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Measuring dynamic efficiency under uncertainty

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Abstract

In this paper we develop a theoretical model that links dynamic efficiency measurement and optimal investment under uncertainty. It is widely acknowledged that uncertainty has an impact on the optimal factor use of a profit maximizing firm. This is particularly true for the optimal adjustment of the firm's capital stock. While uncertainty has been considered in the static efficiency measurement literature it has been ignored so far in the context of dynamic efficiency measurement. This paper targets at closing this gap. For that purpose we take up a dynamic efficiency model which embeds a stochastic intertemporal duality model into a shadow cost framework and allows for measuring technical and allocative inefficiency. We derive hypotheses on how uncertainty affects the measurement of efficiency. The factor demand equations, which we derive, may serve as a starting point for an empirical validation of these hypotheses.

Keywords: efficiency, shadow cost approach, dynamic duality, uncertainty

JEL-codes: D61, D81, Q12

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Zusammenfassung

Ziel dieses Beitrages ist es, ein theoretisches Modell zur dynamischen Effizienzmessung so zu erweitern, dass unsichere Preise für Produktionsfaktoren bei der optimalen Faktoreinsatzmenge variabler und fixer Faktoren berücksichtigt werden. Insbesondere bei der Anpassung quasi-fixer Faktoren mittels Investitionen spielen unsichere Erwartungen über zukünftige Rückflüsse eine große Rolle. Während bereits einige Studien Unsicherheit bei der statischen Effizienzmessung berücksichtigen wird dieser Aspekt bei der dynamischen Effizienzmessung bislang ignoriert. Das hier verwendete dynamische Effizienzmodell basiert auf einer Kostenminimierung für eine gegebene Produktionsmenge und erlaubt, mit Hilfe der jeweiligen Schattenpreise und dazugehörigen Faktoreinsatzmengen sowohl allokative als auch technische Effizienz zu messen. Es werden Hypothesen über die Wirkung von Unsicherheit bei der dynamischen Effizienzmessung abgeleitet. Faktornachfragefunktionen, die aus dem Modell resultieren, können als Grundlage für empirische Analysen dienen.

Schlüsselwörter: Effizienz, Schattenpreise, dynamische Optimierung, Unsicherheit

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1 Introduction

It is widely acknowledged that structural change is closely related to the efficiency of firms within a sector (e.g., Goddard et al. 1993); however, there is a controversial discussion about the direction of causality. The classical structure-conduct-performance (SCP) paradigm states that there is a direct relationship between the market structure (degree of concentration) and the degree of competition among firms (Bain 1951). The SCP paradigm states that a higher degree of competition drives monopoly profits towards zero, leading to a higher degree of (social) efficiency. The SCP has been criticized for the assumption of an exogenous market structure. In reality, market structure is affected by firms' conduct and performance. This criticism has led to the efficiency-structure (ES) hypothesis (Demsetz 1973). According to the ES hypothesis, performance causes structure; firms with superior performance and higher efficiency increase their market share at the expense of less efficient firms, thereby increasing concentration. From this viewpoint, it is essential to understand and empirically analyse efficiency, as it is a major driver of structural change.

Unfortunately, standard efficiency analyses treat the time dimension of structural change in an unsatisfactory way. In the simplest case productivity and efficiency indicators are based on cross-sectional data and ignore time at all. If panel data are available, time varying inefficiency can be estimated in a stochastic production frontier model, examples are given by Kumbakhar (1990) or Battese and Coelli (1992). Ahn et al. (2006) present a flexible specification of the time varying efficiency term leading to a so called "dynamic frontier". Panel data also allow for a calculation of changes in the total factor productivity (TFP) of farms over time, for example by using the Malmqvist TFP index. Moreover, productivity changes can be decomposed into technical changes, changes of the technical and allocative inefficiency as well as scale effects (e.g., Brümmer et al. 2002). Though this kind of decomposition paves the way for a subtle analysis of the economic development of farms it is still not a fully dynamic analysis. The crucial point is whether the analysis of efficiency is based on a theory of intertemporal decision making or not. With a few exceptions to be mentioned below, efficiency analyses do not depart from a static cost minimization or profit maximization problem. Thereby intertemporal dependencies of factor allocations are ignored. In particular, no special attention is given to adjustments of the capital stock. That means no difference is made between investments and adjustments of variable production factors. In fact, standard efficiency analyses assume that capital can be adjusted to an optimal level instantaneously and without other costs than interest. This view, however, ignores the quasi-fixed character of capital. Disregarding possible costs attached to adjusting the capital stock and dynamic constraints may result in biased estimates of efficient frontiers in the sense that firms, which actually behave optimally, may appear inefficient. In what follows we call this phenomenon "seemingly inefficient". For example, it may be optimal for a particular farm to stick to an out-dated technology and sacrifice a gain of productivity if investments costs are irreversible and future returns are random. Similarly, it could be optimal not to reduce the capital stock in response to a decline in marginal capital productivity, because of lacking secondary markets for specific assets. Thus if the role of efficiency is considered in a long-term perspective, i.e., in order to discover its role for structural change a dynamic efficiency measurement is required allowing a distinction between short term and long-term efficiency. This distinction takes into account that decisions on the use of variable inputs are conditional on the endowment with quasi-fixed assets and farms incur adjustment costs when changing the quasi-fixed production factors. In other words, dynamic efficiency measurement takes into account that farmers' decisions are made in the short run with a view to the long run.

Acknowledging the necessity to distinguish between variable and quasi-fixed production factors brings up another challenge, namely the consideration of uncertainty when deriving the "optimal" level of input use. Even in a static framework risk is an issue when measuring efficiency (Kumbakhar 2002). However, risk seems to be even more important for decisions about fixed factors like investments, because risk unfolds over time and farmers have to build expectations on costs and returns over a longer time period. Thus it is not surprising that modern investment theory explicitly accounts for risk (e.g., Abel and Eberly 1994; Dixit and Pindyck 1994). The impact of risk on optimal investment and the adjustment path of the firm's capital can be twofold. In the presence of uncertainty risk-averse decision makers will discount future investment returns at a higher (risk adjusted) discount rate so that investment projects, which appear profitable in a deterministic setting, are rejected. Second, even in the case of risk neutrality uncertainty is relevant for the investment decision since it creates a value of waiting which in turn results in investment reluctance. That means increasing risk may widen the optimal range of inaction for capital and other quasi-fixed factors. Note, this view is emphasized by the real options approach for analysing investments.

It is well known that uncertainty affects the optimal adjustment path of quasi-fixed inputs over time, though the sparse literature on dynamic efficiency analysis invariably assumes static expectations on future costs and returns. The objective of this paper is to incorporate risk into a dynamic efficiency model. To our best knowledge such an attempt has not been made so far and thus it closes a gap in the existing literature. The basic idea, which we will pursue for this purpose, is to bridge models of investment under uncertainty and (deterministic) dynamic efficiency analysis.

The next section provides an overview of the existing literature related to the consideration of risk in efficiency measurement and non-parametric as well as parametric measurement of dynamic efficiency. Section 3 presents a theoretical framework of a dynamic efficiency model under uncertainty and hypotheses. Section 4 concludes and discusses proposals for further research.

2 Literature review

The impacts of risk or uncertainty on firms' decision making and their efficiency has been an issue for a long time, for instance Kumbhakar (1993; 2002), Caudill and Ford (1993), Battese et al. (1997) or Wang (2002). All these studies share the aim to account for production risk in the efficiency measurement approach. Kumbhakar (1993) measures production risk and technical efficiency using a static Stochastic Frontier Analysis (SFA) where risk may also cause a deviation from the efficient frontier. Kumbhakar (2002) generalizes this model by distinguishing between production risk and risk aversion. Thereby the risk preference function consists of two parts, one associated with production risk and one with technical inefficiency. The findings reveal these two parts with a stronger production risk part. Alternative specifications have been investigated by Caudill and Ford (1993), Wang (2002) or Battese et al. (1997). Besides this strand, Chambers and Quiggin (2002) use a state-contingent approach to account for risk since producers may face possible states characterized by different price levels. Stochastic production frontiers presume one common state of nature and deviations are assigned to random errors or inefficiency. Thereby, possible responses to a set of states are disregarded through which differing responses the producers may appear inefficient even though the decision has been rational. Empirical evidence is found by O'Donnell and Griffiths (2006) and Chavas (2008). Also Nauges et al. (2009) find evidence on the defined states of risk while rejecting the classical SFA model. O'Donnell et al. (2010) even show using simulated data that under state-contingent risk the classical SFA approach leads to wrong estimates of the technical inefficiency.

Even though there is large literature about how risk may be considered in the efficiency measurement approaches, various shortcomings remain. First, there is no demarcation between inputs in the production process and all inputs are treated like variable inputs. Second, the explicit role of time and the adjustment of the farms over time are not considered in static efficiency models. Third, it is not taken into account whether quasi-fixed inputs have to be adjusted in the long run. The adjustment process of quasi-fixed inputs over time may generate additional transitory costs in the decision making process. Static efficiency approaches assume that firms adjust to the long-term optimal values immediately and efficiency is measured by relating the observed input and the optimal long-term value. Disregarding the long-term optimal adjustment to the optimal input level may cause inaccurate measures of efficiency. Dynamic efficiency approaches account explicitly for the optimal path of adjustment over time and measure efficiency by relating the observed input and the optimal adjustment path of the input over time. Though, it is not surprising that Gardebroek and Oude Lansink (2008) suggest that dynamic efficiency measurement is more appropriate than the static one.

While static efficiency measurement has a long history, dynamic efficiency measurement is a rather novel research area. Dynamic efficiency measurement strives for a cross-fertilization of dynamic models of decision making and traditional efficiency analysis. Nemoto and Goto (1999, 2003) develop a dynamic DEA model that takes into account adjustment costs. Ouellette and Yan (2008) take up this model and generalize it. Their model distinguishes between variable inputs that can vary in the short run and quasi-fixed (nondiscretionary) inputs that can vary only in the long run. Inter-temporal adjustment restrictions are incorporated into a static cost-minimizing DEA model. These restrictions reflect an optimization over several periods where a DMU (decision making unit) balances the cost of an investment (acquisition costs plus adjustment costs) and the expected reduction of variable costs due to this investment. The resulting dynamic DEA allows for a decomposition of overall economic efficiency into static and dynamic efficiency.

Silva and Stefanou (2003, 2007) develop non-parametric dynamic measures of technical, allocative and economic efficiency in the short run and in the long run. Short run measures indicate whether variable inputs are employed efficiently in the production process, whereas long run efficiency captures both variable and quasi-fixed factors. The starting point of their model is an inter-temporal cost minimization problem in which capital is treated as a quasi-fixed factor. The dynamic nature of the decision problem is addressed in the production technology specification via a convex adjustment cost function for a change in quasi-fixed factors. The authors derive lower and upper bounds for each efficiency-measure and apply their model to a panel data set of U.S. dairy farms. They show that the allocation decisions of inputs with adjustment processes over time such as capital or labour are a main source of farm specific inefficiency. Oude Lansink and Silva (2006) refer to the theoretical framework of Silva and Stefanou (2003) and measure dynamic efficiency in the short and long run by means of a directional distance function approach. They apply their model to horticultural firms and their findings reveal that in the short and long run allocative efficiency is higher than technical efficiency with lower values in the long run. They further show that the allocation of quasi-fixed factors is less optimal than the allocation of variable factors which provides evidence for the presence of adjustment costs.

