(\Gamma) \)).

Next, by elementary integral calculus it is easily seen that there exists a positive constant \( K_0 \) such that
\[ \int_{\partial \Omega_i} \frac{1}{|x-y|^{2(1-\delta)}} dS_y \leq K_0 \quad \forall \ x \in \partial \Omega_i \quad (i = 1, 2) \]

\((K_0 = K_0(\delta) \to +\infty \text{ as } \delta \to 0)\). Then the second double integral on the right hand side of (A1.11) can be estimated as follows

\[ \int_{\Gamma} \int_{\Gamma} \frac{(|\mathbf{v}_1(x) - \mathbf{v}_2(x)| + |\mathbf{v}_1(y) - \mathbf{v}_2(y)|)^4}{|x-y|^{2(1-\delta)}} dS_x dS_y \leq \]
\[ \leq 16 \int_{\Gamma} |\mathbf{v}_1(x) - \mathbf{v}_2(x)| \left( \int_{\Gamma} \frac{1}{|x-y|^{2(1-\delta)}} dS_y \right) dS_x \]
\[ + 16 \int_{\Gamma} |\mathbf{v}_1(y) - \mathbf{v}_2(y)| \left( \int_{\Gamma} \frac{1}{|x-y|^{2(1-\delta)}} dS_x \right) dS_y \]
\[ \leq 32 K_0 \| \mathbf{v}_1 - \mathbf{v}_2 \|_{L^4(\Gamma)} ^{4/3} \]

Inserting this estimate into (A1.11) we find (A1.10) \((c = c(\delta) \to +\infty \text{ as } \delta \to 0)\).

**Step 2** From Theorem A1.1 (with \( \Omega = \Omega_i \), \( s = \frac{1-\delta}{2} \), \( q = \frac{4}{3} \)) and (A1.10) it follows that

\[ \tilde{g} \in W^{\frac{1-\delta}{2}, \frac{4}{3}}(\partial \Omega_i), \]
\[ \| \tilde{g} \|_{W^{\frac{1-\delta}{2}, \frac{4}{3}}(\partial \Omega_i)} \leq c \| g \|_{W^{\frac{1-\delta}{2}, \frac{4}{3}}(\Gamma)} = \]
\[ = c \left( \| g \|_{L^\frac{8}{3}(\Gamma)} ^{\frac{4}{3}} + \int_{\Gamma} \int_{\Gamma} \frac{|\tilde{g}(x) - \tilde{g}(y)|^\frac{4}{3}}{|x-y|^{2(1-\delta) + \frac{4}{3}}} dS_x dS_y \right)^{\frac{3}{4}} \]
\[ \leq \left( \| \mathbf{v}_1 - \mathbf{v}_2 \|_{L^\frac{8}{3}(\Gamma)} ^{\frac{4}{3}} + \| \mathbf{v}_1 - \mathbf{v}_2 \|_{W^{\frac{1}{2},2}(\Gamma)} \| \mathbf{v}_1 - \mathbf{v}_2 \|_{L^4(\Gamma)} \right) \]

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To proceed, we notice the continuous embedding

\[(A1.13) \quad W^{\frac{1-d}{2}, \frac{1}{q}}(\partial \Omega_i) \subset W^{\frac{1-d}{2} - \alpha, \frac{1}{2q}}(\partial \Omega_i) \quad \left(0 < \alpha < \frac{1-\delta}{2}\right)\]

(see, e.g., [1], [25; p. 328, \(n = d - 1 = 2\) in (8)]).

Now, consider \(q\) such that \(\frac{4}{3} < q < \frac{3}{2}\). Define

\[\delta := \frac{2(3 - 2q)}{q}, \quad \alpha := \frac{1 - 2\delta}{6}.\]

It follows

\[\frac{1 - \delta}{2} - \alpha = 1 - \frac{1}{q}, \quad \frac{4}{3 - 2\alpha} = q.\]

By combining (A1.12) and (A1.13) we obtain the statement of Theorem A1.2 when \(d = 3\).

\[\square\]

**Corollary A1.1** Suppose that \(\Gamma \cap (\partial \Omega_i \setminus \Gamma) (i = 1, 2)\) satisfies assumption (A). Let be \(u_i \in W^{1,2}(\Omega_i)\) such that

\[\gamma_i(u_i) = 0 \quad a. e. \ on \quad \partial \Omega_i \setminus \Gamma.\]

Define

\[\bar{h}_i := \begin{cases} G_i(\|\gamma_1(u_1) - \gamma_2(u_2)\|_{W^{1,2}}^2) & a. e. \ on \ \Gamma, \\ 0 & a. e. \ on \ \partial \Omega_i \setminus \Gamma \end{cases}\]

\((G_i \ as \ in \ (1.6); \ i = 1, 2)\).

