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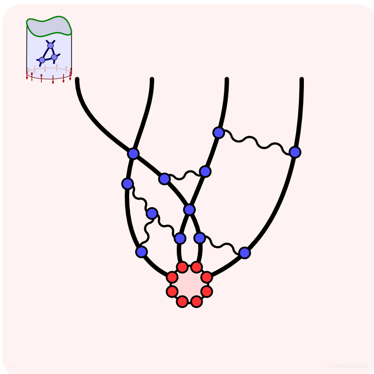
Review of AdS/CFT Integrability, Chapter I.2: The spectrum from perturbative gauge theory

C. Sieg

Institut für Mathematik und Institut für Physik, Humboldt-Universität zu Berlin
Johann von Neumann Haus, Rudower Chaussee 25, 12489 Berlin, Germany

Niels Bohr International Academy, Niels Bohr Institute
Blegdamsvej 17, 2100 Copenhagen, Denmark

csieg@math.hu-berlin.de



Abstract: We review the constructions and tests of the dilatation operator and of the spectrum of composite operators in the flavour $SU(2)$ subsector of $\mathcal{N} = 4$ SYM in the planar limit by explicit Feynman graph calculations with emphasis on analyses beyond one loop. From four loops on, the dilatation operator determines the spectrum only in the asymptotic regime, i.e. to a loop order which is strictly smaller than the number of elementary fields of the composite operators. We review also the calculations which take a first step beyond this limitation by including the leading wrapping corrections.

1 Introduction

In the context of the AdS/CFT correspondence [1, 7, 18], the discovery of integrability is a key ingredient towards finding the exact spectrum of strings in $AdS_5 \times S^5$ and of composite operators in $\mathcal{N} = 4$ SYM theory with gauge group $SU(N)$ in the planar limit, i.e. for $N \to \infty$. As reviewed in chapters [II.1] and [II.2], on the string side of the duality the spectrum is accessible order by order as a strong coupling expansion in terms of the 't Hooft coupling by a (semi)classical analysis of string states with large quantized charges. It is also described in terms of respective string Bethe ansätze which are reviewed in chapter [III.1].

In the $\mathcal{N} = 4$ SYM theory, the weak coupling expansion of the planar spectrum, i.e. the conformal dimensions of composite operators, can be obtained by direct perturbative calculations of various correlation functions. The appearing UV divergences require renormalization, which then leads to a mixing among operators with the same bare conformal dimension. The eigenvalues of the new eigenstates under conformal rescalings are given as the sum of the bare scaling dimension and an individual anomalous dimension. The operator mixing can be extracted e.g. from the correlation functions involving two composite operators. Alternatively, one can directly calculate the diagrams which contribute to the renormalization of these operators. This directly allows to obtain an expression for the dilatation operator, whose eigenvalues are the anomalous dimensions.

Direct perturbative calculations become very cumbersome at high loop orders and can be avoided, if the observed integrability at one loop, which is reviewed in chapter [I.1], also persists to higher loop orders. The dilatation operator can then be determined, using some very general structural information from the underlying Feynman graphs only and some data from the gauge Bethe ansätze. The respective details are reviewed in chapter [I.3]. Direct Feynman graph calculations of the dilatation operator in the flavour $SU(2)$ subsector to three loops and of some of its eigenvalues and of parts of the Bethe ansätze also to higher loops provide important checks for the assumed integrability.

Even if integrability holds to all loop orders, the respective Bethe ansätze and planar dilatation operator can only be applied to compute the anomalous dimension in the asymptotic regime. In this regime, the loop order of the result is constrained to be strictly smaller than the length (the number of elementary fields) of the shortest composite operator involved. At loop orders which are equal to or exceed this number, the so called wrapping interactions [28, 37] have to be considered. They are corrections due to the finite size of the composite operators and have their origin in the neglected higher genus contributions to the dilatation operator [45]. In the dual string theory the counterparts of the wrapping interactions are corrections due to the finite circumference of the closed string worldsheet cylinder [55]. Their analyses are reviewed in chapters [III.5] and [III.6].

In this chapter we review the explicit Feynman graph calculations in $\mathcal{N} = 4$ SYM theory in the planar limit beyond one loop. It is organized as follows:

In section 2 we give with a short summary of how composite operators are renormalized, and how the dilatation operator is defined in terms of the renormalization constants.

In section 3 we then review the explicit calculations and tests of the dilatation operator with particular focus on calculations beyond the first order in perturbation theory.[1] Only the flavour $SU(2)$ subsector will be considered, since most higher loop calculations are performed within this subsector. As examples we recalculate in detail the respective one- and two-loop dilatation operator in $\mathcal{N}=1$ superfield formalism. This approach is much more efficient than the originally used component formalism, and it yields more direct relations between the dilatation operator and the underlying Feynman graphs. We then display the result of a three-loop computation and also summarize the existing checks of the magnon dispersion relation, of the structure of the dilatation operator and of some of its eigenvalues in the asymptotic regime beyond at three and higher loops.

In section 4, we review the perturbative calculations which consider the first wrapping corrections and hence yield results beyond the asymptotic regime. The general strategy of these calculations will be explained at the example of the four-loop calculation for the length four Konishi descendant in the flavour $SU(2)$ subsector. Further results for different operators and for the terms of highest transcendentality are then summarized briefly.

In section 5 we give a concluding summary, and in two appendices we present the explicit D-algebra manipulations for the one- and two-loop calculation and the expressions for the relevant integrals.

2 Renormalization of composite operators

The dilatation operator and anomalous dimensions can be obtained from a perturbative calculation of the correlation functions which involve the composite operators $\mathcal{O}_a$, where a labels the different operators. The encountered UV divergences require a renormalization of the composite operators as

$$\mathcal{O}_{a,\mathrm{ren}}(\phi_{i,\mathrm{ren}}) = \mathcal{Z}_a{}^b(\lambda,\varepsilon)\mathcal{O}_{b,\mathrm{bare}}(\phi_{i,\mathrm{bare}}) \ , \qquad \phi_{i,\mathrm{ren}} = \mathcal{Z}_i^{1/2}\phi_{i,\mathrm{bare}} \tag{2.1}$$

where in an appropriate basis $\mathcal{Z} = \mathbb{1} + \delta\mathcal{Z}$, and the matrix $\delta\mathcal{Z}$ is of order $\mathcal{O}(\lambda)$ in the renormalized coupling constant λ. It also depends on the regulator ε and is in general non-diagonal and thus leads to mixing between the different composite operators. The matrix element $\delta\mathcal{Z}_a{}^b$ is given by the negative of the sum of the overall UV divergences of the Feynman diagrams in which the vertices of the theory lead to interactions between the elementary fields of operator $\mathcal{O}_b$, such that the resulting external field flavour and ordering coincide with the ones of the operator $\mathcal{O}_a$. One also has to consider contributions from wave function renormalization of the elementary fields ϕ_i the operators are composed of. Respective factors $\mathcal{Z}_i^{1/2}$ are included within $\mathcal{Z}$.