Rungsuriyawiboon and Stefanou (2007, 2008) pursue a similar approach which will be explained in detail in the next section, since their model forms the starting point for our exposition. The authors establish a dynamic efficiency model by integrating the static shadow cost approach and the dynamic dual model of inter-temporal decision-making. Based on an inter-temporal cost minimization problem they derive the optimal dynamic factor demand functions for the variable inputs and quasi-fixed inputs. The incorporation and the decomposition of economic efficiency are achieved by a shadow cost approach. In essence, it is distinguished between actual costs and behavioural (or shadow) costs of a firm. The actual cost function refers to the perfect minimization of cost with respect to the observed prices, whereas the behavioural or shadow costs are associated with the observed input levels of the firm chosen to be the cost-minimizing level with respect to the shadow prices. In the presence of inefficiencies, shadow costs for production factors will deviate from actual (market) prices. They find overcapitalization of U.S. electric utilities and a relative underuse of the variable production factors.

We can resume that several attempts for measuring dynamic efficiency exist. The aforementioned contributions to dynamic efficiency measurement share one important feature, namely the assumption of static expectations of future prices and returns. This basically means that current prices and outputs contain all relevant information and will persist in the future. Decision makers are not allowed to anticipate revisions in their expectations and uncertainty does not play a role at all. This is, of course, a highly unrealistic assumption. Actually uncertainty turned out to be an important determinant for investment demand and production decisions (e.g., Dixit and Pindyck 1994). Contrarily, studies that incorporate risk into efficiency measurement are purely static. The model, which we develop in the next section, merges these two aspects.

3 A dynamic efficiency model with uncertainty

The model that we propose rests upon two building blocks: first, a static shadow cost approach to efficiency measurement and second, the dynamic dual model of inter-temporal decision making. We follow Rungsuriyawiboon and Stefanou (2007) in combining these two components. The main contribution of this paper is to enhance the deterministic dynamic dual model by a stochastic one. In what follows, we first outline the idea of dynamic efficiency measurement and the general measurement procedure using a parametric shadow cost approach (3.1). Afterwards we derive the theoretical dynamic efficiency model under uncertainty (3.2). Finally, we suggest a functional form for the underlying value function that may serve as a vantage point for the derivation of an empirical model (3.3) and present possible hypotheses how risk may affect the measurement of efficiency (3.4).

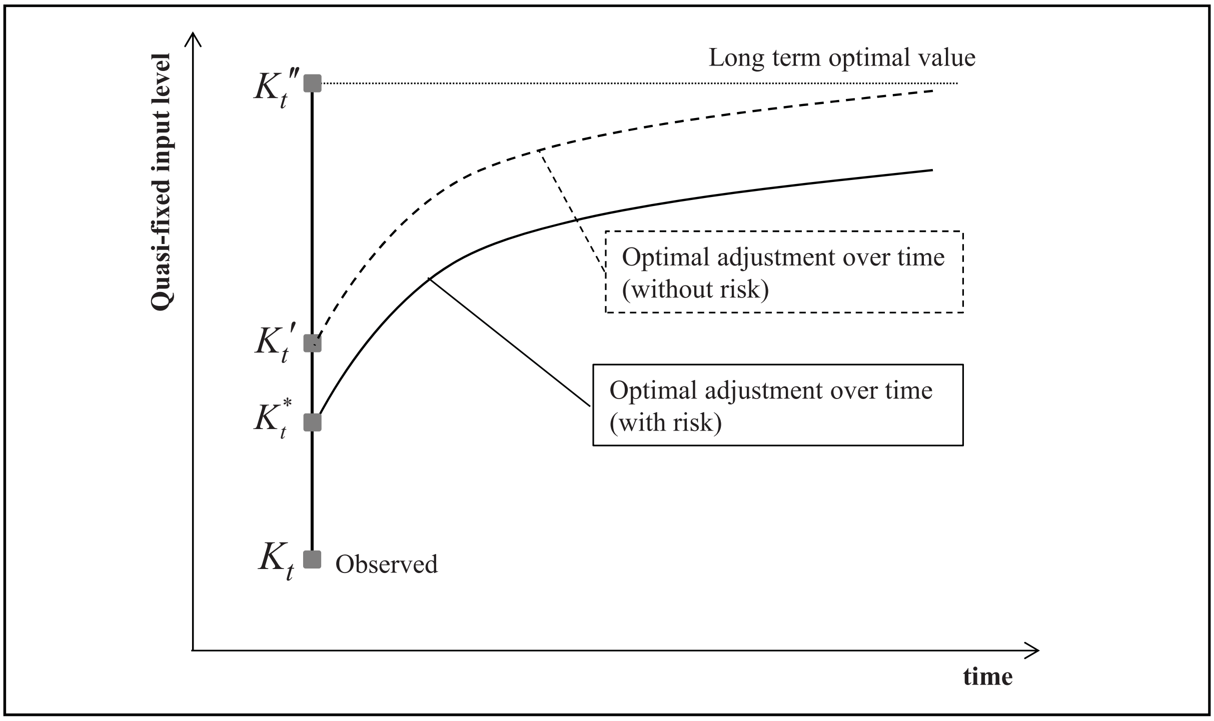
3.1 The general idea of dynamic efficiency measurement

As mentioned above, the majority of efficiency studies ignore the existence of adjustment cost and the interdependence of production decisions over time. As Gardebroek and Oude Lansink (2008) point out, this ignorance may cause inaccurate measures of efficiency and firms may be seemingly inefficient. When explicitly considering the role of time and referring to a dynamic context, it is useful to distinguish two types of inputs, namely variable inputs and quasi-fixed inputs. For the latter a costly adjustment process to the optimal long-run level has to be taken into account. Costs attached to the adjustment process may for instance be temporary losses in production or transaction costs. If these are not appropriately considered in the efficiency measurement they may contribute to the occurrence of seeming inefficiency.[1] They may also add to inefficiency. For instance, in the case of a too rapid adjustment of the capital stock a firm may incur higher adjustment costs than compared to the optimal adjustment rate (Rungsuriyawiboon and Stefanou 2007).

Figure 1 illustrates the difference between static and dynamic efficiency measurement. Therein K_t is the observed level of a quasi-fixed factor for a farm in a particular time period *t*. The curve starting in $K_t^{'}$ represents the optimal adjustment path of the quasi-fixed input level over time and $K_t^{''}$ denotes the long-term optimal value. *Static* efficiency approaches assume that firms adjust to the long-term optimal values immediately and measure efficiency as the ratio of observed input (K_t) and the long-term optimal value ($K_t^{''}$). In contrast, *dynamic* efficiency approaches account for the optimal path of adjustment and measure efficiency by referring to the ratio of observed input (K_t) and optimal adjustment of the input over time ($K_t^{'}$). Obviously, referring to $K_t^{'}$ instead of $K_t^{''}$ as the cost minimal level of the capital stock, overestimates inefficiency, at least in the short run. This effect could be even more pronounced if uncertainty of future costs or revenues is taken into account because uncertainty may also increase investment reluctance. As a result the optimal adjustment path shifts downwards and the capital stock at time *t* is K^*.

[1] Note, a detailed overview about adjustment costs in the context of quasi-fixed factors can be found for instance in Hamermesh and Pfann (1996).

Figure 1. Efficiency measurement over time



Source: Adopted from Gardebroek and Oude Lansink (2008).

Shadow cost approach

Now we turn to the first block in order to measure firm specific efficiency in a dynamic context. The parametric shadow cost approach as initially introduced by Lau and Yotopoulos (1971) and later generalized to panel data setting by Atkinson and Cornwell (1994) has the advantage that the decomposition of the cost efficiency into its technical and allocative components is not that cumbersome as in the SFA approach (Kumbhakar and Lovell 2003). Moreover, within the shadow cost approach no distributional assumptions are necessary with respect to the inefficiency term. It is assumed that a representative firm minimizes its shadow or behavioural costs. Thereby, the shadow prices are defined as input prices that force a technically efficient input choice to be the cost-minimizing choice. In the presence of inefficiency, the shadow input prices will differ from the actual prices (that may be observed). Since the shadow prices are not observable it is not possible to directly estimate the shadow cost function. For this reason the shadow prices are directly related to the actual (observed) prices by means of the inefficiency terms. Technical inefficiency is introduced as a deviation from the shadow cost optimal input choice. Thus, the firm's actual cost functions and the observed input demand equations are expressed in terms of shadow input prices including technical and allocative inefficiency terms. Empirical applications of the shadow cost approach in agricultural economics can be found for instance in Maietta (2000), Reinhard and Thijssen (2000) or Mosheim and Lovell (2009).

Outline of the model derivation

The major aim of our paper is to combine the shadow cost approach with a stochastic dynamic dual model. This involves several steps which are depicted in a flow chart as shown in Figure 2. We start from a cost minimization problem for a representative firm assuming that it is optimal to minimize factor inputs for a given output level. Such behaviour seems reasonable for example for dairy farms under the milk quota regulation. The optimization is subject to the equation of motion of capital, the stochastic price development over time and the production sequence. Using dynamic programming solves the decision problem and applying Shephard's Lemma yields the optimal net investment and optimal variable input demand functions under uncertainty. This standard procedure of the dynamic dual model, however, has to be extended in order to capture inefficiency effects. This is achieved by means of the shadow cost approach. For this purpose the behavioural or shadow value function using shadow prices and quantities and the actual value function using actual prices and quantities are defined. The shadow value function guarantees the optimality with respect to the shadow prices that may differ from the actual prices by inefficiencies, whereas the actual value function gives the condition for optimality under full efficiency.