Then, for every \(1 \leq q < \frac{d}{d-1}\),

\[\bar{h}_i \in W^{1-\frac{1}{q}}(\partial \Omega_i), \quad \|\bar{h}_i\|_{W^{1-\frac{1}{q}, q}(\partial \Omega_i)} \leq c \sum_{j=1}^{2} \|u_j\|_{W^{1,2}(\Omega_j)}^2,\]

\[33\]
where $c = c(q) \to +\infty$ as $q \to \frac{d}{d-1}$.

**Proof** As above, for notational simplicity, set $v_i := \gamma_i(u_i)$ and $h_i := G_i(|v_1 - v_2|^2)$ a.e. on $\Gamma$ $(i = 1, 2)$. Then

$$
\tilde{h}_i := \begin{cases} 
  h_i & \text{a.e. on } \Gamma, \\
  0 & \text{a.e. on } \partial \Omega_i \setminus \Gamma 
\end{cases}
$$

and

$$
|h_i(x) - h_i(y)| \leq c_0 \left| v_1(x) - v_2(x) \right|^2 - \left| v_1(y) - v_2(y) \right| \quad \text{for a.e. } x, y \in \Gamma.
$$

It is readily seen that the proof of Theorem A1.2 can be repeated word by word with $h_i$ and $\tilde{h}_i$ in place of $g$ and $\tilde{g}_i$, respectively. We obtain

$$
\tilde{h}_i \in \bigcap_{1 \leq q < \frac{d}{d-1}} W^{1 - \frac{1}{q}, q}(\partial \Omega_i),
$$

$$
\|\tilde{h}_i\|_{W^{1 - \frac{1}{q}, q}(\partial \Omega_i)} \leq c \|v_1 - v_2\|_{W^{\frac{1}{2}, q}(\Gamma)} \|v_1 - v_2\|_{L^r(\Gamma)},
$$

where $r$ is as in Theorem A1.2.

Combining this and the continuity of the trace mapping $\gamma_i : W^{1, 2}(\Omega_i) \to W^{\frac{1}{2}, 2}(\partial \Omega_i)$ we get the assertion of the corollary.

**Appendix 2. The inhomogeneous Dirichlet problem for the Poisson equation with right hand side in $L^1$**

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with boundary $\partial \Omega \in C^1$. We consider the following boundary value problem:

(A2.1) \quad \begin{align*}
-\Delta u &= f \quad \text{in} \quad \Omega, \\
\end{align*}

(A2.2) \quad \begin{align*}
u &= g \quad \text{on} \quad \partial \Omega.
\end{align*}

Our basic existence result concerning weak solutions to this problem is

**Theorem A2.1** Assume

$$
f \in L^1(\Omega), \quad g \in W^{1 - \frac{1}{q}, q}(\partial \Omega) \quad \left(1 < q < \frac{N}{N-1}\right)
$$

Then, there exists exactly one $u \in W^{1, q}(\Omega)$ such that
\( \int_{\Omega} \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi \quad \forall \ \varphi \in W^{1,q'}_0(\Omega), \)

(A2.4) \( u = g \quad \text{on} \quad \partial \Omega, \)

(A2.5) \( \|u\|_{W^{1,q}} \leq c(\|f\|_{L^1} + \|g\|_{W^{1-\frac{1}{q}}}) \)

Moreover, for every \( \Omega' \subset \subset \Omega \) and every \( \delta > 0 \) there holds

\[
\begin{cases}
\frac{\nabla u}{(1 + |u|)^{\frac{1+\delta}{2}}} \in L^2(\Omega'), \\
\int_{\Omega'} \frac{|\nabla u|^2}{(1 + |u|)^{1+\delta}} \leq \frac{c}{\delta}(\|f\|_{L^1} + \|g\|_{W^{1-\frac{1}{q}}})
\end{cases}
\]

where \( c \to +\infty \) as \( \text{dist} (\Omega', \partial \Omega) \to 0. \)

If, in addition, \( f \in L^r_{\text{loc}}(\Omega) \) (\( r > 1 \)) then

(A2.7) \( u \in W^{2,r}_{\text{loc}}(\Omega). \)

**Proof** We begin by noting the following result. For every \( 1 < q < +\infty \) there exists a positive constant \( C_q \) such that, for any \( v \in W^{1,q}_0(\Omega), \)

\[
\|\nabla v\|_{L^q} \leq C_q \sup_{\varphi \in W^{1,q'}_0(\Omega), \ \varphi \neq 0} \left\{ \frac{\int_{\Omega} \nabla v \cdot \nabla \varphi}{\|\nabla \varphi\|_{L^{q'}}} \right\}
\]

(see. [21; Thm. 4.2, p. 191]).