$\mathcal{N}=4$ SYM theory can be regularized by supersymmetric dimensional reduction [59] in $D=4-2\varepsilon$ dimensions. The coupling constant g_{YM} is then accompanied by the 't Hooft mass μ in the combination $g_{\mathrm{YM}}\mu^{\varepsilon}$ to restore the mass dimension of the loop integrals. The coupling constant itself is not renormalized and hence does not explicitly depend on μ, such that superconformal invariance is preserved. This was explicitly found to three loops by computing the vanishing of the β-function in an $\mathcal{N}=1$ superfield

[1]The one-loop results are reviewed in chapter [I.1].

formulation [60, 2]. The finiteness of $\mathcal{N} = 4$ SYM theory was then later shown to all orders [3]. A first argument was given in [4]. Wave function renormalization of the superfields is hence finite, i.e. $\mathcal{Z}_i^{1/2}$ is trivial.[2] In the planar limit, where the coupling constant is $\lambda = g_{\mathrm{YM}}^2 N$, the dilatation operator is then extracted from the renormalization constant of the composite operators in (2.1) as

$$\mathcal{D} = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} \ln \mathcal{Z}(\lambda \mu^{2\varepsilon}, \varepsilon) = \lim_{\varepsilon \to 0} \left[2\varepsilon\lambda \frac{\mathrm{d}}{\mathrm{d}\lambda} \ln \mathcal{Z}(\lambda, \varepsilon) \right] . \tag{2.2}$$

The logarithm of $\mathcal{Z} = \mathbb{1} + \delta\mathcal{Z}$ has to be understood as a formal series in powers of $\delta\mathcal{Z}$. All poles of higher order in ε must cancel in $\ln \mathcal{Z}$, such that it only contains simple $\frac{1}{\varepsilon}$ poles. In effect, the above description extracts the coefficient of the $\frac{1}{\varepsilon}$ pole of $\mathcal{Z}$, and at a given loop order K multiplies it by a factor $2K$. This then yields the dilatation operator as a power series

$$\mathcal{D} = \sum_{k \geq 1} g^{2k} \mathcal{D}_k , \qquad g = \frac{\sqrt{\lambda}}{4\pi} , \tag{2.3}$$

where for later convenience we have absorbed powers of 4π into the definition of a new coupling constant g.

3 Dilatation operator in the $SU(2)$ subsector

$\mathcal{N} = 4$ SYM theory contains six real scalar fields which can be complexified and combined together with the fermions into three chiral and antichiral $\mathcal{N} = 1$ superfields ϕ_i and $\bar{\phi}_i$, $i = 1, 2, 3$, respectively. The remaining gauge field and fermion are combined into an $\mathcal{N} = 1$ vector multiplet V. In the following, we denote the three flavours by (ϕ, ψ, Z). They are transformed into each other by an $SU(3)$ subgroup of the $SU(4)$ R-symmetry group. All fields are in the adjoint representation of the gauge group $SU(N)$. Our superspace conventions are as in [5]. In particular, we use the Feynman rules in which the Wick rotation is included. They can be found e.g. in [6].

The flavour $SU(2)$ subsector contains operators which are composed of only two of the three chiral superfields, e.g. ϕ and Z. Their color indices are all contracted with each other to yield a gauge invariant object. This in general leads to multiple traces of products of the elementary fields, but in the planar limit only operators where all fields appear within a single trace are relevant. It suffices to consider operators which contain a number of fields ϕ which does not exceed the number of fields Z, since the results for the remaining operators follow immediately by an exchange of the role of the two fields. Usually, the fields ϕ are then denoted as impurities which appear between fields of type Z within the single trace over the gauge group. The total number of fields inside the trace defines the length L of the composite operator.

The composite operators have to be renormalized due to the emergence of UV divergences in the loop integrals of the Feynman diagrams in which the elementary fields interact via the elementary vertices of the theory. Such interactions can change the

[2]This holds apart form gauge artefacts that are not relevant here.

flavour of the elementary fields and even the length of the composite operators. However, in the flavour $SU(2)$ subsector they can only affect the order in which the two types of elementary fields appear inside the trace over the gauge group. Since the composite operators do not contain the adjoint fields $(\bar{\phi}, \bar{\psi}, \bar{Z})$ they are free of flavour subtraces, i.e. flavour contractions cannot appear. Only the permutation and the identity remain as relevant operators in flavour space. Mixing then occurs only within subsets of operators with fixed numbers of both types of fields and thus with fixed length L. The $SU(2)$ subsector itself is closed under renormalization, at least perturbatively [8]. The operators

$$\operatorname{tr}\left(Z^L\right) , \qquad \operatorname{tr}\left(\phi Z^{L-1}\right) \tag{3.1}$$

which are the ground state and a state with a single impurity are protected and do not acquire anomalous dimensions. Operators which contain more than a single impurity ϕ undergo non-trivial mixing.

The flavour permutations can be written as products of permutations acting on nearest neighbour sites. This allows for a representation of the renormalization constant $\mathcal{Z}$ and the dilatation operator $\mathcal{D}$ for composite operators of fixed length L in terms of the permutation structures [9]

$$\{a_1, \dots, a_n\} = \sum_{r=0}^{L-1} \mathrm{P}_{a_1+r\; a_1+r+1} \cdots \mathrm{P}_{a_n+r\; a_n+r+1} \tag{3.2}$$

and the identity $\{\}$ in flavour space. The structures consider the insertion of the Feynman subdiagrams in which elementary fields interact, at all possible positions within the single trace of the composite operator by the summation. Periodicity with period L is thereby understood. In particular, $\{\}$ then measures the length L of the composite operator it is applied to. The range of the interaction in flavour space κ is given by the number of nearest neighbours on which the permutation structures act. It is extracted from the list of integers $a_1, \dots, a_n$ as

$$\kappa = \max_{a_1,\dots,a_n} - \min_{a_1,\dots,a_n} + 2 \; . \tag{3.3}$$

At a given loop order K the relevant Feynman diagrams can only generate the permutation structures which obey the following conditions

$$n \leq K \; , \qquad \kappa \leq K + 1 \; . \tag{3.4}$$

The first inequality considers that each nearest-neighbour permutation is associated with at least one loop in the underlying Feynman graph. The second condition ensures that the range of the interaction in flavour space does not exceed the limit on the range of the Feynman graphs, defined as the number R of elementary fields of the composite operator which can interact with each other. It is restricted to $R \leq K + 1$. Since the summation in (3.2) runs over all insertion points with periodicity L, the smallest integer entry can always be fixed e.g. to 1 by shifting all a_i by a common integer. Further relations between the structures (3.2) can be found in [10]. The independent permutation structures which obey (3.4) then form a basis in which the K-loop dilatation operator can be written down.