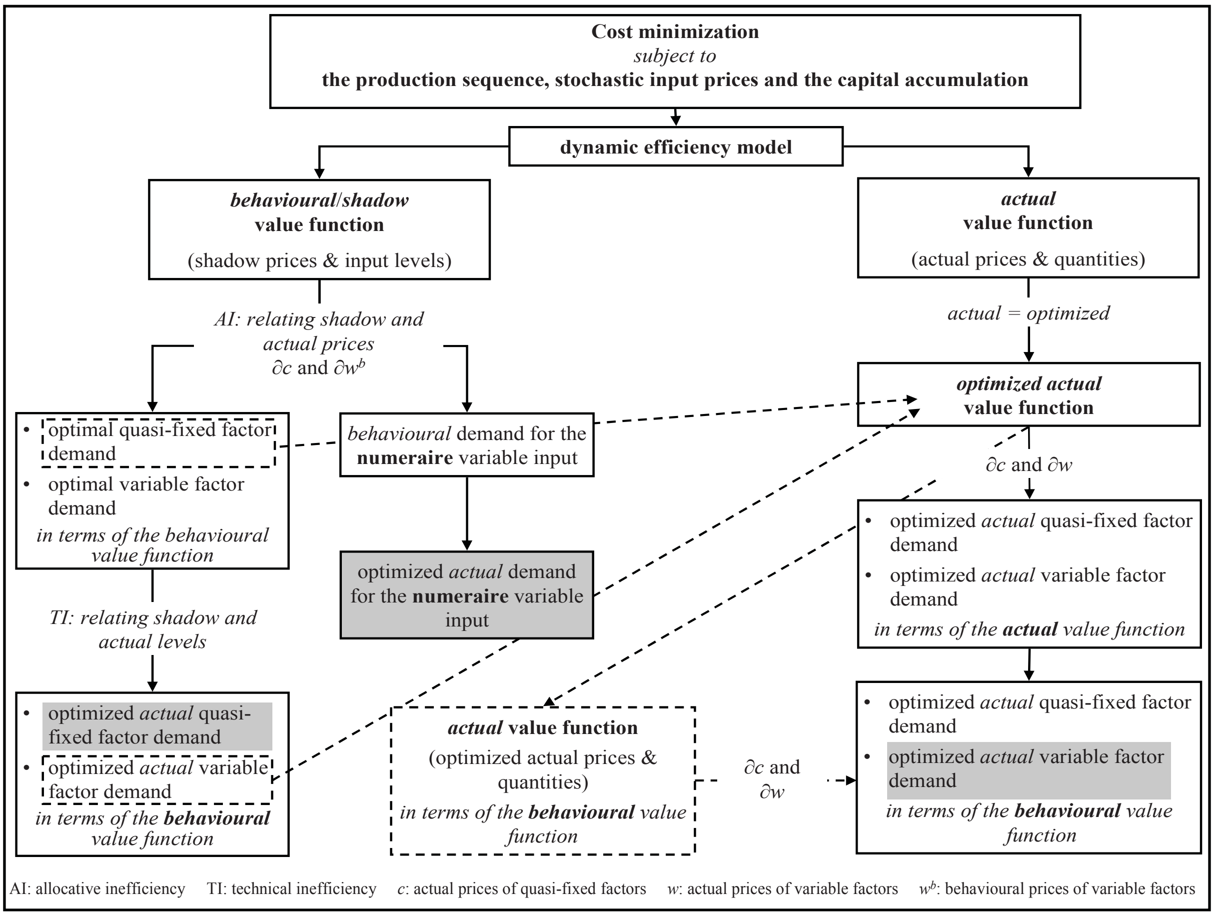
First, the behavioural value function (left hand side of the flow chart) is defined using shadow prices for the variable inputs (w^b) and the shadow factor demand for variable (x^b) and quasi-fixed factors ($\dot{K}^b$). Shadow prices are related to observed input prices (w) by means of an allocative inefficiency term λ. Prices of the variable factors are normalized to a numeraire variable such that allocative inefficiency is interpreted as price distortions in relation to the numeraire variable in order to identify over- or underuse of variable input factors. The shadow factor demand is related to the actual factor level by means of the technical inefficiency term, for both the variable and the fixed factors.

In order to get the variable and fixed factor demand equations in terms of the behavioural value function with both inefficiency terms, the behavioural value function is differentiated with respect to (a) the shadow prices (w^b), thereby the allocative inefficiency is incorporated into the factor demand equations and (b) the price for the quasi-fixed input (c), thereby technical inefficiency has to be incorporated since the factor demand (variable and quasi-fixed) optimized with respect to the shadow prices differs from the actual/observed level by a technical inefficiency term. Carrying out these two steps yields the optimized actual quasi-fixed and variable factor demand equations in terms of the behavioural value function (bottom of the left hand side in the flow chart in Figure 2).

Second, the cost minimization problem is solved under actual conditions, i.e. observable prices and quantities (right hand side of the flow chart in Figure 2). The resulting optimized actual value function may be interpreted as the long-run cost function. The latter is differentiated with respect to actual prices in order to get the optimized actual quasi-fixed and variable factor demand in terms of the actual value function.

Then, in a final step, the optimized factor demand equations expressed in terms of the behavioural value function are incorporated into the actual value function with the respective inefficiency terms (dotted lines) yielding the actual value function with optimized actual prices and quantities in terms of the behavioural value function. The latter value function is differentiated with respect to the actual prices. Using the optimized actual factor demand equations in terms of the actual value function gives the optimized actual factor demand equations expressed in terms of the behavioural value function (bottom of the right hand side of the flow chart). This step is necessary to identify the inefficiency parameters in terms of observable input prices and levels. The grey shaded boxes thereby indicate the final equations that may serve as a base for empirical work.

Figure 2. Procedure of dynamic efficiency measurement under uncertainty



3.2 Theoretical model

Cost minimization under uncertainty

We refer to the dynamic intertemporal duality model of Epstein (1981) wherein a firm minimizes its variable costs for a planned level output. The dual variable cost function is given by

$$C_i\left(w_{ni}(t), y_i(t), K_{mi}(t), \dot{K}_{mi}(t), t\right) = \min_{x_{ni}(t)} \left\{ \sum_n \left(w_{ni}(t) \cdot x_{ni}(t) \right) \right\} \tag{1}$$

subject to

$$y_i(t) \le F\left(x_{ni}(t), K_{mi}(t), \dot{K}_{mi}(t)\right) \tag{2}$$

where i indexes individuals, t denotes time[2], $x_n(t)$ denotes the level of n^{th} variable factor in use where $x_1(t), x_2(t), ..., x_{\bar{n}}(t) \in \Re_+^{\bar{n}}$ with the respective factor prices $w_1(t), w_2(t), ..., w_{\bar{n}}(t) \in \Re_{++}^{\bar{n}}$. In addition to the variable costs, the firm also incurs costs of quasi-fixed factors by purchasing quasi-fixed factors represented by $\sum_m \left(c_m(t) \cdot K_m(t)\right)$, where $K_m(t)$ denotes the level of the m^{th} quasi-fixed factor with $K_1(t), K_2(t), ..., K_{\bar{m}}(t) \in \Re_+^{\bar{m}}$ and $c_1(t), c_2(t), ..., c_{\bar{m}}(t)$ being the prices of the quasi-fixed factors respectively. $y(t)$ represents the expected production level of a single output at time t and the representative firm's technology is described by the production function $F\left(x(t), K(t), \dot{K}(t)\right)$. Therein $\dot{K}_m(t)$ refers to the net investment into the respective m^{th} quasi-fixed factor. The inclusion of the net investment in the production function allows to account for the dependency of the output on the size of adjustments in the stock of the quasi-fixed factors and reflects the presence of internal adjustment cost in terms of foregone output (Stefanou 1989). The production function is assumed to be concave in $\dot{K}_m(t)$ which implies increasing marginal adjustment costs: the loss in production is assumed to be larger for faster adjustments in the capital stock and as a result, the firm will tend to adjust more slowly such that $\dot{K}_m F_{\dot{K}_m} < 0$ and $F_{\dot{K}_m \dot{K}_m} < 0$ holds (Stefanou 1989).

The variable cost function $C(\cdot)$ as given in (1) reflects the least cost variable input combination for each quantity of output $y(t)$. In line with Epstein and Denny (1983) we assume that the producer takes the factor prices as well as the level of output as given in the base period $t = 0$. The firm is assumed to minimize its expected discounted sum of all future cost over an infinite planning horizon. In contrast to Epstein and Denny (1983), we assume that the future costs are uncertain. Formally, the optimization problem is given by the value function $J(\cdot)$.

$$J\left(w(0), c(0), y(0), K(0)\right) = \min_{I_m(t)} E_0 \int_0^\infty e^{-rt} \left[C\left(w_n(t), y(t), K_m(t), \dot{K}_m(t), t\right) + \sum_m \left(c_m(t) \cdot K_m(t)\right) \right] \cdot dt \tag{3}$$

where E_0 denotes the expectation operator conditional on information available at present time and $r > 0$ is a discount rate. This cost minimization is subject to the evolution of the capital stock described by

[2] The subscript i is suppressed and the time dependency left out for notational convenience wherever possible.

$$\dot{K}_m(t) = \left(I_m(t) - \delta \cdot K_m(t)\right) \tag{4}$$

where $K_m(t) > 0$. $I_m(t)$ denotes gross investment, and δ refers to the depreciation rate which is assumed to be constant over time. Moreover, the optimization in equation (3) is subject to uncertain dynamics in all input prices and in the output level. Thus, we summarize the latter in a state vector $z(t) = \left(\ln y(t),\ \ln w_1(t),...,\ln w_{\bar{n}}(t),\ \ln c_1(t),...,\ln c_{\bar{m}}(t)\right)'$ containing the logarithms of the output quantity, the variable factor prices and the quasi-fixed factor prices. The evolution of this state vector follows an arithmetic Brownian Motion as follows[3]:

$$dz = \alpha \cdot dt + \psi \cdot dv \tag{5}$$

where α denotes the drift parameter, $\Sigma = \psi\psi'$ represents the variance-covariance matrix and dv is an identically, independently and normally distributed vector term with $E(dv) = 0$, $E\left[(dv)^2\right] = dt$ and $E\left(dv_i, dv_j\right) = 0$ for all $i \neq j$. The firm's stochastic optimization problem as given in equation (3), subject to the constraints (4) and (5) is solved using stochastic dynamic programming. The Hamilton-Jacobi-Bellman (HJB) equation for this problem is given by (Pietola and Myers 2000):

$$\begin{aligned} rJ(z, K) = \\ \min_{x,\, I} \Big\{ &\sum_n \left(w_n \cdot x_n(t)\right) + \sum_m \left(c_m \cdot K_m(t)\right) + \sum_m \left(J_{K_m} \cdot \left(I_m(t) - \delta \cdot K_m(t)\right)\right) \\ &+ \gamma(t) \cdot \left[y(t) - F\left(x_n(t), K_m(t), \dot{K}_m(t)\right)\right] + J_z \cdot \alpha + \frac{1}{2} \cdot \Omega \Big\} \end{aligned} \tag{6}$$

where $\gamma(t) = \partial J / \partial y$ is the so called co-state variable associated with the production target constraint and is interpreted as the long-run marginal cost (savings) by increasing or decreasing the level of the planned output at time *t* (Stefanou 1989). $J_{K_m} = \partial J / \partial K_m$ denotes the partial derivative of *J* with respect to the m^{th} quasi-fixed factor. These partial derivatives can be interpreted as shadow values: the change in the value function induced by a change in the firm's initial stock of quasi-fixed factors. $J_z = \partial J / \partial z$ denotes the partial derivatives of *J* with respect to the state vector z including the factor prices and the logarithm of the output which may also be interpreted as a shadow value. For instance, with respect to the input prices they reflect a change in the value function caused by a change in the firm's initial price level (note, the derivative with respect to the logarithm of the output is not explicitly interpreted since the shadow value of the output constraint is already accounted for by the co-state variable γ). Finally, $\Omega = \sum_{j=1}^{1+\bar{n}+\bar{m}} \sum_{j'=1}^{1+\bar{n}+\bar{m}} J_{z_j z_{j'}} \sigma_{jj'}$ wherein J_{zz} refers to the Hessian matrix of *J*

[3] Note that this implies a geometric Brownian motion for $y(t)$, $w(t)$, and $c(t)$.

containing the second order derivatives with respect to $z(t)$. *j* and *j'* index the respective elements of $z(t)$.