Next, by the inverse trace theorem, there exists \( h \in W^{1,q}(\Omega) \) such that

\[
\gamma(h) = g \quad \text{a. e. on} \quad \partial \Omega, \quad \|h\|_{W^{1,q}} \leq c\|g\|_{W^{1-\frac{1}{q}}}
\]

Then we can find functions \( f_m, h_m \in C^\infty(\bar{\Omega}) \) (\( m \in \mathbb{N} \)) such that

\[
f_m \to f \quad \text{strongly in} \quad L^1(\Omega), \quad h_m \to h \quad \text{strongly in} \quad W^{1,q}(\Omega)
\]

as \( m \to \infty. \) The Riesz representation theorem for linear continuous functionals on the Hilbert space \( W^{1,2}_0(\Omega) \) provides the existence and uniqueness of a \( v_m \in W^{1,2}_0(\Omega) \) satisfying

\[
\int_{\Omega} \nabla v_m \cdot \nabla \varphi = \int_{\Omega} (f_m \varphi + (\partial_i h_m) \partial_i \varphi) \quad \forall \ \varphi \in W^{1,2}_0(\Omega).
\]
Now, let $1 < q < \frac{N}{N-1}$. Observing that $W^{1,q}(\Omega) \subset C(\overline{\Omega})$ we obtain

$$\left| \int_{\Omega} (f_m \phi + (\partial_i h_m) \partial_i \phi) \right| \leq c(\|f_m\|_{L^1} + \|h_m\|_{W^{1,q}})\|\phi\|_{W^{1,q}} \quad \forall \phi \in W^{1,q}_0(\Omega).$$

Combining this estimate and (A2.8), (A2.9) gives

$$\|\nabla v_m\|_{L^q} \leq c(\|f_m\|_{L^1} + \|h_m\|_{W^{1,q}}).$$

Define $u_m := v_m + h_m$ ($m \in \mathbb{N}$). Then $u_m \in W^{1,2}(\Omega)$ and

(A2.10) $\int_{\Omega} \nabla u_m \cdot \nabla \phi = \int_{\Omega} f_m \phi \quad \forall \phi \in W^{1,2}_0(\Omega),$

(A2.11) $u_m = h_m \text{ a. e. on } \partial \Omega \quad \text{[in the sense of traces]},$

(A2.12) $\|\nabla u_m\|_{L^q} \leq c(\|f_m\|_{L^q} + \|h_m\|_{W^{1,q}}).$

From (A2.12) we conclude (by passing to a subsequence if necessary) that $u_m \to u$ weakly in $W^{1,q}(\Omega)$ as $m \to \infty$. By a routine argument, $u = g$ a. e. on $\partial \Omega$ (in the sense of traces).

The passage to the limit $m \to \infty$ in (A2.10), (A2.11) gives (A2.3), (A2.4), respectively.

Finally, taking the lim inf on both sides of (A2.12) provides (A2.5).

The uniqueness of $u$ follows from (A2.5).

To prove the interior estimate (A2.6), let $\delta > 0$. We consider the function

$$\phi(t) = \phi_\delta(t) := \left(1 - \frac{1}{(1 + |t|)^\delta}\right) \text{ sign } t, \quad t \in \mathbb{R}.$$  

Clearly,

$$|\phi(t)| \leq 1, \quad \phi'(t) = \frac{\delta}{(1 + |t|)^{1+\delta}} \quad \forall t \in \mathbb{R}.$$  