The basis with elements (3.2) is not the best choice to express the result of an explicit Feynman diagram calculation, since a single Feynman diagram generates a contribution which contains several permutation structures (3.2) with fixed relative coefficients. If instead the generated combinations themselves are used as basis elements, each Feynman diagram is associated with only one of them [11, 12]. The basis elements obtained from supergraphs are called chiral functions and are defined as

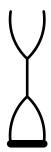
$$\chi(a_1, \dots, a_n) = \{a_1, \dots, a_n\}\big|_{\mathrm{P} \to \mathrm{P} - \mathbb{1}} \ , \tag{3.5}$$

where $\mathrm{P} \to \mathrm{P} - \mathbb{1}$ denotes a replacement of all permutations in (3.2) by the fixed combination of permutation and identity. The expansion of the resulting products yields χ in terms of linear combinations of permutation structures. The chiral functions capture the structure of the chiral and anti-chiral superfield lines of the underlying supergraphs. Hence, all supergraphs which only differ by the arrangement of the flavour-neutral vector fields generate contributions with the same chiral function. In particular, at loop order K the chiral functions $\chi(a_1, \dots, a_n)$ with $n = K$ are associated each with a single Feynman graph since they do not contain any vector fields. Except of $\chi() = \{\}$, all chiral functions (3.5) yield zero when they are applied to one of the states in (3.1). The expression of the dilatation operator in terms of chiral functions should hence not explicitly depend on $\chi()$. We will come back to this statement at the end of section 3.4.2.

3.1 One-loop dilatation operator

The one-loop calculation in the $SU(2)$ subsector was addressed by Berenstein, Maldacena and Nastase in [13]. They used component fields to compute the term involving the permutation structure $\{1\}$, which permutes the flavour between two neighboured fields. It is the maximum shuffling term at one loop, since it shifts the position of the impurity by the maximum number of one site at this loop order. Its generalization to higher loops will be discussed in section 3.4.1. The remaining Feynman diagrams all contribute to the identity operator $\{\}$ and were not computed explicitly. Instead, their contribution was reconstructed from the fact that the eigenvalue for the ground state in (3.1) should be zero.

Using instead $\mathcal{N} = 1$ superfields for the one-loop calculation, only a single Feynman diagram contains a UV divergence and hence contributes to the renormalization constant in (2.1). It is evaluated as



$$= +\lambda I_1 \chi(1) \ , \tag{3.6}$$

where the bold horizontal line represents the interacting fields of a composite operator of arbitrary length $L \geq 2$. The D-algebra manipulations are trivial in this case as explicitly displayed in appendix A. The resulting loop integral is given in appendix B. The further one-loop diagram of gluon exchange is finite, and the one-loop wave function renormalization vanishes. This is different from their behaviour in component formalism, where they have to be considered. According to the description (2.2), the

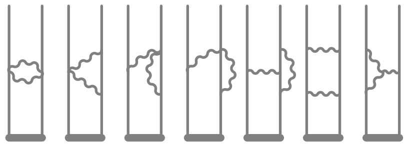
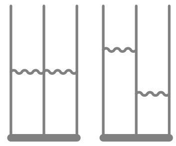
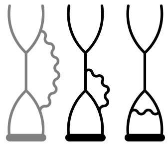
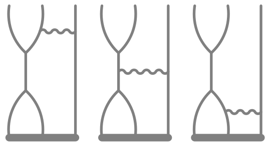
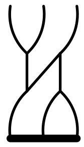
	$R=1$	$R=2$	$R=3$
$\chi()$			
$\chi(1)$	–		
$\chi(1,2)$	–	–	

Table 1: Diagrams in $\mathcal{N}=1$ superfields (apart from eventual reflections) which can in principle contribute to the two-loop dilatation operator. Graphs which contain the vanishing one-loop self energies are not drawn. It turns out that all diagrams depicted in gray are also irrelevant. The two-loop chiral self energy is finite, and the remaining range $R \geq 2$ diagrams are irrelevant due to generalized finiteness conditions [6].

one-loop dilatation operator hence reads

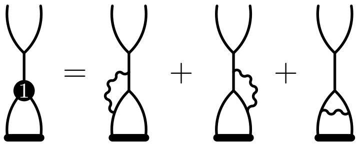
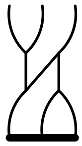
$$\mathcal{D}_1 = -2\chi(1) \ . \tag{3.7}$$

Including also the contributions to the trace operator in flavour space, which extends the result to the flavour $SO(6)$ subsector, the full one-loop calculation in component fields was performed in [14], and the result was recognized as the Hamiltonian of a respective integrable Heisenberg spin chain.

3.2 Two-loop dilatation operator

The two-loop calculation of the dilatation operator was performed in [15] in component formalism and for composite operators in the $SU(2)$ subsector. As in the one-loop case [13] only the diagrams which contribute to genuine flavour permutations were explicitly calculated, and the coefficient of the identity operation was then determined by the condition of a vanishing eigenvalue of the ground state (3.1).

The relevant diagrams for the complete two-loop computation in terms of $\mathcal{N}=1$ superfields are given in table 1. The chiral self energy is identically zero at one loop and finite at higher loops. According to the generalized finiteness conditions derived in [6], all range $R \geq 2$ diagrams, in which all vertices appear in loops are also finite. This concerns all remaining diagrams in the first line and in the second line the respective first diagram in the second and third column. The pole parts of the last two diagrams in this line in the third column cancel against each other [11,12]. This cancellation is based on the fact, that all spinor derivative D_α have to be kept inside the loops for the divergent contributions when the D-algebra is performed. These constraints on the D-algebra manipulations also follow from the finiteness conditions [6]. All these irrelevant diagrams are depicted in gray. We only have to compute the remaining diagrams and consider also their reflections where necessary. The substructures in the relevant range $R = 2$ diagrams with chiral function $\chi(1)$ combine into the one-loop chiral vertex correction that is explictly given in (A.2). We then find

 $= -2\lambda^2 I_2\chi(1)$,  $= +\lambda^2 I_2\chi(1,2)$, (3.8)

where we have to consider also the reflection of the last diagram which contributes with chiral function $\chi(2,1)$. According to the description (2.2), the two-loop dilatation operator is then obtained by extracting the $\frac{1}{\varepsilon}$ pole of the sum of these diagrams and multiplying it by -4. With the pole part of the respective integral I_2 given in (B.4) this then yields

$$\mathcal{D}_2 = 4\chi(1) - 2[\chi(1,2) + \chi(2,1)] \ . \tag{3.9}$$

An explicit demonstration of the cancellation of the double poles in $\ln\mathcal{Z}$ as mentioned after (2.2) can be found in [6], where the one- and two-loop calculations were presented as a demonstration for the efficiency of the used approach.

3.3 Three-loop dilatation operator

At three-loop order a computation of the dilatation operator directly from Feynman graphs of $\mathcal{N} = 1$ superfields was recently performed in [6]. The result reads

$$\begin{aligned}\mathcal{D}_3 = &-4(\chi(1,2,3) + \chi(3,2,1)) + 2(\chi(2,1,3) - \chi(1,3,2)) - 4\chi(1,3)\\ &+ 16(\chi(1,2) + \chi(2,1)) - 16\chi(1) - 4(\chi(1,2,1) + \chi(2,1,2)) \ .\end{aligned} \tag{3.10}$$

It determines the planar spectrum in the $SU(2)$ subsector to three loops and hence goes beyond an earlier test of two eigenvalues [16], which employs Anselmi's trick [17] to reduce the calculation to two loops. The three-loop results confirm the prediction from integrability in [9]. Earlier checks of three-loop eigenvalues are summarized in section 3.4.3.