The equilibrium condition in equation (6) states that the variable inputs *x* and investments *I* should be chosen at each time such that the variable production costs $\sum_n \left(w_n \cdot x_n(t) \right)$, the costs of quasi-fixed factors $\sum_m \left(c_m \cdot K_m(t) \right)$, the gain from changing the stock of the quasi-fixed factors $\sum_m \left(J_{K_m} \cdot \left(I_m(t) - \delta \cdot K_m(t) \right) \right)$ and the instantaneous change in the long-run cost given by $\gamma(t) \cdot \left[y(t) - F\left(x_n(t), K_m(t), \dot{K}_m(t) \right) \right]$ are minimized. Note, the last two terms, $J_z \cdot \alpha$ and $\frac{1}{2} \cdot \Omega$, arise from the stochastic evolution of the logarithms of the output and the input prices.

According to Shephard's Lemma, differentiating the HJB equation given in (6) with respect to $\ln c_m$ and $\ln w_n$ yields the respective conditional factor demand equations. Note, because of the assumption of an arithmetic Brownian Motion for the state vector we need to differentiate with respect to logarithms of the factor prices while applying Shephard's Lemma. Referring to Pietola and Myers (2000), the optimal net investment demand is given by

$$\dot{K}_m^* = \left(J_{K_m, \ln c_m} \right)^{-1} \left(rJ_{\ln c_m} - c_m \cdot K_m - \sum_{m' \neq m} \left(\dot{K}_{m'} \cdot J_{K_{m'}, \ln c_m} \right) - \alpha \cdot J_{z, \ln c_m} - \frac{1}{2} \cdot \Omega_{\ln c_m} \right) \tag{7}$$

where index m' indicates the quasi-fixed factors other than *m* with $m' = 1, 2, \ldots, \overline{m} \quad \forall m' \neq m$. Thus, according to equation (7), the optimal investment demand for the m^{th} quasi-fixed factor is a function of all investments in other quasi-fixed factors indicated by $\sum_{m' \neq m} \left(\dot{K}_{m'} \cdot J_{K_{m'}, \ln c_m} \right)$.

Furthermore, the optimal variable factor demand for the n^{th} variable input is given by

$$x_n^* = \frac{1}{w_n} \cdot \left(r \cdot J_{\ln w_n} - \sum_m \left(J_{K_m, \ln w_n} \cdot \dot{K}_m \right) - \alpha \cdot J_{z, \ln w_n} - \frac{1}{2} \cdot \Omega_{\ln w_n} \right) \tag{8}$$

From equations (7) and (8) it becomes apparent that uncertain input prices will affect the optimal decisions with respect to the variable and quasi-fixed inputs over time.

Incorporation of technical and allocative inefficiency

Now we integrate the static shadow cost approach into the intertemporal dual model of cost minimization in order to measure economic inefficiency under uncertainty in a dynamic context. For this purpose we create two types of cost functions: the behavioural and the actual cost function. Using the shadow input prices and quantities we set up the behavioural value function which guarantees the cost minimized relation under shadow prices. Further, the actual (may be observed) input prices and quantities are used to set up the actual value function. By considering the actual used input quantities as the optimal ones, the actual value function becomes the optimized actual value function and represents a fully efficient input use. That is, in the presence of perfect efficiency the actual value function is equivalent to the behavioural value function, whereas in the presence of inefficiency they differ. As a result, the optimized actual value function is expressed in terms of behavioural value function and this allows to express the optimized actual factor demand equations in terms of the behavioural value function including several inefficiency terms.

In a first step, we define a behavioural value function (indicated by the superscript '*b*') quite similar to equation (6), but variable costs are now calculated with the shadow input prices and quantities. Minimizing the behavioural value function leads to the behavioural HJB equation that can be written in terms of the shadow input prices and quantities as follows[4].

$$\begin{aligned} & rJ^b\left(w_n^b(t), c_m(t), K_m(t), y(t)\right) = \\ & \sum_n\left(w_n^b(t)\cdot x_n^b(t)\right) + \sum_m\left(c_m(t)\cdot K_m(t)\right) + \sum_m\left(J_{K_m}^b\cdot\left(I_m(t)-\delta\cdot K_m(t)\right)\right) \\ & +\gamma^b(t)\cdot\left[y(t) - F\left(x_n^b(t), K_m(t), \dot{K}_m^b(t)\right)\right] + J_z^b\cdot\alpha + \frac{1}{2}\cdot\Omega^b \end{aligned} \tag{9}$$

where $w_n^b = \lambda_n w_n = \lambda_1 w_1, \lambda_2 w_2, ..., \lambda_{\bar{n}} w_{\bar{n}}$ denotes the shadow prices of the variable factors. λ_n denotes the firm specific allocative inefficiency parameter for the n^{th} variable factor (AE parameter). If $\lambda_n = 1$, then the n^{th} variable factor is allocative efficiently used. Values of $\lambda_n > 1$ (<1) indicate that the decision maker distributes less (more) of the n^{th} input compared to the cost-minimizing allocation. $J_{K_m}^b$ denotes the marginal value of the behavioural capital stock which can be related to the marginal value of actual capital, $J_{K_m}^a$, by the following definition: $J_{K_m}^b = \mu_m \cdot \left(J_{K_m}^a\right)$. Herein μ_m indicates the allocative inefficiency parameter of net investments in quasi-fixed factors. $\gamma^b(t) \geq 0$ is the behavioural short run marginal cost of production and gives the value of relaxing the production target constraint in terms of the shadow (behavioural) factor prices. $\Omega^b = \sum_{j=1}^{1+\bar{n}+\bar{m}} \sum_{j'=1}^{1+\bar{n}+\bar{m}} J_{z_j z_{j'}}^b \sigma_{jj'}$ is the stochastic term in the behavioural HJB equation.

x_n^b in equation (9) stands for the behavioural variable factor demand which is assumed to be the technically efficient level. The optimized behavioural level is assumed to be the technically and allocatively efficient level of the respective variable factor. However, the observed variable factor demand (x_n) may deviate from the behavioural demand due to technically inefficient choices. Introducing $\tau_{x_n} \geq 1$ as a measure of input-oriented technical efficiency of variable factor use, behavioural factor demand $\left(x_n^b\right)$ is related to the actual (x_n)

4 Note, equation (9) is the optimized version of the behavioural HJB equation, thus the min-operator does not appear here.

factor demand such that $x_n^b = \left(1/\tau_{x_n}\right) \cdot x_n$. Likewise the demand for quasi-fixed inputs may be technically inefficient, thus: $\dot{K}_m^b = \left(1/\tau_{K_m}\right) \cdot \dot{K}_m$, where $\dot{K}_m^b$ and $\dot{K}_m$ denotes the level of behavioural and actual net investments in the stock of quasi-fixed factors, respectively and $\tau_{K_m} \geq 1$ is the input-oriented measure of the technical efficiency in the quasi-fixed factors.

In order to reduce the complexity of our model we assume for the subsequent steps that the drift rate of the Brownian Motion is zero such that $\alpha = 0$. Thus, the behavioural value function in (9) can be rewritten as[5]

$$
\begin{aligned}
& rJ^b\left(\lambda_n w_n, c_m, K_m, y\right) \ = \\
& \sum_n \left(\left(\lambda_n w_n\right) \cdot x_n^b\right) + \sum_m \left(c_m \cdot K_m\right) + \sum_m \left(\left(J_{K_m}^b\right) \cdot \left(I_m - \delta \cdot K_m\right)\right) \\
& + \gamma^b \cdot \left(y - F\left(x_n^b, K_m, \dot{K}_m^b\right)\right) + \frac{1}{2} \cdot \Omega^b
\end{aligned}
\tag{10}
$$

Carrying out the same steps as in the basic cost minimization problem, i.e., applying Shephard's Lemma and differentiating equation (10) with respect to $\ln c_m$ and $\ln \lambda_n w_n$, we obtain the optimal factor demand equations in terms of the shadow prices and shadow quantities. The optimal behavioural net investment demand function for the m^{th} quasi-fixed input is given by

$$
\dot{K}_m^b = \left(J_{K_m, \ln c_m}^b\right)^{-1} \left(rJ_{\ln c_m}^b - c_m \cdot K_m - \sum_{m' \neq m} \left(\dot{K}_{m'}^b \cdot J_{K_{m'}, \ln c_m}^b\right) - \frac{1}{2} \cdot \Omega_{\ln c_m}^b \right)
\tag{11}
$$

and the optimal behavioural variable factor demand for the n^{th} variable input reads as follows:

$$
x_n^b = \frac{1}{w_n^b} \cdot \left(rJ_{\ln w_n}^b - \sum_m \left(J_{K_m, \ln w_n}^b \cdot \dot{K}_m^b\right) - \frac{1}{2} \cdot \Omega_{\ln w_n}^b \right)
\tag{12}
$$

In the presence of technical inefficiency, the optimal behavioural factor demand equations can be expressed in terms of the optimized actual net investment and variable factor demand equations (based on the behavioural value function). As these are observed we indicate 'observed' by the superscript ^{o}. Note, optimized actual and observable are used interchangeably here.

$$
\begin{aligned}
& \dot{K}_m^o = \tau_{K_m} \cdot \dot{K}_m^b \\
& \dot{K}_m^o = \tau_{K_m} \cdot \left(J_{K_m, \ln c_m}^b\right)^{-1} \left(rJ_{\ln c_m}^b - c_m \cdot K_m - \sum_{m' \neq m} \left(\dot{K}_{m'}^b \cdot J_{K_{m'}, \ln c_m}^b\right) - \frac{1}{2} \cdot \Omega_{\ln c_m}^b \right)
\end{aligned}
\tag{13}
$$

[5] The time dependency is suppressed for notational convenience where possible.

$$x_n^o = \tau_{x_n} \cdot \left(x_n^b \right)$$

$$x_n^o = \frac{\tau_{x_n}}{w_n^b} \cdot \left(rJ_{\ln w_n}^b - \sum_m \left(J_{K_m,\ \ln w_n}^b \cdot \dot{K}_m^b \right) - \frac{1}{2} \cdot \Omega_{\ln w_n}^b \right) \tag{14}$$

Next, we introduce the actual value function (indicated by the superscript ^{a}) of the firm's cost minimization problem in order to get the optimized actual factor demand equations (recall Figure 2). Here, the optimized version of the actual HJB equation in the presence of uncertainty corresponding to the actual input prices and quantities can be written as

$$rJ^a =$$

$$\sum_n \left(w_n \cdot x_n \right) + \sum_m \left(c_m \cdot K_m \right) + \sum_m \left(J_{K_m}^a \cdot \dot{K}_m \right) + \gamma^a \cdot \left(y - F\left(x_n, K_m, \dot{K}_m \right) \right) + \frac{1}{2} \cdot \Omega^a \tag{15}$$

We assume that the actual input levels are the optimal ones and we replace the actual input levels by the optimized actual ones denoted by $\dot{K}_m^o$ and x_n^o. This further implies that actual output is the optimized output such that $y - F\left(x_n^o, K_m, \dot{K}_m^o \right) = 0.$ Note, $\Omega^a = \sum_{j=1}^{1+\bar{n}+\bar{m}} \sum_{j'=1}^{1+\bar{n}+\bar{m}} J_{z_j z_{j'}}^a \sigma_{jj'}$ represents the uncertainty in the actual HJB equation.