Let $\zeta \in C^1_c(\Omega)$ be a cut-off function for $\Omega'$, i. e. $\zeta \equiv 1$ on $\Omega'$ and $0 \leq \zeta \leq 1$ in $\Omega$. Then the function $\varphi = \phi(u_m)\zeta^2$ is admissible in (A2.10). By (A2.12),

$$\delta \int_{\Omega'} \frac{\|
abla u_m\|^2}{(1 + |u_m|)^{1+\delta}} \leq \|f_m\|_{L^1} + 2 \max_{\Omega} |\nabla \zeta| \int_{\Omega} |\nabla u_m| \leq \|f_m\|_{L^1} + 2 \max_{\Omega} |\nabla \zeta| (\text{mes } \Omega)^{\frac{1}{q'}} \cdot c(\|f_m\|_{L^1} + \|h_m\|_{W^{1,q}}).$$

Thus,
\( \int \frac{\nabla u_m^2}{(1 + |u_m|)^{1+\delta}} \leq \frac{C}{\delta} \quad \forall \ m \in \mathbb{N} \quad (C = \text{const}). \)

As above, we may assume that \( u_m \rightarrow u \) weakly in \( W^{1,q}(\Omega) \) and, in addition, \( u_m \rightarrow u \) a.e. in \( \Omega \). These convergence properties together with (A2.13) imply

\[
\frac{\nabla u_m}{(1 + |u_m|)^{\frac{1+r}{2}}} \rightharpoonup \frac{\nabla u}{(1 + |u|)^{\frac{1+r}{2}}} \quad \text{weakly in} \quad L^2(\Omega') \quad \text{as} \quad m \rightarrow \infty.
\]

Whence (2.6).

To prove (A2.7), we first note that \( W^{1,q}(\Omega) \subset L^{\frac{Nq}{N-q}}(\Omega) \). Now, let \( B_R \) be a ball such that \( B_{2R} \subset \Omega \). Let \( 1 < r \leq \frac{Nq}{N-q} \). Then \( u \in L^r(B_{2R}) \) and

\[
\left| \int_{B_{2R}} u \Delta \varphi \right| \leq \|f\|_{L^r(B_{2R})} \|\varphi\|_{L^r(B_{2R})} \quad \forall \ \varphi \in C_c(\mathbb{B}(2R)) \quad [\text{by (A2.3)}].
\]

From [20; Thm. 9.5 (3), p. 144] it follows

\[
u \in W^{2,r}(B_R), \quad \|u\|_{W^{2,r}(B_R)} \leq c(\|f\|_{L^r(B_{2R})} + \|u\|_{L^r(B_{2R})}).
\]

Hence, (A2.7) holds for all values of \( r \) satisfying \( 1 < r \leq \frac{Nq}{N-q} \). By a bootstrapping argument, (A2.7) can be proved for any \( r > \frac{Nq}{N-q} \). \( \blacksquare \)

**Remark A2.1** We notice that the existence and uniqueness result stated in Theorem A2.1, follows from the \( L^p \)-theory of linear elliptic boundary value problems developed in [15], provided the boundary \( \partial \Omega \) is sufficiently smooth. Theorem A2.1 is also an immediate consequence of [20; Thm. 10.7, pp. 181-182; \( \partial \Omega \in C^1 \)].

On the other hand, the existence of a weak solution \( u \in \bigcap_{1 < q < \frac{N}{N-1}} W^{1,q}(\Omega) \) to linear elliptic equations in divergence form with bounded measurable coefficients, right hand sides in \( L^1 \) and zero boundary condition has been proved in [23] by a duality argument.

**Remark A2.2** Our approximation procedure for solving boundary value problem (A2.1), (A2.2) permits to prove additional properties of the weak solution \( u \in W^{1,q}(\Omega) \) (for instance, the interior estimate (A2.6)). Moreover, we have

**Theorem A2.2** Let the assumptions of Theorem A2.1 hold. Let \( u \in W^{1,q}(\Omega) \) satisfy (A2.3)-(A2.5). Then

1° if \( f \geq 0 \) a.e. in \( \Omega \) and \( g \geq 0 \) a.e. on \( \partial \Omega \), then

\[ u \geq 0 \quad \text{a. e. in} \quad \Omega; \]
2° if \( f \in L^r_{\text{loc}}(\Omega) \ (r > \frac{N}{2}) \), then
\[
\text{ess sup}_{\Omega'} |u| < +\infty \quad \forall \ \Omega' \subset\subset \Omega;
\]

3° if \( f \in L^r(\Omega) \ (r > \frac{N}{2}) \), \( \text{ess sup}_{\partial\Omega} |g| < +\infty \), then
\[
\text{ess sup}_{\Omega} |u| < +\infty.
\]

This theorem can be proved by the methods developed in [5] and [23].

References