3.4 Partial tests at higher loops

To three-loop order and also beyond, certain parts of the respective Bethe ansatz and dilatation operator have been checked by direct Feynman diagram calculations. This concerns the so-called maximum shuffling terms, which contribute to the dispersion relation of the Bethe ansatz. Moreover, certain terms in the expressions of the dilatation operator have been tested explicitly.

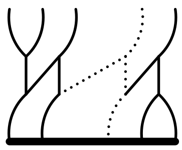
3.4.1 Tests of the magnon dispersion relation

Even if explicit expressions for the $SU(2)$ dilatation operator have been obtained only to the first few loop orders [37], the magnon dispersion relation of the Bethe ansatz is an all-order expression and directly related to certain Feynman diagrams. For a single magnon of momentum p it is given by [37]

$$E(p) = \sqrt{1 + 16g^2 \sin^2 \tfrac{p}{2}} - 1 \ , \tag{3.11}$$

and it is fixed by the underlying symmetry algebra up to an unknown function of the coupling constant [19], which is trivial here and has already been substituted accordingly.[3]

At a fixed loop order K in the expansion of the above relation, the momentum dependence can be expressed as linear combination of the elements $\cos(k-1)p \sin^2 \frac{p}{2}$ with $1 \leq k \leq K$. In particular, the term with $k = K$ is generated by the so-called maximum shuffling diagrams, which include shifts of the position of a single impurity (which is a magnon in the spin chain notation) by the maximum number of K neighbouring sites. The relevant diagrams are given by



$$\rightarrow \lambda^K I_K \chi(1, 2, \ldots, K-1, K) \tag{3.12}$$

and by its reflection. When the resulting sum from the above diagram and from its reflection are applied to the eigenstate of a single magnon with momentum p, it yields the eigenvalue $\lambda^K I_K(-8)\cos(K-1)p \sin^2 \frac{p}{2}$. According to the description (2.2), the $\frac{1}{\varepsilon}$ pole of this expression has to be multiplied by $-2K$ to obtain its contribution to the magnon dispersion relation. A comparison with the respective term in the expansion of (3.11), thereby taking into account the relation (2.3) between the couplings, then makes a prediction for the $\frac{1}{\varepsilon}$ pole of the integral I_K as

$$\operatorname{Res}_0(\mathrm{K}\,\mathrm{R}(I_K)) = \frac{1}{(4\pi)^{2K}} \frac{(2K-2)!}{(K-1)!K!} \frac{1}{K} \ . \tag{3.13}$$

The explicit expressions for the poles of I_K for some K are listed in (B.4). They are consistent with this result.

In [20] it was shown that at generic loop order the pole structure of the maximum shuffling diagrams in component fields is in accord with the square root formula of BMN [13]. This formula follows from the dispersion relation (3.11) for magnon momenta $p_j = \frac{2\pi n_j}{L} \ll 1$, where L is the length of the operator and n_j are mode numbers. The respective momenta are obtained from the Bethe equations in [37], which are incomplete since they do not consider the so-called dressing phase [21]. The dressing phase first appeared at strong coupling [22] but it is important also at weak coupling [23, 21], where

[3] The explicit Feynman diagram computation in [6] confirms that this function is trivial to three loops. It is non-trivial in the AdS_4/CFT_3 correspondence that is reviewed in chapter [IV.3].

it alters the magnon momenta at order $\mathcal{O}(g^6)$.[4] Therefore, the BMN square root formula fails to describe the anomalous dimensions of two-impurity operators beyond three loops.[5] In any case, the dressing phase does not affect the maximum shuffling terms, and hence the test of the BMN formula in [20] is insensitive to the necessary modification.

An all order derivation of the BMN square root formula employing $\mathcal{N} = 1$ superfield formalism is presented in [24]. Since it is based on the study of the two-point functions of operators with a single impurity, it is insensitive to the effects of the dressing phase which appears in the magnon S-matrix. It would be more appropriate to say that in the analysis the magnon dispersion relation in the BMN limit is obtained.

The magnon dispersion relation (3.11) describes the free propagation of one magnon. It it thus built up from all Feynman diagrams with chiral functions that do not yield a vanishing result when applied to the single magnon momentum eigenstate. The number of impurities of the composite operator sets an upper bound on the number of bubbles formed by two neighboured lines of the composite operator inside the Feynman diagrams. Such a bubble appears for example in the lower right corner of the graph in (3.12), and it vanishes unless the two involved field flavours are different. The diagrams contributing to the magnon dispersion relation hence must not contain more than one of these bubbles. This restrict their chiral functions to $\chi(1,\dots,k)$ and $\chi(k,\dots,1)$ after the identities for the permutation structures (3.2) found in [10] have been used to simplify the chiral functions, e.g. as $\chi(1,2,1) = \chi(2,1,2) = \chi(1)$ for the three loop result (3.10). All-order expressions for the coefficients of these terms in the dilatation operator then follow directly from the magnon dispersion relation (3.11) and can be found in [6]. It should be stressed that the above described contributions also yield non-vanishing results when additional magnons are present outside of the $k+1$ interacting legs. They therefore also contribute to the magnon S-matrix.

3.4.2 Tests of magnon scattering

The Feynman diagrams that vanish for a single magnon state, but are non-vanishing if two or more magnons are present within their respective interaction ranges should exclusively be associated with the magnon S-matrix. Their contributions appear together with the ones of the above described maximum and non-maximum shuffling terms in the dilatation operator as constructed from the underlying integrability. In the $SU(2)$ subsector some of them have been verified by direct Feynman diagram calculations.[6]

As a concrete example, we consider the four-loop dilatation operator. It can be determined from the underlying integrability as reviewed in chapter [I.3]. In the basis of

[4] The dressing phase is reviewed in chapter [III.3].

[5] This breakdown is independent of the general restriction of the Bethe ansatz to the asymptotic regime that requires a termination of the expansion at a loop order $K \leq L-1$ to avoid the wrapping corrections.