The optimized version of the actual HJB equation in (15) is now expressed in terms of the actual prices and the optimized actual input levels resulting in an optimized actual HJB equation. The optimized actual HJB equation states the perfect efficiency condition, hence without inefficiency and is given by

$$rJ^a =$$

$$\sum_n \left(w_n \cdot x_n^o \right) + \sum_m \left(c_m \cdot K_m \right) + \sum_m \left(J_{K_m}^a \cdot \dot{K}_m^o \right) + \frac{1}{2} \cdot \Omega^a \tag{16}$$

Taking the derivatives of the optimized actual HJB equation in (16) with respect to $\ln c_m$ and $\ln w_n$ yields the optimized actual net investment demand function for the m^{th} quasi-fixed input and the optimized actual variable factor demand function for the n^{th} variable input under perfect efficiency as follows:

$$\dot{K}_m^o = \left(J_{K_m, \ln c_m}^a \right)^{-1} \left(rJ_{\ln c_m}^a - c_m \cdot K_m - \sum_{m' \neq m} \left(\dot{K}_{m'}^o \cdot J_{K_{m'}, \ln c_m}^a \right) - \frac{1}{2} \cdot \Omega_{\ln c_m}^a \right) \tag{17}$$

$$x_n^o = \frac{1}{w_n} \cdot \left(r \cdot J_{\ln w_n}^a - \sum_m \left(J_{K_m, \ln w_n}^a \cdot \dot{K}_m^o \right) - \frac{1}{2} \cdot \Omega_{\ln w_n}^a \right) \tag{18}$$

In the presence of perfect efficiency, the optimized actual value function is equivalent to the behavioural value function. Whereas in the presence of inefficiency, the optimized actual value function is not equivalent to the behavioural value function and the deviation is accounted in the inefficiency parameters.

So finally, we need to express the optimized actual HJB equation in (16) in terms of the behavioural value function (recall Figure 2). This step is also necessary to introduce the allocative inefficiency parameter of the net investment denoted by μ_m into the model. For that purpose we express the actual and optimized actual terms in the equation (16) by corresponding behavioural terms, i.e., we substitute $\dot{K}_m^o$ by $\tau_{K_m} \cdot \dot{K}_m^b$, J_{zz}^a by J_{zz}^b and $J_{K_m}^a$ by $(1/\mu_m) \cdot J_{K_m}^b$. In equation (16), we further replace (x_n^o) by the optimized actual variable factor demand defined in equation (14) and ($\dot{K}_m^b$) is substituted using the behavioural net investment demand equation given in (11). The optimized actual HJB in equation (16) is now rewritten in terms of the behavioural value function as:

$$
\begin{aligned}
rJ^a =& \\
&\sum_n \left\{ \frac{\tau_{x_n}}{\lambda_n} \left[rJ^b_{\ln w_n} - \sum_m \left\{ \left(J^b_{K_m,\ \ln w_n}\right) \cdot \left(J^b_{K_m,\ln c_m}\right)^{-1} \left(rJ^b_{\ln c_m} - c_m \cdot K_m \right. \right. \right. \right. \\
&\left. \left. \left. \left. - \sum_{m' \neq m} \left(\dot{K}^b_{m'} \cdot J^b_{K_{m'},\ln c_m}\right) - \frac{1}{2} \cdot \Omega^b_{\ln c_m} \right) \right\} - \frac{1}{2} \cdot \Omega^b_{\ln w_n} \right] \right\} \\
&+ \sum_m \left(c_m \cdot K_m\right) + \sum_m \left\{ \frac{\tau_{K_m}}{\mu_m} \cdot J^b_{K_m} \cdot \left(J^b_{K_m,\ln c_m}\right)^{-1} \left(rJ^b_{\ln c_m} - c_m \cdot K_m \right. \right. \\
&\left. \left. - \sum_{m' \neq m} \left(\dot{K}^b_{m'} \cdot J^b_{K_{m'},\ln c_m}\right) - \frac{1}{2} \cdot \Omega^b_{\ln c_m} \right) \right\} + \frac{1}{2} \cdot \Omega^b
\end{aligned}
\tag{19}
$$

In equation (17) the optimized actual net investment demand under perfect efficiency is expressed in terms of $J^a_{K_m,\ln c_m}$, $J^a_{\ln c_m}$, $J^a_{K_{m'},\ln c_m}$ and $\Omega^a_{\ln c_m}$. To obtain these derivatives we make use of the optimized actual HJB equation which is expressed in terms of the behavioural value function in equation (19). Taking the derivatives of equation (19) with respect to $\ln c_m$, with respect to first K_m and then $\ln c_m$ and finally first with respect to $K_{m'}$ and then $\ln c_m$. These are then substituted into equation (17) yielding the m^{th} optimized actual net investment demand in terms of the behavioural value function under uncertainty. It is given by the following equation, note that we have ignored higher than second order derivatives of $J^b(\cdot)$ and in order to improve the readability we indicate the factor in use in bold letters.

$$
\begin{aligned}
&\Bigg\{ \frac{1}{r}\sum_n \left[\frac{\tau_{x_n}}{\lambda_n}\cdot\left(\left(J^b_{K_\mathbf{m},\ln c_\mathbf{m}}\right)^{-1}\cdot J^b_{K_\mathbf{m},\ \ln w_n}\cdot c_\mathbf{m}\right)\right] + \frac{1}{r}\left(1-\frac{\tau_{K_\mathbf{m}}}{\mu_\mathbf{m}}\right)c_\mathbf{m} \\
&\qquad +\sum_m\left[\frac{\tau_{K_m}}{\mu_m}\left(J^b_{\ln c_m,K_\mathbf{m}}\cdot\left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m,\ln c_\mathbf{m}} + \left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m,K_\mathbf{m}}\cdot J^b_{\ln c_m,\ln c_\mathbf{m}}\right)\right] \\
&\qquad -\frac{1}{r}\frac{\tau_{K_\mathbf{m}}}{\mu_\mathbf{m}}\cdot\left(\left(J^b_{K_\mathbf{m},\ln c_\mathbf{m}}\right)^{-1}\cdot J^b_{K_\mathbf{m}}\cdot c_\mathbf{m}\right) - \frac{1}{r}\left[\frac{\tau_{K_\mathbf{m}}}{\mu_\mathbf{m}}\cdot\left(\left(J^b_{K_\mathbf{m},\ln c_\mathbf{m}}\right)^{-1}\cdot J^b_{K_\mathbf{m},K_\mathbf{m}}\cdot\left(c_\mathbf{m}\cdot K_\mathbf{m}\right)\right)\right] \\
&\qquad -\frac{1}{2r}\sum_m\left[\frac{\tau_{K_m}}{\mu_m}\cdot\left(\Omega^b_{\ln c_m,K_\mathbf{m}}\cdot\left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m,\ln c_\mathbf{m}} + \left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m,K_\mathbf{m}}\cdot\Omega^b_{\ln c_m,\ln c_\mathbf{m}}\right)\right] \\
&+\frac{1}{2r}\cdot\Omega^b_{K_\mathbf{m},\ln c_\mathbf{m}}\Bigg\}\dot{K}^o_\mathbf{m} = \\
&\sum_n\frac{\tau_{x_n}}{\lambda_n}\cdot rJ^b_{\ln w_n,\ln c_\mathbf{m}} - \sum_n\left[\frac{\tau_{x_n}}{\lambda_n}\cdot r\sum_m\left(\left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m,\ \ln w_n}\cdot J^b_{\ln c_m,\ln c_\mathbf{m}}\right)\right] \\
&+\sum_n\left[\frac{\tau_{x_n}}{\lambda_n}\cdot\left(\left(J^b_{K_\mathbf{m},\ln c_\mathbf{m}}\right)^{-1}\cdot J^b_{K_\mathbf{m},\ \ln w_n}\cdot\left(c_\mathbf{m}\cdot K_\mathbf{m}\right)\right)\right] \\
&+\sum_n\left[\frac{\tau_{x_n}}{\lambda_n}\cdot\frac{1}{2}\cdot\sum_m\left(\left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m,\ \ln w_n}\cdot\Omega^b_{\ln c_m,\ln c_\mathbf{m}}\right)\right] - \sum_n\frac{\tau_{x_n}}{\lambda_n}\cdot\frac{1}{2}\cdot\Omega^b_{\ln w_n,\ln c_\mathbf{m}} \\
&+\sum_m\left[\frac{\tau_{K_m}}{\mu_m}\cdot r\left(\left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m}\cdot J^b_{\ln c_m,\ln c_\mathbf{m}} + J^b_{\ln c_m}\cdot\left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m,\ln c_\mathbf{m}}\right)\right] \\
&-\frac{\tau_{K_\mathbf{m}}}{\mu_\mathbf{m}}\cdot\left(\left(J^b_{K_\mathbf{m},\ln c_\mathbf{m}}\right)^{-1}\cdot J^b_{K_\mathbf{m}}\cdot\left(c_\mathbf{m}\cdot K_\mathbf{m}\right)\right) - \sum_m\left[\frac{\tau_{K_m}}{\mu_m}\cdot\left(\left(c_m\cdot K_m\right)\cdot\left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m,\ln c_\mathbf{m}}\right)\right] \\
&-\sum_m\left[\frac{\tau_{K_m}}{\mu_m}\cdot\left(\left(\sum_{m'\neq\mathbf{m}}\left(\dot{K}^b_{m'}\cdot J^b_{K_{m'},\ln c_\mathbf{m}}\right)\right)\cdot\left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m,\ln c_\mathbf{m}}\right)\right] \\
&-\sum_m\left[\frac{\tau_{K_m}}{\mu_m}\cdot\frac{1}{2}\cdot\left(\left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m}\cdot\Omega^b_{\ln c_m,\ln c_\mathbf{m}} + \Omega^b_{\ln c_m}\cdot\left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m,\ln c_\mathbf{m}}\right)\right] \\
&-\sum_{\substack{m'=1\\ \forall m'\neq m}}^{\bar{m}}\Bigg\{\dot{K}^o_{m'}\cdot\Bigg[\sum_m\left[\frac{\tau_{K_m}}{\mu_m}\cdot\left(J^b_{\ln c_m,K_{m'}}\cdot\left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m,\ln c_\mathbf{m}} + \left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m,K_{m'}}\cdot J^b_{\ln c_m,\ln c_\mathbf{m}}\right)\right] \\
&\qquad -\frac{1}{r}\frac{\tau_{K_{m'}}}{\mu_{m'}}\cdot\left(c_{m'}\cdot\left(J^b_{K_{m'},\ln c_{m'}}\right)^{-1}\cdot J^b_{K_{m'},\ln c_\mathbf{m}}\right) - \frac{1}{r}\left[\frac{\tau_{K_\mathbf{m}}}{\mu_\mathbf{m}}\cdot\left(\left(J^b_{K_\mathbf{m},\ln c_\mathbf{m}}\right)^{-1}\cdot J^b_{K_\mathbf{m},K_{m'}}\cdot\left(c_\mathbf{m}\cdot K_\mathbf{m}\right)\right)\right] \\
&\qquad -\frac{1}{2r}\cdot\sum_m\left[\frac{\tau_{K_m}}{\mu_m}\cdot\left(\Omega^b_{\ln c_m,K_{m'}}\cdot\left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m,\ln c_\mathbf{m}} + \left(J^b_{K_m,\ln c_m}\right)^{-1}\cdot J^b_{K_m,K_{m'}}\cdot\Omega^b_{\ln c_m,\ln c_\mathbf{m}}\right)\right] \\
&\qquad +\frac{1}{2r}\cdot\Omega^b_{K_{m'},\ln c_\mathbf{m}}\Bigg]\Bigg\}
\end{aligned}
\tag{20}
$$