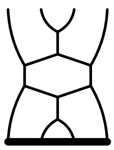
[6] A two-loop test of the S-matrix of the $SL(2)$ subsector can be found in [25].

the chiral functions (3.5) it reads

$$\begin{aligned}\mathcal{D}_4 = \;& + 200\chi(1) - 150[\chi(1,2)+\chi(2,1)] + 8(10+\epsilon_{3a})\chi(1,3) - 4\chi(1,4)\\ & + 60[\chi(1,2,3)+\chi(3,2,1)]\\ & + (8+2\beta+4\epsilon_{3a}-4i\epsilon_{3b}+2i\epsilon_{3c}-4i\epsilon_{3d})\chi(1,3,2)\\ & + (8+2\beta+4\epsilon_{3a}+4i\epsilon_{3b}-2i\epsilon_{3c}+4i\epsilon_{3d})\chi(2,1,3)\\ & - (4+4i\epsilon_{3b}+2i\epsilon_{3c})[\chi(1,2,4)+\chi(1,4,3)]\\ & - (4-4i\epsilon_{3b}-2i\epsilon_{3c})[\chi(1,3,4)+\chi(2,1,4)]\\ & - (12+2\beta+4\epsilon_{3a})\chi(2,1,3,2)\\ & + (18+4\epsilon_{3a})[\chi(1,3,2,4)+\chi(2,1,4,3)]\\ & - (8+2\epsilon_{3a}+2i\epsilon_{3b})[\chi(1,2,4,3)+\chi(1,4,3,2)]\\ & - (8+2\epsilon_{3a}-2i\epsilon_{3b})[\chi(2,1,3,4)+\chi(3,2,1,4)]\\ & - 10[\chi(1,2,3,4)+\chi(4,3,2,1)]\;.\end{aligned} \tag{3.14}$$

The coefficients ϵ_i, $i = 3a, 3b, 3c, 3d$ in the above result are not fixed by the construction. They correspond to similarity transformations, i.e. changes in the basis of operators [9, 10], that do not affect the eigenvalues. The coefficient β is the leading term of the previously mentioned dressing phase. The magnon dispersion relation is encoded in the first two terms in the first line, the second line and the last line. The further contributions should be associated with magnon scattering. As the contributions from the maximum shuffling diagrams (3.12) in the last line, also the other terms in the last four lines have chiral functions that satisfy both bounds (3.4). They can hence be calculated from Feynman diagrams as easily as the maximum shuffling terms.

The term in (3.14) with chiral function $\chi(2,1,3,2)$ only satisfies the first bound in (3.4), i.e. the underlying Feynman diagram is chiral but it is not of maximum range. It involves the leading coefficient β of the dressing phase, which can be determined from an evaluation of the respective diagram



$$\rightarrow \lambda^4 I_\beta \chi(2,1,3,2) \tag{3.15}$$

if the coefficient ϵ_{3a} of the similarity transformations is known. One finds $\epsilon_{3a} = -4$ for example by computing the diagram which generates $\chi(1,3,2,4)$ or $\chi(2,1,4,3)$. With the pole part of the integral I_β given in (B.5), the leading coefficient of the dressing phase is then determined as $\beta = 4\zeta(3)$. The result was obtained in [26], using component formalism. It agrees with one of the proposals in [21] and with the result extracted from a four-loop calculation of a four-point amplitude in [23].

It is also relatively easy to compute the terms with chiral functions which only satisfy the second bound of (3.4), i.e. all terms in (3.14) with chiral functions that contain 1 and 4 in their lists of arguments and hence only stem from Feynman diagrams with maximum interaction range $R = 5$. This calculation was performed in [11, 12] in $\mathcal{N} = 1$ supergraph formalism in the context of calculating the first wrapping correction to be discussed below. The results yield an overdetermined system of equations that uniquely fixes the coefficients ϵ_i and provides non-trivial checks of the remaining coefficients that are fixed by the underlying integrability. The analogous calculation of the $R = 6$ diagrams at five loops can be found in [27].

The expressions (3.7), (3.9), (3.10) and (3.14) do not depend on the identity $\chi()$. This guarantees that the anomalous dimension of the BPS operators (3.1) are zero. The generalized finiteness conditions in [6] predict this to all orders and relate it to the finiteness of the chiral self energy, i.e. to the preservation of conformal invariance.

3.4.3 Checks of eigenvalues

To three loops the results (3.7), (3.9) and (3.10) for the dilatation operator have been obtained by direct Feynman diagram calculations. At higher loops, only the terms that saturate at least one of the bounds in (3.4) have been tested as describe above. Further checks involve the eigenvalues of the dilatation operator for some composite operators. They should match with the anomalous dimensions obtained in direct Feynman diagram calculations.

Of particular interest is thereby the Konishi supermultiplet. As superconformal primary it contains the $\mathcal{N} = 1$ Konishi operator [29] that has bare scaling dimension $\Delta_0 = 2$ and reads

$$\mathcal{K} = \operatorname{tr}\left(\mathrm{e}^{-g_{\mathrm{YM}}V}\, \bar{\phi}_i\, \mathrm{e}^{g_{\mathrm{YM}}V}\, \phi^i \right) . \tag{3.16}$$

This operator is not chiral, and hence all its superfield components lie beyond the $SU(2)$ subsector. However, the Konishi supermultiplet also contains an operator of this subsector. To find it, one first has to select the level four descendant of bare dimension $\Delta_0 = 4$, which is chiral and pick out the relevant $SU(4)$ R-symmetry component given by

$$\operatorname{tr}\left([\phi,Z]\,[\phi,Z] \right) . \tag{3.17}$$

It contains as lowest superfield component the respective operator built out of the two scalar fields of the flavour $SU(2)$ subsector.

All members of a superconformal multiplet acquire the same anomalous dimension. For the Konishi multiplet it is given to four loops in (4.1). The one- and two-loop contributions were obtained by explicit Feynman diagram calculations in [30] and [31], and then also by an OPE analysis in [32], see also [33]. These results are also found in the special case $S = 2$ from the conformal dimensions of twist-two operators of general conformal spin S that belong to the $SL(2)$ subsector. A respective state appears as component in another level four descendant in the Konishi multiplet. From Feynman diagrams the result to two loops has been obtained in [34]. At three loops it could be extracted [35] as the terms with highest transcendentality, i.e. with highest degrees of the harmonic sums, from the NNLO QCD result for the non-singlet splitting functions of QCD [36]. The truncation of the QCD result is based on the observation [38] that due to special properties of the DGLAP and BFKL equations in $\mathcal{N} = 4$ SYM theory a mixing between functions of different transcendentality degrees does not occur. Specializing to $S = 2$, the extracted result yields the three-loop contribution in (4.1). The dilatation operator (3.7), (3.9) and (3.10) correctly yields the result in (4.1) when applied to the state (3.17).[7] In fact, the three-loop term was first predicted in [9], where the dilatation operator is constructed from integrability. Later, an explicit Feynman diagram calculation [16], which employs Anselmi's trick [17] to reduce the calculation to two loops led to the same result. It also immediately follows from the three-loop Feynman diagram calculation of the dilatation operator in [6] as one of its eigenvalues.

The previously mentioned twist-two operators of the $SL(2)$ sector are very important for tests of the AdS/CFT correspondence and the underlying integrability. These tests are reviewed in chapter [III.4]. In particular, the results in the strict $S \to \infty$ limit are not modified by wrapping interactions. At finite S such modifications occur. The simplest example is $S = 2$, i.e. the operator which appears in the Konishi multiplet. Its anomalous dimension is affected by wrapping interactions at four loops and beyond.

4 Wrapping interactions

In the following we briefly summarize the calculations of the previously mentioned wrapping interactions. A more detailed review is given by [39].

The Bethe ansätze or the dilatation operator yield reliable results for the anomalous dimensions in the asymptotic limit only. The origin and precise form of this restriction can be understood by recalling the construction from Feynman diagrams. In section 3 it was argued that at a given loop order K the dilatation operator is determined from Feynman diagrams with range $R \leq K + 1$, which lead to flavour permutations with range $\kappa \leq K + 1$. For the construction of the diagrams it is thereby implicitly assumed, that the length L of the involved composite operator is at least as big as the maximal interaction range $K + 1$. Therefore, the K-loop dilatation operator can in general only yield the correct anomalous dimension when it is applied in the asymptotic limit, i.e. to a state of length $L \geq K + 1$.

In order to obtain the spectrum beyond the asymptotic limit, corrections due to the finite length L of the composite operator have to be considered. In the dilatation operator, the contributions of all Feynman diagram with range $R > L$ that were constructed with the assumption of a sufficiently long operator have to be replaced by new diagrams that contain the operator of lower length L. The new diagrams are called wrapping diagrams since, due to the insufficient length of the composite operator, the interactions wrap around it. See figure 4.2 for two examples of such diagrams. This procedure introduces an explicit dependence of the dilatation operator on the length L of the composite operator it is applied to. More precisely, the coefficients of the chiral functions in the expression of the dilatation operator become functions of L, while in the asymptotic limit they are constants, and the dilatation operator depends on the length only via the permutation structures (3.2).

The appearance of wrapping interactions is closely connected to the truncation of the genus h expansion of the dilatation operator beyond the planar $h = 0$ contribution [45]. If in a planar wrapping diagram the composite operator is replaced by a longer operator, the additional fields lines cannot leave the diagram without crossing any other lines, i.e.

[7] At four and higher loops this is no longer the case since the wrapping interactions have to be considered. This will be discussed in section 4.

it becomes a diagram of genus $h = 1$. The appearing wrapping diagrams hence come from certain genus $h = 1$ contributions to the dilatation operator, which become planar when it is applied to a sufficiently short composite operator. Wrapping diagrams appear at all orders in the genus expansion of the dilatation operator. They are of genus $h+1$ in the asymptotic regime and encode the finite size effects at genus h. The planar wrapping diagrams are special since they can be projected out of all genus one contributions by introducing spectator fields [45]. While in general for higher genus diagrams the notion of the range of the interaction is not meaningful, it is still well defined for the subset of genus one diagrams when they become the planar wrapping diagrams. Integrability seems to persist, even if in general at higher genus its breakdown is expected [9].[8]

In order to obtain the anomalous dimensions beyond the asymptotic regime, one should not abandon the dilatation operator of the asymptotic regime and compute all Feynman diagrams. As mentioned above, the dilatation operator is still useful, since it can be corrected for an application to the composite operator of shorter length L. First, at each loop order K all contributions from Feynman graphs of longer range $K+1 \leq R \leq L$ have to be removed. Then, contributions from the wrapping interactions have to be added. This procedure is particularly powerful at the critical order $K = L$ where wrapping arises for the first time, since only relatively few Feynman diagrams of restricted topology have to be computed explicitly. Most diagrams are captured automatically by those terms in the dilatation operator that are not removed in the modification process.

This procedure has been first initiated in [11], with the details given in [12] for the case $K = L = 4$, i.e. for the four-loop anomalous dimension of the Konishi operator. In $\mathcal{N} = 4$ SYM theory it is the simplest case where wrapping arises.

One starts from the four-loop asymptotic dilatation operator (3.14) and modifies it for an application to the length four Konishi descendant (3.17). First, all contributions from Feynman diagrams which are of range $R = 5$ have to be identified and removed. The subtraction of a contribution from a Feynman diagram with $R = 5$ affects the coefficients of several permutation structures (3.2) in the dilatation operator.[9] The basis of chiral functions (3.5) was introduced to ensure that this does not happen in the case $R = \kappa = 5$. One still has to remove all contributions from Feynman diagrams with $R = 5$ but $\kappa \leq 4$. Such Feynman diagrams contain a chiral structure with interaction range κ, and the remaining $R-\kappa$ neighboured field lines are connected with it and with each other only by vector fields. Since the latter are flavour neutral, these $R = 5$ diagrams appears to be of lower range $\kappa \leq 4$ and are not captured when all contributions with $\kappa = 5$ are removed. It can be shown [12] that in $\mathcal{N} = 1$ supergraph formalism all these diagrams are finite or their divergences cancel against each other, and that therefore they do not contribute to the dilatation operator in the first place. This is also an implication of the generalized finiteness conditions derived in [6]. In section 3.2 we have already used the results to argue for the finiteness of the two-loop diagrams in the first two rows of the last column of table 1. These diagrams have $R = 3$ but $\kappa \leq 2$ and are the counterparts

[8] In chapter [IV.1] the analyses of higher genus contributions are reviewed.

[9] In the context of the BMN matrix model such a subtraction attempt was made in [40]. It does not lead to the correct result, since the necessary modifications of the contributions with permutation structures of lower range and the addition of the wrapping diagrams was not performed.

to some of the four-loop diagrams that appear here.

At critical wrapping order $K = L$ the use of chiral functions and the finiteness theorem make the subtraction procedure almost trivial: one just has to remove all contributions with chiral functions with range $\kappa = K + 1$. The subtraction procedure itself does not require the calculation of any Feynman diagrams. However, the eigenvalues of the subtracted dilatation operator are no longer independent of the scheme coefficients ϵ_i, which have to be fixed by calculating at least some of the diagrams with range $R = K+1$. If one could compute the wrapping interactions also as functions of ϵ_i and add them to the subtracted dilatation operator, the eigenvalues of this sum should not depend on the ϵ_i. However, the calculation of the wrapping interactions takes place in a fixed scheme related to the use of $\mathcal{N} = 1$ supergraphs, and therefore the ϵ_i in the subtracted dilatation operator have to assume the respective values.

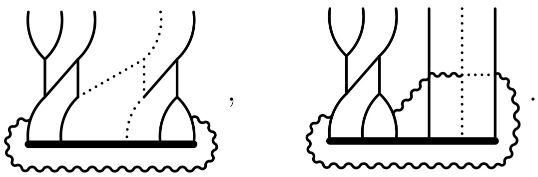
Applying the modified four-loop dilatation operator to the $SU(2)$ descendant of the Konishi supermultiplet yields the four-loop anomalous dimension [11,12]. Including also the lower orders, it reads

$$\gamma = 12g^2 - 48g^4 + 336g^6 + (-2496 + 576\zeta(3) - 1440\zeta(5))g^8 \ , \tag{4.1}$$

where the full conformal dimension is obtained as $\Delta = \Delta_0 + \gamma$ with the naive scaling dimension Δ_0 as described in section 3.4.3. The four-loop contribution has also been obtained from a generalized Lüscher formula [41]. The approach is reviewed in chapter [III.5]. Furthermore, it was later also found in a computer-based calculation in component formalism [42]. The matching between the Feynman diagram and Lüscher based calculations provides the first test of AdS/CFT and the underlying integrability beyond the asymptotic limit. It is also reproduced by the recently proposed Y-system [43], which is derived from the thermodynamic Bethe ansatz (TBA) [44] and is a candidate to capture the full planar spectrum of $\mathcal{N} = 4$ SYM theory. The TBA and Y-system are reviewed respectively in chapters [III.6] and [III.7]. Earlier attempts to describe the wrapping effects are included in chapter [I.3].