Under perfect efficiency, the optimized actual variable input demand equation (18) is expressed in terms of $J^a_{\ln w_n}, J^a_{K_m,\ln w_n}$ and $\Omega^a_{\ln w_n}$. Note, these derivatives are obtained from optimized actual HJB equation as given in equation (19) that is differentiated with respect to $\ln w_n$ as well as with respect to K_m and $\ln w_n$. These are inserted into equation (18) yielding the optimized actual variable factor demand for the n^{th} input factor expressed in terms of the behavioural value function. Also here higher than second order derivatives of $J^b(\cdot)$ are ignored and we indicate the respective factor in use in bold letters to improve the readability.

$$
\begin{aligned}
x^o_{\boldsymbol{n}} = \frac{1}{w_{\boldsymbol{n}}} \cdot \Bigg\{ & r \cdot \sum_n \frac{\tau_{x_n}}{\lambda_n} \cdot J^b_{\ln w_n, \ln w_{\boldsymbol{n}}} - r \cdot \sum_n \Bigg[\frac{\tau_{x_n}}{\lambda_n} \cdot \sum_m \Big(\big(J^b_{K_m,\ln c_m}\big)^{-1} \cdot J^b_{K_m,\ln w_n} \cdot J^b_{\ln c_m, \ln w_{\boldsymbol{n}}} \Big) \Bigg] \\
& + \frac{1}{2} \cdot \sum_n \Bigg[\frac{\tau_{x_n}}{\lambda_n} \cdot \sum_m \Big(\big(J^b_{K_m,\ln c_m}\big)^{-1} \cdot J^b_{K_m,\ln w_n} \cdot \Omega^b_{\ln c_m, \ln w_{\boldsymbol{n}}} \Big) \Bigg] - \frac{1}{2} \cdot \sum_n \frac{\tau_{x_n}}{\lambda_n} \cdot \Omega^b_{\ln w_n, \ln w_{\boldsymbol{n}}} \\
& + r \cdot \sum_m \Bigg[\frac{\tau_{K_m}}{\mu_m} \cdot \Big(\big(J^b_{K_m,\ln c_m}\big)^{-1} \cdot J^b_{K_m} \cdot J^b_{\ln c_m, \ln w_{\boldsymbol{n}}} + J^b_{\ln c_m} \cdot \big(J^b_{K_m,\ln c_m}\big)^{-1} \cdot J^b_{K_m, \ln w_{\boldsymbol{n}}} \Big) \Bigg] \\
& - \sum_m \Bigg[\frac{\tau_{K_m}}{\mu_m} \cdot \Big((c_m \cdot K_m) \cdot \big(J^b_{K_m,\ln c_m}\big)^{-1} \cdot J^b_{K_m, \ln w_{\boldsymbol{n}}} \Big) \Bigg] \\
& - \sum_m \Bigg[\frac{\tau_{K_m}}{\mu_m} \cdot \Big(\sum_{m' \neq m} \big(\dot{K}^b_{m'} \cdot J^b_{K_{m'},\ln c_m} \big) \Big) \cdot \big(J^b_{K_m,\ln c_m}\big)^{-1} \cdot J^b_{K_m, \ln w_{\boldsymbol{n}}} \Bigg] \\
& - \frac{1}{2} \cdot \sum_m \Bigg[\frac{\tau_{K_m}}{\mu_m} \cdot \Big(\big(J^b_{K_m,\ln c_m}\big)^{-1} \cdot J^b_{K_m} \cdot \Omega^b_{\ln c_m, \ln w_{\boldsymbol{n}}} + \Omega^b_{\ln c_m} \cdot \big(J^b_{K_m,\ln c_m}\big)^{-1} \cdot J^b_{K_m, \ln w_{\boldsymbol{n}}} \Big) \Bigg] \\
& - \sum_{\boldsymbol{m}} \Bigg[\Bigg(\sum_m \Bigg[\frac{\tau_{K_m}}{\mu_m} \Big(J^b_{\ln c_m, K_{\boldsymbol{m}}} \cdot \big(J^b_{K_m,\ln c_m}\big)^{-1} \cdot J^b_{K_m, \ln w_{\boldsymbol{n}}} \Big) \Bigg] \Bigg) \cdot \dot{K}^o_{\boldsymbol{m}} \Bigg] \\
& - \sum_{\boldsymbol{m}} \Bigg[\Bigg(\sum_m \Bigg[\frac{\tau_{K_m}}{\mu_m} \Big(\big(J^b_{K_m,\ln c_m}\big)^{-1} \cdot J^b_{K_m, K_{\boldsymbol{m}}} \cdot J^b_{\ln c_m, \ln w_{\boldsymbol{n}}} \Big) \Bigg] \Bigg) \cdot \dot{K}^o_{\boldsymbol{m}} \Bigg] \\
& + \frac{1}{r} \cdot \sum_m \Bigg[\Bigg(\frac{\tau_{K_m}}{\mu_m} \cdot \Big(c_m \cdot \big(J^b_{K_m,\ln c_m}\big)^{-1} \cdot J^b_{K_m, \ln w_{\boldsymbol{n}}} \Big) \Bigg) \cdot \dot{K}^o_m \Bigg] \\
& + \frac{1}{2r} \cdot \sum_{\boldsymbol{m}} \Bigg[\Bigg(\sum_m \Bigg[\frac{\tau_{K_m}}{\mu_m} \cdot \Big(\Omega^b_{\ln c_m, K_{\boldsymbol{m}}} \cdot \big(J^b_{K_m,\ln c_m}\big)^{-1} \cdot J^b_{K_m, \ln w_{\boldsymbol{n}}} \Big) \Bigg] \Bigg) \cdot \dot{K}^o_{\boldsymbol{m}} \Bigg] \\
& + \frac{1}{2r} \cdot \sum_{\boldsymbol{m}} \Bigg[\Bigg(\sum_m \Bigg[\frac{\tau_{K_m}}{\mu_m} \cdot \Big(\big(J^b_{K_m,\ln c_m}\big)^{-1} \cdot J^b_{K_m, K_{\boldsymbol{m}}} \cdot \Omega^b_{\ln c_m, \ln w_{\boldsymbol{n}}} \Big) \Bigg] \Bigg) \cdot \dot{K}^o_{\boldsymbol{m}} \Bigg] \\
- \frac{1}{2r} \cdot \sum_m & \Big[\big(\Omega^b_{K_m, \ln w_{\boldsymbol{n}}} \big) \cdot \dot{K}^o_m \Big] \Bigg\}
\end{aligned}
\tag{21}
$$

For the subsequent steps, we need to normalize the factor prices by one of the variable factor prices to satisfy the property of linear homogeneity in prices of the cost function.

For this purpose, the shadow prices for the variable factors, defined as $w_n^b = \lambda_n w_n$ with $\lambda_1 w_1, \lambda_2 w_2, ..., \lambda_{\bar{n}} w_{\bar{n}}$, are redefined using the first variable factor price as a numeraire variable: $w_n^b = (\lambda_n w_n / \lambda_1 w_1)$ with $((\lambda_1 w_1 / \lambda_1 w_1), (\lambda_2 w_2 / \lambda_1 w_1), ..., (\lambda_{\bar{n}} w_{\bar{n}} / \lambda_1 w_1))$. In order to measure the deviation from the actual prices, the allocative efficiency (AE) parameter $\lambda_{\bar{n}1}$ is introduced and denotes the price distortions of the $\bar{n}^{th}$ variable factor relative to the 1st variable factor: $w_n^b = \lambda_{n1} w_{n1}$ with $(1, \lambda_{21} w_{21}, ..., \lambda_{\bar{n}1} w_{\bar{n}1})$. An estimate of the AE parameter for the variable factor demand $\lambda_{\bar{n}1} > 1 (< 1)$ means that the ratio of the shadow price of the $\bar{n}^{th}$ variable factor relative to the 1st variable factor is higher (lower) than the respective actual prices ratio. This implies under-use (over-use) of $\bar{n}^{th}$ variable factor in relation to the 1st (numeraire) variable factor (see Maietta (2000) or Rungsuriyawiboon and Stefanou (2007) for a similar procedure).

Based on the normalized optimization problem we derive the factor demand for the numeraire variable input. In order to achieve this, we single out the numeraire variable in the optimized version of the behavioural HJB equation (9):

$$rJ^b(\lambda_n w_n, c_m, K_m, y) = x_1^b + \sum_{n=2}^{\bar{n}} (w_n^b \cdot x_n^b) + \sum_m (c_m \cdot K_m) + \sum_m (J_{K_m}^b \cdot \dot{K}_m^b) + \frac{1}{2} \cdot \Omega^b \tag{22}$$

where x_1^b denotes the behavioural demand for the numeraire variable factor and x_n^b is the behavioural demand for the other variable factors. The conditional behavioural demand for the numeraire variable factor under uncertainty can then be expressed as follows:

$$x_1^b = rJ^b - \sum_{n=2}^{\bar{n}} (w_n^b \cdot x_n^b) - \sum_m (c_m \cdot K_m) - \sum_m (J_{K_m}^b \cdot \dot{K}_m^b) - \frac{1}{2} \cdot \Omega^b \tag{23}$$

Accordingly, the optimized actual (observed) demand for the numeraire variable factor in the presence of uncertainty reads as

$$\begin{aligned} x_1^o &= \tau_{x_1} \cdot (x_1^b) \\ &= \tau_{x_1} \cdot \left(rJ^b - \sum_{n=2}^{\bar{n}} (w_n^b \cdot x_n^b) - \sum_m (c_m \cdot K_m) - \sum_m (J_{K_m}^b \cdot \dot{K}_m^b) - \frac{1}{2} \cdot \Omega^b \right) \end{aligned} \tag{24}$$

3.3 Specification of the value function

The derivation of estimable decision rules from equations (13), (21) and (24) requires a choice of the functional form for the behavioural value function J^b and the drift function of the GBM α. Since we refer to stochastic transition equations in our cost minimization problem it is not possible to directly refer to the approach presented by Rungsuriyawiboon and Stefanou (2007) for measuring dynamic efficiency. As shown by Pietola and Myers (2000) the convexity properties of the cost function require fourth order curvature properties on the value function. Otherwise the output and price uncertainty would have no influence on the optimal decision rules. According to Pietola and Myers (2000), the functional form of the value function needs to fulfil the following properties[6]:

i. J is concave in (w,c),

ii. J_K is linear in (w,c),

iii. J_{zz} is linear in (w,c),

iv. α is non-increasing and convex in (w,c).