In [46] the result (4.1) which also holds for the earlier mentioned twist-two operator with conformal spin $S = 2$ has been generalized to arbitrary S. When analytically continued to $S = -1$, it yields the correct pole structure as predicted from the BFKL equation.

A result for the five-loop anomalous dimension of the Konishi operator has been obtained in impressive calculations on the basis of the generalized Lüscher formula [47] and the TBA [48]. Also this result has been generalized to arbitrary spin S, and it is in accord with the pole structure from the BFKL equation [49]. To obtain the five-loop result for the Konishi multiplet from a Feynman diagram calculation is very difficult, even with the universal cancellation mechanisms discovered in [6]. Instead, a five-loop result for the $L = 5$ operator $\operatorname{tr}\big([\phi,Z]\,[\phi,Z]\,Z\big)$ which is in the same supermultiplet as certain twist-three operators has been computed [27], and it agrees with results from the generalized Lüscher formula [50] and the Y-system [51]. The six-loop results for the twist-three operators with generic conformal spin S has recently become available [52].

 (4.2)

Beyond the asymptotic limit the contributions of highest transcendentality, i.e. which contain the ζ-function with biggest argument, are generated entirely by the wrapping interactions. In the four loops result in (4.1) this is the term with $\zeta(5)$. Its generalization to twist-two operators with generic conformal spin S has been obtained from a Feynman diagram computation in component formalism in [53]. At generic loop and critical wrapping order $K = L$ the highest transcendentality degree of the wrapping diagrams is $2K - 3$ compared to $2K - 5$ of the dressing phase in the asymptotic Bethe ansatz. A clean setup to study the transcendentality structure without admixtures from the dressing phase is given by single impurity operators in the β-deformed $\mathcal{N} = 4$ SYM theory.[10] The leading wrapping corrections have been calculated up to 11 loops [54, 56] and were confirmed in [50, 51]. A clear pattern emerges also for the terms of lower transcendentality. The diagrams in figure 4.2 are responsible for the highest transcendentality contribution involving $\zeta(2K - 3)$. The respective term can be traced back to a component $\frac{1}{2}P_K$ in the decomposition of the integrals, where P_K is the K-loop cake integral P_K given in (B.6).

5 Conclusions

We have reviewed the explicit Feynman diagram calculations which at small 't Hooft coupling determine the planar spectrum of composite operators in the flavour $SU(2)$ subsector of $\mathcal{N} = 4$ SYM theory and test the underlying integrability. The use of $\mathcal{N} = 1$ superspace techniques and of chiral functions as operators in flavour space allowed for a direct connection of Feynman diagram calculations and the results from integrability. We have presented the calculations up to two-loops in detail, and summarized the calculations and partial checks at higher loops.

Then we reviewed how a anomalous dimensions beyond the asymptotic limit can be obtained by computing the leading wrapping corrections and which properties and interpretation these interactions have. The existing tests in these setups have been summarized.

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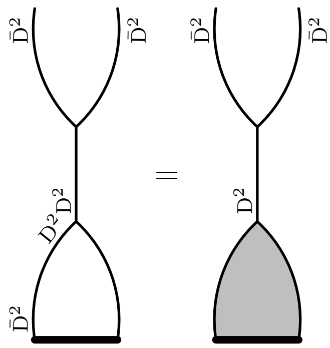
[10] Among other deformations the β-deformation is reviewed in chapter [IV.2].

Mauri, Joseph Minahan, Alberto Santambrogio, Olof Ohlsson Sax, Gabriele Tartaglino-Mazzucchelli, Alessandro Torielli and Daniela Zanon for very pleasant collaborations on some of the papers reviewed here and in other chapters of this review.

A D-algebra

The propagators and vertices of superfields depend not only on the bosonic, but also on the fermionic coordinates θ^α, $\bar\theta^{\dot\alpha}$ of the superspace and carry covariant spinor derivatives D_α, $\bar{\mathrm{D}}_{\dot\alpha}$. By the D-algebra manipulation which consists of transfers, partial integrations and the use of (anti)-commutation relations between products of these spinor derivatives, the underlying expression is transformed, such that the final result is localized at a single point in the coordinates θ^α, $\bar\theta^{\dot\alpha}$. We refer the reader to [5] for our conventions and to [6] for an explicit presentation of the relevant Feynman rules. Here, we only recall that in each loop two D_α and two $\bar{\mathrm{D}}_{\dot\alpha}$ have to remain to obtain a non-vanishing result. The loop is then localized in the fermionic coordinates. We indicated this by filling it gray. Also, we recall the relations $\mathrm{D}^2\,\bar{\mathrm{D}}^2\,\mathrm{D}^2 = \Box\,\mathrm{D}^2$ and $\bar{\mathrm{D}}^2\,\mathrm{D}^2\,\bar{\mathrm{D}}^2 = \Box\,\bar{\mathrm{D}}^2$, which transform spinor derivatives into spacetime derivatives $\Box = \partial^\mu\partial_\mu$.

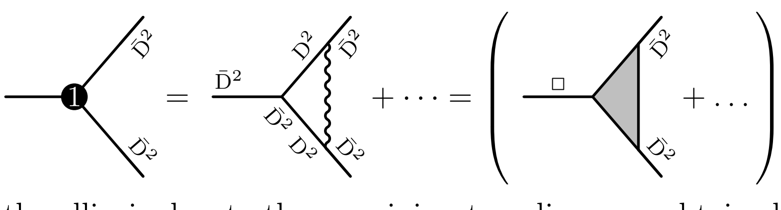
The one-loop diagram (3.6) requires no D-algebra manipulations, and one directly obtains



$\to -I_1$, (A.1)

where we have not considered the non-trivial prefactors of the propagators and vertices in the final result. They are contained within the color- and flavour factors (chiral functions) that enter the complete result. There appears an additional factor -1 in front of I_1: we have to transform the full fermionic measure to the chiral measure of the source term that for the chiral composite operator we have added to the action. This yields $\mathrm{d}^4\theta = \mathrm{d}^2\theta\,\bar{\mathrm{D}}^2$ with extra derivatives $\bar{\mathrm{D}}^2$ that combine with the remaining D^2 in the above diagram to $\Box$, such that the propagator which connects the chiral and antichiral cubic vertex is cancelled, thereby yielding the factor -1.