Past studies frequently used a simple quadratic approximation of the value function even though it is not flexible. Flexibility can be attained by adding additional terms and parameters (cf. Epstein 1981) and is required to achieve theoretical consistency of the parameter estimates. Thus we suggest the following specification for the value function satisfying the four aforementioned properties:

$$J^b(\mathbf{z},\mathbf{K}) = a_0 + \begin{pmatrix} b'_K & b'_y & b'_w & b'_c \end{pmatrix} \begin{pmatrix} \mathbf{K} \\ \ln \mathbf{y} \\ \ln \mathbf{w^b} \\ \ln \mathbf{c} \end{pmatrix}$$

$$+\frac{1}{2}\begin{pmatrix} \mathbf{K}' & (\ln \mathbf{y})' & (\ln \mathbf{w^b})' & (\ln \mathbf{c})' \end{pmatrix} \begin{bmatrix} A_{KK} & A'_{yK} & 0 & 0 \\ A_{yK} & A_{yy} & A'_{wy} & A'_{cy} \\ 0 & A_{wy} & A_{ww} & A'_{cw} \\ 0 & A_{cy} & A_{cw} & A_{cc} \end{bmatrix} \begin{pmatrix} \mathbf{K} \\ \ln \mathbf{y} \\ \ln \mathbf{w^b} \\ \ln \mathbf{c} \end{pmatrix} \qquad (25)$$

$$+\mathbf{c}'\mathbf{M}^{-1}\mathbf{K} + \mathbf{w^b}'\mathbf{A}_{wK}\mathbf{K}$$

Therein refers $\mathbf{K}$ to the $(\overline{m}\times 1)$ vector of quasi-fixed inputs, $\mathbf{y} = (\ln y)$ denotes the scalar of the log output and a_0 is an unknown constant scalar parameter. Note, in the normalised specification of the long-run cost function we considered the 1st variable input price as a numeraire. Now we consider all variable input prices $\mathbf{w^b}$ as normalized. Accordingly, $\ln \mathbf{w^b}$

6 A detailed proof can be found in Pietola and Myers (2000: 966).

denotes the $(\bar{n}\times 1)$ vector containing the logarithms of the normalized variable input prices and $\ln\mathbf{c}$ denotes the $(\bar{m}\times 1)$ price vector of the logarithmic prices for the quasi-fixed factors.

The elements of the vector $\mathbf{b}'=\begin{pmatrix} b_K^{'} & b_y^{'} & b_w^{'} & b_c^{'}\end{pmatrix}_{(1\times(2\bar{m}+\bar{n}+1))}$ represent the first order parameters of the respective Taylor series expansion of the value function. Thereby denotes $b_K^{'}$ the parameters for the elements of $\mathbf{K}$ with dimension $(\bar{m}\times 1)$, analogously refers $b_y^{'}$ to the parameter vector with respect to log output and reduces to a scalar. $b_w^{'}$ and $b_c^{'}$ contain the input price parameters and are of dimension $(\bar{n}\times 1)$ and $(\bar{m}\times 1)$, respectively. Matrix $\mathbf{A}$ is defined as $\mathbf{A}=\begin{bmatrix} A_{KK} & A'_{yK} & 0 & 0 \\ A_{yK} & A_{yy} & A'_{wy} & A'_{cy} \\ 0 & A_{wy} & A_{ww} & A'_{cw} \\ 0 & A_{cy} & A_{cw} & A_{cc}\end{bmatrix}_{((2\bar{m}+\bar{n}+1)\times(2\bar{m}+\bar{n}+1))}$ and contains the second order parameters where the elements except A_{yy} are itself matrices. A_{KK} is a symmetric $(\bar{m}\times\bar{m})$-matrix, A_{yK} is a $(1\times\bar{m})$-dimensional vector, A_{yy} reduces to a scalar, A_{wy} and A_{cy} are the vectors of dimension $(\bar{n}\times 1)$ and $(\bar{m}\times 1)$ respectively. A_{ww} and A_{cc} are symmetric matrices of dimension $(\bar{n}\times\bar{n})$ and $(\bar{m}\times\bar{m})$, respectively, and finally, A_{cw} is a $(\bar{m}\times\bar{n})$-matrix. The zero restrictions in matrix $\mathbf{A}$ guarantee that J_K and J_{zz} are linear in the quasi-fixed input prices (see property ii. and iii.).

In the stochastic model compared to deterministic models, we require additional restrictions, namely J_z to be quadratic in $(\mathbf{w},\mathbf{c})$ and J_{zz} to be linear in $(\mathbf{w},\mathbf{c})$ (see property iii). This is ensured by adding the last two terms $\mathbf{c}'\mathbf{M}^{-1}\mathbf{K}$ and $\mathbf{w}^{\mathbf{b}'}\mathbf{A}_{Kw}\mathbf{K}$. Herein $\mathbf{M}$ is a symmetric $(\bar{m}\times\bar{m})$-matrix of so called adjustment rates which can be interpreted as costs or losses in production attached to an adjustment in the stock of quasi-fixed factors and represent indirectly adjustment costs. $\mathbf{A}_{Kw}$ is a $(\bar{m}\times\bar{n})$-matrix.

Using this specification of the value function we are able to derive the factor demand equations that may serve as a starting point for an empirical specification. The m^{th} optimized actual net investment demand using the behavioural value function as given by equation (13) is now specified in terms of the value function (25) as follows:

$$\begin{aligned}\dot{K}_{\boldsymbol{m}}^{o}=\tau_{K_{\boldsymbol{m}}}\cdot\left(M_{c_{\boldsymbol{m}}K_{\boldsymbol{m}}}\cdot c_{\boldsymbol{m}}^{-1}\right)\cdot\Bigg\{ r\Bigg[b_{c_{\boldsymbol{m}}}+A_{c_{\boldsymbol{m}}y}\cdot\ln y+\sum_{n=2}^{\bar{n}}A_{c_{\boldsymbol{m}}w_n}\cdot\ln w_n^b+\sum_{m=1}^{\bar{m}}A_{c_{\boldsymbol{m}}c_m}\cdot\ln c_m \\ +c_{\boldsymbol{m}}\cdot\left(\sum_{m=1}^{\bar{m}}M_{c_{\boldsymbol{m}}K_m}^{-1}\cdot K_m\right)\Bigg]-c_{\boldsymbol{m}}\cdot K_{\boldsymbol{m}} \\ -\sum_{m'=1,m'\neq\boldsymbol{m}}^{\bar{m}}\dot{K}_{m'}^{b}\cdot\left(M_{c_{\boldsymbol{m}}K_{m'}}^{-1}\cdot c_{\boldsymbol{m}}\right)-\frac{1}{2}\cdot\left(\sum_{m=1}^{\bar{m}}M_{c_{\boldsymbol{m}}K_m}^{-1}\cdot K_m\right)\cdot c_{\boldsymbol{m}}\cdot\sigma_{\ln c_{\boldsymbol{m}}}^{2}\Bigg\}\end{aligned}\tag{26}$$

where τ_{K_m} denotes the technical inefficiency parameter of the m^{th} quasi-fixed input. Note, also here we mark the factor in use boldly to improve the readability. $M_{c_m K_m}$ represents the diagonal elements of the matrix $\mathbf{M}$ and $\sum_{m=1}^{\bar{m}} M_{c_m K_m}^{-1}$ represents the elements of the m^{th} row of the inverse of matrix $\mathbf{M}$. $\sigma_{\ln c_m}^2$ denotes the variance term of the respective m^{th} quasi-fixed input price.

Accordingly, the specified version of the n^{th} optimized actual variable input demand function using the behavioural value function (21) is given by (note, also here we use the bold index to improve the readability):

$$
\begin{aligned}
x_n^o = \frac{1}{w_n} \cdot \Bigg\{ & r \cdot \frac{\tau_{x_n}}{\lambda_n} \cdot \left(A_{w_n w_n} + \sum_{m=1}^{\bar{m}} \left(A_{w_n K_m} \cdot K_m \right) \cdot w_n^b \right) - \sum_{n=2}^{\bar{n}} \left[\frac{\tau_{x_n}}{\lambda_n} \cdot r \sum_{m=1}^{\bar{m}} \left(\frac{M_{c_m K_m}}{c_m} \cdot \left(A_{w_n K_m} \cdot w_n^b \right) \cdot A_{c_m w_n} \right) \right] \\
& - \frac{1}{2} \cdot \frac{\tau_{x_n}}{\lambda_n} \cdot \left(\sum_{m=1}^{\bar{m}} A_{w_n K_m} \cdot K_m \right) \cdot w_n^b \cdot \sigma_{\ln w_n}^2 + r \cdot \sum_{m=1}^{\bar{m}} \left[\frac{\tau_{K_m}}{\mu_m} \cdot \left(\frac{M_{c_m K_m}}{c_m} \right) \cdot \right. \\
& \left. \left(b_{K_m} + \sum_{m=1}^{\bar{m}} \left(A_{K_m K_m} \cdot K_m \right) + A_{y K_m} \cdot \ln y + \sum_{m=1}^{\bar{m}} \left(M_{c_m K_m}^{-1} \cdot c_m \right) + \sum_{n=2}^{\bar{n}} \left(A_{w_n K_m} \cdot w_n^b \right) \right) \cdot A_{c_m w_n} \right] \\
& + r \cdot \sum_{m=1}^{\bar{m}} \left[\frac{\tau_{K_m}}{\mu_m} \cdot \left(b_{c_m} + A_{c_m y} \cdot \ln y + \sum_{n=2}^{\bar{n}} \left(A_{c_m w_n} \cdot \ln w_n^b \right) + \sum_{m=1}^{\bar{m}} \left(A_{c_m c_m} \cdot \ln c_m \right) \right. \right. \\
& \left. \left. + c_m \cdot \sum_{m=1}^{\bar{m}} \left(M_{c_m K_m}^{-1} \cdot K_m \right) \right) \cdot \left(\frac{M_{c_m K_m}}{c_m} \right) \cdot \left(A_{w_n K_m} \cdot w_n^b \right) \right] \\
& - \sum_{m=1}^{\bar{m}} \left[\frac{\tau_{K_m}}{\mu_m} \cdot \left(K_m \cdot M_{c_m K_m} \cdot \left(A_{w_n K_m} \cdot w_n^b \right) \right) \right] \\
& - \sum_{m=1}^{\bar{m}} \left[\frac{\tau_{K_m}}{\mu_m} \cdot \left(\sum_{m'=1, m' \neq m}^{\bar{m}} \dot{K}_{m'}^b \cdot M_{c_{m'} K_m}^{-1} \right) \cdot M_{c_m K_m} \cdot \left(A_{w_n K_m} \cdot w_n^b \right) \right] \\
& - \frac{1}{2} \cdot \sum_{m=1}^{\bar{m}} \left[\frac{\tau_{K_m}}{\mu_m} \cdot \left(\sum_{m=1}^{\bar{m}} M_{c_m K_m}^{-1} \cdot K_m \right) \sigma_{\ln c_m}^2 \cdot M_{c_m K_m} \cdot \left(A_{w_n K_m} \cdot w_n^b \right) \right] \\
& - \sum_{m=1}^{\bar{m}} \left[\sum_{m=1}^{\bar{m}} \left[\frac{\tau_{K_m}}{\mu_m} \cdot \left(\left(c_m \cdot M_{c_m K_m}^{-1} \right) \cdot \left(A_{w_n K_m} \cdot w_n^b \right) + A_{K_m K_m} \cdot A_{c_m w_n} \right) \cdot \frac{M_{c_m K_m}}{c_m} \right] \cdot \dot{K}_m^o \right] \\
& + \frac{1}{r} \cdot \sum_{m=1}^{\bar{m}} \left[\frac{\tau_{K_m}}{\mu_m} \cdot \left(M_{c_m K_m} \cdot \left(A_{w_n K_m} \cdot w_n^b \right) \right) \cdot \dot{K}_m^o \right] \\
& + \frac{1}{2r} \cdot \sum_{m=1}^{\bar{m}} \left[\left[\sum_{m=1}^{\bar{m}} \left(\frac{\tau_{K_m}}{\mu_m} \cdot \left(M_{c_m K_m}^{-1} \cdot c_m \cdot \sigma_{\ln c_m}^2 \right) \cdot \frac{M_{c_m K_m}}{c_m} \cdot \left(A_{w_n K_m} \cdot w_n^b \right) \right) \right] \cdot \dot{K}_m^o \right] \\
& - \frac{1}{2r} \cdot \sum_{m=1}^{\bar{m}} \left[\left(A_{w_n K_m} \cdot w_n^b \cdot \sigma_{\ln w_n}^2 \right) \cdot \dot{K}_m^o \right] \Bigg\}
\end{aligned}
\tag{27}
$$

where τ_{x_n}, λ_n are technical and allocative inefficiency parameters of the n^{th} variable input, $\sum_{m=1}^{\bar{m}} M_{c_m K_m}$ represents the diagonal elements of matrix $\mathbf{M}$ and the technical and allocative inefficiency parameters of the quasi-fixed inputs are given by $\sum_{m=1}^{\bar{m}} \left(\tau_{K_m} / \mu_m \right)$. $\sum_{m=1}^{\bar{m}} M_{c_m K_m}^{-1}$ denotes the diagonal elements of the inverse of matrix $\mathbf{M}$, $\sum_{m=1}^{\bar{m}} \sigma_{\ln c_m}^2$ is the variance term of the respective quasi-fixed input prices, $\dot{K}_m^o$ denotes the optimized actual net investment demand for the m^{th} factor, $\sigma_{\ln w_n}^2$ represents the variance of the n^{th} variable input price and $\sum_{m'=1, m' \neq m}^{\bar{m}} \dot{K}_{m'}^b$ denotes the behavioural net investment demand function for quasi-fixed inputs other than m.

Finally the specified version of the optimized actual demand for the numeraire variable input (24) is given by

$$
\begin{aligned}
x_1^o = \tau_{x_1} \cdot r \Bigg\{ & a_0 + \sum_{m=1}^{\bar{m}} \left(b_{K_m} \cdot K_m \right) + b_y \cdot \ln y + \sum_{n=2}^{\bar{n}} \left(b_{w_n} \cdot \ln w_n^b \right) + \sum_{m=1}^{\bar{m}} \left(b_{c_m} \cdot \ln c_m \right) \\
& + \frac{1}{2} \cdot \sum_{m=1}^{\bar{m}} \left(\sum_{m=1}^{\bar{m}} K_m \cdot A_{K_m K_m} \cdot K_m \right) + \sum_{m=1}^{\bar{m}} K_m \cdot A_{yK_m} \cdot \ln y \\
& + \frac{1}{2} \cdot \ln y \cdot A_{yy} \cdot \ln y + \sum_{n=2}^{\bar{n}} \ln y \cdot A_{w_n y} \cdot \ln w_n^b \\
& + \frac{1}{2} \cdot \sum_{n=2}^{\bar{n}} \left(\sum_{n=2}^{\bar{n}} \ln w_n^b \cdot A_{w_n w_n} \cdot \ln w_n^b \right) + \sum_{m=1}^{\bar{m}} \ln y \cdot A_{c_m y} \cdot \ln c_m \\
& + \sum_{n=2}^{\bar{n}} \left(\sum_{m=1}^{\bar{m}} \ln w_n^b \cdot A_{c_m w_n} \cdot \ln c_m \right) + \frac{1}{2} \cdot \sum_{m=1}^{\bar{m}} \left(\sum_{m=1}^{\bar{m}} \ln c_m \cdot A_{c_m c_m} \cdot \ln c_m \right) \\
& + \sum_{m=1}^{\bar{m}} \left(\sum_{m=1}^{\bar{m}} c_m \cdot M_{c_m K_m}^{-1} \cdot K_m \right) + \sum_{n=2}^{\bar{n}} \left(\sum_{m=1}^{\bar{m}} w_n^b \cdot A_{w_n K_m} \cdot K_m \right) \Bigg\} \\
- \tau_{x_1} \sum_{n=2}^{\bar{n}} & \left(w_n^b \cdot \frac{x_n^o}{\tau_{x_n}} \right) - \tau_{x_1} \cdot \sum_{m=1}^{\bar{m}} \left(c_m \cdot K_m \right) - \tau_{x_1} \cdot \sum_{m} \Bigg\{ \Bigg(b_{K_m} + \sum_{m=1}^{\bar{m}} \left(A_{K_m K_m} \cdot K_m \right) \\
& + A_{yK_m} \cdot \ln y + \sum_{m=1}^{\bar{m}} \left(M_{c_m K_m}^{-1} \cdot c_m \right) + \sum_{n=2}^{\bar{n}} \left(A_{w_n K_m} \cdot w_n^b \right) \Bigg) \cdot \frac{\dot{K}_m^o}{\tau_{K_m}} \Bigg\} \\
- \tau_{x_1} \cdot \frac{1}{2} \cdot \Bigg[& A_{yy} \cdot \sigma_{\ln y}^2 + 2 \cdot \sum_{n=2}^{\bar{n}} A_{w_n y} \cdot \sigma_{\ln w_n, \ln y} + 2 \cdot \sum_{m=1}^{\bar{m}} A_{c_m y} \cdot \sigma_{\ln c_m, \ln y} \\
& + \sum_{n=2}^{\bar{n}} \left(A_{w_n w_n} \cdot \sigma_{\ln w_n}^2 + \sum_{m=1}^{\bar{m}} \left(A_{w_n K_m} \cdot K_m \right) \cdot w_n^b \cdot \sigma_{\ln w_n}^2 \right) \\
& + \sum_{n=2}^{\bar{n}} \left(\sum_{m=1}^{\bar{m}} A_{c_m w_n} \cdot \sigma_{\ln c_m, \ln w_n} \right) + \sum_{m=1}^{\bar{m}} \left(\sum_{n=2}^{\bar{n}} A_{c_m w_n} \cdot \sigma_{\ln c_m, \ln w_n} \right) \\
& + \sum_{m=1}^{\bar{m}} \left(A_{c_m c_m} \cdot \sigma_{\ln c_m}^2 + \sum_{m=1}^{\bar{m}} \left(M_{c_m K_m}^{-1} \cdot K_m \right) \cdot c_m \cdot \sigma_{\ln c_m}^2 \right) \Bigg]
\end{aligned}
\tag{28}
$$

where $\sigma^2_{\ln y}$ is the variance of the logarithmic output, $\sigma_{\ln w_n, \ln y}$ and $\sigma_{\ln c_m, \ln y}$ are the respective covariances of the output and input prices. Similarly denotes $\sigma_{\ln c_m, \ln w_n}$ the covariance of the quasi-fixed and variable input prices.

3.4 Hypotheses

The main motivation of the derivation of stochastic factor demand equations was the conjecture that uncertainty affects the optimal factor demand which, in turn, might have an impact on estimates of a firm's efficiency. Equations (26) to (28) actually reveal the importance of factor price uncertainty in this context. To be specific, the negative sign of the last term in equation (26) indicates that volatility in prices of quasi-fixed factors, $\sigma^2_{\ln c_m}$, reduces optimal investment (i.e. increases disinvestments). A negative investment-uncertainty relationship was also derived by Dixit and Pindyck (1994) and was empirically confirmed, for example, by Pietola and Myers (2000) and Hinrichs et al. (2008). It is not straightforward to deduce how this impact of uncertainty will affect the parameter estimates of technical and allocative efficiency. Clearly, there will be an omitted variable bias with respect to the inefficiency parameters if uncertainty is not included in an econometric model, but its direction and magnitude are hard to tell a priori. However, an intuitive conjecture is that ignoring uncertainty leads to an overestimation of inefficiency parameters. The reason is that actual capital stocks spuriously appear too small (or too large in the case of disinvestment) if the optimal speed of adjustment is overestimated. Referring to Figure 1 this means that the optimal adjustment path is shifted downwards due to uncertainty.

Equation (27) reveals that two different sources of uncertainty play a role for the variable factor demand, namely the variance of the quasi-fixed input price and the variable input prices. The effect of $\sigma^2_{\ln c_m}$ is apparently ambiguous. In contrast, the effect of $\sigma^2_{\ln w_n}$ is again negative, provided that net investment is nonnegative. In the case of disinvestments this negative effect is dampened.

Inspection of equation (28) shows the variances of the (log) output and the (log) prices of variable and quasi fixed factors have a negative impact on the demand of the numeraire variable input. This effect can be either amplified or attenuated by positive or negative covariances between the stochastic variable, so that it is difficult to affect the net effect of uncertainty on the factor demand.

4 Concluding remarks

Summing up we have derived a parametric model of dynamic efficiency in a shadow cost framework. Our model extends existing approaches since it accounts for non-static expectations of factor prices. We provide stochastic factor demand equations which can serve as a starting point for the econometric estimation of technical and allocative inefficiency as well as short run and long run input demand elasticities. Nevertheless, one has to admit that much of the complexity of recent investment models cannot be captured by our approach. For example, adjustment costs are expressed by simple adjustment rates that may be transformed into a linear accelerator model though it has been emphasized in the investment literature that more sophisticated adjustment cost functions are required for an appropriate specification of investment demand functions (cf. Huettel et al. 2010). We suggest this cross fertilization of investment models and efficiency models a direction for future research.

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