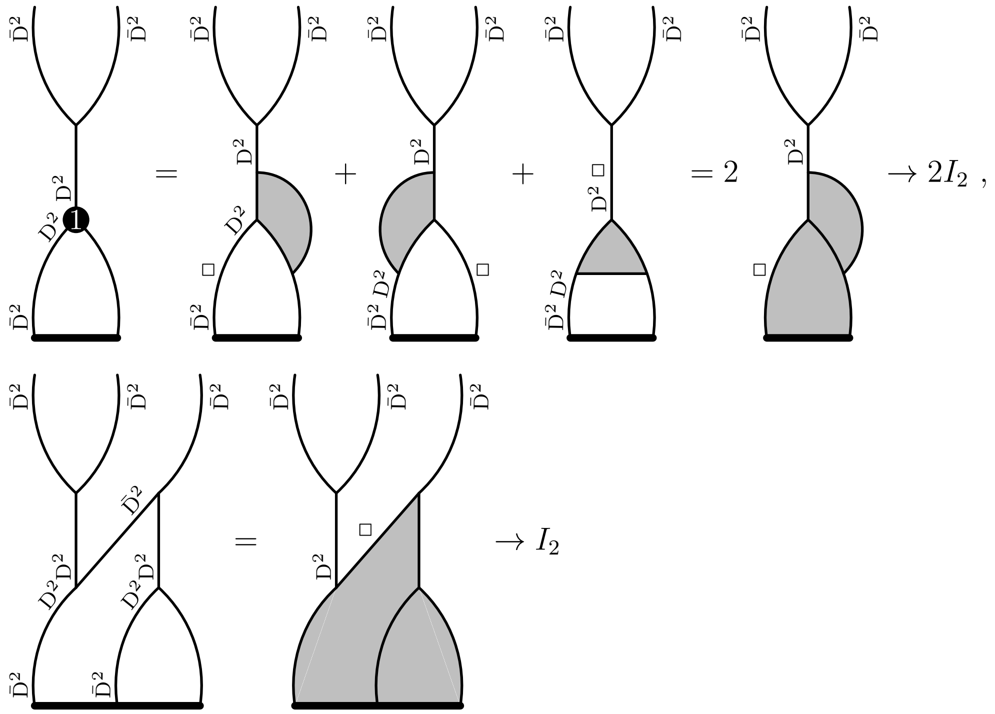
The one-loop correction to the chiral vertex that enters (3.8) is easily evaluated as



$i\lambda g_{\mathrm{YM}}\epsilon_{ijk}$, (A.2)

where the ellipsis denote the remaining two diagrams obtained by cyclic permutations of the external legs, and we have omitted the color trace but included the other color and flavour factors. Also in this case the $\Box$ is produced after reducing the full fermionic measure to the chiral measure as mentioned above. When $\Box$ cancels the propagator a factor -1 is produced.

The D-algebra manipulations for the diagrams (3.8) contributing to the two-loop dilatation operator are



(A.3)

where equalities hold up to disregarded finite contributions, and the final expressions in terms of the integral I_2 consider the factor -1 discussed above.

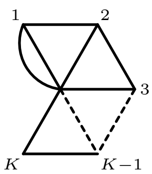
B Integrals

Using the scalar G-function defined as

$$G(\alpha,\beta) = \frac{\Gamma(\frac{D}{2}-\alpha)\Gamma(\frac{D}{2}-\beta)\Gamma(\alpha+\beta-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}\Gamma(\alpha)\Gamma(\beta)\Gamma(D-\alpha-\beta)} , \tag{B.1}$$

in D-dimensional Euclidean space, the following integrals can be found exactly to all loop orders

$$I_K =$$



$$= \prod_{k=0}^{K-1} G(1-(\tfrac{D}{2}-2)k, 1) . \tag{B.2}$$

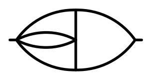
They are logarithmically divergent in $D = 4-2\varepsilon$ dimensions, and their overall UV divergence is obtained with the operations K to extract the pole part and R to subtract subdivergences as

$$\mathrm{K}\,\mathrm{R}(I_K) = \mathrm{K}\left(I_K - \sum_{k=1}^{K-1} \mathrm{K}\,\mathrm{R}(I_k) I_{K-k}\right) . \tag{B.3}$$

To the first few loop orders, one finds

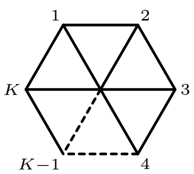
$$\begin{aligned}
\mathrm{K}\,\mathrm{R}(I_1) &= \frac{1}{(4\pi)^2}\frac{1}{\varepsilon} ,\\
\mathrm{K}\,\mathrm{R}(I_2) &= \frac{1}{(4\pi)^4}\Big(-\frac{1}{2\varepsilon^2}+\frac{1}{2\varepsilon}\Big) ,\\
\mathrm{K}\,\mathrm{R}(I_3) &= \frac{1}{(4\pi)^6}\Big(\frac{1}{6\varepsilon^3}-\frac{1}{2\varepsilon^2}+\frac{2}{3\varepsilon}\Big) ,\\
\mathrm{K}\,\mathrm{R}(I_4) &= \frac{1}{(4\pi)^8}\Big(-\frac{1}{24\varepsilon^4}+\frac{1}{4\varepsilon^3}-\frac{19}{24\varepsilon^2}+\frac{5}{4\varepsilon}\Big) ,\\
\mathrm{K}\,\mathrm{R}(I_5) &= \frac{1}{(4\pi)^{10}}\Big(\frac{1}{120\varepsilon^5}-\frac{1}{12\varepsilon^4}+\frac{11}{24\varepsilon^3}-\frac{19}{12\varepsilon^2}+\frac{14}{5\varepsilon}\Big) ,\\
\mathrm{K}\,\mathrm{R}(I_6) &= \frac{1}{(4\pi)^{12}}\Big(-\frac{1}{720\varepsilon^6}+\frac{1}{48\varepsilon^5}-\frac{25}{144\varepsilon^4}+\frac{47}{48\varepsilon^3}-\frac{1313}{360\varepsilon^2}+\frac{7}{\varepsilon}\Big) .
\end{aligned} \tag{B.4}$$

The pole parts of the integrals which determine the four-loop dressing phase or come from the wrapping interactions at critical wrapping order can very efficiently be computed by using a modified and extended version of the Gegenbauer polynomial x-space technique [57, 12]. The integral of the simplest contribution that allows to determine the leading four-loop coefficient of the dressing phase reads

$I_\beta =$  ,

$$\mathrm{K}\,\mathrm{R}(I_\beta) = \frac{1}{(4\pi)^8}\Big(-\frac{1}{12\varepsilon^4}+\frac{1}{3\varepsilon^3}-\frac{5}{12\varepsilon^2}-\frac{1}{\varepsilon}\Big(\frac{1}{2}-\zeta(3)\Big)\Big) . \tag{B.5}$$

The terms of highest transcendentality from wrapping corrections at critical order are determined by the cake integral. This integral is logarithmically divergent for $K \geq 4$ loops and reads

$P_K =$ 

,

$$\mathrm{K}(P_K) = \frac{1}{(4\pi)^{2K}}\frac{1}{\varepsilon}\frac{2}{K}\binom{2K-3}{K-1}\zeta(2K-3) , \tag{B.6}$$

where the pole part has been obtained in [58] at generic loop order.

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